Optimal Self-Enforcing and Termination

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Abstract

We study a dynamic principal-agent relationship in which the agent receives a stochastic outside opportunity/offer each period and he cannot commit to not leaving the ongoing relationship. Termination, while costly, allows the principal to go to an external market to hire a new agent. We treat self-enforcing as a choice variable by letting the principal respond strategically to the agent's outside offers. Starting initially from a sufficiently low expected utility of the agent (so the commitment constraint is binding, initially), the continuation of the optimal contract converges to Burdett (1978) where each period the agent quits whenever his outside offer is above the utility the current principal offers, and he stays to receive the same constant expected utility otherwise. On the path of convergence, termination occurs whenever the agent's outside offer exceeds a constant ceiling, and the principal acts to match the agent's outside offer if it is below that constant ceiling but better than his current promised utility. The convergence is monotonic. Conditional on continuation, over time the agent's expected utility converges monotonically to its limiting level while the commitment constraint binds monotonically less; and the limiting utility of the agent is the lowest level of his expected utility at which the commitment constraint is not binding.

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1 Introduction

Models of dynamic contracts with limited commitment (e.g., Thomas and Worrall, 1988; Phelan, 1995; Kocherlakota, 1996) study problems of optimal contracting in environments where one or more contracting parties of the relationship are exposed to outside opportunities that generate temptations for them to leave the ongoing relationship. The solution concept the existing models all use is to impose universal self-enforcing: the contract must be designed to be such that in all ex post states of the world the agents never have incentives to leave the ongoing relationship which, therefore, lasts forever.

In this paper, we study a model that has in essence the same economic tension that the models in the above literature possess, but we approach the problem differently. Instead of imposing universally a self-enforcing constraint in each and every state of the world that the contract may dictate, we let the contract choose, optimally, whether or not to impose such a constraint in each individual state of the world. In other words, we make self-enforcing endogenous, by choosing optimally a subset of the states of the world to impose the self-enforcing constraint, while letting the contract terminate (letting the agent pursue the outside opportunity he receives) in the states outside this subset.

Specifically, we study a problem of dynamic contracting between a risk neutral principal and a risk averse agent where each period a stochastic outside opportunity (offer), drawn independently from a given distribution, arises for the agent. There is less than full commitment: the agent in each period is free to walk away from the ongoing contract to pursue the outside offer he receives. Thus, upon the realization of any specific outside opportunity, the contract must decide whether the agent should take it, in which case the current relationship must be terminated; or not to take it, in which case the current relationship continues, but the continuation of the contract must give the agent a value higher than his current outside offer – the contract must be self-enforcing. Termination is costly: there is a fixed cost that the principal must incur each time termination occurs. Upon any termination, the agent leaves to pursue the outside opportunity he receives, and the principal then goes back to an external market to find and start a new relationship with a new agent.

Suppose the cost of termination is infinitely high and so termination is never optimal. Then in order to enforce continuation – to cope with the problem of the lack of commitment – the optimal contract must match the agent's outside offer whenever it exceeds his current promised utility, pushing up monotonically the agent's expected utility over time. This results in an efficiency loss, as optimality under full commitment would dictate perfect intertemporal consumption smoothing and a time-invariant expected utility for the agent.

When the cost of termination is finite, termination not only allows the principal to utilize the outside offers as an external means for compensating the agent, but can also be used as a device for mitigating the effects of the lack of commitment. Specifically, in any state where the lack of commitment may be a binding constraint, instead of imposing self-enforcing in that state (matching the agent's outside offer to induce him to stay), the contract could choose to terminate the agent. Termination undoes the effects of the lack of commitment, allowing the principal to avoid the cost of self-enforcing in that state. It then follows that higher outside offers generate stronger incentives for termination, since the agent receiving a higher outside offer must be promised a higher utility in order to induce him to stay. Higher outside offers also make termination more worthwhile because of the higher values the agent receives upon termination. With the optimal contract then, sufficiently high outside offers always induce termination. It also follows that a higher expected utility for the agent dictates less termination. This is because the problem of the lack of commitment diminishes as the agent's expected utility increases, generating weaker demand for using termination as a device for alleviating the effects of the lack of commitment. Higher expected utilities for the agent require higher utilities be promised to the agent in all states of his outside offers, relaxing the self-enforcing constraint.

With the optmal contract, a binding self-enforing constraint is imposed in a state of the agent's outside offer (the principal matches the agent's outside offer) if and only if the offer is neither sufficiently high to warrant termination, nor sufficiently low (below the agent's current expected utility) to make retention voluntary under full commitment. Since the problem of the lack of commitment diminishes as the agent's expected utility increases, the size of the set of the agent's outside offers where self-enforing is imposed is diminishing in the agent's expected utility. Moreover, for sufficiently high levels of the agent's expected utility, self-enforcing is not imposed in any of the agent's outside offers – the principal never matches the outside offers of an agent with a sufficiently high expected utility. For the agent with a sufficiently high expected utility, termination is sufficient for taking care of the problem of the lack of commitment.

The model generates interesting dynamics. With the optimal contract, starting initially anywhere in the domain of the agent's expected utility the contract can achieve, the continuation of the optimal contract converges stochastically but monotonically to a stationary dynamic contract. This stationary dynamic contract consists of a sequence of static contracts where the agent quits whenever his outside offer is above the utility the current principal offers, and he stays to receive the same constant expected utility otherwise. The convergence occurs immediately if the agent starts with a sufficiently high promised utility. Suppose not. Then each time the agent receives an outside offer above his current promised utility but below a constant upper bound, the principal matches the agent's outside offer and the agent stays with the principal; otherwise he is either let go to pursue an outside offer that is strictly better than the current contract, or his outside offer is disregarded by the principal. Thus, conditional on that the agent stays with the principal, over time his expected utility increases and converges stochastically to a constant upper bound. On the path of convergence, higher and higher outside offers are matched, with lower and lower probabilities. In the limit, that is, at the upper bound, the optimal contract is such that the agent stays whenever his outside offer is below his constant value of continuation with the current contract, and quits otherwise. There, the principal never reacts to the agent's outside offers, and the expected utility for the agent is the lowest at which termination is sufficient for handling the problem of the lack of commitment and self-enforcing need no longer be imposed.

Because termination may help mitigate the problem of the lack of commitment, a lower termination cost, which facilitates more efficient use of termination as a device for mitigating limited commitment, should give rise to less efficiency losses from the lack of commitment. Specifically, compared to the case where termination is infinitely costly, the optimal contract in the case where termination is feasible at a finite cost dictates a smaller set of the states in which a binding self-enforcing constraint must be imposed for optimality. Moreover, the size of this set shrinks monotonically and converges to zero as the cost of termination decreases and converges to zero.

One interpretation of our model is to view it as a model of on-the-job search. The job is a dynamic contract between a firm and a worker. The search is a sequence of stochastic draws of outside opportunities that the employed worker receives. The worker is risk averse. The worker is free to walk away from the firm to pursue any outside offer he receives - limited commitment. Termination is feasible but costly to the firm. After terminating an incumbent worker, the firm goes back to an external labor market to hire a new worker.

Most existing models of on-the-job search do not model how the principal responds optimally to the agent's current outside offers. In Burdett (1978), for example, the employment contract is governed by a fixed wage while on-the-job search is a sequence of costly random draws from a given distribution of outside wage offers. Separation (voluntary quit) occurs whenever the worker receives an outside offer that is better than the fixed wage. The more recent work of Burdett and Coles (2003) studies an equilibrium labor market where labor contracts are designed efficiently with respect to the worker's tenure with the firm, but not to his outside opportunities. On-the-job search then implies it is optimal for the principal to backload wages over the agent's tenure. Backloaded wages increase an agent's value of staying with the principal, reducing his quit probability. Moscarini (2005) analyzes the role of on-the-job search for job separation and equilibrium wage distribution in a Jovanovic (1984) type learning model that is nested in the Mortensen and Pissarides (1994) framework of search and matching. In Moscarini, when an employed agent has a successful contact with an outside vacancy, the incumbent principal and the poaching principal play a sequential auction game to determine a unique outcome for the agent: either the incumbent principal does not respond to the outside offer and the agent quits, or the current principal and the agent disregard the outside offer. There, again, unlike in our model, the principal never matches the agent's outside offers.

We model explicitly the principal's optimal response to the agent's outside offers, in fact the whole history of the agent's outsider offers. This difference in modelling generates a richer set of optimal strategies the principal uses in reacting to the agent's outside offers. Specifically, the optimal contract permits three and only three ways in which the principal responds to the agent's outside offer. One, the principal does not respond to the agent's outside offer (offering him the same continuation contract) but the agent chooses to stay, voluntarily, with the principal. Two, retain the agent by matching his outside offers. Three, terminate the agent by letting him quit voluntarily, or involuntarily. In an involuntary termination, the agent must accept an outside offer that is inferior to his current contract. Existing models of on-the-job search generate only voluntary retentions and voluntary quits.

Section 2 lays out the model. Section 3 studies the optimal contract under the assumption that the cost of termination is infinite and so self-enforcing must be implemented in all states of the world. Section 4 studies the optimal contract for the case where the cost of termination is finite. Section 5 concludes the paper. The proofs of lemmas, theorems and corollaries are in the Appendix.

2 Model

Let t denote time: $t = 1, 2, \cdots$ There is a single perishable consumption good in the model. There is one principal but a competitive supply of agents. The principal's objective is to maximize his expected discounted payoffs. Agents are identical and have the following preferences:

$$\mathbb{E}_\tau \left[\sum_{t=\tau}^{\infty} \beta^{t-\tau} u(c_t) \right],$$

where E_τ denotes the agent's expectation conditional on information available at the beginning of period τ, $\tau \geq 1$; $\beta \in (0,1)$ is the discount factor which is shared by the principal and the agent; c_t and $u(c_t)$ denote, respectively, the agent's consumption and utility in period t. Assume $c_t \in \mathbb{R}_+$ for all t. That is, consumption must be non-negative.[1] Last, assume the utility function

[1] As will be clear later, what is important is that the agent's consumption is bounded from below, but not specifically by zero.

u is bounded, strictly increasing, strictly concave, twice differentiable, and satisfies the Inada conditions.

Each period, the principal could hire an agent to produce. Each agent, when employed, produces a constant output of $\theta \geq 0$. After producing the output, the agent draws a random value ξ from a set Σ of real numbers. This ξ represents the value of an outside opportunity for the agent, Σ being the set of all such values. Specifically, the agent's expected utility would be ξ if he quits his current job to take this outside offer. Assume

$$\Sigma = [V_{\min}, V_{\max}) \text{ where } V_{\min} \equiv \frac{u(0)}{1-\beta} \text{ and } V_{\max} \equiv \frac{u(\infty)}{1-\beta}.$$

Here, $\xi = V_{\min}$ is interpreted as the expected utility of the agent in the state in which if he leaves his current job, then he will enter a state of permanent unemployment where his consumption is constantly zero; and $V_{\max}$ is the agent's expected utility in the state in which he is entitled to an infinite amount of consumption each period. Last, let $F : \Sigma \to [0, 1]$ denote the stationary distribution function for the random outside offer ξ the employed agent receives each period.[2] The agent's outside offers are perishable - there is no recall of past offers.

The principal and a newly hired agent can enter into a contract that is fully dynamic. Termination is feasible, but costly. There is a fixed cost $C_0 (\geq 0)$ that the principal must incur for terminating and replacing any incumbent agent. Upon termination, the agent leaves the principal to pursue the outside offer he receives and the principal then goes to the external market to hire a new agent. The new agent's reservation utility is $V_0 \in \Sigma$.

We make the following assumptions on contracting. First, we assume limited liability on the agent's part: compensation to the agent must be non-negative. In other words, it is only feasible that the principal pays the agent (a non-negative amount of compensation), the agent cannot pay the principal. Second, we assume limited commitment on the agent's part: each period, the agent is free to walk away from the contract, before and after receiving his outside offer. This implies that, if the principal wants to retain an agent who has an outside value of ξ, then the continuation of the contract must promise the agent expected utility of at least ξ. The principal, on the other hand, is able to fully commit. Specifically, the principal can commit to the terms of the continuation of the contract in any ex post state of the world.

[2]This specification of Σ, which imposes that outside offers are bounded from below by $V_{\min}$, is without loss of generality. Obviously, if an agent is free to receive an outsider offer each period, he must also have the option to quit his current job and then stay permanently out of the labor force. This implies that any outside offer below $V_{\min}$ need not be considered. Note also that we do not impose any restrictions on the function F. This allows for example the realizations of ξ to be actually taken from a subset of Σ.

To close this section, as a benchmark for the analysis in the sections to follow, consider the case where either the agent never receives any outside offers or the agent can fully commit and the cost of termination is infinite so termination never occurs at the optimum. Then with the optimal contract the agent's compensation and expected utility stay constant over time at its initial level:

$$u(c_t) = (1-\beta)V, \quad \forall t, \tag{1}$$

where V is the agent's expected utility at the start of the contract. In other words, the agent's consumption is perfectly smooth across current states of the world and over time.

3 The Case $C_0 = \infty$

Suppose it is physically infeasible for the principal to terminate and replace any agent once he is employed (the cost of termination is infinitely high). As discussed earlier, this assumption is made implicitly in most of the existing models of dynamic contracting with limited commitment (e.g., Thomas and Worrall, 1988; Phelan, 1995; Kocherlakota, 1996).

Under the assumption of $C_0 = \infty$, following the tradition of Green (1987) and Spear and Srivastava (1987), a contract between the principal and the agent, written recursively, takes the form of

$$\sigma \equiv \{c(V), V_r(\xi; V) : \ \xi \in \Sigma \text{ and } V \in \Omega\},$$

where V denotes the expected utility of the agent the contract promises to deliver at the beginning of a period - the "state" variable that summarizes the history of what happened up to that period, and Ω is the space for V to take values from - the state space. More precisely, Ω is the set of all expected utilities that a feasible contract in this environment (the exact meaning of this to be specified shortly) can deliver to the agent, at the beginning of any period. Clearly, this state space is an endogenous variable of the model. Then, for any given $V \in \Omega$, $c(V)$ is the agent's consumption in that period, $V_r(\xi; V)$ is the agent's expected utility next period that the contract promises to deliver, conditional on the agent's current outside offer ξ, and on V.

Given the definition of V and the model's assumptions, σ is required to satisfy the following constraints:

$$u(c(V)) + \beta \int_{\Sigma} V_r(\xi; V) dF(\xi) = V, \, \forall V \in \Omega, \tag{2}$$

$$c(V) \geq 0, \, \forall V \in \Omega, \tag{3}$$

$$V_r(\xi; V) \geq \xi, \, \forall \xi \in \Sigma, \, \forall V \in \Omega, \tag{4}$$

$$V_r(\xi; V) \in \Omega, \forall \xi \in \Sigma, \forall V \in \Omega, \tag{5}$$

where the state space Ω is the largest set of V that is consistent with the above constraints, or more precisely, Ω is the largest set of V for which there exists a contract that satisfy (2)-(5), for the same Ω.[3]

In the above, (2) requires that the choice of $c(V)$ and $V_r(\cdot; V)$ be consistent with the definition of V. Equation (3) requires compensation be non-negative - the limited liability constraint. Equation (4) is the self-enforcing constraint that requires that, upon the realization of any of the agent's possible outside offers, the contract promises to give the agent more so he would choose to stay with the principal. Finally, equation (5) says that the utility that the principal promises must be deliverable.

Lemma 1 $\Omega = [\overline{V}, V_{\max})$ *where* $\overline{V}$ *solves*

$$u(0) + \beta \left(F(\overline{V})\overline{V} + \int_{\overline{V}}^{V_{\max}} \xi dF(\xi) \right) = \overline{V}. \tag{6}$$

Lemma 1 says that in order for any V to be deliverable, it must be sufficiently high: higher than or equal to $\overline{V}$. Equation (6) gives a Bellman equation for $\overline{V}$, it says that $\overline{V}$ can be delivered in the following way: give the agent zero consumption in the first period, promises him $\overline{V}$ for the next period if his current period outside offer is below $\overline{V}$, match the agent's outside offer if it is above $\overline{V}$. It is straightforward to show that (6) has a unique solution in the interval $(V_{\min}, V_{\max})$.

Notice quickly that if $V_0 \leq \overline{V}$, then the principal would not need to pay any cost of compensation throughout his life and would then attain a value equal to $\overline{U} \equiv \theta/(1-\beta)$.

Now for any $V \in \Omega$, let $U^*_{NT}(V)$ denote the maximum value for the principal that can be attained by a contract satisfying (2)-(5). Next, let $B(\Omega)$ denote the space of real-valued functions on Ω that are bounded from above by $\overline{U}$, endowed with the topology of pointwise convergence. Then $U^*_{NT} \in B(\Omega)$ must satisfy the following Bellman equation:

$$U^*_{NT} = \Lambda U^*_{NT}, \tag{7}$$

where $\Lambda : B(\Omega) \to B(\Omega)$ is a mapping defined as follows: for all $U \in B(\Omega)$ and all $V \in \Omega$,

$$\Lambda U(V) \equiv \max_{c, V_r(\xi):\xi \in \Sigma} \left\{ (\theta - c) + \beta \int_{\Sigma} U(V_r(\xi)) dF(\xi) \right\}$$

[3] This gives Ω a recursive definition, in a fixed point argument. In the language of APS (1990), Ω must be the largest self-generating set with respect to (2)-(5). For this, see Wang (1995).

subject to

$$u(c) + \beta \int_{\Sigma} V_r(\xi) dF(\xi) = V, \tag{8}$$

$$c \geq 0, \tag{9}$$

$$V_r(\xi) \geq \xi, \ \forall \xi \in \Sigma, \tag{10}$$

$$V_r(\xi) \in \Omega, \ \forall \xi \in \Sigma. \tag{11}$$

Theorem 1 *(i) The principal's value function U^*_{NT} is strictly decreasing, strictly concave, and can be attained asymptotically by applying the mapping Λ repeatedly on $\overline{U}$. (ii) The optimal contract has: for all $V \in \Omega$,*

$$V_r(\xi, V) = \max\{\xi, V\}, \ \forall \xi. \tag{12}$$

The key part of the proof of Theorem 1 is to show that the value function U^*_{NT} is concave, which is achieved by constructing a sequence of concave functions that converges to it. Theorem 1 says that, with the optimal contract, either the principal ignores the agent's outside offer - when it is below his current promised utility, or the principal matches the agent's outside offer to retain him - when it exceeds his current promised utility. The logic behind this is simple. (a) If the self-enforcing constraint (10) is not binding, then $V_r(\xi) = V$ - the first best intertemporal smoothing at (V, ξ) is attained. (b) If (10) is binding, then of course $V_r(\xi) = \xi$. (c) (10) binds if and only if $\xi \geq V$.

Figure 1 depicts the optimal contract in a two dimensional graph where the horizontal axis represents V and the vertical axis ξ. By Theorem 1, in any state (V, ξ) in the area below the 45 degree line (and with $V \geq \overline{V}$), the self-enforcing constraint is not binding and the agent is given a promised utility equal to his current utility V; in the states above the 45 degree line, the self-enforcing constraint is binding and the optimal contract specifies $V_r(\xi) = \xi$.

Figure 1 gives an obvious sense that the commitment problem diminishes in the agent's expected utility V, and increases in his current outside offer ξ. Specifically, for any given ξ, the self-enforcing constraint is binding if and only if V is sufficiently low (below ξ); and for any given V, the self-enforcing constraint is binding if and only if ξ is sufficiently high (above V). This is intuitive. A higher ξ requires a higher promised utility to be given to the agent in order to induce him to stay (it increases the value of the right hand side of the constraint (10)). A higher

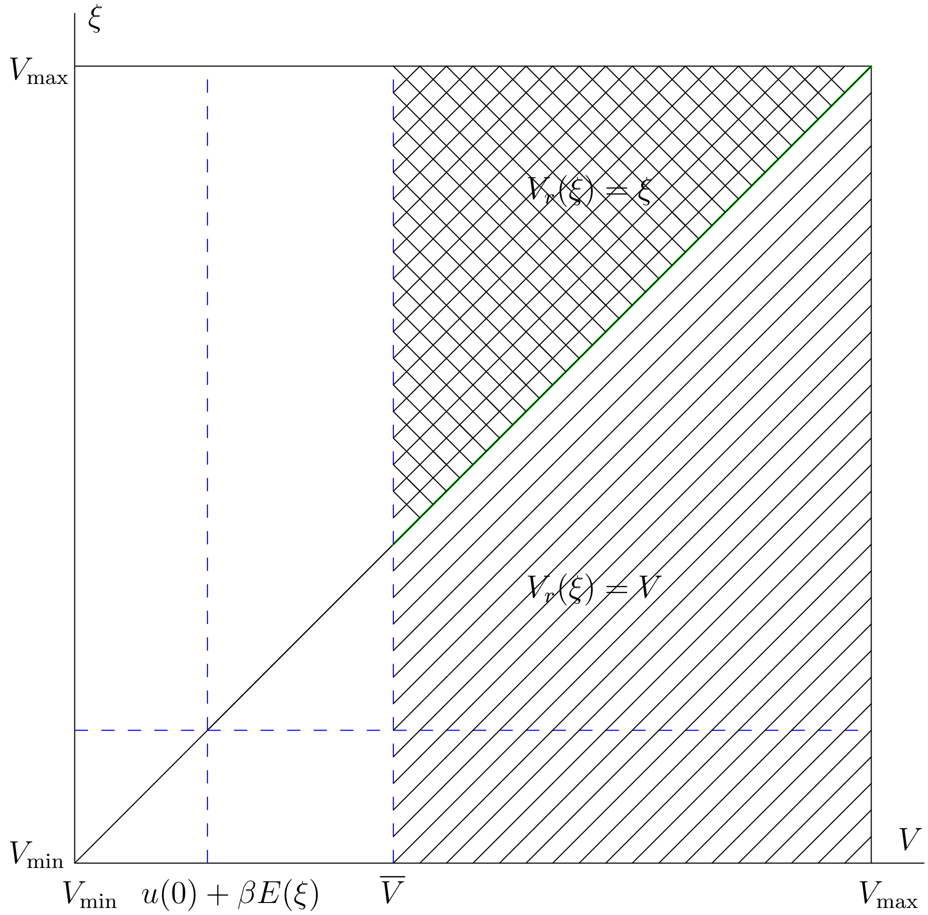

Figure 1: The Optimal Contract with $C_0 = \infty$

V dictates higher utilities be promised to the agent in all states of the agent's outside offer ξ, relaxing the self-enforcing constraint.

Given the above, the dynamics the optimal contract generates should over time push the agent up, from lower to higher Vs, so as to reduce the problem of the lack of commitment monotonically over time. In fact this is what the model predicts. Over time, higher and higher outside offers are matched to push the agent's expected utility up monotonically to higher and higher levels. Meanwhile, over time, the probability the principal matches the agent's outside offer gets lower and lower and eventually converges to zero as the problem of the lack of commitment become less and less severe and eventually diminishes to null.

Corollary 1 *The agent's compensation is back-loaded with the optimal contract: for all V, $u(c(V)) < (1-\beta)V$.*

The agent's compensation is backloaded because his utility is backloaded. His utility is back-loaded because over time the self-enforcing constraint forces the principal to match higher and higher outside offers of the agent so that he has the incentives to stay. Back-loading as a means for providing incentives is an old idea in economics. In two of the most related examples, Edwards (1981) and Burdett and Coles (2003), back-loading in compensation builds a collateral against worker quitting. So far in the literature though, back-loading has not been modeled as an outcome of the principal's optimal response to the agent's outside offers. In Burdett and Coles (2003) for example, wage increases optimally with the worker's tenure but is not directly contingent on his outside offers.

4 The Case $C_0 < \infty$

When the cost of termination is finite, termination maybe optimal. There are three motives for termination in our model. (a) Any outside offer represents an opportunity the principal could use as an external means for fulfilling his promise to the agent, and termination of the current relationship makes this possible. (b) Terminating the current relationship may allow the principal to start a new relationship with a less expensive agent. (c) Termination can be used as a means for mitigating the problem of the lack of commitment - as discussed in the introduction of the paper. In this section, we study how these motives for termination work jointly to determine the dynamics of the optimal contract.

Use the agent's beginning-of-period expected utility, denoted V, as a state variable, a dynamic contract, defined recursively, is

$$\sigma \equiv \{c(V), I(\xi; V), V_r(\xi; V) : \xi \in \Sigma \text{ and } V \in \Phi\},$$

where $c(V)$ is the agent's compensation in the current period; $I(\xi; V)$ is the probability with which the agent is retained conditional on his current outside offer being ξ; $V_r(\xi; V)$ is the agent's expected utility next period if his current outside offer is ξ and he is retained; and finally, Φ is the space for V - the set of all expected utilities that the contract can attain - which is also an endogenous variable of the model.

A contract σ is feasible if it satisfies the following constraints:

$$u(c(V)) + \beta \int_{\Sigma} [I(\xi; V)V_r(\xi; V) + (1 - I(\xi; V))\xi] dF(\xi) = V, \ \forall V \in \Phi, \tag{13}$$

$$c(V) \geq 0, \ \forall V \in \Phi, \tag{14}$$

$$I(\xi; V)(1 - I(\xi; V)) \geq 0, \ \forall \xi \in \Sigma, \ \forall V \in \Phi, \tag{15}$$

$$V_r(\xi; V) \geq \xi, \ \forall \xi \in \Sigma, \ \forall V \in \Phi, \tag{16}$$

$$V_r(\xi; V) \in \Phi, \ \forall \xi \in \Sigma, \ \forall V \in \Phi, \tag{17}$$

where Φ is the largest set of the Vs that is consistent with the above constraints (the largest self-generating set with respect to the constraints).

In the above, equation (13) is the promise-keeping constraint that requires that the choices of the current variables be consistent with the definition of V; Equation (14) is the limited liability constraint; Equation (15) holds if and only if $I(\xi; V) \in [0, 1]$, for all $\xi \in \Sigma$ and $V \in \Phi$.[4][5] Equation (16) is the self-enforcing constraint which says that if the principal retains the agent, then the agent must be given no less than ξ. Equation (17) requires that if the agent stays, then he must be given a promised utility that the contract can deliver.

Notice that the self-enforcing constraint (16) and the admissibility constraint (17) are imposed only in the states of retention. This immediately gives a sense that (weakly) less self-enforcing are imposed if the contract prescribes more termination. Remember in the case of $C_0 = \infty$, since termination is not feasible, (16) and (17) must imposed in all states of the world.

Lemma 2 $\Phi = [u(0) + \beta E(\xi), V_{\max})$.

So, relative to the case where the cost of termination is infinite, termination expands the set of expected utilities deliverable, allowing the contract to attain any level of expected utility below $\overline{V}$ but above $u(0) + \beta E(\xi)$. To understand this, compare what happens in the states of termination and retention. Suppose termination occurs at an arbitrary ξ. Then the agent receives ξ, leaves, his relationship with the current principal ends. Suppose the agent is retained. Then he must be given an expected utility that's not only better than ξ but also consistent with the fact that retention gives him opportunities to receive more outside offers which he can potentially use for higher utilities. This immediately implies that the contract must promise weakly more utilities to the agent in the state of retention than termination. And this should especially be so when ξ is low. In fact, as we will see, this very logic could dictate that the optimal contract prescribes termination in a lower rather than a higher state of ξ when the problem of the lack of commitment is sufficiently severe.[6]

[4] To formulate $I(\xi; V) \in [0, 1]$ as (15) has a technical advantage: it reduces the number of Lagrangian multipliers by one in the analysis to follow.

[5] Note that randomized termination is allowed under (15). In the case of deterministic termination where $I(\xi)$ must be either 0 or 1, we need only replace (15) with $I(\xi)(1 - I(\xi)) = 0$, $\forall \xi \in \Sigma$.

[6] Note that termination allows a lower promised utility to be attained *not* because it allows the contract to avoid matching the more expensive outside offers.

For all $V \in [u(0) + \beta E(\xi), V_{\max})$, let $U^*(V)$ denote the principal's value conditional on V. Call $U^* : [u(0) + \beta E(\xi), V_{\max}) \to \mathbb{R}$ the principal's value function. This function must satisfy the following Bellman equation:

$$U^* = \Gamma U^*,$$

where Γ is defined as follows: for all $U \in B(\Phi)$ and all $V \in \Phi$,

$$\Gamma U(V) \equiv \max_{c, I(\xi), V_r(\xi): \xi \in \Sigma} \left\{ (\theta - c) + \beta \int_{\Sigma} [I(\xi) U(V_r(\xi)) + (1 - I(\xi))(\rho U_0(U) - C_0)] dF(\xi) \right\} \tag{18}$$

subject to

$$u(c) + \beta \int_{\Sigma} [I(\xi) V_r(\xi) + (1 - I(\xi))\xi] dF(\xi) = V, \tag{19}$$

$$c \geq 0, \tag{20}$$

$$I(\xi)(1 - I(\xi)) \geq 0, \, \forall \xi \in \Sigma, \tag{21}$$

$$V_r(\xi) \geq \xi, \, \forall \xi \in \Sigma, \tag{22}$$

$$V_r(\xi) \in \Phi, \, \forall \xi \in \Sigma, \tag{23}$$

where

$$U_0(U) \equiv \max_{\widetilde{V} \in \Phi} \{U(V)\} \text{ s.t. } \widetilde{V} \geq V_0. \tag{24}$$

Above, (22) is the self-enforcing constraint, (23) is the deliverability constraint requiring that the agent's promised utility be deliverable. Equation (24) says that after the termination the principal would go back to the external market to hire a new agent who will be given an expected utility $\widetilde{V}$ that maximizes the principal's value, subject to the condition that this expected utility is above the agent's reservation V_0 and is deliverable with a feasible contract ($\widetilde{V} \in \Phi$).

Theorem 2 *(i) The principal's value function U^* is concave, attains its maximum at $V = \overline{V}$, and can be attained asymptotically by applying the mapping Γ repeatedly on $\overline{U}$.*

(ii) The following contract is optimal: for all $V \in \Phi$, there exist $\overline{\xi}_1 \in [V_{\min}, \overline{V}]$ and $\overline{\xi}_2 \in [\overline{V}, V_{\max}]$ ($\overline{\xi}_1$ and $\overline{\xi}_2$ depending on V) such that

$$I(\xi) = \begin{cases} 1 & , \text{for } \xi \in (\overline{\xi}_1, \overline{\xi}_2) \\ 0 & , \text{for } \xi < \overline{\xi}_1 \text{ and } \xi > \overline{\xi}_2 \end{cases} ; \tag{25}$$

$$V_r(\xi) = \max\{\xi, V\}, \; \forall \xi. \tag{26}$$

(iii) $\bar{\xi}_1 > V_{\min}$ *for all* $V < \overline{V}$*. Moreover,* $\bar{\xi}_1$ *is strictly decreasing in* V *over the interval* $[u(0) + \beta E(\xi), \overline{V}]$*; and* $\bar{\xi}_1 = V_{\min}$ *for all* $V \geq \overline{V}$*.*

(iv) There exists $V'' \geq \overline{V}$ *with* $U^*(V'') = \rho U_0(U^*) - C_0$ *such that* $\bar{\xi}_2 = V''$ *for all* $V \leq V''$ *and* $\bar{\xi}_2$ *is strictly increasing in* V *for* $V > V''$*.*

(v) With the optimal contract, the principal matches the agent's outside offer to retain him (the self-enforcing constraint is binding) if and only if $V \leq \xi \leq V''$*.*

Thus, with the optimal contract, except at the cutoffs $\bar{\xi}_1$ and $\bar{\xi}_2$, termination is deterministic, not stochastic. And the optimal contract, by Theorem 2, is fully characterized by these two cutoffs, where $\bar{\xi}_1$ is the cutoff level of the agent's outside offer below which he is terminated, $\bar{\xi}_2$ is the cutoff level of his outside offer above which he is terminated. Figure 2 depicts these cutoffs, each as a function of V. The principal's value function U^* is depicted in Figure 3. The principal's value $U^*(V)$ attains its maximum value at $V = \overline{V}$. Remember the principal achieves its maximum value at the same $\overline{V}$ in the case of infinite termination cost.

Theorem 2 states that for all V, termination occurs if and only if the agent's outside offer is sufficiently good (above $\bar{\xi}_2$) or sufficiently bad (below $\bar{\xi}_1$). A larger ξ makes termination more worthwhile because accepting it allows the agent to catch a better value which, in turn, implies the principal would save more on the cost of compensating the agent. A larger ξ also makes self-enforcing more costly: in order to induce the agent to stay the principal must now give him more, and this makes termination a more attractive alternative than continuation. These two effects work in the same direction to explain why sufficiently high outside offers induce termination. What is somewhat surprising from Theorem 2, though, is that a sufficiently low ξ may not guarantee retention, noticing, specifically, that for any $V < \overline{V}$, $\bar{\xi}_1 > V_{\min}$, and thus any $\xi \in [V_{\min}, \bar{\xi}_1)$ would trigger termination. If a small ξ does not offer much benefit for the principal to use it as an external means of compensation, why terminating at it? We will answer this question later.

Observe next that $\bar{\xi}_2$ is increasing in the agent's promised utility V. Put differently, termination that occurs because the agent's outside offer is sufficiently high diminishes as V increases. Since a lower V is associated with a more severe commitment problem (remember a lower V in the case of full commitment would dictate lower promised utilities for the agent in the states of retention, and this makes self-enforcing in the case of limited commitment less likely to be consistent with the first best allocation, or the self-enforcing constraint is more likely to be binding), there is higher demand for utilizing termination to avoid a binding self-enforcing constraint, termination then occurs with a higher probability at a lower rather than a higher V.

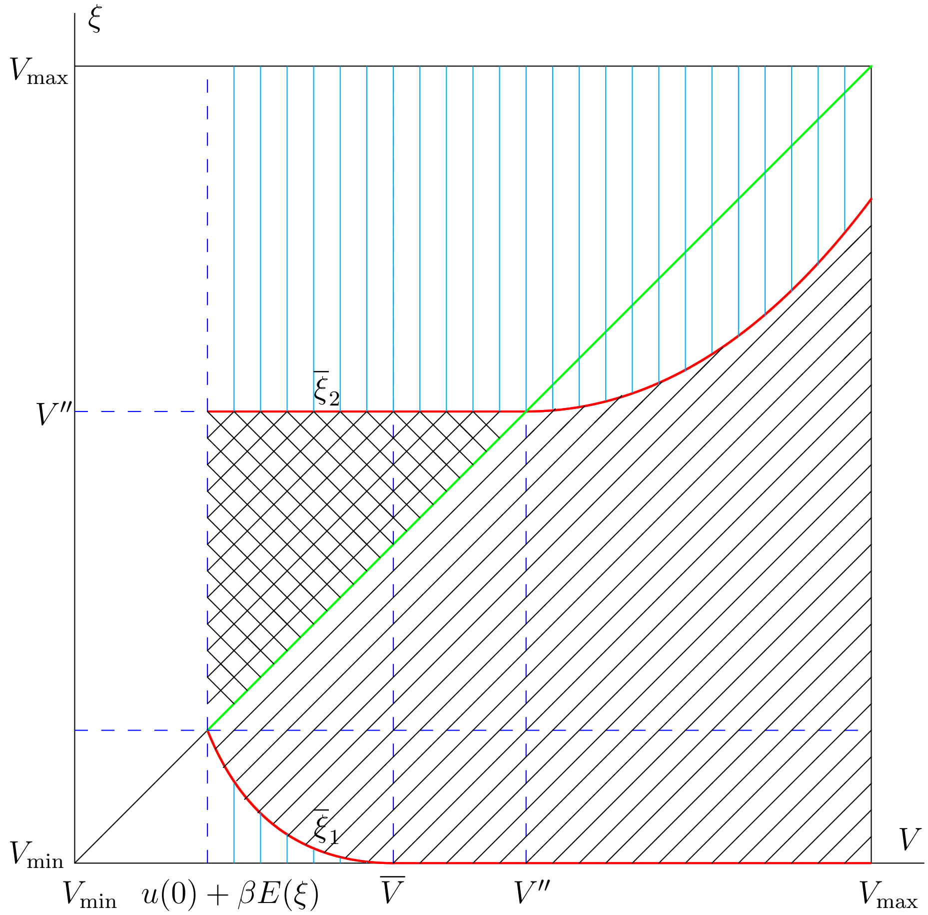

Figure 2: The Optimal Contract with $C_0 < \infty$

As in the case of $C_0 = \infty$, if the agent is retained in a state (ξ, V) with $\xi \leq V$, then the self-enforcing constraint is not binding, first best intertemporal allocation is attained in this state with $V_r(\xi; V) = V$. That is, if the agent is retained upon an outside offer inferior to his current promised utility V, then his promised utility next period will continue to be V. On the other hand, if the agent is retained at a $\xi > V$, the full-commitment optimality condition $V_r(\xi) = V$ must be violated in order to provide incentives for the agent to stay and, as a result, the self-enforcing constraint is binding and the optimal contract dictates that the principal match the agent's outside offer to induce him to stay.

Thus, with the optimal contract, the self-enforcing constraint is imposed and binding in a specific state of the world if and only if the contract finds it optimal for the principal to match the agent's outside offer in that state. This, in turn, occurs if and only if (i) termination is not optimal and (ii) the agent's outside offer exceeds his current promised utility. Obviously, (ii) occurs with a lower probability for a higher V. Figure 2 shows that (i) and (ii) never occur simultaneously for Vs sufficiently high: higher than V''. Put differently, for a "richer" agent (with a higher V), his outside offers are less likely to be matched; and the optimal contract never seeks to match the agent's any outside offer if he is sufficiently rich.

Observe that the principal never offers to match the agent's outside offer beyond the constant upper bound V'': $\overline{\xi}_2$ is flat in V for $V \leq V''$. This is interesting. This implies that the agent's expected utility on the path of retention is capped by V'', provided that he starts the contract below V''. To explain this, for any V, let $\bar{\xi}$ denote the cutoff at which the contract is indifferent between termination and retention (retention achieved by matching the agent's outside offer). Then it must hold that, at the margin between termination and retention, the net gains from the act of switching from retention to termination be zero at $\bar{\xi}$:

$$(\rho U_0(U^*) - C_0) - U^*(\bar{\xi}) = 0, \tag{27}$$

where remember $U_0(U^*)$ is the principal's value with the new agent hired to replace the existing one. Note that at $\bar{\xi}$, the agent is indifferent between being terminated and being retained: he receives the same $\bar{\xi}$, terminated or retained. For the principal, switching from retention to termination gives him a net benefit that consists of two components: one measures the net benefit from switching to a new agent $(\rho U_0(U^*) - U^*(\bar{\xi}))$, the other the fixed cost C_0 of termination.

Obviously, the solution of $\bar{\xi}$ to (27) is constant in V and it is just our V''. Notice that V'' is the critical level of V below which the self-enforcing constraint is binding with a positive probability (with respect to the random outside offer ξ) and beyond which the self-enforcing constraint is not binding at all. This gives us a sense that V'' is a measure for how serious the problem of the lack of commitment is.

Over the interval $[u(0) + \beta E(\xi), \overline{V}]$, the principal's value function U^* is increasing. Over the same interval, termination occurs not only for ξ sufficiently high (above V'') but also for ξ sufficiently low (below $\overline{\xi}_1(> V_{\min})$). Obviously, these features of the optimal contract must be connected, and we explain how. To do so, imagine lowering the value of V from $\overline{V}$ to its minimum level deliverable with a feasible contract, i.e., $u(0) + \beta E(\xi)$. From promise-keeping, a lower V forces the contract to reduce the values of $u(c)$ and $V_r(\xi)$. But, for V below $\overline{V}$, there are two factors that limit the contract's ability of doing so:

One. For all $\xi \geq V$, limited commitment dictates that the value of $V_r(\xi)$ must be at least ξ, whether the agent is retained, or terminated.

Two. For all $\xi \leq V$ where the agent is retained, optimal risk sharing requires $V_r(\xi) = V$. That is, for these ξ, optimal risk sharing requires that the agent be paid more than his outside offers.

It then follows that, for any V sufficiently close to $u(0) + \beta E(\xi)$, termination must occur for at least some $\xi \in [V_{\min}, V)$.[7] Given this, it is optimal to let termination occur at a lower, rather than a higher ξ. Why? There is a trade off. On the one hand, when the agent is retained at a lower ξ, a greater amount of extra value (a larger $V - \xi$) must be given him. That is, including a lower ξ in the retention region makes optimal risk sharing more costly. On the other hand, terminating the agent at a low ξ (a ξ that is near $V_{\min}$) brings the principal less benefits, for the same fixed cost of termination. Remember termination allows the ξ to be used as an external means for compensating the agent. Theorem 2 says that the first effect dominates.

An alternative way to think about why a lower rather than a higher ξ should be terminated is as follows. Given that a lower V, by way of tightening up the self-enforcing constraint, imposes more termination on the contract, the principal's value is lower, not higher, for lower values of V. Given this, terminating the agent at a lower ξ has advantages: by giving the agent a lower utility in the state of termination allows the principal to promise the agent a higher utility in the state of retention which, in turn, because of the upward sloping value function, gives the principal a higher value.

Does the optimal contract still show back-loading in the agent's utility over time? Under a specified condition, the answer is yes.

Corollary 2 *Suppose*

$$-\frac{u''(c)c}{u'(c)} \geq \frac{1}{1-\beta}, \ \forall c \geq u^{-1}((1-\beta)\overline{V}).$$

Then the optimal contract has $u(c(V)) < (1-\beta)V$ *for all* $V \in \Phi$

Remember if the agent were not to receive any outside offers in the model, then his compensation and utility would be loaded evenly over time. Here, there are two reasons why the optimal contract is back-loaded. First, backloading is an outcome of the principal's optimal response to the agent's outside offers, conditional on retention, as in the case of $C_0 = \infty$ in Section 3. Second, with a back-loaded contract, compensation is delayed, maybe avoided. A back-loaded compensation scheme allows the contract to shift the burden of compensation from the his current principal to the future, increasing the current principal's value.

[7] In particular, at $V = u(0) + \beta E(\xi)$, if there is no termination over the interval $[V_{\min}, V)$. Then the minimum expected utility the contract can deliver is

$$u(0) + \beta\left(F(u(0)+\beta E(\xi))V + \int_{u(0)+\beta E(\xi)}^{V_{\max}} \xi dF(\xi)\right) > u(0) + \beta E(\xi).$$

The remainder of this section highlights the paper's main ideas and results, as well as the economic logic behind them.

4.1 Termination as a Device for Mitigating the Problem of the Lack of Commitment

In any given state of the world in the model, if termination occurs, then the contract need not enforce continuation in that state, then it does not matter whether the agent can commit to staying with the principal in that state, and it is through this termination helps alleviate the problem of the lack of commitment. We now summarize several aspects of the optimal contract that illustrate this role termination plays.

(i) Termination, because it helps alleviate the problem of the lack of commitment, expands the set of expected utilities the contract can deliver (Lemma 2 and the remark following it).

(ii) For any attainable V, termination reduces the set of the agent's outside offers at which a binding self-enforcing constraint is imposed with the optimal contract. Compare Figures 1 and 2. In particular, termination reduces the problem of the lack of commitment to zero - no self-enforcing is needed - for all V above V''.

(iii) The ability with which termination can be used as a device for mitigating limited commitment depends on the cost of termination. When the cost of termination is lower, termination is used more and the optimal contract then need only impose self-enforcing over a smaller set of the agent's outside offers. Observe from equation (27) that if $C_0 = 0$, then $V'' = \overline{V}$. In general, equation (27) indicates that V'' is increasing in C_0. When $C_0 \to \infty$, $V'' \to V_{\max}$ and the optimal contract then converges to that for $C_0 = \infty$ where the commitment constraint is binding for all $\xi \geq V$ for all V.

4.2 Dynamics and Convergence

The model generates dynamics that the literature has not seen. Starting initially from any $V < V''$ (so the lack of commitment is a binding constraint), so long as the relationship continues, it moves up monotonically in expected utility and, over time, converges to, but never actually attains, V''. Over time, the agent's promised utility increases each time he receives an outside offer that exceeds his current promised utility and the principal matches that offer. Along the path of the optimal contract, the principal never offers any voluntary raises to the agent.

In the limit at V'', the continuation of the optimal contract is such that it is composed of a sequence of static contracts each of which offers retention at the same promised utility V'' but never responds to any of the agent's outside offers. Thus, in the limit, the agent either stays voluntarily with the principal and continues to receive the constant promised utility, or he quits voluntarily to pursue a better outside opportunity. This limiting contract is exactly Burdett (1978).

That the convergence occurs on the path of retention is because the optimal contract generates a monotonic and bounded sequence of promised utilities. That the sequence is monotonic is because the agent's promised utility moves only when his outside offer exceeds his current promised utility and the principal acts to match that offer, and this results from the interaction between the nature of the commitment constraint and the desire for the optimal contract to allocate utility evenly over time. That the dynamics is bounded from above and away from $V_{\max}$ is because of the use of termination in the optimal contract. Since sufficiently high outside offers (higher than V'' specifically) always trigger termination, the agent, upon retention, never would be given a level of promised utility that exceeds V''.

4.3 Outside Opportunities: Good or Bad?

As discussed in the introduction, existing models of limited commitment assume that termination is not feasible and impose a self-enforcing constraint in all states of history. In these models then, the agent's outside opportunities are always *bad* from the principal's perspective: they could never be used but they do impose a cost on the contract. In contrast to the implications of these models, our paper illustrates a sense that, from the principal's perspective, the outside offers that the agent is exposed to may not be a bad thing - they can be either *good* or *bad*. A specific outside offer can be good because it offers an opportunity for the principal to use it as an external means of fulfilling the principal's promises to the agent. It can also be *bad* because it may give the agent leverage against the principal, forcing the principal to match his outside offer.

Our model offers a clear characterization for when the outside offers are good and when they are bad. To obtain this characterization, compare the principal's value function in this section (random outside offers, limited commitment, costly termination), U^*, with that in an assumed scenario in which the agent is not exposed to any outside offers (and so all compensation to the agent - which is constant across period as in equation (1) - must come internally from the principal's pocket), $\hat{U}$. Figure 3 illustrates this comparison. It shows that the outside opportunities are good for the principal when V is sufficiently high (above V'), bad when the V is sufficiently low (below V').

The intuition for this result should now be clear. A higher V allows the principal to gain in the case with outside offers than with not in two ways. First, a higher V, by relaxing the sefl-enforcing constraint, allows the contract to achieve more intertemporal utility smoothing (in particular, first-best intertemporal allocation is achieved in more states of the agent's outside offer). Second, a higher V, since it is associated with a less severe commitment problem, reduces the burden on termination as a device for tackling no commitment, resulting in the more efficient use of the agent's outside offers as an external means of compensation. For this, notice $\bar{\xi}_2$ increases in V for $V \geq V''$. With a higher V, less terminations occur, higher outside offers are taken.

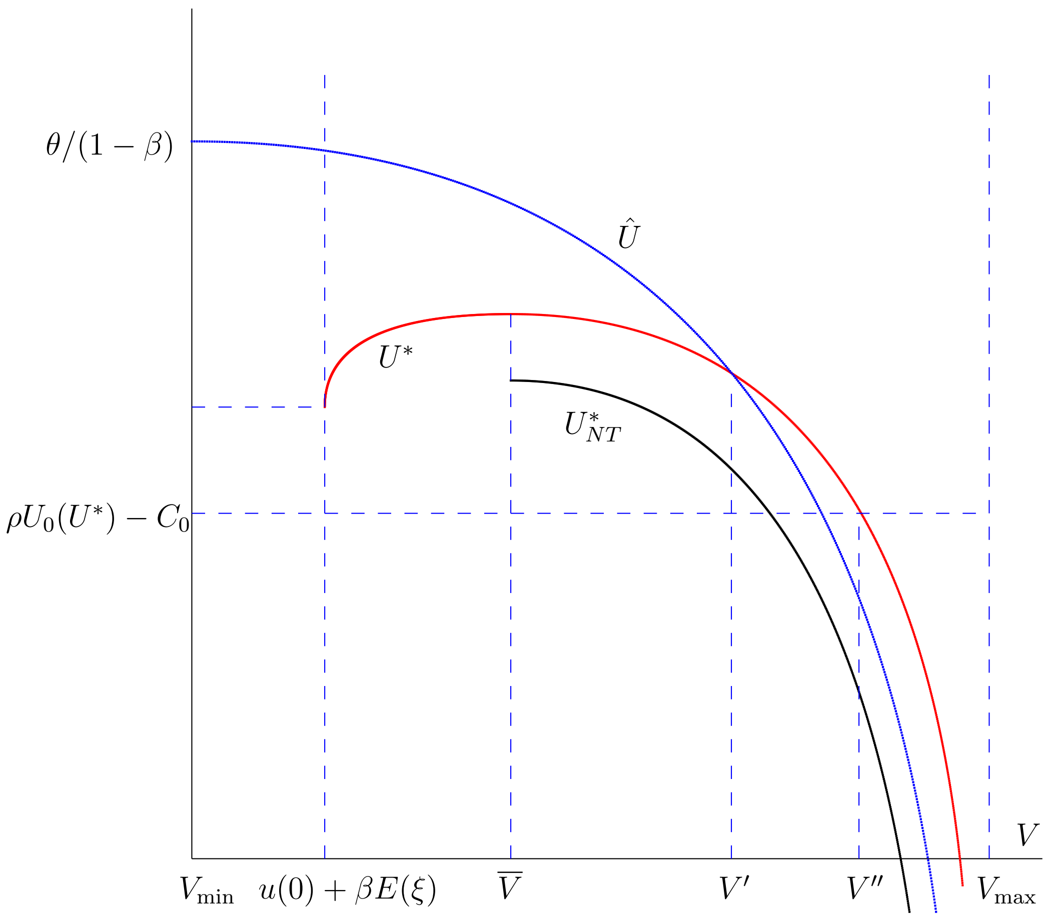

Figure 3: The Value Functions

5 Conclusion

In this paper, we have constructed and studied a model in which an ongoing principal-agent relationship is subject to a sequence of stochastic outside opportunities for the agent. The agent cannot commit to staying with the principal - he is free to pursue any outside offer he receives. Termination allows the principal to utilize the agent's outside offers as an external means for providing compensation to the agent. Termination is also used as a device for alleviating the problem of the lack of commitment. Optimality is achieved through simultaneously determining where to enforce continuation in the space of the agent's history of outside offers, and where to call for termination of the relationship. It turns out that optimal self-enforcing and termination generate interesting dynamics that provides a theoretical justification for the contract used in Burdett (1978)'s classic work of on-the-job search.

In the real world, many long-term economic relationships, like the one we have studied here, are subject to external shocks that generate temptations for one or more parties of the relationship to leave to pursue the values of an outside opportunity. Our paper presents one theoretical attempt to study the dynamics of these relationships. Obviously, many extensions to our model are possible as interesting future research topics, including taking the self-enforcing/termination issue to a full equilibrium framework.

Appendix

A Proof of Lemma 1

Step 1 We show that if $V \in \Omega$, then $V \geq \overline{V}$. Let $\widetilde{V} \equiv \min\{\Omega\}$. It suffices to show $\widetilde{V} \geq \overline{V}$. Given $\widetilde{V} \in \Omega$, there exists a feasible contract $\sigma = \{c(V), V_r(\xi; V)\colon \xi \in \Sigma \text{ and } V \in \Omega\}$ satisfying (2)-(5) such that

$$
\begin{aligned}
\widetilde{V} &= u(c(\widetilde{V})) + \beta \int_{\Sigma} V_r(\xi; \widetilde{V}) dF(\xi) \\
&\geq u(0) + \beta \int_{\Sigma} \max\{\xi, \widetilde{V}\} dF(\xi) \\
&= u(0) + \beta \left(F(\widetilde{V})\widetilde{V} + \int_{\widetilde{V}}^{V_{\max}} \xi dF(\xi) \right),
\end{aligned}
$$

where the first equality follows from (2), and the inequality follows from $c(\widetilde{V}) \geq 0$ by (3) and $V_r(\xi; \widetilde{V}) \geq \max\{\xi, \widetilde{V}\}$ for all $\xi \in \Sigma$ by (4), (5), and $\widetilde{V} \equiv \min\{\Omega\}$. Thus,

$$
Y(\widetilde{V}) \equiv \widetilde{V} - \left[u(0) + \beta \left(F(\widetilde{V})\widetilde{V} + \int_{\widetilde{V}}^{V_{\max}} \xi dF(\xi) \right) \right] \geq 0.
$$

Given $Y'(\widetilde{V}) = 1 - \beta F(\widetilde{V}) > 0$ and $Y(\overline{V}) = 0$ by (6), it must then hold that $\widetilde{V} \geq \overline{V}$.

Step 2 We show that for any $V \in [\overline{V}, V_{\max})$, there exists a feasible contract that attains it. Consider contract $\sigma = \{c(V'), V_r(\xi, V') : \xi \in \Sigma, V' \in [\overline{V}, V_{\max})\}$ where, for each ξ and V',

$$
V_r(\xi; V') = \max\{\xi, V'\},
$$

and $c(V')$ is chosen to be such that $c(V') \geq 0$ and

$$u(c(V')) + \int_{\Sigma} V_r(\xi, V')d\xi = V'.$$

It is straightforward to show that such $c(V')$ exists for all $V' \in [\overline{V}, V_{\max})$. As such, σ is feasible and attains any $V' \in [\overline{V}, V_{\max})$, including V. This proves the lemma.

B Proof of Theorem 1

By Lemma 1, constraints (10) and (11) can be combined to read

$$V_r(\xi) \geq \max\{\xi, \overline{V}\}, \forall \xi \in \Sigma. \tag{28}$$

The first prove (i) of the theorem. This takes five steps.

Step 1 We show that if $U \in B(\Omega)$ is decreasing, then ΛU is decreasing. Let $U \in B(\Omega)$ be decreasing. Let $V^1, V^2 \in \Omega$ with $V^1 < V^2$. Suppose $\{c^2, V_r^2(\xi) : \forall \xi \in \Sigma\}$ is optimal at $V = V^2$. Let

$$k \equiv \frac{V^1 - \overline{V}}{V^2 - \overline{V}} \in (0, 1],$$

$$c^1 = u^{-1}((1-k)u(0) + ku(c^2)) \leq c^2,$$

$$V_r^1(\xi) = (1-k)\max\{\xi, \overline{V}\} + kV_r^2(\xi) \leq V_r^2(\xi), \ \forall \xi \in \Sigma.$$

Then it is straightforward to show that $\{c^1, V_r^1(\cdot)\}$ satisfy (8), (9), and (28) at $V = V^1$ and that

$$\begin{aligned}
\Lambda U(V^1) &\geq (\theta - u^{-1}((1-k)u(0) + ku(c^2))) + \beta \int_{\Sigma} U((1-k)\max\{\xi, \overline{V}\} + kV_r^2(\xi))dF(\xi) \\
&\geq (\theta - c^2) + \beta \int_{\Sigma} U(V_r^2(\xi))dF(\xi) \\
&= \Lambda U(V^2),
\end{aligned}$$

where the second inequality follows from the assumption that U is decreasing. So ΛU is decreasing.

Step 2 We show that if $U \in B(\Omega)$ is concave, then ΛU is concave. Suppose U is concave. For any given $V^1, V^2 \in \Omega$, suppose $\{c^i, V_r^i(\xi) : \forall \xi \in \Sigma\}$ is optimal at $V = V^i$, $i = 1, 2$. Then it is straightforward to show that $\{c, V_r(\xi), \xi \in \Sigma\}$, where

$$c = u^{-1}(ku(c^1) + (1-k)u(c^2)),$$

$$V_r(\xi) = kV_r^1(\xi) + (1-k)V_r^2(\xi), \ \forall \xi \in \Sigma,$$

satisfy (8), (9), and (28) at $V = kV^1 + (1-k)V^2$ for all $k \in [0,1]$. Moreover,

$$\begin{aligned} \Lambda U(V) &\geq (\theta - u^{-1}(ku(c^1) + (1-k)u(c^2))) + \beta \int_\Sigma U(kV_r^1(\xi) + (1-k)V_r^2(\xi))dF(\xi) \\ &\geq [\theta - (kc^1 + (1-k)c^2)] + \beta \int_\Sigma [kU(V_r^1(\xi)) + (1-k)U(V_r^2(\xi))]dF(\xi) \\ &= k\Lambda U(V^1) + (1-k)\Lambda U(V^2), \end{aligned}$$

where the second inequality holds because both $-u^{-1}$ and U are concave.

Step 3 We show that $\{\Lambda^n \overline{U}\}_{n=0}^\infty \subseteq B(\Omega)$ is a sequence of decreasing and concave functions converging monotonically to U_{NT}^*, which is then decreasing and concave.

First, we show that U_{NT}^* is the *maximum* fixed point of Λ in the following sense: there does not exist any fixed point of Λ denoted $\widetilde{U}$ such that $\widetilde{U}(V) > U_{NT}^*(V)$ for some V. Suppose otherwise. Fix any V with $\widetilde{U}(V) > U_{NT}^*(V)$. A contract can be constructed recursively to deliver the agent expected utility V and the principal expected payoff $\widetilde{U}(V) > U_{NT}^*(V)$, which contradicts with the definition of $U_{NT}^*(V)$.

Second, we show $U_{NT}^*(V) \leq \Gamma^{n+1}\overline{U}(V) \leq \Gamma^n \overline{U}(V) \leq \overline{U}(V) \equiv \theta/(1-\beta)$ for all V and $n = 1, 2, \cdots$. It is straightforward to show $U_{NT}^*(V) \leq \Gamma\overline{U}(V) \leq \overline{U}(V)$ for all V. Given that Λ is monotonic, the desired result then follows.

Third, we show that the function U_{NT}^* is decreasing and concave. Since $\Gamma^\infty \overline{U}$ is a fixed point of Γ with $\Gamma^\infty\overline{U}(V) \geq U_{NT}^*(V)$ for all V, by Step 1 we must have $U_{NT}^* = \Gamma^\infty\overline{U}$. Moreover, given $\overline{U}$ is decreasing and concave, $\Gamma^n\overline{U}$ is decreasing and concave for all n, by Steps 1 and 2. Hence, $\Gamma^\infty\overline{U}$ or U_{NT}^* is decreasing and concave.

Step 4 We show that U_{NT}^* is differentiable. The proof applies Theorem 4.10 of Stokey and Lucas (1989). Basically, given that U_{NT}^* is concave, to show it is differentiable at any given $\widetilde{V}$, we need only construct a concave function $W(V)$ which takes as its domain a neighbourhood of $\widetilde{V}$, is uniformly below U_{NT}^* but attains the same value as does U_{NT}^* at $\widetilde{V}$. In the following, we ignore the case $\widetilde{V} = V$ and take as given that $\widetilde{V} > \overline{V}$.

Let $c^*(\widetilde{V})$ is the optimal c at $\widetilde{V}$.

Case 1 $c^*(\widetilde{V}) > 0$. The function W is constructed as: for all V,

$$W(V) \equiv (\theta - u^{-1}(u(c^*(\widetilde{V})) + (V - \widetilde{V})) + \beta \int_\Sigma U_{NT}^*(V_r(\xi; \widetilde{V}))dF(\xi).$$

This function is concave and differentiable, given that u is concave and differentiable. Moreover, $W(V) \leq U_{NT}^*(V)$ and $W(\widetilde{V}) = U_{NT}^*(\widetilde{V})$, given that $c \equiv u^{-1}(u(c^*(\widetilde{V})) + (V - \widetilde{V}))$ and $V_r(\xi; \widetilde{V})$ for all ξ satisfy (8)-(11) at $\widetilde{V}$. Thus U_{NT}^* is differentiable at $\widetilde{V}$ by Stokey and Lucas, 1989).

Case 2 $c^*(\widetilde{V}) = 0$. In this case, for all V, let

$$W(V) \equiv (\theta - u^{-1}(u(c(\widetilde{V})) + \max\{0, V - \widetilde{V}\})) + \beta \int_\Sigma U^*_{NT}(V_r(\xi; \widetilde{V}))dF(\xi).$$

The function W such defined is concave, given that u is increasing and concave.

For $V < \widetilde{V}$, W is differentiable with $W(V) = U^*_{NT}(\widetilde{V}) \leq U^*_{NT}(V)$, given that U^*_{NT} is decreasing by Step 3.

For $V > \widetilde{V}$, W is differentiable given that u is differentiable, and $W(V) \leq U^*_{NT}(V)$ given that $c = u^{-1}(u(c^*(\widetilde{V})) + (V - \widetilde{V}))$ and $V_r(\xi; \widetilde{V})$ for all ξ satisfy (8)-(11).

For $V = \widetilde{V}$, W is left differentiable with $W'(V^-) = 0$, and right differentiable with $W'(V^+) = 0$ given $u'(0) = \infty$ by the Inada conditions. Thus W is differentiable at V.

To summarize, W is concave and differentiable with $W(V) \leq U^*_{NT}(V)$ for all V, and $W(\widetilde{V}) = U^*_{NT}(\widetilde{V})$. Apply again Stokey and Lucas (1989) and the desired result follows.

Step 5 We show that U^*_{NT} is strictly decreasing and strictly concave.

Let $\alpha(V)$, $\mu(V)$, and $\beta\gamma(\xi; V)$ be the Lagrangian multipliers for (8), (9), and (28) respectively. Then the Kuhn-Tucker conditions are as follows: for all V and for all ξ,

$$-1 + \alpha(V)u'(c(V)) + \mu(V) = 0, \tag{29}$$

$$(U^{*\prime}_{NT}(V_r(\xi; V)) + \alpha(V))f(\xi) + \gamma(\xi; V) = 0, \tag{30}$$

$$\mu(V)c(V) = 0, \tag{31}$$

$$\gamma(\xi; V)(V_r(\xi; V) - \max\{\xi, \overline{V}\}) = 0, \tag{32}$$

$$\mu(V), \gamma(\xi; V) \geq 0. \tag{33}$$

Furthermore, the Envelope Theorem gives

$$U^{*\prime}_{NT}(V) = -\alpha(V). \tag{34}$$

Given that U^*_{NT} is concave by Step 3, (34) implies that α is increasing. To obtain the desired results for this step, we need only show that α is strictly increasing, which implies that U^*_{NT} is strictly concave by (34), which in turn implies that U^*_{NT} is strictly decreasing given that U^*_{NT} is decreasing by Step 3.

Suppose not. That is, suppose there exist $V^1, V^2 \in \Omega$ with $V^1 < V^2$ such that $\alpha(V) \equiv \widetilde{\alpha}$, for some $\widetilde{\alpha}$ and for all $V \in [V^1, V^2]$. Without loss of generality, suppose $V^1 = \inf\{V \in \Omega : \alpha(V) = \widetilde{\alpha}\}$ and $V^2 = \sup\{V \in \Omega : \alpha(V) = \widetilde{\alpha}\}$. We take the following three steps to derive a contradiction.

First, we show that for all $V \in [V^1, V^2]$, $c(V) \equiv \widetilde{c}$, for some $\widetilde{c}$.

Suppose $\widetilde{\alpha} = 0$. Then for all $V \in [V^1, V^2]$, if $c(V) > 0$, then $\mu(V) = 0$ by (31), which implies that the LHS of (29) is -1, which contradicts with (29). We then conclude $c(V) = 0$.

Suppose $\widetilde{\alpha} > 0$. Then for all $V \in [V^1, V^2]$, if $c(V) = 0$, then the LHS of (29) is $-1 + \widetilde{\alpha} u'(0) + \mu(V) > 0$ by $u'(0) = \infty$ and $\mu(V) \geq 0$ by (33), which contradicts with (29). Thus $c(V) > 0$, which implies $\mu(V) = 0$ by (31), which in turn implies $c(V) = u'^{-1}(1/\widetilde{\alpha}) \equiv \widetilde{c}$ by (29) where, given that u is strictly concave, the inverse function is well defined.

Second, we show that for all $V \in [V^1, V^2]$,

$$V_r(\xi; V) \begin{cases} \in [V^1, V^2], & \text{for } \xi \leq V^2 \\ = \xi, & \text{for } \xi > V^2 \end{cases}. \tag{35}$$

For all $\xi \leq V^2$, we show $\gamma(\xi; V) = 0$, which implies $\alpha(V_r(\xi; V)) = -U^{*\prime}_{NT}(V_r(\xi; V)) = \alpha(V) = \widetilde{\alpha}$ by (30) and (34), which in turn implies $V_r(\xi; V) \in [V^1, V^2]$. Suppose $\gamma(\xi; V) > 0$. Then (32) implies $V_r(\xi; V) = \max\{\xi, \overline{V}\} \leq V^2$, which in turn implies that the LHS of (30) is strictly positive given that α is increasing, which contradicts with (30).

For all $\xi > V^2$, we show $\gamma(\xi; V) > 0$, which implies $V_r(\xi; V) = \max\{\xi, \overline{V}\} = \xi$ by (32). Suppose $\gamma(\xi; V) = 0$. Then (30) and (34) imply $\alpha(V_r(\xi; V)) = \widetilde{\alpha}$, which in turn implies $V_r(\xi; V) \leq V^2 < \xi$, which contradicts with (4).

Third, we show $V^1 = V^2$, which implies that α is strictly increasing.

Given (4) and (35), (8) implies

$$u(\widetilde{c}) + \beta \left(F(V^1)V^1 + \int_{V^1}^{V_{\max}} \xi dF(\xi) \right) \leq V^1 \leq V^2 \leq u(\widetilde{c}) + \beta \left(F(V^2)V^2 + \int_{V^2}^{V_{\max}} \xi dF(\xi) \right),$$

which in turn implies $Y(V^1) \geq 0 \geq Y(V^2)$ where

$$Y(V) \equiv V - \left[u(\widetilde{c}) + \beta \left(F(V)V + \int_{V}^{V_{\max}} \xi dF(\xi) \right) \right],$$

which in turn implies $V^1 \geq V^2$ given $Y'(V) = 1 - \beta F(V) > 0$. Thus we conclude $V^1 = V^2$.

This completes Step 5 and hence the proof of Part (i) of the theorem. Part (ii) of the theorem follows from (35) and $V^1 = V = V^2$ by Step 5 from the above proof of Part (i). The proof of Theorem 1 is complete.

C Proof of Corollary 1

Given (ii) of Theorem 1, equation (8) implies that for all V,

$$u(c(V)) = V - \beta \int_{\Sigma} \max\{\xi, V\} dF(\xi) < V - \beta V = (1 - \beta)V.$$

The lemma is proved.

D Proof of Lemma 2

Step 1 We show that if $V \in \Phi$, then $V \geq u(0) + \beta E(\xi)$. Suppose $V \in \Phi$. Then there exists a feasible contract $\sigma = \{c(V'), I(\xi; V'), V_r(\xi; V')$: $\xi \in \Sigma$ and $V' \in \Phi\}$ satisfying (13)-(17) such that

$$\begin{aligned} V &= u(c(V)) + \beta \int_{\Sigma} [I(\xi; V) V_r(\xi; V) + (1 - I(\xi; V))\xi] dF(\xi) \\ &\geq u(0) + \beta \int_{\Sigma} [I(\xi; V)\xi + (1 - I(\xi; V))\xi] dF(\xi) \\ &= u(0) + \beta E(\xi), \end{aligned}$$

where the first equality follows from (13), and the inequality follows from $c(V) \geq 0$ by (14) and $V_r(\xi; V) \geq \xi$ for all $\xi \in \Sigma$ by (16).

Step 2 We show that there is a feasible contract that attains all $V \in [u(0) + \beta E(\xi), V_{\max})$. This contract is constructed as follows. For all $\xi \in \Sigma$ and all $V \in [u(0) + \beta E(\xi), V_{\max})$, let

$$I(\xi; V) = 1, \text{ if } \xi < \overline{\xi}(V); \ 0, \text{ otherwise},$$

$$V_r(\xi; V) = \max\{\xi, V\},$$

where for all V, $\overline{\xi}(V) \in \Sigma$ is set, together with $c(V) \geq 0$, to be such that (13) holds. It is straightforward to show that such $c(V)$ and $\overline{\xi}(V) \in \Sigma$ exist and the lemma is proven.

E Proof of Theorem 2

By Lemma 2, constraints (22) and (23) can be combined to read as

$$V_r(\xi) \geq \max\{\xi, u(0) + \beta E(\xi)\}, \forall \xi \in \Sigma. \tag{36}$$

Proof of (i) & (ii) of the theorem Fix V. Let α, μ, $\beta\lambda(\xi)$, and $\beta\gamma(\xi)$ be the Lagrangian multipliers for (19)-(21) and (36) respectively. Then the Kuhn-Tucker conditions are as follows: for all ξ,

$$-1 + \alpha u'(c) + \mu = 0 \tag{37}$$

$$L(\xi) f(\xi) + \lambda(\xi)(1 - 2I(\xi)) = 0 \tag{38}$$

$$I(\xi)(U'(V_r(\xi)) + \alpha) f(\xi) + \gamma(\xi) = 0 \tag{39}$$

$$\mu c = 0 \tag{40}$$

$$\lambda(\xi) I(\xi)(1 - I(\xi)) = 0 \tag{41}$$

$$\gamma(\xi)(V_r(\xi) - \max\{\xi, u(0) + \beta E(\xi)\}) = 0 \tag{42}$$

$$\mu, \lambda(\xi), \gamma(\xi) \geq 0 \tag{43}$$

where

$$L(\xi) \equiv U(V_r(\xi)) - (\rho U_0(U) - C_0) + \alpha(V_r(\xi) - \xi). \tag{44}$$

In addition, the Envelope Theorem gives

$$(\Gamma U)'(V) = -\alpha. \tag{45}$$

The remainder of the proof takes 4 steps, as organized in Lemmas 3-6, respectively.

Lemma 3 *For all ξ, $I(\xi) = 1$ if $L(\xi) > 0$, and $I(\xi) = 0$ if $L(\xi) < 0$.*

Proof. Fix ξ. Suppose $L(\xi) > 0$. Then (38) implies $\lambda(\xi)(1 - 2I(\xi)) < 0$, which in turn implies $\lambda(\xi) > 0$ and $1 - 2I(\xi) < 0$ given $\lambda(\xi) \geq 0$ by (43). Furthermore, $\lambda(\xi) > 0$ implies $I(\xi) = 0$ or 1 by (41), and $1 - 2I(\xi) < 0$ implies $I(\xi) > 1/2$. Hence, we conclude $I(\xi) = 1$. The proof of the second part of Lemma 3 is similar. ■

In words, $L(\xi)$ is the net marginal benefit of retention given outside offer ξ.

Lemma 4 *Suppose that $U \in B(\Phi)$ is concave with $U'(u(0) + \beta E(\xi)) \geq 0$ and $U'(V) \leq 0$ for some V. Denote*

$$V^*(U) \equiv \sup \left\{ \arg\max_{V \in \Phi} \{U(V)\} \right\} \in [u(0) + \beta E(\xi), V_{\max}]^8$$

$$\overline{V}(U) \equiv u(0) + \beta \left(F(V^*(U))V^*(U) + \int_{V^*(U)}^{V_{\max}} \xi dF(\xi) \right) \in (u(0) + \beta E(\xi), V_{\max}).$$

Then the solution to the problem (37)-(43) has:

(i) For all V, there exist $\overline{\xi}_1 \in [V_{\min}, V^*(U)]$, $\overline{\xi}_2 \in [V^*(U), V_{\max}]$, and $\overline{V}_r \in [u(0)+\beta E(\xi), V_{\max}]$ ($\overline{\xi}_1$, $\overline{\xi}_2$, and $\overline{V}_r$ depending on V) such that

$$I(\xi) = \begin{cases} 1 & , \text{for } \xi \in (\overline{\xi}_1, \overline{\xi}_2) \\ 0 & , \text{for } \xi < \overline{\xi}_1 \text{ and } \xi > \overline{\xi}_2 \end{cases}; \tag{a}$$

$$V_r(\xi) = \max\{\xi, \overline{V}_r\}, \ \forall \xi. \tag{b}$$

(ii) $\Gamma U \in B(\Phi)$ is concave, and attains its maximum at $V = \overline{V}(U)$ with $(\Gamma U)'(u(0)+\beta E(\xi)) \geq 0$ and $(\Gamma U)'(V) \leq 0$ for some V.

Proof of (i) of Lemma 4 Fix V. We show that the desired result holds in the two cases $\alpha \geq 0$ and $\alpha < 0$ respectively.

[8]Note that $\arg\max_{V \in \Phi}\{U(V)\}$ is nonempty since U is concave with $U'(V) \leq 0$ for some V.

Case 1 $\alpha \leq 0$. The proof of (i) in this case takes four steps.

Step 1 We show $c = 0$.

Equation (37) implies $\mu = 1 - \alpha u'(c) > 0$, which in turn implies $c = 0$ by (40).

Step 2 We show that there exists $\overline{V}_r \in [u(0) + \beta E(\xi), V^*(U)]$ with $U'(\overline{V}_r) \leq -\alpha$ (where the equality holds if $\overline{V}_r > u(0) + \beta E(\xi)$) such that (b) holds.

Suppose $\int_\Sigma I(\xi) dF(\xi) = 0$. Then the choice of $V_r(\xi)$ is arbitrary for all ξ, and the existence of $\overline{V}_r \in [u(0) + \beta E(\xi), V^*(U)]$ with $U'(\overline{V}_r) \leq -\alpha$ is guaranteed by that U is concave with $U'(u(0) + \beta E(\xi)) \geq 0$ and $U'(V^*(U)) = 0$.

Suppose $\int_\Sigma I(\xi) dF(\xi) > 0$. Then for all ξ with $I(\xi) > 0$, if $\gamma(\xi) = 0$, then (39) implies $U'(V_r(\xi)) = -\alpha$; if $\gamma(\xi) > 0$, then (39) and (42) imply $V_r(\xi) = \max\{\xi, u(0) + \beta E(\xi)\}$ with $U'(V_r(\xi)) < -\alpha$. Hence, given that U is concave, the result follows.

Step 3 We show that there exist $\overline{\xi}_1 \in [V_{\min}, V^*(U)]$ and $\overline{\xi}_2 \in [V^*(U), V_{\max}]$ ($\overline{\xi}_2$ is independent of V) such that (a) holds.

Given $V_r(\xi) = \max\{\xi, \overline{V}_r\}$ for all ξ by Step 2, (44) implies

$$L(\xi) = U(\max\{\xi, \overline{V}_r\}) - (\rho U_0(U) - C_0) + \alpha(\max\{\xi, \overline{V}_r\} - \xi),$$

which is increasing on $[V_{\min}, V^*(U)]$, and strictly decreasing on $[V^*(U), V_{\max})$ with

$$L(V^*(U)) = U(V^*(U)) - (\rho U_0(U) - C_0) \geq 0$$

given $\overline{V}_r \leq V^*(U)$ by Step 2. The result then follows from Lemma 3.

Step 4 We show $V \leq \overline{V}(U)$.

The result follows from Step 1-2 and (19).

Case 2 $\alpha > 0$. The proof of (i) in this case takes four steps.

Step 1 We show $u'(c) = 1/\alpha$.

If $c = 0$, then $-1 + \alpha u'(0) + \mu > 0$ given $u'(0) = \infty$ and $\mu \geq 0$ by (43), which contradicts with (37). We then proceed by assuming $c > 0$, which implies $\mu = 0$ by (40), which in turn implies $u'(c) = 1/\alpha$ by (37).

Step 2 We show that there exists $\overline{V}_r \in [V^*(U), V_{\max}]$ with $U'(\overline{V}_r) \geq -\alpha$ (where the equality holds if $\overline{V}_r < V_{\max}$) such that (b) holds.

Suppose $\int_\Sigma I(\xi) dF(\xi) = 0$. Then the choice of $V_r(\xi)$ is arbitrary for all ξ, and the existence of $\overline{V}_r \in [V^*(U), V_{\max}]$ with $U'(\overline{V}_r) \geq -\alpha$ is guaranteed by that U is concave with $U'(V^*(U)) = 0$.

Suppose $\int_\Sigma I(\xi) dF(\xi) > 0$. Then for all ξ with $I(\xi) > 0$, if $\gamma(\xi) = 0$, then (39) implies $U'(V_r(\xi)) = -\alpha$; if $\gamma(\xi) > 0$, then (39) and (42) imply $V_r(\xi) = \max\{\xi, u(0) + \beta E(\xi)\}$ with $U'(V_r(\xi)) < -\alpha$. Hence, given that U is concave, the result follows.

Step 3 We show that there exist $\overline{\xi}_1 = V_{\min}$ and $\overline{\xi}_2 \in [V^*(U), V_{\max}]$ such that (a) holds.

Given $V_r(\xi) = \max\{\xi, \overline{V}_r\}$ for all ξ by Step 2, (44) implies

$$L(\xi) = U(\max\{\xi, \overline{V}_r\}) - (\rho U_0(U) - C_0) + \alpha(\max\{\xi, \overline{V}_r\} - \xi),$$

which is strictly decreasing with

$$\begin{aligned} L(V^*(U)) &= U(\overline{V}_r) - (\rho U_0(U) - C_0) + \alpha(\overline{V}_r - V^*(U)) \\ &\geq U(\overline{V}_r) - (\rho U_0(U) - C_0) - U'(\overline{V}_r)(\overline{V}_r - V^*(U)) \\ &\geq U(V^*(U)) - (\rho U_0(U) - C_0) \\ &\geq 0, \end{aligned}$$

where the equality follows from $\overline{V}_r \geq V^*(U)$ by Step 2, the first inequality follows from $\overline{V}_r \geq V^*(U)$ and $U'(\overline{V}_r) \geq -\alpha$ by Step 2, and the second inequality follows from that U is concave. The result then follows from Lemma 3.

Step 4 We show $V > \overline{V}(U)$. This follows follows from Step 1-3 and (19).

Proof of (ii) of Lemma 4 The proof takes two steps.

Step 1 We show that ΓU is increasing and concave on $[u(0) + \beta E(\xi), \overline{V}(U)]$.

Given what have been shown in Step 1-4 of Case 1, we have

$$(\overline{\xi}_1 - V_{\min})L(\overline{\xi}_1) = 0,$$

$$[\overline{V}_r - (u(0) + \beta E(\xi))](U'(\overline{V}_r) + \alpha) = 0,$$

$$u(0) + \beta\left(\int_{V_{\min}}^{\overline{\xi}_1} \xi dF(\xi) + \int_{\overline{\xi}_1}^{\overline{\xi}_2} \max\{\xi, \overline{V}_r\} dF(\xi) + \int_{\overline{\xi}_2}^{V_{\max}} \xi dF(\xi)\right) = V^{9}.$$

Totally differentiating the equations above gives

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ a_{31} & a_{32} & 0 \end{bmatrix} \begin{bmatrix} d\overline{\xi}_1 \\ d\overline{V}_r \\ d\alpha \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} dV,$$

where

$$a_{11} = L(\overline{\xi}_1) - \alpha(\overline{\xi}_1 - V_{\min}) \geq 0,$$

$$a_{12} = (\overline{\xi}_1 - V_{\min})(U'(\overline{V}_r) + \alpha) \leq 0,$$

$$a_{13} = (\overline{\xi}_1 - V_{\min})(\overline{V}_r - \overline{\xi}_1) \geq 0,$$

$$a_{22} = (U'(\overline{V}_r) + \alpha) + U''(\overline{V}_r)[\overline{V}_r - (u(0) + \beta E(\xi))] \leq 0,$$

[9] For $V \in [u(0) + \beta E(\xi), \overline{V}(U)]$, $\overline{\xi}_2$ is independent of V by Step 3 of Case 1. Therefore, it can be treated as a constant. In addition, it is straightforward to show $\overline{\xi}_1 \leq \overline{V}_r \leq \overline{\xi}_2$.

$$a_{23} = \overline{V}_r - (u(0) + \beta E(\xi)) \geq 0,$$

$$a_{31} = -\beta f(\overline{\xi}_1)(\overline{V}_r - \overline{\xi}_1) \leq 0,$$

$$a_{32} = \beta(F(\overline{V}_r) - F(\overline{\xi}_1)) \geq 0.$$

Furthermore, by Cramer's rule,

$$\frac{d\overline{\xi}_1}{dV} = \frac{\|\mathbf{A}_1\|}{\|\mathbf{A}\|} \leq 0, \; \frac{d\overline{V}_r}{dV} = \frac{\|\mathbf{A}_2\|}{\|\mathbf{A}\|} \geq 0, \text{ and } \frac{d\alpha}{dV} = \frac{\|\mathbf{A}_3\|}{\|\mathbf{A}\|} \geq 0,$$

where

$$\|\mathbf{A}\| = -a_{11}a_{23}a_{32} - a_{13}a_{22}a_{31} \leq 0,$$

$$\|\mathbf{A}_1\| = -a_{13}a_{22} \geq 0,$$

$$\|\mathbf{A}_2\| = -a_{11}a_{23} \leq 0,$$

$$\|\mathbf{A}_3\| = a_{11}a_{22} \leq 0.$$

Thus, ΓU is concave given $(\Gamma U)''(V) = -d\alpha/dV \leq 0$ by (45).

Step 2 We show that ΓU is strictly decreasing and concave on $(\overline{V}(U), V_{\max})$.

Given what have been shown in Step 1-4 of Case 2, we have

$$L(\overline{\xi}_2)(\overline{\xi}_2 - V_{\max}) = 0,$$

$$(U'(\overline{V}_r) + \alpha)(\overline{V}_r - V_{\max}) = 0,$$

$$u\left(u'^{-1}\left(\frac{1}{\alpha}\right)\right) + \beta\left(\int_{V_{\min}}^{\overline{\xi}_2} \max\{\xi, \overline{V}_r\} dF(\xi) + \int_{\overline{\xi}_2}^{V_{\max}} \xi dF(\xi)\right) = V.$$

Totally differentiating the equations above gives

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} d\overline{\xi}_2 \\ d\overline{V}_r \\ d\alpha \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} dV,$$

where

$$a_{11} = \min\{-\alpha, U'(\overline{\xi}_2)\}(\overline{\xi}_2 - V_{\max}) + L(\overline{\xi}_2) \geq 0,$$

$$a_{12} = (U'(\overline{V}_r) + \alpha)(\overline{\xi}_2 - V_{\max}) \leq 0,$$

$$a_{13} = (\max\{\overline{\xi}_2, \overline{V}_r\} - \overline{\xi}_2)(\overline{\xi}_2 - V_{\max}) \leq 0,$$

$$a_{22} = U''(\overline{V}_r)(\overline{V}_r - V_{\max}) + (U'(\overline{V}_r) + \alpha) \geq 0,$$

$$a_{23} = \overline{V}_r - V_{\max} \leq 0,$$

$$a_{31} = \beta f(\overline{\xi}_2)(\max\{\overline{\xi}_2, \overline{V}_r\} - \overline{\xi}_2) \geq 0,$$

$$a_{32} = \beta F(\min\{\bar{\xi}_2, \overline{V}_r\}) \geq 0,$$

$$a_{33} = -u'(c)/(\alpha^2 u''(c)) \geq 0.$$

Furthermore, by Cramer's rule,

$$\frac{d\bar{\xi}_2}{dV} = \frac{\|\mathbf{A}_1\|}{\|\mathbf{A}\|} \geq 0,\ \frac{d\overline{V}_r}{dV} = \frac{\|\mathbf{A}_2\|}{\|\mathbf{A}\|} \geq 0, \text{ and } \frac{d\alpha}{dV} = \frac{\|\mathbf{A}_3\|}{\|\mathbf{A}\|} \geq 0,$$

where

$$\|\mathbf{A}\| = a_{11}(a_{22}a_{33} - a_{23}a_{32}) - a_{13}a_{22}a_{31} \geq 0,$$

$$\|\mathbf{A}_1\| = -a_{13}a_{22} \geq 0,$$

$$\|\mathbf{A}_2\| = -a_{11}a_{23} \geq 0,$$

$$\|\mathbf{A}_3\| = a_{11}a_{22} \geq 0.$$

Thus, ΓU is concave given $(\Gamma U)''(V) = -d\alpha/dV$ by (45).

To summarize, ΓU is concave, and for all $V \in \Phi$,

$$\frac{d\bar{\xi}_1}{dV} \leq 0,\ \frac{d\bar{\xi}_2}{dV} \geq 0, \text{ and } \frac{d\overline{V}_r}{dV} \geq 0. \tag{46}$$

■

Lemma 5 *Let $U_0 = \overline{U} \in B(\Phi)$ and $U_{n+1} = \Gamma U_n$ for $n = 0, 1, \cdots$. Then the sequence $\{U_n\}_{n=0}^{\infty} \subseteq B(\Phi)$ converges pointwisely and monotonically to the principal's value function U^* as n goes to infinity. In addition, $V^*(\Gamma^\infty \overline{U}) = \overline{V}(\Gamma^\infty \overline{U}) = \overline{V}$.*

Proof. Given that the mapping Γ is monotonic and preserves concavity by (ii) of Lemma 4, it is straightforward to show that $\Gamma^n \overline{U}$ is concave with $\overline{U}(V) \geq \Gamma^n \overline{U}(V) \geq \Gamma^{n+1}\overline{U}(V) \geq U^*(V)$ for all V and $n = 1, 2, \cdots$. The result then follows. In addition, $V^*(\Gamma^\infty \overline{U}) = \overline{V}(\Gamma^\infty \overline{U}) = \overline{V}$. ■

Lemma 6 *For all V, $\overline{V}_r = V$.*

Proof. Fix V. The proof takes two steps.

Step 1 We show $U^{*\prime}(V) = U^{*\prime}(\overline{V}_r) = -\alpha$.

The result follows from (45) and Step 2 of Case 1-2 of the proof of (i) of Lemma 4.

Step 2 We show $\overline{V}_r = V$.

By Step 1, $U^{*\prime}(V) = U^{*\prime}(\overline{V}_r^t(V)) = -\alpha$ for $t = 1, \cdots$ where $\overline{V}_r^t(V)$ denotes the agent's continuation expected utility starting from period $t+1$ conditional on retention and $\xi \leq \overline{V}_r^\tau(V)$ in period $\tau = 1, \cdots, t$. It implies $c(V) = c(\overline{V}_r^t(V)) \equiv c$ by Step 1 of the proof of (i) of Lemma

4, $\bar{\xi}_1(V) = \bar{\xi}_1(\overline{V}_r^n(V)) \equiv \bar{\xi}_1$, and $\bar{\xi}_2(V) = \bar{\xi}_2(\overline{V}_r^t(V)) \equiv \bar{\xi}_2$ by (44) and Lemma 3. Hence, for $t = 1, \cdots$, (19) implies

$$u(c) + \beta \left(\int_{V_{\min}}^{\bar{\xi}_1} \xi dF(\xi) + \int_{\bar{\xi}_1}^{\bar{\xi}_2} \max\{\xi, \overline{V}_r^t(V)\} dF(\xi) + \int_{\bar{\xi}_2}^{V_{\max}} \xi dF(\xi) \right) = \overline{V}_r^{t-1}(V),$$

where $V \equiv \overline{V}_r^0(V)$. Suppose $V \neq \overline{V}_r(V)$. Given $\beta \in (0,1)$, $\{\overline{V}_r^{t-1}(V)\}_{t=1}^{\infty}$ is either an increasing sequence converging to $\infty > V_{\max}$, or a decreasing sequence converging to $-\infty < \overline{V}$. Therefore, we conclude $V = \overline{V}_r(V)$. This completes the proof of Lemma 6 and hence the proof of parts (i) and (ii) of the theorem. ■

We now proceed to prove the last part of the theorem. Given (ii) of the theorem which we have just proved, the promise-keeping constraint (19) can be rewritten as

$$u(c) + \beta \left(\int_{V_{\min}}^{\bar{\xi}_1} \xi dF(\xi) + \int_{\bar{\xi}_1}^{\bar{\xi}_2} \max\{\xi, V\} dF(\xi) + \int_{\bar{\xi}_2}^{V_{\max}} \xi dF(\xi) \right) = V; \tag{47}$$

the net marginal benefit of retention (44) can be rewritten as

$$L(\xi; V) = U^*(\max\{\xi, V\}) - (\rho U_0(U^*) - C_0) + \alpha(\max\{\xi, V\} - \xi); \tag{48}$$

and the value function (18) can be rewritten as

$$U^*(V) = (\theta - c) + \beta \left\{ \int_{\bar{\xi}_1}^{\bar{\xi}_2} U^*(\max\{\xi, V\}) dF(\xi) + [F(\bar{\xi}_1) + (1 - F(\bar{\xi}_2))](\rho U_0(U^*) - C_0) \right\}. \tag{49}$$

Proof of (iii) of the theorem Let $V < \overline{V}$. Given $\alpha = -U^{*\prime}(V)$ by (45), (47) implies

$$L(\xi) = U^*(\max\{\xi, V\}) - (\rho U_0(U^*) - C_0) - U^{*\prime}(V)(\max\{\xi, V\} - \xi),$$

as illustrated in the Figure 4.

For $\xi \geq V$, $L(\xi) = U^*(\xi) - (\rho U_0(U^*) - C_0)$. This part of the function L is depicted by the curved line in the figure. For $\xi < V$, $L(\xi) = U^*(V) - (\rho U_0(U^*) - C_0) - U^{*\prime}(V)(V - \xi)$. This part of L is depicted by the straight line (with a constant slope of $U^{*\prime}(V)$) in the figure. Apparently, L is continuous at V.

Given $c = 0$ by Step 1 of Case 1 of the proof of (i) of Lemma 4 and $\xi_2 \geq \overline{V} > V$ by (ii) of Theorem 2, (47) implies

$$u(c) + \beta \left(\int_{V_{\min}}^{\bar{\xi}_1} \xi dF(\xi) + \int_{\bar{\xi}_1}^{V_{\max}} \max\{\xi, V\} dF(\xi) \right) = V.$$

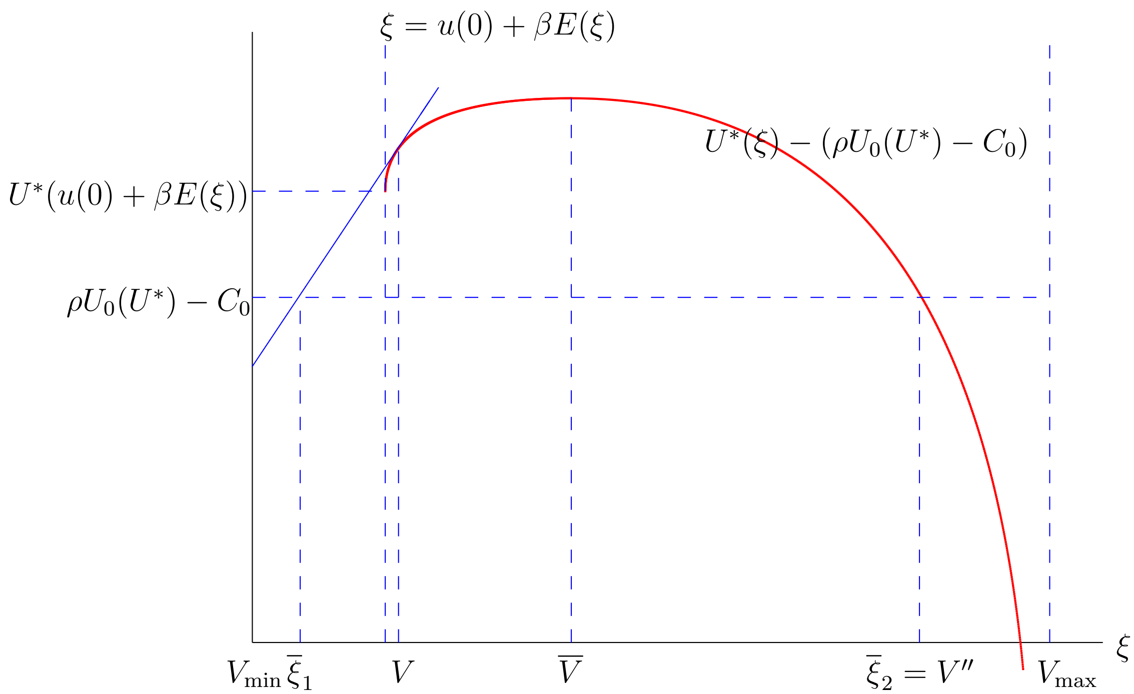

Figure 4: The Retention/Termination Region for $V < \overline{V}$

Suppose $\bar{\xi}_1 = V_{\min}$. Then $V = \overline{V}$ by (6), which contradicts with $V < \overline{V}$. Hence, we conclude $\bar{\xi}_1 > V_{\min}$. Moreover, given (46) and (ii) of Theorem 2, and as is obvious from the figure, $\bar{\xi}_1$ as a function of V is decreasing, as long as $V < \overline{V}$ (and strictly decreasing if V is sufficiently small so that $L(\xi)$ cuts the horizointal line at a level of ξ great than $V_{\min}$).

Last, suppose $V \geq \overline{V}$. Then the slope of the function L over the interval $[V_{\min}, V]$ would be negative and it should be obvious that $\bar{\xi}_1$ should be constant at $V_{\min}$.

Proof of (iv) of the theorem Fix V. The proof takes three steps.

Step 1 We show that with the optimal contract, an outside offer $\xi \geq V$ is matched if and only if $U^*(\xi) \geq \rho U_0(U^*) - C_0$.

The *only if* part is obvious. The *if* part: For all $\xi \geq V$, (48) implies $L(\xi; V) = U^*(\xi) - (\rho U_0(U^*) - C_0)$. The desired result follows by applying Lemma 3.

Step 2 We show that for any $\xi \geq V$, the inequality $U^*(\xi) \geq \rho U_0(U^*) - C_0$ holds if and only if $\xi \leq V''$.

Consider $V = u(0)+\beta E(\xi)$. This valuve of the agent's expected utility can be achieved with a feasible contract that gives the agent first period consumption $c(V) = 0$, first period probability of retention $I(\xi, V) = 0$, and $V_r(\xi) = \max\{\xi, V\} : \forall \xi\}$. But this implies

$$U^*(u(0) + \beta E(\xi)) \geq \theta + \beta(\rho U_0(U^*) - C_0) \geq \rho U_0(U^*) - C_0.$$

The desired result then follows from the fact that U^* is concave by (i) of the theorem, which was already proven, and the definition of V''.

Step 3 We show that $\overline{\xi}_2$ as a function of V is increasing, but constant at V'' for $V \leq V''$. This follows directly from (46) and Step 2.

The proof of Theorem 2 is now complete.

F Proof of Corollary 2

Observe that for all $V \leq \overline{V}$, $u(c) = u(0) \leq (1-\beta)V$, by Step 1 of Case 1 of the proof of (i) of Lemma 4. Observe next that for all $V > \overline{V}$, given $\overline{\xi}_1 = V_{\min}$ by (iii) of Theorem 2, (47) implies

$$u(c) + \beta\left(\int_{V_{\min}}^{\overline{\xi}_2} \max\{\xi, V\} dF(\xi) + \int_{\overline{\xi}_2}^{V_{\max}} \xi dF(\xi)\right) = V. \tag{50}$$

Suppose $\overline{\xi}_2 \geq V$. Then (50) implies

$$\begin{aligned} u(c) &= V - \beta\left(\int_{V_{\min}}^{\overline{\xi}_2} \max\{\xi, V\} dF(\xi) + \int_{\overline{\xi}_2}^{V_{\max}} \xi dF(\xi)\right) \\ &= V - \beta \int_{\Sigma} \max\{\xi, V\} dF(\xi) \\ &< (1-\beta)V, \end{aligned}$$

where the second equality follows from $\xi = \max\{\xi, V\}$ for all $\xi \geq \overline{\xi}_2$ given $\overline{\xi}_2 \geq V$.

Suppose $\overline{\xi}_2 < V$. Then (50) implies

$$u(c) + \beta\left(F(\overline{\xi}_2)V + \int_{\overline{\xi}_2}^{V_{\max}} \xi dF(\xi)\right) = V. \tag{51}$$

Furthermore, (48) implies

$$L(\overline{\xi}_2; V) = U^*(V) - (\rho U_0(U^*) - C_0) + (1/u'(c))(V - \overline{\xi}_2) = 0, \tag{52}$$

given $\overline{\xi}_2 < V$ and $\alpha = 1/u'(c)$ by Step 1 of Case 2 of the proof of (i) of Lemma 4. And (49) implies

$$U^*(V) = \frac{(\theta - c) + \beta(1 - F(\overline{\xi}_2))(\rho U_0(U^*) - C_0)}{1 - \beta F(\overline{\xi}_2)} \tag{53}$$

given $\overline{\xi}_1 = V_{\min}$ by (iii) of Theorem 2 and $\overline{\xi}_2 < V$. With these, (51) implies

$$\begin{aligned}
0 &= u(c) - (1-\beta F(\overline{\xi}_2))V + \beta \int_{\overline{\xi}_2}^{V_{\max}} \xi dF(\xi) \\
&= u(c) - (1-\beta F(\overline{\xi}_2))\{\overline{\xi}_2 - u'(c)[U^*(V) - (\rho U_0(U^*) - C_0)]\} + \beta \int_{\overline{\xi}_2}^{V_{\max}} \xi dF(\xi) \\
&= u(c) + u'(c)[(\theta - c) - (1-\beta)(\rho U_0(U^*) - C_0)] - (1-\beta F(\overline{\xi}_2))\overline{\xi}_2 + \beta \int_{\overline{\xi}_2}^{V_{\max}} \xi dF(\xi) \\
&\geq u(c) - u'(c)c - (1-\beta F(\overline{\xi}_2))\overline{\xi}_2 + \beta \int_{\overline{\xi}_2}^{V_{\max}} \xi dF(\xi) \\
&> u(c) - u'(c)c - (1-\beta F(V))V + \beta \int_{V}^{V_{\max}} \xi dF(\xi),
\end{aligned} \tag{54}$$

where the second equality follows from (52), the third equality follows from (53), the first inequality follows from $\rho U_0(U^*) - C_0 \leq \theta/(1-\beta)$, and the second inequality follows from that the last two terms together on the fourth line is a strictly decreasing function of $\overline{\xi}_2$ and $\overline{\xi}_2 < V$. Furthermore, given that $u(c) - u'(c)c$ is a strictly increasing function of c, $u(c) < (1-\beta)V$ if $K(V) \geq 0$ where

$$K(V) \equiv u(u^{-1}((1-\beta)V)) - u'(u^{-1}((1-\beta)V))u^{-1}((1-\beta)V) - (1-\beta F(V))V + \beta \int_{V}^{V_{\max}} \xi dF(\xi). \tag{55}$$

In fact, we have $K(V) \geq 0$ for all $V > \overline{V}$. This holds because

$$\begin{aligned}
K'(V) &= (1-\beta)\left(-\frac{u''(u^{-1}((1-\beta)V))u^{-1}((1-\beta)V)}{u'(u^{-1}((1-\beta)V))}\right) - (1-\beta F(V)) \\
&\geq \beta F(V) \\
&\geq 0,
\end{aligned}$$

where the first inequality follows from that the relative risk aversion coefficient is greater than $1/(1-\beta)$ for all $c \geq u^{-1}((1-\beta)\overline{V})$ and $V > \overline{V}$, and

$$\begin{aligned}
&K(\overline{V}) \\
=\ & u(u^{-1}((1-\beta)\overline{V})) - u'(u^{-1}((1-\beta)\overline{V}))u^{-1}((1-\beta)\overline{V}) - (1-\beta F(\overline{V}))\overline{V} + \beta \int_{\overline{V}}^{V_{\max}} \xi dF(\xi) \\
=\ & u(u^{-1}((1-\beta)\overline{V})) - u'(u^{-1}((1-\beta)\overline{V}))u^{-1}((1-\beta)\overline{V}) - u(0) \\
\geq\ & 0,
\end{aligned}$$

where the second equality follows from (6), and the inequality follows from that u is concave. This completes the proof of the corollary.

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