

PLEDGABILITY AND LIQUIDITY ^{*}

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Abstract

This paper models the role of assets in facilitating intertemporal exchange: because limited commitment precludes unsecured credit, buyers need to pledge assets as collateral. We develop a general equilibrium model where assets differ in terms of pledgability, and put it to work in applications to finance and macroeconomics. The framework nests standard growth and asset-pricing models as special cases. We can price fiat currency as well as real assets, and analyze how monetary policy affects interest rates, generalizing Fishers approach. We also deliver a Tobin effect of inflation on capital accumulation. We study liquidity differentials along both extensive and intensive margins, making pledgability endogenous, while determining the terms of trade in a general way that captures standard pricing mechanisms as special cases.

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1 Introduction

Collateral is, after all, only good if a creditor can get his hands on it.

Niall Ferguson, *The Ascent of Money: A Financial History of the World*

The goal of this project is to develop a tractable theory of the role of assets in the exchange process. Our approach is based on the premise that credit is hindered by various imperfections, or frictions, including limited commitment. Interacted with some notion of imperfect monitoring or record keeping, a strong form of which is anonymity, limited commitment implies that assets may be needed to facilitate intertemporal trade¹ The first requirement of a theory meant to address this is that agents should trade with one another, and not simply against their budget lines, as in classical general equilibrium theory. It is only when one has such a theory, where agents trade with each other, that one can reasonably ask how they trade, i.e., one can study whether they use barter, fiat or commodity money, secured or unsecured credit, and so on. In this setting we analyze the role of assets as collateral.

By way of example, suppose that you want either a consumption or a production good from someone now, but you have neither consumption nor production goods that they want at the moment, so you cannot simply barter. Perhaps you will have something a later date that they will want, maybe a good, or a claim to some goods, or a claim to general purchasing power anything, really and consider trading on credit. That is, consider promising them that if they give you what you want now you will transfer something of value to them in the future. The problem, of course, is that they may be worried you may renege on this obligation. This is what the lack of- commitment friction means. One way to try to get you to honor your repayment obligation is to use rewards and punishments, such as allowing you to continue using credit in the future if you do not default on your obligation, and excluding you from using credit in the future if you do default. But when agents are anonymous, or more generally, when monitoring or record keeping is difficult, this may not work very well, and that leads to a role for assets in the facilitation of exchange.

There are two ways in which assets can help in this regard. First, if you want to acquire something, and have assets on hand, you can turn them over to the seller immediately and directly. In this case, assets serve as a medium of exchange. Alternatively, you can give the seller the right

¹As shown rigorously by Kocherlakota (1998), one needs imperfect monitoring or record keeping what he calls imperfect memory to preclude the use of punishment strategies that can support credit even without commitment. See Wallace (2010) for an updated presentation.

to seize the assets in the event that you renege on any promise to deliver something of value in the future. In this case, the assets serve as collateral. Collateral is useful in the presence of commitment problems because it helps ensure compliance: if you fail to honor an obligation for opportunistic reasons - i.e., because you would rather not - the creditor can seize the collateral. To the extent that you value the collateral, losing it constitutes a punishment that helps deter opportunistic behavior. For this to work it is not even necessary that the counterparty values the collateral; it can be enough that you do, since this alone makes a promise to repay your debts to avoid losing the collateral more credible. However, if it is also valued by your counterparty, the situation can be all the better, since this eliminates his risk that you may renege - if you default, for any reason, he gets the collateral.

These two ways in which assets facilitate intertemporal exchange - serving as a medium of exchange and serving as collateral - are not dissimilar. As David Andolfatto put it in a recent blog: " On the surface, these two methods of payment look rather different. The first entails immediate settlement, while the second entails delayed settlement. To the extent that the asset in question circulates widely as a device used for immediate settlement, it is called money (in this case, backed money). To the extent it is used in support of debt, it is called collateral. But while the monetary and credit transactions just described look different on the surface, they are equivalent in the sense that capital is used to facilitate transactions that might not otherwise have taken place. " To make this concrete, suppose that you have assets that are currently worth x , and will be worth x' at some later date. You can turn over to a seller some of these assets now, or you can promise to give him something worth y later. But he does not trust you, so you need to secure this pledge by offering assets as collateral.

How big a repayment can you credibly pledge? If no punishments are available except for seizing the collateral, you can get credit up to the limit $y = x'$, since it is clear you prefer honoring an obligation $y < x'$ over forfeiting the assets, and you prefer forfeiting over delivering $y > x'$. It is equally clear that instead of pledging the assets as collateral you may as well turn them over to the seller now. There may be complications that affect this conclusion, but they have to be somewhat complicated complications. Maybe, e.g., the assets only increase in value from x to x' while in your possession, because they need your tender loving care. In this case, one could argue that the seller does not want the assets now, and prefers deferred settlement. But this is not quite right, since, absent other frictions, he could take possession now and pay you, or someone else, to take care of assets and still get the yield x' . While most economic models that assume a role for collateral do not specify explicitly what other frictions do the trick, one can imagine that a specification may

exist that delivers a preference for deferred settlement, and hence for the use of assets as collateral. Alternatively, maybe the assets only yields x' if one makes an effort to provide the requisite care, e.g., and maybe you are inclined to shirk in that regard, preferring to forfeit the neglected assets. This simple moral hazard story suggests a preference for immediate settlement, which means using assets as a medium of exchange².

This paper pursues the idea that assets used in one of these ways are essential for intertemporal trade. In terms of the literature, as a canonical model of assets being used as a medium of exchange, we have in mind Kiyotaki and Wright (1989, 1993). As a canonical model of assets being used as collateral to secure credit we have in mind Kiyotaki and Moore (1997, 2005). For many applications it does not matter whether the assets are used as a medium of exchange or as collateral; for others it may matter. What does matter is how much of an asset, or of ones portfolio, one can use. If interpreted as a medium of exchange, it matters how much of an asset a counterparty is willing to accept, with a special case being the one where he does not accept it at all, although more generally he may accept some but not an unlimited amount, perhaps because he is worried the assets may be counterfeits or lemons (see below for a discussion of this approach in the literature). Similarly, if interpreted as collateral, again it matters how much a seller is willing to accept as a pledge, as emphasized by Holmstrom and Tirole (2011). Rather than dwelling on the distinction, or lack thereof, between Kiyotaki-Wright and Kiyotaki-Moore, we propose a framework where assets can differ in terms of pledgability, and put it to work in several substantive applications in finance and macro economics.

Our setup nests as special cases standard growth and asset-pricing models, the workhorses of modern macro and finance, and can be interpreted as a version of the New Monetarist approach, recently surveyed by Williamson and Wright (?) and Nosal and Rocheteau (2011). As such, it allows us to price fiat currency, as well as real assets, and hence allows us to analyze the effects of monetary policy rigorously (i.e., in a model with microfoundations for money). In particular, we present a generalized version of Fishers theory of interest rates, showing how the effect of inflation on real and nominal returns depends on the pledgability of any given asset. We also develop a microfounded version of Tobins theory of the effect of inflation on capital accumulation. We can also use the model to discuss financial markets where agents swap assets, because what is pledgable for one may not be as pledgable for the another agent (e.g., someone holding American assets who

²Then can also be more complicated arrangements, such as repurchase agreements, whereby you give the counterparty assets now and he gives them back later at a predetermined price; see Monnet and Narajabad (2011) and the references therein.

wants to trade with a Canadian may swap with someone holding Canadian assets wants to trade with a American). We can also allow pledgability to be endogenous, depending on equilibrium investments.

Some similar results can be found in Lester et al. (2011), where differential liquidity across assets is modeled using information frictions: certain agents are not able to recognize whether certain assets are of high or low quality, where low quality means the asset is a worthless counterfeit or lemon. For tractability, that paper assumes agents either recognize quality perfectly or not at all, and in latter case they reject assets outright. This avoids the delicate problem of bargaining over the terms of trade under asymmetric information, but also precludes any discussion of partial liquidity in the sense that a seller might accept some but not unlimited quantities of an asset.

To put it differently, in that model, liquidity differs only along the extensive margin (an asset can be accepted by more or fewer counterparties) and not the intensive margin (a given counterparty accepts all or nothing, never some). Now, one can tackle bargaining under asymmetric information in the model using the approach suggested by Rocheteau (2011), Li and Rocheteau (2008, 2009) and Li et al. (2011). This does deliver liquidity differentials along the intensive margin, but relies on a special bargaining protocol, where asset holders, who have the informational advantage, make take-it-or-leave-it offers. While there is nothing wrong per se with take-it-or-leave-it offers, in this context, they unfortunately eliminate any incentive for agents to invest in information, which is the main focus of Lester et al. (2011).

By way of contrast, we use a general mechanism for determining the terms of trade, capturing generalized Nash bargaining, Kalai's proportional bargaining solution, Walrasian price taking and some other commonly used pricing mechanisms as special cases. Because our approach is based on pledgability rather than recognizability i.e., on commitment frictions rather than information frictions price determination is much easier, and there is no problem having liquidity differ along both intensive and extensive margins. This allows us to consider applications beyond those discussed in the earlier papers. Our approach of course is similar to the large body of work emanating from Kehoe and Levine (1993, 2001), Alvarez and Jermann (2000), and others on commitment issues, although that literature typically does not take up applications like the ones studied here, and it focuses primarily on competitive (price-taking) behavior, while we explicitly have agents trading with each other, and consider general pricing mechanisms³. Our paper is obviously also related to

³For a sample of other work that uses limited commitment and goes beyond price-taking behavior, and also provides related applications to money, credit and asset markets, see Sanches and Williamson (2010), Gu and Wright (2010) or Monnet et al. (2009).

a growing body of work on finance used search theory, some of which we discuss more below⁴.

2 Model

The model environment closely follows the framework used by the recent New Monetarist literature, following Lagos and Wright (2005). Time is infinite and discrete. The economy is populated by a unit measure of infinitely-lived agents. Every period, these agents trade goods, labor and assets (including fiat money) with each other in a frictionless market, denoted CM (for centralized market). Before this market (denoted CM, for centralized market) opens, each agent faces a probability $\tilde{\alpha}$ of entering a decentralized market, denoted DM, where all exchange takes place in bilateral meetings. With the complementary probability, $1 - \tilde{\alpha}$, the agent skips this round of trading and moves to the following period. Making the usual independence and Law of Large Numbers assumptions, $\tilde{\alpha}$ also denotes the fraction of agents entering the DM. Conditional on entering the DM, each agent comes into contact with another agent in a type- s with probability ϖ_s . In each such meeting, only one of the agents (chosen at random and called the buyer) is interested in consuming the output of the other (called the seller). Thus, $\alpha_s \equiv \frac{1}{2}\tilde{\alpha}\varpi_s$ denotes the probability that any given agent in the economy trades in the DM as a buyer in a type- s meeting. Note that it is also probability of becoming a seller.

We start with the description of preferences, technology and asset markets in the CM. There is a single good that is produced using labor and productive capital as inputs. The total amount of this good available for consumption and investment is given by:

$$x = F(k, \ell) \equiv f(k, \ell) + (1 - \vartheta)k$$

where the second term is the stock of undepreciated capital. The function f is assumed to be strictly increasing in both arguments and strictly concave. The good can be transformed into consumption or physical capital. The usual optimality conditions of the representative firm operating this technology yield the following expressions for factor prices:

$$\omega = F_2(k, \ell)$$

$$\rho = F_1(k, \ell)$$

⁴For a sample of this work that is related to the current paper, see Duffie et al. (2005), Geromichalos et al. (2007), Ferraris and Watanabe (2008), Lagos and Rocheteau (2008, 2009), Lagos (2010, 2011) and Li and Li (2010). For a more extensive review of the relevant literature, see the above-mentioned surveys on New Monetarist Economics.

Every agent enjoys a period utility from consumption and leisure according to

$$U(x, \ell) = \mathcal{U}(x) - B\ell$$

Apart from physical capital and labor, the agents also trade a number of assets, indexed by $j = 1, 2, 3, \dots, J$. Asset j pays a dividend stream δ_j in the centralized market every period and is in fixed supply, with the total available quantity denoted A_j . Agents also can trade fiat money, i.e. an asset which pays no dividend. The evolution of the total stock of fiat money will be determined by monetary policy, which we will discuss later in this section. Every agent enters the centralized market with holdings of assets $\mathbf{a}$, physical capital k , money holdings m and a vector of promised transfers of assets, money and capital, denoted $\mathbf{P}$, P_m and P_k respectively.

The agent maximizes expected utility, discounted by β , subject to a budget constraint:

$$x = \omega\ell + (k - P_k)\rho - \hat{k} + (\mathbf{a} - \mathbf{P})(\delta + \psi) - \hat{\mathbf{a}}\psi + (m - P_m)\phi - \phi\hat{m} + T \quad (1)$$

where $\hat{k}$, $\hat{\mathbf{a}}$ and $\hat{m}$ are capital, asset and money holdings at the end of trading⁵, δ and ψ denote the $J \times 1$ vectors of dividends asset prices respectively, T is government transfers and ϕ is the real value of money (the inverse of price level).

Throughout this paper, we restrict attention to stationary equilibria, i.e. equilibria where all real variables (output, consumption, real balances, asset and factor prices) are constant. Let $V(\hat{\mathbf{a}}, \hat{k}, \hat{m})$ denote the value at the end of the centralized market environment with asset position $\hat{\mathbf{a}}$, capital $\hat{k}$ and money holdings $\hat{m}$. Given V , the value of entering the current period with asset holdings $\mathbf{a}$, capital k , money holdings m and obligations $\{\mathbf{P}, P_k, P_m\}$ is

$$W(\mathbf{a}, k, m, \mathbf{P}, P_k, P_m) = \max_{x, \ell, \hat{\mathbf{a}}, \hat{k}, \hat{m}} \mathcal{U}(x) - B\ell + \beta V(\hat{\mathbf{a}}, \hat{k}, \hat{m})$$

subject to the budget constraint (1). Given the quasi-linear specification of preferences, the marginal utility of consumption (or equivalently, the value of the multiplier on the budget constraint) is $\frac{B}{\omega}$, independent of the agent's history/asset position. Let x^* solve $\mathcal{U}'(x^*) = \frac{B}{\omega}$. Substituting for ℓ using the budget constraint,

⁵The notation xy , where x and y are $N \times 1$ vectors represents $x' \cdot y = \sum_{n=1}^N x(n)y(n)$.

$$\begin{aligned}
W(\mathbf{a}, k, m, \mathbf{P}, P_k, P_m) = & (\mathbf{a} - \mathbf{P})(\delta + \psi)\frac{B}{\omega} + (m - P_m)\phi\frac{B}{\omega} + (k - P_k)\rho\frac{B}{\omega} \\
& \mathcal{U}(x^*) - \frac{B}{\omega}x^* + T\frac{B}{\omega} \\
& + \max_{\hat{\mathbf{a}}, \hat{k}, \hat{m}} \beta V(\hat{\mathbf{a}}, \hat{k}, \hat{m}) - \hat{\mathbf{a}}\psi\frac{B}{\omega} - \phi\hat{m}\frac{B}{\omega} - \hat{k}\frac{B}{\omega}
\end{aligned}$$

We can simplify this further by noting that this value function depends only on net wealth, defined as follows:

$$y \equiv (\mathbf{a} - \mathbf{P})(\delta + \psi) + (m - P_m)\phi + (k - P_k)\rho \quad (2)$$

Using this, we can rewrite the value function

$$\begin{aligned}
W(\mathbf{a}, k, m, \mathbf{P}, P_k, P_m) = & y\frac{B}{\omega} + \mathcal{U}(x^*) - x^*\frac{B}{\omega} + T\frac{B}{\omega} \\
& + \max_{\hat{\mathbf{a}}, \hat{k}, \hat{m}} \beta V(\hat{\mathbf{a}}, \hat{k}, \hat{m}) - \hat{\mathbf{a}}\psi\frac{B}{\omega} - \phi\hat{m}\frac{B}{\omega} - \hat{k}\frac{B}{\omega}
\end{aligned}$$

with y defined by 2. It is easy to see that W is linear in y with slope $\frac{B}{\omega}$ and that the choice variables $\hat{\mathbf{a}}, \hat{k}$ and $\hat{m}$ are independent of wealth. As a result, end-of-period asset holdings are history independent and therefore, degenerate. This is the key feature of the Lagos-Wright setup that keeps the whole model tractable. In a slight abuse of notation, we will use $W(y)$ to represent the value function $W(\mathbf{a}, k, m, \mathbf{P}, P_k, P_m)$.

We now turn to the characterization of V , the continuation value function at the end of centralized trading. This value has 2 parts - the first arises from the possibility of trading in the decentralized market and the second from trading in the following centralized market. Recall that, before the centralized market in any period opens, every agent has a probability of trading in decentralized, bilateral interactions. In each such meeting, one agent is the ‘buyer’ and the other the ‘seller’. The seller can produce a non-storable good at a utility cost $c(q)$. Analogously, the buyer gets a utility value of $u(q)$ from this DM good.

Each agent is a buyer in a type- s meeting with probability α_s . Making the standard law of large numbers assumption, α_s is also the fraction of agents who become buyers in a type s meeting. Since we need an equal number of buyers and sellers⁶, this is also the probability that she is a seller in such a meeting.

⁶This is not crucial - we could have meetings where both agents are buyers. Clearly, such meetings witness no trade and therefore, do not have any effect on allocations/prices.

A type $s \in \{1, 2, \dots, S\}$ meeting is characterized by a set of pledgability restrictions⁷. We model these frictions in a general way - the restrictions on asset j is represented by 2 functions of the buyer's asset holdings, $d_j^s(a_j)$ and $D_j^s(\mathbf{a})$. Note that the first depends only on the buyer's holding of asset j , whereas the second is a function of the entire vector of asset holdings. Formally, the transfers (or the promised transfers) in a type- s meetings are subject to the following constraints:

$$P_j^s(\delta_j + \psi_j') \frac{B}{\omega'} \leq d_j^s(a_j) \quad j = 1, 2, \dots, J \quad (3)$$

$$\mathbf{P}^s(\delta + \psi') \frac{B}{\omega'} \leq D^s(\mathbf{a}) \quad (4)$$

Similarly, the restrictions on pledging physical capital to a type- s seller are summarized in a function $d^s(k)$:

$$P_k^s \rho' \frac{B}{\omega'} \leq d^s(k)$$

For the rest of the exposition in this section, we focus on the case where $S = 1$, i.e. there is only 1 type of meeting. It is straightforward to extend our key results to a case with multiple meetings.

Rather than assume a specific bargaining game or trading protocol, we represent the outcome of the bilateral interaction as a general mechanism $(q, g(q))$, where q is quantity of the DM good transferred from the seller to the buyer and $g(q)$ is the total compensation to the seller (expressed in utility terms). This compensation can be in the form of transfers of money (denoted P_m), capital (denoted P_k) and/or assets, denoted by the vector $\mathbf{P}$. The value of these transfers (to either party) depends on prices in the immediately following centralized market. Note that this is true whether these transfers take place immediately (i.e. during the decentralized interaction) or are effected in the following CM. We require that the mechanism be incentive compatible i.e. both the buyer and seller benefit from participating. Formally, this amounts to imposing the following constraints on the mechanism:

$$\begin{aligned} u(q) - \mathbf{P}(\delta + \psi') \frac{B}{\omega'} - P_m \phi' \frac{B}{\omega'} - P_k \rho' \frac{B}{\omega'} &\geq 0 \\ -c(q) + \mathbf{P}(\delta + \psi') \frac{B}{\omega'} + P_m \phi' \frac{B}{\omega'} + P_k \rho' \frac{B}{\omega'} &\geq 0 \end{aligned}$$

where ψ', ω', ρ' and ϕ' denote asset prices, the real wage rate, the rental rate of capital and the inverse of the price level in the centralized market immediately following the bilateral interaction⁸.

⁷For now, we treat these as completely exogenous. In Section ??, we present a model where pledgeability is a function of endogenous investment choices.

⁸In steady state, the first three objects will be constant over time and the last grows at a constant rate.

All the mechanisms we consider will satisfy these constraints, hence we will ignore them in the formal analysis that follows. We also restrict our attention to Pareto-optimal mechanisms INSERT DISCUSSION ON MECHANISM HERE.

The value to a buyer who enters the bilateral interaction with asset position $(\mathbf{a}, k, m)$ and leaves it with obligations $(\mathbf{P}, P_k, P_m)$ is given by

$$u(q) + W(y)$$

where $y = (\mathbf{a} - \mathbf{P})(\delta + \psi') + (m - P_m)\phi' + (k - P_k)\rho'$. Using the fact that W is linear with slope $\frac{B}{\omega'}$, we can rewrite this value as

$$u(q) - \underbrace{(\mathbf{P}(\delta + \psi') + P_m\phi' + P_k\rho') \frac{B}{\omega'}}_{g(q)} + W(y_0)$$

where $y_0 = \mathbf{a}(\delta + \psi') + m\phi' + k\rho'$, i.e. the net wealth of an agent with zero pledges. More succinctly,

$$u(q) - g(q) + W(y_0)$$

Similarly, the value to a seller who enters the DM meeting with $(\mathbf{a}, k, m)$ is

$$g(q) - c(q) + W(y_0)$$

Thus, the continuation value function V is given by

$$V(\mathbf{a}, k, m) = \alpha[u(q) - g(q)] + \alpha[g(q) - c(q)] + W(y_0)$$

Given the transfer to the seller $g(q)$ specified under the mechanism, an agent's portfolio problem is summarized by the following program:

$$\mathcal{L} = \max_{\mathbf{a}, k, m, q, \mathbf{P}, P_k, P_m} -\frac{B}{\omega}[\mathbf{a}\psi + m\phi + k] + \beta\alpha(u(q) - g(q)) \quad (5)$$

$$+ \beta\frac{B}{\omega'} [\mathbf{a}(\delta + \psi') + k\rho' + m\phi'] \quad (6)$$

subject to

$$g(q) \leq \mathbf{P}(\delta + \psi') \frac{B}{\omega'} + P_m \phi' \frac{B}{\omega'} + P_k \rho' \frac{B}{\omega'} \quad (7)$$

$$P_j(\delta_j + \psi'_j) \frac{B}{\omega'} \leq d_j(a_j) \quad j = 1, 2, \dots, J \quad (8)$$

$$\mathbf{P}(\delta + \psi') \frac{B}{\omega'} \leq D(\mathbf{a}) \quad (9)$$

$$P_k \rho' \frac{B}{\omega'} \leq d(k) \quad (10)$$

$$P_m \frac{B}{\omega'} \leq m \frac{B}{\omega'} \quad (11)$$

The first constraint requires that the seller is compensated in accordance with the mechanism. The next 3 represent the various pledgeability restrictions. The last restriction is a feasibility constraint on the monetary transfer⁹.

We associate multipliers $\lambda_0, \lambda_j, \lambda_T, \lambda_m$ and λ_k with the above constraints and derive the following necessary conditions for optimality:

$$-\psi_j \frac{B}{\omega} + \beta(\delta_j + \psi_j) \frac{B}{\omega'} + \lambda_j \frac{\partial d_j}{\partial a_j} + \lambda_T \frac{\partial D}{\partial a_j} = 0 \quad (12)$$

$$-\phi \frac{B}{\omega} + \beta \phi' \frac{B}{\omega'} + \lambda_m \frac{B}{\omega'} \leq 0 \quad \text{with equality if } m > 0 \quad (13)$$

$$-\frac{B}{\omega} + \beta \rho' \frac{B}{\omega'} + \lambda_k \frac{\partial d}{\partial k} = 0 \quad (14)$$

$$\beta \alpha[u'(q) - g'(q)] - \lambda_0 g'(q) = 0 \quad (15)$$

$$-\lambda_j - \lambda_T + \lambda_0 \leq 0 \quad \text{with equality if } P_j > 0 \quad (16)$$

$$-\lambda_m + \phi' \lambda_0 \leq 0 \quad \text{with equality if } P_m > 0 \quad (17)$$

$$-\lambda_k + \lambda_0 \leq 0 \quad \text{with equality if } P_k > 0 \quad (18)$$

The model is closed by the assumption of a central bank or monetary authority, which chooses the nominal interest rate $1 + i$ (and adjusts the stock of fiat money as needed). In other words, the central bank sets the (nominal) return on a unit of money invested in a nominal bond, which pays off in the following CM (but cannot be used in the intervening DM). By no arbitrage, the nominal and real interest rates must satisfy

$$1 + i = (1 + r) \frac{\phi}{\phi'}$$

⁹Technically, there are feasibility restrictions on the asset and capital transfers as well, but they are subsumed within the pledgeability restrictions. Similarly, we have abstracted from pledgeability restrictions on money - an issue we will return to in Section ???.

where r , the (steady state) real interest rate, is given by:

$$r \equiv \frac{1}{\beta} - 1 .$$

Therefore, the nominal interest rate and inflation are linked by the following relationship

$$1 + i = \frac{1}{\beta} \frac{\phi}{\phi'}$$

In other words, choosing the level of the nominal interest rate in a stationary equilibrium is equivalent to chooses the (constant) rate of inflation. In such a steady state, real balances are also constant, so this is equivalent to picking a growth rate for the stock of fiat money¹⁰. Substituting in the FOC for money holdings, we get

$$1 + i \geq 1 + \alpha \left(\frac{u'(q)}{g'(q)} - 1 \right)$$

with equality if $m > 0$.

3 Risk of Diversion a la Holmstrom-Tirole

We analyze the case where the pledgeability restriction takes the form of a constant fraction of each asset, i.e. $\frac{\partial d_j}{\partial a_j} = \mu_j(\delta_j + \psi'_j) \frac{B}{\omega'}$ and $\frac{\partial d(k)}{\partial a_k} = \mu_k(\rho' + 1 - \vartheta) \frac{B}{\omega'}$. DISCUSS MICROFOUNDATIONS HERE. Note that, in this case, the aggregate restriction D is redundant - only the individual restrictions d_j are relevant.

There are 3 possibilities for the nature of the steady state equilibrium:

- Case 1: Liquid
- Case 2: Illiquid non-monetary
- Case 3: Monetary

3.1 Case 1: Liquid

If the supply of liquidity from real assets and physical capital¹¹ is sufficient to achieve q^* , the unconstrained quantity choice, then all assets trade at their fundamental price, i.e. there is no

¹⁰There is an equivalent formulation in which the central bank chooses the growth rate of the stock of fiat money and lets the rate of inflation and nominal interest rate be determined in equilibrium.

¹¹We can also have a liquid equilibrium if the cost of holding money balances is 0, i.e. the nominal interest is set to 0.

liquidity premium. The equilibrium is then characterized by the following system of equations:

$$\mathcal{U}'(x)F_2(k, \ell) = 1 \quad (19)$$

$$x + k = F(k, \ell) + \sum_{j=1}^J A_j \delta_j \quad (20)$$

$$F_1(k, \ell) = 1 + r \quad (21)$$

$$u'(q^*) = g'(q^*) \quad (22)$$

The first 3 expressions are the standard steady state equations of the neoclassical model. The last equation defines the unconstrained quantity choice, q^* . Finally, the following expression provides a necessary and sufficient condition for the solution to the above system to constitute a valid equilibrium:

$$g(q^*) \leq \frac{B}{F_2(k, \ell)} \left(\sum_{j=1}^J A_j \mu_j \frac{1+r}{r} \delta_j + \mu_k k F_1(k, \ell) \right) \quad (23)$$

If this condition holds, all assets will trade at their fundamental value and earn a rate of return equal to the stationary real interest rate, i.e.

$$\psi_j = \frac{\delta_j}{r} \quad j = 1, 2, \dots, J \quad (24)$$

$$\Rightarrow \quad 1 + r_j \equiv \frac{\delta_j + \psi_j}{\psi_j} = 1 + r \quad (25)$$

3.2 Case 2: Illiquid Non-monetary

If condition (23) does not hold, then assets will play a role in facilitating exchange. We start with the case where real assets and physical capital are the only sources of liquidity, i.e. money is not valued in equilibrium.

The liquidity value of owning an asset leads to a reduction in its measured rate of return (i.e. the return on asset without taking into account the liquidity value). To see this, note that the steady state versions of the optimality conditions for asset holdings and physical capital are given by:

$$1 + r_j = \frac{1 + r}{1 + \lambda_0 \mu_j} \leq 1 + r \quad (26)$$

$$F_1(k, \ell) = \frac{1 + r}{1 + \lambda_0(q) \mu_k} \leq 1 + r \quad (27)$$

Thus, assets which facilitate exchange (i.e. $\mu_j > 0$) earn an accounting rate of return less than the real interest rate. Higher the degree of pledgeability μ_j , the greater this discount in rates of return (or equivalently, the greater the price premium). This generalizes Fisher's theory of the equalization of rates of returns across assets. When assets are valued not just for their dividend flows, but also for the liquidity they provide, it is the *liquidity-adjusted* returns that are equalized, not the *accounting* rates of return.

Note, however, that this liquidity premium takes a slightly different form for Lucas trees and physical capital. For the former, which are in fixed supply, this premium manifests itself as a price premium.

$$\psi_j = \frac{\delta_j(1 + \lambda_0 \mu_j)}{r - \lambda_0 \mu_j} \geq \frac{\delta_j}{r} \quad (28)$$

However, the technology for accumulation of physical capital pins down its relative price at 1. Therefore, the liquidity value induces an adjustment along the quantity margin. In other words, the lower rate of return implies a higher capital-labor ratio compared to the equilibrium where liquidity is abundant.

For completeness, we give below the system of equations which characterizes the full equilibrium:

$$\mathcal{U}'(x)F_2(k, \ell) = 1 \quad (29)$$

$$x + k = F(k, \ell) + \sum_j A_j \delta_j \quad (30)$$

$$F_1(k, \ell) = \frac{1 + r}{1 + \lambda_0 \mu_k} \quad (31)$$

$$g(q) = \frac{A_j \mu_j \delta_j (1 + r)}{r - \lambda_0 \mu_j} \frac{B}{F_2(k, \ell)} + k \mu_k (F_1(k, \ell) + 1 - \vartheta) \frac{B}{F_2(k, \ell)} \quad (32)$$

Recall that $\lambda_0 = \alpha \left(\frac{u'(q)}{g'(q)} - 1 \right)$, so this is a system of 4 equations in 4 unknowns - x, k, ℓ, q .
DISCUSS EXISTENCE/UNIQUENESS¹².

¹²An interesting special case is one where capital cannot be used as a medium of exchange i.e. $\mu_k = 0$. In that case, CM consumption, labor supply and physical capital are as in the liquid equilibrium and equation (32) uniquely pins down q .

This is the only equilibrium if the nominal interest rate is sufficiently high. In particular, if the following condition holds,

$$i > \alpha \left(\frac{u'(q)}{g'(q)} - 1 \right) \quad (33)$$

the liquidity value of money is not sufficient to outweigh the opportunity cost of holding it, so it cannot command a positive value in equilibrium.

3.3 Case 3: Monetary

If (33) does not hold, then there exists an equilibrium in which money is valued. In this case, the quantity produced in the DM is a function of just the nominal interest rate:

$$i = \alpha \left(\frac{u'(q)}{g'(q)} - 1 \right) \quad (34)$$

Rates of return and asset prices are given by

$$1 + r_j = \frac{1 + r}{1 + i\mu_j} \leq 1 + r \quad (35)$$

$$\psi_j = \frac{\delta_j(1 + i\mu_j)}{r - i\mu_j} \geq \frac{\delta_j}{r} \quad (36)$$

The relation (35), another version of the generalized Fisher theory in the previous subsection, shows how monetary policy affects the measured rates of return on real assets. Higher interest rates increase the cost of holding money and therefore, increases the role of real assets in facilitating exchange. This increased liquidity value manifests itself as a reduction in the accounting rate of return of these assets.

Define the (gross) nominal and real interest rate as follows:

$$\begin{aligned} \hat{r} &\equiv 1 + r \\ \hat{\pi} &\equiv \frac{\phi}{\bar{\phi}} \\ \hat{i} &\equiv 1 + i = \hat{r}\hat{\pi} \end{aligned}$$

Analogously, the (gross) real and nominal rates of return on an asset with pledgeability μ are

defined as follows:

$$\begin{aligned}\hat{r}_\mu &\equiv \frac{1+r}{1+i\mu_j} = \frac{\hat{r}}{1-\mu+\hat{i}\mu} \\ \hat{i}_\mu &\equiv \hat{r}_\mu \hat{\pi}\end{aligned}$$

INSERT FIGURE PLOTTING $\hat{r}_\mu$ and $\hat{i}_\mu$ AS FUNCTION OF $\hat{\pi}$ or $\hat{i}$.

As the figure shows, monetary policy has no effect on the real rate of return of an asset with no liquidity value ($\mu = 0$). The nominal return on the asset, however, increases 1-for-1 with inflation. At the other extreme, an fully liquid asset ($\mu = 1$) earns the same rate of return as money - a gross nominal return of 1 (independent of the rate of inflation) and a real rate of return that is inversely proportional to the inflation rate. For intermediate levels of pledgeability μ , inflation increases the nominal rate of return on the asset, but reduces the measured real rate of return.

The elasticity of an asset's real rate of return to the inflation rate is given by

$$\mathbf{e}_{\hat{r}_\mu, \hat{\pi}} = - \left(\frac{\hat{r} \hat{\pi} \mu}{1 - \mu + \hat{r} \hat{\pi} \mu} \right)$$

It is easy to see that this is decreasing (i.e. becoming more negative) in μ . In other words, the more liquid the asset, the more sensitive its return to monetary policy. For a fully liquid asset ($\mu = 1$), this elasticity equals -1 , the same as that of money.

Consumption and capital accumulation choices in the CM in such a monetary equilibrium solve

$$1 = \mathcal{U}'(x) F_2(k, \ell) \tag{37}$$

$$x + k = F(k, \ell) + \sum_j A_j \delta_j \tag{38}$$

$$F_1(k, \ell) = \frac{1+r}{1+i\mu_k} \tag{39}$$

3.4 Effect of Monetary Policy

In this subsection, we examine the implications of monetary policy for long-run (i.e. steady state) levels of activity. It is immediate from () that DM activity decreases with inflation (or equivalently, the nominal interest rate). In the CM, monetary policy affects the level of capital accumulation as well as employment. As the following proposition shows, when capital provides liquidity (i.e. $\mu_k > 0$), an increase in inflation leads to a rise in the steady state level of capital, a version of the so-called Mundell-Tobin effect. In other words, an increase in the opportunity cost of holding money causes agents to substitute towards other sources of liquidity - in this case, physical capital.

Proposition 1 1. (*The Tobin Effect:*) The steady state level of capital is increasing in the nominal interest rate, i.e. $\frac{\partial k}{\partial i} > 0$

2. The steady state level of capital is increasing in degree of pledgability, i.e. $\frac{\partial k}{\partial \mu_k} > 0$

Proof.

$$i \uparrow \text{ or } \mu_k \uparrow \Rightarrow F_1 \downarrow \Rightarrow F_2 \uparrow \Rightarrow x \uparrow$$

Rearranging the resource constraint,

$$k = \frac{x - \sum_j A_j \delta_j}{\frac{f(k, \ell)}{k} - \vartheta}$$

The numerator increases and denominator decreases with i or μ_k , establishing the result. ■

Next, we study the effect on employment in the CM. The change in capital stock in response to a change in the nominal interest rate generates wealth and substitution effects, which work in opposite directions. As a result, in general, the response of labor to monetary policy is of ambiguous sign. To put it differently, when capital provides liquidity in addition to being a factor of production, the slope of long-run Phillips curve can be positive or negative, depending on parameters.

To investigate this relationship, we impose additional structure on the model. In particular, we make assumptions about the form of utility and production functions in the CM - CM utility exhibits constant relative risk aversion and the production function is Cobb-Douglas, i.e. $\mathcal{U}'(x) = x^{\frac{-1}{\sigma}}$ and $F(k, \ell) = k^\nu \ell^{1-\nu} + (1 - \vartheta)k$. We assume that there are no other assets and that we are in a region of the parameter space where a monetary equilibrium exists. The following result shows that the substitution effect from a monetary policy change dominates the income effect, i.e. CM employment increases with inflation, if the intertemporal elasticity of substitution is sufficiently low.

Proposition 2 (*The long-run Phillips Curve:*) With CRRA utility and Cobb-Douglas production functions in the CM, steady state employment increases with inflation if, and only if, $\sigma \geq \sigma^*$, where σ^* is bounded above by 1.

Proof. From the resource constraint,

$$\begin{aligned}
\frac{c}{\ell} + \frac{k}{\ell} &= \frac{F(k, \ell)}{\ell} \\
\Rightarrow \frac{\omega^\sigma}{\ell} &= \frac{\omega}{1-\nu} - \vartheta \left(\frac{\omega}{1-\nu} \right)^{\frac{1}{\nu}} \\
\Rightarrow \ell &= \frac{\omega^\sigma}{\frac{\omega}{1-\nu} - \vartheta \left(\frac{\omega}{1-\nu} \right)^{\frac{1}{\nu}}} \\
\Rightarrow \ell &= \frac{\omega^{\sigma-1}}{\frac{1}{1-\nu} - \vartheta \left(\frac{1}{1-\nu} \right)^{\frac{1}{\nu}} \omega^{\frac{1-\nu}{\nu}}} \\
\Rightarrow \frac{\partial \ell}{\partial \omega} &\approx^s \frac{\sigma-1}{1-\nu} + \vartheta \left(\frac{1}{1-\nu} \right)^{\frac{1}{\nu}} \omega^{\frac{1-\nu}{\nu}} \left(\frac{1}{\nu} - \sigma \right)
\end{aligned}$$

where $\approx^s$ denotes equivalence in sign. Rearranging yields the result with σ^* given by

$$\sigma^* = \frac{\frac{1}{1-\nu} - \vartheta \left(\frac{1}{1-\nu} \right)^{\frac{1}{\nu}} \omega^{\frac{1-\nu}{\nu}}}{\frac{1}{1-\nu} - \vartheta \left(\frac{1}{1-\nu} \right)^{\frac{1}{\nu}} \omega^{\frac{1-\nu}{\nu}}}$$

Since $\nu \in (0, 1)$, this is bounded above by 1.

■

4 Endogenous Liquidity

In this section, we make the degree of pledgeability endogenous. For expositional simplicity, we focus on a monetary equilibrium with a single asset in fixed supply and pledgeability restrictions modeled as in Section 3, i.e. only a constant fraction of the asset's value can be pledged.

4.1 Specification I: Fixed cost

We assume that all agents have access to two technologies for enforcement of debt contracts collateralized by claims on this asset. The first is free and allows sellers to appropriate a fraction of μ_1 of the asset holdings of a defaulting debtor. The second involves incurring a fixed cost in the preceding CM, but the enables the appropriation of a greater fraction $\mu_2 > \mu_1$ of the asset upon default. The cost is agent-specific, i.e. agent $s \in [0, 1]$ needs to invest $\kappa_s \in [\underline{\kappa}, \bar{\kappa}]$ to be able to take advantage of the superior enforcement technology.

Let χ denote the (endogenous) fraction of agents who choose to invest in the μ_2 -technology. Let σ denote the probability that any given agent will be a seller (and symmetrically, a buyer) in the

DM. Conditional on being a seller, her type is determined by her decision to invest in the superior enforcement technology in the preceding CM. Conditional on being a buyer, χ also represents the fraction of her meetings that are of type 2, i.e. with pledgeability μ_2 . Then, from the buyer's perspective, the probability of a type 1 meeting, α_1 , is given by $\alpha(1 - \chi)$ and that of a type 2 meeting is $\alpha\chi$.

Let $\Delta(\chi)$ denote the benefit to the seller from having access to the superior enforcement technology, where we emphasize the dependence of the benefit on χ , the fraction of sellers with access to the μ_2 technology.

$$\Delta \equiv (g(q_2) - c(q_2)) - (g(q_1) - c(q_1))$$

where q_1, q_2 represent DM production levels at the two types of meetings.

Without loss of generality, we order agents so that κ_s is increasing in s and let $H(\cdot)$ denote the corresponding cdf, i.e. $H(s)$ is the fraction of agents with a cost less than or equal to κ_s . Then, in equilibrium, the fraction of agents who chose to invest in the μ_2 technology, denoted χ , is given by

$$\chi = 0 \quad \text{if} \quad \beta\alpha\Delta < \underline{\kappa} \quad (40)$$

$$= H(s) \quad \text{if} \quad \beta\alpha\Delta = \kappa_s \quad (41)$$

$$= 1 \quad \text{if} \quad \beta\alpha\Delta > \bar{\kappa} \quad (42)$$

Define

$$T(\chi) = H(\max\{s : \kappa_s \leq \beta\alpha\Delta(\chi)\})$$

The characterization of the equilibrium with endogenous enforcement capacity is completed by the following fixed point relationship:

$$\chi = T(\chi) \quad (43)$$

The following assumption lists conditions which guarantee that there is at least one solution to (44).

Assumption 1 *In the region $(0, q^*)$, the function g satisfies both the following conditions*

1. $g(q) - c(q)$ is increasing.

2. For any $q_1 < q_2$, we have $\frac{g(q_2)}{g(q_1)} \leq \frac{c(q_2)}{c(q_1)}$

The requirements of this condition are met, for example, under proportional bargaining a la Kalai. In this case, the seller's net surplus from a type j meeting is simply $(1 - \theta) \cdot (u(q_j) - c(q_j))$, where θ , a parameter, indexes the share of the total surplus appropriated by the buyer. The next lemma establishes monotonicity of Δ when Assumption 1 holds.

Lemma 1 $\Delta(\chi)$ is increasing.

Proof. In the Appendix. ■

This result allows us to apply Tarski's fixed point theorem to (44) and thus, ensures existence of an equilibrium. Note that, in general, there can be multiple solutions to (44). In particular, depending on the support of κ_s , we can have equilibria where no one invests in the μ_2 -technology (i.e. $\chi = 0$) or everyone does ($\chi = 1$). There could be multiple interior fixed points as well.

4.2 Specification II: Continuous cost function

Here, we consider an alternative technology for investing in pledgability. Instead of a fixed cost, the ex-ante investment required is now described by a continuous function, $\Gamma(\mu)$, increasing and convex. Note that the marginal value to a seller of increasing μ is simply $\psi + \delta$. Then, an equilibrium with an interior choice for μ is characterized by the following fixed point problem:

$$\Gamma'(\mu) = \psi(\mu) + \delta \tag{44}$$

where we make explicit the dependence of asset prices on μ . Note that, in general, this relationship is non-monotonic. DISCUSS EXISTENCE AND UNIQUENESS OF EQUILIBRIUM.

5 Risky Dividends

Now, suppose the dividend stream δ is an iid random variable, whose realization is known during the bilateral interaction preceding the centralized market. Then, in a monetary equilibrium,

$$i = \mathbb{E}\lambda_0(q(\delta)) \tag{45}$$

$$1 + r_j \equiv \frac{\phi_j + \bar{\delta}_j}{\phi_j} = \frac{1 + r - \mu_j \text{Cov}\left(\lambda_0, \frac{\delta_j}{\psi_j}\right)}{1 + i\mu_j} \tag{46}$$

In other words, assets which have higher dividend payouts (and therefore higher liquidity value) in states when liquidity is most needed (λ_0 is high) have lower expected accounting returns.

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Appendix A Proofs of Results

A.1 Proof of Lemma 1

We consider 3 cases

- Case I: $q_1 = q_2 = q^*$
- Case II: $q_1 < q_2 = q^*$

- Case III: $q_1 < q_2 < q^*$

Obviously, in case I, $\Delta = 0$ for all χ . In case II, note that

$$g(q_1) = \frac{A\delta(1+r)\mu_1}{r - \beta\alpha(1-\chi)\tilde{\lambda}(q_1)\mu_1}$$

where $\tilde{\lambda}(q) = \frac{u'(q)}{g'(q)} - 1$. Then, q_1 decreases when more sellers have the superior enforcement technology. Suppose otherwise, i.e. $q_1 \uparrow$. Then, $g(q_1) \uparrow \Rightarrow \frac{A\delta(1+r)\mu_1}{r - \beta\alpha(1-\chi)\tilde{\lambda}(q_1)\mu_1} \uparrow \Rightarrow \beta\alpha(1-\chi)\tilde{\lambda}(q_1)\mu_1 \uparrow \Rightarrow \tilde{\lambda}(q_1) \uparrow \Rightarrow q_1 \downarrow$, contradicting the original claim. Therefore, an increase in χ leads to an decrease in q_1 . Since $g(q) - c(q)$ is increasing by assumption, it is easy to see that $\Delta = (g(q^*) - c(q^*)) - (g(q_1) - c(q_1))$ rises with χ .

In case III, the quantities q_1 and q_2 are the solution to the following system of equations

$$i = \alpha(1-\chi)\tilde{\lambda}(q_1) + \alpha\chi\tilde{\lambda}(q_2) \quad (47)$$

$$g(q_2) - g(q_1) = \frac{A\delta(1+r)(\mu_2 - \mu_1)}{r - \alpha(1-\chi)\tilde{\lambda}(q_1)\mu_1 - \alpha\chi\tilde{\lambda}(q_2)\mu_2} \quad (48)$$

where $\tilde{\lambda}(q) = \frac{u'(q)}{g'(q)} - 1$. Taking the derivative with respect to χ , and rewriting in matrix notation,

$$\begin{pmatrix} \Omega_{11} & \Omega_{12} \\ \Omega_{21} & \Omega_{22} \end{pmatrix} \begin{pmatrix} \frac{\partial q_1}{\partial \chi} \\ \frac{\partial q_2}{\partial \chi} \end{pmatrix} = \begin{pmatrix} \tilde{\lambda}(q_1) - \tilde{\lambda}(q_2) \\ \frac{g(q_2) - g(q_1)}{r - \mathbb{E}\mu\tilde{\lambda}} (\tilde{\lambda}'(q_1)\mu_1 - \tilde{\lambda}'(q_2)\mu_2) \end{pmatrix}$$

where

$$\begin{aligned} \Omega_{11} &= (1-\chi)\tilde{\lambda}'(q_1)\alpha \\ \Omega_{12} &= \chi\tilde{\lambda}'(q_2)\alpha \\ \Omega_{21} &= g'(q_1) + \frac{g(q_2) - g(q_1)}{r - \mathbb{E}\mu\tilde{\lambda}} \alpha(1-\chi)\tilde{\lambda}'(q_1)\mu_1 \\ \Omega_{22} &= -g'(q_2) + \frac{g(q_2) - g(q_1)}{r - \mathbb{E}\mu\tilde{\lambda}} \alpha\chi\tilde{\lambda}'(q_2)\mu_2 \end{aligned}$$

It is straightforward to show that the determinant $\mathbb{D}$ of the coefficient matrix on the left-hand side is positive. Solving,

$$\mathbb{D} \frac{\partial q_1}{\partial \chi} = -g'(q_2) (\tilde{\lambda}(q_1) - \tilde{\lambda}(q_2)) + \frac{g(q_2) - g(q_1)}{r - \mathbb{E}\mu\tilde{\lambda}} \sigma\chi\tilde{\lambda}'(q_2)\tilde{\lambda}(q_1)(\mu_2 - \mu_1) \quad (49)$$

$$\mathbb{D} \frac{\partial q_2}{\partial \chi} = -g'(q_1) (\tilde{\lambda}(q_1) - \tilde{\lambda}(q_2)) - \frac{g(q_2) - g(q_1)}{r - \mathbb{E}\mu\tilde{\lambda}} \sigma(1-\chi)\tilde{\lambda}'(q_1)\tilde{\lambda}(q_2)(\mu_2 - \mu_1) \quad (50)$$

Next, we note that

$$\frac{\partial \Delta}{\partial \chi} = (g'(q_2) - c'(q_2)) \frac{\partial q_2}{\partial \chi} - (g'(q_1) - c'(q_1)) \frac{\partial q_1}{\partial \chi}$$

Substituting,

$$\frac{\partial \Delta}{\partial \chi} = (g'(q_1)c'(q_2) - g'(q_2)c'(q_1)) \left(\tilde{\lambda}(q_1) - \tilde{\lambda}(q_2) \right) \quad (51)$$

$$- (g'(q_2) - c'(q_2)) \frac{g(q_2) - g(q_1)}{r - \mathbb{E}\mu\tilde{\lambda}} \alpha(1 - \chi) \tilde{\lambda}'(q_1) \tilde{\lambda}(q_2) (\mu_2 - \mu_1) \quad (52)$$

$$- (g'(q_1) - c'(q_1)) \frac{g(q_2) - g(q_1)}{r - \mathbb{E}\mu\tilde{\lambda}} \alpha \chi \tilde{\lambda}'(q_2) \tilde{\lambda}(q_1) (\mu_2 - \mu_1) \quad (53)$$

Now, $\tilde{\lambda}'(q) < 0$. Also, by the first statement in Assumption 1, the seller's surplus is increasing in q , i.e. $g'(q) - c'(q) > 0$, making the second and third terms positive. The second statement of Assumption 1 makes the first term positive as well, completing the proof.