Equilibrium Collateral Constraints

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Preliminary and Incomplete - Please, do not cite or distribute.

February 13, 2012

Abstract

I study a model in which banks need to borrow to make risky loans whose return is private information known only by the bank who made the loan. To raise funds, banks can either sell assets or pledge them as collateral. I show that collateral contracts arise in equilibrium even though all agents would value the asset the same in autarky. The persistence in the role as borrowers or lenders and the banks' ability to make a profit from loans imply that banks will value the asset more than lenders. On top of paying dividends, the asset resolves the banks' maturity mismatch problem and, since it is used as collateral, it relaxes a borrowing constraint. The amount that can be borrowed against the asset is determined in equilibrium. I show that increases in risk may decrease the asset's debt capacity and, thus, the level of intermediation in the economy.

1 Introduction

Collateralized debt is a widely used form of financing. Trillions of dollars are traded daily in debt collateralized by diverse financial assets, such as sale and repurchase agreements (repos) and collateralized over-the-counter derivative trades. The terms under which collateralized debt is issued affect the availability of funds in the economy and, as experienced in the 2007-2008 financial crisis, can affect systemic risk. Some of these terms include which assets are accepted as collateral, and how much can be borrowed against them. There is empirical evidence that these terms varied substantially during the last financial crisis.[1]

In this paper, I will focus on the amount that can be borrowed against an asset (i.e., its debt capacity). I provide a discrete-time, infinite-horizon model in which collateral contracts arise as equilibrium outcomes, even though all agents would value the asset the same in autarky. There are two types of risk neutral agents, banks and lenders, and one durable asset which pays dividends each period. Banks can invest in risky loans but they need external funds to do so. Since the return of the loans is private information known by the bank who invested in them, banks can either sell their

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†I am specially indebted with Ricardo Lagos and Tom Sargent for their advice in this project. I also thank Douglas Gale for very helpful comments and suggestions. I thank Saki Bigio, Alberto Bisin, V.V. Chari, Gian Luca Clementi, Todd Keister, Emiliano Marambio Catan, Antoine Martin, Juan Pablo Nicolini and Michal Szkup for their insightful remarks.

[1] See Gorton, G. and Metrick (2010a, 2010b) and Copeland, Martin, and Walker (2010).

assets or pledge them as collateral to raise funds. Banks value the asset more than lenders, and, therefore, as shown by Lacker (2001), choose to use collateral contracts.

In my paper, the difference in valuations arises in equilibrium from the persistence of the roles as borrowers and lenders. This persistence is consistent with the observation that, in many collateralized markets, different types of institutions specialize in borrowing or lending. For example, in the repo market, money market funds are usually lenders whereas hedge funds and specialty lenders are usually borrowers. The larger specialty lenders are either captive subsidiaries of major manufacturing firms that provide financing for the purchase of the parents' products or entities that provide a variety of consumer and commercial business credit lines. Specialty lenders are borrowers in markets for secured and unsecured debt and they are lenders to consumers, businesses, and households.[2] Banks in my model resemble specialty lenders.

In my model, banks have the investment opportunity before the asset's dividends are paid and, thus, cannot invest without external financing. This timing implies a maturity mismatch for the bank between the need of funds and their availability. Lenders do not have access to investment opportunities. Therefore, if left in autarky, both banks and lenders would value the asset as the expected discounted sum of dividends. When banks can borrow from lenders, they value the asset not only because of its dividend stream but also because it allows them to invest in the risky loans. This extra value can be divided in two: a private liquidity premium and a private collateral premium. The private liquidity premium arises from being able to sell the asset and use the funds to invest in the loans. The private collateral premium is the additional value banks can obtain by using the asset as collateral instead of selling it.

By solving the model in closed form, I am able to characterize the asset's debt capacity completely. In the model, the amount that can be borrowed against an asset depends on the riskiness of the loans, on the asset's liquidity and on the asset's dividend structure: the current level of dividends, and the correlation between the borrower's ability to repay and the asset's future dividend stream. I show that when all banks can invest in the same type of loans, i.e., all loans have the same correlation with the asset's future dividends, an increase in this correlation increases the asset's debt capacity. The bank has incentives to lie when the return of the loans is high. An asset that has a higher value when the borrower has incentives to lie makes it easier to motivate the borrower to tell the truth, and hence is better collateral. Moreover, when the return of the loans are positively correlated with future dividends, a mean preserving spread on the distribution of the loan's return decreases the asset's debt capacity and the amount intermediated in the economy. When banks can invest in different types of loans, two regimes may arise in equilibrium: one in which all banks use the asset as collateral and one in which some banks use the asset as collateral and other banks choose to sell the asset to raise funds. The effect of changes in the correlation structure on the overall amount intermediated in the economy depends on the distribution of loan types across banks and on the regime the economy is in.

There is a large literature that analyzes collateral contracts. Lacker (2001) shows that in a multiple-good risk-sharing environment with ex-post private information, collateralized debt is the optimal contract when the borrower values the collateral good more than the lender. He shows this in a two-good, two-period model with two agents, a risk-averse

[2] Financial System Oversight Council annual report, Chapter 5, Financial Developments, http://www.treasury.gov/initiatives/fsoc/Documents/Financial%20Developments.pdf.

borrower and a risk-neutral lender. The amount of payment good that the borrower receives in the second period, when the loan repayment is due, is random, and its realization is private information known by the borrower. The lender's marginal valuation of capital is exogenous and constant.

In a similar environment with non-pecuniary costs of default, Rampini (2005) studies how default varies with aggregate income when individual income is privately observed by the agents. The optimal risk sharing contract allows for default. Default penalties ensure that only agents with low realizations of income default. These default penalties are modeled as transfers of agent-specific goods which are only valued by the agent who is endowed with it, and can be interpreted as collateral which is only valued by the original holder.

My model resembles these two papers in that the realization of the amount of payment good the borrower has when he has to repay the loan is private information known by the borrower. In contrast, my model is in infinite horizon, all agents are risk neutral, and the marginal value of the collateral good both for the lender and the borrower is determined in equilibrium. In my model, collateral contracts are always optimal since, in equilibrium, the borrower values the collateral good more than the lender.

Kocherlakota (2001) analyzes optimal repayment contracts in the presence of collateral when there are enforcement frictions. Collateral contracts are optimal in Kocherlakota's model, provided that lenders are assumed to be less willing than borrowers to substitute consumption for collateral goods. In his setup, collateral is used to force the borrower to share the returns on the project he invested in with the lender.

In the general equilibrium literature, Geanakoplos (2003*a*, 2003*b*) , and Geanakoplos and Zame (2007) study the use of durable assets as collateral. They show that when collateral is needed, the set of traded assets will be determined endogenously. Araujo, Orrillo and Pascoa (1994) and Araujo, Fajardo and Pascoa (2005) assume collateral contracts and make collateral endogenous by allowing each seller of assets to fix the level of collateral or the bundle used as collateral. In all these papers, collateral contracts are assumed.

In the macro literature, Kiyotaki and Moore (1997) analyze the effects of collateral constraints on aggregate fluctuations. In their model, the value of collateral is endogenous, and this endogeneity plays a crucial role in generating large effects of collateral constraints on output. Jermann and Quadrini (2011) find that a tightening of the firms' financing conditions contributed significantly to the 2008-2009 recession and to the downturns in 1990-91 and 2001. In their paper, collateral (borrowing) constraints arise from an enforcement problem: at the time of repayment, the lender can recover only a fraction of the collateral value. This fraction is stochastic and depends on (unspecified) market conditions. In these papers, durable assets are used not only as collateral for loans, but also as inputs for production.

In the model presented in this paper, the asset used as collateral is not an input in production and there is perfect enforcement. The asset is used as collateral as an optimal response to asymmetric information about the resources available to the borrower at the time of repayment. The amount that can be borrowed against the asset is determined in equilibrium, and, thus, changes in the collateral constraints faced by banks reflect changes in the economy's fundamentals.

The rest of the paper is organized as follows. Section 2 presents the model. Section 3 defines and characterizes equilibrium. Comparative statics results are presented in section 4. Section 5 extends the model in section 2, allowing for heterogeneity among banks. Section 6 concludes.

2 Model

Time is discrete, starts at $t = 0$, and goes on forever. Each period t is divided in two subperiods, morning and afternoon. There are two types of consumption, non-storable good, morning and afternoon specific. There is one asset that lasts forever and which is in fixed supply $\bar{k}$. k units of capital yield $d_t k$ units of (afternoon) consumption good as dividend at the end of each afternoon t. The dividend d_t is known at the beginning of morning t. There are two types of infinitely-lived, risk-neutral agents, banks and lenders. There is a measure 1 of each type and all agents have the same discount factor $\beta \in (0, 1)$. Each agent starts his life with an endowment of the asset. Each subperiod lenders are endowed with a large amount of consumption good, e_L^m in the morning and e_L^a in the afternoon. Each morning t banks receive a large endowment of (morning) consumption good, and each afternoon t banks can make short-term loans on which they can make profits. Loans are risky: some loans may not be repaid. Thus, one unit of (afternoon) consumption lent by the bank at the beginning of afternoon t by bank j yields a random payoff θ_t^j in (afternoon) consumption good at the end of the period. $\theta_t^j \in \{\theta_L, \theta_H\}$. The return of the loans is i.i.d. across banks and time.

There is an underlying unobservable i.i.d. aggregate state $\omega_t \in (\omega_0, \omega_1)$ that determines the probability of success of the investment in loans.[3] Let $p_i^n := \Pr\left(\theta_t^j = \theta_i | \omega_t = \omega_n\right)$, and $p_i := \Pr(\omega_t = \omega_1) p_i^1 + \Pr(\omega_t = \omega_2) p_i^2 = \Pr\left(\theta_t^j = \theta_i\right)$, $i = L, H, \forall j, .\forall t \geq 0$. I assume that $E\left(\theta_t^j\right) > 1 > \theta_L, \forall j, .\forall t \geq 0$, i.e., that the loans are expected to be profitable but that they incur losses if the low state is realized, and that $1 > \beta \frac{E(\theta) - \theta_L}{1 - \theta_L}$. The return of the loans is private information known by the bank who made the loans, and it is potentially correlated with the dividend paid by the asset the following period.[4] The joint distribution of returns in period t and dividends in period $t + 1$ is stationary, i.e.,

$$E\left(d_{t+1} | \theta_t^j = \theta_i\right) = \sum_{n=1,2} \Pr\left(\omega_t = \omega_n | \theta_t^j = \theta_i\right) E\left(d_{t+1} | \omega_t = \omega_n\right) := d_i \text{ for } i = L, H, \forall j, \forall t \geq 0,$$

where

$$\Pr\left(\omega_t = \omega_n | \theta_t^j = \theta_i\right) = \frac{p_i^n \Pr(\omega = \omega_n)}{p_i}.$$

The dividend is paid after the investment in the loans is made. Therefore, dividends cannot be lent by the bank. This timing implies that banks need to borrow to make risky loans, and creates a maturity mismatch between the need of funds (liquidity) and the availability of funds. Moreover, it implies that if agents were left in autarky, they would all value the asset as the discounted sum of expected dividends.

Each morning, after the dividend level d_t is known, an asset market opens. Banks and lenders meet randomly. Trades made in this market are bilateral and the terms of the transaction are determined through Nash bargaining. At the beginning of each afternoon, a funds market opens: each bank is randomly paired with a lender and there is a bilateral funding contract in which all bargaining power is given to the borrower.

The timing is illustrated in figure 1.

[3] I assume the law of large number, and thus the fraction of successful investmens in loans will also depend on the state realized.

[4] One can think of the asset as mortgage backed security or as an asset backed security backed by loans made by banks.

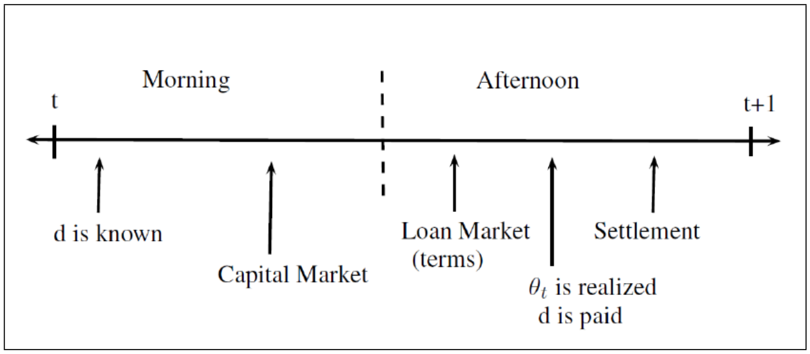

Figure 1: Timing

2.1 Asset Market

At the beginning of each morning, each bank is randomly matched with a lender in the asset market. The terms of trade, quantity and price, are determined through Nash bargaining. The bargaining power of banks is equal to $\gamma \in (0,1]$.[5] Since matches are random both in the asset market and in the loan market, the aggregate state of the economy is $\phi = (d, F_L^m, F_B^m)$, where F_L^m and F_B^m are the distribution of assets in the hands of lenders in the morning, and the distribution of assets in the hands of banks in the morning, respectively. I assume that banks have enough consumption good each morning to buy all assets from the non-producer they meet with. By making this assumption, the model abstracts from borrowing constraints in the capital market. Since the main focus of the paper is to understand collateralized loan contracts, I believe that considering such borrowing constraints, though interesting and realistic, will make it harder to disentangle the forces at work without adding much to the analysis.

Let $k^T(k_B, k_L; \phi)$, and $P(k_B, k_L; \phi)$ be the quantity and price that result from the encounter of a bank with k_B units of the asset and a lender with k_L units of the asset.

Then, the value of a bank in the morning, before entering the asset market, is

$$V_B^m(k_B;\phi) = \left(\int \left[\bar{V}_B^a\left(k_B + k^T(k_B,k_L;\phi);\phi\right) - P(k_B,k_L;\phi) \right] dF_L^m(k_L) \right) \tag{1}$$

where $\bar{V}_B^a(k_B;\phi) = \int V_B^a(k_B,k_L;\phi)\, dF_a(k_L)$, and $V_B^a(k_B,k_L;\phi)$ is the value of a bank with assets k_B who enters a loan contract with a lender with assets k_L.

The value of a lender in the morning, before entering the asset market, is

$$V_L^m(k_L;\phi) = \int \left(P(k_B,k_L;\phi) + d\left(k_L - k^T(k_B,k_L;\phi)\right) + \beta E_{\phi'}\left(V_L^m\left(k_L - k^T(k_B,k_L;\phi);\phi'\right)\right)\right) dF_B^m(k_B) \tag{2}$$

Therefore, since the terms of trade are determined by Nash bargaining, $P(k_B,k_L;\phi)$ and $k^T(k_B,k_L;\phi)$ solve

$$\begin{aligned} &\max_{P_0,k_0} \left(P_0 - dk_0 - \beta E_{\phi'}\left(V_L^m(k_L;\phi') - V_L^m(k_L - k_0;\phi')\right)\right)^{1-\gamma} \\ &\times \left(\bar{V}_B^a(k_B + k_0;\phi) - \bar{V}_B^a(k_B;\phi) - P_0\right)^{\gamma} \end{aligned} \tag{3}$$

[5] If $\gamma = 0$ both agents value the asset the same in equilibrium and the optimal debt contract is indeterminate.

2.2 Funds (Loan) Market

Every afternoon, a bilateral funds market opens. Each bank is matched randomly with a lender and the terms of the loan contract are determined by the borrower (the bank)[6]. Loan contracts are one-subperiod contracts and they consist of a loan amount in terms of (afternoon) consumption good, q, and contingent repayments in terms of (afternoon) consumption, r_i, and in terms of assets, t_i, $i = L, H$.[7]

A contract (q, r_L, r_H, t_L, t_H) is feasible at time t when the dividend level is d if

$$\begin{aligned} 0 &\leq q \leq e_L^a \\ 0 &\leq r_i \leq dk_{Bt} + \theta_i q \quad i = L, H \\ 0 &\leq t_i \leq k_{Bt} \qquad\qquad i = L, H \end{aligned}$$

where k_{Bt} is the amount of assets held by the bank in afternoon t.

Since θ_t is only observed by the bank, for contracts to be credible, they must be incentive compatible.

A contract (q, r_L, r_H, t_L, t_H) is incentive compatible if

$$-r_L + \beta E_{\phi'}\left(\bar{V}_B^m\left(k_B - t_L; \phi'\right)|\theta_t = \theta_H\right) \leq -r_H + \beta E_{\phi'}\left(\bar{V}_B^m\left(k_B - t_H; \phi'\right)|\theta_t = \theta_H\right) \tag{4}$$

and whenever $r_H \leq \theta_L\left(dk_B + q\right)$

$$-r_L + \beta E_{\phi'}\left(\bar{V}_B^m\left(k_B - t_L; \phi'\right)|\theta_t = \theta_L\right) \geq -r_H + \beta E_{\phi'}\left(\bar{V}_B^m\left(k_B - t_H; \phi'\right)|\theta_t = \theta_L\right) \tag{5}$$

where $\bar{V}_B^m\left(k_B - t; \widetilde{d}, \phi'\right) = \int V_B^m\left(k_B - t, k_L; \widetilde{d}, \phi'\right)\, dF_L^{m\prime}\left(k_L\right)$

The first constraint states that, in the high state, it is always at least as good for the bank (borrower) to tell the truth and report that the high state has occurred than to lie and report a low realization of θ. The second constraint states the analogous for the low state with the restriction that this constraint is only active when lying in the low state is feasible, i.e., when there are enough resources in the low state to match the contingent repayment in terms of goods in the high state, r_H. I assume that the amount of the loan in equilibrium is never restricted by the amount of consumption good owned by lenders, i.e., that $q^* < e_L^a$.[8]

Let $V_B^a\left(k_B, k_L; \phi\right)$ be the value of a bank with assets k_B who is matched with a lender with assets k_L in the loan market. Then, $V_B^a\left(k_B, k_L; \phi\right) =$

$$\begin{aligned} \sup_{q, r_L, r_H, t_H, t_L} \quad & E\left(\theta\right) q - p_H r_H - p_L r_L + dk_P \\ & + \beta p_H E_{\phi'}\left(V_B^m\left(k_B - t_H; \phi'\right)|\theta_H\right) + \beta p_L E_{\phi'}\left(V_B^m\left(k_B - t_L; \phi'\right)|\theta_L\right) \end{aligned} \tag{6}$$

s.t.

$$\left(q, r_L, r_H, t_L, t_H\right) \text{ is feasible and IC}$$

[6] The shape of the contract is the same if all bargaining power is assigned to the lender.

[7] One sub-period contracts implement long-term contracts. Say more about this!!!?

[8] If this was not the case, the incentive compatibity constraints would not bind, the size of the loan would be efficient and debt would be riskless.

$$\begin{aligned}\beta E_{\phi'}\left(V_L^m\left(k_L;\phi'\right)\right) &\leq -q+p_Hr_H+\beta p_H E_{\phi'}\left(V_L^m\left(k_L+t_H;\phi'\right)|\theta_H\right) \\ &\quad +p_Lr_L+\beta p_L E_{\phi'}\left(V_L^m\left(k_L+t_L;\phi'\right)|\theta_L\right) \qquad (7)\\ \phi' &= H\left(\phi;\theta\right) \qquad (8)\end{aligned}$$

The bank chooses a feasible and incentive compatible contract to maximize his expected utility subject to the participation constraint (7) and given the perceived law of motion for the aggregate state (8). (7) states that the lender has to be at least as good participating in the contract as he would be if he didn't participate in it. The borrower invests all the loan amount, q, in the risky technology since $E(\theta)>1$, and gets an expected return $E(\theta)q$. He expects to repay $p_Hr_H+p_Lr_L$ in terms of consumption good and he gets dividends dk_P from his stock of assets at the beginning of the afternoon. Finally, his expected continuation value in the morning depends on the contingent transfers of assets t_L and t_H that are part of the contract.

The following three lemmas, and the proposition in this section simplify the bank's problem in the loan market.

Lemma 1 *Without loss of generality,* (7) *can be replaced by*

$$\begin{aligned}\beta E_{\phi'}\left(V_L^m\left(k_L;\phi'\right)\right) &= -q+p_Hr_H+\beta p_H E_{\phi'}\left(V_L^m\left(k_L+t_H;\phi'\right)|\theta_H\right) \\ &\quad +p_Lr_L+\beta p_L E_{\phi'}\left(V_L^m\left(k_s+t_L;\phi'\right)|\theta_L\right). \qquad (9)\end{aligned}$$

in the bank's problem.

Proof. Let V^* be the solution to the bank's problem. Let $\{V_j\}_{j\geq 0}$ be such that $\lim_{j\to\infty}V_j=V^*$, where

$$\begin{aligned}&V_j=E(\theta)q_j-p_Hr_{Hj}-p_Lr_{Lj}+dk_B \qquad (10)\\ &+\beta p_H E_{\phi'}\left(V_B^m\left(k_B-t_{Hj};\phi'\right)|\theta_t^i=\theta_H\right)+\beta p_L E_{\phi'}\left(V_B^m\left(k_B-t_{Lj};\phi'\right)|\theta_t^i=\theta_L\right)\end{aligned}$$

for some feasible and incentive compatible $\{q_j,r_{Lj},r_{Hj},t_{Lj},t_{Hj}\}$ that satisfies the participation constraint (7). Suppose that for some $j\geq 0$, $(q_j,r_{Lj},r_{Hj},t_{Lj},t_{Hj})$ is such that (7) is slack. Then, one could increase q_j and increase V_j to V_j^0 still satisfying all the other constraints. Let $\{V_j'\}_{j\geq 0}$ be a sequence identical to $\{V_j\}_{j\geq 0}$ if at $(q_j,r_{Lj},r_{Hj},t_{Lj},t_{Hj})$ (7) holds with equality and V_j^0 otherwise. Then, by construction, $V_j^{'}\geq V_j$ and therefore,

$$\lim_{j\to\infty}V_j'\geq\lim_{j\to\infty}V_j=V^*.$$

Therefore, we can replace (7) by (9) in the bank's problem. ■

Lemma 2 *Without loss of generality, the incentive compatibility constraints can be replaced by*

$$-r_L+\beta E_{\phi'}\left(\bar{V}_B^m\left(k_B-t_L;\phi'\right)|\theta_t^i=\theta_H\right)=-r_H+\beta E_{\phi'}\left(\bar{V}_B^m\left(k_B-t_H;\phi'\right)|\theta_t^i=\theta_H\right).$$

in the bank's problem.

Proof. Let V^* be the solution to the bank's problem. Let $\{V_j\}_{j\geq 0}$ be such that $\lim_{j\to\infty}V_j=V^*$, where

$$\begin{aligned}&V_j=E(\theta)q_j-p_Hr_{Hj}-p_Lr_{Lj}+dk_P \qquad (11)\\ &+\beta p_H E_{\phi'}\left(V_B^m\left(k_B-t_{Hj};\phi'\right)|\theta_t^i=\theta_H\right)+\beta p_L E_{\phi'}\left(V_B^m\left(k_B-t_{Lj};\phi'\right)|\theta_t^i=\theta_L\right)\end{aligned}$$

for some feasible and incentive compatible $\{q_j, r_{Lj}, r_{Hj}, t_{Lj}, t_{Hj}\}$ that satisfies the participation constraint (9). Suppose that for some $s \geq 0$, no incentive compatibility constraint binds. Then, there exists $\varepsilon_s > 0$ such that

$$\beta E_{\phi'}\left(\bar{V}_B^m\left(k_B - t_{Hs};\phi'\right) - \bar{V}_B^m\left(k_B - t_{Ls};\phi'\right)|\theta_L\right) \leq r_{Hs} - r_{Ls} + \left(\theta_H - \theta_L\right)\varepsilon_s \leq \beta E_{\phi'}\left(\bar{V}_B^m\left(k_B - t_{Hs};\phi'\right) - \bar{V}_B^m\left(k_B - t_{Ls};\phi'\right)|\theta_H\right).$$

Replace $\{q_s, r_{Ls}, r_{Hs}, t_{Ls}, t_{Hs}\}$ by $\{q_s + \varepsilon_s + \varepsilon_0, r_{Ls} + \theta_L\varepsilon_s, r_{Hs} + \theta_H\varepsilon_s, t_{Lj}, t_{Hj}\}$ where $\varepsilon_0 > 0$ is such that the participation constraint binds. This contract still satisfies all the constraints, but it attains a value $V_s^0 > V_s$.

If $r_{Hs} > r_{Ls}$, (4) is the only relevant incentive compatibility constraint. For all s such that $r_{Hs} > r_{Ls}$ and (4) is not binding, the previous argument applies and a value $V_s^0 > V_s$ can be attained.

Now consider those $s \geq 0$ such that $r_H s \leq r_L s < \theta_L q_s + dk_P$. If (5) binds, one could keep $r_{Hs} - r_{Ls_s}$ constant by increasing both r_{Ls} and r_{Hs} and by increasing q_s to keep the participation constraint binding which would result in an increase in the objective function. Let this new value be V_s^0. If for $s \geq 0$, $r_{Hs} \leq r_{Ls} = \theta_L q_s + dk_P$, (5) doesn't bind unless (4) binds. Suppose (5) binds and (4) doesn't. Then, one could increase r_{Hs} still satisfying incentive compatibility and relaxing the participation constraint. Therefore, one could increase q_s which would increase the objective function and give a value $V_s^0 \geq 0$. Therefore, once can construct a new sequence $\left\{V_j'\right\}_{j\geq 0}$, $V_j' \geq V_j$ such that $V_j' = V_j$ is the incentive compatibility constraint in the high state binds and $\mathrm{V}_j' = V_j^0$ if it doesn't. By construction,

$$\lim_{j\to\infty} V_j^{'} \geq \lim_{j\to\infty} V_j = V^*.$$

Therefore, without loss of generality one can concentrate on those sequences that are feasible in which (4) holds with equality, i.e.,

$$r_{H_j} - r_{L_j} = \beta E_{\phi'}\left(\bar{V}_B^m\left(k_B - t_{Hj};\phi'\right) - \bar{V}_B^m\left(k_B - t_{Lj};\phi'\right)|\theta_H\right) \text{ for all } j. \tag{12}$$

■

Lemma 3 *Without loss of generality, the feasibility constraints on contingent transfers in consumption good can be replaced by*

$$r_L = \theta_L q + dk_B \textit{ and } r_H \geq 0$$

in the bank's problem.

Proof. By assumption $q \leq e_L^a$ will not bind in a solution to the bank's problem. Using lemma 1, the participation constraint can be assumed to hold with equality, and using this in the objective function one can see that the objective function is always increasing in the amount of the loan q. Using lemma 2, the incentive compatibility constraint holds with equality which implies that the upper bound for q is given by the maximum amount that can be repaid in the low state, i.e., by $r_L = \theta_L q + dk_P$.

Let V^* be a solution to the bank's problem. Let $\{V_j\}$ be a sequence such that $lim_{j\to\infty} V_j = V^*$ and where

$$\begin{aligned} V_j &= \left(E\left(\theta\right) - 1\right) q_j + \beta p_H E_{\phi'}\left(V_L^m\left(k_L + t_{Hj};\phi'\right)|\theta_t^i = \theta_H\right) + \beta p_L E_{\phi'}\left(V_L^m\left(k_L + t_{Lj};\phi'\right)|\theta_t^i = \theta_L\right) + dk_P \\ &+ \beta p_H E_{\phi'}\left(V_B^m\left(k_B - t_{Hj};\phi'\right)|\theta_t^i = \theta_H\right) + \beta p_L E_{\phi'}\left(V_B^m\left(k_B - t_{Lj};\phi'\right)|\theta_t^i = \theta_L\right) - \beta E_{\phi'}\left(V_L^m\left(k_L;\phi'\right)\right) \end{aligned} \tag{13}$$

for some feasible and incentive compatible that satisfies (12) , and (7) with equality, that is that,

$$\begin{aligned} r_{Lj} &= q^* - \beta\left(p_H E_{\phi'}\left(V_L^m\left(k_L + t_{Hj};\phi'\right)|\theta_H\right) + p_L E_{\phi'}\left(V_L^m\left(k_L + t_{Lj};\phi'\right)|\theta_L\right)\right) \\ &\quad -\beta p_H\left(E_{\phi'}\left(\bar{V}_B^m\left(k_B - t_{Hj};\phi'\right) - \bar{V}_B^m\left(k_B - t_{Lj};\phi'\right)|\theta_H\right)\right) + \beta E_{\phi'}\left(V_L^m\left(k_L;\phi'\right)\right) \qquad (14) \\ r_{Hj} &= q^* - \beta\left(p_H E_{\phi'}\left(V_L^m\left(k_L + t_{Hj};\phi'\right)|\theta_H\right) + p_L E_{\phi'}\left(V_L^m\left(k_L + t_{Lj};\phi'\right)|\theta_L\right)\right) \\ &\quad -\beta p_L\left(E_{\phi'}\left(\bar{V}_B^m\left(k_B - t_{Lj};\phi'\right) - \bar{V}_B^m\left(k_B - t_{Hj};\phi'\right)|\theta_H\right)\right) + \beta E_{\phi'}\left(V_L^m\left(k_{NP};\phi'\right)\right) \qquad (15) \end{aligned}$$

Then, for all j, the contract can be summarized by $\{q_j, t_{Lj}, t_{Hj}\}$. The feasibility constraints for r_L and r_H imply the following constraints

$$\begin{aligned} q_j &\leq \frac{dk_B + \beta\left(p_H E_{\phi'}\left(V_L^m\left(k_L + t_{Hj};\phi'\right)|\theta_H\right) + p_L E_{\phi'}\left(V_L^m\left(k_L + t_{Lj};\phi'\right)|\theta_L\right)\right)}{1-\theta_L} \\ &\quad + \frac{\beta p_H\left(E_{\phi'}\left(\bar{V}_B^m\left(k_B - t_{Hj};\phi'\right) - \bar{V}_B^m\left(k_B - t_{Lj};\phi'\right)|\theta_H\right)\right) - \beta E_{\phi'}\left(V_L^m\left(k_{NP};\phi'\right)\right)}{1-\theta_L}, \qquad (16) \\ q_j &\geq \beta p_H E_{\phi'}\left(V_L^m\left(k_L + t_{Hj};\phi'\right)|\theta_H\right) + \beta p_L E_{\phi'}\left(V_L^m\left(k_L + t_{Lj};\phi'\right)|\theta_L\right) \\ &\quad + p_H \beta E_{\phi'}\left(\bar{V}_B^m\left(k_B - t_{Hj};\phi'\right) - \bar{V}_B^m\left(k_B - t_{Lj};\phi'\right)|\theta_H\right) - \beta E_{\phi'}\left(V_L^m\left(k_L;\phi'\right)\right), \qquad (17) \\ q_j &\geq \frac{dk_P + \beta p_H E_{\phi'}\left(V_L^m\left(k_L + t_{Hj};\phi'\right)|\theta_H\right) + \beta p_L E_{\phi'}\left(V_L^m\left(k_L + t_{Lj};\phi'\right)|\theta_L\right)}{1-\theta_H} \\ &\quad + \frac{p_L \beta E_{\phi'}\left(\bar{V}_B^m\left(k_B - t_{Lj};\phi'\right) - \bar{V}_B^m\left(k_B - t_{Hj};\phi'\right)|\theta_H\right) - \beta E_{\phi'}\left(V_L^m\left(k_L;\phi'\right)\right)}{1-\theta_H} \qquad (18) \\ q_j &\geq \beta p_H E_{\phi'}\left(V_L^m\left(k_L + t_{Hj};\phi'\right)|\theta_H\right) + \beta p_L E_{\phi'}\left(V_L^m\left(k_L + t_{Lj};\phi'\right)|\theta_L\right) \\ &\quad - p_L \beta E_{\phi'}\left(\bar{V}_B^m\left(k_B - t_{Hj};\phi'\right) - \bar{V}_B^m\left(k_B - t_{Lj};\phi'\right)|\theta_H\right) - \beta E_{\phi'}\left(V_L^m\left(k_L;\phi'\right)\right). \qquad (19) \end{aligned}$$

Construct the following sequence $\{V_j'\}$: if $\{q_j, t_{Lj}, t_{Hj}\}$ is such that (16) holds with equality, set $V_j' = V_j$. If $\{q_j, t_{Lj}, t_{Hj}\}$ is such that (16) is slack let V_j' be the value attained by the contract that satisfies (16) with equality. Since the transfers in terms of consumption good are defined by (14) and (15), this contract is still incentive compatible and feasible. Moreover, $q_j' > q_j$ and $V_j' > V_j$. Therefore,

$$\lim_{j\to\infty} V_j' \geq \lim_{j\to\infty} V_j = V^*$$

and without loss of generality one can concentrate on the sequences $\{V_j\}$ as defined above in (13), such that the loan quantities $\{q_j\}$ satisfy (16) with equality. Having this constraint hold with equality implies $r_{Lj} = \theta_L q_j + dk_P$. Since $q_j \geq 0$ always, this implies that all contracts along this sequence satisfy $r_L > 0$ which is the same as satisfying (17) with strict inequality.

Suppose that for some j (18) holds with equality. This implies $r_{Hj} = \theta_H q_j + dk_P$ and since $r_{Lj} = \theta_L q_j + dk_P$ this would imply that participation constraint is slack, and that the producer is giving the non-producer all the gains from the project. From lemma 1, there exists a feasible and incentive compatible contract that attains a higher value that contract j and therefore, without loss of generality we can ignore sequences in which for some elements j, (18) holds with equality. ■

Proposition 4 *The bank's problem can be rewritten as* $V_B^a(k_B,k_L;\phi) =$

$$\sup_{(t_H,t_L)\in[0,k_B]^2} \frac{(E(\theta)-\theta_L)}{1-\theta_L}\left(dk_B+\beta\left(p_H E_{\phi'}\left(V_L^m\left(k_L+t_H;\phi'\right)|\theta_H\right)+p_L E_{\phi'}\left(V_L^m\left(k_L+t_L;\phi'\right)|\theta_L\right)-E_{\phi'}\left(V_L^m\left(k_L;\phi'\right)\right)\right)\right)$$
$$+\frac{(E(\theta)-1)}{1-\theta_L}\beta p_H\left(E_{\phi'}\left(\bar{V}_B^m\left(k_B-t_H;\phi'\right)|\theta_H\right)-E_{\phi'}\left(\bar{V}_B^m\left(k_B-t_L;\phi'\right)|\theta_H\right)\right)$$
$$+\beta p_L E_{\phi'}\left(\bar{V}_B^m\left(k_B-t_L;\phi'\right)|\theta_L\right)+\beta p_H E_{\phi'}\left(\bar{V}_B^m\left(k_B-t_H;\phi'\right)|\theta_H\right)$$

subject to

$$\begin{aligned} q^* &= \frac{dk_B+\beta\left(p_H E_{\phi'}\left(V_L^m\left(k_L+t_H;\phi'\right)|\theta_H\right)+p_L E_{\phi'}\left(V_L^m\left(k_L+t_L;\phi'\right)|\theta_L\right)\right)}{1-\theta_L} \\ &\quad+\frac{\beta p_H\left(E_{\phi'}\left(\bar{V}_B^m\left(k_B-t_H;\phi'\right)-\bar{V}_B^m\left(k_B-t_L;\phi'\right)|\theta_H\right)\right)-\beta E_{\phi'}\left(V_L^m\left(k_L;\phi'\right)\right)}{1-\theta_L} \\ q^* &\geq \max\{0,\beta p_H E_{\phi'}\left(V_L^m\left(k_L+t_H;\phi'\right)|\theta_H\right)+\beta p_L E_{\phi'}\left(V_L^m\left(k_L+t_L;\phi'\right)|\theta_L\right) \\ &\quad -p_L\beta E_{\phi'}\left(\bar{V}_B^m\left(k_B-t_H;\phi'\right)-\bar{V}_B^m\left(k_B-t_L;\phi'\right)|\theta_H\right)-\beta E_{\phi'}\left(V_L^m\left(k_{NP};\phi'\right)\right)\}\end{aligned}$$

Proof. It follows from lemmas 1, 2 and 3 using the equality constraints in the objective function. ■

3 Equilibrium

Definition 5 *A recursive equilibrium in this economy is a pair of value functions for banks, in the morning and in the afternoon,* $V_B^m(k_B;\phi)$ *and* $V_B^a(k_B,k_L;\phi)$*, a value function for lenders* $V_L^m(k_L;\phi)$*, price and quantity functions in the bilateral asset market,* $P(k_B,k_L;\phi)$ *and* $k^T(k_B,k_L;\phi)$*, a loan contract*

$$(q(k_B,k_L;\phi), r_L(k_B,k_L;\phi), r_H(k_B,k_L;\phi), t_L(k_B,k_L;\phi), t_H(k_B,k_L;\phi))$$

and a law of motion for ϕ*,* $H(\phi,\omega)$*, such that* (1)*,* (2)*,* (3)*, and* (6) *are satisfied and the law of motion for* ϕ *satisfies:*

$$\begin{aligned} F_L^{m\prime}(k,\omega_i) &= p_H^i\left(\int F_L^a\left(k-t_H(k_B,k;d,\phi),\omega_{-1}\right)dF_B^a(k_B,\omega_{-1})\right)+p_L^i\left(\int F_L^a\left(k-t_L(k_B,k;d,\phi),\omega_{-1}\right)dF_B^a(k_B,\omega_{-1})\right) \\ F_B^a(k,\omega_{-1}) &= \int F_B^m\left(k-k^T(k,k_L;d,\phi)\right)dF_L^m(k_L,\omega_{-1}) \\ F_B^{m\prime}(k,\omega_i) &= p_H^i\left(\int F_B^a\left(k+t_H(k,k_L;d,\phi),\omega_{-1}\right)dF_L^a(k_L,\omega_{-1})\right)+p_L^i\left(\int F_B^a\left(k+t_L(k,k_L;d,\phi),\omega_{-1}\right)dF_L^a(k_L,\omega_{-1})\right) \\ F_L^{a\prime}(k,\omega_i) &= \int F_L^{m\prime}\left(k+k^T(k_B,k;d,\phi),\omega_i\right)dF_B^{m\prime}(k_B,\omega_i)\end{aligned}$$

of distributions where ω_{-1} *is the realized aggregate state the period before.*

3.1 Affine Equilibrium

The mapping that characterizes an equilibrium is not a contraction mapping. Therefore I cannot show that the equilibrium is unique. Nevertheless, there is a unique recursive affine equilibrium, i.e., a unique equilibrium in which the value functions are affine in asset holdings. I will use the guess an verify method to show that such equilibrium exists and that it is unique.

Assume

$$\begin{aligned}\bar{V}_B^a\left(k_B;\phi\right) &= V_B^a\left(k_B,k_L;\phi\right)=c_B\left(d\right)k_B+a_B\left(\phi\right)\\ V_L^m\left(k_L;\phi\right) &= c_L\left(d\right)k_L+a_L\left(\phi\right)\end{aligned}$$

$c_B\left(d\right)$ and $c_L\left(d\right)$ are affine in d.

Under these assumption, the terms of trade in the asset market are the solution to

$$\max_{P_0,k_0}\left(P_0-dk_0-\beta c_L\left(\bar{d}\right)k_0\right)^{1-\gamma}\times\left(c_B\left(d\right)k_0-P_0\right)^{\gamma}$$

where $\bar{d}=E_{d'}\left(d'\right)$ .Then,

$$P_0=\left(1-\gamma\right)c_B\left(d\right)k_0+\gamma\left(dk_0+\beta c_L\left(\bar{d}\right)k_0\right)$$

and $k^T\left(k_B,k_L;\phi\right)=$

$$\arg\max_{k_0\in[-k_P,k_{NP}]}\left(c_B\left(d\right)-d-\beta c_L\left(\bar{d}\right)\right)k_0$$

which implies $k^T\left(k_B,k_L;\phi\right)\in\left\{-k_B,k_L\right\}$. As it is usually the case with linear value functions, the quantity traded through Nash bargaining maximizes surplus and thus trades are efficient. Since the bank can wait until the afternoon and sell the asset in the loan market, by setting $t_L=t_H=k_B$, he will never choose to sell in the morning. Selling in the afternoon, allows the bank to invest the proceeds of the sale in the risky loans which has a higher expected return than consuming the goods in the morning. Therefore, the terms of trade in the asset market are independent of the asset stock with which the bank (buyer) enters the market, k_B, and $k^T\left(k_B,k_L;\phi\right)=k_L$, for all k_B,k_L, and ϕ.

Given this structure, the value functions for lenders and banks in the morning, before entering the asset market are, respectively,

$$V_L^m\left(k_L;\phi\right)=P\left(k_B,k_L;\phi\right)=\left[\left(1-\gamma\right)c_B\left(d\right)+\gamma\left(d+\beta c_L\left(\bar{d}\right)\right)\right]k_L$$

and

$$V_B^m\left(k_B;\phi\right)=\gamma\left[c_B\left(d\right)-\left(d+\beta c_L\left(\bar{d}\right)\right)\right]\int kdF_L^m\left(k\right)+c_B\left(d\right)k_B.$$

Thus, using the guessed functional form for the value functions gives

$$c_L\left(d\right)=\left(1-\gamma\right)c_B\left(d\right)+\gamma\left(d+\beta c_L\left(\bar{d}\right)\right)$$

and

$$c_L\left(\bar{d}\right)=\frac{\left[\left(1-\gamma\right)c_B\left(\bar{d}\right)+\gamma\bar{d}\right]}{1-\gamma\beta}.$$

Therefore,

$$c_L\left(d\right)=\left[\left(1-\gamma\right)c_B\left(d\right)+\gamma\left(d+\beta\frac{\left(1-\gamma\right)c_B\left(\bar{d}\right)+\gamma\bar{d}}{1-\gamma\beta}\right)\right].$$

Finally, the value of a bank who enters the loan market with k_B units of asset and meets a lender with k_L units of assets when the state is ϕ is, using proposition 4,

$$\begin{aligned} V_B^a(k_B, k_L; \phi) &= \max_{t_H, t_L \in [0, k_P]^2} (E(\theta) - 1) q^* + dk_B + \beta (p_H c_L(d_H) t_H + p_H c_L(d_L) t_L) \\ &\quad + \beta p_H c_B(d_H)(k_B - t_H) + \beta p_L c_B(d_L)(k_B - t_L) \\ &\quad + \beta E_{\phi'} \left(\gamma \left[c_B(d') - \left(d' + \beta c_L(\bar{d}) \right) \right] \int k dF_L^{m'}(k) \right) \end{aligned}$$

$$\begin{aligned} \text{s.t. } q^* &= \frac{dk_B + \beta (p_H c_L(d_H) t_H + p_L c_L(d_L) t_L) - p_H \beta c_B(d_H)(t_H - t_L)}{1 - \theta_L} && (20) \\ q^* &\geq \max \{ \beta (p_H c_L(d_H) t_H + p_L c_L(d_L) t_L) + p_L \beta c_B(d_H)(t_H - t_L), 0 \} && (21) \\ r_H &= q^* - \beta p_H c_L(d_H) t_H - \beta p_L c_L(d_L) t_L + p_L \beta c_B(d_H)(t_H - t_L) \\ r_L &= q^* - \beta p_H c_L(d_H) t_H - \beta p_L c_L(d_L) t_L + p_H \beta c_B(d_H)(t_H - t_L) \\ & LOM \text{ for } F \end{aligned}$$

Rewriting the problem in this way shows that the producer will choose the same contract a planner who is subject to the asymmetric information problem and put all weight on the borrower would choose. The equilibrium contract is constrained Pareto efficient.

Given the affine specification of the utility functions, the solution to the producers' problem in the afternoon will be in a corner. If the constraints (21) are ignored, there are four possible solutions: $t_L = 0 = t_H$, $t_L = 0$ and $t_H = k_B$, $t_L = k_B$ and $t_H = 0$, and $t_L = k_B = t_H$. If $t_L \geq t_H$ then the constraints on q are satisfied. If $t_L < t_H$, (21) might bind. In the appendix I show that (21) can't bind in equilibrium. Therefore, ignoring the constraints on q^*, (21),and using the guessed functional form for $V_P^a(k_P, k_{NP}; \phi)$, one can match coefficients and get

$$\begin{aligned} c_B(d) &= \frac{(E(\theta) - \theta_L)}{1 - \theta_L} \left(d + \beta \left(p_H c_L(d_H) \frac{\partial t_H}{\partial k_P} + p_L c_L(d_L) \frac{\partial t_L}{\partial k_P} \right) \right) - \frac{(E(\theta) - 1)}{1 - \theta_L} p_H \beta c_B(d_H) \left(\frac{\partial t_H}{\partial k_P} - \frac{\partial t_L}{\partial k_P} \right) \\ &\quad + \beta p_H c_B(d_H) \left(1 - \frac{\partial t_H}{\partial k_P} \right) + \beta p_L c_B(d_L) \left(1 - \frac{\partial t_L}{\partial k_P} \right) && (22) \\ a_B(\phi) &= \beta E_{\phi'} \left(\gamma \left[c_B(d') - \left(d' + \beta c_L(\bar{d}) \right) \right] \int k dF_{NP}^{m'}(k) \right). \end{aligned}$$

Proposition 6 *In the only affine equilibrium,* $t_L^* = k_B$ *and* $t_H^* = 0$.

The proof of this proposition can be found in the appendix.

3.2 Implementation of the Optimal Loan Contract

Using the results from the previous section, the optimal loan contract $(q^*, r_L^*, r_H^*, t_L^*, t_H^*)$ is given by

$$q^*(d) = \frac{(d + \beta p_L c_L(d_L) + \beta p_H c_B(d_H))}{1 - \theta_L} k_B$$

$$\begin{aligned} r_L^*(d) &= \frac{(\theta_L q^*(d) + d)}{1 - \theta_L} k_B \\ r_H^*(d) &= r_L(d) + \beta c_B(d_H) k_B \\ t_L^* &= k_P,\ t_H^* = 0 \end{aligned}$$

The optimal loan contract only depends on the aggregate state through the current dividend level d. Banks are collateral constrained: the maximum amount that the banks are able to borrow depends linearly on the amount of assets they have. The collateral constraint is an equilibrium outcome and it depends on how much the expected holder values the asset. With probability p_L the return of the loans is low and the borrower transfers all his asset holdings to the lender whose expected discounted value of one unit of asset is $\beta c_L(d_L)$. With probability p_H the return of the loans is high and the borrower keeps all his assets which he values $\beta c_B(d_H)$ per unit.

The optimal loan contract can be implemented using two different debt contracts simultaneously: riskless debt and collateralized debt. This implementation is not unique. I am going to describe the implementation with the maximum amount of riskless debt, which is the one that includes riskless debt at 0 interest rate.

The maximum amount that can be repaid independently of the realized state, risklessly, is $r_L^*(d)$. Therefore, the amount of riskless debt in this implementation is $r_L^*(d)$. The remaining part of the loan amount $q^*(d)$ is repaid in consumption goods only if the return of the project is high, whereas if it is low, the lender receives an asset transfer from the borrower. I will refer to this fraction of the loan amount as collateralized debt.

Then, the amount of collateralized debt in this implementation is

$$q_c := q(d) - r_L^*(d) = \beta\left(p_L c_L(d_L) + p_H c_B(d_H)\right) k_P$$

which is independent of the dividend level. The interest rate on this loan can be computed as

$$\begin{aligned} \frac{r_H(d) - r_L(d)}{q_c k_b} - 1 &= \frac{\beta c_P(d_H) - q_c}{q_c} \\ &= \beta \frac{p_L\left(c_B(d_H) - c_L(d_L)\right)}{q_c} \\ &= \frac{p_L\left(c_B(d_H) - c_L(d_L)\right)}{\left(p_L c_L(d_L) + p_H\left(c_B(d_H)\right)\right)}. \end{aligned}$$

The expected return on collateralized debt is

$$\frac{\beta p_L c_L(d_L) t_L^* + p_H\left(r_H^* - r_L^*\right)}{q_c} - 1 = \frac{\beta p_L c_L(d_L) + p_H \beta c_B(d_H)}{\beta\left(p_L c_L(d_L) + p_H c_B(d_H)\right)} - 1 = 0.$$

The maximum amount that can be borrowed against the asset, its debt capacity, is given by

$$D = \beta\left(p_L c_L(d_L) + p_H c_B(d_H)\right).$$

The asset's debt capacity depends on the value of collateral for lenders when there is default (insurance) and on the value of collateral for banks when there isn't default (incentives). Ceteris paribus, a higher value of collateral for lenders increases the loan amount since they can recover more when there is default, while a higher value of collateral for banks decreases the producers' incentives to lie and therefore allows them to borrow more.

3.3 Premia

The difference between the agents' valuation of assets and the fundamental value of the asset can be decomposed in several premia. The borrower values the asset more than the expected discounted dividend stream for two reasons. The first reason is that the asset serves a liquidity transformation device, it allows the borrower to solve the maturity mismatch problem he faces. The extra value due to this function is captured by the private liquidity premium, which I define as the difference between how much the borrower would value the asset if he chose to sell it to get funds and the fundamental value of the asset. If the borrower chose to sell the asset in the afternoon, he would get $d+\beta c_L\left(\bar{d}\right)$ per unit of asset, which is the maximum amount the lender would be willing to pay for it. With this funds, the bank would be able to invest in loans and he would get the return on equity $\frac{E(\theta)-\theta_L}{1-\theta_L}$ on them. Therefore, the bank would value each unit of asset $\frac{E(\theta)-\theta_L}{1-\theta_L}\left(d+\beta c_L\left(\bar{d}\right)\right)$, and the private liquidity premium is defined as

$$\frac{E(\theta)-\theta_L}{1-\theta_L}\left(d+\beta c_L\left(\bar{d}\right)\right)-\left(d+\beta\frac{\bar{d}}{1-\beta}\right)=\frac{E(\theta)-1}{1-\theta_L}\left(d+\beta\frac{\bar{d}}{1-\beta}\right)+\frac{E(\theta)-\theta_L}{1-\theta_L}\beta\left(c_L\left(\bar{d}\right)-\frac{\bar{d}}{1-\beta}\right).$$

The second reason why the bank values the asset more than the lender is that he expects to use it as collateral the following period. I define the private collateral premium as the extra value a bank gets from using the asset as collateral instead of selling it to raise funds. From the characterization of the affine equilibrium, a bank who chooses to use the asset as collateral values it $c_B(d)$ per unit. Then, the private collateral premium is

$$c_B(d)-\frac{E(\theta)-\theta_L}{1-\theta_L}\left(d+\beta c_L\left(\bar{d}\right)\right)=\frac{E(\theta)-\theta_L}{1-\theta_L}\beta p_H\left(c_B\left(d_H\right)-c_L\left(d_H\right)\right).$$

This premium depends on the difference in valuations for the bank and the lender. If both agents value the asset the same, the private collateral premium is 0. In this case, the borrower would be indifferent between selling the asset and pledging it as collateral. When the bank values the asset more than the lender, the private collateral premium is positive and the bank chooses to use the asset as collateral.

Finally, the lender may value the asset more than the expected discounted sum of its dividends if he has some bargaining power in the capital market. By being able to sell the asset to agents that value the asset more than themselves, lenders can extract some of this extra value whenever their bargaining power is positive, i.e., $\gamma<1$. This extra value is what I call a liquidity premium and it is defined as

$$c_L(d)-\left(d+\beta\frac{\bar{d}}{1-\beta}\right)=(1-\gamma)\frac{E(\theta)-1}{1-\theta_L}\left(\frac{\beta p_H\left(c_B\left(d_H\right)-c_L\left(d_H\right)\right)+\left(\gamma+\frac{E(\theta)-\theta_L}{1-\theta_L}(1-\gamma)\right)\left(\bar{d}-d\right)+\left(d+\beta\frac{\bar{d}}{1-\beta}\right)}{1-\beta\left(\gamma+\frac{E(\theta)-\theta_L}{1-\theta_L}(1-\gamma)\right)}\right).$$

As I mentioned above, this premium is 0 when the banks have all bargaining power in the capital market.

4 Comparative Statics

4.1 Different correlation structures

The debt capacity of the asset depends on the correlation between future dividends and the success of the investment made by banks. In this subsection, I will show that, keeping the (unconditional) expected return of the asset, $\bar{d}$, fixed, an increase in the correlation between the success of the investment made by banks and the future dividends paid by the asset increases the asset's debt capacity.

$$
\begin{aligned}
\frac{\partial D}{\partial d_L|_{\bar{d}}} &= \beta \frac{\partial \left(p_H c_B\left(d_H\right) + p_L c_L\left(d_L\right)\right)}{\partial d_L|_{\bar{d}}} \\
&\propto -\beta p_L \left(\frac{\left(E\left(\theta\right) - \theta_L\right)}{1-\theta_L} - \frac{\partial c_L\left(d_L\right)}{\partial d_L}\right) \\
&\propto -\beta p_L \frac{\left(1-\gamma\beta\right)\gamma \frac{\left(E(\theta)-\theta_L\right)}{1-\theta_L}\left(1 - \beta p_H \frac{\left(E(\theta)-\theta_L\right)}{1-\theta_L}\right)}{1 - \gamma\beta - \frac{\left(E(\theta)-\theta_L\right)}{1-\theta_L}\beta\left(1-\gamma\left(\beta p_H + p_L\right)\right)}
\end{aligned}
$$

Therefore,

$$
sign\left(\frac{\partial D}{\partial d_L|_{\bar{d}}}\right) = -sign\left(1 - \gamma\beta - \frac{\left(E\left(\theta\right)-\theta_L\right)}{1-\theta_L}\beta\left(1-\gamma\left(\beta p_H + p_L\right)\right)\right).
$$

Let $f\left(\gamma\right) := 1 - \gamma\beta - \frac{\left(E(\theta)-\theta_L\right)}{1-\theta_L}\beta\left(1-\gamma\left(\beta p_H + p_L\right)\right)$. Since by assumption $\left(1 - \beta\frac{E(\theta)-\theta_L}{1-\theta_L}\right) > 0,\ f\left(\gamma\right) > 0\ \forall\gamma \in [0,1]$ and

$$
\frac{\partial D}{\partial d_L|_{\bar{d}}} < 0 \Longleftrightarrow \frac{\partial D}{\partial d_H|_{\bar{d}}} > 0
$$

This result implies that a higher d_H, increases the asset's debt capacity and it makes it better collateral. The asset is worth more to the borrower when he has more incentives to default, in the event of a successful investment, and therefore decreases his incentives to lie.

4.2 Change in risk

Without loss of generality, θ_L can be set to 0. Then, the variance of the project is given by

$$
\begin{aligned}
V\left(\theta\right) &= p_H\left(\theta_H - p_H\theta_H\right)^2 + p_L\left(p_H\theta_H\right)^2 \\
&= p_H\left(1-p_H\right)\theta_H^2 \\
&= E\left(\theta\right)\theta_H - E\left(\theta\right)^2
\end{aligned}
$$

Therefore, one can get a mean preserving spread by increasing θ_H and setting

$$
p_H = \frac{E\left(\theta\right)}{\theta_H}.
$$

Proposition 7 *If $d_H \geq d_{\min}$, where $d\min < \bar{d}$, an increase in risk decreases the debt capacity of the asset, i.e.*

$$
\frac{\partial D}{\partial p_H|_{E(\theta)}} > 0.
$$

Proof. From the definition of debt capacity

$$
\begin{aligned}
\frac{\partial D}{\partial p_H|_{E(\theta)}} &= \beta\frac{\partial\left(p_H c_B\left(d_H\right) + p_L c_L\left(d_L\right)\right)}{\partial p_H|_{E(\theta)}} \\
&= \beta\frac{\left(c_B\left(d_H\right) - c_L\left(d_L\right) + p_L\frac{\partial c_L\left(d_L\right)}{\partial p_H|_{E(\theta)}}\right)}{1-\beta p_H\frac{E(\theta)-\theta_L}{1-\theta_L}} \\
&= \beta\frac{\left(\frac{E(\theta)-\theta_L}{1-\theta_L}\left(d_H - d_L\right) + \gamma\left(c_B\left(d_L\right) - \left(d_L + \beta c_L\left(\bar{d}\right)\right)\right) + p_L\frac{\partial c_L\left(d_L\right)}{\partial p_H|_{E(\theta)}}\right)}{1-\beta p_H\frac{E(\theta)-\theta_L}{1-\theta_L}}
\end{aligned}
$$

Differentiating the expression for $c_L(d_L)$ found in the appendix with respect to p_H keeping $E(\theta)$ fixed, one can see that

$$\frac{\partial c_L(d_L)}{\partial p_H|_{E(\theta)}} > 0.$$

Therefore, if $d_H \geq d_{\min}$

$$\frac{\partial D}{\partial p_H|_{E(\theta)}} > 0,$$

where $d_{\min} < \bar{d}$ and $d_{\min}$ solves

$$\frac{E(\theta) - \theta_L}{1 - \theta_L} \frac{-\bar{d} + d_{\min}}{p_L} + \gamma \left(c_B \left(\frac{\bar{d} - p_H d_{\min}}{p_L} \right) - \left(\frac{\bar{d} - p_H d_{\min}}{p_L} + \beta c_L(\bar{d}) \right) \right) + p_L \frac{\partial c_L \left(\frac{\bar{d} - p_H d_{\min}}{p_L} \right)}{\partial p_H|_{E(\theta)}} = 0$$

■

When the return of the loans and the future dividend level are sufficiently positively correlated, an increase in the riskiness of the loans decreases the debt capacity of the asset. In this setup, an increase in risk decreases the probability of success of the project and, therefore, increases the probability of default. This shift in probabilities has a direct and an indirect effect on the debt capacity of the asset. The direct effect is that now the debt capacity of the asset puts more weight on the lender's value of collateral in the event of default at the cost of decreasing the weight on borrower's value of collateral in the event of no default. The sign of this direct effect depends on whether the lender values the asset more in the default state, than does the borrower when he gets to keep it. The indirect effect comes from the change in the asset's valuation for the lender. An increase in the probability of default decreases the lender's valuation of the asset. When $d_H - d_L \geq 0$, both effects are negative. If $d_H - d_L < 0$, the net effect then depends on which of these effects is larger. When $d_H > d_{\min}$, the increase in the default probability is the dominating effect.

5 Multiple Loan Types - Incomplete

In this section I extend the benchmark model and introduce heterogeneity among banks. Each bank is characterized by the returns of the loans he is able to invest in. Loans made by different banks differ in their correlation with the dividends paid by the asset but they share the same success probability and expected return. There are J types of loans and a fraction μ_j of banks can invest in loans of type j, $j = 1, ..., J$.

In this case, in an affine equilibrium, the marginal value of assets for a lenders is

$$c_L(d) = (1 - \gamma) \sum_{i \in I} \mu_i c_B(d) + \gamma \left(d + \beta c_{NP}(\bar{d}) \right).$$

The banks's problem remains unchanged, though now the value for lenders depends on the average valuation among banks. Depending on the parameters of the model, two different kinds of regimes might arise. In one, all banks choose to use the asset as collateral. This is clearly the case when $\gamma = 1$ since the multiple loan type model and the benchmark model give the same contract for each agent. The existence of other types of banks only matters through the resale value of the asset in the asset market. If the sellers don't get any surplus from this sale, then the price of the asset in the capital market will be equal to the expected discounted value of dividend and, thus, it would be independent of the distribution of bank types in the economy.

If $\gamma < 1$ there might be banks who choose to sell the asset at the beginning of the afternoon and invest those funds rather than pledging it as collateral. Since the expected price at which lenders can sell the asset in the capital market depends on the banks' average valuation of the asset, it may be the case that this average valuation is high enough to motivate some of the banks who value the asset the least to sell the asset to raise funds. Whether there are some banks that don't use the asset as collateral or not depends on the value of γ. For high values of γ everybody uses the asset as collateral in the only symmetric equilibrium. For lower values of γ multiple equilibrium may arise, in principle (I am still working on this - so the equilibrium with sales might not exist), and some banks may choose to sell the asset instead of using it as collateral. Complementarity through the price when $\gamma < 1$.

Remark 8 *When $\gamma = 1$ or the asset's dividends are uncorrelated with the loans made by banks (e.g. risk free assets) all banks choose to use the asset as collateral (same as benchmark case).*

In both cases, the asset will always be used as collateral by at least one type of bank.

Proposition 9 *The banks with the highest marginal valuation of the asset will always use it as collateral.*

The proof of this proposition can be found in the appendix. Banks whose loans have the highest positive correlation with the dividend paid by the asset value the asset the most. The higher this correlation, the larger the amount that can be borrowed against the asset since it is better at providing incentives to solve the asymmetric information problem.

To be completed.

5.1 Example: Two Regimes(??)

TO BE WRITTEN.

5.2 Changes in correlation structure.

In this subsection I present an example to illustrate how aggregate quantities respond to changes in the correlation structure and risk when there are multiple types of loans in the economy and all banks choose to use the asset as collateral.

There are $J = 2$ types of banks. There is a population μ_j of type j banks, $j = 1, 2$. Let θ^j be the return of the loans in which a bank of type j can invest. The unconditional probability of success is equal for both types of loans but conditional on the aggregate state, the success probabilities are given by

$$\begin{aligned}
\Pr\left(\theta^1 = \theta_H | \omega_1\right) &= \pi, \\
\Pr\left(\theta^2 = \theta_H | \omega_1\right) &= (1 - \pi), \\
\Pr\left(\theta^1 = \theta_H | \omega_2\right) &= (1 - \pi), \\
\Pr\left(\theta^2 = \theta_H | \omega_2\right) &= \pi,
\end{aligned}$$

where $\Pr(\omega_1) = \Pr(\omega_2) = 0.5$. Let $d_i = E(d_{t+1} | \omega_i)$, $i = 1, 2$, where $d_1 > d_2$.

Then, the expected value of the dividend conditional on the return of project j is

$$\begin{aligned} E\left(d_{t+1}|\theta_t^1 = \theta_H\right) &= \frac{\pi d_1 + (1-\pi) d_2}{2p_H}, \\ E\left(d_{t+1}|\theta_t^2 = \theta_H\right) &= \frac{(1-\pi) d_1 + \pi d_2}{2p_H}. \end{aligned}$$

Given this correlation structure, an increase in π, increases (decreases) the correlation between the return of type 1 (type 2) projects and the future dividend level while keeping the unconditional probability of success, p_H, and the unconditional expected dividend, $\bar{d}$, constant.

Let $\bar{D} = \mu_1 D^1 + \mu_2 D^2$ be the average debt capacity of the asset. This is the debt capacity a representative bank would have in this economy.

Proposition 10 *An increase in π has the following effects.*

- *If $\mu_1 = \mu_2$, the economy's average debt capacity remains unchanged, type 1 banks collateral constraint is relaxed and type 2 banks collateral constraint is tightened.*
- *if $\mu_1 > \mu_2$, the economy's average debt capacity increases, and type 1 banks collateral constraint is relaxed.*
- *if $\mu_1 < \mu_2$, the economy's average debt capacity decreases, and type 2 banks collateral constraint is tightened.*

This proposition states that, when there are multiple types of loans, the effect of changes in the correlation structure on how much can be borrowed in the economy depends on the distribution of loan types among banks. When there are multiple projects, an increase in the correlation between the return of the loan a bank invests in and the future dividend has two effects. On the one hand, an increase in this correlation makes the asset more valuable for the bank making it better collateral. On the other hand, this change in correlation decreases the value of the asset for other types of banks and thus affects the resale value of the asset. The net effect depends on the distribution of types of loans across banks.

INCOMPLETE

6 Conclusion

In this paper I showed that when the roles as borrowers and lenders are persistent, and there is ex-post asymmetric information about the borrower's ability to repay, collateralized debt is the optimal way for borrowers to raise funds. Borrowers would rather offer their asset as collateral than sell it since they value it more than lenders. If they sold it, they would get at most the valuation of lenders, whereas by offering it as collateral they keep it when there is no default.

This difference in marginal valuations of the asset between borrowers and lenders is an equilibrium outcome. In autarky, both borrowers and lenders value the asset as the expected discounted sum of the dividend stream. When the agents are able to trade, the borrower values the asset more then the lender and they both value the asset (weakly) more than its fundamental value. The borrowers' excess valuation can be divided in two premia: a private liquidity premium and a private collateral premium. The first comes from the asset solving a maturity mismatch for the borrower: the asset pays dividends in the future but the borrower has an investment opportunity today. Being able to sell the asset provides the borrower with funds in the moment he needs them. The private collateral premium is the extra value the borrower assigns to the asset due to its role as collateral. Since the borrower will use collateral contracts in the future, the asset relaxes a borrowing constraint and thus has more value.

Since collateral contracts are optimal in this setup, and the marginal valuations of both borrowers and lenders are endogenous, the maximum amount that can be borrowed against the asset, its debt capacity, is also an equilibrium outcome. I am able to solve the model in closed form which, in turn, allows me to compute some comparative statics with respect to the correlation structure, and the riskiness of the loans in which banks invest. I find that a higher correlation between the success of the loans and future dividends makes the asset better collateral since it is better at motivating the borrower to tell the truth when he has incentives to lie. If the return of the risky loans and the asset's dividends are sufficiently positively correlated, an increase in the riskiness of the loans decreases the asset's debt capacity.

I finally extended the model to include heterogeneity among banks. (To be completed)

We know from previous literature that changes in financial constraints played an important role in recent crises.[9] Having a model that characterizes these constraints as equilibrium outcomes is important both from a positive and a normative point of view. In positive terms, it is interesting to see where the financial shocks come from and how they interact with the fundamentals of the economy. From the normative side, policies that aim at stabilizing the cycle and preventing financial crises should take into account what drives changes in the financing conditions faced by banks, firms, and households. This paper is an attempt to deliver some of the insights needed to understand collateralized debt markets better.

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8 Appendix

8.1 Uniqueness

Lemma 11 (21) *can't bind in equilibrium.*

Proof. Suppose (21) binds, then,

$$\frac{dk_B + \theta_L \beta \left(p_H c_L \left(d_H\right) t_H + p_L c_L \left(d_L\right) t_L\right)}{\beta c_B \left(d_H\right)} = t_H - t_L. \tag{23}$$

If $r_H = 0$ binds,

$$q^* = \left(\beta p_H c_L \left(d_H\right) t_H - \beta p_L c_L \left(d_L\right) t_L\right)\left(1-\theta_L\right) + p_L dk_B$$

and using the definition of q^* together with (**??**) this implies

$$q^* = \frac{p_L dk_B + \beta \left(p_H c_L \left(d_H\right) t_H + p_L c_L \left(d_L\right) t_L\right)\left(1 - p_H \theta_L\right)}{1-\theta_L}.$$

Putting these last two equations together gives

$$\beta \left(p_H c_L \left(d_H\right) t_H + p_L c_L \left(d_L\right) t_L\right)\left(2 - p_H - \theta_L\right)\theta_L = -\theta_L p_L dk_B$$

which implies $p_H c_L \left(d_H\right) t_H + p_L c_L \left(d_L\right) t_L < 0$, a contradiction. ■

Lemma 12 *In equilibrium,* $t_L \neq 0$

Proof. Suppose $t_L = 0$ in equilibrium. If the objective function is decreasing in t_L, then it is also decreasing in t_H. Therefore, $t_L = 0$ implies $t_H = 0$. The coefficients of the value functions in an affine equilibrium would then become

$$\begin{aligned} c_B(d) &= \frac{\left(E(\theta) - \theta_L\right)}{1-\theta_L} d + \beta c_P\left(\bar{d}\right) \\ c_B(d) &= \frac{\left(E(\theta) - \theta_L\right)}{1-\theta_L}\left(d + \beta \frac{\bar{d}}{1-\beta}\right), \end{aligned}$$

and

$$\begin{aligned} c_L(d) &= (1-\gamma) c_B(d) + \gamma\left(d + \beta c_{NP}\left(\bar{d}\right)\right) \\ c_L\left(\bar{d}\right) &= \frac{(1-\gamma)\frac{\left(E(\theta)-\theta_L\right)}{1-\theta_L}\frac{\bar{d}}{1-\beta} + \gamma \bar{d}}{1-\gamma\beta}. \end{aligned}$$

The derivative of the objective function with respect to t_L is

$$\frac{\left(E(\theta) - \theta_L\right)}{1-\theta_L} \beta p_L c_L\left(d_L\right) + \frac{\left(E(\theta) - 1\right)}{1-\theta_L} p_H \beta c_P\left(d_H\right) - \beta p_L c_B\left(d_L\right).$$

If $t_L = 0$ and $t_H = 0$ this becomes

$$\begin{aligned} &\frac{\left(E(\theta) - \theta_L\right)}{1-\theta_L}\frac{\beta p_L}{1-\gamma\beta}\left((1-\gamma\beta)(1-\gamma)\frac{\left(E(\theta)-1\right)}{1-\theta_L} d_L + \left(\frac{\left(E(\theta)-\theta_L\right)}{1-\theta_L} + \left(\gamma^2 - 1 + \gamma\beta(1-\gamma)\right)\beta\right)\frac{\bar{d}}{1-\beta}\right) \\ &+ \frac{\left(E(\theta) - \theta_L\right)}{1-\theta_L}\frac{\left(E(\theta)-1\right)}{1-\theta_L} p_H \beta\left(d_H + \beta\frac{\bar{d}}{1-\beta}\right) \end{aligned}$$

which is > 0 and therefore contradicts $t_L = 0$. ■

Lemma 13 *$t_L = k_P = t_H$ is not an equilibrium.*

Proof. Suppose $t_L = k_P = t_H$ in equilibrium. Then,

$$\begin{aligned} c_B(d) &= \frac{(E(\theta)-\theta_L)}{1-\theta_L}\left(d+\beta c_{NP}\left(\bar{d}\right)\right) \\ c_L(d) &= \left[(1-\gamma)c_B(d)+\gamma\left(d+\beta\frac{(1-\gamma)c_B\left(\bar{d}\right)+\gamma\bar{d}}{1-\gamma\beta}\right)\right] \end{aligned}$$

Therefore,

$$c_L\left(\bar{d}\right) = \frac{(1-\gamma)c_B\left(\bar{d}\right)+\gamma\bar{d}}{1-\gamma\beta},$$

and

$$c_B\left(\bar{d}\right) = \frac{\frac{(E(\theta)-\theta_L)}{1-\theta_L}\bar{d}}{\left(1-\beta\left(\gamma+\frac{(E(\theta)-\theta_L)}{1-\theta_L}(1-\gamma)\right)\right)}$$

which implies

$$c_L\left(\bar{d}\right) = \frac{\left(\gamma+\frac{(E(\theta)-\theta_L)}{1-\theta_L}(1-\gamma)\right)}{\left(1-\beta\left(\gamma+\frac{(E(\theta)-\theta_L)}{1-\theta_L}(1-\gamma)\right)\right)}\bar{d}$$

and

$$c_B(d) = \frac{(E(\theta)-\theta_L)}{1-\theta_L}\left(d+\beta\frac{\left(\gamma+\frac{(E(\theta)-\theta_L)}{1-\theta_L}(1-\gamma)\right)\bar{d}}{\left(1-\beta\left(\gamma+\frac{(E(\theta)-\theta_L)}{1-\theta_L}(1-\gamma)\right)\right)}\right).$$

Thus,

$$c_L(d)-c_B(d) = \gamma\left(-\frac{(E(\theta)-1)}{1-\theta_L}\left(d+\beta\frac{\left(\gamma+\frac{(E(\theta)-\theta_L)}{1-\theta_L}(1-\gamma)\right)}{\left(1-\beta\left(\gamma+\frac{(E(\theta)-\theta_L)}{1-\theta_L}(1-\gamma)\right)\right)}\bar{d}\right)\right) < 0$$

what would imply that the derivative of the objective function is decreasing in t_H and, thus, $t_H = 0$, which is a contradiction. ■

Proposition ?? In the only affine equilibrium, $t_L^* = k_B$ and $t_H^* = 0$.

Proof. From the previous lemmas in this section, the only candidate left for equilibrium is $t_L^* = k_B$ and $t_H^* = 0$. Assume $t_L^* = k_B$ and $t_H^* = 0$. Then, (22) becomes

$$\begin{aligned} c_B(d) &= \frac{(E(\theta)-\theta_L)}{1-\theta_L}\left(d+\beta\left(p_Lc_B(d_L)+p_Hc_B(d_H)\right)\right), \text{ and} \\ c_B(d_H) &= \frac{\frac{(E(\theta)-\theta_L)}{1-\theta_L}\left(d_H+\beta p_Lc_B(d_L)\right)}{1-\beta p_H\frac{(E(\theta)-\theta_L)}{1-\theta_L}}. \end{aligned}$$

Therefore

$$c_B(d) = \frac{(E(\theta)-\theta_L)}{1-\theta_L}\left(d+\frac{\beta\left(p_Lc_L(d_L)+p_H\frac{(E(\theta)-\theta_L)}{1-\theta_L}d_H\right)}{1-\beta p_H\frac{(E(\theta)-\theta_L)}{1-\theta_L}}\right). \tag{24}$$

To have $t_L^* = k_P$ and $t_H^* = 0$ be chosen by the producer, the objective function must be increasing in t_L and decreasing in t_H, i.e.,

$$\frac{E(\theta)-\theta_L}{1-\theta_L}\beta p_Lc_L(d_L)-\beta p_Lc_B(d_L)+\frac{E(\theta)-1}{1-\theta_L}\beta p_Hc_B(d_H) > 0 \tag{25}$$

and

$$\frac{E(\theta)-\theta_L}{1-\theta_L}\beta\left(c_L(d_H)-c_B(d_H)\right) < 0. \tag{26}$$

Using (24) evaluated at $d = d_L$ and $d = d_H$ the derivative of the objective function with respect to t_L becomes

$$\frac{\frac{E(\theta)-\theta_L}{1-\theta_L}}{1-\beta p_H\frac{(E(\theta)-\theta_L)}{1-\theta_L}}\left((1-\beta)\,p_L c_L\,(d_L)-\left(1-\beta p_H\frac{(E\,(\theta)-\theta_L)}{1-\theta_L}\right)p_L d_L-\beta p_L\frac{(E\,(\theta)-\theta_L)}{1-\theta_L}p_H d_H+\frac{E\,(\theta)-1}{1-\theta_L}p_H d_H\right)$$

$$\frac{\frac{E(\theta)-\theta_L}{1-\theta_L}}{1-\beta p_H\frac{(E(\theta)-\theta_L)}{1-\theta_L}}\left((1-\beta)\,p_L c_L\,(d_L)-\left(1-\beta p_H\frac{(E\,(\theta)-\theta_L)}{1-\theta_L}\right)p_L d_L+(1-\beta p_L)\,\frac{(E\,(\theta)-\theta_L)}{1-\theta_L}p_H d_H-p_H d_H\right)$$

$$\frac{\frac{E(\theta)-\theta_L}{1-\theta_L}}{1-\beta p_H\frac{(E(\theta)-\theta_L)}{1-\theta_L}}\left((1-\beta)\,p_L c_L\,(d_L)-\bar{d}+\beta p_H\frac{(E\,(\theta)-\theta_L)}{1-\theta_L}p_L d_L+(1-\beta p_L)\,\frac{(E\,(\theta)-\theta_L)}{1-\theta_L}p_H d_H\right)$$

$$\frac{\frac{E(\theta)-\theta_L}{1-\theta_L}\beta\,(1-\beta)}{1-\beta p_H\frac{(E(\theta)-\theta_L)}{1-\theta_L}}\left(p_L\left(c_L\,(d_L)-\left(d_L+\beta\frac{\bar{d}}{(1-\beta)}\right)\right)-p_H\left(d_H+\beta\frac{\bar{d}}{(1-\beta)}\right)+\frac{(E\,(\theta)-\theta_L)}{1-\theta_L}p_H\frac{(1-\beta p_L)\,d_H+\beta p_L d_L}{(1-\beta)}\right)$$

Using that

$$(1-\beta p_L)\,d_H+\beta p_L d_L=(1-\beta+\beta p_H)\,d_H+\beta p_L d_L=(1-\beta)\,d_H+\beta\bar{d}$$

the expression above becomes

$$\frac{\frac{E(\theta)-\theta_L}{1-\theta_L}\beta\,(1-\beta)}{1-\beta p_H\frac{(E(\theta)-\theta_L)}{1-\theta_L}}\left(p_L\left(c_L\,(d_L)-\left(d_L+\beta\frac{\bar{d}}{(1-\beta)}\right)\right)+\frac{(E\,(\theta)-1)}{1-\theta_L}p_H\left(d_H^i+\frac{\beta\bar{d}}{(1-\beta)}\right)\right)>0$$

since $c_L\,(d_L)-\left(d_L+\beta\frac{\bar{d}}{(1-\beta)}\right)\geq 0$. Therefore, the derivative of the objective function is positive under this guess which in turn implies $t_L > 0$.

The sign of the derivative of the objective function with respect to t_H depends on the sign of the difference in marginal valuations between the borrower (bank) and the lender. Using the definition of $c_L\,(d)$ and $c_B\,(d)$ this difference can be rewritten as

$$\begin{aligned}
c_L\,(d)-c_B\,(d) &= \left[(1-\gamma)\,c_B\,(d)+\gamma\left(d+\beta\frac{(1-\gamma)\,c_B\,(\bar{d})+\gamma\bar{d}}{1-\gamma\beta}\right)\right]-c_B\,(d)\\
&= \gamma\left(d+\beta\frac{(1-\gamma)\,c_B\,(\bar{d})+\gamma\bar{d}}{1-\gamma\beta}-c_B\,(d)\right)\\
&= \gamma\left(-\frac{(E\,(\theta)-1)}{1-\theta_L}d+\frac{(1-\beta)}{1-\gamma\beta}\left(\left((1-\gamma)\,\frac{(E\,(\theta)-\theta_L)}{1-\theta_L}+\gamma\right)\frac{\beta\bar{d}}{1-\beta}-c_B\,(\bar{d})+\frac{(E\,(\theta)-\theta_L)}{1-\theta_L}\bar{d}\right)\right)\\
&\leq \gamma\left(-\frac{(E\,(\theta)-1)}{1-\theta_L}d+\frac{(1-\beta)}{1-\gamma\beta}\left(\frac{(E\,(\theta)-\theta_L)}{1-\theta_L}\frac{\bar{d}}{1-\beta}-c_B\,(\bar{d})\right)\right)
\end{aligned}$$

But

$$\begin{aligned}
c_B\,(\bar{d}) &> \frac{(E\,(\theta)-\theta_L)}{1-\theta_L}\frac{\bar{d}}{1-\beta}\\
p_L c_L\,(d_L)+p_H\frac{(E\,(\theta)-\theta_L)}{1-\theta_L}d_H &> \frac{\bar{d}}{1-\beta}\left(1-\beta p_H\frac{(E\,(\theta)-\theta_L)}{1-\theta_L}\right)\\
p_H\frac{(E\,(\theta)-\theta_L)}{1-\theta_L}\left(d_H+\frac{\beta\bar{d}}{1-\beta}\right) &> \frac{\bar{d}}{1-\beta}-p_L c_L\,(d_L)\\
p_H\frac{(E\,(\theta)-\theta_L)}{1-\theta_L}\left(d_H+\frac{\beta\bar{d}}{1-\beta}\right) &> p_H\left(d_H+\frac{\beta\bar{d}}{1-\beta}\right)=\frac{\bar{d}}{1-\beta}-p_L\left(d_L+\beta\frac{\bar{d}}{1-\beta}\right)\geq\frac{\bar{d}}{1-\beta}-p_L c_L\,(d_L)
\end{aligned}$$

since $\frac{(E(\theta)-\theta_L)}{1-\theta_L}>1$.

Then,

$$c_L(d_H) - c_B(d_H) \leq \gamma\left(-\frac{(E(\theta)-1)}{1-\theta_L}d_H + \frac{(1-\beta)}{1-\gamma\beta}\left(\frac{(E(\theta)-\theta_L)}{1-\theta_L}\frac{\bar{d}}{1-\beta} - c_B(\bar{d})\right)\right) < 0,$$

and this implies $t_H = 0$. Therefore, if the contract implied by $t_L^* = k_B$ and $t_H^* = 0$ is a solution to the bank's problem in the loan market. ■

8.2 Value function coefficients with one project.

From section 3, the marginal value of assets for lenders and producers are given by

$$\begin{aligned} c_L(d) &= \left[(1-\gamma)c_B(d) + \gamma\left(d + \beta\frac{(1-\gamma)c_B(\bar{d}) + \gamma\bar{d}}{1-\gamma\beta}\right)\right] \\ c_B(d) &= \frac{(E(\theta)-\theta_L)}{1-\theta_L}\left(d + \frac{\beta\left(p_L c_L(d_L) + p_H\frac{(E(\theta)-\theta_L)}{1-\theta_L}d_H\right)}{1-\beta p_H\frac{(E(\theta)-\theta_L)}{1-\theta_L}}\right) \end{aligned}$$

which in turn are all functions of $c_L(d_L)$. Using the equations above, one can get

$$\begin{aligned} c_L(d_L) &= \left[(1-\gamma)c_B(d_L) + \gamma\left(d_L + \beta\frac{(1-\gamma)c_B(\bar{d}) + \gamma\bar{d}}{1-\gamma\beta}\right)\right] \\ c_L(d_L) &= \left[(1-\gamma)\left(c_B(\bar{d}) + \frac{E(\theta)-\theta_L}{1-\theta_L}(d_L-\bar{d})\right) + \gamma\left(d_L + \beta\frac{(1-\gamma)c_B(\bar{d}) + \gamma\bar{d}}{1-\gamma\beta}\right)\right] \\ c_L(d_L) &= \left[\frac{(1-\gamma)}{1-\gamma\beta}c_B(\bar{d}) + \left((1-\gamma)\frac{E(\theta)-\theta_L}{1-\theta_L} + \gamma\right)d_L + \left(\frac{\gamma^2\beta}{1-\gamma\beta} - (1-\gamma)\frac{E(\theta)-\theta_L}{1-\theta_L}\right)\bar{d}\right]. \end{aligned}$$

Using the definition of $c_B(\bar{d})$ gives

$$\begin{aligned} &\left((1-\gamma\beta)\left(1-\beta p_H\frac{(E(\theta)-\theta_L)}{1-\theta_L}\right) - (1-\gamma)\frac{(E(\theta)-\theta_L)}{1-\theta_L}\beta p_L\right)c_L(d_L) \\ =\ &\left(\left((1-\gamma)\frac{E(\theta)-\theta_L}{1-\theta_L} + \gamma\right)\left(1-\beta p_H\frac{(E(\theta)-\theta_L)}{1-\theta_L}\right)\gamma\beta + (1-\gamma)\left(\frac{(E(\theta)-\theta_L)}{1-\theta_L}\right)^2\beta\right)\bar{d} \\ &+\left(\left((1-\gamma)\frac{E(\theta)-\theta_L}{1-\theta_L} + \gamma\right)\left(1-\beta p_H\frac{(E(\theta)-\theta_L)}{1-\theta_L}\right)(1-\gamma\beta) - (1-\gamma)\left(\frac{(E(\theta)-\theta_L)}{1-\theta_L}\right)^2\beta p_L\right)d_L \end{aligned}$$

8.3 Equilibrium with multiple loan types.

Proposition 14 *There can't be an equilibrium in which some type of banks don't use collateral contracts and some do, nor one in which no one uses collateral.*

Proof. If a bank chooses not to use the asset as collateral, his marginal value of assets is

$$c_B^j(d) = \frac{E(\theta)-\theta_L}{1-\theta_L}(d + \beta c_L(d)) = c_B^{NC}(d)$$

which is independent of the correlation structure of his own project with the dividend paid by the asset. From the benchmark model, we know that the marginal value of a bank with investment opportunity j that chooses to use the asset as collateral is

$$c_B^j(d) = \frac{E(\theta)-\theta_L}{1-\theta_L}\left(d + \beta\frac{p_L c_L\left(d_L^j\right) + p_H\frac{E(\theta)-\theta_L}{1-\theta_L}d_H^j}{1-\beta p_H\frac{E(\theta)-\theta_L}{1-\theta_L}}\right).$$

If some banks $i \in J^{NC}$ are choosing not to use the asset as collateral, it must be the case that

$$c_L\left(d_H^i\right) - c_B^{NC}\left(d_H^i\right) \geq 0 \text{ for all } i \in J^{NC}.$$

This difference in marginal valuations at any d can be rewritten as

$$c_L(d) - c_B^{NC}(d) = (1-\gamma)\sum_{j\in J^C}\mu_j\left(c_B^j(d) - c_B^{NP}(d)\right) - \frac{E(\theta)-1}{1-\theta_L}\gamma\left(d + \beta c_L\left(\bar{d}\right)\right) \tag{27}$$

where J^C is the set of all bank types that use the asset as collateral. Thus, using the definition of $c_B^j(d)$ for $j \in J^C$, this becomes

$$\begin{aligned}
c_L(d) - c_B^{NC}(d) &= \frac{E(\theta)-\theta_L}{1-\theta_L}(1-\gamma)\beta\frac{\sum_{j\in J^C}\mu_j\left(p_L c_L\left(d_L^j\right) + p_H\frac{E(\theta)-\theta_L}{1-\theta_L}d_H^j - \left(1-\frac{E(\theta)-\theta_L}{1-\theta_L}\beta p_H\right)c_L\left(\bar{d}\right)\right)}{1-\beta p_H\frac{E(\theta)-\theta_L}{1-\theta_L}} \\
&\quad -\gamma\frac{E(\theta)-1}{1-\theta_L}\left(d+\beta c_L\left(\bar{d}\right)\right) \\
c_L(d) - c_B^{NC}(d) &= \frac{E(\theta)-\theta_L}{1-\theta_L}(1-\gamma)\beta p_H\frac{\sum_{j\in J^C}\mu_j\left(\frac{E(\theta)-\theta_L}{1-\theta_L}\left(d_H^j+\beta c_L\left(\bar{d}\right)\right) - c_L\left(d_H^j\right)\right)}{1-\beta p_H\frac{E(\theta)-\theta_L}{1-\theta_L}} \\
&\quad -\gamma\frac{E(\theta)-1}{1-\theta_L}\left(d+\beta c_L\left(\bar{d}\right)\right) \\
c_L(d) - c_B^{NC}(d) &= \frac{E(\theta)-\theta_L}{1-\theta_L}(1-\gamma)\beta p_H\frac{\sum_{j\in J^C}\mu_j\left(c_B^{NC}\left(d_H^j\right) - c_L\left(d_H^j\right)\right)}{1-\beta p_H\frac{E(\theta)-\theta_L}{1-\theta_L}} - \gamma\frac{E(\theta)-1}{1-\theta_L}\left(d+\beta c_L\left(\bar{d}\right)\right) \\
c_L(d) - c_B^{NC}(d) &= \frac{E(\theta)-\theta_L}{1-\theta_L}(1-\gamma)\beta p_H\frac{\sum_{j\in J^C}\mu_j\left(c_B^{NC}\left(d_H^j\right) - c_L\left(d_H^j\right)\right)}{1-\beta p_H\frac{E(\theta)-\theta_L}{1-\theta_L}} - \gamma\frac{E(\theta)-1}{1-\theta_L}\left(d+\beta c_L\left(\bar{d}\right)\right)
\end{aligned}$$

Setting $d = d_H^j$ and summing over $j \in J^C$,

$$\begin{aligned}
\sum_{j\in J^C}\mu_j\left(c_L\left(d_H^j\right) - c_B^{NC}\left(d_H^j\right)\right) &= \sum_{j\in J^C}\mu_j\frac{E(\theta)-\theta_L}{1-\theta_L}(1-\gamma)\beta p_H\frac{\sum_{j\in J^C}\mu_j\left(c_B^{NC}\left(d_H^j\right) - c_L\left(d_H^j\right)\right)}{1-\beta p_H\frac{E(\theta)-\theta_L}{1-\theta_L}} \\
&\quad -\gamma\frac{E(\theta)-1}{1-\theta_L}\sum_{i\in I^C}\mu_i\left(d_H^i+\beta c_L\left(\bar{d}\right)\right) \\
\sum_{j\in J^C}\mu_j\left(c_L\left(d_H^j\right) - c_B^{NC}\left(d_H^j\right)\right) &= \frac{-\gamma\frac{E(\theta)-1}{1-\theta_L}\sum_{j\in J^C}\mu_j\left(d_H^j+\beta c_L\left(\bar{d}\right)\right)\left(1-\beta p_H\frac{E(\theta)-\theta_L}{1-\theta_L}\right)}{\left(1+\frac{E(\theta)-\theta_L}{1-\theta_L}\beta p_H\left(\sum_{i\in I^C}\mu_i(1-\gamma)-1\right)\right)} < 0.
\end{aligned}$$

But this implies

$$\sum_{j\in J^C}\mu_j\left(c_B^j(d) - c_B^{NP}(d)\right) = \sum_{j\in J^C}\mu_j\left(c_B^{NC}\left(d_H^j\right) - c_L\left(d_H^j\right)\right) > 0$$

and

$$\sum_{j\in J^C}\mu_j c_B^j(d) > \sum_{j\in J^C}\mu_j c_B^{NP}(d).$$

The average marginal valuation of banks conditional on them using the asset as collateral is higher than the marginal valuation of banks choosing not to use the asset as collateral, if there are any. ■

Proposition 15 *The banks with the highest marginal valuation of the asset always use it as collateral.*

Proof. Let $c_B^{\max}$ be the marginal valuation of the bank who values the asset the most. Then,

$$c_L(d) - c_B^{\max}(d) = (1-\gamma)\sum_{j\in J}\mu_j\left(c_B^j(d) - c_B^{\max}(d)\right) + \gamma\left(d + \beta c_L\left(\bar{d}\right) - c_B^{\max}(d)\right) \text{ for all } d.$$

By definition of $c_B^{\max}$ the first term is (weakly) negative. Moreover, we know that $c_B^{\max} \geq \frac{E(\theta)-\theta_L}{1-\theta_L}\left(d+\beta c_L\left(\bar{d}\right)\right) > d+\beta c_L\left(\bar{d}\right)$ since the bank can always choose to sell the asset in the afternoon and invest the proceeds in the risky loans. Therefore, the bank with the maximum marginal valuation of the asset chooses not to set transfers of capital to 0 when the realization of the return of the loans is high.

(Need to show that derivative wrt tL >0 always)

$$\frac{E(\theta)-\theta_L}{1-\theta_L}\beta\left(p_L c_L\left(d_L^j\right) - p_L\left(d_L^i + \beta\frac{p_L c_L\left(d_L^j\right) + p_H\frac{E(\theta)-\theta_L}{1-\theta_L}d_H^j}{1-\beta p_H\frac{E(\theta)-\theta_L}{1-\theta_L}}\right) + \frac{E(\theta)-1}{1-\theta_L}p_H\left(d_H^j + \beta\frac{p_L c_L\left(d_L^j\right) + p_H\frac{E(\theta)-\theta_L}{1-\theta_L}d_H^j}{1-\beta p_H\frac{E(\theta)-\theta_L}{1-\theta_L}}\right)\right)$$

$$= \frac{\frac{(E(\theta)-\theta_L)}{1-\theta_L}\beta}{1-\frac{(E(\theta)-\theta_L)}{1-\theta_L}\beta p_H}\left(p_L(1-\beta)c_L\left(d_L^j\right) - \bar{d} + (1-\beta p_L)\frac{(E(\theta)-\theta_L)}{1-\theta_L}p_H d_H^j + \beta p_H\frac{E(\theta)-\theta_L}{1-\theta_L}p_L d_L^j\right)$$

$$= \frac{\frac{(E(\theta)-\theta_L)}{1-\theta_L}}{1-\frac{(E(\theta)-\theta_L)}{1-\theta_L}\beta p_H}\frac{\beta}{1-\beta}\left(p_L\left(c_L\left(d_L^j\right) - \left(d_L^j + \beta\frac{\bar{d}}{1-\beta}\right)\right) + p_H\frac{E(\theta)-1}{1-\theta_L}\left(d_H^j + \beta\frac{\bar{d}}{1-\beta}\right)\right) > 0$$

■

8.4 Value function coefficients with multiple loan types

8.4.1 All banks use the asset as collateral

From sections 3 and 5, the marginal value of capital for a producer of type j is given .

$$c_B^j(d) = \frac{(E(\theta)-\theta_L)}{1-\theta_L}\left(d + \beta\left(p_L c_L\left(d_L^j\right) + p_H c_B^j\left(d_H^j\right)\right)\right)$$

which gives

$$c_B^j\left(d_H^j\right) = \frac{\frac{(E(\theta)-\theta_L)}{1-\theta_L}\left(d_H^j + \beta p_L c_L\left(d_L^j\right)\right)}{1-\frac{(E(\theta)-\theta_L)}{1-\theta_L}\beta p_H}$$

and

$$c_B^j(d) = \frac{(E(\theta)-\theta_L)}{1-\theta_L}\left[d + \beta\frac{p_L c_L\left(d_L^j\right) + p_H\frac{(E(\theta)-\theta_L)}{1-\theta_L}d_H^j}{1-\frac{(E(\theta)-\theta_L)}{1-\theta_L}\beta p_H}\right]$$

Moreover, the marginal value of assets for lenders is

$$c_L(d) = (1-\gamma)\sum_{j\in J}\mu_j c_B^j(d) + \gamma\left(d + \beta c_{NP}\left(\bar{d}\right)\right).$$

Then,

$$c_L\left(\bar{d}\right) = \frac{(1-\gamma)\sum_{j\in J}\mu_j c_B^j\left(\bar{d}\right) + \gamma\bar{d}}{1-\gamma\beta}$$

and

$$c_L(d) = \frac{(1-\gamma\beta)(1-\gamma)\sum_{j\in J}\mu_j c_B^j(d) + (1-\gamma\beta)\gamma d + \gamma\beta(1-\gamma)\sum_{j\in J}\mu_j c_B^j(\bar{d}) + \gamma\beta\gamma\bar{d}}{1-\gamma\beta}.$$

Using the definition of c_B^j gives

$$\begin{aligned}(1-\gamma\beta)c_L(d) &= (1-\gamma)\beta\frac{(E(\theta)-\theta_L)}{1-\theta_L}\frac{p_L c_L\left(\sum_{j\in J}\mu_j d_L^j\right) + p_H\frac{(E(\theta)-\theta_L)}{1-\theta_L}\sum_{j\in J}\mu_j d_H^j}{1-\frac{(E(\theta)-\theta_L)}{1-\theta_L}\beta p_H} \\ &\quad + \left((1-\gamma)\frac{(E(\theta)-\theta_L)}{1-\theta_L}+\gamma\right)\left((1-\gamma\beta)d+\gamma\beta\bar{d}\right).\end{aligned}$$

Rearranging terms and setting $d=\sum_{j\in J}\mu_j d_L^j$,

$$\begin{aligned}&\left((1-\gamma\beta)\left(1-\beta p_H\frac{(E(\theta)-\theta_L)}{1-\theta_L}\right)-(1-\gamma)\frac{(E(\theta)-\theta_L)}{1-\theta_L}\beta p_L\right)c_L\left(\sum_{j\in J}\mu_j d_L^j\right) \\ =&\left(\left(1-\frac{(E(\theta)-\theta_L)}{1-\theta_L}\beta p_H\right)\left((1-\gamma)\frac{(E(\theta)-\theta_L)}{1-\theta_L}+\gamma\right)\gamma+(1-\gamma)\left(\frac{(E(\theta)-\theta_L)}{1-\theta_L}\right)^2\right)\beta\bar{d} \\ &+\begin{pmatrix}\left(1-\frac{(E(\theta)-\theta_L)}{1-\theta_L}\beta p_H\right)\left((1-\gamma)\frac{(E(\theta)-\theta_L)}{1-\theta_L}+\gamma\right)(1-\gamma\beta) \\ -(1-\gamma)p_L\left(\frac{(E(\theta)-\theta_L)}{1-\theta_L}\right)^2\beta\end{pmatrix}\sum_{j\in J}\mu_j d_L^j.\end{aligned}$$

The guess is correct, the coefficients are affine in the dividend level.

8.4.2 Some banks don't use the asset as collateral

From previous sections the marginal value of a bank type $j\in J^C$ that uses the asset as collateral is given by

$$c_B^j(d) = \frac{(E(\theta)-\theta_L)}{1-\theta_L}\left[d+\beta\frac{p_L c_L\left(d_L^j\right)+p_H\frac{(E(\theta)-\theta_L)}{1-\theta_L}d_H^j}{1-\frac{(E(\theta)-\theta_L)}{1-\theta_L}\beta p_H}\right].$$

The marginal value of a bank type $j\in J^{NC}$ who chooses not to use the asset as collateral is

$$c_B^j(d) = \frac{(E(\theta)-\theta_L)}{1-\theta_L}\left[d+\beta c_L(\bar{d})\right] = c_B^{NC}(d).$$

Moreover, the marginal value of assets for lenders is

$$c_L(d) = \frac{(1-\gamma\beta)(1-\gamma)\sum_{j\in J}\mu_j c_B^j(d) + (1-\gamma\beta)\gamma d + \gamma\beta(1-\gamma)\sum_{j\in J}\mu_j c_B^j(\bar{d}) + \gamma\beta\gamma\bar{d}}{1-\gamma\beta}.$$

Using the definition of c_B^j gives

$$\begin{aligned}(1-\gamma\beta)c_L(d) &= (1-\gamma)\frac{(E(\theta)-\theta_L)}{1-\theta_L}\beta\left(\frac{\sum_{j\in J^C}\mu_j p_L c_L\left(d_L^j\right)+p_H\frac{(E(\theta)-\theta_L)}{1-\theta_L}\sum_{j\in J^C}\mu_j d_H^j}{1-\frac{(E(\theta)-\theta_L)}{1-\theta_L}\beta p_H}+\sum_{j\in J^{NC}}\mu_j c_L(\bar{d})\right) \\ &\quad+\left(\gamma+\frac{(E(\theta)-\theta_L)}{1-\theta_L}(1-\gamma)\right)\left(\gamma\beta\bar{d}+(1-\gamma\beta)d\right).\end{aligned}$$

Using that

$$c_L(\bar{d}) = \frac{(1-\gamma)\frac{(E(\theta)-\theta_L)}{1-\theta_L}\beta\left(\frac{p_L\sum_{j\in J^C}\mu c_L\left(_j d_L^j\right)+p_H\frac{(E(\theta)-\theta_L)}{1-\theta_L}\sum_{j\in J^C}\mu_j d_H^j}{1-\frac{(E(\theta)-\theta_L)}{1-\theta_L}\beta p_H}\right)+\left(\gamma+\frac{(E(\theta)-\theta_L)}{1-\theta_L}(1-\gamma)\right)\bar{d}}{\left((1-\gamma\beta)-(1-\gamma)\sum_{j\in J^{NC}}\mu_j\frac{(E(\theta)-\theta_L)}{1-\theta_L}\beta\right)},$$

the coefficients can be recovered using the solution to

$$
\begin{bmatrix}
1+\frac{\sum_{j\in J^{NC}}\mu_j p_L(1-\gamma)\frac{(E(\theta)-\theta_L)}{1-\theta_L}\beta}{1-\beta p_H\frac{(E(\theta)-\theta_L)}{1-\theta_L}} & -\sum_{j\in J^C}\mu_j\beta\left(\gamma+(1-\gamma)\frac{(E(\theta)-\theta_L)}{1-\theta_L}\frac{\left(1-\beta p_H\frac{(E(\theta)-\theta_L)}{1-\theta_L}\sum_{j\in J^{NC}}\mu_j\right)}{1-\beta p_H\frac{(E(\theta)-\theta_L)}{1-\theta_L}}\right) \\
\frac{(1-\gamma)\frac{(E(\theta)-\theta_L)}{1-\theta_L}\beta p_L}{1-\beta p_H\frac{(E(\theta)-\theta_L)}{1-\theta_L}} & 1-\beta\left(\gamma+(1-\gamma)\frac{(E(\theta)-\theta_L)}{1-\theta_L}\frac{\left(1-\beta p_H\frac{(E(\theta)-\theta_L)}{1-\theta_L}\sum_{j\in J^{NC}}\mu_j\right)}{1-\beta p_H\frac{(E(\theta)-\theta_L)}{1-\theta_L}}\right)
\end{bmatrix}
\begin{bmatrix}
\sum_{j\in J^C}\mu_j c_L\left(d_L^j\right) \\
c_L\left(\bar{d}\right)
\end{bmatrix}
$$

$$
=\begin{bmatrix}
\left((1-\gamma)\frac{(E(\theta)-\theta_L)}{1-\theta_L}+\gamma\right)\sum_{j\in J^C}\mu_j d_L^j+\frac{\sum_{j\in J^C}\mu_j(1-\gamma)\left(\frac{(E(\theta)-\theta_L)}{1-\theta_L}\right)^2\beta p_H\sum_{j\in J^C}\mu_j d_H^j}{1-\beta p_H\frac{(E(\theta)-\theta_L)}{1-\theta_L}} \\
\left((1-\gamma)\frac{(E(\theta)-\theta_L)}{1-\theta_L}+\gamma\right)\bar{d}+\frac{(1-\gamma)\left(\frac{(E(\theta)-\theta_L)}{1-\theta_L}\right)^2\beta p_H\sum_{j\in J^C}\mu_j d_H^j}{1-\beta p_H\frac{(E(\theta)-\theta_L)}{1-\theta_L}}
\end{bmatrix}
$$

The guess is correct, the coefficients are affine in the dividend level.

8.5 Changes in correlation when there are multiple types of loans and everyone uses collateral

Proposition 10 An increase in π has the following effects.

- If $\mu_1 = \mu_2$, the economy's average debt capacity remains unchanged, type 1 banks collateral constraint is relaxed and type 2 banks collateral constraint is tightened.
- If $\mu_1 > \mu_2$, the economy's average debt capacity increases, and type 1 banks collateral constraint is relaxed.
- If $\mu_1 < \mu_2$, the economy's average debt capacity decreases, and type 2 banks collateral constraint is tightened.

Proof. From the characterization of the value function coefficients in the appendix, for all $j = 1, 2$,

$$
c_B^j(d)=\frac{(E(\theta)-\theta_L)}{1-\theta_L}\left[d+\beta\frac{p_L c_L\left(d_L^j\right)+p_H\frac{(E(\theta)-\theta_L)}{1-\theta_L}d_H^j}{1-\frac{(E(\theta)-\theta_L)}{1-\theta_L}\beta p_H}\right]
$$

and

$$
\begin{aligned}
(1-\gamma\beta)\,c_L(d) &= (1-\gamma)\,\beta\frac{(E(\theta)-\theta_L)}{1-\theta_L}\frac{p_L c_L\left(\sum_{j\in J}\mu_j d_L^j\right)+p_H\frac{(E(\theta)-\theta_L)}{1-\theta_L}\sum_{j\in J}\mu_j d_H^j}{1-\frac{(E(\theta)-\theta_L)}{1-\theta_L}\beta p_H} \\
&\quad+\left((1-\gamma)\frac{(E(\theta)-\theta_L)}{1-\theta_L}+\gamma\right)\left((1-\gamma\beta)\,d+\gamma\beta\bar{d}\right),
\end{aligned}
$$

where

$$
\begin{aligned}
&\left((1-\gamma\beta)\left(1-\beta p_H\frac{(E(\theta)-\theta_L)}{1-\theta_L}\right)-(1-\gamma)\frac{(E(\theta)-\theta_L)}{1-\theta_L}\beta p_L\right)c_L\left(\sum_{j\in J}\mu_j d_L^j\right) \\
=&\left(\left(1-\frac{(E(\theta)-\theta_L)}{1-\theta_L}\beta p_H\right)\left((1-\gamma)\frac{(E(\theta)-\theta_L)}{1-\theta_L}+\gamma\right)\gamma+(1-\gamma)\left(\frac{(E(\theta)-\theta_L)}{1-\theta_L}\right)^2\right)\beta\bar{d} \\
&+\left(\left(1-\frac{(E(\theta)-\theta_L)}{1-\theta_L}\beta p_H\right)\left((1-\gamma)\frac{(E(\theta)-\theta_L)}{1-\theta_L}+\gamma\right)(1-\gamma\beta)-(1-\gamma)\,p_L\left(\frac{(E(\theta)-\theta_L)}{1-\theta_L}\right)^2\beta\right)\sum_{j\in J}\mu_j d_L^j.
\end{aligned}
$$

The debt capacity of the asset for a bank of type j is given by

$$D^j = \beta \left(p_H c_B \left(d_H^j \right) + p_L c_L \left(d_L^j \right) \right).$$

$$\begin{aligned}
\frac{\partial D^j}{\partial \pi} &= \beta \frac{\partial \left(p_H c_B \left(d_H^j \right) + p_L c_L \left(d_L^j \right) \right)}{\partial \pi} \\
&\propto \left(p_H \frac{(E(\theta) - \theta_L)}{1 - \theta_L} \frac{\partial d_H^j}{\partial \pi} + p_L \frac{\partial c_L \left(d_L^j \right)}{\partial \pi} \right) \\
&\propto \left(\frac{(E(\theta) - \theta_L)}{1 - \theta_L} (-1)^{j-1} \frac{d_1 - d_2}{2} + p_L \frac{\partial c_L \left(d_L^j \right)}{\partial \pi} \right)
\end{aligned}$$

But,

$$(1 - \gamma\beta) c_L(d) = (1 - \gamma) \frac{(E(\theta) - \theta_L)}{1 - \theta_L} \bar{D} + \left((1 - \gamma) \frac{(E(\theta) - \theta_L)}{1 - \theta_L} + \gamma \right) \left((1 - \gamma\beta) d + \gamma\beta\bar{d} \right),$$

where $\bar{D} = \beta \frac{\left(p_L c_L \left(\sum_{j \in J} \mu_j d_L^j \right) + p_H \frac{(E(\theta) - \theta_L)}{1 - \theta_L} \sum_{j \in J} \mu_j d_H^j \right)}{1 - \frac{(E(\theta) - \theta_L)}{1 - \theta_L} \beta p_H}$ is the economy's aggregate debt capacity. Then,

$$\frac{\partial D^j}{\partial \pi} \propto (-1)^{j-1} \frac{d_1 - d_2}{2} (1 - \gamma) \frac{(E(\theta) - 1)}{1 - \theta_L} + p_L \frac{(1 - \gamma)}{(1 - \gamma\beta)} \frac{(E(\theta) - \theta_L)}{1 - \theta_L} \frac{\partial \bar{D}}{\partial \pi}.$$

But

$$\begin{aligned}
\frac{\partial \bar{D}}{\partial \pi} &= \frac{\beta p_L}{1 - \frac{(E(\theta) - \theta_L)}{1 - \theta_L} \beta p_H} \frac{\partial \left(\sum_{j \in J} \mu_j d_L^j \right)}{\partial \pi} \left(\frac{\partial c_L \left(\sum_{j \in J} \mu_j d_L^j \right)}{\partial \sum_{j \in J} \mu_j d_L^j} - \frac{(E(\theta) - \theta_L)}{1 - \theta_L} \right) \\
&= -\frac{\beta p_L}{1 - \frac{(E(\theta) - \theta_L)}{1 - \theta_L} \beta p_H} \frac{(\mu_1 - \mu_2)(d_1 - d_2)}{2 p_L} \left(\frac{\left(1 - \frac{(E(\theta) - \theta_L)}{1 - \theta_L} \beta p_H \right) \left((1 - \gamma) \frac{(E(\theta) - \theta_L)}{1 - \theta_L} + \gamma \right) (1 - \gamma\beta) - (1 - \gamma) p_L \left(\frac{(E(\theta) - \theta_L)}{1 - \theta_L} \right)^2 \beta \cdots}{\left((1 - \gamma\beta) \left(1 - \beta p_H \frac{(E(\theta) - \theta_L)}{1 - \theta_L} \right) - (1 - \gamma) \frac{(E(\theta) - \theta_L)}{1 - \theta_L} \beta p_L \right)} \right. \\
&= \beta \frac{(\mu_1 - \mu_2)(d_1 - d_2)}{2} \frac{\gamma (1 - \gamma\beta) \frac{(E(\theta) - 1)}{1 - \theta_L}}{\left((1 - \gamma\beta) \left(1 - \beta p_H \frac{(E(\theta) - \theta_L)}{1 - \theta_L} \right) - (1 - \gamma) \frac{(E(\theta) - \theta_L)}{1 - \theta_L} \beta p_L \right)}.
\end{aligned}$$

An increase in π will have no effect whatsoever on the aggregate debt capacity if $\mu_1 = \mu_2$, It will increase it if $\mu_1 > \mu_2$, and it will decrease it if $\mu_2 > \mu_1$.

Then,

$$\frac{\partial D^j}{\partial \pi} \propto \frac{(E(\theta) - 1)}{1 - \theta_L} \frac{d_1 - d_2}{2} (1 - \gamma) \left[(-1)^{j-1} + \frac{\gamma p_L \frac{(E(\theta) - \theta_L)}{1 - \theta_L} \beta \frac{(\mu_1 - \mu_2)}{2}}{\left((1 - \gamma\beta) \left(1 - \beta p_H \frac{(E(\theta) - \theta_L)}{1 - \theta_L} \right) - (1 - \gamma) \frac{(E(\theta) - \theta_L)}{1 - \theta_L} \beta p_L \right)} \right].$$

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