

Liquidity Misallocation in an Over-The-Counter Market*

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Abstract

We show that the dispersion of private valuation reduces market liquidity and allocative efficiency of a dynamic OTC market. In this decentralized market, traders have time varying and heterogeneous private value over the asset and dealers act as competing mechanism designers. We characterize the optimal liquidity provision with endogenous valuation, outside options and type distributions. Depending on traders' value, liquidity can be distorted in three ways, trade breakdown, trade delay or price distortion. The three distortions coexist in the equilibrium. Trade with small gain breaks down. Trade with intermediate gain is delayed. And trade with large gain faces largest distortion in price. As the dispersion of private valuation increases, price dispersion increases and trade is more likely to be delayed or break down for any type. Welfare loss increases as dispersion of private value increases. Quantitatively, welfare loss from liquidity misallocation could reach 5%.

Key words: *Liquidity; Misallocation; Durable good/asset; Random search frictions; Private value; Over-the-counter markets*

JEL: D82, G1

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1 Introduction

Evidences show that uncertainty about asset quality may not be so large as to cause a crisis like the recent one. This is especially true for many financial markets not troubled by heterogeneity in asset quality. So could inefficient market outcome arise even without lemon in the market? As an attempt to answer this question, we single out an alternative channel through which financial markets may malfunction during economic turmoil: as uncertainty arises, dispersion of private value of a homogeneous asset may arise, leading to market breakdown, delay in trade and price distortion. We show that all the three distortions coexist in the equilibrium of a dynamic over-the-counter market with random matching friction.

In the over-the-counter market we study, trade is bilateral, search is costly and matching is random, traders have private value over the asset, and prices and immediacy of trade are posted by profit-maximizing dealers. We show that dealers' profit seeking behavior will affect market liquidity under private value. When dealers know traders' preference about the asset, delaying trade hurt both parties since future profit and utility are discounted. The profit maximizing behavior of a dealer has no effect on market liquidity. With asymmetric information, however, dealers might find it profitable to delay a trader if by doing so, they can better screen traders of different type. Market liquidity will therefore respond to dispersion in value.

The distortion comes from dealers' incentive to screen traders to extract surplus from those with large gain from trade. Traders lose more from delay if the gain from trade is larger. In other words, traders with large private gain from trade have a less elastic demand for the immediacy of trade. So to screen the profitable trade, dealers deter them from lying about their preferences by delaying trade with smaller gain.

Applying tools from optimal mechanism design to the dynamic matching market, we solve for the optimal contract menu and show that the solution is unique. Each contract in the menu specifies price, bid price for sellers and ask price for buyers, and immediacy of the trade. Traders of different preference over the asset self select into the contract they prefer. In the optimal contract menu, dealers ask a large pie from trade with a large gain but finish transaction more quickly, and delay the trade of small gain but ask for a smaller pie out of it.

We find that value specific liquidity can be distorted in three ways, trade breakdown, trade delay or price distortion and the three distortions coexist in the equilibrium. If we classify trade by gain from trade, then roughly speaking, trade with small gain breaks down completely, trade with intermediate gain is delayed and trade with large gain is executed without delay. In terms of price distortion, more surplus is extracted from trade with large gain and price distortion is largest when trade is carried out without delay. Therefore, trade at all level is distorted by the profit seeking motive of dealers, one way or the other.

The result comes from the combination of mechanism design and careful analysis of equilibrium type distribution, outside options and continuation values. In a typical mechanism design literature, the type distribution, continuation value and outside option of traders are all exogenous. In the market equilibrium, each dealer acts as a mechanism designer. All these key ingredients are endogenous and will affect the optimal contract menu designed by dealers. We show that this has a profound effect on equilibrium allocation and optimal contract menu. Specifically, when dealers are indifferent between offering a trading term with more immediacy or less in equilibrium, a bang-bang solution is likely to show up in the mechanism design problem of a dealer and there will exist multiple equilibria because of the indifference. But in a market equilibrium, the trading frequencies are uniquely pinned down by integral equations that links equilibrium type distribution to equilibrium continuation values and outside options. The coexistence of three distortions also rely on the combination. This is a technical contribution of this paper.

Distortion to liquidity allocation is tightly linked with dispersion of value, search friction and competition among dealers. As heterogeneity of preference over the asset increases, price elasticity to immediacy decreases, and trading delay and price distortion weakly increases for all traders. Competition among dealers drives the allocation close to the efficient one. But the effect of competition is not uniform across trade of different types. As the degree of competition increases, the distortions to trade with intermediate and small gain are decreases more. Since dealers derive most of their profit from trade with large gains, distortions to these trades are least likely to disappear. Search friction has two effects. Costly search itself destroys value. An indirect effect is through competition. A higher search friction decreases the value from outside option and increases the monopoly power of each individual dealer. Liquidity misallocation in the market leads to

asset misallocation.

Literature Review

Grossman and Miller (1988) [8] model market liquidity as being determined by supply and demand of immediacy. Market makers compete to match the supply and demand of immediacy. Some trade is delayed because traders do not have a strong demand for immediacy. Our concept of liquidity is consistent with theirs.

Most recent papers that study the effect of asymmetric information on liquidity study an environment with adverse selection of common value, such as Chiu and Koepl (2011)[3], Carmago and Lestor (2011)[16], Guerrieri and Shimer (2011)[9] and Chang (2011)[2]. We want to fill the gap to properly study the effect of dispersed private value on liquidity allocation.

This paper builds on the recent framework of studying liquidity in over-the-counter markets, a prevalent market microstructure in financial markets and in durable goods markets. Duffie, Garleanu and Pedersen (2005) [6] and Lagos and Rochetau (2010)[14] studies the effect of search friction on liquidity in OTC markets. Duffie, Garleanu and Pedersen (2005) Lagos and Rochetau (2010) studies a dealer-driven over-the-counter market and allows continuous asset holdings. Duffie (2012) [5] gives a detailed review of the progress on this topic. In this framework, value fluctuates over time and resale is allowed. We believe that friction from dispersed private value should be properly studied in this rich framework, without making simplifying assumptions.

Guerrieri, Shimer and Wright (2010)[10] and Chang (2011)[2], combine directed search friction and asymmetric information. But their models are not about an over-the-counter market with random search frictions, as modeled by [6, 14]. They show that adverse selection in an environment with directed search friction might lead to endogenous market segmentation and endogenous trading friction across submarkets. We study a different environment, which has random search friction and private information about a private value, which is closer to OTC markets. There are also notable differences in terms of findings. Results in Guerrieri, Shimer and Wright (2010) and Chang (2011) are vulnerable to perturbation to type distribution and the response

to changes in type distribution is not satisfactory. Even if the environment is close to complete information, the equilibrium allocation will look the same as an environment with severe adverse selection, as long as the support of distribution remains the same. This is because their model is a signaling game and they select the a Pareto optimal equilibrium. In contrast, dealers in our environment offer screening contracts and take the effect of type distribution on their profit into consideration when they optimize the menu of contracts they offer. The equilibrium allocation will converge to that of a Walrasian market with complete information as the dispersion of private information diminishes. Also, their results depend on matching functions, while our results come from optimal mechanism design of dealers and depends on matching frictions only through the Poisson meeting rate between traders and dealers.

Mechanism design in the random matching market imbeds mechanism design problem into a Nash equilibrium for games with a continuum of agents. The solution concept is laid out by Mas-colell (1984)[17] and Schmeidler (1973) [19].

We also contribute to a growing literature on asymmetric information in decentralized and dynamic markets. Lauermann and Wolinsky (2011)[15], Wolinsky (1990)[20] and Blouin Serrano (2001)[1] study information aggregation in the lemon market and study social welfare as friction vanishes. Golosov, Lorenzoni, and Tsyvinski (2009)[7] and Carmago and Lestor (2011)[16] are concerned with trading dynamics. Horner and Vieille (2009)[12] studies the interaction of strategies of sequentially arriving short-lived buyers in a dynamic lemon market. Inderst (2005)[13] studies limiting property of a matching market with adverse selection. Hendel and Lizzeri (1999)[11] studies market for lemons with durable goods. All these papers study an environment with asymmetric information about common value of the goods or asset traded, while we study an environment with private values. While most papers focus on limiting results, our focus is to characterize carefully the equilibrium with frictions.

2 Model

Environment, preference and endowment

Our model builds on the continuous time search model of Duffie, Garleanu and Pedersen (2005). There is a continuum of infinitely lived traders of total measure 1. And there is a continuum of dealers.¹ There are an asset and a numeraire good in the economy.

The asset is a Lucas tree, which bears a unit flow of fruit per unit. The total asset supply is $A \in (0, 1)$. Each trader can hold either zero or one unit of asset. The asset is homogeneous in quality.

Traders' valuation of the asset is heterogeneous and independent from each other. Given preference x , a trader enjoys flow utility x from holding one unit of the asset. The asset holding of traders is observable. The arrival of traders are observable to dealers. But the anonymity of traders implies that the arrival and reentry of a trader are not discernible. But traders' value over the asset is their private information. With Poisson rate δ , a trader experiences a preference shock. The shock is i.i.d. After the shock, a trader draws randomly a preference from a continuous distribution $G(\cdot)$ with a connected support $\mathbb{X} \subset \mathbb{R}_+$. Denote $\underline{x} = \inf \mathbb{X}$, $\bar{x} = \sup \mathbb{X}$.

Dealers' direct utility from holding the asset is zero.

All agents are risk neutral and derive 1 unit of utility from holding 1 unit of the numeraire good. All payment in the trade are in terms of the numeraire good. And we assume that agents have enough numeraire goods in their pockets so they are not constrained when they want to trade the good for the asset. All agents discount at rate r .

¹The profit of a dealer is constant return to scale. And we use the Law of Large Numbers when calculating dealers' profit. So the total measure of dealers does not matter.

Over the counter market between dealers and traders

Dealers and traders match randomly. With Poisson rate α , a trader meets a dealer and starts a long term relationship.² A trader stays with the same dealer until a shock breaks them apart. At any moment, a relationship breaks up with Poisson rate γ . The exogenous break-up rate is meant to be a neat way to capture dynamic competition between dealers.

When dealers and traders meet, the identity of traders is anonymous to dealers. So dealers cannot condition their strategies on traders' identity. And dealers cannot use history dependent strategies in the long run strategies as well. Otherwise, traders will walk away from an existing contract and pretend to be a new comer. The anonymity assumption is standard in dynamic random matching models.

Dealers are connected with a continuum of traders. So dealers do not care about their profit from a trader of a specific type, but care about total expected profit from all traders over whom they have bargaining power. The contract dealers offer is a menu of bid/ask trading terms to dealers. Each trading term or contract specifies bid/ask price and trading frequency, which is a Poisson rate of finishing a transaction. The rate at which a transaction is carried out is measures immediacy. Trading frequency, denoted q , is bounded from above, $q \in [0, \bar{q}]$. This characterizes the highest speed with which dealer can deliver an asset to a trader.

³The contract is contingent on reported type of a trader and whether the trader holds asset or not. The contracts for asset holders are bid contracts. And the contracts for non asset holders are ask contracts.

Each contract is a commitment for the dealer to deliver the asset at Poisson rate q and if a trading occasion arises, the dealer is committed to trade at the price specified in the contract. The contract does

²To be accurate, the rate at which traders meet dealers should come from a matching function, and therefore depends on the search population on both the dealer side and the trader side. But since the measure of dealers are exogenous and the break up rate of a relationship is exogenous too, we have enough degree of freedom to equivalently impose an exogenous rate for the random matching.

³This assumption is not essential for our results. It is out of technical convenience for analytical solution and numerical computation of the optimal contract menus. As trading frequencies increases, the marginal return from increasing trading frequencies decreases dramatically. Without an upper bound on trading frequency, return will be overwhelmed by numerical errors, causing unnecessary noises. With a bound high enough, equilibrium allocation is not affected much. But on the other hand, it is not unreasonable to assume that transaction takes time, especially if we are talking about liquidity within a day.

not require commitment from traders' side. After signing a contract, a trader will only trade if there is gain from trade ex post, at the price specified in the contract.

Inter-dealer market

The inter-dealer market is competitive and frictionless. Dealers can buy and sell any amounts of asset at the market price without any delay. Denote the market clearing price to be P . Without capacity constraint or trading frictions, dealers will not hold any inventory.

3 Equilibrium

3.1 Definition of equilibrium

Actions

Traders can take three actions. When facing a dealer with a menu of contracts, they can choose the contract that delivers highest expected payoff to them from the menu. The choice depends on their asset holding and their preference. While trading mechanism can vary, the contract menu can always be equivalently thought of as a direct mechanism. So traders' contract choice can be thought of as reporting their preference type to the dealer, who in turn delivers the contract for the reported type. If no contract in the menu is more favorable than their outside option, to walk away from a relationship to find another dealer, they can choose to do so. The third action is an enforcement constraint related to the anonymity of traders. Since trading opportunities arrive at a Poisson rate specified by a contract, there is a difference between ex ante individual rationality constraint, whether a contract delivers expected gain from trade, and ex post individual rationality constraint, whether upon arrival of each trading opportunity, the trading delivers gain from trade. This second constraint is important if traders are anonymous: traders can choose to walk away from a contract without incurring search cost. Instead, they will reject the unfavorable trading opportunity, walk away, and then walk back, pretending to be a new customer, waiting for a better trading opportunity. Denote the probability of accepting a trading opportunity at moment t to be $a_t \in [0, 1]$. Denote the

probability of choosing contract m in a dealer's menu $\mathbb{M}_i$ to be $s_{it}(m)$ and the probability of choosing the outside option to be $s_{it}(\emptyset)$. $s_{it}(\emptyset) + \sum_{m \in \mathbb{M}} s_{it}(m) = 1, \forall t, i \in \{a, b\}$.

The most important action of a dealer is to choose the menu of trading terms. Denote trading terms at moment t to be $\{(q_{amt}, p_{amt}), (q_{bmt}, p_{bmt})\}_{\forall m \in \mathbb{X}}$. The menu for asset holders is $\mathbb{M}_{bt} = \{(q_{bm}, p_{bm})\}_{\forall m \in \mathbb{X}}$ and the menu of non-asset holders is $\mathbb{M}_{at} = \{(q_{amt}, p_{amt})\}_{\forall m \in \mathbb{X}}$. Each contract m in the menu of contracts, $\mathbb{M}$, specifies two components: price, p_{imt} , and trading frequency, q_{imt} , which is a Poisson rate of transaction. ($i \in \{a, b\}$ denotes whether the contract is for a “bid”, if traders sell to dealers, or an “ask”, if traders buy from dealers.) To be complete, trading in the competitive inter-dealer market is also dealers' action. But the action is mechanical since they act non-strategically in the competitive inter-dealer market.

We focus on symmetric equilibrium, where in equilibrium dealers post the same contract menu.

Traders' problem

Traders expected payoffs depends on their choice, outside options and continuation values after shocks or transactions. On the other hand, outside options and continuation values are also expected payoffs that depends on traders' payoff from their contract choice and trading.

Denote the expected payoff of an asset holder connected with a dealer at t to be $\tilde{M}_{1xt}(s_t)$, with value $x \in \mathbb{X}$, action $s \in \Delta(\{\emptyset, \mathbb{M}_{bt}\})$, outside options $U_{1x\tau}, \forall \tau \geq t$, continuation value $M_{0x\tau}, M_{1x\tau}, \forall \tau > t, \forall x \in \mathbb{X}$.

$$\begin{aligned} \tilde{M}_{1xt}(s_t) &= s_t(\emptyset)U_{1xt} \\ &+ \int \left\{ \int_t^\infty e^{-(\delta+\gamma)(\tau-t) - \int_t^\tau q_{bmz} dz} \left[\begin{aligned} &\int_t^\tau x e^{-r(z-t)} dz + e^{-r(\tau-t)} \\ &\left(\begin{array}{cc} a_\tau (p_{bm\tau} + M_{0x\tau}) & + \delta E M_{1\tau} \\ q_{bm\tau} \max_{a_\tau} \left(\begin{array}{c} a_\tau (p_{bm\tau} + M_{0x\tau}) \\ +(1-a_\tau)M_{1x\tau} \end{array} \right) & + \gamma U_{1x\tau} \end{array} \right) \end{aligned} \right] d\tau \right\} ds_t(m) \end{aligned} \quad (1)$$

Anonymity of traders implies that traders can get

Likewise, denote the payoff of a non asset holder to be $\tilde{M}_{0xt}(s_t)$, with value $x \in \mathbb{X}$, action $s \in \Delta(\{\emptyset, \mathbb{M}_{at}\})$, outside options $U_{0x\tau} \forall \tau \geq t$ and continuation value $M_{0x\tau}, M_{1x\tau}, \forall \tau > t, \forall x \in \mathbb{X}$.

$$\begin{aligned}
\tilde{M}_{0xt}(s_t) &= s_t(\emptyset)U_{0xt} \\
&+ \int \left\{ \int_t^\infty e^{-(\delta+\gamma+r)(\tau-t)-\int_t^\tau q_{amz}dz} \begin{pmatrix} q_{am\tau} \max_{a_\tau} \begin{pmatrix} a_\tau (-p_{am\tau} + M_{1x\tau}) \\ +(1-a_\tau)M_{0x\tau} \end{pmatrix} + \delta E M_{0\tau} \\ + \gamma U_{0x\tau} \end{pmatrix} d\tau \right\} ds_t(m)
\end{aligned} \tag{2}$$

The expected payoff of an optimal choice from the contract menu at t is M_{1xt} and M_{0xt} . So the continuation value M_{1xt} and M_{0xt} can be written as follows.

$$\begin{aligned}
M_{1xt} &= \max_{s_t} \tilde{M}_{1xt}(s_t) \\
M_{0xt} &= \max_{s_t} \tilde{M}_{0xt}(s_t)
\end{aligned} \tag{3}$$

Given traders' value when matched with dealers, we can further define the value from outside options U_{1xt} and U_{0xt} ,

$$\begin{aligned}
U_{1xt} &= \int_t^\infty e^{-\alpha(\tau-t)} \left[\int_t^\tau x e^{-r(z-t)} dz + \alpha M_{1x\tau} e^{-r(\tau-t)} \right] d\tau \\
U_{0xt} &= \int_t^\infty e^{-\alpha\tau} \alpha M_{0x\tau} e^{-r(\tau-t)} d\tau
\end{aligned} \tag{4}$$

We can see from the payoff of traders the dynamic features of our model: traders choose optimally from the contract menu at t , taking outside options and continuation values as given; continuation values and outside options are then pinned down, given payoffs from the optimal choice of contract. The endogeneity of continuation values and outside options distinguishes our analysis from static or partial equilibrium analysis of mechanism design.

Dealers' problem

Each dealer acts as a mechanism designer, offering a menu of contracts to discriminate traders of different type, in an effort to maximize their profit. Dealers' expected payoff at t , denoted M_t^d , depends on profit from each transaction and the distribution of transactions of different types. Denote the population density of asset holders of type x to be $\tilde{n}_{1xt}^m$, and the population density of non asset holders of type x to be $\tilde{n}_{0xt}^m$.

$$M_t^d = \max_{\{\mathbb{M}_\tau\}_{\tau \geq t}} \int e^{-r(\tau-t)} \int [q_{ax\tau}(p_{ax\tau} - P_\tau) \tilde{n}_{0x\tau}^m + q_{bx\tau}(P_\tau - p_{bx\tau}) \tilde{n}_{1x\tau}^m] dx d\tau \quad (5)$$

subject to

$$\begin{aligned} \dot{\tilde{n}}_{1xt}^m &= \alpha n_{1xt}^u - \delta \tilde{n}_{1xt}^m + \delta g(x) \tilde{N}_{1t}^m - \gamma \tilde{n}_{1xt}^m - q_{bx} \tilde{n}_{1xt}^m + q_{ax} \tilde{n}_{0xt}^m, \forall x \\ \dot{\tilde{n}}_{0xt}^m &= \alpha n_{0xt}^u - \delta \tilde{n}_{0xt}^m + \delta g(x) \tilde{N}_{0t}^m - \gamma \tilde{n}_{0xt}^m + q_{bx} \tilde{n}_{1xt}^m - q_{ax} \tilde{n}_{0xt}^m, \forall x \\ \{x\} &\in \arg \max_{s_t \in \Delta(\{\emptyset, \mathbb{M}\})} \tilde{M}_{1x\tau}(s_t), \forall x, \forall \tau \geq t \\ \{x\} &\in \arg \max_{s_t \in \Delta(\{\emptyset, \mathbb{M}\})} \tilde{M}_{0x\tau}(s_t), \forall x, \forall \tau \geq t \\ p_{ax\tau} &\leq \tilde{M}_{1x\tau}(\{x\}) - \tilde{M}_{0x\tau}(\{x\}) \\ p_{bx\tau} &\geq \tilde{M}_{1x\tau}(\{x\}) - \tilde{M}_{0x\tau}(\{x\}) \\ \tilde{M}_{1xt}(\{x\}) &\geq U_{1xt}, \forall x \\ \tilde{M}_{0xt}(\{x\}) &\geq U_{0xt}, \forall x \end{aligned}$$

The first two constraints define the response of type distribution of the dealer's customer base to changes in trading frequencies, $q_{ix\tau}, i \in \{a, b\}$, in the menu of contracts, assuming that the contract menu is incentive compatible and individually rational. The population density is affected by inflow of traders from the pool of unmatched traders, n_{ixt}^u , preference shock, breakup of dealer-customer match and trading frequencies. $\tilde{N}_{it}^m = \int \tilde{n}_{ixt}^m dx, \forall i \in \{0, 1\}$. Notice that choice of trading frequencies will affect the type distribution and therefore payoff. The response of distribution to optimal mechanism is a feature not in partial equilibrium mechanism design.

The third and fourth constraints are incentive compatibility (IC) constraints for traders to choose the trading term dealers expect them to choose. The fifth and sixth constraints are enforcement constraints of contracts. Since the commitment of traders required by a contract is limited, any price with which a trade is carried out must leave agents with positive gain from trade, so agents will accept the transaction willingly. The last two constraints are traders' individually rational (IR) constraint. Since we focus on symmetric equilibrium and it is costly to search for a new dealer, we would expect that the optimal contract menu lies in the interior defined by IR constraints. Therefore, when we solve for the optimal mechanisms, IR

constraints can be safely ignored.

Note that although the outside options U_{ixt} and customer inflow from the unmatched population n_{ix}^u are endogenous, individual dealers take them as given when choosing their optimal contract menu. This is because individual dealers are infinitesimal in the random matching market. Since outside options and distribution unmatched population is determined only by equilibrium contract menu of other dealers, out of equilibrium belief about outside options and composition of customer inflow is well restricted. This is the feature of Nash equilibrium with a continuum of agents.

Laws of Motion

Denote the density of the measure of traders with type ix and matched with a dealer at moment t to be n_{ixt}^m , $\forall i \in \{0, 1\}$ and those unmatched to be n_{ixt}^u . $N_{1t}^m = \int n_{1xt}^m dx$, $N_{0t}^m = \int n_{0xt}^m dx$, $N_{1t}^u = \int n_{1xt}^u dx$, $N_{0t}^u = \int n_{0xt}^u dx$.

Forces that affect the laws of motion of the population density of unmatched traders are: the rate of meeting a dealer, the preference shock, and the breakup rate, which determines the speed at which matched traders switch back to the pool of unmatched traders.

$$\begin{aligned}\dot{n}_{1xt}^u &= -\alpha n_{1xt}^u - \delta n_{1xt}^u + \delta g(x) N_{1t}^u + \gamma n_{1xt}^m \\ \dot{n}_{0xt}^u &= -\alpha n_{0xt}^u - \delta n_{0xt}^u + \delta g(x) N_{0t}^u + \gamma n_{0xt}^m\end{aligned}\tag{6}$$

Forces that affect the laws of motions of the population density of matched traders are: the rate for the unmatched agents to meet dealers, the preference shock, the breakup rate of a match, and then the trading frequency, which affect the rate at which traders switch between holding an asset or not.

$$\begin{aligned}\dot{n}_{1xt}^m &= \alpha n_{1xt}^u - \delta n_{1xt}^m + \delta g(x) N_{1t}^m - \gamma n_{1xt}^m - q_{bxt} n_{1xt}^m + q_{axt} n_{0xt}^m \\ \dot{n}_{0xt}^m &= \alpha n_{0xt}^u - \delta n_{0xt}^m + \delta g(x) N_{0t}^m - \gamma n_{0xt}^m + q_{bxt} n_{1xt}^m - q_{axt} n_{0xt}^m\end{aligned}\tag{7}$$

Since preference shock is independent of the arrival rate of meeting and breaking up with a dealer, we assume that the total population of traders matched or unmatched with a dealer is stationary.

$$N_{0t}^m + N_{1t}^m = \frac{\alpha}{\alpha+\gamma}$$

$$N_{0t}^u + N_{1t}^u = \frac{\gamma}{\alpha+\gamma}$$

Likewise by assuming the distribution of agents with preference type x is stationary, we have,

$$n_{0xt}^m + n_{1xt}^m + n_{0xt}^u + n_{0xt}^u = g(x) \quad (8)$$

We also have,

$$N_{0t}^m + N_{0t}^u = 1 - A$$

$$N_{1t}^m + N_{1t}^u = A$$

Since the matching and separation of traders and dealers are independent from the asset holding of an agent,

$$N_{1t}^u = \frac{\gamma}{\alpha+\gamma} A$$

$$N_{1t}^m = \frac{\alpha}{\alpha+\gamma} A$$

$$N_{0t}^u = \frac{\gamma}{\alpha+\gamma} (1 - A)$$

$$N_{0t}^m = \frac{\alpha}{\alpha+\gamma} (1 - A)$$

(9)

The restrictions on laws of motion are summarized in the following lemma.

Lemma 1. *Laws of motion are constrained by the following restrictions*

$$n_{1x}^u + n_{0x}^u = \frac{\gamma}{\alpha+\gamma} g(x)$$

$$n_{1x}^m + n_{0x}^m = \frac{\alpha}{\alpha+\gamma} g(x)$$

$$N_{1t}^u = \frac{\gamma}{\alpha+\gamma} A$$

$$N_{1t}^m = \frac{\alpha}{\alpha+\gamma} A$$

$$N_{0t}^u = \frac{\gamma}{\alpha+\gamma} (1 - A)$$

$$N_{0t}^m = \frac{\alpha}{\alpha+\gamma} (1 - A)$$

Dealers solve for the optimal menu of contracts taking the market price of the asset in the inter-dealer market as given. The market price is pinned down by the following market clearing condition. Since no dealers hold inventories, the demand and supply of asset in the inter-dealer market is related to the bid and ask transaction flow at any moment.

$$\int q_{axt} n_{0xt}^m dx = \int q_{bxt} n_{1xt}^m dx \quad (10)$$

Definition of equilibrium

Definition 1. An equilibrium is value functions, M_{1xt} , M_{0xt} , U_{1xt} , U_{0xt} , and M_t^d , laws of motion for n_{1xt}^m , n_{0xt}^m , n_{1xt}^u and n_{0xt}^u , the contract menu, $\mathbb{M}_{at} = \{(q_{ax}, p_{ax})\}_{\forall x \in \mathbb{X}}$, $\mathbb{M}_{bt} = \{(q_{bx}, p_{bx})\}_{\forall x \in \mathbb{X}}$, and market price P_t in the inter-dealer market such that

(i) Given $\{U_{1x\tau}, U_{0x\tau}\}_{\forall \tau \geq t}$, $M_{0x\tau}$, $s_\tau(\{(q_{ax}, p_{ax})\}) = 1$, $a_\tau = 1$, $\forall \tau \geq t$, solves the problem of a non asset holder of type x and $M_{1x\tau}$, $s_\tau(\{(q_{bx}, p_{bx})\}) = 1$, $a_\tau = 1$, $\forall \tau \geq t$, solves the problem of an asset holder of type x .

(ii) Given $\{M_{1x\tau}, M_{0x\tau}\}_{\forall \tau \geq t}$, U_{1xt} , U_{0xt} , $\forall t$ are solved by equations (4).

(iii) Given laws of motion for $\{n_{1x\tau}^u, n_{0x\tau}^u\}_{\forall x \in \mathbb{X}, \forall \tau \geq t}$ and $\{U_{1x\tau}, U_{0x\tau}\}_{\forall \tau \geq t}$, $\mathbb{M}_{a\tau}$, $\mathbb{M}_{b\tau}$, $\forall \tau \geq t$ solves a dealer's problem at t .

(iv) laws of motions are consistent with the equilibrium contract menus over time, as specified in equations (21) and (22).

(v) the inter-dealer market clears at any moment t , with market clearing condition specified by equation (10).

3.2 Characterization of stationary equilibrium

To characterize the equilibrium allocation, we focus on the stationary equilibrium.

In a stationary equilibrium, the population distribution rests in the stationary distribution of the Markov process defined by the equilibrium, so $\dot{n}_{ixt}^j = 0$, $\forall i \in \{0, 1\}, j \in \{m, u\}$, $\dot{M}_{1xt} = \dot{M}_{0xt} = \dot{U}_{1xt} = \dot{U}_{0xt} =$

$\dot{M}_t^d = 0, \forall x, t$ and the equilibrium contract menu is not time varying. So we omit the time index in the notation.

Trader's problem

$$rM_{1x} = \max_{\hat{x}} x + \delta(EM_1 - M_{1x}) + q_{b\hat{x}}(p_{b\hat{x}} + M_{0x} - M_{1x}) + \gamma(U_{1x} - M_{1x})$$

$$rU_{1x} = x + \alpha(M_{1x} - U_{1x}) + \delta(EU_1 - U_{1x})$$

$$rM_{0x} = \max_{\hat{x}} \delta(EM_0 - M_{0x}) + q_{a\hat{x}}(-p_{a\hat{x}} + M_{1x} - M_{0x}) + \gamma(U_{0x} - M_{0x})$$

$$rU_{0x} = \alpha(M_{0x} - U_{0x}) + \delta(EU_0 - U_{0x})$$

What matters for trading decision is not continuation values and outside options themselves, but the difference a trading will make to the values. So we define $d_x = M_{1x} - M_{0x}$ and $e_x = U_{1x} - U_{0x}$. Denote $Ee = \int e_x dG(x)$, $Ed = \int d_x dG(x)$, $Ex = \int x dG(x)$, which are expectations conditional on being hit by a preference shock.

$$d_x = \frac{x + \delta Ed + q_{bx}p_{bx} + q_{ax}p_{ax} + \gamma e_x}{r + \delta + q_{ax} + q_{bx} + \gamma}$$

$$e_x = \frac{x + \alpha d_x}{r + \alpha + \delta} + \frac{\delta}{r + \alpha + \delta} \frac{Ex + \alpha Ed}{r + \alpha}$$

Enforcement constraint and the threshold type for buying and selling

Lemma 2. *There exists a cutoff κ such that only types $x \geq \kappa$ are willing to sell and types $x \leq \kappa$ are willing to buy. So $q_{bx} = 0, \forall x \geq \kappa$ and $q_{ax} = 0, \forall x \leq \kappa$.*

Proof. By the enforcement constraint, So $p_{bx} \geq M_{1x} - M_{0x}$ for all type x willing to sell. $M_{1x} - M_{0x} \geq p_{ax}$ for all type x willing to buy. Since $p_{bx} \leq P \leq p_{ax}$, the proof will be done if we can show there exists a unique

type κ such that $d_\kappa = P$. This will be true if d_x is strictly increasing in x . The rest of the proof is about the monotonicity of d_x .⁴

Using envelope theorem, we know that $M'_{ix} = \frac{\partial}{\partial x} M_{ix}, \forall i \in \{a, b\}$. So M'_{ix} and U'_{ix} can be written as follows,

$$rM'_{1x} = 1 - \delta M'_{1x} - q_{bx}d'_x + \gamma(U'_{1x} - M'_{1x})$$

$$rU'_{1x} = 1 + \alpha(M'_{1x} - U'_{1x}) - \delta U'_{1x}$$

$$rM'_{0x} = -\delta M'_{0x} + q_{ax}d'_x + \gamma(U'_{0x} - M'_{0x})$$

$$rU'_{0x} = \alpha(M'_{0x} - U'_{0x}) - \delta U'_{0x}$$

So d'_x and e'_x are as follows,

$$\begin{aligned} rd'_x &= 1 - \delta d_x - (q_{bx} + q_{ax})d'_x + \gamma(e'_x - d'_x) \\ d'_x &= \frac{1 + \gamma e'_x}{r + \delta + q_{bx} + q_{ax} + \gamma} \\ re'_x &= 1 + \alpha(d'_x - e'_x) - \delta e'_x \\ e'_x &= \frac{1 + \alpha d'_x}{r + \alpha + \delta} \\ rd'_x &= 1 - \delta d_x - (q_{bx} + q_{ax})d'_x + \gamma\left(\frac{1 + \alpha d'_x}{r + \alpha + \delta} - d'_x\right) \\ d'_x &= \frac{1 + \gamma \frac{1}{r + \alpha + \delta}}{r + \delta + q_{bx} + q_{ax} + \gamma \frac{r + \delta}{r + \alpha + \delta}} \\ &> 0 \end{aligned}$$

Market clearing rules that there must exist a κ such that $d_\kappa = P$. By monotonicity of d_x , κ is unique. $\square$

⁴So the claim in the lemma can also be applied to characterize a non-stationary equilibrium if we assume that d_{xt} is strictly increasing at any moment in the non-stationary equilibrium.

Dealer's problem

$$\begin{aligned}
M^d &= \max_{\{(q_{ax}, p_{ax}), (q_{bx}, p_{bx})\}_{\forall x}} \frac{1}{r} \int q_{ax}(p_{ax} - P) \tilde{n}_{0x}^m + q_{bx}(P - p_{bx}) \tilde{n}_{1x}^m dx \\
&\text{s.t.} \\
\tilde{n}_{1x}^m &= \begin{cases} \frac{1}{\delta + \gamma + q_{bx}} \left(\alpha n_{1x}^u + \delta g(x) \tilde{N}_1^m \right) & \text{if } x < \kappa \\ \alpha n_{1x}^u - \delta \tilde{n}_{1x}^m + \delta g(x) \tilde{N}_1^m - \gamma \tilde{n}_{1x}^m + q_{ax} \tilde{n}_{0x}^m & \text{if } x \geq \kappa \end{cases} \\
\tilde{n}_{0x}^m &= \begin{cases} \alpha n_{0x}^u - \delta \tilde{n}_{0x}^m + \delta g(x) \tilde{N}_0^m - \gamma \tilde{n}_{0x}^m + q_{bx} \tilde{n}_{1x}^m & \text{if } x < \kappa \\ \frac{1}{\delta + \gamma + q_{ax}} \left(\alpha n_{0x}^u + \delta g(x) \tilde{N}_0^m \right) & \text{if } x \geq \kappa \end{cases} \quad (11) \\
\tilde{M}_{1xx} &\geq \tilde{M}_{1xx'}, \forall x \\
\tilde{M}_{0xx} &\geq \tilde{M}_{0xx'}, \forall x \\
\tilde{M}_{1xx} &\geq U_{1x}, \forall x \\
\tilde{M}_{0xx} &\geq U_{0x}, \forall x
\end{aligned}$$

The response of population density to contract menu choice is simplified by the existence of threshold type κ . Note that since we know that dealers will choose $q_{ax} = 0, \forall x \leq \kappa$ and $q_{bx} = 0, \forall x \geq \kappa$, we only need to pay attention to the response of distribution in region $x > \kappa$ for $\tilde{n}_{0x}^m$, and $x < \kappa$ for $\tilde{n}_{1x}^m$. So the information about the response for type $1x$ when $x \geq \kappa$ and for type $0x$ when $x \leq \kappa$ is redundant for the optimization problem. We can see that as trading frequencies decrease, the population of that type increases and thus canceling out the loss of profit from decreasing trading frequencies. What hurts the profit of a dealer is the delay caused by decreasing trading frequencies.

Laws of motion

The laws of motion of equilibrium population density can also be simplified because of the existence of the threshold type κ .

The laws of motion in a stationary environment is as follows

$$\begin{aligned}
\dot{n}_{1x}^u &= -\alpha n_{1x}^u - \delta n_{1x}^u + \delta g(x) N_1^u + \gamma n_{1x}^m \\
\dot{n}_{0x}^u &= -\alpha n_{0x}^u - \delta n_{0x}^u + \delta g(x) N_0^u + \gamma n_{0x}^m
\end{aligned}$$

$$\begin{aligned}\dot{n}_{1x}^m &= \begin{cases} \alpha n_{1x}^u - \delta n_{1x}^m + \delta g(x) N_1^m - \gamma n_{1x}^m - q_{bx} n_{1x}^m & \text{if } x < \kappa \\ \alpha n_{1x}^u - \delta n_{1x}^m + \delta g(x) N_1^m - \gamma n_{1x}^m + q_{ax} n_{0x}^m & \text{if } x \geq \kappa \end{cases} \\ \dot{n}_{0x}^m &= \begin{cases} \alpha n_{0x}^u - \delta n_{0x}^m + \delta g(x) N_0^m - \gamma n_{0x}^m + q_{bx} n_{1x}^m & \text{if } x < \kappa \\ \alpha n_{0x}^u - \delta n_{0x}^m + \delta g(x) N_0^m - \gamma n_{0x}^m - q_{ax} n_{0x}^m & \text{if } x \geq \kappa \end{cases}\end{aligned}$$

By imposing $\dot{n}_{ixt}^j = 0$, $\forall i \in \{0, 1\}$, $j \in \{m, u\}$, we can derive the stationary distribution. The derivation is in the appendix.

3.3 Characterization

Assumption 1. *equilibrium trading frequency q_{ax} and q_{bx} are continuous in x .*

Theorem 1. *Under assumption 1, there exists a stationary equilibrium characterized by the following system of equations:*

(1) *Given e_x and n_{1x}^u , n_{0x}^u , q_{ax} , q_{bx} in equilibrium contract menu are solved by the following integral equations,*

$$\begin{aligned}C_b(x) &= x - \kappa + \gamma(e_x - e_\kappa) + r \int_x^\kappa \frac{1 + \gamma e'_s}{r + \delta + q_{bs} + \gamma} ds \\ D_b(x) &= \int_x^\kappa \frac{r + \delta + q_{bs} + \gamma}{\delta + q_{bs} + \gamma} (\alpha n_{1s}^u + \delta g(s) N_1^m) ds \\ \frac{d}{dx} (C_b(x) D_b(x)) &= 0, \forall x \in \{z : q_{bz} \in (0, \bar{q})\} \\ C_a(x) &= -\kappa + x - \gamma(e_\kappa - e_x) - r \int_\kappa^x \frac{1 + \gamma e'_s}{r + \delta + q_{as} + \gamma} ds \\ D_a(x) &= \int_x^{\bar{x}} \frac{r + \delta + q_{as} + \gamma}{\delta + q_{as} + \gamma} (\alpha n_{0s}^u + \delta g(s) N_0^m) ds \\ \frac{d}{dx} (C_a(x) D_a(x)) &= 0, \forall x \in \{z : q_{az} \in (0, \bar{q})\}\end{aligned} \tag{12}$$

$$q_{ax} = 0, \quad \forall x \leq \inf \{z : q_{az} \in (0, \bar{q})\}$$

$$q_{ax} = \bar{q}, \quad \forall x \geq \sup \{z : q_{az} \in (0, \bar{q})\}$$

$$q_{bx} = 0, \quad \forall x \geq \sup \{z : q_{bz} \in (0, \bar{q})\}$$

$$q_{bx} = \bar{q}, \quad \forall x \leq \inf \{z : q_{bz} \in (0, \bar{q})\}$$

(2) Given Ed , e_x and P , prices in the equilibrium contract menu can be solved by the following equations,

$$\begin{aligned} p_{bx} &= \frac{r+\delta+q_{bx}+\gamma}{q_{bx}} \left[P - \int_x^\kappa \frac{1+\gamma e'_s}{r+\delta+q_{bs}+\gamma} ds \right] - \frac{x+\delta Ed+\gamma e_x}{q_{bx}} \\ p_{ax} &= \frac{r+\delta+q_{ax}+\gamma}{q_{ax}} \left[P + \int_\kappa^x \frac{1+\gamma e'_s}{r+\delta+q_{as}+\gamma} ds \right] - \frac{x+\delta Ed+\gamma e_x}{q_{ax}} \end{aligned} \quad (13)$$

(3) Given q_{ax} , q_{bx} ,

$$\begin{aligned} n_{1x}^u &= \frac{\delta}{\alpha+\delta} \frac{\gamma}{\alpha+\gamma} Ag(x) + \frac{\gamma}{\alpha+\delta} n_{1x}^m \\ n_{0x}^u &= \frac{\delta}{\alpha+\delta} \frac{\gamma}{\alpha+\gamma} (1-A)g(x) + \frac{\gamma}{\alpha+\delta} n_{0x}^m \end{aligned} \quad (14)$$

$$\begin{aligned} n_{1x}^m &= \begin{cases} g(x) \frac{\alpha}{\alpha+\gamma} \frac{\alpha+\delta+\gamma}{\alpha+\delta+\gamma+\frac{\alpha+\delta}{\delta} q_{bx}} A & \text{if } x < \kappa \\ g(x) \frac{\alpha}{\alpha+\gamma} \left[1 - \frac{\alpha+\delta+\gamma}{\alpha+\delta+\gamma+\frac{\alpha+\delta}{\delta} q_{ax}} (1-A) \right] & \text{if } x \geq \kappa \end{cases} \\ n_{0x}^m &= \begin{cases} g(x) \frac{\alpha}{\alpha+\gamma} \left[1 - \frac{\alpha+\delta+\gamma}{\alpha+\delta+\gamma+\frac{\alpha+\delta}{\delta} q_{bx}} A \right] & \text{if } x < \kappa \\ g(x) \frac{\alpha}{\alpha+\gamma} \frac{\alpha+\delta+\gamma}{\alpha+\delta+\gamma+\frac{\alpha+\delta}{\delta} q_{ax}} (1-A) & \text{if } x \geq \kappa \end{cases} \end{aligned} \quad (15)$$

(4) Given trading terms, value functions are determined by

$$d_x = \frac{x + \delta Ed + q_{bx} p_{bx} + q_{ax} p_{ax} + \gamma e_x}{r + \delta + q_{ax} + q_{bx} + \gamma} \quad (16)$$

$$e_x = \frac{x + \alpha d_x}{r + \alpha + \delta} + \frac{\delta}{r + \alpha + \delta} \frac{X + \alpha Ed}{r + \alpha} \quad (17)$$

$$P = d_\kappa \quad (18)$$

(5) Given n_{0x}^m and n_{1x}^m , market clearing condition determines κ

$$\int_{x \geq \kappa} \frac{q_{ax}}{\delta + \gamma + q_{ax}} n_{0x}^m dx = \int_{x \leq \kappa} \frac{q_{bx}}{\delta + \gamma + q_{bx}} n_{1x}^m dx \quad (19)$$

Proof. The proof is left to the Appendix. In the Appendix, we first show part (3) of the theorem, how we solve for the stationary distribution from laws of motion, in section A. Then we go back to show part (1) of the theorem in section B, which comes from mechanism design problem of dealers. The results are unique

to the market equilibrium with endogenous distribution. Part (3) is a by product of the mechanism design problem. Part (4) and part (5) is obvious. What is left to show is the existence of the equilibrium. This is an application of Macolell (1984) [17]. $\square$

The theorem is presented in a constructive way to make it clear how we solve the model recursively. In the following sections, we will first derive some qualitative characterization of the equilibrium and then characterize numerically the equilibrium allocation in more detail.

4 Liquidity

In this section, we first discuss qualitative predictions of our model on liquidity. And then we further study quantitatively how the equilibrium allocation of liquidity is affected by frictions in the market, such as search friction, competition and dispersion of private value.

4.1 Qualitative results

Proposition 1. *Distortion to liquidity is type specific, there exist threshold types $\underline{x}^b, \bar{x}^b, \underline{x}^a, \bar{x}^a$, $\underline{x} < \underline{x}^b < \bar{x}^b < \kappa < \underline{x}^a < \bar{x}^a < \bar{x}$ such that*

- (1) *Trade for sellers of type $x \in [\bar{x}^b, \kappa]$ and buyers of type $x \in [\kappa, \underline{x}^a]$ breaks down.*
- (2) *Trade for sellers of type $x \in (\underline{x}^b, \bar{x}^b)$ and buyers of type $x \in (\underline{x}^a, \bar{x}^a)$ is delayed. Trading frequency is lower than $\bar{q}$. Trading frequency for type x decreases as x comes closer to κ .*
- (3) *Trade for sellers of type $x \in [\underline{x}, \underline{x}^b]$ and buyers of type $x \in [\bar{x}^a, \bar{x}]$ is executed without delay. But the bid price for sellers of type $x \in [\underline{x}, \underline{x}^b]$ is the lowest in the market and the ask price for buyers of type $x \in [\bar{x}^a, \bar{x}]$ is the highest.*

Proof. The proposition is a direct application of the equilibrium characterization from Theorem 1. The reason why region mentioned in point (1) and point (3) must exist in equilibrium is that the indifference condition, equations (12), cannot hold at either the boundary of the support of type distribution or at types close to κ . This is because $C_i(x)$ converges to zero as x converges to κ while $D_i(x)$ converges to zero as x

converges to $\underline{x}$ for $i = b$ and to $\bar{x}$ for $i = a$. This cannot hold when the indifference condition in equations (12) requires that $C_i(x)D_i(x)$ is a constant as long as this constant is nonzero. But since we assume q_{ax} and q_{bx} are continuous functions, $C_i(x)D_i(x)$ has to be nonzero for certain type, the constant is nonzero. So dealers strictly prefer offering highest possible trading frequencies for value close enough to the extreme. And they strictly prefer offering no trade for value close enough to κ . This proves the proposition. $\square$

While trade delays or breakdown distorts liquidity allocation directly, bid/ask spread also affect liquidity allocation indirectly through its effect on the gain from trade. Since dealers take away more profit from traders with extreme valuation of the assets, this leaves traders less incentive to trade.

Proposition 2. *Compared with the efficient allocation, the equilibrium allocation has the following qualitative properties. As dispersion increases,*

- (1) *bid-ask spread arises and increases*
- (2) *delay due to trading frictions increases*
- (3) *aggregate trading flow decreases*
- (4) *depending on asset supply, dispersion of the private value, liquidity distortion on the seller and buyer side does not have to be symmetric.*

Proof. From Theorem 1, we can show that it must be the case that $\bar{x}^b < \kappa < \underline{x}^a$. This means that dealers must earn positive profit. As long as dealers earn positive profit, there must exist non-trivial bid-ask spread. No trade takes place for all type $x \in [\bar{x}^b, \underline{x}^a]$, this shows that there must be delay due to endogenous trading frictions and trading volume must decrease.

Under complete information or perfect competition among dealers, the trading flow is $\delta \frac{\alpha}{\alpha+\gamma} AG(\kappa^*)$, where κ^* is the threshold type for efficient allocation. Note threshold type κ^* differs from threshold type κ . With private value dispersion, the aggregate trade flow is $\delta \frac{\alpha}{\alpha+\gamma} AG(\bar{x}^b)$. Market clearing condition rules that $AG(\bar{x}^b) = (1 - A)G(\underline{x}^a)$, for $\bar{x}^b < \underline{x}^a$. From feasibility condition of efficient allocation, we have $AG(\kappa^*) = (1 - A)G(\kappa^*)$. Since $G(x)$ is a strictly increasing function, contradiction will be reached if κ^* does not lie in the interval $(\bar{x}^b, \underline{x}^a)$. Therefore, $G(\kappa^*) > G(\bar{x}^b)$, which proves that aggregate trading volume

Preference			Matching		
shock arrival rate	δ	0.01	meeting	α	0.1
upper bound	$\bar{x}$	5	breakup	γ	0.1
lower bound	$\underline{x}$	15	Market		
mean		$\frac{\bar{x}+\underline{x}}{2}$	asset supply	A	0.5
standard deviation		$\frac{\bar{x}-\underline{x}}{9}$	trading frequency	$\bar{q}$	1
discount rate	r	0.02			

Table 1: Parameter value for benchmark case

decreases because of the existence of dispersed private value.

Depending on asset supply, it is clear that liquidity distortions may not be symmetric, since the allocation also depends on distribution of preference. This point can be illustrated by numerical examples. $\square$

To sum up, our results are qualitatively different from results in [10, 2]. We now go to numerical experiments to find out more features of the model. We will claims to label conclusions only tested in numerical experiments and use propositions, lemmas or corollaries to label conclusions proved rigorously.

4.2 Numerical results

Depending on application, we can calibrate our model with different parameter values. The parameter values used in the benchmark case are shown in table 1. We use truncated normal distribution with finite support as type distribution. The mean and standard deviation of the normal distribution are shown in the table. Without otherwise specified, we will use parameter values in the table for illustration.

Figure 1 illustrates the contract menu with parameter values in table 1.

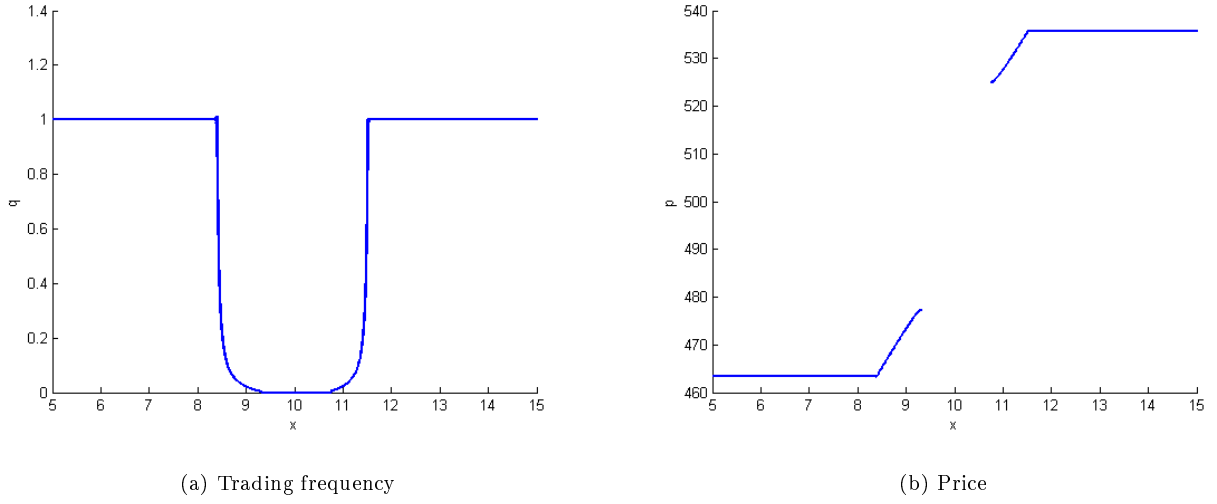


Figure 1: Equilibrium contract menu

Response of equilibrium contract menu to increasing dispersion

Claim 1. Both average bid ask spread and maximum and minimum bid ask spread increases with increasing dispersion. Delay increases uniformly for all types as

We also do some numerical experimentation with changes in distribution and the claim is in general true. Figure ? illustrates the change of bid-ask spread as we increase the standard deviation from 1.125 to 2.25⁵

Observed Price Elasticity to Immediacy

Individual investors are infinitesimal, so they take the menu of trading terms as given. The menu of trading terms implies price for different immediacy of service. So price elasticity for immediacy can be implied from the optimal contract.

Claim 2. Observed price elasticity to $\frac{\partial \ln |p-P|}{\partial \ln q}$ immediacy decreases as trading frequency increases.

This is because traders with less elastic demand self select into contracts with higher trading frequency. So as trading frequencies increases, prices are closer to monopoly price

⁵More specifically, we change rescale the type distribution and leave the mean of the distribution unchanged.

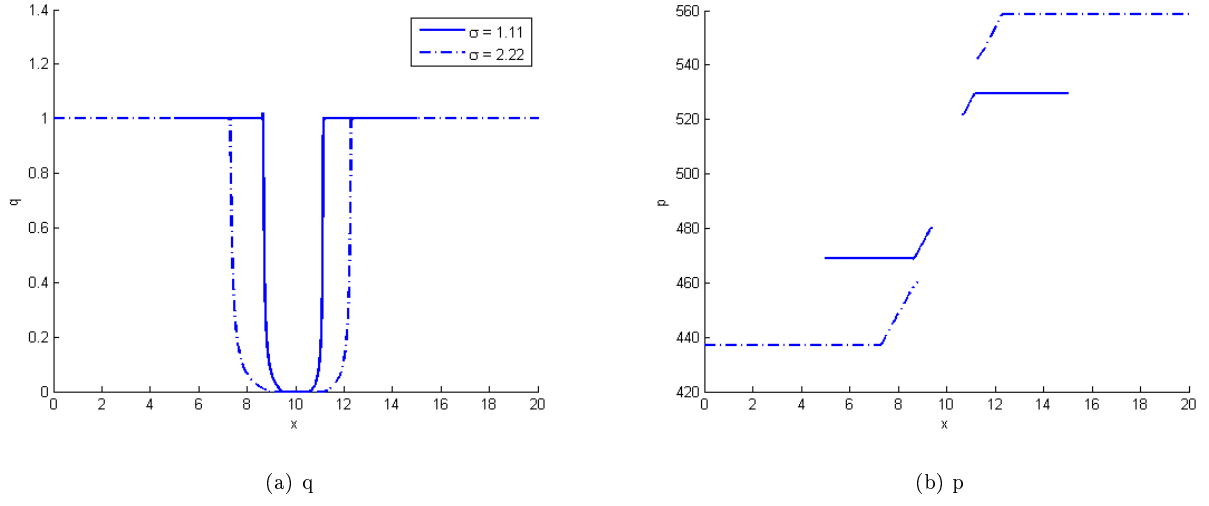


Figure 2: Response of the equilibrium contract menu to dispersion of private value

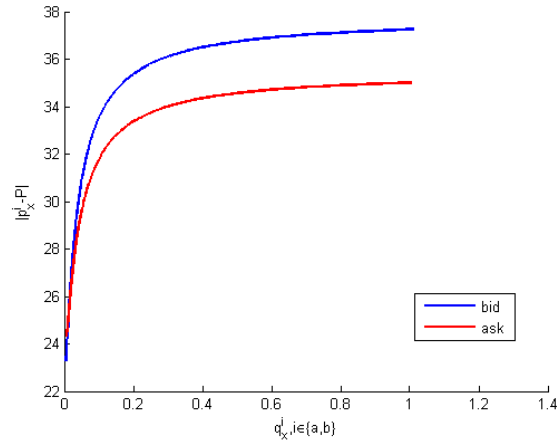


Figure 3: Observed price elasticity to immediacy of trade is decreasing in immediacy

Price in the Inter-dealer Market

The effect of private value dispersion on the price in the competitive inter-dealer market depends on the asset supply.

Claim 3. The market price is suppressed as the dispersion increases if total asset supply is large; the market price rises as the dispersion increases if total asset supply is small.

Effect of competition

Competition has a nonuniform effect on immediacy of trade for different type of traders. Traders with extreme value are less affected by competition.

Corollary 1. *The equilibrium allocation will not converge to efficient allocation even as γ converge to infinity.*

Proof. This is a corollary of 1. The intervals $[\underline{x}, \underline{x}^b]$ and $[\bar{x}^a, \bar{x}]$ will not be empty for any finite value of γ . $\square$

Claim 4. As γ increase, competition will drive the equilibrium allocation close to efficient allocation.

Figure 4 illustrates the point of Claim 4. All parameter values are benchmark value except those parameters mentioned in the figure. We intentionally raise α to a very high level to highlight the effect of competition. When α is low, breakup from a dealer not just introduces competition but also increases search cost. Interestingly, competition has an asymmetric effect on the contract menu for buyers and sellers. In this example, trading frequency does not weakly increase for all type of traders as competition among dealers becomes more intensive.

5 Efficient allocation and its decentralization

Before showing the welfare consequences of the liquidity distortions in our model. In this section, we will study the efficient allocation and what market microstructure can help decentralize the efficient allocation. The decentralization highlights the effect of imperfect competition among dealers on efficiency.

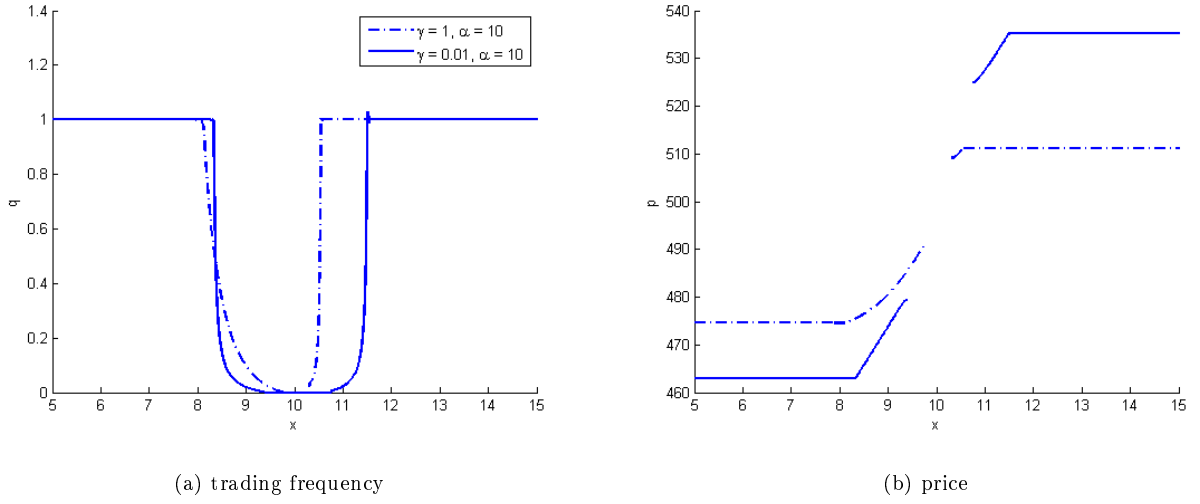


Figure 4: The effect of competition on liquidity

5.1 Social planner's solution

Suppose the social planner replaces dealers to allocate resources. The constrained efficient allocation with search friction should be such that when asset holders and non asset holders are in touch with the dealers, .

Definition 2. An *allocation* is a mapping from the state of an agent to decisions on asset allocation and allocation frequency, $\Gamma : (x, a) \mapsto (a', q)$, which applies to an agent when she meets the planner.

The allocation takes into consideration the fact that asset may be delivered with delay. Let $\Gamma(x, a) = (a'(x, a), q(x, a))$, then we have

Definition 3. A *feasible allocation* is an allocation that satisfies the following restrictions: (1) $q \leq \bar{q}$; (2) $\int q(x, 0) [0 - a'(x, 0)] n_{x0}^m dx + \int q(x, 1) [1 - a'(x, 1)] n_{x1}^m dx \geq 0$

An allocation is feasible if the frequency of delivery is lower than the upper bound and it satisfies the flow resource constraint. We assume that the planner does not keep inventories so that the flow supply of asset unit will be greater or equal to the total amount of asset allocated.

Definition 4. A *constrained efficient allocation* is a feasible allocation that maximizes social welfare.

Let the value of a trader with n asset and value x be $V(x, n)$. Then the social planner solves the following problem,

$$\begin{aligned} & \max_{\Gamma} \int M_{a'(x,1),x} n_{x1}^m + M_{a'(x,0),x} n_{x1}^m dx \\ \text{s.t} \quad & q(x, n) \leq \bar{q}, \forall x, n \\ & \int q(x, 0) [0 - a'(x, 0)] n_{x0}^m dx + \int q(x, 1) [1 - a'(x, 1)] n_{x1}^m dx \geq 0 \end{aligned}$$

Proposition 3. *The constrained efficient allocation is characterized by a cutoff value κ such that $G(\kappa) = (1 - A)$ and*

$$\begin{aligned} a'(x, 0) &= \begin{cases} 1 & \text{if } x \geq \kappa \\ 0 & \text{if } x < \kappa \end{cases} \\ a'(x, 1) &= \begin{cases} 1 & \text{if } x \geq \kappa \\ 0 & \text{if } x < \kappa \end{cases} \\ q(x, 0) &= \begin{cases} \bar{q} & \text{if } x \geq \kappa \\ 0 & \text{if } x < \kappa \end{cases} \\ q(x, 1) &= \begin{cases} 0 & \text{if } x \geq \kappa \\ \bar{q} & \text{if } x < \kappa \end{cases} \end{aligned}$$

Proof. First, we look at social planner's problem without trading frequency constraint. For any agent with value higher than κ , the allocation from the social planner is first best. And because the social planner's resource of asset is used up on agents with value above κ , to transform the current allocation to any alternative allocation, the social planner will have to move some asset from agents with value above κ to agents with value below κ . But any such reallocation decreases social welfare.

Since, the social planner knows the first best allocation, then there is no reason for her to delay the allocation. So all allocation takes place at the highest trading frequency.

Therefore, the cutoff threshold κ can be backed out from the resource constraint of the social planner, where n_{x0}^m and n_{x1}^m can be solved from laws of motion given allocation decisions. The description of the allocation of a social planner's solution is not complete. The endogenous state distribution is not specified yet. We defer the specification to the proof for decentralization results in the next subsection. $\square$

The social planner's solution is first best if she can observe the type of traders. But it is also incentive compatible for traders to truthfully report their type to the planner under the efficient allocation. So the allocation is efficient with or without private information. In the next subsection, we will show that the first best allocation can be decentralized in an economy with private value and trading through dealers.

5.2 Decentralization of the first best solution

In this subsection, our goal is to decentralize the first best allocation in an OTC market. To decentralize the first best allocation, we must incorporate the following features: (1) there is no delay in trade; (2) all asset holders with value below κ are willing to sell and all non asset holders with value above κ are willing to buy. Another requirement from the market structure of the OTC market is that dealers must earn non-negative profit.

To satisfy the first requirement, we need the trading to take place as fast as possible. If all trade takes place at the same frequency, then there is no way for a dealer to price discriminate traders of different type. So trade must take place in a pooling equilibrium. Suppose bid/ask in the pooling equilibrium is p^b and p^a respectively.

To satisfy the second requirement, the value of no trade for all asset holders with value below κ shall be below the revenue from trade, the bid price of dealers. $p^b + M_{x0} \geq M_{x1}, \forall x \leq \kappa$

And the gain from trade for non asset holders with value above κ shall be positive. $V(x, 1) - p^a \geq V(x, 0), \forall x \geq \kappa$.

Therefore, we have $p^a \leq M_{x1} - M_{x0}, \forall x \geq \kappa$, and $p^b \geq M_{x1} - M_{x0}, \forall x \leq \kappa$. But for dealers to non-negative profit, $p^b \leq p^a$. Therefore, we know that as long as the value function are continuous and increasing

in value, it must be that $p^b = p^a = M_{\kappa 1} - M_{\kappa 0}$.

And the allocation is incentive compatible. This is because for asset holders with value above κ and non asset holder with value below κ , the option value of trade is negative.

The allocation can be decentralized in such an OTC market: every time when a trader meets a dealer, traders always get the chance to propose a trading term. Expecting that the market price in the inter-dealer market to be P , traders will always leave zero profit to the dealer. Therefore, trade always take place as quick as possible at the competitive market price of the inter-dealer market.

Proposition 4. *The constrained efficient allocation can be decentralized in an OTC market by the following trading term: $p^b = p^a = P$, $q^b = q^a = \bar{q}$. The inter-dealer market is cleared by the price $P = p^b = p^a$. The equilibrium allocation can be characterized by equations in Theorem 1 except that*

$$q_{ax} = \begin{cases} \bar{q} & \text{if } x \geq \kappa \\ 0 & \text{if } x < \kappa \end{cases}$$

$$q_{bx} = \begin{cases} 0 & \text{if } x \geq \kappa \\ \bar{q} & \text{if } x < \kappa \end{cases}$$

and

$$p_{ax} = P, \forall x \geq \kappa$$

$$p_{bx} = P, \forall x \leq \kappa$$

In the decentralization of the first best allocation, bid ask spread is zero and no delay takes place in trade. This is not what we usually see. In the next section, we will show how the monopoly power of dealers will generate an endogenous distribution of bid ask spread, delay in trade and at the same time, bring distort to the economy, pushing it away from the first best allocation.

The decentralization of the efficient allocation and the allocation from the dealer-proposing market game we study can be thought as two extreme cases in the bargaining problem between dealers and traders. If

traders propose with probability 1, the efficient allocation can be decentralized by the market game. If dealers propose with probability 1, the strategic behavior of dealers leads to distortion in liquidity allocation in the trading game. In the extensions, we discuss an extension of our model to introduce bargaining power of traders, by allowing both sides to make take-it-or-leave offer with some probability. The equilibrium allocation of the extended model would be between the allocation of our model and the efficient allocation. And characterization of the allocation will remain qualitatively similar.

6 Asset misallocation and welfare

Liquidity misallocation also leads to misallocation of the asset across traders of different value. In this section, we characterize the equilibrium allocation of the asset and study efficiency loss from both the liquidity distortion and asset misallocation.

From Theorem 2, we can see that the equilibrium allocation of liquidity is not efficient. So in this section, we first study how the allocative efficiency of liquidity varies across types of traders. And then we study quantitatively how efficiency is related to such market conditions as competition intensity between dealers and total asset supply.

6.1 Allocative efficiency and asset misallocation

Proposition 5. *Asset misallocation comes from two sources of liquidity misallocation: trade breakdown and trade delay*

- (1) *Distortion to asset allocation for $x \in [\bar{x}^b, \underline{x}^a]$ are due to trade breakdown.*
- (2) *Distortion to asset allocation for $x \in (\underline{x}^b, \bar{x}^b)$ and $x \in (\underline{x}^a, \bar{x}^a)$ are due to trade delay.*

Figure 5 illustrates the difference between the efficient asset allocation and the actual asset allocation predicted by our model. The distortion between the two kinks in the actual allocation is due to trade breakdown. Misallocation of asset across agents not matched with dealers is similar. The additional misallocation for searching agents comes from the fact that searching for a dealer takes time. Preference shock before

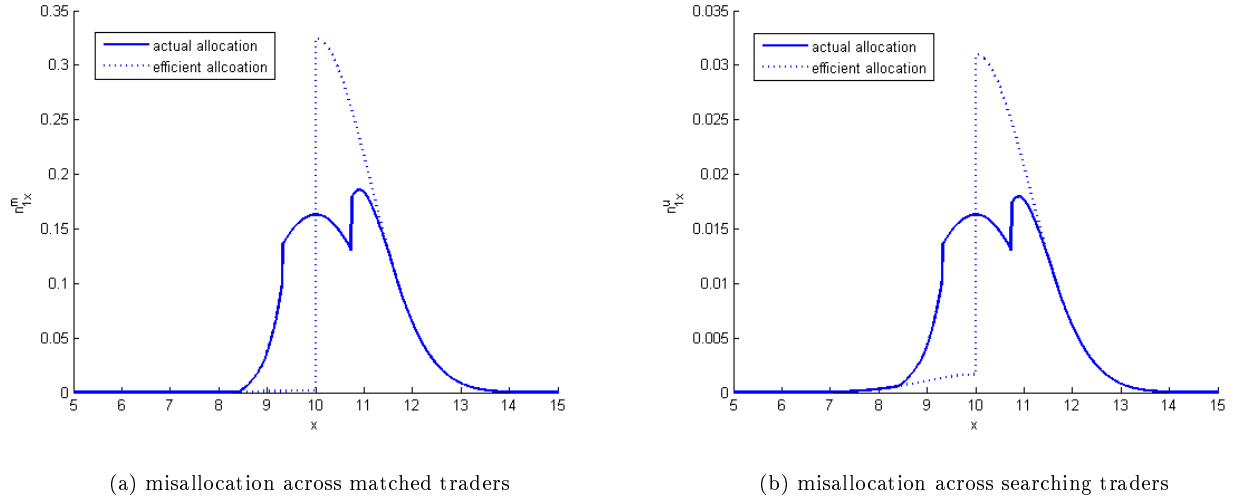


Figure 5: Distortion to asset allocation due to misallocation of liquidity

finding a dealer leads to the additional welfare loss.

6.2 Distortion to aggregate welfare

Lemma 3. *In a stationary equilibrium, the social welfare is determined by the efficiency of asset allocation, the social welfare S can be solved by the following equation*

$$S = \frac{1}{r} \int x(n_{1x}^m + n_{1x}^u) dx \quad (20)$$

Proof. The utility of agents in the economy is comprised of transfer and direct utility from owning the asset. Transfers from traders to dealers do not matter for aggregate welfare since they add up to zero. What matters is only asset allocation. □

So as a corollary of Proposition 1, we know that private value dispersion decreases social welfare.

Corollary 2. *The social welfare of the equilibrium allocation with dispersed private value is lower than that of the efficient allocation.*

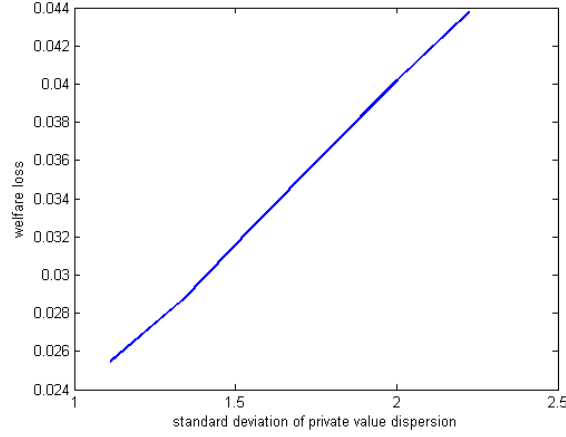


Figure 6: Welfare loss as a percentage of the efficient level increases as the dispersion of private value increases.

Quantitatively, with benchmark parameter values, the social welfare of the equilibrium allocation is 2.5% lower than the level of the efficient allocation.

Claim 5. As dispersion of privation value increases, welfare loss increases.

As illustrated in Figure 6, when we use the benchmark parameter values, as we increase the standard deviation from 1.125 to 2.25, welfare loss rises from 2.5% to nearly 4.5%.

7 Comparison with complete information counterpart

To see more clearly the effect of private value, we study the complete information counterpart of our model in this section.

In the complete information model, Diamond Paradox[4] will kick in. So the result is trivial: all surplus from trade is taken by dealers. And the allocation would be efficient.

Proposition 6. *Under complete information,*

$$M_{1x} = \frac{rx + \delta Ex}{r(r + \delta)}$$

$$M_{0x} = 0$$

$$U_{1x} = \frac{rx + \delta Ex}{r(r + \delta)}$$

$$U_{0x} = 0$$

Proof. Under complete information, all gain from trade is taken from traders given continuation value M_{1x} , M_{0x} . So traders' value can be written as follows,

$$M_{1x} = \frac{x + \delta EM_1 + \gamma U_{1x}}{r + \delta + \gamma}$$

$$M_{0x} = \frac{\delta EM_0 + \gamma U_{0x}}{r + \delta + \gamma}$$

$$U_{1x} = \frac{x + \delta EU_1 + \alpha M_{1x}}{r + \delta + \alpha}$$

$$U_{0x} = \frac{\delta EU_0 + \alpha M_{0x}}{r + \delta + \alpha}$$

□

Proposition 6 shows the forces behind Diamond Paradox: dealers have monopoly power at any moment, leaving traders no gain from trade. So traders get Autarky values.

Interestingly, the Diamond Paradox will not show up with incomplete information. This is because dealers, even if they are monopolist, cannot extract all surplus from traders under asymmetric information. At least some information rent has to go to traders. But whenever some rent is left to traders, dynamic competition between dealers will show up.

In comparison with results in our model, we can see that our results on endogenous delay and bid/ask spread come solely from the information friction of private value.

8 Extensions

Bargaining power of dealers

Alternatively, we can characterize the competition effect by allowing traders to make a take-it-or-leave-it offer to dealers with some probability λ . Since dealers' reservation price is P for bidding or asking, traders

will always propose to trade with price P if they have the opportunity to do so.

At the beginning of a new relationship, traders have all the bargaining power with probability λ and dealers are assigned all bargaining power with probability $1 - \lambda$. The party assigned the bargaining power makes a take-it-or-leave-it contract. In the benchmark model, we assume $\lambda = 0$.

Competition through on-the-job search

Another way to introduce competition among dealers is to allow on-the-job search. Then each dealer knows that they only have monopoly power over a fraction of their customer pool. The tradeoff for them is whether to extract monopoly profit from the fraction or to compete with other dealers by offering a more favorable menu of contracts so they can attract more customers from the pool.

9 Conclusion

We study in a random matching environment the effect of private value on liquidity, asset allocation and social welfare. We find that the dispersion of private value could have a large impact on all these dimensions. This is especially important for an economy in turmoil, when heterogeneity in private value may explode.

As Figure 5 shows, this paper also implies a theory of mismatch between asset holding and private value. This is worth further exploring.

Depending on interpretation, our model can be applied to many specific markets with similar market microstructure. We think that the market structure is especially similar to that in a housing market, where our model implies that even if the housing quality is homogeneous, there might still be misallocation across agents of different values. If agents hold an asset for liquidity purposes, we can study the effect of private information on liquidity allocation. Gale and Yorulmazer (2011) study liquidity hoarding of ex ante homogeneous banks. With modification, we intend to study the trading and liquidity of the market for the liquidity asset, with ex ante heterogeneous banks. Our results can also be applied to study a durable goods market, allowing resale and fluctuating tastes.

References

- [1] M.R. Blouin and R. Serrano. A decentralized market with common values uncertainty: Non-steady states. *The Review of Economic Studies*, 68(2):323, 2001.
- [2] B. Chang. Adverse selection and liquidity distortion in decentralized markets. *working paper*, 2011.
- [3] J. Chiu and T. Koepl. Trading dynamics with adverse selection and search: Market freeze, intervention and recovery. 2011.
- [4] P.A. Diamond. A model of price adjustment. *Journal of Economic Theory*, 3(2):156–168, 1971.
- [5] D. Duffie. *Dark Markets: Asset Pricing and Information Transmission in Over-The-Counter Markets*. Princeton Univ Pr, 2012.
- [6] D. Duffie, N. Gârleanu, and L.H. Pedersen. Over-the-counter markets. *Econometrica*, 73(6):1815–1847, 2005.
- [7] M. Golosov, G. Lorenzoni, and A. Tsyvinski. Decentralized trading with private information, forthcoming.
- [8] S.J. Grossman and M.H. Miller. Liquidity and market structure. *The Journal of Finance*, 43(3), 1988.
- [9] V. Guerrieri and R. Shimer. Dynamic adverse selection: A theory of illiquidity, fire sales, and flight to quality. 2011.
- [10] Veronica Guerrieri, Robert Shimer, and Randall Wright. Adverse selection in competitive search equilibrium. *Econometrica*, 78(6):1823–1862, 2010.
- [11] I. Hendel and A. Lizzeri. Adverse selection in durable goods markets. *American Economic Review*, 89(5):1097–1115, 1999.
- [12] J. Hörner and N. Vieille. Public vs. private offers in the market for lemons. *Econometrica*, 77(1):29–69, 2009.

- [13] Roman Inderst. Matching markets with adverse selection. *Journal of Economic Theory*, 121(2):145–166, 2005.
- [14] R. Lagos and G. Rocheteau. Liquidity in asset markets with search frictions. *Econometrica*, 77(2):403–426, 2009.
- [15] S. Lauermann and A. Wolinsky. Search with adverse selection. *working paper*, 2011.
- [16] B. Lester and B. Camargo. Trading dynamics in decentralized markets with adverse selection. 2011.
- [17] A. Mas-Colell. On a theorem of schmeidler. *Journal of Mathematical Economics*, 13(3):201–206, 1984.
- [18] P. Milgrom and I. Segal. Envelope theorems for arbitrary choice sets. *Econometrica*, 70(2):583–601, 2002.
- [19] D. Schmeidler. Equilibrium points of nonatomic games. *Journal of Statistical Physics*, 7(4):295–300, 1973.
- [20] A. Wolinsky. Information revelation in a market with pairwise meetings. *Econometrica*, pages 1–23, 1990.

A Stationary Distribution

To solve for the stationary distribution, we first write down the laws of motion,

$$\begin{aligned}\dot{n}_{1x}^u &= -\alpha n_{1x}^u - \delta n_{1x}^u + \delta g(x)N_1^u + \gamma n_{1x}^m \\ \dot{n}_{0x}^u &= -\alpha n_{0x}^u - \delta n_{0x}^u + \delta g(x)N_0^u + \gamma n_{0x}^m\end{aligned}\tag{21}$$

$$\begin{aligned}\dot{n}_{1x}^m &= \begin{cases} \alpha n_{1x}^u - \delta n_{1x}^m + \delta g(x)N_1^m - \gamma n_{1x}^m - q_{bx}n_{1x}^m & \text{if } x < \kappa \\ \alpha n_{1x}^u - \delta n_{1x}^m + \delta g(x)N_1^m - \gamma n_{1x}^m + q_{ax}n_{0x}^m & \text{if } x \geq \kappa \end{cases} \\ \dot{n}_{0x}^m &= \begin{cases} \alpha n_{0x}^u - \delta n_{0x}^m + \delta g(x)N_0^m - \gamma n_{0x}^m + q_{bx}n_{1x}^m & \text{if } x < \kappa \\ \alpha n_{0x}^u - \delta n_{0x}^m + \delta g(x)N_0^m - \gamma n_{0x}^m - q_{ax}n_{0x}^m & \text{if } x \geq \kappa \end{cases}\end{aligned}\tag{22}$$

When the economy rests in the stationary distribution, $\dot{n}_{ix}^j = 0$, so we have,

$$\begin{aligned}(\alpha + \delta)n_{1x}^u &= \delta g(x)N_1^u + \gamma n_{1x}^m \\ (\alpha + \delta)n_{0x}^u &= \delta g(x)N_0^u + \gamma n_{0x}^m\end{aligned}$$

By Lemma 1, we have,

$$\begin{aligned}n_{1x}^u &= \frac{\delta}{\alpha + \delta} \frac{\gamma}{\alpha + \gamma} Ag(x) + \frac{\gamma}{\alpha + \delta} n_{1x}^m \\ n_{0x}^u &= \frac{\delta}{\alpha + \delta} \frac{\gamma}{\alpha + \gamma} (1 - A)g(x) + \frac{\gamma}{\alpha + \delta} n_{0x}^m\end{aligned}\tag{23}$$

Substituting equations (23) into equations (22) and also making use of Lemma 1, we have,

$$\begin{aligned}0 &= \begin{cases} \alpha \left[\frac{\delta}{\alpha + \delta} \frac{\gamma}{\alpha + \gamma} Ag(x) + \frac{\gamma}{\alpha + \delta} n_{1x}^m \right] - \delta n_{1x}^m + \delta g(x) \frac{\alpha}{\alpha + \gamma} A - \gamma n_{1x}^m - q_{bx}n_{1x}^m & \text{if } x < \kappa \\ \alpha \left[\frac{\delta}{\alpha + \delta} \frac{\gamma}{\alpha + \gamma} Ag(x) + \frac{\gamma}{\alpha + \delta} n_{1x}^m \right] - \delta n_{1x}^m + \delta g(x) \frac{\alpha}{\alpha + \gamma} A - \gamma n_{1x}^m + q_{ax}n_{0x}^m & \text{if } x \geq \kappa \end{cases} \\ 0 &= \begin{cases} \alpha \left[\frac{\delta}{\alpha + \delta} \frac{\gamma}{\alpha + \gamma} (1 - A)g(x) + \frac{\gamma}{\alpha + \delta} n_{0x}^m \right] - \delta n_{0x}^m + \delta g(x) \frac{\alpha}{\alpha + \gamma} (1 - A) - \gamma n_{0x}^m + q_{bx}n_{1x}^m & \text{if } x < \kappa \\ \alpha \left[\frac{\delta}{\alpha + \delta} \frac{\gamma}{\alpha + \gamma} (1 - A)g(x) + \frac{\gamma}{\alpha + \delta} n_{0x}^m \right] - \delta n_{0x}^m + \delta g(x) \frac{\alpha}{\alpha + \gamma} (1 - A) - \gamma n_{0x}^m - q_{ax}n_{0x}^m & \text{if } x \geq \kappa \end{cases}\end{aligned}$$

$$\begin{aligned}
n_{1x}^m &= \begin{cases} \frac{1}{\delta + q_{bx} + \gamma - \frac{\alpha}{\alpha+\delta}\gamma} \left\{ \alpha \left[\frac{\delta}{\alpha+\delta} \frac{\gamma}{\alpha+\gamma} Ag(x) \right] + \delta g(x) \frac{\alpha}{\alpha+\gamma} A \right\} & \text{if } x < \kappa \\ \frac{1}{\delta + \gamma - \frac{\alpha}{\alpha+\delta}\gamma} \left\{ \alpha \left[\frac{\delta}{\alpha+\delta} \frac{\gamma}{\alpha+\gamma} Ag(x) \right] + \delta g(x) \frac{\alpha}{\alpha+\gamma} A + q_{ax} n_{0x}^m \right\} & \text{if } x \geq \kappa \end{cases} \\
n_{0x}^m &= \begin{cases} \frac{1}{\delta + \gamma - \frac{\alpha}{\alpha+\delta}\gamma} \left\{ \alpha \left[\frac{\delta}{\alpha+\delta} \frac{\gamma}{\alpha+\gamma} (1-A)g(x) \right] + \delta g(x) \frac{\alpha}{\alpha+\gamma} (1-A) + q_{bx} n_{1x}^m \right\} & \text{if } x < \kappa \\ \frac{1}{\delta + \gamma - \frac{\alpha}{\alpha+\delta}\gamma + q_{ax}} \left\{ \alpha \left[\frac{\delta}{\alpha+\delta} \frac{\gamma}{\alpha+\gamma} (1-A)g(x) \right] + \delta g(x) \frac{\alpha}{\alpha+\gamma} (1-A) \right\} & \text{if } x \geq \kappa \end{cases}
\end{aligned}$$

After simplification,

$$\begin{aligned}
n_{1x}^m &= \begin{cases} \frac{g(x)}{1 + \frac{\gamma}{\alpha+\delta} + \frac{q_{bx}}{\delta}} \frac{\alpha+\delta+\gamma}{\alpha+\delta} \frac{\alpha}{\alpha+\gamma} A & \text{if } x < \kappa \\ \frac{g(x)}{1 + \frac{\gamma}{\alpha+\delta}} \frac{\alpha+\delta+\gamma}{\alpha+\delta} \frac{\alpha}{\alpha+\gamma} A + \frac{q_{ax}}{1 + \frac{\gamma}{\alpha+\delta}} n_{0x}^m & \text{if } x \geq \kappa \end{cases} \\
n_{0x}^m &= \begin{cases} \frac{g(x)}{1 + \frac{\gamma}{\alpha+\delta}} \frac{\alpha+\delta+\gamma}{\alpha+\delta} \frac{\alpha}{\alpha+\gamma} (1-A) + \frac{q_{bx}}{1 + \frac{\gamma}{\alpha+\delta}} n_{1x}^m & \text{if } x < \kappa \\ \frac{g(x)}{1 + \frac{\gamma}{\alpha+\delta} + \frac{q_{ax}}{\delta}} \frac{\alpha+\delta+\gamma}{\alpha+\delta} \frac{\alpha}{\alpha+\gamma} (1-A) & \text{if } x \geq \kappa \end{cases}
\end{aligned}$$

Reorganizing the equations,

$$\begin{aligned}
n_{1x}^m &= \begin{cases} \frac{1 + \frac{\gamma}{\alpha+\delta}}{1 + \frac{\gamma}{\alpha+\delta} + \frac{q_{bx}}{\delta}} g(x) \frac{\alpha}{\alpha+\gamma} A & \text{if } x < \kappa \\ g(x) \frac{\alpha}{\alpha+\gamma} A + \frac{q_{ax}}{1 + \frac{\gamma}{\alpha+\delta} + \frac{q_{bx}}{\delta}} g(x) \frac{\alpha}{\alpha+\gamma} (1-A) & \text{if } x \geq \kappa \end{cases} \\
n_{0x}^m &= \begin{cases} g(x) \frac{\alpha}{\alpha+\gamma} (1-A) + \frac{q_{bx}}{1 + \frac{\gamma}{\alpha+\delta} + \frac{q_{bx}}{\delta}} g(x) \frac{\alpha}{\alpha+\gamma} A & \text{if } x < \kappa \\ \frac{1 + \frac{\gamma}{\alpha+\delta}}{1 + \frac{\gamma}{\alpha+\delta} + \frac{q_{ax}}{\delta}} g(x) \frac{\alpha}{\alpha+\gamma} (1-A) & \text{if } x \geq \kappa \end{cases} \\
n_{1x}^m &= \begin{cases} g(x) \frac{\alpha}{\alpha+\gamma} \frac{\alpha+\delta+\gamma}{\alpha+\delta+\gamma+\frac{\alpha+\delta}{\delta}q_{bx}} A & \text{if } x < \kappa \\ g(x) \frac{\alpha}{\alpha+\gamma} \left[1 - \frac{\alpha+\delta+\gamma}{\alpha+\delta+\gamma+\frac{\alpha+\delta}{\delta}q_{ax}} (1-A) \right] & \text{if } x \geq \kappa \end{cases} \\
n_{0x}^m &= \begin{cases} g(x) \frac{\alpha}{\alpha+\gamma} \left[1 - \frac{\alpha+\delta+\gamma}{\alpha+\delta+\gamma+\frac{\alpha+\delta}{\delta}q_{bx}} A \right] & \text{if } x < \kappa \\ g(x) \frac{\alpha}{\alpha+\gamma} \frac{\alpha+\delta+\gamma}{\alpha+\delta+\gamma+\frac{\alpha+\delta}{\delta}q_{ax}} (1-A) & \text{if } x \geq \kappa \end{cases}
\end{aligned} \tag{24}$$

One can check that Lemma 1 holds with these solutions. This proves part (3) of Theorem 1.

B Solution to the mechanism design problem of a dealer

Dealer's objective is to maximize total expected profit given constraints specified in equations (11). Before solving the maximization problem, we need to explain: (1) how we solve for the response of customer type distribution, as shown in equations (11); (2) how we reduce the constrained problem to be an unconstrained problem.

Response of customer distributions to choice of contract menus

Taking n_{1x}^u and n_{0x}^u as given, what is the response of $\tilde{n}_{ix}^m$ of a dealer to her contract menu?

The laws of motion for $\tilde{n}_{ix}^m$ are

$$\begin{aligned}\dot{\tilde{n}}_{1x}^m &= \begin{cases} \alpha n_{1x}^u - \delta \tilde{n}_{1x}^m + \delta g(x) \tilde{N}_1^m - \gamma \tilde{n}_{1x}^m - q_{bx} \tilde{n}_{1x}^m & \text{if } x < \kappa \\ \alpha n_{1x}^u - \delta \tilde{n}_{1x}^m + \delta g(x) \tilde{N}_1^m - \gamma \tilde{n}_{1x}^m + q_{ax} \tilde{n}_{0x}^m & \text{if } x \geq \kappa \end{cases} \\ \dot{\tilde{n}}_{0x}^m &= \begin{cases} \alpha n_{0x}^u - \delta \tilde{n}_{0x}^m + \delta g(x) \tilde{N}_0^m - \gamma \tilde{n}_{0x}^m + q_{bx} \tilde{n}_{1x}^m & \text{if } x < \kappa \\ \alpha n_{0x}^u - \delta \tilde{n}_{0x}^m + \delta g(x) \tilde{N}_0^m - \gamma \tilde{n}_{0x}^m - q_{ax} \tilde{n}_{0x}^m & \text{if } x \geq \kappa \end{cases}\end{aligned}$$

From the laws of motion we can solve for $\tilde{n}_{ix}^m$ in the steady state.

$$\begin{aligned}\tilde{n}_{1x}^m &= \begin{cases} \frac{1}{\delta + \gamma + q_{bx}} \left(\alpha n_{1x}^u + \delta g(x) \tilde{N}_1^m \right) & \text{if } x < \kappa \\ \alpha n_{1x}^u - \delta \tilde{n}_{1x}^m + \delta g(x) \tilde{N}_1^m - \gamma \tilde{n}_{1x}^m + q_{ax} \tilde{n}_{0x}^m & \text{if } x \geq \kappa \end{cases} \\ \tilde{n}_{0x}^m &= \begin{cases} \alpha n_{0x}^u - \delta \tilde{n}_{0x}^m + \delta g(x) \tilde{N}_0^m - \gamma \tilde{n}_{0x}^m + q_{bx} \tilde{n}_{1x}^m & \text{if } x < \kappa \\ \frac{1}{\delta + \gamma + q_{ax}} \left(\alpha n_{0x}^u + \delta g(x) \tilde{N}_0^m \right) & \text{if } x \geq \kappa \end{cases}\end{aligned}\tag{25}$$

We assume that firms only do local perturbation to their equilibrium menu of contracts, so that when the perturbation is carried out, the aggregate population of asset holders and non asset holders connected with the dealer does not change. That is, $\tilde{N}_1^m = \frac{\alpha}{\alpha + \gamma} A$, $\tilde{N}_0^m = \frac{\alpha}{\alpha + \gamma} (1 - A)$.

Reducing the number of constraints

We assume that apart from the constraints on the response of customer distributions, only downward incentive compatibility constraints is binding, that is, the constraints for traders with small gain not to pretend to be traders with larger gain. After we solve the problem, we will verify that this is indeed the case. Since in equilibrium, it is always better off for agents to stay with the current dealer rather than quit and search for a new dealer. We only need to verify if IC constraints and enforceability constraints are binding. We will check that numerically. The monotonicity condition that secures incentive compatibility of the contract menu holds in the numerical example we use in the paper. For seq_{ax} is increasing in x and q_{bx} is decreasing in x .

$$d_x = \frac{x + \delta E d + q_{bx} p_{bx} + q_{ax} p_{ax} + \gamma e_x}{r + \delta + q_{ax} + q_{bx} + \gamma}$$

$$= \begin{cases} \frac{x + \delta E d + q_{ax} p_{ax} + \gamma e_x}{r + \delta + q_{ax} + \gamma} & \text{if } x > \kappa \\ \frac{x + \delta E d + q_{bx} p_{bx} + \gamma e_x}{r + \delta + q_{bx} + \gamma} & \text{if } x \leq \kappa \end{cases} \quad (26)$$

$$rM_{1x} = \max_{\hat{x}} x + \delta(EM_1 - M_{1x}) + q_{b\hat{x}}(p_{b\hat{x}} + M_{0x} - M_{1x}) + \gamma(U_{1x} - M_{1x})$$

$$rM_{0x} = \max_{\hat{x}} \delta(EM_0 - M_{0x}) + q_{a\hat{x}}(-p_{a\hat{x}} + M_{1x} - M_{0x}) + \gamma(U_{0x} - M_{0x})$$

Since it is incentive compatible for type $1x$ to choose (q_{bx}, p_{bx}) and for type $0x$ to choose (q_{ax}, p_{ax}) , by envelope theorem[18], we have

$$rM'_{1x} = 1 - \delta M'_{1x} - q_{bx} d'_x + \gamma(U'_{1x} - M'_{1x})$$

$$rM'_{0x} = -\delta M'_{0x} + q_{ax} d'_x + \gamma(U'_{0x} - M'_{0x})$$

From them we can derive,

$$\begin{aligned}
d'_x &= \frac{1+\gamma e'_x}{r+\delta+q_{bx}+q_{ax}+\gamma} \\
&= \begin{cases} \frac{1+\gamma e'_x}{r+\delta+q_{ax}+\gamma} & \text{if } x > \kappa \\ \frac{1+\gamma e'_x}{r+\delta+q_{bx}+\gamma} & \text{if } x \leq \kappa \end{cases}
\end{aligned} \tag{27}$$

Since $d_\kappa = P$ by definition, we can use d_κ as a parameter in dealers' problem (it will be endogenized after we solve the whole equilibrium and obtain the market clearing price P),

$$d_x = d_\kappa + \int_\kappa^x d'_s ds \tag{28}$$

From equations (26), (27) and (28), we can express prices in terms of trading frequencies in the whole contract menu.

$$\begin{aligned}
\frac{q_{bx}p_{bx}}{r+\delta+q_{bx}+\gamma} &= P - \int_x^\kappa \frac{1+\gamma e'_s}{r+\delta+q_{bs}+\gamma} ds - \frac{x+\delta Ed+\gamma e_x}{r+\delta+q_{bx}+\gamma}, \forall x \leq \kappa \\
\frac{q_{ax}p_{ax}}{r+\delta+q_{ax}+\gamma} &= P + \int_\kappa^x \frac{1+\gamma e'_s}{r+\delta+q_{as}+\gamma} ds - \frac{x+\delta Ed+\gamma e_x}{r+\delta+q_{ax}+\gamma}, \forall x \geq \kappa \\
p_{bx} &= \frac{r+\delta+q_{bx}+\gamma}{q_{bx}} \left[d_\kappa - \int_x^\kappa \frac{1+\gamma e'_s}{r+\delta+q_{bs}+\gamma} ds \right] - \frac{x+\delta Ed+\gamma e_x}{q_{bx}} \\
P - p_{bx} &= \frac{x-\kappa+\gamma(e_x-e_\kappa)}{q_{bx}} + \frac{r+\delta+q_{bx}+\gamma}{q_{bx}} \int_x^\kappa \frac{1+\gamma e'_s}{r+\delta+q_{bs}+\gamma} ds \\
p_{ax} &= \frac{r+\delta+q_{ax}+\gamma}{q_{ax}} \left[P + \int_\kappa^x \frac{1+\gamma e'_s}{r+\delta+q_{as}+\gamma} ds \right] - \frac{x+\delta Ed+\gamma e_x}{q_{ax}} \\
p_{ax} - P &= \frac{\kappa-x+\gamma(e_\kappa-e_x)}{q_{ax}} + \frac{r+\delta+q_{ax}+\gamma}{q_{ax}} \int_\kappa^x \frac{1+\gamma e'_s}{r+\delta+q_{as}+\gamma} ds \\
q_{bx}(P - p_{bx}) &= x - \kappa + \gamma(e_x - e_\kappa) + (r + \delta + q_{bx} + \gamma) \int_x^\kappa \frac{1+\gamma e'_s}{r+\delta+q_{bs}+\gamma} ds \\
q_{ax}(p_{ax} - P) &= \kappa - x + \gamma(e_\kappa - e_x) + (r + \delta + q_{ax} + \gamma) \int_\kappa^x \frac{1+\gamma e'_s}{r+\delta+q_{as}+\gamma} ds
\end{aligned} \tag{29}$$

Unconstrained optimization

Substituting results in equations (25) and equations (29) into the dealers' objective function, we now can solve an unconstrained problem with control variables q_{ax}, q_{bx} with restrictions that $q_{ax}, q_{bx} \in [0, \bar{q}], \forall x$

$$\begin{aligned}
& \int q_{ax}(p_{ax} - P)n_{0x}^m + q_{bx}(P - p_{bx})n_{1x}^m dx \\
&= \int_{x \geq \kappa} \left[\kappa - x + \gamma(e_\kappa - e_x) + (r + \delta + q_{ax} + \gamma) \int_\kappa^x \frac{1+\gamma e'_s}{r+\delta+q_{as}+\gamma} ds \right] n_{0x}^m dx \\
&\quad + \int_{x \leq \kappa} \left[x - \kappa + \gamma(e_x - e_\kappa) + (r + \delta + q_{bx} + \gamma) \int_x^\kappa \frac{1+\gamma e'_s}{r+\delta+q_{bs}+\gamma} ds \right] n_{1x}^m dx \\
&= \int_{x \geq \kappa} \left[\kappa - x + \gamma(e_\kappa - e_x) + (r + \delta + q_{ax} + \gamma) \int_\kappa^x \frac{1+\gamma e'_s}{r+\delta+q_{as}+\gamma} ds \right] \frac{1}{\delta+\gamma+q_{ax}} (\alpha n_{0x}^u + \delta g(x) N_0^m) dx \\
&\quad + \int_{x \leq \kappa} \left[x - \kappa + \gamma(e_x - e_\kappa) + (r + \delta + q_{bx} + \gamma) \int_x^\kappa \frac{1+\gamma e'_s}{r+\delta+q_{bs}+\gamma} ds \right] \frac{1}{\delta+\gamma+q_{bx}} (\alpha n_{1x}^u + \delta g(x) N_1^m) dx
\end{aligned}$$

In general this is an optimal control problem where the control variables are two functionals. But the problem for this objective function is particularly simple, we can directly take first order condition with respect to every value q_{ax} and q_{bx} , $\forall x$ such that $q_{ix} \in (0, \bar{q})$.

The first order condition with respect q_{ax} for $q_{ax} \in (0, \bar{q})$

$$\begin{aligned}
& - \left[\kappa - x + \gamma(e_\kappa - e_x) + r \int_\kappa^x \frac{1+\gamma e'_s}{r+\delta+q_{as}+\gamma} ds \right] \frac{1}{(\delta+\gamma+q_{ax})^2} (\alpha n_{0x}^u + \delta g(x) N_0^m) \\
& - \frac{1+\gamma e'_x}{(r+\delta+q_{ax}+\gamma)^2} \int_x^{\bar{x}} \frac{r+\delta+q_{as}+\gamma}{\delta+q_{as}+\gamma} (\alpha n_{0s}^u + \delta g(s) N_0^m) ds = 0
\end{aligned}$$

The first order conditions for q_{ax} , $\forall x \in \{z : q_{ax} \in (0, \bar{q})\}$ can be summarized by the following integral equation,

$$\begin{aligned}
0 &= C'_a(x) D_a(x) + C_a(x) D'_a(x) \\
C_a(x) &= -\kappa + x - \gamma(e_\kappa - e_x) - r \int_\kappa^x \frac{1+\gamma e'_s}{r+\delta+q_{as}+\gamma} ds \\
&= \int_\kappa^x C'_a(s) ds \\
D_a(x) &= \int_x^{\bar{x}} \frac{r+\delta+q_{as}+\gamma}{\delta+q_{as}+\gamma} (\alpha n_{0s}^u + \delta g(s) N_0^m) ds \\
&= - \int_x^{\bar{x}} D'(s) ds \\
C'_a(x) &= \left(-\frac{r}{r+\delta+q_{ax}+\gamma} + 1 \right) (1 + \gamma e'_x) \\
&= \frac{\delta+q_{ax}+\gamma}{r+\delta+q_{ax}+\gamma} (1 + \gamma e'_x) \\
D'_a(x) &= -\frac{r+\delta+q_{ax}+\gamma}{\delta+q_{ax}+\gamma} (\alpha n_{0x}^u + \delta g(x) N_0^m)
\end{aligned}$$

FOC wrt q_{bx} for $q_{bx} \in (0, \bar{q})$,

$$\begin{aligned}
& - \left[x - \kappa + \gamma(e_x - e_\kappa) + r \int_x^\kappa \frac{1 + \gamma e'_s}{r + \delta + q_{bs} + \gamma} ds \right] \frac{1}{(\delta + \gamma + q_{bx})^2} (\alpha n_{1x}^u + \delta g(x) N_1^m) \\
& - \frac{1 + \gamma e'_x}{(r + \delta + q_{bx} + \gamma)^2} \int_{\underline{x}}^x \frac{r + \delta + q_{bs} + \gamma}{\delta + q_{bs} + \gamma} (\alpha n_{1s}^u + \delta g(s) N_1^m) ds = 0
\end{aligned}$$

The first order conditions for q_{bx} , $\forall x \in \{z : q_{bx} \in (0, \bar{q})\}$ can be summarized by the following integral equation,

$$\begin{aligned}
0 &= C'_b(x) D_b(x) + C_b(x) D'_b(x) \\
C_b(x) &= x - \kappa + \gamma(e_x - e_\kappa) + r \int_x^\kappa \frac{1 + \gamma e'_s}{r + \delta + q_{bs} + \gamma} ds \\
D_b(x) &= \int_{\underline{x}}^x \frac{r + \delta + q_{as} + \gamma}{\delta + q_{as} + \gamma} (\alpha n_{1s}^u + \delta g(s) N_1^m) ds \\
C'_b(x) &= \frac{\delta + q_{bx} + \gamma}{r + \delta + q_{bx} + \gamma} (1 + \gamma e'_x) \\
D'_b(x) &= \frac{r + \delta + q_{bx} + \gamma}{\delta + q_{bx} + \gamma} (\alpha n_{1x}^u + \delta g(x) N_1^m)
\end{aligned}$$

This proves part (1) and (2) of Theorem 1.