

Asset Pricing with Heterogeneous Investors and Portfolio Constraints*

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June 2011

*I am especially grateful to Suleyman Basak and Anna Pavlova for extensive discussions and comments. I am also grateful to Gurdip Bakshi, Harjoat Bhamra, Sudipto Bhattacharya, Andrea Buraschi, Jerome Detemple (discussant), Bernard Dumas, Wayne Ferson, Vito Gala, Nicolae Gârleanu (discussant), Francisco Gomes, Christopher Hennessy, Steven Heston, Julien Hugonnier, Pete Kyle, Mark Loewenstein, Igor Makarov, Antonio Mele, Antonio Mello, Anthony Neuberger, Anna Obizhaeva, Stavros Panageas, Marcel Rindisbacher, Raman Uppal, Dimitri Vayanos, Pietro Veronesi, Fernando Zapatero and seminar participants at European Finance Association Meetings (Bergen), Western Finance Association Meetings (Santa Fe), Woolley Centre for the Study of Capital Markets Dysfunctionality Workshop (Toulouse), Boston University, Cornell University, Imperial College London, London Business School, London School of Economics, University of Lausanne, University of Maryland, University of Southern California, University of Toronto, University of Warwick, and University of Zurich for helpful comments. The financial support from the Paul Woolley Centre at LSE is gratefully acknowledged. All errors are my responsibility. The previous version of this paper was circulated under the title “Asset Pricing in General Equilibrium with Constraints.”

Asset Pricing with Heterogeneous Investors and Portfolio Constraints

Abstract

We evaluate the impact of portfolio constraints on financial markets in a dynamic equilibrium pure exchange economy with one consumption good and two CRRA investors that may differ in risk aversions, beliefs regarding the dividend process and portfolio constraints. Despite numerous applications, portfolio constraints are notoriously difficult to incorporate into dynamic equilibrium analysis without the restrictive assumption of logarithmic preferences. We provide a tractable solution method that yields new insights on the asset pricing implications of portfolio constraints such as limited stock market participation, margin requirements and short sales prohibition without restricting risk aversion parameters. We demonstrate that in a setting where one investor is unconstrained while the other faces an upper bound constraint on the proportion of wealth that can be invested in stocks the model generates countercyclical market prices of risk and stock return volatilities, procyclical price-dividend ratios, excess volatility and other patterns consistent with empirical findings. In a setting with margin requirements we demonstrate that under plausible parameters tighter constraints decrease stock return volatilities during the times when the constraints are likely to bind.

Journal of Economic Literature Classification Numbers: D52, G12.

Keywords: asset pricing, dynamic equilibrium, heterogeneous investors, portfolio constraints, stock return volatility.

Portfolio constraints and market frictions have long been considered among key contributors towards understanding investor behavior and equilibrium asset prices. In particular, dynamic equilibrium models with heterogeneous investors facing portfolio constraints have extensively been employed by financial economists to confront a wide range of phenomena such as the equity premium puzzle, mispricing of redundant assets, role of arbitrageurs, impact of heterogeneous beliefs on asset prices, and stock comovements [e.g., among others, Detemple and Murthy (1997); Basak and Cuoco (1998); Basak and Croitoru (2000, 2006); Gallmeyer and Hollifield (2008); Pavlova and Rigobon (2008)]. However, tractable characterizations of equilibria are only obtained assuming that a constrained investor has logarithmic preferences which simplifies the analysis at the cost of assuming investor's myopia. Despite recent developments in portfolio optimization, such as the duality method of Cvitanic and Karatzas (1992), portfolio constraints are notoriously difficult to incorporate into general equilibrium analysis as well as portfolio choice when constrained investors have more general preferences inducing hedging demands.

The assumption of logarithmic preferences is not innocuous and impedes the evaluation of the impact of constraints on stock prices and stock return volatilities. Thus, in economic settings with two logarithmic investors and single consumption good [e.g., Detemple and Murthy (1997); Basak and Cuoco (1998); Basak and Croitoru (2000, 2006)] stock prices and hence stock return volatilities are unaffected by constraints since the income and substitution effects perfectly offset each other. When the constrained investor is logarithmic, the volatility effects of constraints have been studied in specific settings where the other (unconstrained) investor has different preferences [e.g., Gallmeyer and Hollifield (2008)], which requires further justification. To the best of our knowledge this paper is the first to provide a tractable methodology for evaluating the impact of portfolio constraints on stock prices that allows for heterogeneous non-logarithmic investors and many different types of portfolio constraints such as limited stock market participation, margin requirements, borrowing and short-sale constraints.

Our solution method yields new insights on the impact of portfolio constraints on stock prices. Specifically, we focus on limited stock market participation and margin constraints, while other portfolio constraints can be addressed along the same lines. In various settings we investigate how the tightness of constraints affects equilibrium parameters and highlight the role of constraints in explaining empirically observed procyclical variation of price-dividend ratios and countercyclical variation of stock return volatilities (i.e., positive shocks to dividend growth rates lead to higher price-dividend ratios and lower stock return volatilities), excess volatility, negative correlation between price-dividend ratios and risk premia. Furthermore, we demonstrate that margin requirements significantly reduce the stock market volatility.

We solve for the equilibrium in a continuous-time pure exchange economy with one consumption good and two CRRA investors which differ in their risk aversions, beliefs regarding mean dividend growth rates and portfolio constraints. Our model provides a tractable asset pricing framework for evaluating the interactions of different types of investor heterogeneity and portfolio constraints. For general CRRA preferences and constraints we provide tractable characteriza-

tions of interest rates and market prices of risk which highlight the directions in which various portfolio constraints push the equilibrium parameters. Based on these results, we develop a solution method for the efficient computation of equilibria in economies with constraints. Specifically, we derive stock price-dividend ratios, stock return volatilities and other parameters in terms of wealth-consumption ratios that can be computed numerically via a simple iterative procedure with fast convergence. By employing this methodology we study the equilibrium with limited participation, and then with borrowing and margin constraints.

We start with the limited participation model where one investor is unconstrained while the other faces an upper bound $\bar{\theta} < 1$ on the proportion of wealth invested in stocks.¹ In this specific setting we assume that the investors have identical risk aversions and beliefs and differ only in portfolio constraints, which allows us to examine the pure effect of constraints and to eliminate unnecessary correlation between being constrained and having specific risk aversions and beliefs. First, we demonstrate that tighter constraints give rise to lower and procyclical interest rates, and higher and countercyclical market prices of risk, consistently with previous theoretical studies [e.g., Basak and Cuoco (1998)] and empirical literature [e.g., Ferson and Harvey (1991)]. We also show that bad dividend shocks shift the distribution of consumption from the unconstrained to the constrained investor since the latter is less exposed to stock market fluctuations.

The effect of constraints on price-dividend ratios and stock-return volatilities depends on the relative strength of classical income and substitution effects. When the intertemporal elasticity of substitution (IES) is greater than one and the substitution effect is stronger, price-dividend ratios decrease while stock return volatilities increase with tighter constraints, and vice versa when IES is less than one and the income effect is stronger.² The effects of constraints are more pronounced in bad times when dividends are hit by adverse shocks than in good times. Consequently, when the substitution effect is stronger our straightforward extension of classical Lucas economy [Lucas (1978)] generates procyclical price-dividend ratios, countercyclical stock return volatilities and risk premia, negatively correlated risk premia and price-dividend ratios, as well as excess stock return volatility, consistently with empirical findings [e.g., Shiller (1981); Campbell and Shiller (1988); Schwert (1989); Campbell and Cochrane (1999)].

To understand the intuition we note that the investment opportunities of the constrained investor deteriorate with tighter constraints since the interest rate falls and the investor is unable

¹Srinivas, Whitehouse and Yermo (2000) show that limits on both domestic and foreign equity holdings of pension funds are in place in a number of OECD countries such as Germany (30% on EU and 6% on non-EU equities), Switzerland (30% on domestic and 25% on foreign equities) and Japan (30% on domestic and 30% on foreign equities), among others. Moreover, our approach allows to study the impact of passive investors that hold a fixed fraction of their wealth in stocks, as in Chien, Cole and Lustig (2008). Samuelson and Zeckhauser (1988) document the popularity of this strategy due to “status quo bias”, while Campbell (2006) points out that households may limit their stock market participation in stock market due to the lack of necessary skills. Important special case of our framework is stock market non-participation which in year 2002 accounted for 50% of U.S. households [e.g., Guvenen (2006, 2009)].

²When the investment opportunities worsen, the income effect induces investors to decrease consumption and save more while the substitution effect induces them to consume more and save less due to cheaper current consumption. For CRRA preferences with risk aversion γ , $IES=1/\gamma$, the income effect dominates for $IES < 1$ and the substitution effect dominates for $IES > 1$ while for $IES = 1$ both effects perfectly offset each other.

to benefit from the increase in the market price of risk, and the effects are stronger in bad times. As a result, the constrained investor's wealth-consumption ratio decreases when the substitution effect dominates and increases when the income effect dominates. Assuming further that the substitution effect dominates [as in Bansal and Yaron (2004)], price-dividend ratio decreases in bad times. This decrease is due to the fact that in a pure-exchange economy the price-dividend ratio equals the ratio of aggregate wealth to aggregate consumption which in bad times approximately equals the wealth-consumption ratio of the constrained investor since the latter holds a large fraction of the aggregate wealth and consumption. The effect of constraints is weaker in good times when the unconstrained investor dominates and hence the equilibrium is closer to the equilibrium in the unconstrained economy. Therefore, the tighter constraints decrease price-dividend ratios more in bad times than in good times, leading to a procyclical pattern. Furthermore, since by the definition of procyclicality both bad and good shocks change the price-dividend ratios and dividends in the same direction stocks are more volatile than the dividends and the excess volatility is larger in bad times.

We next evaluate the impact of borrowing and margin constraints when the investors can borrow up to a certain fraction of wealth or asset holdings, using stocks as a collateral. To make the constraints binding we allow the investors to differ both in risk aversions and beliefs regarding the mean dividend growth rate. Under our parametrization one investor is more optimistic and less risk averse than the other, and hence the latter never binds on the constraints. As in the case of limited participation the portfolio constraints decrease the interest rates and increase the market prices of risk since the tighter constraints decrease the demand for borrowing pushing the interest rates down, and increase the stock market exposure of the unconstrained investor pushing the market prices of risk up to provide the compensation for excessive risk taking.

For plausible model parameters the price-dividend ratios are procyclical and stock returns are more volatile than dividends, as in the limited participation case. We further demonstrate that tighter constraints reduce stock price volatilities, consistently with empirical evidence in Hardouvelis and Theodossiou (2002), and the effect is stronger in bad times when the constraints are binding. Intuitively, tighter constraints limit the investors' ability to trade on their heterogeneity. Therefore, their stock holdings look more homogeneous and the equilibrium parameters move closer to the values in a homogeneous investor economy. In particular, the stock return volatility decreases towards the volatility in the homogeneous investor economy, given by the volatility of dividends. Whether constraints are binding or not is determined by the amount of liquidity available for borrowing. The liquidity is supplied by the pessimist and hence its availability depends on the pessimist's share in aggregate wealth and consumption.

Finally, we study the probability density functions (p.d.f.) for the optimist's consumption share and investigate the long-run survival of the constrained and the unconstrained investors. In a setting where the optimist has correct beliefs regarding the consumption growth we show that in the course of time the pessimist's consumption share declines under our model parameters, even though it remains significant for very long periods of time. In the unconstrained benchmark

case where the investors differ in risk aversions and beliefs we provide a closed-form expression for the p.d.f. functions while the previous studies either generate them via Monte-Carlo simulations or study their asymptotic properties [Yan (2008); Bhamra and Uppal (2010); Cvitanić, Jouini, Malamud, and Napa (2010)]. Moreover, in the unconstrained benchmark economy we derive the equilibrium parameters in closed form in terms of familiar hypergeometric functions, widely employed in the literature. Our solution generalizes the solution in Longstaff and Wang (2008) that studies the equilibrium in an unconstrained economy where one investor is twice more risk averse than the other. Bhamra and Uppal (2010) provide an alternative closed-form characterization of the unconstrained equilibrium in terms of infinite series.

The methodological contribution of this paper is the tractable solution method that allows to compute the equilibrium in economies with heterogeneous investors that have different risk aversions, beliefs and constraints. The tractability of our solution method comes from the fact that it avoids solving the duality problem of Cvitanić and Karatzas (1992) which is difficult to solve unless the investor is logarithmic or stocks follow geometric Brownian motions [e.g. Cvitanić and Karatzas (1992)].³ First, following Cvitanić and Karatzas (1992) we derive optimal consumptions in terms of the state price densities in equivalent unconstrained fictitious economies in which the interest rates and market prices of risk are adjusted to account for the difference in investors' behavior in constrained economies. Then, market clearing for consumption yields expressions for equilibrium parameters in terms of the adjustments that solve a fixed point problem. The adjustments to interest rates and market prices of risk can be derived in terms of investors' wealth-consumption ratios that satisfy a system of quasilinear Hamilton-Jacobi-Bellman equations. We solve this system of equations numerically via a simple iterative procedure with fast convergence that requires solving a simple system of linear equations at each step.

There is a growing literature studying dynamic equilibria in continuous-time economies with heterogeneous investors and portfolio constraints assuming that constrained investors have logarithmic preferences. Basak and Cuoco (1998), Detemple and Murthy (1997), Basak and Croitoru (2000, 2006) present equilibrium models with constrained logarithmic investors, heterogeneous beliefs and portfolio constraints. They derive closed-form expressions for the equilibrium parameters but in contrast to our work all the above papers do not find the impact of constraints on stock prices and their moments. Kogan, Makarov and Uppal (2007) derive equilibrium parameters in an economy with a logarithmic investor facing no-borrowing constraint and find that all equilibrium parameters are time-deterministic. When little borrowing is permitted they numerically find interest rates and market prices of risk as functions of wealth distributions but do not consider the volatilities of stock returns. Hugonnier (2008) considers a limited participation model with two logarithmic investors similar model and shows that under restricted participation the stock prices implied by market clearing may contain a bubble and in the setting with multiple stocks the equilibrium might not be unique.

³Detemple and Rindisbacher (2005) provide a methodology for solving portfolio choice problems in a partial equilibrium setting by characterizing Cvitanić- Karatzas adjustments as solutions of backward stochastic equations.

Gallmeyer and Hollifield (2008) study the asset pricing with short-sale constraints in the presence of heterogeneous beliefs when the pessimist and the optimist have logarithmic and CRRA utilities respectively. They derive asset-pricing implications of constraints by employing Monte-Carlo simulations. Gârleanu and Pedersen (2009) study the deviations of asset prices from the law of one price in a model with CRRA and logarithmic investors facing margin requirements. In contrast to these works, our model does not rely on the restrictive assumption of logarithmic preferences or specific portfolio constraints and provides the implications of portfolio constraints for explaining dynamic patterns of price-dividend ratios, stock return volatilities, market prices of risk and other equilibrium parameters. Guvenen (2009) studies restricted participation in a model with heterogeneous investors with Epstein-Zin preferences and demonstrates that the equilibrium parameters in a simulated economy are consistent with empirical findings. In contrast to Guvenen (2009) our model allows for more general constraints, investigates the impact of the tightness of constraints on the equilibrium parameters, and provides a tractable analysis in terms of exact solutions rather than simulations.

Other related works include Dumas and Maenhout (2002), He and Krishnamurthy (2008), Wu (2008) that study various equilibrium models with specific preferences and constraints. Danielsson, Shin and Zigrand (2009) and Rytchkov (2009) study the equilibrium implications of risk management constraints. Gromb and Vayanos (2002, 2009), Brunnermeier and Pedersen (2009), Fostel and Geanakoplos (2008), Geanakoplos (2009) study the implications of margin requirements and risk management constraints for the liquidity provision, while Gromb and Vayanos (2010) survey the literature on limits of arbitrage. Pavlova and Rigobon (2008), and Schornick (2009) study models with constrained logarithmic investors in international finance model and multiple Lucas trees. Longstaff (2009) studies the asset pricing implications of illiquidity in the presence of multiple Lucas trees. Cuoco and He (2001), Coen-Pirani (2005), Guvenen (2006, 2009), Chien, Cole and Lustig (2008), Gomes and Michaelides (2008), Dumas and Lysaoff (2008) solve for equilibrium in various discrete-time incomplete market settings with CRRA and Epstein-Zin preferences. Our paper also contributes to growing literature that studies the equilibrium in the economies with heterogeneous unconstrained investors, such as Dumas (1989), Wang (1996), Zapatero (1998), Chan and Kogan (2002), Longstaff and Wang (2008), Yan (2008), Berrada, Hugonnier and Rindisbacher (2007), Gârleanu and Panageas (2008), Berrada (2009), Bhamra and Uppal (2009, 2010), Borovička (2009), Dumas, Kurshev and Uppal (2009), Weinbaum (2009), Xiong and Yan (2009), Cvitanić, Jouini, Malamud and Napp (2010), Cvitanić and Malamud (2010a, 2010b), Xiouros and Zapatero (2010).

The remainder of the paper is organized as follows. Section 1 provides the economic setup. In Section 2, we derive interest rates, market prices of risk and other equilibrium parameters and discuss their properties. In Section 3 we illustrate our solution method by computing the equilibrium in models with limited stock market participation, and with margin requirements. Section 4 concludes, Appendix A provides the proofs and Appendix B provides further details for our numerical method.

1. Economic Setup

We consider a continuous-time infinite horizon economy with two heterogeneous investors and one consumption good generated by a Lucas tree. The investors, in general, differ in their risk aversions, beliefs about the dividend growth rates, and portfolio constraints. In this Section we discuss the information structure of the economy, the investors' optimization, and define the equilibrium.

1.1. Information Structure and Securities Market

The uncertainty is represented by a filtered probability space $(\Omega, \{\mathcal{F}_t\}, \mathbb{P})$, on which is defined a Brownian motion w . The stochastic processes are adapted to the filtration $\{\mathcal{F}_t, t \in [0, \infty)\}$ generated by w . There are two heterogeneous investors that trade continuously in two securities, a riskless bond in zero net supply with instantaneous interest rate r_t and a stock in positive net supply, normalized to one unit. The stock is a claim to an exogenous strictly positive stream of dividends δ_t following a stochastic process:

$$d\delta_t = \delta_t[\mu_{\delta t}dt + \sigma_{\delta t}dw_t], \quad (1)$$

where $\mu_{\delta t}$ and $\sigma_{\delta t}$ in general can be stochastic. Even though the investors observe the same dividend process they have different beliefs about growth rate $\mu_{\delta t}$ but agree on the volatility $\sigma_{\delta t}$ which, unlike $\mu_{\delta t}$, is easier to estimate from the quadratic variation [e.g., Merton (1980)].⁴ We assume that one investor is optimistic ($i = o$) while the other is pessimistic ($i = p$), so that the optimist has a higher prior regarding growth rate $\mu_{\delta t}$ at the initial date.

Effectively, investors have their own probability spaces $(\Omega, \{\mathcal{F}_t^i\}, \mathbb{P}^i)$ endowed with *subjective probability measures* $\mathbb{P}^i$ which are equivalent to probability measure $\mathbb{P}$. The investors infer the information by observing a stream of dividends δ_t , so that their filtrations $\{\mathcal{F}_t^i\}$ coincide with the augmented filtration $\{\mathcal{F}_t^\delta\}$, generated by δ_t . From investor i 's perspective the dividends evolve as follows:

$$d\delta_t = \delta_t[\mu_{\delta t}^i dt + \sigma_{\delta t}dw_t^i], \quad i = o, p, \quad (2)$$

where w_t^i denotes a Brownian motion under the investor's probability measure $\mathbb{P}^i$, and $\mu_{\delta 0}^o \geq \mu_{\delta 0}^p$. If the investors update their beliefs in a Bayesian fashion parameters $\mu_{\delta t}^i$ are given by $\mu_{\delta t}^i = E^i[\mu_{\delta t} | \mathcal{F}_t^\delta]$, where $E^i[\cdot]$ denotes the expectation under the subjective probability measure. From Girsanov's Theorem and the filtering theory in Lipster and Shiryaev (1977) it follows that Brownian motions w_t^i are related as follows:

$$dw_t^o = dw_t^p - \Delta_{\delta t}dt, \quad \text{with } \Delta_{\delta t} = \frac{\mu_{\delta t}^o - \mu_{\delta t}^p}{\sigma_{\delta t}}. \quad (3)$$

⁴The assumption that the investors agree to disagree on dividend growth rates has been commonly made in the literature in the context of general equilibrium analysis [e.g., Detemple and Murthy (1997), Zapatero (1998), Basak (2000, 2005), Gallmeier and Hollifield (2008), Yan (2008), Xiong and Yan (2009), Bhamra and Uppal (2010)].

We consider equilibria in which bond prices, B_t , and stock prices, S_t , follow processes:

$$dB_t = B_t r_t dt, \quad (4)$$

$$\begin{aligned} dS_t + \delta_t dt &= S_t [\mu_t dt + \sigma_t dw_t], \\ &= S_t [\mu_t^i dt + \sigma_t dw_t^i], \quad i = o, p, \end{aligned} \quad (5)$$

where interest rate r_t , stock mean return μ_t , and volatility σ_t are stochastic processes determined in equilibrium, and bond price at time 0 is normalized so that $B_0 = 1$. The investors agree on the stock and bond prices but disagree on expected stock mean return. From (3) and (5) it can easily be shown that the relation (3) between Brownian motions w_t^o and w_t^p imposes the following *consistency condition* on the beliefs regarding stock and dividend growth rates:

$$\frac{\mu_t^o - \mu_t^p}{\sigma_t} = \frac{\mu_{\delta t}^o - \mu_{\delta t}^p}{\sigma_{\delta t}} \equiv \Delta_{\delta t}. \quad (6)$$

The consistency condition (6) allows us to formulate the following definition.

Definition 1. *Stochastic process $\Delta_{\delta t}$ which quantifies the disagreement on stock returns and dividend growth rates is called the disagreement process.*

1.2. Portfolio Constraints and Investors' Optimization

The optimist ($i = o$) is endowed with s units of stock and $-b$ units of bond, while the pessimist ($i = p$) is endowed with $1-s$ units of stock and b units of bond. The investors choose consumption, c_{it} , and an investment policy, $\{\alpha_{it}, \theta_{it}\}$, where α_{it} and θ_{it} denote the fractions of wealth invested in bonds and stocks, respectively, and hence, $\alpha_{it} + \theta_{it} = 1$. Investor i 's wealth process W_{it} evolves as:

$$dW_{it} = \left[W_{it} \left(r_t + \theta_{it} (\mu_t^i - r_t) \right) - c_{it} \right] dt + W_{it} \theta_{it} \sigma_t dw_t^i, \quad i = o, p, \quad (7)$$

and the investment policies are subject to portfolio constraints:

$$\theta_{it} \in \Theta_i = [\underline{\theta}_i, \bar{\theta}_i], \quad i = o, p. \quad (8)$$

The special cases of constraint (8) include restricted participation ($\underline{\theta} = 0, \bar{\theta} = 0$), limited participation ($\underline{\theta} = 0, \bar{\theta} < 1$), borrowing constraints ($\bar{\theta} \geq 1$), short-sale constraints ($\underline{\theta} = 0$), and margin requirement constraint $m_+ \theta^+ + m_- \theta^- \leq 1$ [e.g., Brunnermeier and Pedersen (2009)], where $m_{\pm} \geq 0$, $\theta^+ = \max(\theta, 0)$, and $\theta^- = -\min(\theta, 0)$.⁵

The investors have CRRA utility function $u_i(c)$ over consumption, given by:

$$u_i(c_t) = \frac{c_t^{1-\gamma_i}}{1-\gamma_i}, \quad i = o, p, \quad (9)$$

⁵For simplicity, we assume that $\underline{\theta}_i$ and $\bar{\theta}_i$ in (8) are constants. However, our solution method can also handle the case when these parameters are time-varying, e.g. depend on stock volatility σ_t , as in risk management constraints in Rytchkov (2009).

and are, in general, heterogeneous in risk aversion coefficients γ_i . The case $\gamma_i = 1$ corresponds to logarithmic utility $\ln(c_t)$. Each investor i ($i = o, p$) maximizes expected discounted utility over a stream of consumption c_{it} with time discount $\rho > 0$:

$$\max_{c_{it}, \theta_{it}} E \left[\int_0^\infty e^{-\rho t} u_i(c_{it}) dt \right], \quad (10)$$

subject to budget constraint (7), no-bankruptcy constraint $W_t \geq 0$, and portfolio constraints (8).

1.3. Equilibrium

Definition 2. *An equilibrium is a set of parameters $\{r_t, \mu_t^i, \sigma_t\}_{i \in \{o, p\}}$ and of consumption and investment policies $\{c_{it}^*, \alpha_{it}^*, \theta_{it}^*\}_{i \in \{o, p\}}$ such that consumption and investment policies solve dynamic optimization problem (10) for each investor, given price parameters $\{r_t, \mu_t^i, \sigma_t\}_{i \in \{o, p\}}$, and consumption and financial markets clear, i.e.,*

$$\begin{aligned} c_{ot}^* + c_{pt}^* &= \delta_t, \\ \alpha_{ot}^* W_{ot}^* + \alpha_{pt}^* W_{pt}^* &= 0, \\ \theta_{ot}^* W_{ot}^* + \theta_{pt}^* W_{pt}^* &= S_t, \end{aligned} \quad (11)$$

where W_{ot}^* and W_{pt}^* denote optimal wealths of the optimist and pessimist, respectively.

2. General Equilibrium with Constraints

This Section characterizes the equilibrium in economies with constrained investors. In Subsection 2.1 by employing the duality method of Karatzas and Cvitanic (1992) we recover expressions for equilibrium interest rates and market prices of risk. These expressions highlight the impact of risk-sharing and attitude towards risk on the equilibrium parameters. In Subsection 2.2 we characterize the remaining equilibrium parameters, such as price-dividend ratios and stock return volatilities, in terms of investors' wealth-consumption ratios that we derive numerically by solving a system of PDEs via an efficient numerical procedure.

2.1. Characterization of Equilibrium

We start by characterizing optimal consumptions of investors in a partial equilibrium setting in which the investment opportunities are taken as given, and then obtain the interest rate r_t , and the *market price of risk* κ_t , defined as Sharpe ratio $(\mu_t - r_t)/\sigma_t$, from the consumption clearing condition. Solving the optimization problem of a constrained investor is a challenging task even at a partial equilibrium level. We here follow the approach of Cvitanic and Karatzas (1992) and characterize constrained investors' optimal consumptions by embedding the partial

Table 1
Effective Domains and Support Functions

Case	Constraint	Υ	$f(\nu \nu \in \Upsilon)$
(a)	$\theta \in \mathbb{R}$	0	0
(b)	$\theta = 0$	$\mathbb{R}$	0
(c)	$\theta \leq \bar{\theta}, \bar{\theta} > 0$	$\nu \leq 0$	$-\nu\bar{\theta}$
(d)	$\theta \geq \underline{\theta}, \underline{\theta} < 0$	$\nu \geq 0$	$-\nu\underline{\theta}$
(e)	$\underline{\theta} \leq \theta \leq \bar{\theta}, \underline{\theta} \leq 0$	$\mathbb{R}$	$\max(-\nu, 0)\bar{\theta} - \max(\nu, 0)\underline{\theta}$

equilibrium economy into an equivalent *fictitious* complete-market economy with bond and stock prices following dynamics with adjusted parameters:

$$dB_t = B_t[r_t + f(\nu_{it}^*)]dt, \quad (12)$$

$$dS_t + \delta_t dt = S_t[(\mu_t^i + \nu_{it}^* + f_i(\nu_{it}^*))dt + \sigma_t dw_t^i], \quad i = o, p, \quad (13)$$

where $f_i(\nu)$ is the *support function* for the set of portfolio constraints Θ_i , defined as:

$$f_i(\nu) = \sup_{\theta \in \Theta_i} (-\nu\theta), \quad (14)$$

and ν_{it}^* solve so called *dual optimization problem*, defined in Cvitanić and Karatzas (1992), and lie in the *effective domains* for the support function, given by:

$$\Upsilon_i = \{\nu \in \mathbb{R} : f_i(\nu) < \infty\}. \quad (15)$$

As demonstrated in Cvitanić and Karatzas (1992) the constrained investor's optimal consumptions and investment strategies in the fictitious economy (12)–(13) coincide with those in the original constrained economy. Table 1 presents the effective domains and the support functions on the effective domains for various portfolio constraints.

It follows from the dynamics of bond and stock prices in fictitious economy (12)–(13) that the corresponding state prices of investors, ξ_{it} , evolve as:

$$d\xi_{it} = -\xi_{it}[r_{it}dt + \kappa_{it}dw_t^i], \quad i = o, p, \quad (16)$$

where r_{it} and κ_{it} denote the adjusted riskless rate and market price of risk in fictitious economy i , given by:

$$r_{it} = r_t + f_i(\nu_{it}^*), \quad \kappa_{it} = \frac{\mu_t^i - r_t}{\sigma_t} + \frac{\nu_{it}^*}{\sigma_t}, \quad i = o, p. \quad (17)$$

Throughout this Section we assume that the solutions to dual optimization problems exist and since the fictitious economies are complete, the marginal utilities of optimal consumption are given by [e.g., Huang and Pagés (1992); Cuoco (1997)]:

$$e^{-\rho t}(c_{it}^*)^{-\gamma_i} = \psi_i \xi_{it}, \quad i = o, p, \quad (18)$$

for some constants $\psi_i > 0$. As shown in Cvitanic and Karatzas (1992), optimal consumption and investment decisions of the constrained investor are equivalent to those of the unconstrained investor in the fictitious economy with adjusted parameters.

Next we characterize the equilibrium in terms of *adjustments* ν_{it}^* and the optimist's share in the aggregate consumption, defined as:

$$y_t = \frac{c_{ot}^*}{\delta_t}. \quad (19)$$

Since the investors are heterogeneous in utilities, beliefs, and constraints, the consumption share y_t is in general stochastic, following the dynamics:

$$\begin{aligned} dy_t &= -y_t[\mu_{yt}dt + \sigma_{yt}dw_t] \\ &= -y_t[\mu_{yt}^i dt + \sigma_{yt}dw_t^i], \quad i = o, p. \end{aligned} \quad (20)$$

To determine the interest rates and market prices of risk both in the original and fictitious economies we substitute optimal consumptions from first order condition (18) into consumption clearing condition in (11), apply Itô's Lemma to both sides and recover equilibrium parameters in terms of adjustments ν_{it}^* by matching the drift and volatility terms. Similarly, we derive the parameters of consumption share process y_t by substituting the optimal consumption c_{ot}^* from first order conditions (18) in equation (19) for y_t , applying Itô's Lemma to both sides and matching the terms.

This approach is similar to the approach in Basak and Cuoco (1998), Cuoco and He (2001), Basak (2000, 2005) among others that characterize the equilibrium in terms of the ratio of marginal utilities, which serves as a convenient state variable in their models. The following Proposition summarizes our results.

Proposition 1. *If there exists an equilibrium, interest rate r_t , market price of risk $\kappa_t = (\mu_t - r_t)/\sigma_t$, drift μ_{yt} and volatility σ_{yt} of optimistic investor's consumption share y_t in the original economy under true probability measure $\mathbb{P}$ are given by:*

$$\begin{aligned} r_t &= \rho + \Gamma_t \bar{\mu}_{\delta t} - \frac{\Gamma_t \Pi_t}{2} \sigma_{\delta t}^2 \\ &\quad + \Gamma_t^3 \frac{y_t(1-y_t)}{\gamma_o^2 \gamma_p^2} \left[\left((\gamma_o - \gamma_p) \sigma_{\delta t} - \Delta_{\delta t} \right) \Delta_{\delta t} + \frac{1}{2} \left(1 + \frac{\gamma_o \gamma_p}{\Gamma_t} \right) \Delta_{\delta t}^2 \right] \\ &\quad + \Gamma_t^3 \frac{y_t(1-y_t)}{\gamma_o^2 \gamma_p^2} \left[\left((\gamma_o - \gamma_p) \sigma_{\delta t} - \Delta_{\delta t} \right) \frac{\nu_{ot}^* - \nu_{pt}^*}{\sigma_t} - \frac{1}{2} \left(1 + \frac{\gamma_o \gamma_p}{\Gamma_t} \right) \left(\frac{\nu_{ot}^* - \nu_{pt}^*}{\sigma_t} \right)^2 \right] \\ &\quad - \Gamma_t \frac{y_t}{\gamma_o} f_o(\nu_{ot}^*) - \Gamma_t \frac{1-y_t}{\gamma_p} f_p(\nu_{pt}^*), \end{aligned} \quad (21)$$

$$\kappa_t = \Gamma_t \sigma_{\delta t} + \frac{\mu_{\delta t} - \bar{\mu}_{\delta t}}{\sigma_{\delta t}} - \Gamma_t \frac{y_t}{\gamma_o} \frac{\nu_{ot}^*}{\sigma_t} - \Gamma_t \frac{1-y_t}{\gamma_p} \frac{\nu_{pt}^*}{\sigma_t}, \quad (22)$$

$$\sigma_{yt} = \Gamma_t \frac{1-y_t}{\gamma_o \gamma_p} \left((\gamma_o - \gamma_p) \sigma_{\delta t} - \Delta_{\delta t} - \frac{\nu_{ot}^* - \nu_{pt}^*}{\sigma_t} \right), \quad (23)$$

$$\mu_{yt} = \mu_{\delta t} - \sigma_{yt} \sigma_{\delta t} - \frac{r_t + f(\nu_{ot}^*) - \rho}{\gamma_o} - (\sigma_{\delta t} - \sigma_{yt}) \frac{\mu_{\delta t} - \mu_{\delta t}^o}{\sigma_{\delta t}} - \frac{1 + \gamma_o}{2} (\sigma_{\delta t} - \sigma_{yt})^2, \quad (24)$$

where Γ_t and Π_t denote relative risk aversion and prudence parameters of a representative investor, given by:⁶

$$\Gamma_t = \frac{\gamma_o \gamma_p}{\gamma_o(1 - y_t) + \gamma_p y_t}, \quad \Pi_t = \Gamma_t^2 \left(\frac{1 + \gamma_o}{\gamma_o^2} y_t + \frac{1 + \gamma_p}{\gamma_p^2} (1 - y_t) \right), \quad (25)$$

and $\bar{\mu}_{\delta t}$ is a weighted average of the investors' subjective dividend growth rates $\mu_{\delta t}^i$, given by:

$$\bar{\mu}_{\delta t} = \frac{\gamma_p y_t}{\gamma_o(1 - y_t) + \gamma_p y_t} \mu_{\delta t}^o + \frac{\gamma_o(1 - y_t)}{\gamma_o(1 - y_t) + \gamma_p y_t} \mu_{\delta t}^p. \quad (26)$$

The equilibrium parameters in fictitious unconstrained economies under investors' subjective beliefs are given by:

$$r_{ot} = r_t + f_o(\nu_{ot}^*), \quad r_{pt} = r_t + f_p(\nu_{pt}^*), \quad (27)$$

$$\kappa_{ot} = \gamma_o(\sigma_{\delta t} - \sigma_{yt}), \quad \kappa_{pt} = \gamma_p \left(\sigma_{\delta t} + \sigma_{yt} \frac{y_t}{1 - y_t} \right), \quad (28)$$

$$\mu_{yt}^o = \mu_{\delta t}^o - \sigma_{yt} \sigma_{\delta t} - \frac{r_t + f(\nu_{ot}^*) - \rho}{\gamma_o} - \frac{1 + \gamma_o}{2} (\sigma_{\delta t} - \sigma_{yt})^2, \quad \mu_{yt}^p = \mu_{yt}^o - \Delta_{\delta t} \sigma_{yt}. \quad (29)$$

Proposition 1 provides the equilibrium parameters as functions of consumption share y_t , adjustment parameters ν_{it}^* , and stock return volatility σ_t . In the unconstrained case the adjustments ν_{it}^* are equal to zero [case (a) in Table 1] and the expressions in Proposition 1 give the closed form expressions for the equilibrium parameters in the unconstrained economy, previously derived in Basak (2000, 2005). When the investors face binding portfolio constraints the terms involving ν_{it}^* capture the impact of constraints on the risk sharing and the demand for financial securities.

Expression (21) decomposes the interest rate into two groups of terms. First four terms give the closed-form expression for the interest rate in an unconstrained economy with heterogeneous investors. Specifically, the first three terms look similar to the interest rate in a standard Lucas economy, and capture the effects of risk aversion Γ and prudence Π of the representative investor. The remaining terms capture the impact of heterogeneous beliefs and portfolio constraints. Similarly, the expressions for the market price of risk (22) and the volatility of optimist's consumption share (23) explicitly separate the effects of heterogeneity in preferences and beliefs and portfolio constraints.

Even though finding the adjustments ν_{it}^* is a challenging problem (addressed in Subsection 2.2) their signs and hence the directions in which they affect the equilibrium can explicitly be identified by computing the effective domains Υ_i , as demonstrated in Table 1. As an illustration

⁶Similarly to Basak (2000, 2005) it can be demonstrated that the equilibrium in this economy is equivalent to the equilibrium in an economy with a representative investor with a utility function given by:

$$u(c; \lambda) = \max_{c_o + c_p = c} u_o(c_o) + \lambda u_p(c_p),$$

where $\lambda = \xi_o/\xi_p$. The expressions for the relative risk aversion Γ_t and prudence Π_t of the representative investor in (25) are special cases of those in Basak (2000, 2005), derived for general utility functions.

consider the case where the optimist faces a borrowing constraint ($\theta_{ot} \leq \bar{\theta}$, $\bar{\theta} > 1$) while the pessimist faces a short-sale constraint ($\theta_{pt} \geq 0$), and hence $\nu_{ot}^* \leq 0$ and $\nu_{pt}^* \geq 0$ [cases (c) and (d) in Table 1]. Moreover, we will assume that stock return volatility σ_t is positive (i.e. stock returns are positively correlated with dividend growth) and $(\gamma_o - \gamma_p)\sigma_{\delta t} - \Delta_{\delta t} < 0$, which means that the optimist's risk aversion is not very high while the pessimist's risk aversion is not very low, so that the constraints are likely to bind.

The impact of constraints on interest rates is ambiguous since r_t is a quadratic function of the adjustments. Under the assumptions above, in the expression for interest rates in (21) the fifth quadratic term is positive when the adjustments ν_{it}^* are small and negative when they are large, the sixth term is negative and the last term is equal to zero [case (d) in Table 1]. Therefore the sum of terms involving ν_{it}^* can both be positive or negative. In the numerical examples in Section 3 we demonstrate that the interest rates always go down with tighter borrowing constraint (i.e. when $\bar{\theta}$ decreases). Intuitively, when $\bar{\theta} < 1$ tighter constraint $\theta_{ot} \leq \bar{\theta}$ increases the constrained investor's demand for bonds pushing the interest rates downwards. When $\bar{\theta} > 1$ tighter constraint reduces the constrained investor's leverage, decreasing the interest rates. We note that maximizing r_t with respect to ν_{it}^* and subject to constraints $\nu_{it}^* \in \Upsilon_i$ gives an explicit upper bound for interest rates in terms of exogenous model parameters, which however is beyond our scope.

The signs of the adjustments in Table 1 imply that binding borrowing constraints increase while the short-sale constraints decrease the market price of risk κ_t . Intuitively, binding borrowing constraint causes the optimist to hold less stocks, as compared to the unconstrained case, holding the consumption share y_t fixed. Therefore, for the market to clear, the market price of risk increases in order to induce the pessimist to increase the exposure to stocks. Similarly, if the pessimist cannot short-sale stocks the optimist should hold less stocks, and hence the market price of risk goes down to equilibrate the supply and demand. Furthermore, expression (23) demonstrates that the optimist's consumption share volatility σ_{yt} goes up if either of the constraints binds.

Proposition 1 also allows to obtain the expressions for consumption growth volatilities of investors which capture the effect of risk sharing. Suppose, under the true probability measure $\mathbb{P}$ the optimal consumptions evolve as follows:

$$dc_{it}^* = c_{it}^*[\mu_{c_i^*t}dt + \sigma_{c_i^*t}dw_t], \quad i = o, p. \quad (30)$$

The expressions for the volatilities $\sigma_{c_i^*t}$ can be obtained by applying Itô's Lemma to optimal consumptions in (18). Corollary 1 reports the results.

Corollary 1. *The optimal consumption growth volatilities of the investors are given by:*

$$\sigma_{c_o^*t} = \sigma_{\delta t} - \sigma_{yt}, \quad \sigma_{c_p^*t} = \sigma_{\delta t} + \frac{y_t}{1 - y_t}\sigma_{yt}. \quad (31)$$

Section 3 provides further analysis of consumption growth volatilities $\sigma_{c_i^*t}$ in specific equilibrium settings.

2.2. Characterization of Adjustments and Stock Return Volatilities

In this subsection we tackle the problem of finding the adjustments ν_{it}^* . We then demonstrate how to derive all the equilibrium parameters if the adjustments are known. On the other hand, we argue that if the equilibrium parameters are known the adjustments can be obtained from the complementary slackness conditions in Karatzas and Shreve (1998). Therefore, the pair $\{\nu_{ot}^*, \nu_{pt}^*\}$ emerges as a fixed point of a non-linear mapping which can be found by solving a system of two Hamilton-Jacobi-Bellman (HJB) equations for investors' wealth-consumption ratios. Even though in equilibrium the coefficients of HJB equations themselves depend on the sensitivities of wealth-consumption ratios with respect to state variable y_t , we demonstrate that the time-independent solutions can easily be obtained via an iterative procedure with fast convergence that at each step requires solving a simple system of linear algebraic equations. This approach avoids solving the dual optimization problem in Cvitanić and Karatzas (1992) which is very difficult unless investors are logarithmic or stock prices follow process (5) with deterministic coefficients.

Our solution method does not rely on a widely used assumption of a logarithmic constrained investor [e.g., Detemple and Murthy (1997); Basak and Cuoco (1998); Basak and Croitoru (2000, 2006); Kogan, Makarov and Uppal (2003); Gallmeyer and Hollifield (2008); Hugonnier (2008); Pavlova and Rigobon (2008); Schornick (2009)] which allows for tractability in specific settings at the cost of investor's myopia inherent in logarithmic preferences.⁷ We note that the tractability of logarithmic preferences is limited to specific settings and constraints (discussed in Section 3.2) where the HJB equations are linear. However, in a model where the unconstrained investor has CRRA utility while the logarithmic investor faces restricted participation constraint or margin requirements the HJB equation for the unconstrained investor remains nonlinear.

In Subsection 2.1 we have obtained the equilibrium parameters in terms of adjustments ν_{it}^* and highlighted the effect of constraints on equilibrium. While the analysis in Subsection 2.1 goes through for general dividend processes and beliefs, in order to recover the adjustments ν_{it}^* , following the literature we make three simplifying assumptions.

Assumption 1. *The dividend process (1) follows a geometric Brownian motion under the true probability measure $\mathbb{P}$:*

$$d\delta_t = \delta_t[\mu_\delta dt + \sigma_\delta dw_t],$$

where μ_δ and σ_δ are constants.

Assumption 2. *The investors do not update their beliefs, and each of them perceives the dividend process as a geometric Brownian motion:*

$$d\delta_t = \delta_t[\mu_\delta^i dt + \sigma_\delta dw_t^i], \quad i = o, p. \quad (32)$$

⁷The tractability of logarithmic preferences comes from the fact that the optimal investment policies θ_{it}^* of constrained logarithmic investors can easily be computed in closed form both in the original and fictitious economies.

The optimist ($i = o$) rationally believes that $\mu_{\delta t}^o = \mu_\delta$ while the pessimist ($i = p$) permanently underestimates the expected dividend growth rate so that $\mu_\delta > \mu_\delta^p$.

Assumption 3. The optimist ($i = o$) faces constraint:

$$\theta_{ot} \leq \bar{\theta}, \quad (33)$$

where $\bar{\theta}$ can be any non-negative number, while the pessimist ($i = p$) is unconstrained.

Assumptions 1 and 2 guarantee that the equilibrium parameters depend only on one state variable, the optimist's consumption share y_t , which significantly facilitates the tractability. Both assumptions are commonly employed in the literature [e.g., Chan and Kogan (2002), Gallmeyer and Hollifield (2008), Bhamra and Uppal (2009, 2010), Yan (2008) and Yan (2008), Bhamra and Uppal (2009, 2010), respectively]. The fact that investors do not update their beliefs is unlikely to affect our results for plausible model parameters due to very slow convergence of beliefs in the case of Bayesian updating.⁸

Assumption 3 for simplicity restricts the analysis to constraint (33) while short-sale constraints can be analyzed similarly. Since in our analysis below it turns out that in equilibrium $\theta_{ot}^* \geq 0$, the case $\bar{\theta} < 1$ describes limited market participation while the case $\bar{\theta} \geq 1$ corresponds to borrowing constraints or margin requirements for collateralized borrowing (e.g. in repo markets). Specifically, in the latter case constraint (33) is equivalent to constraint $\theta_{ot} \leq 1 + m\theta_{ot}$ which limits the *leverage ratio* (i.e. debt to wealth ratio $\theta_{ot} - 1$) by a proportion $m < 1$ of the stock holding [e.g., Brunnermeier and Pedersen (2009)].

For convenience, we solve the optimization problem of the constrained optimist in an equivalent fictitious unconstrained economy in which the investor maximizes objective function (10) subject to the budget constraint:

$$dW_{ot} = \left[W_{ot} \left(r_t + f(\nu_{ot}^*) + \theta_{ot}(\mu_t^o - r_t + \nu_{ot}^*) \right) - c_{ot} \right] dt + W_{ot} \theta_{ot} \sigma_t dw_t^o, \quad (34)$$

where ν_{ot}^* and $f(\nu_{ot}^*)$ are adjustments to stock mean returns and riskless rates respectively. By applying dynamic programming we find that the *indirect utility functions* of investors, which we denote as $J_{it} = J_i(W_{it}, y_t, t)$, satisfy the following HJB equations:

$$0 = \max_{c_i, \theta_i} \left\{ e^{-\rho t} \frac{c_{it}^{1-\gamma_i}}{1-\gamma_i} + \frac{\partial J_{it}}{\partial t} + \left[W_{it} \left(r_{it} + \theta_{it} \sigma_t \kappa_{it} \right) - c_{it} \right] \frac{\partial J_{it}}{\partial W_{it}} - y_t \mu_{yt}^i \frac{\partial J_{it}}{\partial y_t} + \frac{1}{2} \left[W_{it}^2 \theta_{it}^2 \sigma_t^2 \frac{\partial^2 J_{it}}{\partial W_{it}^2} - 2W_{it} \theta_{it} \sigma_t y_t \sigma_{yt} \frac{\partial^2 J_{it}}{\partial W_{it} \partial y_t} + y_t^2 \sigma_{yt}^2 \frac{\partial^2 J_{it}}{\partial y_t^2} \right] \right\}, \quad (35)$$

⁸If dividends follow a geometric Brownian motion, investors update their beliefs in a Bayesian fashion and have normally distributed initial priors $\mu_{\delta 0}^i \sim N(\hat{\mu}_{\delta}^i, \hat{\sigma}_{\delta}^i)$ with $\hat{\sigma}_{p\delta} = \hat{\sigma}_{o\delta}$ it turns out that the disagreement process is explicitly given by [e.g., Lipster and Shiryaev (1977); Basak (2000, 2005)]:

$$\Delta_{\delta t} = \Delta_{\delta 0} \left(\frac{\sigma_{\delta}^2}{\hat{\sigma}_{p\delta} t + \sigma_{\delta}^2} \right)^{\sigma_{\delta}}.$$

Assuming further that $\hat{\sigma}_{p\delta} = \sigma_{\delta}$ and taking $\sigma_{\delta} = 3.2\%$ as in Campbell (2003), we obtain that it takes 100 years for the disagreement $\Delta_{\delta t}$ to decrease by 20%.

subject to transversality condition $E_t[J_{iT}] \rightarrow 0$ as $T \rightarrow \infty$, which guarantees the convergence of the integral in investors' optimization (10). The HJB equations in (35) are standard except for the fact that they are in terms of the parameters of the fictitious unconstrained economy.

We conjecture that the indirect utility functions admit the following representation:

$$J_i(W_{it}, y_t, t) = e^{-\rho t} \frac{W_{it}^{1-\gamma_i}}{1-\gamma_i} H_i(y_t, t)^{\gamma_i}, \quad i = o, p. \quad (36)$$

Then, from the first order conditions with respect to consumption we obtain:

$$c_{it}^* = \frac{W_{it}}{H_{it}}, \quad i = o, p, \quad (37)$$

where H_{it} is a shorthand notation for investor i 's *wealth-consumption ratio* $H_i(y, t)$. By substituting indirect utility functions (36) into HJB equations it can be verified that wealth-consumption ratios satisfy the following PDEs:

$$\frac{\partial H_{it}}{\partial t} + \frac{y_t^2 \sigma_{yt}^2}{2} \frac{\partial^2 H_{it}}{\partial y_t^2} - y_t \left(\mu_{yt}^i + \frac{1-\gamma_i}{\gamma_i} \kappa_{it} \sigma_{yt} \right) \frac{\partial H_{it}}{\partial y_t} + \left(\frac{1-\gamma_i}{2\gamma_i} \kappa_{it}^2 + (1-\gamma_i)r_{it} - \rho \right) \frac{H_{it}}{\gamma_i} + 1 = 0, \quad i = o, p, \quad (38)$$

where κ_{it} and r_{it} denote riskless rate and price of risk in a fictitious economy as defined in (17). Moreover, optimal investment policies for investors 1 and 2 are given by:

$$\theta_{it} = \frac{1}{\gamma_i \sigma_t} \left(\kappa_{it} - \gamma_i \sigma_{yt} \frac{\partial H_{it}}{\partial y_t} \frac{y_t}{H_{it}} \right), \quad i = o, p. \quad (39)$$

Since the horizon is infinite we look for *time-independent* solutions $H_i(y)$ of equations (38). Conveniently, since the fictitious economy is complete the equations for wealth-consumption ratios in (38) are linear if the equilibrium parameters κ_{it} , r_{it} , and μ_{yt}^i are known. However, as demonstrated in Proposition 1, these parameters depend on adjustments ν_{it}^* and stock return volatility σ_t , which in turn are the functions of wealth-consumption ratios H_{it} , making the HJB equations quasilinear. Proposition 2 summarizes our results and provides a characterization of consumption share and stock return volatilities, σ_{yt} and σ_t , *price-dividend ratio* Ψ_t , and the adjustments ν_{it}^* in terms of wealth-consumption ratios.

Proposition 2. *If there exists an equilibrium such that $\sigma_t > 0$ and σ_{yt} is a continuous function of y_t in $(0, 1)$, the volatility of consumption share σ_{yt} is given by:*

$$\sigma_{yt} = \begin{cases} \max \left\{ \frac{((\gamma_o - \gamma_p)\sigma_\delta - \Delta_\delta)(1 - y_t)}{\gamma_o(1 - y_t) + \gamma_p y_t}, \frac{(1 - \bar{\theta})\sigma_\delta}{G_t} \right\}, & \text{if } G_t > 0, \\ \frac{((\gamma_o - \gamma_p)\sigma_\delta - \Delta_\delta)(1 - y_t)}{\gamma_o(1 - y_t) + \gamma_p y_t}, & \text{if } G_t \leq 0, \end{cases} \quad (40)$$

where G_t is defined as:

$$G_t = 1 + \frac{\partial H_{ot}}{\partial y_t} \frac{y_t}{H_{ot}} - \bar{\theta} \frac{\partial \Psi_t}{\partial y_t} \frac{y_t}{\Psi_t}.$$

Price-dividend ratio $\Psi_t \equiv S_t/\delta_t$ and stock return volatility σ_t are given by:

$$\Psi_t = y_t H_{ot} + (1 - y_t) H_{pt}, \quad \sigma_t = \sigma_\delta - \sigma_{yt} \frac{\partial \Psi_t}{\partial y_t} \frac{y_t}{\Psi_t}. \quad (41)$$

The adjustment parameters are given by:

$$\nu_{ot}^* = \sigma_t \left((\gamma_o - \gamma_p) \sigma_\delta - \Delta_\delta - \sigma_{yt} \frac{\gamma_o(1 - y_t) + \gamma_p y_t}{1 - y_t} \right), \quad \nu_{pt}^* = 0. \quad (42)$$

Wealth-consumption ratios H_{it} satisfy HJB equations (38), while constrained investor's consumption share y_0 at time $t = 0$ solves the equation:

$$s(1 - y_0) H_p(y_0, 0) \delta_0 - (1 - s) y_0 H_o(y_0, 0) \delta_0 = b. \quad (43)$$

Proposition 2 completes the description of equilibrium. It identifies the adjustment parameter ν_{ot}^* and stock return volatility σ_t in terms of wealth-consumption ratios satisfying HJB equations (38), while Proposition 1 identifies the other equilibrium parameters in terms of ν_{ot}^* and σ_t . We derive volatilities σ_t and σ_{yt} assuming that $\sigma_t > 0$ and σ_{yt} is continuous. After computing the equilibrium we verify that these conditions are indeed satisfied and the parameters comprise the equilibrium. The characterization of all equilibrium parameters in terms of wealth-consumption ratios implies that the coefficients in HJB equations (38) themselves depend on wealth-consumption ratios H_{it} , which gives a system of two quasilinear HJB equations.

Appendix A provides the proof of Proposition 2. We here just note that price dividend ratio Ψ_t in (41) is derived from the market clearing conditions (11), while volatility σ_t is derived by applying Itô's Lemma to stock price $S_t \equiv \Psi_t \delta_t$. The adjustment parameter ν_{pt}^* is zero since the pessimist is unconstrained [case (a) in Table 1], while ν_{ot}^* can be found from a complementary slackness condition $\theta_{ot}^* \nu_{ot}^* + f(\nu_{ot}^*) = 0$ derived in Karatzas and Shreve (1998) which for the case of constraint (33) takes the form:

$$(\theta_{ot}^* - \bar{\theta}) \nu_{ot}^* = 0. \quad (44)$$

Therefore, if the constraint is not binding condition (44) implies that $\nu_{ot}^* = 0$, while if the constraint is binding ν_{ot}^* can be found from the condition $\theta_{ot}^* = \bar{\theta}$. Consequently, the expression for volatility σ_{yt} in terms of adjustments in (23) on one hand gives expression (40) for σ_{yt} , and on the other hand gives the adjustment in terms of σ_{yt} in (42).

Remark 1 (Two Constrained Investors). In a model where the pessimist faces a short-sale constraint while the optimist faces a margin requirement, volatility σ_{yt} can be derived along the same lines as in Proposition 2. However, when both constraints bind simultaneously, from the expression for σ_{yt} in terms of adjustments given in (23) it is only possible to pin down $(\nu_{ot}^* - \nu_{pt}^*)$, but not the individual adjustments ν_{it}^* , which leads to the multiplicity of equilibrium adjustments. From the expressions for equilibrium parameters in (27)–(29) it follows that parameters κ_{it} and μ_{yt}^i can be uniquely determined if σ_{yt} is known while fictitious interest rates r_{it} require

the individual adjustments ν_{it}^* . Therefore, as demonstrated in Detemple and Murthy (1997), simultaneously binding constraints generate discontinuous interest rates. In a setting with logarithmic investors in Detemple and Murthy there is no impact of constraints and interest rate discontinuities on stock prices. However, in our setting the multiplicity of adjustments is likely to lead to endogenous jumps in stock prices, which is a challenging direction for future research. We also note that in a setting where investors differ only in risk aversions and have identical beliefs the short-sale constraint never binds while margin requirement constraint binds only for a less risk averse investor, which endogenously leads to a setting with one constrained investor.

2.3. Computation of Equilibrium

We next solve for time-independent solutions of PDEs (38) $H_i(y_t)$ which correspond to the infinite horizon case. Before proceeding to the general case, we note that there are four important special cases where the equilibrium interest rate r_t , market prices of risk κ_t , drift μ_{yt}^i and volatility σ_{yt} can be obtained in closed form as functions of consumption share y_t both in the original and fictitious economies, and hence the HJB equations (38) are linear and easier to solve. First case is where both investors are logarithmic. In this case, the wealth-consumption ratios are given by $H_{it} = 1/\rho$, $i = o, p$ [e.g., Detemple and Murthy (1997); Basak and Cuoco (1998)], and there is no impact of constraints on stock prices.

Second case is the case of no-borrowing constraint ($\bar{\theta} = 1$) assuming $(\gamma_o - \gamma_p)\sigma_\delta - \Delta_\delta \leq 0$ (for the constraint to be binding). From formulas (40) and (41) it can be seen that $\sigma_{yt} = 0$ and $\sigma_t = \sigma_\delta$. Therefore, the HJB equations (38) become first order linear ODEs that can be solved in closed form. Then, price-dividend ratio Ψ_t can be obtained from (41) while formulas (28)–(29) provide the remaining equilibrium parameters. Since $\sigma_{yt} = 0$, consumption share y_t and hence all the equilibrium parameters are deterministic.⁹

Third case is when the constrained investor has logarithmic utility and faces specific constraints, e.g. restricted stock market participation (i.e. $\gamma_o = 1$, $\bar{\theta} = 0$) or short-sale constraints. Since for the logarithmic investor $H_{ot} = 1/\rho$ it can easily be verified that σ_{yt} in (40) depends only on y_t whenever $\bar{\theta} = 0$. The closed-form expressions for market prices of risk κ_{it} , interest rate r_t and drift parameters μ_{yt}^i as functions of consumption share y_t are then obtained from formulas (28)–(29), similarly to Basak and Cuoco (1998). The case of short-sale constraints can be analyzed in a similar way. We note also that for the case of margin requirements the HJB equations remain nonlinear.

Finally, the HJB equations are linear when the constraint never binds (i.e. $\bar{\theta}$ very large)

⁹In the economy with $\bar{\theta} = 1$ Kogan, Makarov, and Uppal (2007) provide solutions in the case when constrained investor is logarithmic and the investors have identical beliefs. They find that constraint $\theta_t \leq 1$ decreases interest rates and increases market prices of risk. However, this constraint does not generate stochastic variation in equilibrium parameters, which is in contrast to our model with $\bar{\theta} \neq 1$. Bhamra and Uppal (2009) consider an economy with heterogeneous risk aversion and demonstrate that in the absence of a risk-free asset the equilibrium parameters are deterministic.

and hence $\nu_{ot}^* = 0$. The unconstrained equilibrium is a convenient benchmark against which we compare our main results. We here provide a tractable solution in terms of familiar hypergeometric functions commonly used in the literature [e.g., Ingersoll and Ross (1992); Cochrane, Longstaff and Santa-Clara (2008); Longstaff and Wang (2008); Longstaff (2009); Martin (2009)] by extending the approach of Longstaff and Wang (2008).¹⁰ For brevity, Proposition 3 below reports only the price-dividend ratio, while the interest rates and market prices of risk can be obtained from the expressions in Proposition 1, and σ_t can be obtained in closed form from (41).

Proposition 3. *If there exists an equilibrium in the economy with two unconstrained heterogeneous investors the equilibrium price-dividend ratio is given by:*

$$\begin{aligned} \Psi_t = \frac{1}{|a_2|\sqrt{2b}} \Big[& - \frac{1}{\gamma_o + \varphi_-} {}_2F_1\left(\left(1 - \frac{\gamma_p}{\gamma_o}\right)\varphi_- - \gamma_p, 1, 1 - \gamma_p - \frac{\gamma_p}{\gamma_o}\varphi_-; 1 - y_t\right) \\ & + \left(1 - \frac{\gamma_p}{\gamma_o}\right) \frac{1 - y_t}{1 - \gamma_p - \frac{\gamma_p}{\gamma_o}\varphi_-} {}_2F_1\left(\left(1 - \frac{\gamma_p}{\gamma_o}\right)\varphi_- + 1 - \gamma_p, 1, 2 - \gamma_p - \frac{\gamma_p}{\gamma_o}\varphi_-; 1 - y_t\right) \\ & + \frac{\gamma_p}{\gamma_o} \frac{1}{\varphi_+} {}_2F_1\left(\left(1 - \frac{\gamma_p}{\gamma_o}\right)\varphi_+ - \gamma_p, 1, 1 + \varphi_+; y_t\right) \\ & + \left(1 - \frac{\gamma_p}{\gamma_o}\right) \frac{1 - y_t}{\varphi_+} {}_2F_1\left(\left(1 - \frac{\gamma_p}{\gamma_o}\right)\varphi_+ + 1 - \gamma_p, 1, 1 + \varphi_+; y_t\right) \Big], \end{aligned} \quad (45)$$

where

$$\begin{aligned} \varphi_{\pm} &= \frac{a_1 \pm |a_2|\sqrt{2b}}{a_2^2}, \\ a_1 &= \frac{1}{\gamma_o} \left[(\gamma_p - \gamma_o) \left(\mu_{\delta}^p - \frac{\sigma_{\delta}^2}{2} \right) - \frac{\Delta_{\delta}^2}{2} - ((\gamma_o - \gamma_p)\sigma_{\delta} - \Delta_{\delta})(1 - \gamma_p)\sigma_{\delta} \right], \\ a_2 &= -\frac{(\gamma_o - \gamma_p)\sigma_{\delta} - \Delta_{\delta}}{\gamma_o}, \quad b = \rho - (1 - \gamma_p)\mu_{\delta}^p + \frac{\gamma_p(1 - \gamma_p)\sigma_{\delta}^2}{2} + \frac{a_1^2}{2a_2^2}, \end{aligned} \quad (46)$$

and ${}_2F_1(x_1, x_2, x_3; y)$ denotes a hypergeometric function, given in Appendix A.

To solve the HJB equations (38) in the general case we first obtain boundary conditions at $y = 0$ and $y = 1$ for HJB equations (38) by passing to the limit when y_t tends to 0 or 1, respectively, as discussed in Appendix B. Next, there are two ways to solve for equilibrium. The first one is the fixed point iteration method, widely used in the literature [e.g., Chien, Cole and Lustig (2008), Gomes and Michaelides (2008), Guvenen (2009)], in which we use candidate wealth-consumption ratios $H_{i,n}(y)$ at step n to compute all the equilibrium parameters, and hence the coefficients of equations (38), and then solve numerically the resulting linear HJB equations to obtain wealth consumption ratios $H_{i,n+1}(y)$ at step $n + 1$. This method gives very

¹⁰Yan (2008) derives closed-form solutions when investors have identical integer risk aversions, while Longstaff and Wang (2008) derive the equilibrium in terms of hypergeometric functions when one investor is twice more risk averse than the other and they have homogeneous beliefs. Chabakauri (2009) generalizes the result in Yan (2008) for any positive risk aversion. Bhamra and Uppal (2010) derive analytical infinite series expansions for equilibrium parameters when the investors additionally differ in risk aversions.

fast convergence if the initial conjectured wealth-consumption ratios at step 0 are not very far from the equilibrium ones.

The second way is to keep time-dependence in HJB equations (38), fix a large horizon parameter T , choose a starting value for $H_i(y, T)$ and then solve the equations backwards in time using a modification of Euler's finite-difference method until the solutions converge to time-independent functions $H_i(y)$. This approach is similar to the successive iterations method for solving Bellman equations in discrete time [e.g., Ljungqvist and Sargent (2004)] when the value function is set equal to a certain function (usually zero) at a distant time in the future and then the value functions at earlier dates are obtained by solving equations backwards. To solve equations (38) we replace the derivatives by their finite-difference analogues letting the time and state variable increments denote $\Delta t \equiv T/M$ and $\Delta y \equiv 1/N$, where M and N are integer numbers. To solve the equations backwards in time, sitting at time t we compute the coefficients of finite-difference analogues of PDEs (38) using the solutions $H_i(y, t + \Delta t)$ obtained from the previous step $t + \Delta t$. As a result, the coefficients of the equations for $H_i(y, t)$ are known at time t , and hence $H_i(y, t)$ can be found by solving a system of linear finite-difference equations with three-diagonal matrix.

In most of the cases studied in this paper we use fixed point iterations while in certain cases we use a combination of two methods to facilitate the convergence. Appendix B provides further details of the numerical algorithm and discusses the speed of convergence as well as the optimality of consumptions c_{it}^* and investment policies θ_{it}^* . The wealth-consumption ratios then allow us to derive all the equilibrium parameters. The numerical analysis shows that the function on the left-hand side of the equation for y_0 in (43) is a monotone function of y_0 and maps interval $(0, 1)$ into (C_0, C_1) , where $C_0 < 0$ and $C_1 > 0$ are constants. Therefore, if $b \in (C_0, C_1)$ there exists the unique solution y_0 of equation (43). Similarly to Basak and Cuoco (1998), if $b \notin (C_0, C_1)$ the solution to (43) does not exist since the borrower is leveraged to such an extent that will never be able to repay the debt.

3. Analysis of Equilibrium

In this section we analyze the impact of portfolio constraints in various general equilibrium settings. In Subsection 3.1 we study the general equilibrium with limited stock market participation ($\bar{\theta} < 1$). In this setting we assume that both the constrained and the unconstrained investors have identical CRRA preferences and beliefs so that there is no unnecessary correlation between being constrained and having specific risk aversions or beliefs, which allows to assess the pure effect of constraints. We demonstrate that this smallest deviation from standard Lucas economy can simultaneously generate countercyclical market prices of risk, procyclical price-dividend ratios, countercyclical stock-return volatilities and risk premia, as well as excess

volatility, consistently with the empirical findings.¹¹

In Subsection 3.2 we evaluate the impact of margin requirements and borrowing constraints ($\bar{\theta} > 1$). To make constraint (33) with $\bar{\theta} > 1$ binding we allow the investors to be heterogeneous in risk aversions and beliefs.¹² We demonstrate that constraints produce equilibrium parameters consistent with empirical findings and decrease the volatility of stock returns. We acknowledge that both limited stock market participation and margin requirements simultaneously play important roles in the formation of asset prices, even though for tractability we study these two constraints separately in Sections 3.1 and 3.2.

Furthermore, our analysis identifies which equilibrium parameters are procyclical or countercyclical, where following the literature [e.g., Chan and Kogan (2002); Longstaff and Wang (2008); Bhamra and Uppal (2009)] we call a stochastic Itô's process X_t *procyclical* (*countercyclical*) if instantaneous changes dX_t and $d\delta_t$ are positively (negatively) correlated, and hence X_t increases in good times (when dividend growth shocks are positive) and decreases in bad times (when dividend growth shocks are negative). In our calibrations the parameters of the dividend process $\mu_\delta = 1.8\%$ and $\sigma_\delta = 3.2\%$ are taken from the estimates in Campbell (2003), based on consumption data in 1891–1998 years, and the time discount parameter is $\rho = 0.01$.

3.1. Equilibrium with Limited Stock Market Participation

Throughout this subsection we study the implications of limited participation constraint when the investor is allowed to invest only up to a fraction $\bar{\theta} < 1$ of wealth in stocks, assuming that the investors have identical risk aversions denoted by γ (i.e., $\gamma_p = \gamma_o = \gamma$) and beliefs regarding the dividend growth rate denoted by μ_δ (i.e., $\mu_\delta^p = \mu_\delta^o = \mu_\delta$). As discussed in the introduction, the limited participation constraints are typical for pension funds [Srinivas, Whitehouse, and Yermo (2000)], and include the restricted participation ($\bar{\theta} = 0$) considered in the literature [e.g., Basak and Cuoco (1998); Guvenen (2006, 2009)] as a special case.

The numerical analysis reveals that constraint (33) always binds when $\bar{\theta} < 1$. Intuitively, if $\bar{\theta} < 1$ and the constrained investor does not find the asset attractive enough to bind on the constraint, the unconstrained investor should also not be willing to allocate more than the fraction of wealth $\bar{\theta} < 1$ to stocks since the investors have identical preferences and beliefs. However, the fact that both investors have $\theta_{it} < 1$ contradicts market clearing conditions (11), and hence the constraint is always binding. For brevity, we do not present the graphs for investment policies in this subsection since the policies of the constrained investor are simply given by $\theta_{ot}^* = \bar{\theta}$,

¹¹Campbell and Cochrane (1999) and Chan and Kogan (2002) present the models with habit formation and “catching up with the Joneses” preferences respectively, that generate the patterns for price-dividend ratios and stock return volatilities.

¹²In a setting with homogeneous preferences and beliefs constraint (33) never binds if $\bar{\theta} > 1$ since the equilibrium coincides with the equilibrium in a standard unconstrained Lucas economy with $\theta_t^* = 1$. We note also, that our results are not driven by a seeming correlation between being constrained and having specific risk aversion or beliefs. In fact, we can assume that both investors face identical constraints $\theta_{it}^* \leq \bar{\theta}$ since under the model parameters in Subsection 3.2 it turns out that in equilibrium the pessimist's constraint never binds.

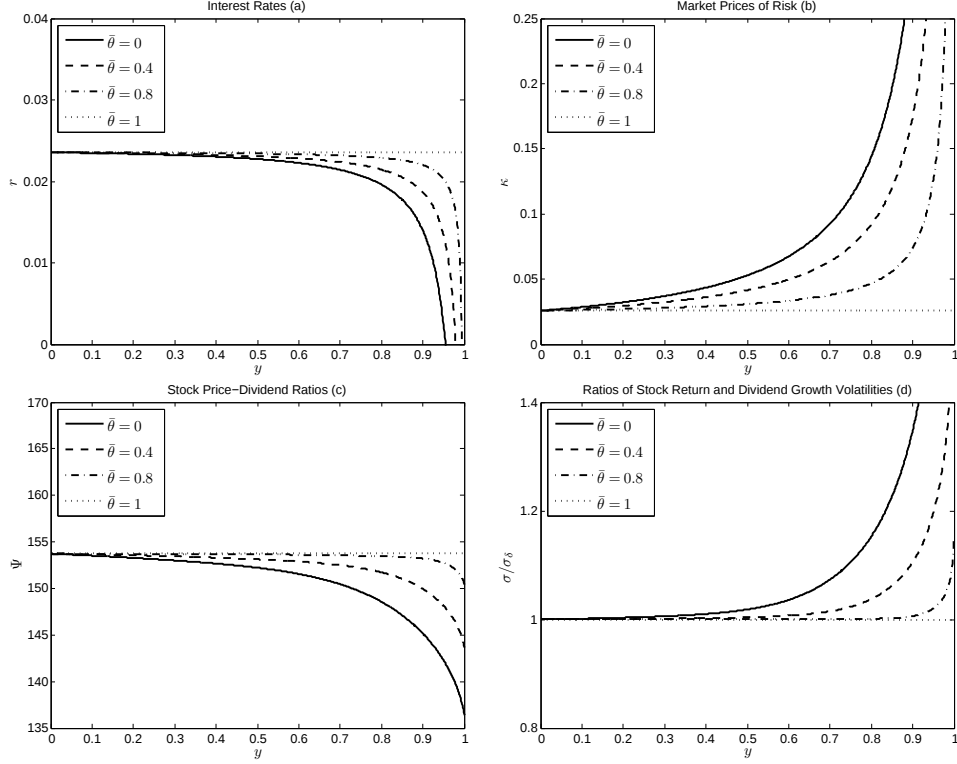


Figure 1: Equilibrium with Constraints, $\gamma < 1$.

Figure 1 presents the equilibrium parameters as functions of consumption share y_t when $\gamma < 1$. Variable y_t is countercyclical and increases in bad times. The parameters are: $\mu_\delta = 1.8\%$, $\sigma_\delta = 3.2\%$ [e.g., Campbell (2003)], $\rho = 0.01$, and $\gamma = 0.8$.

while for the unconstrained investor they behave very similarly to myopic portfolios $\kappa_t/(\gamma\sigma_t)$. In Subsection 3.2 we provide the analysis of optimal portfolios in a richer setting.

We also note that in the limited participation case the constrained investor's consumption share y_t is countercyclical due to the fact that the constrained investor is less exposed to stock market fluctuations, and hence negative (positive) dividend shocks shift relative consumption to the constrained (unconstrained) investor. Consequently, consumption share y_t is higher in bad times and lower in good times, and the equilibrium parameters positively (negatively) correlated with y_t are countercyclical (procyclical). Figures 1 and 2 present equilibrium interest rates, market prices of risk, price-dividend ratios and the ratios of stock return and dividend growth volatilities as functions of constrained investor's consumption share y_t for different levels of constraint tightness $\bar{\theta}$ when risk aversions are less than unity ($\gamma = 0.8$) and greater than unity ($\gamma = 3$), respectively. We first analyze the equilibrium parameters for the case $\gamma < 1$, presented on Figure 1, and then for the case $\gamma > 1$, presented on Figure 2.

Panel (a) of Figure 1 shows interest rates when $\gamma < 1$ and demonstrates that in line with the intuition in Section 2.1 interest rates decrease with tighter constraints, holding consumption

share y_t fixed, since the constrained investor holds more bonds driving interest rates down. Moreover, the interest rates in the constrained economy are lower in bad times when the constrained investor's consumption share y_t is high, and hence the constrained investor is more willing to lend at a lower interest rate. Panel (b) of Figure 1 shows that in line with the intuition in Section 2.1 the market prices of risk increase with tighter constraints. Moreover, consistently with the empirical literature [e.g., Ferson and Harvey (1991)] the market prices of risk are countercyclical since in states with the dominating constrained investor when y_t is high the unconstrained investor possesses less wealth and hence requires higher market prices of risk to clear the market.

Panel (c) of Figure 1 demonstrates that price-dividend ratios Ψ_t are procyclical and decreasing with tighter constraints. The procyclicality of Ψ_t along with other results on Figure 1 implies that price-dividend ratios are negatively correlated with market prices of risk κ_t , risk premia $\mu_t - r_t = \kappa_t \sigma_t$, as well as stock return volatilities, consistently with empirical findings [e.g., Campbell and Shiller (1988); Schwert (1989); Campbell and Cochrane (1999)]. To understand the intuition for the procyclicality of Ψ_t , from the expression for Ψ_t in terms of investors wealth-consumption ratios H_{it} in (41) we observe that the price-dividend ratio is close to the wealth-consumption ratio of the constrained or the unconstrained investor depending on which one of them dominates in the market.

When the unconstrained investor dominates (y_t is low), the equilibrium will be close to that in the unconstrained benchmark economy in which case all equilibrium parameters, including price-dividend ratios, are constant (dotted lines in Figures 1 and 2). However, in states with dominating constrained investor (y_t is high) the price-dividend ratio is close to the constrained investor's wealth-consumption ratio. The investment opportunities for the constrained investor worsen with tighter constraints and higher y_t due to the decline in interest rates and inability to fully benefit from the increase in market prices of risk. Therefore, the constrained investor's wealth-consumption ratio decreases with tighter constraints and higher y_t via classical substitution effect, giving rise to procyclical and decreasing in $\bar{\theta}$ price-dividend ratios. This result is due to the fact that for CRRA investors the intertemporal elasticity of substitution (IES) equals $1/\gamma$, and hence the income effect dominates for $IES < 1$ and the substitution effect dominates for $IES > 1$ while in the case of $IES = 1$ both effects perfectly offset each other. When the investment opportunities worsen, the income effect induces investors to decrease consumption and save more while the substitution effect induces them to do the opposite due to cheaper current consumption.¹³

¹³The relation between wealth-consumption ratios and the attractiveness of investment opportunities can conveniently be illustrated in an unconstrained partial equilibrium economy with constant interest rate r_t and market price of risk $\kappa_t = (\mu_t - r_t)/\sigma_t$, and an investor maximizing the objective (10) subject to budget constraint (7) and no-bankruptcy constraint. It can easily be verified that when condition $\rho - (1 - \gamma)(r + 0.5\kappa^2/\gamma) > 0$ is satisfied, the investor's wealth-consumption ratio is given by:

$$\frac{W}{c} = \frac{\gamma}{\rho - (1 - \gamma)(r + 0.5\kappa^2/\gamma)},$$

Hence, if investment opportunities deteriorate due to decrease of r_t or κ_t , the wealth-consumption ratio increases if the income effect dominates ($\gamma > 1$) and decreases if the substitution effect dominates ($\gamma < 1$).

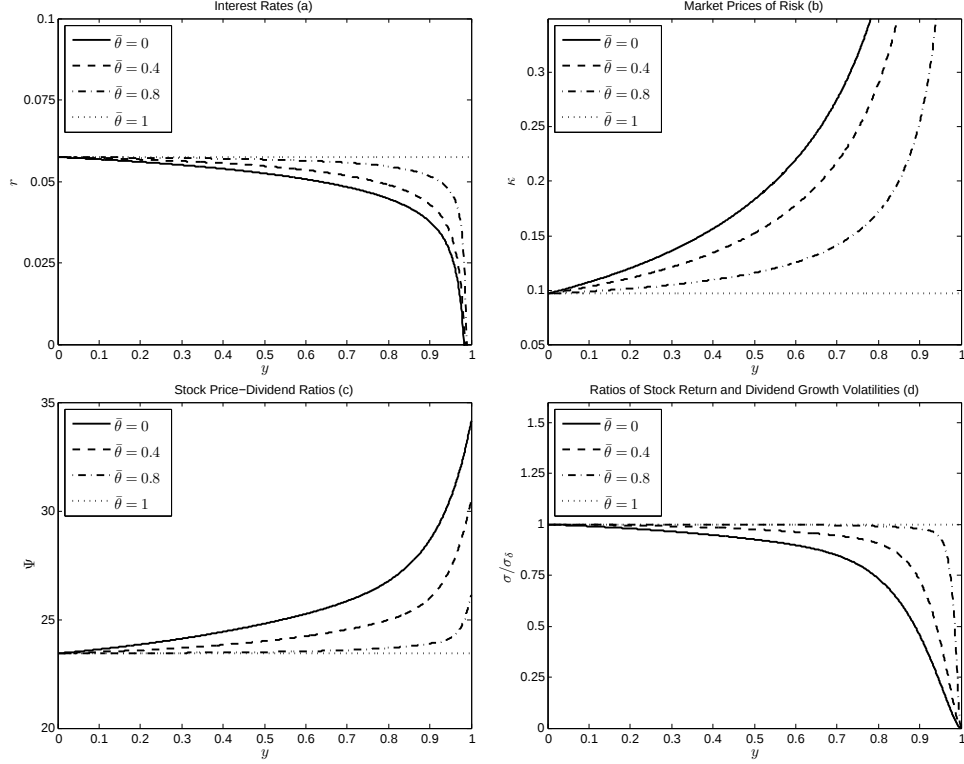


Figure 2: Equilibrium with Constraints, $\gamma > 1$.

Figure 2 presents the equilibrium parameters as functions of consumption share y_t when $\gamma > 1$. Variable y_t is countercyclical and increases in bad times. The parameters are: $\mu_\delta = 1.8\%$, $\sigma_\delta = 3.2\%$ [e.g., Campbell (2003)], $\rho = 0.01$, and $\gamma = 3$.

Panel (d) of Figure 1 shows the ratios σ_t/σ_δ of stock return and dividend growth volatilities. It turns out that stock return volatilities increase with tighter constraints, are countercyclical and more volatile than dividend growth rates, as in the historical data [e.g., Shiller (1981); Schwert (1989); Campbell and Cochrane (1999)]. From the expression for stock return volatilities in terms of price-dividend ratios in (41) it follows that the countercyclical and excess volatility are due to convex and procyclical price-dividend ratios shown in panel (c). Intuitively, since the stock price is given by $S_t = \Psi_t \delta_t$ the procyclicality of price-dividend ratio amplifies dividend shocks, making stocks more volatile. We note that in models with unconstrained investors heterogeneous in risk aversions and beliefs the volatility is not countercyclical since in both limiting cases $y = 0$ or $y = 1$ the equilibrium converges to the equilibrium in a single agent Lucas economy with stock return volatility equal to dividend growth volatility σ_δ .

Turning to the case $\gamma > 1$ we observe from the results shown on Figure 2 that the constraints affect the interest rates and market prices of risk in the same way as in the case $\gamma < 1$. Under plausible parameters described above and $\gamma = 3$, setting $\bar{\theta} = 0$ and $y = 0.7$ [e.g., Mankiw and Zeldes (1991); Guvenen (2006)] we obtain $r = 4.8\%$ and $\kappa = 28\%$, while the volatilities of individual consumptions obtained from (31) are $\sigma_{c_p^*} = 9\%$ and $\sigma_{c_o^*} = 0.7\%$. The estimates in Campbell (2003) show that $r = 2\%$ and $\kappa = 36\%$, while Malloy, Moskowitz, and Vissing-

Jorgensen (2009) show that $\sigma_{c_1} = 3.6\%$ and $\sigma_{c_2} = 1.4\%$. Thus, our model can generate riskless rates and market prices of risk sufficiently close to those in the data for such a simple model. However, by contrast with the case of $\gamma < 1$, due to the dominance of income effect price-dividend ratios increase while stock return volatilities decrease with tighter constraints, and the effects are stronger in bad times, contrary to the empirical evidence. As argued in Remark 2 a model with recursive preferences can reconcile the empirically observed patterns and the values of equilibrium parameters.

Remark 2 (Recursive preferences). The discussion above demonstrates that for risk aversion $\gamma < 1$ the model generates empirically plausible patterns for price-dividend ratios and stock return volatilities while for $\gamma > 1$ it generates high market prices of risk and low interest rates close to those observed in the data. While $\gamma > 1$ is more plausible given the evidence in Mehra and Prescott (1985), we note that the intuition for price-dividend ratios and stock return volatilities relies only on the relative strength of income and substitution effects. Therefore, the inability to match simultaneously the dynamic patterns and the values of equilibrium parameters should be attributed to the fact that for CRRA preferences the intertemporal elasticity of substitution (IES) equals $1/\gamma$ and hence high IES needed to generate the substitution effect is only possible for $\gamma < 1$. However, more general recursive preferences allow for IES independent of risk aversion parameter γ [Epstein and Zin (1989); Duffie and Epstein (1992)]. Our results suggest that in a model with recursive preferences with both IES and risk aversion exceeding unity [as in Bansal and Yaron (2004)] it might be possible to match interest rates and market prices of risk, as well as generate procyclical price-dividend ratios, countercyclical stock return volatilities, excess volatilities and other patterns consistent with the empirical literature.

3.2. Equilibrium with Margin Requirements and Borrowing Constraints

In this subsection we evaluate the effects of constraint $\theta_{ot} \leq \bar{\theta}$ with $\bar{\theta} > 1$, which can be interpreted as a margin constraint or a borrowing constraint, as discussed in Subsection 2.2. To make this constraint binding we allow the investors to be heterogeneous both in risk aversions and beliefs. We also consider the case where the investors have identical risk aversions but heterogeneous beliefs which allows to separate the effects of heterogeneous beliefs and risk aversions. In contrast to the limited participation economy of Subsection 3.1 the consumption share of the constrained investor, y_t , is now procyclical, which we verify numerically by showing that volatility σ_{y_t} is negative, and hence the instantaneous changes $d\delta_t$ and dy_t are positively correlated. Intuitively, in our model the constrained investor is optimistic and not very risk averse, and hence the optimist holds a larger share of wealth in stocks than the pessimist, so that $\theta_{ot}^* \geq 1 \geq \theta_{pt}^*$ [panels (e) and (f) of Figures 3 and 4]. Since the constrained investor is more exposed to stock market fluctuations bad shocks transfer wealth from the constrained to the unconstrained investor, resulting in the procyclicality of the optimist's consumption share y_t .

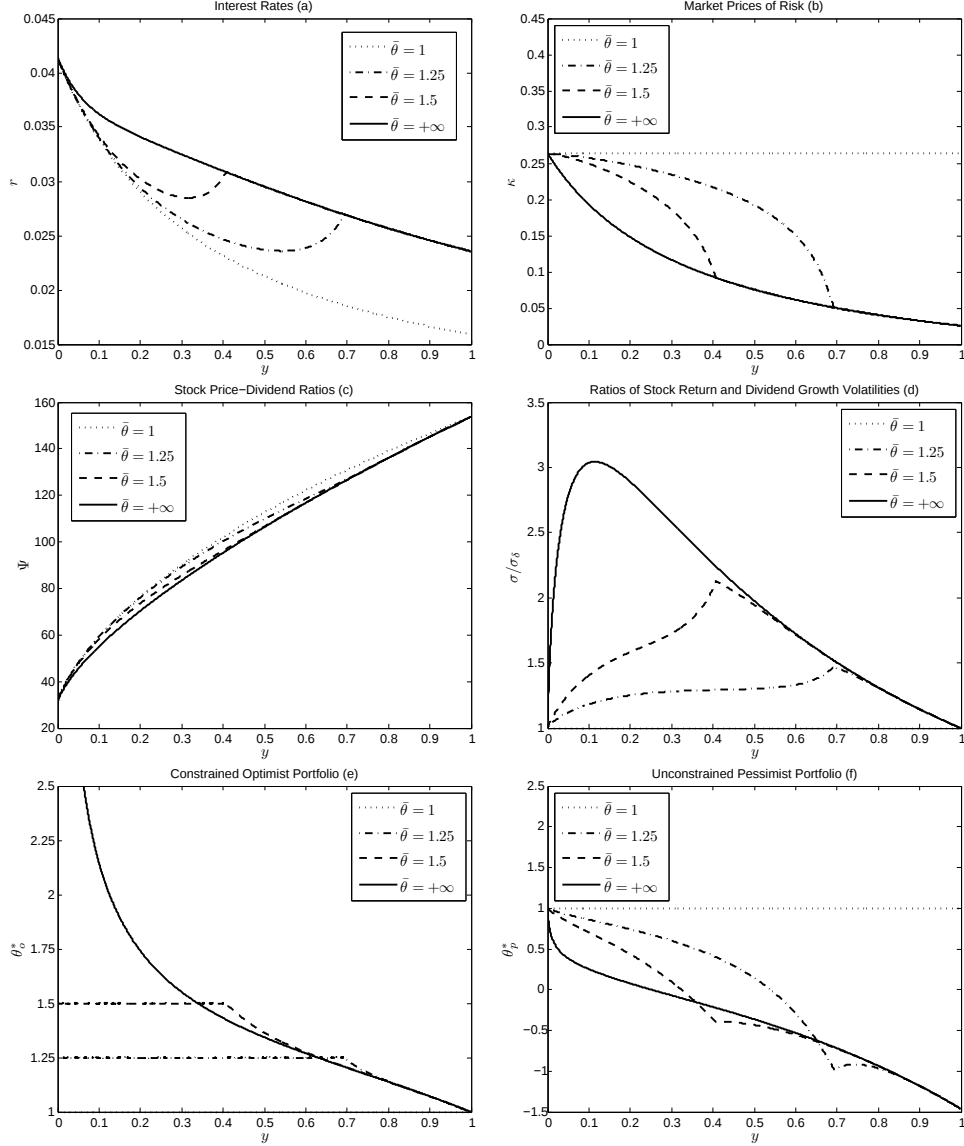


Figure 3: Equilibrium with Constraints, Heterogeneous Beliefs and Risk Aversions.

Figure 3 presents the equilibrium parameters as functions of consumption share y_t . Variable y_t is procyclical and increases in good times. The parameters of dividend process are: $\mu_\delta = 1.8\%$, $\sigma_\delta = 3.2\%$ [e.g., Campbell (2003)]. Time discount, risk aversions, and beliefs are as follows: $\rho = 0.01$, $\gamma_o = 0.8$, $\gamma_p = 3$, $\mu_\delta^o = \mu_\delta$ and $\mu_\delta^p = 0.7\mu_\delta$.

Figure 3 presents the equilibrium parameters as functions of optimist's consumption share y_t for the economy populated by investors with heterogeneous risk aversions ($\gamma_o = 0.8$, $\gamma_p = 3$), and heterogeneous beliefs ($\mu_\delta^o = \mu_\delta$, $\mu_\delta^p = 0.7\mu_\delta$). Thus, the optimist is less risk averse than the pessimist and has correct beliefs, while the pessimist is more risk averse and irrationally underestimates the dividend growth. Figure 3 presents for comparison the equilibrium in the

same economy but assuming that both investors have identical risk aversions ($\gamma_o = 0.8$, $\gamma_p = 0.8$).

Panels (a) and (b) of Figure 3 demonstrate that, holding optimist's consumption share y_t fixed, tighter constraints decrease interest rates r_t and increase market prices of risk κ_t making these parameters closer to observed values of $\kappa = 36\%$ and $r = 2\%$, consistently with the intuition in Subsection 2.1. Moreover, panel (a) identifies a non-monotonic patterns in interest rates for $1 < \bar{\theta} < +\infty$. According to our intuition in Subsection 2.1, the interest rate decreases below the unconstrained benchmark when the constraint is binding. However, in the limit as $y \rightarrow 0$ the economy is dominated by the unconstrained pessimist, and hence the interest rate reverts back to the unconstrained benchmark, giving rise to a non-monotonic pattern.

The market price of risk in panel (b) is countercyclical and decreases with tighter constraints, consistently with the intuition in Section 2.1. Furthermore, panel (c) of Figure 3 demonstrates that the price-dividend ratio is procyclical and increases with tighter constraints. Similarly to the limited participation economy analyzed in Subsection 3.1 the impact of constraints on price-dividend ratios depends on the relative strength of income and substitution effects. For each investor the intertemporal elasticity of substitution determines the pro- or counter- cyclicity of the individual wealth-consumption ratios.¹⁴ The formula for the price-dividend ratio Ψ_t in terms of wealth-consumption ratios in (41) then determines the procyclicality or countercyclicality of Ψ_t .

Panel (d) of Figure 3 presets the ratios of stock return and dividend growth volatilities. In the unconstrained case the investor heterogeneity generates sizable excess volatility which varies countercyclically for a wide range of consumption shares $y \in [0.1, 1]$, as also pointed out in the literature [e.g., Longstaff and Wang (2008); Bhamra and Uppal (2009, 2010)]. The graphs in panel (d) demonstrate that margin constraints significantly decrease the stock market volatility and make it procyclical in those states of the economy where the constraints are binding. This finding is consistent with the empirical evidence in Hardouvelis and Theodossiou (2002) who study 22 episodes of changes in margin requirements by the Federal Reserve between 1934 and 1974 and demonstrate that tighter margins lead to lower stock market volatilities. Intuitively, tighter constraints limit the investors' ability to trade on their heterogeneity making their stock holdings more homogeneous. Consequently, the equilibrium parameters become closer to those in the homogeneous investor economy. In particular, stock return volatility moves closer to the stock return volatility in the homogeneous investor economy, given by the volatility of dividends σ_δ .

Panels (e) and (f) present the optimal investment policies of the optimist and the pessimist, respectively. The graphs in panel (e) demonstrate that since the optimist is not very risk averse, the optimist levers up and allocates a higher fraction of wealth to stocks than the pessimist. Whether the constraint is binding or not in our model is determined by the amount of liquidity

¹⁴Similarly to Mele (2007) it can be shown that the procyclicality (countercyclicality) of wealth-consumption ratios H_{it} is determined by the countercyclicality (procyclicality) of *risk-adjusted discount rates* equal to the coefficient in front of H_{it} in HJB equations (38).

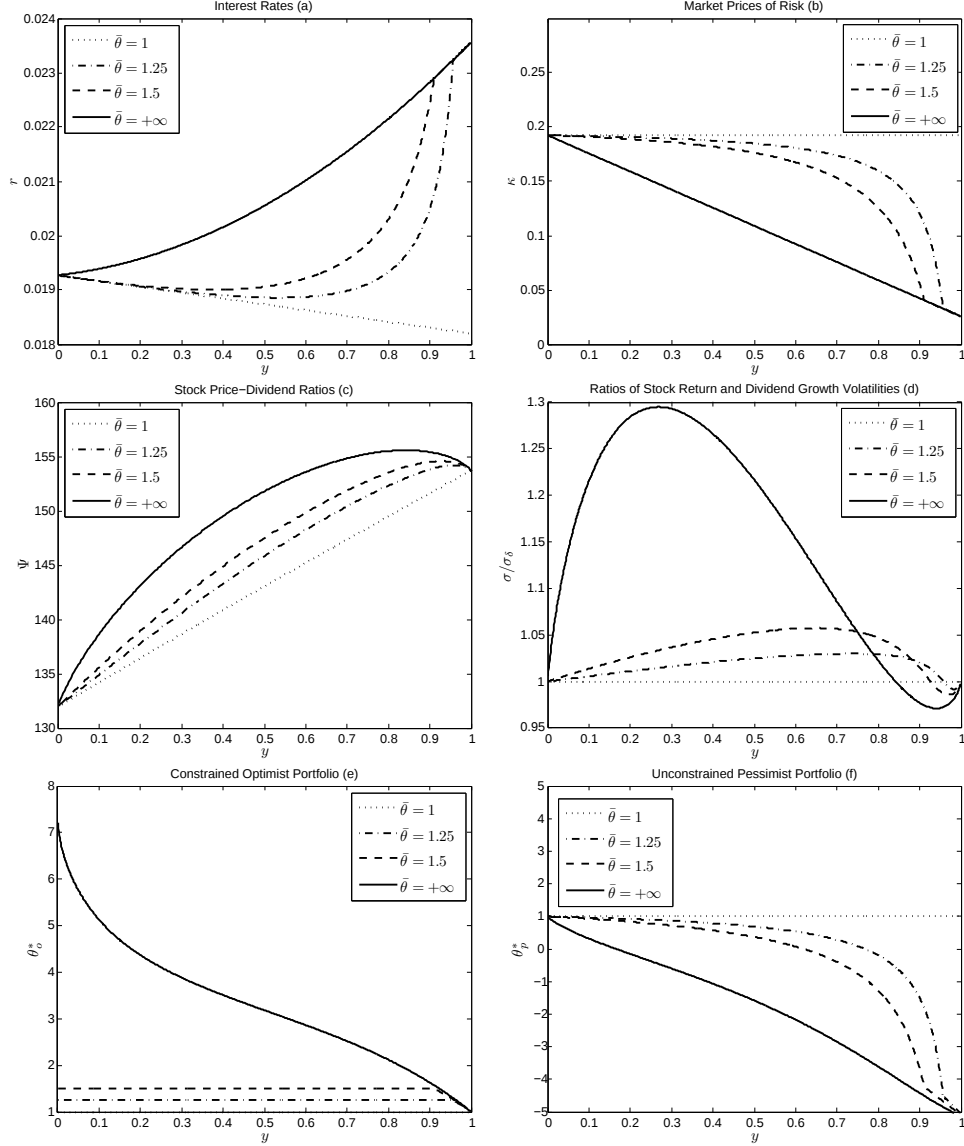


Figure 4: Equilibrium with Constraints and Heterogeneous Beliefs.

Figure 4 presents the equilibrium parameters as functions of consumption share y_t . Variable y_t is procyclical and increases in good times. The parameters of dividend process are: $\mu_\delta = 1.8\%$, $\sigma_\delta = 3.2\%$ [e.g., Campbell (2003)]. Time discount, risk aversions, and beliefs are as follows: $\rho = 0.01$, $\gamma_o = 0.8$, $\gamma_p = 0.8$, $\mu_\delta^o = \mu_\delta$ and $\mu_\delta^p = 0.7\mu_\delta$.

available for borrowing. In particular, since the pessimist is the only provider of liquidity for borrowing, when the economy is dominated by the optimist (i.e. y_t is close to 1) the optimist's leverage ratio declines and the constraint does not bind. Furthermore, when the economy is dominated by the pessimist willing to supply liquidity (i.e. y_t is close to 0) the optimist can easily lever up, and hence margin constraint is more likely to bind.

Figure 4 presents the equilibrium parameters for the case where investors have identical risk

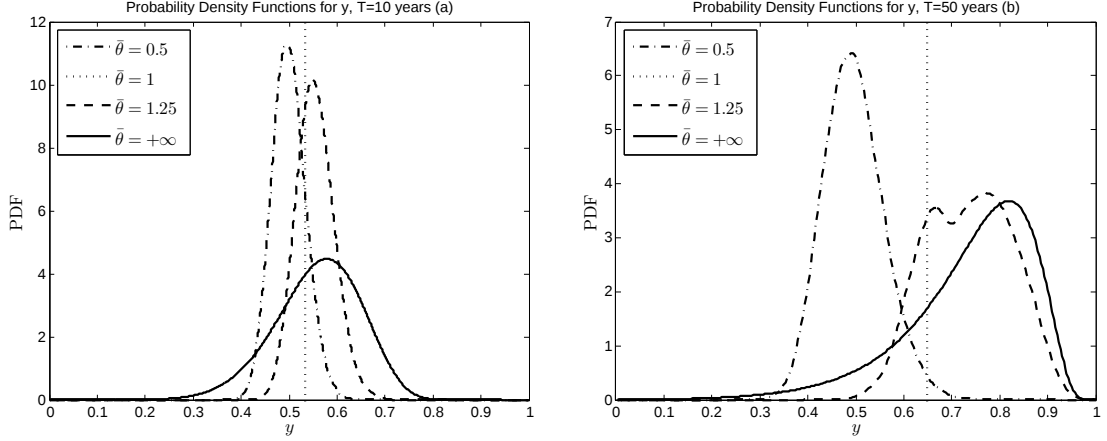


Figure 5: Probability Density Functions for Constrained Rational Investor's Consumption Share y_t .

Figure 5 presents probability density functions for constrained rational investor's consumption share y_t for different constraints. The parameters of dividend process are: $\mu_\delta = 1.8\%$, $\sigma_\delta = 3.2\%$ [e.g., Campbell (2003)]. Moreover, $\rho = 0.01$, $\gamma_o = 0.8$, $\gamma_p = 3$, $\mu_\delta^o = \mu_\delta$ and $\mu_\delta^p = 0.7\mu_\delta$.

aversions. The properties of equilibrium and the effect of portfolio constraints can be analyzed as in the previous case. The comparison of results in Figures 3 and 4 demonstrates that the heterogeneity in risk aversions generates higher stock return volatilities and market prices of risk. The equilibrium in an economy where investors are heterogeneous only in risk aversions has similar properties and can be analyzed in the same way as above. It is worth noting that in this equilibrium the pessimist's optimal investment policy θ_{pt}^* is always positive and hence short-sale constraint never binds.

3.3. Survival of Irrational Investors

Finally, we address the question of how the constraints affect the consumption sharing between the investors. In particular, we study the conditional probability density function (p.d.f.) of consumption share y_t and its evolution in the course of time. There is a growing literature investigating the long-run impact of irrational investors on equilibrium parameters in unconstrained economies [e.g., Kogan, Ross, Wang and Westerfield (2004); Yan (2008); Berrada (2009); Borovička (2009); Dumas, Kurshev and Uppal (2009); Bhamra and Uppal (2010); Cvitanic et al (2010); Cvitanic and Malamud (2010b)], which finds that under plausible model parameters the consumption share of the irrational investor slowly converges to zero. In this subsection we study the effect of portfolio constraints on the survival of investors.

The p.d.f. function in the unconstrained case serves as a benchmark against which we compare the results in the economies with constraints. The methodological contribution of this subsection is that for the case of the unconstrained heterogeneous investor economy we derive the probability

density function in closed form while the existing literature either derives them by Monte Carlo simulations or studies their asymptotic properties over large time horizons. Proposition 4 reports the p.d.f. function for the unconstrained benchmark.

Proposition 4. *If the economy is unconstrained and condition $(\gamma_o - \gamma_p)\sigma_\delta - \Delta_\delta \neq 0$ is satisfied the probability density function of consumption share y_τ at time $\tau > t$ conditional on consumption share y_t at time t is given by:*

$$p(y, \tau; y_t, t) = \frac{1}{\sqrt{2c_2^2(\tau - t)\pi}} \left(\frac{\gamma_o}{y} + \frac{\gamma_p}{1 - y} \right) \exp \left\{ -\frac{\left(\gamma_o \ln \frac{y}{y_t} - \gamma_p \ln \frac{1-y}{1-y_t} - c_1(\tau - t) \right)^2}{2c_2^2(\tau - t)} \right\}, \quad (47)$$

where

$$c_1 = (\gamma_p - \gamma_o) \left(\mu_\delta - \frac{\sigma_\delta^2}{2} \right) + \frac{1}{2} \left[\left(\frac{\mu_\delta^p - \mu_\delta}{\sigma_\delta} \right)^2 - \left(\frac{\mu_\delta^o - \mu_\delta}{\sigma_\delta} \right)^2 \right], \quad c_2 = (\gamma_o - \gamma_p)\sigma_\delta - \Delta_\delta. \quad (48)$$

The expression for the p.d.f. function in (47) demonstrates that the distribution of consumption share y_τ depends on time parameter τ and hence, in general the p.d.f. function is *non-stationary* with the exception of a measure zero case with $c_1 = 0$ and $c_2 = 0$ when the consumption share y_t remains constant. Parameter c_1 is a survival index introduced in Yan (2008), who studies similar economy with homogeneous integer risk aversions and derives p.d.f. functions via Monte Carlo simulations. Clearly, in the limit as τ goes to infinity only the optimist survives if $c_1 > 0$ and only the pessimist survives if $c_1 < 0$. When $c_1 = 0$ it can easily be shown that in the limit each of the investors survives with probability 0.5. We also note that in a similar way it is also possible to derive p.d.f. functions for various equilibrium parameters, which is however beyond our scope.

We next consider the economy of Subsection 3.2 where investors are heterogeneous in risk aversions and beliefs, and face borrowing constraints. We assume for simplicity that the optimist has correct beliefs about the dividend growth while the pessimist irrationally underestimates it. Via Monte Carlo simulations we obtain the p.d.f. functions for the constrained economies. We also note that when $\bar{\theta} = 1$ the consumption share is a deterministic variable, as demonstrated in Subsection 2.3. Figure 5 shows probability density functions of y_t for time horizons of ten years [panel (a)] and fifty years [panel (b)], generated via Monte-Carlo simulations. The results on Figure 5 demonstrate that constraints slow down the elimination of the irrational pessimist whose consumption share remains significant even after 50 years. Moreover, very tight constraints, e.g. $\bar{\theta} = 0.5$, may decrease the optimist's consumption share in the long run. Intuitively, this effect is due to the fact that the portfolio constraint reduces the comparative advantage of the optimist's more accurate information.

4. Conclusion

Despite numerous applications of dynamic equilibrium models with heterogeneous investors facing portfolio constraints, little is known about the equilibrium when we depart from the assumption of logarithmic preferences. In this work we propose a solution method for computing the equilibrium in economies with heterogeneous investors that differ in their risk aversions, beliefs, and portfolio constraints. We completely characterize the equilibrium in terms of investors' wealth-consumption ratios satisfying a system of quasilinear equations, which we solve via an iterative procedure with fast convergence. We evaluate the impact of portfolio constraints on equilibrium parameters in models with limited stock market participation and margin constraints.

In our limited stock market participation model two investors have identical risk aversions and beliefs, one investor is unconstrained while the other faces an upper bound on the proportion of wealth that can be invested in stocks. This minor deviation from the classical Lucas economy has rich asset pricing implications, and when the intertemporal elasticity of substitution is greater than unity the model generates equilibrium parameters consistent with the empirical literature. In a setting with margin constraints we allow the investors to be heterogeneous in risk aversions and beliefs and study the interaction between investor heterogeneity and portfolio constraints. We demonstrate that margin constraints lead to complex patterns in equilibrium parameters, and under plausible model parameters decrease stock return volatilities.

In the unconstrained benchmark economy we derive the equilibrium parameters in terms of hypergeometric functions by generalizing the approach of Longstaff and Wang (2008), and provide closed form expression for the p.d.f. function of optimist's consumption share. Given the tractability of our analysis we believe that our approach for finding equilibria in constrained and unconstrained economies can be employed to study similar problems with recursive preferences and more general dividend processes. The derivation of Cvitanić-Karatzas adjustment parameters might prove useful in solving portfolio choice problems in partial equilibrium settings. Finally, an interesting direction for future research is the equilibrium in the economy with two constrained investors and intermittently binding constraints.

Appendix A: Proofs

Proof of Proposition 1. Taking into account expression (3) for Brownian motion w_t^o in terms of Brownian motion w_t^p we rewrite the processes for state price densities (16) as follows:

$$d\xi_{ot} = -\xi_{ot}[r_{ot}dt + \kappa_{ot}dw_t^o], \quad d\xi_{pt} = -\xi_{pt}[(r_{pt} + \Delta_{\delta t}\kappa_{pt})dt + \kappa_{pt}dw_t^o]. \quad (\text{A.1})$$

Optimal consumptions in fictitious economies can be expressed from first order conditions (18). Substituting c_{it}^* into consumption clearing condition in (11), applying Itô's Lemma to both sides and matching the terms after some algebra we obtain:

$$\begin{aligned} \frac{r_t - \rho}{\Gamma_t} + \frac{y_t}{\gamma_o} f_o(\nu_{ot}^*) + \frac{1 - y_t}{\gamma_p} f_p(\nu_{pt}^*) &+ \frac{1}{2} \left(\frac{1 + \gamma_o}{\gamma_o^2} y_t \kappa_{ot}^2 + \frac{1 + \gamma_p}{\gamma_p^2} (1 - y_t) \kappa_{pt}^2 \right) \\ &= y_t \frac{\kappa_{ot}}{\gamma_o} \frac{\mu_{\delta t}^o}{\sigma_{\delta t}} + (1 - y_t) \frac{\kappa_{pt}}{\gamma_p} \frac{\mu_{\delta t}^p}{\sigma_{\delta t}}, \end{aligned} \quad (\text{A.2})$$

$$y_t \frac{\kappa_{ot}}{\gamma_o} + (1 - y_t) \frac{\kappa_{pt}}{\gamma_p} = \sigma_{\delta t}. \quad (\text{A.3})$$

Next, from the definition of κ_{it} in terms of adjustments ν_{it}^* in (17) and equation (A.3) we obtain:

$$\kappa_{ot} = \Gamma_t \left(\sigma_{\delta t} + \frac{1 - y_t}{\gamma_p} \left(\Delta_{\delta t} + \frac{\nu_{ot}^* - \nu_{pt}^*}{\sigma_t} \right) \right), \quad \kappa_{pt} = \Gamma_t \left(\sigma_{\delta t} - \frac{y_t}{\gamma_o} \left(\Delta_{\delta t} + \frac{\nu_{ot}^* - \nu_{pt}^*}{\sigma_t} \right) \right). \quad (\text{A.4})$$

Substituting κ_{it} from (A.4) into the expression for interest rates (A.2) after some algebra we obtain r_t in (21). To find κ_t , we first note that from the definition of κ_{it} in (17):

$$\kappa_t = \kappa_{ot} + \frac{\mu_t - \mu_t^o}{\sigma_t} - \frac{\nu_{ot}^*}{\sigma_t}. \quad (\text{A.5})$$

Next, to obtain κ_t we express the disagreement $(\mu_t - \mu_t^o)/\sigma_t$ in terms of dividend disagreement $(\mu_{\delta t} - \mu_{\delta t}^o)/\sigma_{\delta t}$. In particular, by comparing the processes for dividends under the true (1) and the subjective (2) probability measures, and then doing the same comparison for stock prices in (5) we find that:

$$dw_t = dw_t^o - \frac{\mu_{\delta t} - \mu_{\delta t}^o}{\sigma_{\delta t}} dt, \quad dw_t = dw_t^o - \frac{\mu_t - \mu_t^o}{\sigma_t} dt. \quad (\text{A.6})$$

Comparing the two expressions we obtain the following consistency condition:

$$\frac{\mu_t - \mu_t^o}{\sigma_t} = \frac{\mu_{\delta t} - \mu_{\delta t}^o}{\sigma_{\delta t}}. \quad (\text{A.7})$$

Substituting (A.7) and κ_{ot} from (A.4) into (A.5) after straightforward algebra we obtain expression (22) for κ_t . We next obtain σ_{yt} and μ_{yt} in (23)–(24) by substituting c_{ot}^* from first order conditions (18) into the definition of y_t in (19), then applying Itô's Lemma (under the true probability measure) to both sides of (19) and matching the terms.

The expressions (27) for r_{it} are obtained directly from their expressions in terms of adjustments in (17), while the expressions (28) for κ_{it} are obtained from (23) and (A.4) by expressing

$\Delta_{\delta t} + (\nu_{ot}^* - \nu_{pt}^*)/\sigma_t$ in terms of σ_{yt} and then substituting into (A.4). Drift parameter μ_{yt}^o in (29) is obtained by substituting c_{ot}^* from first order conditions (18) into the definition of y_t in (19), then applying Itô's Lemma (under the optimist's beliefs) to both sides of (19) and matching the terms. Finally, the parameter μ_{yt}^p is obtained from the following consistency condition, derived similarly to condition (6): $(\mu_{yt}^o - \mu_{yt}^p)/\sigma_{yt} = \Delta_{\delta t}$.

Q.E.D.

Proof of Corollary 1. Applying Itô's Lemma to both sides of the first order conditions for consumption (18) and matching the terms we find that

$$\sigma_{c_{ot}^*} = \frac{\kappa_{ot}}{\gamma_o}, \quad \sigma_{c_{pt}^*} = \frac{\kappa_{pt}}{\gamma_p}. \quad (\text{A.8})$$

Substituting κ_{ot} and κ_{pt} from (28) into (A.8) we obtain expressions (31) for volatilities $\sigma_{c_{it}^*}$.

Q.E.D.

Proof of Proposition 2. We first derive expressions for price-dividend ratio Ψ_t and stock return volatility σ_t . From the market clearing conditions in (11) it follows that $S_t = W_{ot}^* + W_{pt}^*$. The price-dividend ratio is then given by:

$$\Psi_t = \frac{S_t}{\delta_t} = \frac{c_{ot}^*}{\delta_t} \frac{W_{ot}^*}{c_{ot}^*} + \frac{c_{pt}^*}{\delta_t} \frac{W_{pt}^*}{c_{pt}^*},$$

which gives Ψ_t in (41), where H_{it} denote wealth-consumption ratios. The expression for σ_t in (41) is obtained by applying Itô's Lemma to $S_t = \Psi_t \delta_t$.

To obtain σ_{yt} we first substitute κ_{ot} from (28) into investment policy (39) and obtain:

$$\theta_{ot}^* = \frac{1}{\sigma_t} \left(\sigma_{\delta} - \sigma_{yt} \left(1 + \frac{\partial H_{ot}}{\partial y_t} \frac{y_t}{H_{ot}} \right) \right). \quad (\text{A.9})$$

Taking into account that by assumption $\sigma_t > 0$ and σ_t is given by (41), the inequality $\theta_{ot}^* \leq \bar{\theta}$ (which holds because the optimist is constrained) can be rewritten as follows:

$$\sigma_{yt} G_t \geq (1 - \bar{\theta}) \sigma_{\delta}, \quad \text{where} \quad G_t = 1 + \frac{\partial H_{ot}}{\partial y_t} \frac{y_t}{H_{ot}} - \bar{\theta} \frac{\partial \Psi_t}{\partial y_t} \frac{y_t}{\Psi_t}, \quad (\text{A.10})$$

and G_t is a shorthand notation for $G(y_t)$. If the constraint is binding (A.10) holds as equality from which σ_{yt} can be obtained.

On the other hand, from the expression for σ_{yt} in terms of adjustments in (23) and the fact that $\nu_{ot}^* \leq 0$ [case (c) in Table 1] while $\nu_{pt}^* = 0$ [case (a) in Table 1] it follows that:

$$\sigma_{yt} \geq \frac{((\gamma_o - \gamma_p) \sigma_{\delta} - \Delta_{\delta})(1 - y_t)}{\gamma_o(1 - y_t) + \gamma_p y_t}. \quad (\text{A.11})$$

The complementary slackness condition (44) implies that $\nu_{ot}^* = 0$ if the constraint does not bind. Therefore, the expression for σ_{yt} in (23) implies that when constraint does not bind inequality (A.11) holds as equality. If G_t in (A.10) is positive, inequalities (A.10) and (A.11) imply that:

$$\sigma_{yt} = \max\left\{\frac{((\gamma_o - \gamma_p)\sigma_\delta - \Delta_\delta)(1 - y_t)}{\gamma_o(1 - y_t) + \gamma_p y_t}, \frac{(1 - \bar{\theta})\sigma_\delta}{G_t}\right\}, \quad (\text{A.12})$$

which gives the formula for σ_{yt} in Proposition 2 for the case $G_t > 0$.

We now demonstrate that the constraint does not bind when $G(y_t) \leq 0$. Suppose, there exists $\tilde{y}$ such that $G(\tilde{y}) \leq 0$ and the constraint is binding. Assuming $G(y)$ is continuous, when $y \rightarrow 0$ we obtain that $G(y) \rightarrow 1$. Therefore, $G(0) > 0$ and hence $\sigma_y(0)$ can be obtained by passing to limit $y \rightarrow 0$ in (A.12), which yields:

$$\sigma_y(0) = \max\{((\gamma_o - \gamma_p)\sigma_\delta - \Delta_\delta)/\gamma_o, (1 - \bar{\theta})\sigma_\delta\}. \quad (\text{A.13})$$

Next, consider four cases.

1. $\bar{\theta} < 1$.

It follows from (A.13) that $\sigma_y(0) > 0$. However, by assumption, $G(\tilde{y}) \leq 0$ and the constraint is binding. Therefore, inequality (A.10) is satisfied as equality, and hence $\sigma_y(\tilde{y}) = (1 - \bar{\theta})\sigma_\delta/G(\tilde{y}) \leq 0$. Since σ_y is continuous by assumption, and takes different signs at 0 and $\tilde{y}$ there exists y^* such that $\sigma_y(y^*) = 0$. Therefore, it follows from the expression for σ_t in (41) that $\sigma(y^*) = \sigma_\delta$. Substituting $\sigma(y^*)$ into the expression for θ_{ot}^* in (A.9) we obtain that $\theta_o^*(y^*) = 1$, which leads to contradiction since in equilibrium $\theta_{ot}^* \leq \bar{\theta} < 1$.

2. $\bar{\theta} \geq 1$ and $(\gamma_o - \gamma_p)\sigma_\delta - \Delta_\delta < 0$.

It follows from (A.13) that $\sigma_y(0) \leq 0$. Similarly to the previous case, $\sigma_y(\tilde{y}) = (1 - \bar{\theta})\sigma_\delta/G(\tilde{y}) \geq 0$, and there exists y^* such that $\sigma_y(y^*) = 0$ and $\theta_o^*(y^*) = 1 < \bar{\theta}$. Therefore, the constraint is not binding, in which case, as argued above,

$$\sigma_y(y^*) = \frac{((\gamma_o - \gamma_p)\sigma_\delta - \Delta_\delta)(1 - y^*)}{\gamma_o(1 - y^*) + \gamma_p y^*} \neq 0,$$

which leads to contradiction.

3. $\bar{\theta} \geq 1$ and $(\gamma_o - \gamma_p)\sigma_\delta - \Delta_\delta = 0$.

Consider the case $\bar{\theta} = 1$. In this case $\sigma_y = 0$ irrespective of whether constraint binds or not, and hence formula (40) for σ_{yt} holds. Suppose now $\bar{\theta} > 1$. The case with $\sigma_y = 0$ is still an equilibrium. Indeed, if $\sigma_y = 0$ then it follows from (A.9) that $\theta_{ot}^* = 1 < \bar{\theta}$, therefore the constraint is not binding and hence $\sigma_y = 0$ because $(\gamma_o - \gamma_p)\sigma_\delta - \Delta_\delta = 0$.

4. $\bar{\theta} \geq 1$ and $(\gamma_o - \gamma_p)\sigma_\delta - \Delta_\delta > 0$.

In this case we show that the constraint never binds even for $\bar{\theta} = 1$. Clearly, the constraint with $\bar{\theta} > 1$ never binds as well. Suppose, there exists $\tilde{y}$ such that $G(\tilde{y}) < 0$. Then, inequality (A.10) implies $\sigma_y(\tilde{y}) \leq 0$, while (A.11) implies that $\sigma_y(\tilde{y}) > 0$, which leads to contradiction.

As a result, $G(y) \geq 0$ for all $y \in [0, 1]$. From (A.12) it follows that if $G(y) > 0$ σ_{yt} is equal to σ_{yt} in the unconstrained case, and hence the constraint is not binding.

We now analyze the case $G(y) = 0$. Suppose, there exists an interval such that $G(y) = 0$ for all $y \in [y_1, y_2]$. Then, using the definition of G_t in (A.10) and the fact that $\Psi_t = y_t H_{ot} + (1 - y_t) H_{pt}$, by solving the differential equation $G_t = 0$ we find that $H_{ot} = C_0(1 - y_t) H_{pt} / y_t$, where C_0 is a constant. Substituting H_{ot} into the HJB equation (38) we derive a new PDE for H_{pt} , which is different from the HJB equation for H_{pt} . Comparing the two equations we conclude that the same function cannot satisfy them, which leads to contradiction. Therefore, $G(y) = 0$ only in isolated points, in which by continuity σ_{yt} coincides with σ_{yt} in the unconstrained case.

Therefore, in all considered cases the constraint does not bind when $G_t \leq 0$ which leads to formula (40) for σ_{yt} . The formula for ν_{ot}^* in (42) directly follows from (23) and the fact that $\nu_{pt}^* = 0$ since the pessimist is unconstrained. Finally, equation (43) for y_0 is obtained from $t = 0$ budget constraints, given by:

$$s_i S_0 + b_i = W_{i0}^*, \quad i = o, p, \quad (\text{A.14})$$

where s_i and b_i denote time-0 endowments in units of stocks and bonds specified in Subsection 1.2. Substituting $S_0 = \Psi_0 \delta_0$, $W_{o0}^* = y_0 H_o(y_0) \delta_0$, and $W_{p0}^* = (1 - y_0) H_p(y_0) \delta_0$ into time-0 budget constraint (A.14) it can easily be observed that both constraints are satisfied whenever equation (43) for y_0 holds.

Q.E.D.

Proof of Proposition 3. Consider the unconstrained benchmark case. The investors' state prices follow processes:

$$\begin{aligned} d\xi_{pt} &= -\xi_{pt}[r_t dt + \kappa_{pt} dw_t^p], \\ d\xi_{ot} &= -\xi_{ot}[r_t dt + \kappa_{ot} dw_t^o] \\ &= -\xi_{ot}[(r_t - \kappa_{ot} \Delta_\delta) dt + \kappa_{ot} dw_t^p]. \end{aligned} \quad (\text{A.15})$$

From the FOC (18) we find that the ratio of marginal utilities λ_t is given by:

$$\lambda_t \equiv \frac{(c_{pt}^*)^{-\gamma_p}}{(c_{ot}^*)^{-\gamma_o}} = \lambda_0 \frac{\xi_{pt}}{\xi_{ot}}. \quad (\text{A.16})$$

Applying Itô's Lemma to λ_t in (A.16) we find that λ_t is a martingale following a GBM process:

$$d\lambda_t = \lambda_t \Delta_\delta dw_t^p. \quad (\text{A.17})$$

From the definition of λ_t in (A.16), first order conditions (18) and the consumption clearing condition $c_{ot}^* + c_{pt}^* = \delta_t$ we obtain:

$$\frac{c_{pt}^*}{\delta_t} \equiv 1 - y_t = f(\tilde{\lambda}_t), \quad \tilde{\lambda}_t = \lambda_t^{1/\gamma_o} \delta_t^{\gamma_p/\gamma_o - 1}, \quad (\text{A.18})$$

where $f(z)$ is an implicit function satisfying equation:

$$zf(z)^{\gamma_p/\gamma_o} + f(z) = 1, \quad (\text{A.19})$$

and λ_t and δ_t follow GBM processes (A.17) and (32) and hence are explicitly given by:

$$\lambda_\tau = \lambda_t e^{-0.5\Delta_\delta^2(\tau-t) + \Delta_\delta(w_\tau^p - w_t^p)}, \quad \delta_\tau = \delta_t e^{(\mu_\delta^p - 0.5\sigma_\delta^2)(\tau-t) + \sigma_\delta(w_\tau^p - w_t^p)}. \quad (\text{A.20})$$

Since the financial market is complete, the price-dividend ratio can be obtained from the present-value formula:

$$\frac{S_t}{\delta_t} = \frac{1}{\xi_{pt}} E_t^p \left[\int_t^{+\infty} \xi_{p\tau} \frac{\delta_\tau}{\delta_t} d\tau \right], \quad (\text{A.21})$$

where $E_t^p[\cdot]$ denotes the expectation operator under the pessimist's probability measure. Taking into account FOC (18) and expression (A.18) we rewrite present-value formula (A.21) as follows:

$$\begin{aligned} \frac{S_t}{\delta_t} &= \frac{\delta_t^{-\gamma_p}}{(c_{pt}^*)^{-\gamma_p}} E_t^p \left[\int_t^{+\infty} e^{-\rho(\tau-t)} \left(\frac{c_{p\tau}^*}{\delta_\tau} \right)^{-\gamma_p} \left(\frac{\delta_\tau}{\delta_t} \right)^{1-\gamma_p} d\tau \right], \\ &= (1 - y_t)^{\gamma_p} E_t^p \left[\int_t^{+\infty} e^{-\rho(\tau-t)} f \left(\tilde{\lambda}_t \left(\frac{\lambda_\tau}{\lambda_t} \right)^{1/\gamma_o} \left(\frac{\delta_\tau}{\delta_t} \right)^{\gamma_p/\gamma_o - 1} \right)^{-\gamma_p} \left(\frac{\delta_\tau}{\delta_t} \right)^{1-\gamma_p} d\tau \right]. \end{aligned} \quad (\text{A.22})$$

We next substitute λ_τ and δ_τ from (A.20) into (A.22) and rewrite the conditional expectation operator as an integral, noting that $z = (w_\tau^p - w_t^p)/\sqrt{\tau - t}$ is normally distributed as $z \sim N(0, 1)$. After some algebra we obtain:

$$\begin{aligned} \frac{S_t}{\delta_t} &= (1 - y)^{\gamma_p} \int_0^{+\infty} \left[\int_{-\infty}^{+\infty} e^{-\rho\tau} f \left(\tilde{\lambda}_t e^{a_1\tau + a_2\sqrt{\tau}(z - (1-\gamma_p)\sigma_\delta\sqrt{\tau})} \right)^{-\gamma_p} e^{-0.5(z - (1-\gamma_p)\sigma_\delta\sqrt{\tau})^2} dz \right] \times \\ &\quad \frac{1}{\sqrt{2\pi}} e^{((1-\gamma_p)\mu_\delta^p - 0.5\gamma_p(1-\gamma_p)\sigma_\delta^2)\tau} d\tau, \end{aligned} \quad (\text{A.23})$$

where constants a_1 and a_2 are given in (46). Now we change the integration variable z in the inner integral in (A.23) to variable u given by $u = a_1\tau + a_2\sqrt{\tau}(z - (1-\gamma_p)\sigma_\delta\sqrt{\tau})$ and after some algebra and the change in the order of integration we obtain:

$$\begin{aligned} \frac{S_t}{\delta_t} &= \frac{(1 - y_t)^{\gamma_p}}{|a_2|\sqrt{2\pi}} \int_0^{+\infty} \left[\int_{-\infty}^{+\infty} f(\tilde{\lambda}_t e^u)^{-\gamma_p} e^{-\frac{(u - a_1\tau)^2}{2a_2^2\tau}} du \right] \frac{1}{\sqrt{\tau}} e^{-(\rho - (1-\gamma_p)\mu_\delta^p + 0.5\gamma_p(1-\gamma_p)\sigma_\delta^2)\tau} d\tau \\ &= \frac{(1 - y_t)^{\gamma_p}}{|a_2|\sqrt{2\pi}} \int_{-\infty}^{+\infty} f(\tilde{\lambda}_t e^u)^{-\gamma_p} e^{\frac{a_1}{a_2^2}u} F(u) du, \end{aligned} \quad (\text{A.24})$$

where $F(u)$ is given by:

$$F(u) = \int_0^{+\infty} \frac{1}{\sqrt{\tau}} e^{-b\tau - \frac{u^2}{2a_2^2\tau}} d\tau, \quad (\text{A.25})$$

and b is given in (46). Following the derivation of price-dividend ratios in Longstaff and Wang (2008), using formulas 3.471.9 and 8.469.3 from Gradshteyn and Ryzhik (2000) we obtain:

$$F(u) = \sqrt{\frac{\pi}{b}} e^{-\frac{\sqrt{2b}}{|a_2|}|u|}.$$

We then substitute $F(u)$ into expression (A.24), split the integral into a sum of two integrals: from $-\infty$ to 0 and from 0 to $+\infty$ and obtain:

$$\frac{S_t}{\delta_t} = \frac{(1-y_t)^{\gamma_p}}{|a_2|\sqrt{2\pi}} \left[\int_0^{+\infty} f(\tilde{\lambda}_t e^u)^{-\gamma_p} e^{\varphi_- u} du + \int_0^{+\infty} f(\tilde{\lambda}_t e^{-u})^{-\gamma_p} e^{-\varphi_+ u} du \right], \quad (\text{A.26})$$

where $\phi_{\pm}$ are defined in (46). We now demonstrate how to compute the first integral in (A.26) by taking $x = f(\tilde{\lambda}_t e^u)/(1-y_t)$ as a new integration variable, while the second integral is computed analogously. To perform the change of variable we first find $f'(z)$ by differentiating equation (A.19):

$$f'(z) = -\frac{f(z)^{\gamma_p/\gamma_o+1}}{\frac{\gamma_p}{\gamma_o}(zf(z)^{\gamma_p/\gamma_o} + f(z)) + (1 - \frac{\gamma_p}{\gamma_o})f(z)} = -\frac{f(z)^{\gamma_p/\gamma_o+1}}{\frac{\gamma_p}{\gamma_o} + (1 - \frac{\gamma_p}{\gamma_o})f(z)}, \quad (\text{A.27})$$

where the last equality uses equation (A.19) for function $f(z)$ to simplify the denominator. Moreover, from the definition of $f(z)$ in (A.19) setting $z = \tilde{\lambda}_t e^u$ we express e^u in terms of new variable x as follows:

$$e^u = \frac{1-x(1-y_t)}{\tilde{\lambda}_t(x(1-y_t))^{\gamma_p/\gamma_o}}. \quad (\text{A.28})$$

Using (A.28) and (A.27) we obtain:

$$dx = \frac{\tilde{\lambda}_t e^u f'(\tilde{\lambda}_t e^u)}{1-y_t} du = -\frac{(1-x(1-y_t))x}{\frac{\gamma_p}{\gamma_o} + (1 - \frac{\gamma_p}{\gamma_o})x(1-y_t)} du \quad (\text{A.29})$$

We determine the new limits of integration by passing to the limit in equation (A.28). When $u \rightarrow +\infty$ we obtain that $x \rightarrow 0$, and when $u \rightarrow 0$ it follows from (A.18) and (A.19) that $x \rightarrow 1$. Finally, noting from (A.18) and (A.19) that $\tilde{\lambda}_t(1-y_t)^{\gamma_p/\gamma_o} \equiv y_t$, after the change of variable we obtain:

$$\int_0^{+\infty} f(\tilde{\lambda}_t e^u)^{-\gamma_p} e^{\varphi_- u} du = \frac{(1-y_t)^{-\gamma_p}}{y_t^{\varphi_-}} \int_0^1 x^{-\gamma_p-1-\frac{\gamma_p}{\gamma_o}\varphi_-} (1-x(1-y_t))^{\varphi_- - 1} \left(\frac{\gamma_p}{\gamma_o} + \left(1 - \frac{\gamma_p}{\gamma_o}\right)x(1-y_t) \right) dx.$$

This integral which we denote by I_- can be expressed in terms of hypergeometric functions:

$$\begin{aligned} I_- = & \frac{(1-y_t)^{-\gamma_p}}{y_t^{\varphi_-}} \left[-\frac{\gamma_p}{\gamma_o} \frac{1}{\gamma_p + \frac{\gamma_p}{\gamma_o}\varphi_-} {}_2F_1\left(1 - \varphi_-, -\gamma_p - \frac{\gamma_p}{\gamma_o}\varphi_-, 1 - \gamma_p - \frac{\gamma_p}{\gamma_o}\varphi_-; 1-y_t\right) \right. \\ & \left. + \left(1 - \frac{\gamma_p}{\gamma_o}\right) \frac{1-y_t}{1 - \gamma_p - \frac{\gamma_p}{\gamma_o}\varphi_-} {}_2F_1\left(1 - \varphi_-, 1 - \gamma_p - \frac{\gamma_p}{\gamma_o}\varphi_-, 2 - \gamma_p - \frac{\gamma_p}{\gamma_o}\varphi_-; 1-y_t\right) \right], \end{aligned} \quad (\text{A.30})$$

where the hypergeometric function can be defined using the following Euler's formula [e.g., Abramowitz and Stegun (1965)]:

$${}_2F_1(a, b, c; z) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 \frac{x^{b-1}(1-x)^{c-b-1}}{(1-xz)^a} dx.$$

In a similar way we express the second integral in (A.26). We then simplify the resulting expression by employing the following formula from Abramowitz and Stegun (1965):

$${}_2F_1(a, b, c; z)(1-z)^{a+b-c} = {}_2F_1(c-a, c-b, c; z).$$

After some algebra we obtain the expression for price-dividend ratio (45) in Proposition 3.

Q.E.D.

Proof of Proposition 4. We first compute the cumulative conditional distribution function $\text{Prob}(y_\tau \leq y | \mathcal{F}_t)$. From the definition of function $f(\cdot)$ and $\tilde{\lambda}_t$ in (A.18) we find that $y_\tau = 1 - f(\tilde{\lambda}_\tau)$. Similarly to (A.6) we obtain that

$$w_t^p = w_t + \frac{\mu_\delta - \mu_\delta^p}{\sigma_\delta} t.$$

Substituting w_t^p into the expressions for processes λ_τ and δ_τ in (A.20) from the definition of $\tilde{\lambda}_\tau$ in (A.18) we obtain:

$$\tilde{\lambda}_\tau = \tilde{\lambda}_t e^{(c_1(\tau-t) + c_2(w_\tau - w_t))/\gamma_o} = \tilde{\lambda}_t e^{(c_1(\tau-t) + |c_2|z\sqrt{\tau-t})/\gamma_o}, \quad (\text{A.31})$$

where $z \sim N(0, 1)$, and c_1 and c_2 are given in (48).

Then, we compute the cumulative distribution function as follows:

$$\text{Prob}(y_\tau \leq y | \mathcal{F}_t) = \text{Prob}(1 - y \leq f(\tilde{\lambda}_\tau) | \mathcal{F}_t) = \text{Prob}(\tilde{\lambda}_\tau \leq f^{-1}(1 - y) | \mathcal{F}_t), \quad (\text{A.32})$$

where $f^{-1}(\cdot)$ is an inverse of function $f(\cdot)$ defined in (A.19), and the last equality in (A.32) uses the fact that $f(z)$ is a monotonically decreasing function since $f'(z) < 0$, as follows from (A.27). To derive $f^{-1}(1 - y)$ we substitute $z = f^{-1}(1 - y)$ into equation (A.19) and using the fact that by definition $f(f^{-1}(1 - y)) = 1 - y$ we obtain:

$$f^{-1}(1 - y) = \frac{y}{(1 - y)^{\gamma_p/\gamma_o}}. \quad (\text{A.33})$$

Similarly, from the definition of $\tilde{\lambda}_t$ in (A.18) and the equation for $f(z)$ in (A.19) we find that

$$\tilde{\lambda}_t = \frac{y_t}{(1 - y_t)^{\gamma_p/\gamma_o}}. \quad (\text{A.34})$$

Substituting $\tilde{\lambda}_\tau$ from (A.31), $\tilde{\lambda}_t$ from (A.34) and $f^{-1}(1 - y)$ from (A.33) into (A.32) we obtain:

$$\begin{aligned} \text{Prob}(y_\tau \leq y | \mathcal{F}_t) &= \text{Prob}\left(e^{(c_1(\tau-t) + |c_2|z\sqrt{\tau-t})/\gamma_o} \leq \frac{y}{(1 - y)^{\gamma_p/\gamma_o}} \frac{(1 - y_t)^{\gamma_p/\gamma_o}}{y_t} \middle| \mathcal{F}_t\right) \\ &= \text{Prob}\left(z \leq \frac{\gamma_o \ln \frac{y}{y_t} - \gamma_p \ln \frac{1-y}{1-y_t} - c_1(\tau-t)}{|c_2|\sqrt{\tau-t}} \middle| \mathcal{F}_t\right) \\ &= \Phi\left(\frac{\gamma_o \ln \frac{y}{y_t} - \gamma_p \ln \frac{1-y}{1-y_t} - c_1(\tau-t)}{|c_2|\sqrt{\tau-t}}\right), \end{aligned} \quad (\text{A.35})$$

where $\Phi(\cdot)$ denotes the cumulative distribution function for the standard normal distribution. Differentiating the last expression in (A.35) we obtain p.d.f. function (47).

Q.E.D.

Appendix B: Numerical Method

We first demonstrate how to obtain the boundary conditions for the HJB equations, and then how to cope with the nonlinearity of equations by employing the method of successive iterations. There are two ways of addressing the problem of finding boundary conditions. The first one is to replace wealth-consumption ratios $H_i(y, t)$ by $\tilde{H}_i(y, t) = y(1 - y)H_i(y, t)$. Then, after straightforward algebra we obtain PDEs for $\tilde{H}_i(y, t)$ and solve them with boundary conditions $\tilde{H}_i(0, t) = 0$ and $\tilde{H}_i(1, t) = 0$. From $\tilde{H}_i$ we can back out the wealth-consumption ratios.

Another way is to obtain the boundary conditions by passing to the limit in equations (38) when y_t approaches 0 or 1. To compute the limits we assume that the wealth-consumption ratios are such that $H_i(y, t)$ are twice continuously differentiable in the interval $(0, 1)$, and there exist limits $y^2 \partial^2 H_i(y, t) / \partial y^2 \rightarrow 0$ and $y \partial H_i(y, t) / \partial y \rightarrow 0$ as $y \rightarrow 0$, and $(1 - y)^2 \partial^2 H_p(y, t) / \partial y^2 \rightarrow 0$, $(1 - y) \partial^2 H_o(y, t) / \partial y^2 \rightarrow 0$ and $(1 - y) \partial H_p(y, t) / \partial y \rightarrow 0$, as $y \rightarrow 1$. After we compute the solutions we also verify numerically that these assumptions are satisfied for the case when $\gamma_o > 1$ and $\gamma_p > 1$. When $\gamma_i < 1$ it turns out that $H_i(y)$ may become unbounded when y approaches 0 or 1, and hence in this case we always use the first way of dealing with boundary conditions, and after computing the solution we verify that the solution grows very slowly so that boundary conditions $\tilde{H}_i(0, t) = 0$ and $\tilde{H}_i(1, t) = 0$ are indeed satisfied.

After trying both ways of dealing with the boundary conditions the first way appears to be more tractable. However, for completeness we derive the boundary conditions by passing to the limit as y_t converges to 0 or 1 in several important cases since explicit boundary conditions for the wealth-consumption ratios allow to obtain approximate closed-form price-dividend ratios when y is close to 0 or 1. Interestingly, even in the case of boundary condition misspecification the numerical method with incorrect boundary conditions converges in the interior of $(0, 1)$ to the correct solutions. Intuitively, the boundary condition misspecification is alleviated by the fact that the limiting cases $y = 0$ or $y = 1$ are approached with zero probability [see Figure 5].

To find time-independent solutions in the backwards in time iterations method we fix a large horizon T , pick two functions $h_o(y)$ and $h_p(y)$, and specify terminal conditions:

$$H_i(y, T) = h_i(y), \quad i = o, p, \quad (\text{B.1})$$

where functions $h_i(y)$ are chosen to be continuous and differentiable, as discussed below. Assuming $\gamma_o > 1$ and $\gamma_p > 1$ passing to the limit $y \rightarrow 0$ in equations (38) we obtain simple ordinary differential equations for $H_i(0, t)$ solving which yields boundary conditions at $y = 0$:

$$H_i(0, t) = h_i(0)e^{-\beta_i(T-t)} + \frac{1 - e^{-\beta_i(T-t)}}{\beta_i}, \quad i = o, p, \quad (\text{B.2})$$

where

$$\beta_o = \frac{\gamma_o - 1}{2\gamma_o^2} \kappa_{o0}^2 + \frac{\rho - (1 - \gamma_o)r_{o0}}{\gamma_o}, \quad \beta_p = \frac{1}{\gamma_p} \left(\rho + \frac{(\gamma_p - 1)\gamma_p}{2} \sigma_\delta^2 + (\gamma_p - 1) \left[\rho + \gamma_p \mu_\delta^p - \frac{\gamma_p(\gamma_p + 1)}{2} \sigma_\delta^2 \right] \right). \quad (\text{B.3})$$

and

$$\begin{aligned}\kappa_{o0} &= \gamma_o \left(\sigma_\delta - \max \left\{ \frac{(\gamma_o - \gamma_p)\sigma_\delta - \Delta_\delta}{\gamma_o}, (1 - \bar{\theta})\sigma_\delta \right\} \right), \\ r_{o0} &= \rho + \gamma_p \mu_\delta^p - \frac{\gamma_p(\gamma_p + 1)}{2} \sigma_\delta^2 - \bar{\theta} \sigma_\delta \left((\gamma_o - \gamma_p)\sigma_\delta - \Delta_\delta - \max \{ (\gamma_o - \gamma_p)\sigma_\delta - \Delta_\delta, \gamma_o(1 - \bar{\theta})\sigma_\delta \} \right).\end{aligned}$$

To obtain boundary conditions at $y = 1$ we first consider the case when $\bar{\theta} > 1$, and hence the constraint is not binding around $y = 1$ (i.e. $\nu_{ot}^* = 0$) since the economy will converge to the homogeneous investor economy with $\theta_{ot}^* = 1$. Passing to the limit $y \rightarrow 1$ and solving the resulting ODE for $H_i(1, t)$ we obtain:

$$H_i(1, t) = h_i(1)e^{-\hat{\beta}_i(T-t)} + \frac{1 - e^{-\hat{\beta}_i(T-t)}}{\hat{\beta}_i}, \quad i = o, p, \quad (\text{B.4})$$

where

$$\begin{aligned}\hat{\beta}_o &= \frac{1}{\gamma_o} \left(\rho + \frac{(\gamma_o - 1)\gamma_o}{2} \sigma_\delta^2 + (\gamma_o - 1) \left[\rho + \gamma_o \mu_\delta^o - \frac{\gamma_o(\gamma_o + 1)}{2} \sigma_\delta^2 \right] \right), \\ \hat{\beta}_p &= \frac{1}{\gamma_p} \left(\rho + \frac{\gamma_p - 1}{2\gamma_p} (\gamma_o \sigma_\delta - \Delta_\delta)^2 + (\gamma_p - 1) \left[\rho + \gamma_o \mu_\delta^o - \frac{\gamma_o(\gamma_o + 1)}{2} \sigma_\delta^2 \right] \right).\end{aligned} \quad (\text{B.5})$$

Expressions in (B.2), (B.3), (B.4) and (B.5) demonstrate that conditions $\beta_i \geq 0$ and $\hat{\beta}_i \geq 0$ are necessary for the existence of time-independent solutions of equations (38) with finite boundary values. We then simplify the analysis by setting $h_i(y)$ as follows:

$$h_i(y) = \frac{1 - y}{\beta_i} + \frac{y}{\hat{\beta}_i}, \quad i = o, p. \quad (\text{B.6})$$

The terminal functions in (B.6) make $H_i(0, t)$ in (B.2) and $H_i(1, t)$ in (B.4) independent of time t and horizon T , and equal to the limit of $H_i(0, T)$ when $T \rightarrow +\infty$.

As an additional illustration we derive boundary conditions in the case when $\bar{\theta} < 1$ and $\gamma_p = \gamma_o > 1$, and hence the constraint is always binding around $y = 1$. We multiply the equations for $H_p(y, t)$ and $H_o(y, t)$ by $(1 - y)^2$ and $(1 - y)$, respectively, and passing to the limit $y \rightarrow 1$ we obtain:

$$\frac{\partial H_o(1, t)}{\partial y} = (\gamma - 1)H_o(1, t), \quad (1 - \bar{\theta})(\gamma - 1)H_p(1, t) = 0. \quad (\text{B.7})$$

We then simply set $h_i(y)$ as follows:

$$h_o(y) = \frac{1}{\beta_o} + \left(\frac{1}{\hat{\beta}_o} - \frac{1}{\beta_o} \right) y - \left(\frac{1}{\beta_o} + \frac{\gamma_o - 2}{\hat{\beta}_o} \right) y(1 - y), \quad h_p(y) = \frac{1 - y}{\beta_p}. \quad (\text{B.8})$$

The terminal functions in (B.8) are chosen in such a way that they make $H_i(0, t)$ in (B.2) independent of time t and horizon T , and equal to the limit of $H_i(0, T)$ when $T \rightarrow +\infty$, and at the same time $h_i(y)$ satisfy conditions in (B.7).

For simplicity, in the description of the numerical method we omit subscript i . We let the time and state variable increments denote $\Delta t \equiv T/M$ and $\Delta y \equiv 1/N$, where M and N are integer numbers, and index time and state variables by $t = 0, \Delta t, 2\Delta t, \dots, T$ and $y = 0, \Delta y, 2\Delta y, \dots, 1$, respectively. Next, we derive discrete-time analogues of HJB equations and boundary conditions replacing derivatives by their finite-difference analogues as follows:

$$d \frac{H_{n,k+1} - H_{n,k}}{\Delta t} + a_{n,k+1} \frac{H_{n+1,k} - 2H_{n,k} + H_{n-1,k}}{\Delta y^2} + b_{n,k+1} \frac{H_{n+1,k} - H_{n-1,k}}{2\Delta y} + c_{n,k+1} H_{n,k} + 1 = 0, \quad (\text{B.9})$$

$$H_{n,M} = h_n, \quad H_{0,k} = e_{0,k}, \quad H_{N,k} = \hat{e}_{N,k} H_{N-1,k} + \tilde{e}_{N,k}, \quad (\text{B.10})$$

where $n = 1, 2, \dots, N-1$, $k = 1, 2, \dots, M-1$, $H_{n,k} = H(n\Delta y, k\Delta t)$. The coefficients in (B.9) correspond to coefficients in equation (38) and are computed using the solution $H_{n,k+1}$, while coefficients in (B.10) are obtained by replacing terminal condition (B.1) and boundary conditions at $y = 0$ and $y = 1$ by their finite-difference analogues. Parameter d in (B.9) specifies the solution method. The cases $d = 0$ and $d = 1$ correspond to pure fixed point and backwards in time iterations, respectively, while $0 < d < 1$ describes a method in between the two. The system of equations in (B.9)–(B.10) is solved backwards in time, starting at $k = M-1$. Given solution $H_{n,k+1}$ we compute all the coefficients in (B.9) at step $k+1$, and hence at step k function $H_{n,k}$ for fixed k solves a system of linear algebraic equations. We then iterate until convergence.

For the models in Subsection 3.2 we use fixed point method $d = 0$ with step 0 conjectures given by (B.6) and (B.8). For the models in Section 3.1 the fixed point iterations do not converge for these conjectured functions and therefore we use a combination of fixed point and backward iterations by setting $d = 0.01$ and choosing $T = 50$, which significantly improves the convergence. The convergence is assessed by computing the maximum weighted difference between wealth-consumption ratios 10 years (or iterations when $d = 0$) apart: $\varepsilon_1 = 0.5 \max_y |H_o(y, t) - H_o(y, t + 10)| + 0.5 \max_y |H_p(y, t) - H_p(y, t + 10)|$. In the backwards iterations case we also looked at another measure given by $\varepsilon_2 = 0.5 \max_y |(H_o(y, t) - H_o(y, t + \Delta t))/\Delta t| + 0.5 \max_y |H_p(y, t) - H_p(y, t + \Delta t)/\Delta t|$. We solve the model setting $N = 3000$ and for both convergence measures get typical precisions around $\varepsilon \sim 10^{-9}$ after a couple of seconds of Matlab calculations on a PC. In the cases listed in Subsection 2.3 where the closed-form solutions are available we verify that our numerical method converges to the same solutions.

We also verify via Monte-Carlo simulations that the transversality conditions for HJB equations (35) are satisfied. Moreover, for the cases that we study in Section 3 our numerical analysis reveals that whenever $\bar{\theta} > 1$ or $\bar{\theta} < 1$ and $\gamma_i > 1$ for $i = o, p$, the wealth consumption ratios are continuous and smooth function of y so that $H_i(y) \in C^2(0, 1) \cap C[0, 1]$, which we verify for very fine grid steps Δy . By sufficient optimality conditions in Fleming and Soner (2005) [Verification Theorem 5.1] the smoothness and transversality conditions imply the optimality of c_{it}^* and θ_{it}^* . However, when $\gamma_o < 1$ or $\gamma_p < 1$, wealth-consumption ratios turns out to be continuous and twice differentiable in $(0, 1)$ but unbounded when y approaches 0 or 1, violating the conditions in Fleming and Soner, and hence this case requires a more subtle verification theorem.

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