Collateral constraints and macroeconomic volatility*

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Abstract

We show how realistic occasionally binding collateral constraints increase macroeconomic volatility. Collateral constraints imply that the effect of consumers' choices on the price of collateral feeds back into the set of feasible choices, thus giving rise to multiple equilibria. We characterize how the possibility of multiple equilibria depends on aggregate wealth. We find that for low levels of wealth the economy is vulnerable to changes of consumer confidence (sunspots) which increase the volatility of prices and consumption.

Keywords: Collateral constraints, Business cycles, Multiple equilibria, Consumer confidence, Sunspots.
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1 Introduction

The recent recession in the US has been preceded by a period of high leverage among households (see, for example, Mian and Sufi, 2011, and Mian et al., 2011). During the 2007-2009 recession house prices fell by 30%, sharply reducing the value of housing collateral. Consumer spending especially for durables fell by 15% and output contracted by 7% during the same period (see also Heathcote and Perri, 2011, and Ohanian, 2011).

In this paper we construct a model which links these facts. In particular, we show how low levels of wealth can amplify macroeconomic fluctuations due to an occasionally binding collateral constraint. This constraint implies that the effect of consumers' choices on the price of collateral feeds back into the set of feasible choices, thus giving rise to multiple equilibria. The resulting extrinsic uncertainty due equilibrium multiplicity increases macroeconomic volatility, similarly as in recent research by Perri and Quadrini (2011) and Heathcote and Perri (2011). We characterize how the possibility of multiple equilibria in our model depends on aggregate wealth. We find that for low levels of wealth the economy is vulnerable to changes of consumer confidence (sunspots) which increase the volatility of prices and consumption as the economy switches between high and low demand equilibria. Since individual agents do not internalize the effect of their choices on the evolution of aggregate wealth, and thus the likelihood of extrinsic uncertainty due to sunspots, there is scope for policy intervention to mitigate macroeconomic fluctuations.

Most closely related to our research are the papers by Perri and Quadrini (2011) and Heathcote and Perri (2011). Compared with Perri and Quadrini (2011), collateral constraints in our model are not restricting the financing opportunities of firms (see also Kiyotaki and Moore, 1997, or Mendoza, 2010) but the spending of consumers. Furthermore, whereas the possible resale values of collateral in Perri and Quadrini (2011) are exogenous, in our model these values are endogenously determined.[1]

As in Heathcote and Perri (2011) we find that extrinsic uncertainty due to equilibrium multiplicity is more important in economies with low aggregate wealth.[2] Whereas the analysis of the dynamics in Heathcote and Perri (2011) is local around the stable steady state, as in the classic sunspot literature surveyed in Benhabib and Farmer (1999), in our model extrinsic uncertainty, due to random coordination on one of the multiple equilibria, directly enters in the

[1] The effect of leverage on fluctuations has also been analyzed in the literature on leverage cycles. In this literature ex-ante heterogeneity of agents allows to endogenously determine the amount of leverage. See, for example, Geanakoplos (2009) and the references therein.

[2] See also the empirical evidence in Gourinchas and Obstfeld (2012) on the positive association between credit market booms and financial crises for developed and developing countries.

recursive formulation of consumers' maximization problem as in Cole and Kehoe (2000).

The rest of the paper is organized as follows. We present the model and its recursive formulation in Section 2 before we discuss the method used for the solution in Section 3. In Section 4 we study a two-period version of our model to provide analytic results and intuition for the role of equilibrium multiplicity before we conclude.

2 Model

The representative consumer with a (finite or infinite) time horizon T derives utility from housing h_t and consumption c_t, where the period utility $U(c_t, h_t)$ is assumed to be concave in c and h. Whereas all idiosyncratic risk is insured, the representative agent cannot perfectly insure aggregate income shocks. The timing is as follows. After the draw of uncertain exogenous labor income y_t, the agent chooses consumption c_t and the endogenous assets in the next period, i.e., the amount of housing h_{t+1} and the financial risk-free asset a_{t+1}. The agent then derives utility from consumption and housing before the returns of the assets accrue. The financial asset earns risk-free return r, taken as given in the small-open economy, and p_t is relative price of housing.[3]

Markets are incomplete since no assets exist with payoffs that condition on the aggregate shock realization and the agent cannot borrow more than the liquidation value of the housing collateral, denoted by the fraction $\mu \in [0,1)$ (see Kiyotaki and Moore, 1997, for a discussion of the microfoundations). As a consequence, the history of the aggregate shocks matters for the level of the endogenous state variables.

The representative agent maximizes the following problem:

$$\max_{a_{t+1}, h_{t+1}} \sum_{s=t}^{\infty} \beta^{s-t} E_t U(c_s, h_s)$$

subject to

$$\begin{aligned} a_{t+1} + p_t h_{t+1} + c_t &= (1+r)a_t + p_t h_t + y_t \\ (1+r)a_{t+1} &\geq -\mu p_t h_{t+1} \\ h_{t+1} &\geq h_{\min} \\ A_{t+1} &= F(A_t, ...),\ H_{t+1} = 1 \text{ for all } t, \\ \ln y_t &= \rho \ln y_{t-1} + \varepsilon_t, \text{ with } \varepsilon \sim \mathcal{N}(0, \sigma_\varepsilon^2). \end{aligned}$$

[3]For simplicity we abstract from depreciation of the housing stock.

Besides the standard budget constraint, borrowing is constrained by the collateralizable value of housing, $\mu p_t h_{t+1}$, and housing is restricted to have a minimum size $h_{\min} \geq 0$. The timing assumption in the collateral constraint is motivated by existing regulation which allows households to borrow a fraction μ against the current value of the chosen housing stock. This assumption, as in Jeanne and Korinek (2011) or Brumm et al. (2011) p.10, corresponds to a margin requirement where the price p_t is determined by the choices of all households in equilibrium.[4]

As we will see below, household face uncertainty both because of intrinsic uncertainty due to stochastic income, where log-income follows an AR(1)-process with first-order autocorrelation ρ, and because of extrinsic uncertainty due to possible random switches between multiple equilibria. The existence of multiple equilibria depends on the state space and households rationally form expectations by predicting the aggregate states A_{t+1} and H_{t+1}. The latter prediction is trivial since we assume fixed housing supply and normalize the aggregate housing stock to unity.[5]

2.1 Recursive formulation of the household problem

We carefully define auxiliary state variables when formulating the recursive problem in order to simplify the numerical solution of the dynamic stochastic maximization problem with multiple equilibria. The state variables of the consumer consist of the individual state variables and aggregate state variables. We define the equity which the consumer holds voluntarily, beyond the margin requirement, as

$$x_t \equiv (1+r)a_t + \mu p_{t-1} h_t.$$

The collateral constraint requires $x_{t+1} \geq 0$. Formulating the recursive problem with voluntary equity as a state variable has the advantage that we can separately solve for constrained ($x_{t+1} = 0$) and unconstrained ($x_{t+1} > 0$) equilibria in the numerical algorithm which takes advantage of the endogenous gridpoint method. The cost of choosing voluntary equity as state variable is that we need to condition on the price of the previous period p_{t-1} as an additional auxiliary variable for the solution (see also Díaz and Luengo-Prado, 2008, or Bajari et al., 2010).

[4]Alternatively, the timing assumption may be motivated with a moral-hazard problem in which the creditor can observe during the period of loan origination whether the borrower decides to cheat and default in the next period (see Jeanne and Korinek, 2011). Under this "time to cheat" assumption, the creditor can appropriate the collateral good and resell it at current prices until the end of the period if the borrower cheats.

[5]Changes in the aggregate housing stock take time and we are interested in the behavior of the economy at business cycle frequency.

We denote as π_t the sunspot variable that is independently and uniformly distributed on the unit interval. The realization of π_t is a state variable since it determines the equilibrium on which agents coordinate in the presence of multiple equilibria. We summarize the aggregate state variables as $s_t = (y_t, X_t, H_t, \pi_t)$ where the aggregate housing stock is normalized to $H_t = 1$, for all t. Since the equilibrium price in the current period p_t is a function of the aggregate state, the recursive consumer problem is

$$\begin{aligned}
&v_t\left(x_t, h_t, p_{t-1}; s_t\right) \\
&= \max_{a_{t+1}, h_{t+1}} [U(\underbrace{x_t + \left(p_t - \mu p_{t-1}\right) h_t + y_t - a_{t+1} - p_t h_{t+1}}_{c_t}, h_t) \\
&+ \beta E_t v_{t+1}(x_{t+1}, p_t, h_{t+1}; s_{t+1})]
\end{aligned} \tag{1}$$

subject to the constraints

$$\begin{aligned}
a_{t+1} + p_t h_{t+1} + c_t &= x_t + \left(p_t - \mu p_{t-1}\right) h_t + y_t \\
x_t &\equiv (1+r) a_t + \mu p_{t-1} h_t \\
x_{t+1} &\geq 0 \\
h_{t+1} &\geq h_{\min} \\
X_{t+1} &= G(p_{t-1}; s_t) \\
H_{t+1} &= 1, \text{ for all } t.
\end{aligned}$$

The last two constraints require that the consumer rationally predicts the evolution of the aggregate state variables and thus the occurrence of multiple equilibria. Note that E_t is the expectation operator conditional on information available at time t and expectations are formed for intrinsically and extrinsically uncertain events.

2.2 Period utility and derivatives

For later reference we now derive the first-order conditions making common parametric assumptions for the utility function:

$$U(c_t, h_t) = \frac{\psi(c_t, h_t)^{1-\sigma} - 1}{1-\sigma} \text{ and } \psi(c_t, h_t) = c_t^{\theta} h_t^{1-\theta},$$

so that

$$U(c_t, h_t) = \frac{c_t^{\theta(1-\sigma)} h_t^{(1-\theta)(1-\sigma)} - 1}{1-\sigma}.$$

Hence

$$\frac{\partial U(c_t, h_t = 1)}{\partial c_t} = \theta c_t^{\theta(1-\sigma)-1},$$

$$\frac{\partial U(c_t, h_t = 1)}{\partial h_t} = (1-\theta)\, c_t^{\theta(1-\sigma)}.$$

2.3 Envelope conditions

We denote the derivative of the value function with respect to the first argument as

$$\frac{\partial v_t}{\partial x_t} = \frac{\partial U(c_t, h_t = 1)}{\partial c_t} = \theta c_t^{\theta(1-\sigma)-1}, \tag{2}$$

and the derivative with respect to its second argument as

$$\begin{aligned}\frac{\partial v_t}{\partial h_t} &= \frac{\partial U(c_t, h_t = 1)}{\partial h_t} + \frac{\partial U(c_t, h_t = 1)}{\partial c_t}\left(p_t - \mu p_{t-1}\right) \\ &= (1-\theta)\, c_t^{\theta(1-\sigma)} + \theta c_t^{\theta(1-\sigma)-1}\left(p_t - \mu p_{t-1}\right).\end{aligned} \tag{3}$$

2.4 First-order conditions

Assigning the multiplier κ_t to the collateral constraint, the first-order conditions, which characterize equilibrium decisions of problem (1), are

$$-\kappa_t(1+r) + \frac{\partial U(c_t, h_t = 1)}{\partial c_t} = (1+r)\beta E_t \frac{\partial v_{t+1}}{\partial x_{t+1}} \tag{4}$$

for a_{t+1}, and

$$-\kappa_t \mu p_t + \frac{\partial U(c_t, h_t = 1)}{\partial c_t} p_t = \mu p_t \beta E_t \frac{\partial v_{t+1}}{\partial x_{t+1}} + \beta E_t \frac{\partial v_{t+1}}{\partial h_{t+1}} \tag{5}$$

for h_{t+1}, where we use that, in equilibrium, $h_t = H_t = 1 > h_{\min}$ for all t.

2.5 Equilibrium definition

Since the infinite horizon problem is stationary, we drop time indices and denote lags with "$_{-1}$" and leads with a prime "$'$".

An equilibrium consists of a value function v, policy functions a' and h', price function p and an equation of motion for aggregate voluntary equity X' such that:

- v is the value function of the representative consumer with maximizing choices a' and h',
- the price p clears the housing market,

- individual choices are consistent with aggregates: $x' = X'$ and $h' = H' = 1$,
- and the lagged price which enters the recursive consumer problem is the equilibrium price of the previous period $p = (p_{-1})'$.

3 Numerical solution with multiple equilibria

We solve the model by applying the endogenous gridpoint method. More specifically, we extend the algorithm developed in Hintermaier and Koeniger (2010) for our model with endogenous house prices. The endogenous gridpoint method has the major advantage that it conditions on the grid of the future endogenous state variable. This allows to solve separately for the unconstrained and constrained equilibria since $x' = 0$ if the consumers are constrained and $x' > 0$ if they are not. Numerical solution of models with multiple equilibria is very cumbersome instead with methods that condition on the grid of current endogenous state variables (see Feng et al., 2011).

For the numerical implementation with the endogenous gridpoint method, we start with an exogenous grid for the future endogenous state variable x', $G_{x'} \equiv \{x'_1, x'_2, ..., x'_I\}$, the state p_{-1}, $G_{p_{-1}} \equiv \{p_{-1,1}, p_{-1,2}, ..., p_{-1,J}\}$, and discretize the stochastic process for y on the grid $G_y \equiv \{y_1, y_2, ..., y_K\}$ using the Rouwenhorst method described in Kopecky and Suen (2010).

3.1 Unconstrained choices

Since $\kappa = 0$ in the unconstrained case, equations (2), (3), (4) and (5) imply that the price is given by

$$
\begin{aligned}
p_{ik} &= \frac{\beta E\left(\frac{\partial U(c'_{ik},h'=1)}{\partial h'} + \frac{\partial U(c'_{ik},h'=1)}{\partial c'} p'\right)}{\frac{\partial U(c,h=1)}{\partial c}} \qquad (6)\\
&= \frac{\beta E\left(\frac{\partial U(c'_{ik},h'=1)}{\partial h'} + \frac{\partial U(c'_{ik},h'=1)}{\partial c'} p'\right)}{\beta(1+r)E\left(\frac{\partial U(c'_{ik},h'=1)}{\partial c'}\right)}\\
&= \frac{E\left((1-\theta)\left(c'_{ik}\right)^{\theta(1-\sigma)} + \theta\left(c'_{ik}\right)^{\theta(1-\sigma)-1} p'\right)}{(1+r)E\left(\theta\left(c'_{ik}\right)^{\theta(1-\sigma)-1}\right)}.
\end{aligned}
$$

This is a root-finding problem since $c'_{ik}(x'_i, h' = 1, p_{ik}; s')$. For each income state we find the smallest price of the unconstrained solution at $x'_1 = 0$. We call this price $\overline{p}_k$. Positive marginal utility of consumption and housing imply that the relative price $p \geq 0$ (this follows by backward induction, starting from the last period in which $p' = 0$).

Using (4), we then get

$$c_{ik} = \left[\frac{1+r}{\theta}\beta E\frac{\partial v}{\partial x'}\right]^{\frac{1}{\theta(1-\sigma)-1}} \tag{7}$$
$$= \left[\beta\left(1+r\right)E\left(\left(c'_{ik}\right)^{\theta(1-\sigma)-1}\right)\right]^{\frac{1}{\theta(1-\sigma)-1}}$$

and use the definition of voluntary equity to determine

$$a'_{ik} = \frac{x'_i - \mu p_{ik}}{1+r}. \tag{8}$$

The budget constraint then implies

$$x_{ijk} = a'_{ik} + c_{ik} + \mu p_{-1,j} - y_k. \tag{9}$$

3.2 Constrained choices

The lowest price $\overline{p}_k$ for the unconstrained choices, computed above for each income state at $x'_1 = 0$, by continuity is also the highest price of the constrained choices. We specify the grid $G_{p,k} \equiv \{p_1, p_2,\ ...,\ p_{l-1},\ p_l,\ p_{l+1},\ ...,\ \overline{p}_k\}$ for prices $p \in (0, \overline{p}_k]$ that support constrained choices. For $x'_1 = 0$ in the constrained case, it then follows from the collateral constraint that

$$a'_{1l} = -\frac{\mu}{1+r}p_l. \tag{10}$$

Combining equations (4) and (5) by substituting out the current marginal utility of consumption, the multiplier of the collateral constraint is given by

$$\kappa_{1kl} \tag{11}$$
$$= \frac{\beta}{1+r-\mu}\left\{\frac{1}{p_l}E\left(\frac{\partial U(c'_{1kl}, h'=1)}{\partial h'}\right) + E\left(\frac{\partial U(c'_{1kl}, h'=1)}{\partial c'}\left(\frac{p'}{p_l} - (1+r)\right)\right)\right\}$$
$$= \frac{\beta}{1+r-\mu}\left\{\frac{1}{p_l}E\left(\left(1-\theta\right)\left(c'_{1kl}\right)^{\theta(1-\sigma)}\right) + E\left(\theta\left(c'_{1kl}\right)^{\theta(1-\sigma)-1}\left(\frac{p'}{p_l} - (1+r)\right)\right)\right\}.$$

Equation (4) implies for the period utility assumed above that

$$c_{1kl} = \left[\frac{1+r}{\theta}\left(\beta E\frac{\partial v}{\partial x'} + \kappa_{1kl}\right)\right]^{\frac{1}{\theta(1-\sigma)-1}} \tag{12}$$
$$= \left[\frac{1+r}{\theta}\left(\beta\theta E\left[\left(c'_{1kl}\right)^{\theta(1-\sigma)-1}\right] + \kappa_{1kl}\right)\right]^{\frac{1}{\theta(1-\sigma)-1}}.$$

The budget constraint then determines the endogenous grid of the current endogenous state variable x:

$$x_{1jkl} = a'_{1l} + c_{1kl} + \mu p_{-1,j} - y_k.$$

4 Multiple equilibria

Having discussed the numerical solution, we now discuss equilibrium multiplicity more in detail in this section. We first derive analytic results for specific preferences in a two-period model before we illustrate equilibrium multiplicity for more general preferences. Finally, we give an example for how equilibrium multiplicity enters the recursive problem of consumers in a dynamic model with more than two periods. For the purposes of this section, we have to reintroduce time indexes in the notation.

4.1 Analytic example

We show analytically how multiple equilibria may arise if we restrict the model to two periods, T and $T-1$, and assume quasi-linear preferences. This allows insights into the key mechanisms of the model. We assume that period utility is given by

$$U(c_t, h_t) = -\frac{a}{2}(\bar{c} - c_t)^2 + h_t,$$

so that

$$\frac{\partial U(c_t, h_t = 1)}{\partial c_t} = a(\bar{c} - c_t)$$

and

$$\frac{\partial U(c_t, h_t = 1)}{\partial h_t} = 1.$$

The preferences would imply in a static setting that, at an interior optimum with $h_t > 0$ and $c_t > 0$, all additional income is spend on housing h_t. As usual, we assume that $c_t \leq \bar{c}$ to ensure a positive marginal utility of consumption. The quadratic utility function for consumption c_t requires restrictions on the parameter or state space so that $0 < c_t < \bar{c}$ with probability 1 (that is the constraints $c_t \geq 0$ and $c_t \leq \bar{c}$ are not binding for any possible contingency).

Since all is consumed in the last period, $p_T = 0$, $a_{T+1} = 0$ and $h_{T+1} = 0$, so that the budget constraint with $h_T = 1$ implies $c_T = x_T - \mu p_{T-1} + y_T$, where we restrict our attention to the state space where $c_T > 0$. Given our assumption about preferences, the following proposition provides explicit analytic expressions for the constrained and unconstrained choices in the algorithm described in the previous section.

Proposition 1 *If preferences are given by* $U(c_t,h_t) = -\frac{a}{2}(\bar{c}-c_t)^2 + h_t$, $t \in \{T-1, T\}$, *unconstrained choices in period* $T-1$ *with* $x_T > 0$ *are characterized by*

$$p_{T-1} = \frac{-a(1+r)\left(\bar{c} - x_T - E_{T-1}y_T\right) + \sqrt{a^2(1+r)^2\left(\bar{c} - x_T - E_{T-1}y_T\right)^2 + 4a(1+r)\mu}}{2a(1+r)\mu},$$

$$c_{T-1} = \left[1-\beta(1+r)\right]\bar{c} + \beta(1+r)\left(x_T + E_{T-1}y_T - \mu p_{T-1}\right),$$

$$a_T = \frac{x_T - \mu p_{T-1}}{1+r},$$

resulting in

$$\begin{aligned} x_{T-1} &= \bar{c} - \left[\frac{\mu}{1+r} + \beta(1+r)\mu\right] p_{T-1} + \mu p_{T-2} - y_{T-1} - \beta(1+r)\left(\bar{c} - E_{T-1}\left(y_T\right)\right) \\ &\quad + \left[\frac{1}{1+r} + \beta(1+r)\right] x_T. \end{aligned}$$

Constrained choices in period $T-1$ *with* $x_T = 0$*, for* $p_{T-1} \in [\underline{p}_{T-1}, \overline{p}_{T-1}]$ *and* $\overline{p}_{T-1} > \underline{p}_{T-1} > 0$*, are characterized by*

$$\kappa = \frac{\beta}{1+r-\mu}\left[\frac{1}{p_{T-1}} - (1+r)E_{T-1}\left(a(\bar{c} + \mu p_{T-1} - y_T)\right)\right],$$

$$c_{T-1} = \bar{c} - \frac{\beta(1+r)}{a\left[1+r-\mu\right]}\left[\frac{1}{p_{T-1}} - \mu E_{T-1}\left(a(\bar{c} + \mu p_{T-1} - y_T)\right)\right],$$

$$a_T = -\frac{\mu}{1+r}p_{T-1},$$

resulting in

$$\begin{aligned} x_{T-1} = \bar{c} &+ \left[\frac{\beta(1+r)}{1+r-\mu}\mu - \frac{1}{1+r}\right]\mu p_{T-1} + \mu p_{T-2} - y_{T-1} + \frac{\mu\beta(1+r)}{1+r-\mu}\left(\bar{c} - E_{T-1}\left(y_T\right)\right) \\ &- \frac{\beta(1+r)}{a\left[1+r-\mu\right]}\frac{1}{p_{T-1}}, \end{aligned}$$

with

$$\overline{p}_{T-1} \equiv \frac{-a(1+r)\left(\bar{c} - E_{T-1}y_T\right) + \sqrt{\left[a(1+r)\left(\bar{c} - E_{T-1}y_T\right)\right]^2 + 4a(1+r)\mu}}{2a(1+r)\mu},$$

$$\underline{p}_{T-1} \equiv \frac{-\left(\bar{c} + \frac{\beta(1+r)}{1+r-\mu}\mu\left[\bar{c} - E_{T-1}\left(y_T\right)\right]\right) + \sqrt{\left(\bar{c} + \frac{\beta(1+r)}{1+r-\mu}\mu\left[\bar{c} - E_{T-1}\left(y_T\right)\right]\right)^2 + 4\frac{\beta^2(1+r)^2}{a[1+r-\mu]^2}\mu^2}}{2\frac{\beta(1+r)}{1+r-\mu}\mu^2}.$$

The proof in the Appendix also shows which prices $p_{T-1} \in [\underline{p}_{T-1}, \overline{p}_{T-1}]$ map into $x_{T-1} \geq 0$. The analytic expressions in the previous proposition, besides being of interest in their own right, allow us to derive an explicit condition for equilibrium multiplicity in period $T-1$.

In order to establish the existence of multiple equilibria in period $T-1$, we need to find values for x_{T-1} which are attained both by constrained and unconstrained choices when mapping back from the exogenous grid for x_T. That is, we need to show that there is an overlap of the intervals of x_{T-1} that are attained by constrained and unconstrained choices, as is done in the following proposition.

Proposition 2 *If constrained and unconstrained equilibria exist for some $x_{T-1} \geq 0$, there exist parameter values for which multiple equilibria occur in a non-empty interval of x_{T-1}. At most one of the equilibria is unconstrained.*

The proof in the Appendix shows that existence of multiple equilibria does not depend on the state variables p_{T-2} and y_{T-1} since they affect x_{T-1} in the same way for constrained and unconstrained choices. Figures 1 and 2 illustrate equilibrium multiplicity for parameter values that are not meant to be realistic but are entirely chosen for illustration purposes (see the notes in the Figure). Figure 1 illustrates that a necessary condition for multiple equilibria is that the function $x_{T-1}(p_{T-1}, ...)$ for constrained choices is non-monotonic (with continuity this condition is also sufficient). This is because for the assumed utility function, as shown in the proof of the proposition, $p_{T-1}(x_{T-1}, ...)$ is a positive monotonic function for *un*constrained choices. The non-empty interval of x_{T-1} in Figure 1, in which there are two constrained equilibria and one unconstrained equilibrium, requires that the price is very elastic as demand increases and the loan-to-value ratio μ needs to be such that a low-demand-constrained and high-demand-(un)constrained equilibrium can coexist for a given value of the state variable x_{T-1}.[6]

Figure 2 illustrates this further. Multiplicity arises for intermediate values of μ but not for μ close to 0 or 1. If $\mu = 0$, there is no feedback from the price p_{T-1} on borrowing opportunities since the collateral constraint is $a_T \geq 0$ in this case. For large μ close to 1, prices become positive monotonic in x_{T-1} also for constrained choices. Since more demand for housing does not constrain consumption choices if housing can be fully collateralized, the response of the price p_{T-1} to changes in voluntary equity x_{T-1} for constrained choices is then similar to the response of prices for unconstrained choices.

[6]Jeanne and Korinek (2011) offer a preliminary discussion of possible equilibrium multiplicity before focussing on parameter values that imply equilibrium uniqueness.

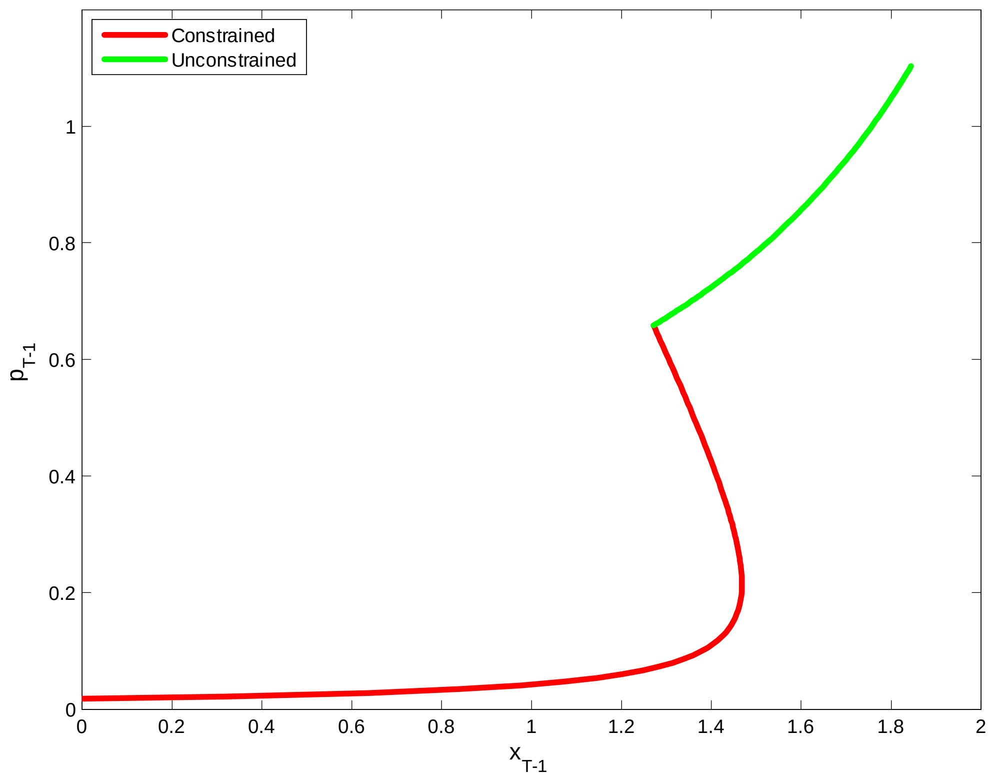

Figure 1: x_{T-1} as a function of p_{T-1}. Notes: The values of the parameters are $\mu = 0.7$, $r = 0.04$, $\beta = 0.01$, $a = 1$, $\bar{c} = 2$. Furthermore, $E_{T-1}y_T = 1$, $y_{T-1} = 0.97$, $p_{T-2} = 1$.

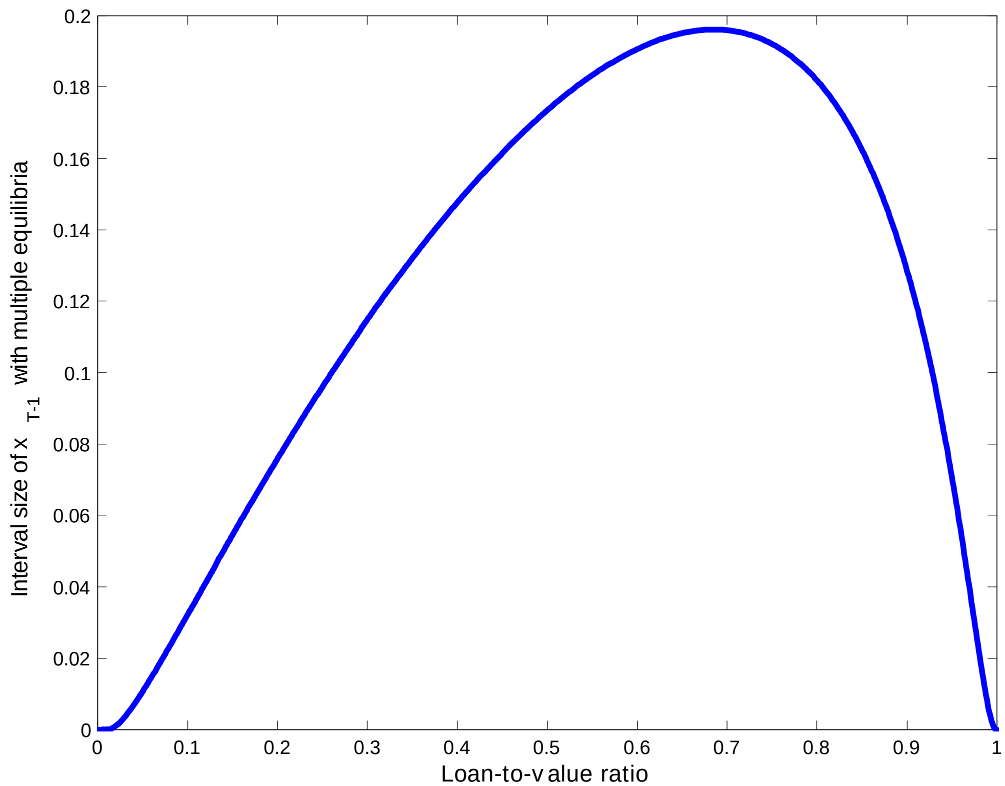

Figure 2: The size of the interval fo x_{T-1} in which multiple equilibria exist, as a function of the loan-to-value ratio μ. Notes: The values of the parameters are $\mu = 0.7$, $r = 0.04$, $\beta = 0.01$, $a = 1$, $\overline{c} = 2$. Furthermore, $E_{T-1}y_T = 1$, $y_{T-1} = 0.97$, $p_{T-2} = 1$.

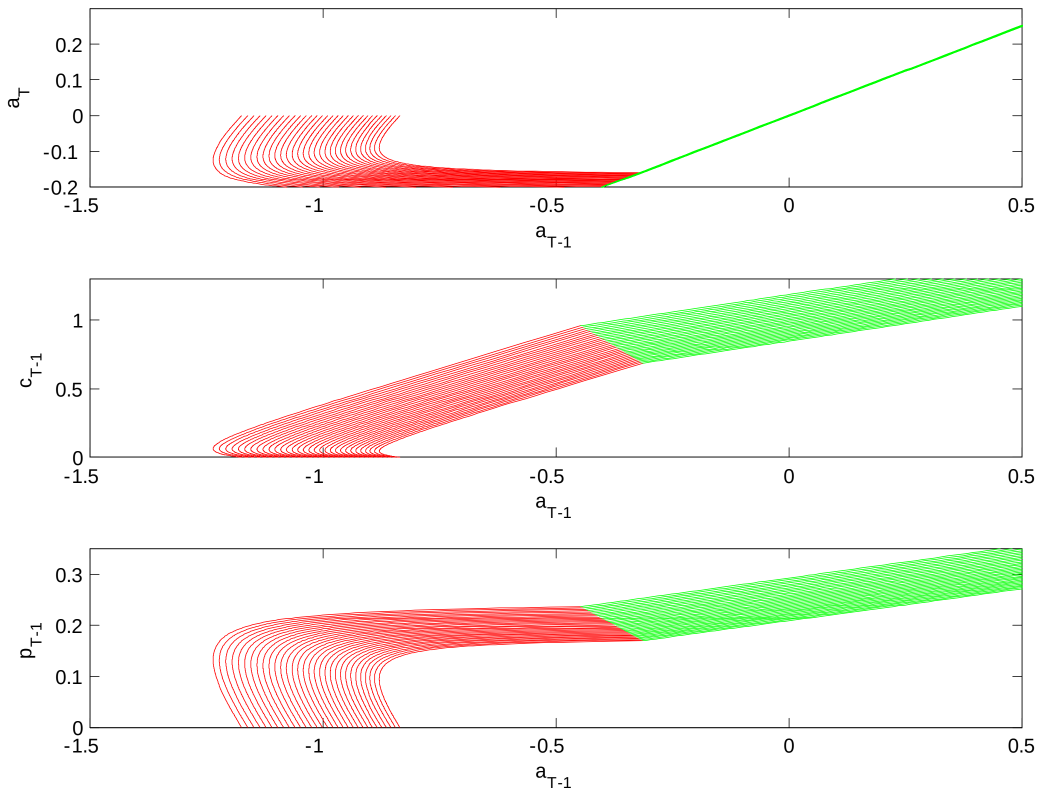

Figure 3: Policy and price correspondences for different values of financial assets a_{T-1}, for 31 income states y_{T-1}. Notes: red loci are constrained choices; green loci are unconstrained choices. The values of the parameters are $\mu = 0.95$, $r = 0.01$, $\beta = 0.99$, $\sigma = 1$, $\theta = 0.8$. Furthermore we assume that log income follows a first-order Markov process with 31 states, persistence $\rho = 0.98$ and a standard deviation of the innovations $\sigma_\epsilon = 0.0062$.

4.2 More general preferences

After these analytic results, we now illustrate equilibrium multiplicity for the more general preferences introduced in Section 2. Although there are no closed form solutions for these preferences, commonly used utility functions with realistic non-linear marginal utility of consumption and the complementarity between housing and consumption generate equilibrium multiplicity for more plausible parameter values.

Figure 3 illustrates that, due to the multiplicity of equilibria, the policies and the price are correspondences and not functions of the endogenous state variable. Note that we have reduced the state space by replacing the two auxiliary variables x_{T-1} and p_{T-2}, which are used for convenience in the numerical solution, with a_{T-1} in order to simplify the visual presentation.[7] The price and policy correspondences are plotted for each of the 31 income states. As in standard models with persistent income shocks, larger income draws increase borrowing, consumption and the price of housing for most values of a_{T-1}. For low levels of financial wealth, however, equilibrium multiplicity arises for constrained choices (which are plotted in red in the figure) and low levels of financial wealth. The extent of extrinsic uncertainty resulting due to multiple equilibria thus depends on the aggregate endogenous state variable financial wealth. This implies that shocks may be propagated endogenously through changes of extrinsic uncertainty. Moreover, there is scope for policy intervention not only due to market incompleteness but also because individual consumers do not internalize the aggregate effect of their choices on extrinsic uncertainty.

Let us now interpret the different equilibria further which arise for low levels of financial wealth. For the example plotted in Figure 3, consumers are at the collateral constraint in both equilibria. In the high-demand equilibrium, the relatively higher consumption and housing are financed with more borrowing. This is feasible since the higher demand drives up the price of the housing collateral which in turn increases the borrowing limit. In the low-demand equilibrium, consumption and housing demand are lower resulting in a lower price. Thus, the value of the collateral is lower and the borrowing limit tighter restricting demand to its lower level.

4.3 Beyond two periods

Since multiple equilibria arise in period $T-1$, the extrinsic uncertainty due to multiple equilibria and its dependance on the state space, enters the recursive problem which consumers solve in period $T-2$. To clarify the role of extrinsic uncertainty, consider the example in which there are at most two equilibria and there is only extrinsic uncertainty (due to random coordination on one of the equilibria) and no intrinsic uncertainty (due to income shocks). The aggregate state then simplifies to $s_t = (X_t, H_t, \pi_t)$. Further assume that π_t is identically distributed on the unit interval across periods. Without loss of generality, for realizations of π_t below threshold $\pi^*(X_{T-1}, p_{T-2}) \in [0;1]$ agents coordinate on the first equilibrium whereas they coordinate on the second equilibrium if $\pi_t > \pi^*(X_{T-1}, p_{T-2})$. Thus, if there exists only the first equilibrium at state variables X_{T-1} and p_{T-2}, $\pi^*(X_{T-1}, p_{T-2}) = 1$ and the probability of the first equilibrium $\eta(X_{T-1}, p_{T-2}) = 1$. If both equilibria exist, then $0 < \pi^*(X_{T-1}, p_{T-2}) < 1$ and

[7] Substituting the definition of x_{T-1} into the budget constraint and solving for a_{T-1}, we have $a_{T-1} = (a_T + c_{T-1} - y_{T-1})/(1+r)$.

$0 < \eta(X_{T-1}, p_{T-2}) < 1$.[8] Then

$$
\begin{aligned}
& v_{T-2}(x_{T-2}, h_{T-2}, p_{T-3}; s_{T-2}) \\
= & \max_{a_{T-1}, h_{T-1}} [U(\underbrace{x_{T-2} + (p_{T-2} - \mu p_{T-3})\, h_{T-2} + y_{T-2} - a_{T-1} - p_{T-2} h_{T-1}}_{c_{T-2}}, h_{T-2}) \\
& + \beta\{\eta(X_{T-1}, p_{T-2}) v_{T-1}(x_{T-1}, h_{T-1}, p_{T-2}; \underbrace{X_{T-1}, H_{T-1}, \pi_{T-1} \leq \pi^*(X_{T-1}, p_{T-2})}_{s_{T-1}}) \\
& + (1 - \eta(X_{T-1}, p_{T-2})) v_{T-1}(x_{T-1}, h_{T-1}, p_{T-2}; \underbrace{X_{T-1}, H_{T-1}, \pi_{T-1} > \pi^*(X_{T-1}, p_{T-2})}_{s_{T-1}})\}].
\end{aligned}
$$

where η is the probability with which one of the possible equilibria occurs. Note that η depends on the aggregate endogenous state variable which agents rationally predict for given state variables in $T-2$. Importantly, the individual consumer takes the law of motion for the aggregate state and prices as given.

5 Quantitative application

To be completed.

6 Conclusion

We have shown how collateral constraints can generate equilibrium multiplicity, especially for low levels of aggregate wealth. The resulting extrinsic uncertainty in certain parts of the state space, due to random coordination on equilibria, introduces the possibility of a propagation mechanism for aggregate shocks which has the potential to amplify macroeconomic fluctuations. Since individual agents do not take into account how their actions affect extrinsic uncertainty, just as they take prices as given, there is scope for policy intervention to mitigate the size of macroeconomic fluctuations.

Appendix

Proofs of Propositions

Proof of Proposition 1

We follow the structure of the algorithm described in Section 2. First we derive the explicit expressions for unconstrained choices and then consider the

[8] If π_t is uniformly distributed, $\eta(X_{T-1}, p_{T-2}) = \pi^*(X_{T-1}, p_{T-2})$.

constrained choices. Finally, we characterize the interval of feasible prices p_{T-1} which support constrained choices.

Unconstrained choices

For given x_T and $\kappa = 0$ in the unconstrained case, the first-order condition for financial assets (7) implies

$$a(\overline{c} - c_{T-1}) = \beta(1+r)E_{T-1}\left(a(\overline{c} - c_T)\right).$$

Since all is consumed in the last period, $p_T = 0$, $a_{T+1} = 0$ and $h_{T+1} = 0$, so that the budget constraint with $h_T = 1$ implies

$$c_T = x_T - \mu p_{T-1} + y_T. \tag{13}$$

Solving the first-order condition for c_{T-1}, we get

$$\begin{aligned} c_{T-1} &= \overline{c} - \frac{\beta}{a}(1+r)E_{T-1}\left(a(\overline{c} - x_T + \mu p_{T-1} - y_T)\right) \qquad (14) \\ &= \left[1 - \beta(1+r)\right]\overline{c} + \beta(1+r)\left(x_T + E_{T-1}y_T - \mu p_{T-1}\right), \end{aligned}$$

where $E_{T-1}x_T = x_T$ since the agent knows that $p_T = 0$ in the last period.

The asset-pricing equation (6) can then be solved for the price p_{T-1}, using $p_T = 0$ and the parametric assumption on the utility function:

$$\begin{aligned} p_{T-1} &= \frac{\beta}{a(\overline{c} - c_{T-1})} \\ &= \frac{\beta/a}{\overline{c} - \left[1 - \beta(1+r)\right]\overline{c} - \beta(1+r)\left(x_T + E_{T-1}y_T - \mu p_{T-1}\right)} \\ &= \frac{1}{a(1+r)}\frac{1}{\overline{c} - x_T + \mu p_{T-1} - E_{T-1}y_T}. \end{aligned}$$

Rearranging this equation to

$$a(1+r)p_{T-1}\left(\overline{c} - x_T + \mu p_{T-1} - E_{T-1}y_T\right) = 1$$

we get a quadratic equation

$$a(1+r)\mu\left(p_{T-1}\right)^2 + a(1+r)\left(\overline{c} - x_T - E_{T-1}y_T\right)p_{T-1} - 1 = 0$$

with solution(s)

$$p_{T-1} = \frac{-a(1+r)\left(\overline{c} - x_T - E_{T-1}y_T\right) \pm \sqrt{a^2(1+r)^2\left(\overline{c} - x_T - E_{T-1}y_T\right)^2 + 4a(1+r)\mu}}{2a(1+r)\mu}. \tag{15}$$

The root is a real number for $\mu \in [0, 1)$ and only the positive root results in a positive price p_{T-1}.[9] Note that for $\mu = 0$ the equation which determines the price is no longer quadratic (since the marginal rate of substitution no longer depends on p_{T-1}) so that

$$p_{T-1} = \frac{1}{a(1+r)\left[\bar{c} - x_T - E_{T-1}y_T\right]}$$

which also equals the limit $\lim_{\mu \to 0} p_{T-1}$, applying L'Hôpital's rule to equation (15).

We then rearrange the definition for x_T to

$$a_T = \frac{x_T - \mu p_{T-1}}{1+r}$$

and use the budget constraint to find x_{T-1} for each x_T, p_{T-2} and y_{T-1}:

$$\begin{aligned}
x_{T-1} &= \frac{x_T - \mu p_{T-1}}{1+r} + c_{T-1} + \mu p_{T-2} - y_{T-1} \\
&= \frac{x_T - \mu p_{T-1}}{1+r} + [1 - \beta(1+r)]\,\bar{c} + \beta(1+r)\left(x_T + E_{T-1}y_T - \mu p_{T-1}\right) \\
&- \left(p_{T-1} - \mu p_{T-2}\right) - y_{T-1} \\
&= \bar{c} - \left[\frac{\mu}{1+r} + \beta(1+r)\mu\right] p_{T-1} + \mu p_{T-2} - y_{T-1} - \beta(1+r)\left(\bar{c} - E_{T-1}\left(y_T\right)\right) \\
&+ \left[\frac{1}{1+r} + \beta(1+r)\right] x_T
\end{aligned} \tag{16}$$

Note that (15) implies that p_{T-1} is a function of x_T.

Constrained choices

If consumers are constrained, $x_T = 0$ in the last period and

$$c_T = -\mu p_{T-1} + y_T. \tag{17}$$

The combinations of debt and prices that are attainable at the constraint, where $x_T = 0$, are

$$a_T = -\frac{\mu}{1+r} p_{T-1}.$$

[9] We get two candidate solutions for p_{T-1} because the price p_{T-1} affects the marginal rate of substitution between housing and consumption in a non-linear way; and the marginal rate substitution in turn determines the price. For a given x_T, a larger p_{T-1} implies lower c_T. Note that there is no multiplicity arising in the mapping from x_T to p_{T-1} since only one solution turns out to be relevant (the one which results in a positive price). For an arbitrary utility function one needs to check that there is not more than one x_{T-1} that can be attained for a given x_T if consumers are unconstrained.

For the assumed utility function and given that $p_T = 0$, it follows from equation (11) that the multiplier κ is

$$\begin{aligned}
&\kappa\left[1+r-\mu\right] \\
&= \beta\left[\frac{1}{p_{T-1}} - (1+r)E_{T-1}\left(a(\overline{c}-c_T)\right)\right].
\end{aligned} \tag{18}$$

The right-hand side of this equation decreases in p_{T-1} and we denote the highest price level at which the collateral constraint is binding as $\overline{p}_{T-1}$. We characterize the feasible prices for constrained choices further below.

Equation (4) then implies for the assumed utility function that

$$\begin{aligned}
a(\overline{c}-c_{T-1}) &= \beta(1+r)E_{T-1}\left(a(\overline{c}-c_T)\right) \\
&\quad + \frac{\beta(1+r)}{1+r-\mu}\left[\frac{1}{p_{T-1}} - (1+r)E_{T-1}\left(a(\overline{c}-c_T)\right)\right] \\
&= \frac{\beta(1+r)\left[1+r-\mu\right] - \beta(1+r)^2}{1+r-\mu}E_{T-1}\left(a(\overline{c}-c_T)\right) + \frac{\beta(1+r)}{1+r-\mu}\frac{1}{p_{T-1}} \\
&= \frac{\beta(1+r)}{1+r-\mu}\left[\frac{1}{p_{T-1}} - \mu E_{T-1}\left(a(\overline{c}-c_T)\right)\right].
\end{aligned}$$

For $p_{T-1} \leq \overline{p}_{T-1}$ and $\mu \in [0,1)$ the right-hand side of this equation is positive and $c_{T-1} \leq \overline{c}$ so that the marginal utility of consumption is positive.

Recalling (17), we then solve the previous equation for c_{T-1} as a non-linear function of the states of period T, $x_T = 0$, y_T and p_{T-1}:

$$c_{T-1} = \overline{c} - \frac{\beta(1+r)}{a\left[1+r-\mu\right]}\left[\frac{1}{p_{T-1}} - \mu E_{T-1}\left(a(\overline{c}-c_T)\right)\right]. \tag{19}$$

Positive consumption, $c_{T-1} > 0$, implies a lower bound $\underline{p}_{T-1}$ which we characterize further below.

We then use the budget constraint to map $x_T = 0$ into x_{T-1}, for all admissible prices p_{T-1} at the collateral constraint. The budget constraint with $h_T = 1$ and $a_T = -\frac{\mu}{1+r}p_{T-1}$ implies

$$x_{T-1} = c_{T-1} - \frac{\mu}{1+r}p_{T-1} + \mu p_{T-2} - y_{T-1}.$$

Substituting out c_{T-1}, using (19), we get

$$\begin{aligned}
x_{T-1} = &-\frac{\mu}{1+r}p_{T-1} + \mu p_{T-2} - y_{T-1} \\
&+ \overline{c} - \frac{\beta(1+r)}{a\left[1+r-\mu\right]}\left[\frac{1}{p_{T-1}} - \mu E_{T-1}\left(a(\overline{c}-c_T)\right)\right],
\end{aligned} \tag{20}$$

where

$$c_T = -\mu p_{T-1} + y_T \leq \overline{c}.$$

Plugging the expression for c_T into (20), we get

$$\begin{aligned}
x_{T-1} &= -\frac{\mu}{1+r} p_{T-1} + \mu p_{T-2} - y_{T-1} \\
&+ \overline{c} - \frac{\beta(1+r)}{a\left[1+r-\mu\right]} \left[\frac{1}{p_{T-1}} - \mu E_{T-1}\left(a(\overline{c} + \mu p_{T-1} - y_T)\right)\right] \\
&= -\frac{\mu}{1+r} p_{T-1} + \mu p_{T-2} - y_{T-1} + \overline{c} + \frac{\beta(1+r)}{1+r-\mu} \mu \overline{c} \\
&+ \frac{\beta(1+r)}{1+r-\mu} \mu^2 p_{T-1} - \frac{\beta(1+r)}{1+r-\mu} \mu E_{T-1}\left(y_T\right) - \frac{\beta(1+r)}{a\left[1+r-\mu\right]} \frac{1}{p_{T-1}} \\
&= \overline{c} + \left[\frac{\beta(1+r)}{1+r-\mu}\mu - \frac{1}{1+r}\right] \mu p_{T-1} + \mu p_{T-2} - y_{T-1} + \frac{\mu\beta(1+r)}{1+r-\mu}\left(\overline{c} - E_{T-1}\left(y_T\right)\right) \qquad (21) \\
&- \frac{\beta(1+r)}{a\left[1+r-\mu\right]} \frac{1}{p_{T-1}}.
\end{aligned}$$

Note that for $\mu = 0$ the equation simplifies to

$$x_{T-1} = \overline{c} - y_{T-1} - \frac{\beta}{a}\frac{1}{p_{T-1}}.$$

The interval of feasible prices for constrained choices

We now characterize the feasible prices which support constrained choices. We denote the highest price level at which the collateral constraint is binding as $\overline{p}_{T-1}$. At this price level, $\kappa = 0$ so that, by equation (18),

$$\overline{p}_{T-1} = \frac{1}{(1+r)E_{T-1}\left(a(\overline{c} - c_T)\right)}. \qquad (22)$$

If the consumer is constrained, i.e., for $\kappa \geq 0$,

$$\frac{1}{p_{T-1}} - (1+r)E_{T-1}\left(a(\overline{c} - c_T)\right) \geq 0, \qquad (23)$$

the marginal utility of an additional unit of housing (in terms of consumption $1/p_{T-1}$) needs to be at least as high as the financing cost, including interest, in terms of marginal utils of consumption.

Note that $\overline{p}_{T-1}$ is also the smallest price attainable with *un*constrained choices at $x_T = 0$ since p_{T-1} in (15) is positive monotonic in x_T, as shown below in (30).

Positive consumption in (19), $c_{T-1} > 0$, implies a lower bound $\underline{p}_{T-1}$ so that

$$\bar{c} - \frac{\beta(1+r)}{a\left[1+r-\mu\right]}\left[\frac{1}{p_{T-1}} - \mu E_{T-1}\left(a(\bar{c}-c_T)\right)\right] > 0. \tag{24}$$

Thus, (17), (23) and (24) result in two equations which determine bounds for the price p_{T-1}. The upper bound $\bar{p}_{T-1}$ is given by

$$\frac{1}{p_{T-1}} - (1+r)E_{T-1}\left(a(\bar{c}+\mu p_{T-1}-y_T)\right) = 0$$

and the lower bound $\underline{p}_{T-1}$ is given by

$$\bar{c} - \frac{\beta(1+r)}{a\left[1+r-\mu\right]}\left[\frac{1}{p_{T-1}} - \mu E_{T-1}\left(a(\bar{c}+\mu p_{T-1}-y_T)\right)\right] = 0.$$

Both equations can be rearranged to standard quadratic equations

$$-a(1+r)\mu\left(\bar{p}_{T-1}\right)^2 - a(1+r)\left(\bar{c}-E_{T-1}y_T\right)\bar{p}_{T-1} + 1 = 0 \tag{25}$$

and

$$\frac{\beta(1+r)}{1+r-\mu}\mu^2\left(\underline{p}_{T-1}\right)^2 + \left(\bar{c} + \frac{\beta(1+r)}{1+r-\mu}\mu\left[\bar{c}-E_{T-1}\left(y_T\right)\right]\right)\underline{p}_{T-1} - \frac{\beta(1+r)}{a\left[1+r-\mu\right]} = 0 \tag{26}$$

with admissible solutions (only the positive roots are relevant)

$$\bar{p}_{T-1} = \frac{-a(1+r)\left(\bar{c}-E_{T-1}y_T\right) + \sqrt{\left[a(1+r)\left(\bar{c}-E_{T-1}y_T\right)\right]^2 + 4a(1+r)\mu}}{2a(1+r)\mu}$$

and

$$\underline{p}_{T-1} = \frac{-\left(\bar{c} + \frac{\beta(1+r)}{1+r-\mu}\mu\left[\bar{c}-E_{T-1}\left(y_T\right)\right]\right) + \sqrt{\left(\bar{c} + \frac{\beta(1+r)}{1+r-\mu}\mu\left[\bar{c}-E_{T-1}\left(y_T\right)\right]\right)^2 + 4\frac{\beta^2(1+r)^2}{a[1+r-\mu]^2}\mu^2}}{2\frac{\beta(1+r)}{1+r-\mu}\mu^2}.$$

Only the positive roots (i) ensure positive price bounds if $\bar{c}-E_{T-1}y_T > 0$, which is required to guarantee positive expected marginal utility in the parameter space that includes $\mu = 0$ (recall $c_T = -\mu p_{T-1} + y_T$), and (ii) ensure that $\lim_{\mu\to 0}\bar{p}_{T-1}$ and $\lim_{\mu\to 0}\underline{p}_{T-1}$ approach the value of the bounds at $\mu = 0$.

For the price interval to be non-empty, it needs to be the case that $\underline{p}_{T-1} < \bar{p}_{T-1}$ which results in a restriction on parameters. For $\mu = 0$ and $\beta(1+r) \leq 1$ the price interval is non-empty, for example, since

$$\underline{p}_{T-1} = \frac{\beta}{a\bar{c}} < \frac{1}{a(1+r)\left(\bar{c}-E_{T-1}y_T\right)} = \bar{p}_{T-1}.$$

Note that these bounds for the price p_{T-1} do not guarantee a mapping into $x_{T-1} \geq 0$. In fact (21) implies that at $\underline{p}_{T-1}$, $x_{T-1} = -y_{T-1}$, and at $\overline{p}_{T-1}$, $x_{T-1} = (1 - \beta(1+r))\overline{c} + (\beta(1+r)E_{T-1}y_T - y_{T-1})$. Thus, for $\mu = 0$ and $\beta(1+r) \leq 1$, a sufficient condition for a constrained equilibrium supported by a positive price is high enough expected income growth $E_{T-1}y_T/y_{T-1} > [\beta(1+r)]^{-1}$.

Let us now determine more generally the interval of prices p_{T-1} which guarantees that x_T maps into $x_{T-1} \geq 0$. To achieve this, we need to find the critical value(s) of p_{T-1} at which the right-hand side of equation (21) is zero. This results in the equation

$$A\,(p_{T-1})^2 + Bp_{T-1} + C = 0,$$

with

$$\begin{aligned}
A &\equiv \mu\left[\frac{\beta(1+r)}{1+r-\mu}\mu - \frac{1}{1+r}\right] \gtrless 0,\\
B &\equiv \overline{c} + \mu p_{T-2} - y_{T-1} + \frac{\mu\beta(1+r)}{1+r-\mu}\left(\overline{c} - E_{T-1}\left(y_T\right)\right) > 0,\\
C &\equiv -\frac{\beta(1+r)}{a\left[1+r-\mu\right]} < 0.
\end{aligned}$$

Note that $A = 0$ if $\mu = 0$ and $\partial A/\partial\mu < 0$ at $\mu = 0$ so that A is negative for small values of μ. Clearly $A > 0$ for some parameter values: for example $\beta(1+r) = 1$ and $\mu = 1$ imply $A = 1/r - 1/(1+r) > 0$. Thus, while $C < 0$ implies that the function $x_{T-1}(p_{T-1})$ has a negative intercept, the slope and curvature depends on parameter values and hence also μ.

The solutions of the quadratic equation are

$$p_{T-1} = \frac{-B \pm \sqrt{B^2 - 4AC}}{2A}.$$

A non-empty interval of positive prices p_{T-1} for which $x_{T-1} \geq 0$ requires the square root to be well defined and thus

$$B^2 - 4AC > 0.$$

Note that for $A < 0$ there may be two positive real roots whereas there is only one positive root for $A > 0$.

Proof of Proposition 2

If a constrained equilibrium exists, then $x_{T-1} \geq 0$ is attained for constrained choices. Analogously, if an unconstrained equilibrium exists, then $x_{T-1} \geq 0$ is attained for unconstrained choices. We proceed in the following steps. We first derive the largest x_{T-1} that is attained with constrained choices, denoted by $\overline{x}^C_{T-1}$. We go on by deriving the smallest x_{T-1} that is attained with unconstrained choices, denoted by $\underline{x}^U_{T-1}$. The positive monotonicity of the mapping $x_{T-1}(x_T, ...)$ for the unconstrained choices, derived below, then implies that a condition for equilibrium multiplicity is $\overline{x}^C_{T-1} > \underline{x}^U_{T-1}$. Furthermore, it follows from the positive monotonicity of $p_{T-1}(x_T, ...)$ and $x_{T-1}(x_T, ...)$ for unconstrained choices, derived below, that $p_{T-1}(x_{T-1}, ...)$ is positive monotonic for unconstrained choices. Thus, if there is equilibrium multiplicity, there is at most one unconstrained equilibrium.

Derivation of $\overline{x}^C_{T-1}$

Consider first the case in which $A < 0$. Let us compute the price at which the right-hand side of equation (21) attains its maximum by

$$\frac{\partial\left[\text{right-hand-side of equation } (21)\right]}{\partial p_{T-1}} \tag{27}$$
$$= \left[\frac{\beta(1+r)}{1+r-\mu}\mu - \frac{1}{1+r}\right]\mu + \frac{\beta(1+r)}{a\left[1+r-\mu\right]}\frac{1}{\left(p_{T-1}\right)^2} = 0.$$

This implies

$$p^{\max} = \sqrt{\frac{\frac{\beta(1+r)}{a[1+r-\mu]}}{\left[\frac{1}{1+r} - \frac{\beta(1+r)}{1+r-\mu}\mu\right]\mu}}.$$

Only the positive root implies a positive price and the root is a real number since $A < 0$ implies

$$1 + r - \mu > \mu\beta(1+r)^2$$

and thus that the denominator is positive. For a non-empty range of feasible prices, $\underline{p}_{T-1} < \overline{p}_{T-1}$, we get

$$p^{\max}_{T-1} = \max\left\{\underline{p}_{T-1}, \min\left\{p^{\max}, \overline{p}_{T-1}\right\}\right\}, \tag{28}$$

since $p^{\max}$ may not be in the interval of feasible prices.

The upper bound for x_{T-1} in a constrained equilibrium with $A < 0$ is thus

$$\begin{aligned}\overline{x}^C_{T-1} = &\left[\frac{\beta(1+r)}{1+r-\mu}\mu - \frac{1}{1+r}\right]\mu p^{\max}_{T-1} + \overline{c} + \mu p_{T-2} \\ &- y_{T-1} + \frac{\mu\beta(1+r)}{1+r-\mu}\left(\overline{c} - E_{T-1}\left(y_T\right)\right) - \frac{\beta(1+r)}{a\left[1+r-\mu\right]}\frac{1}{p^{\max}_{T-1}},\end{aligned}$$

since x_{T-1} attains its maximum at $p^{\max}_{T-1}$.

Now consider the case $A > 0$. The maximum of x_{T-1} is then attained at $\overline{p}_{T-1}$ since x_{T-1} is monotonically increasing in p_{T-1} if $A > 0$. Hence, for $A > 0$,

$$\begin{aligned}\overline{x}^C_{T-1} = & \left[\frac{\beta(1+r)}{1+r-\mu}\mu - \frac{1}{1+r}\right]\mu\overline{p}_{T-1} + \overline{c} + \mu p_{T-2} - y_{T-1} + \frac{\mu\beta(1+r)}{1+r-\mu}\left(\overline{c} - E_{T-1}\left(y_T\right)\right) \\ & - \frac{\beta(1+r)}{a\left[1+r-\mu\right]}\frac{1}{\overline{p}_{T-1}}.\end{aligned}$$

Derivation of $\underline{x}^U_{T-1}$

In order to find $\underline{x}^U_{T-1}$, we need to find the value of the exogenous grid x_T which maps into the smallest x_{T-1}. The mapping $x_{T-1}(x_T, ...)$ for unconstrained choices is given in closed form by equation (16). We now show that the right-hand side of equation (16) is monotonically increasing in x_T so that $\underline{x}^U_{T-1}$ is attained at the smallest gridpoint $x_T = 0$. Formally,

$$\begin{aligned}& \frac{\partial\left\{\begin{array}{c}\left[1-\beta(1+r)\right]\overline{c} + \left[\frac{1}{1+r} + \beta(1+r)\right]x_T - \left[\frac{\mu}{1+r} + \beta(1+r)\mu - 1\right]p_{T-1} \\ +\mu p_{T-2} + \beta(1+r)E_{T-1}y_T - y_{T-1}\end{array}\right\}}{\partial x_T} \\ & = \frac{1}{1+r} + \beta(1+r) - \left[\frac{\mu}{1+r} + \beta(1+r)\mu - 1\right]\frac{\partial p_{T-1}(x_T)}{\partial x_T}.\end{aligned} \tag{29}$$

We now show that this derivative is positive. Equation (15), with the relevant positive root, implies that

$$\begin{aligned}\frac{\partial p_{T-1}(x_T, ...)}{\partial x_T} & = \frac{1}{2\mu} - \frac{a(1+r)\left(\overline{c} - x_T - E_{T-1}y_T\right)}{2\mu\sqrt{a^2(1+r)^2\left(\overline{c} - x_T - E_{T-1}y_T\right)^2 + 4a(1+r)\mu}} \\ & = \frac{\sqrt{a^2(1+r)^2\left(\overline{c} - x_T - E_{T-1}y_T\right)^2 + 4a(1+r)\mu} - a(1+r)\left(\overline{c} - x_T - E_{T-1}y_T\right)}{2\mu\sqrt{a^2(1+r)^2\left(\overline{c} - x_T - E_{T-1}y_T\right)^2 + 4a(1+r)\mu}} \\ & > 0\end{aligned} \tag{30}$$

We distinguish two cases to sign (29). First consider the case

$$\frac{\mu}{1+r} + \beta(1+r)\mu - 1 > 0,$$

which is equivalent to

$$\beta(1+r)^2\mu - (1+r-\mu) > 0,$$

and implies $A > 0$. Noting that

$$\frac{\partial p_{T-1}(x_T, ...)}{\partial x_T} \leq \frac{1}{2\mu},$$

we can substitute $\partial p_{T-1}(x_T, ...)/\partial x_T = 1/(2\mu)$ into (29) to check whether the derivative (29) is always positive (even at the largest value of $\partial p_{T-1}(x_T, ...)/\partial x_T$). This yields

$$\begin{aligned} &\frac{1}{1+r} + \beta(1+r) - \left[\frac{\mu}{1+r} + \beta(1+r)\mu - 1\right]\frac{1}{2\mu} \\ &= \frac{1}{2\,(1+r)} + \frac{\beta(1+r)}{2} + \frac{1}{2\mu} > 0. \end{aligned}$$

Let us now consider the case

$$\frac{\mu}{1+r} + \beta(1+r)\mu - 1 < 0.$$

Since $\partial p_{T-1}(x_T, ...)/\partial x_T > 0$, the derivative (29) is also positive in this case.

Given that the derivative (29) is positive, we know that the smallest value of x_{T-1} is attained by unconstrained choices for $x_T \to 0$. That is,

$$\underline{x}^U_{T-1} = [1-\beta(1+r)]\,\overline{c} - \left[\frac{\mu}{1+r} + \beta(1+r)\mu - 1\right]\underline{p}^U_{T-1} + \mu p_{T-2} + \beta(1+r)E_{T-1}y_T - y_{T-1} \tag{31}$$

where the smallest price for unconstrained choices, $\underline{p}^U_{T-1}$, is equal to the largest feasible price for the constrained choices:

$$\begin{aligned} \underline{p}^U_{T-1} &= \frac{-a(1+r)\left(\overline{c} - E_{T-1}y_T\right) + \sqrt{a^2(1+r)^2\left(\overline{c} - E_{T-1}y_T\right)^2 + 4a(1+r)\mu}}{2a(1+r)\mu} \\ &= \overline{p}_{T-1}. \end{aligned}$$

The condition for $\overline{x}^C_{T-1} > \underline{x}^U_{T-1}$

Multiplicity arises in the interval $[\underline{x}^U_{T-1}; \overline{x}^C_{T-1}]$. This interval is non-empty if $\overline{x}^C_{T-1} > \underline{x}^U_{T-1}$, i.e.,

$$
\begin{aligned}
&\beta(1+r)\left[1+\frac{\mu}{1+r-\mu}\right]\left(\bar{c}-E_{T-1}\left(y_T\right)\right) \\
&> \\
&\frac{\beta(1+r)}{a\left[1+r-\mu\right]}\frac{1}{\overline{p}^C_{T-1}} \\
&+\left[\frac{1}{1+r}-\frac{\beta(1+r)}{1+r-\mu}\mu\right]\mu\overline{p}^C_{T-1} \\
&-\left[\frac{\mu}{1+r}+\beta(1+r)\mu\right]\underline{p}^U_{T-1},
\end{aligned}
\tag{32}
$$

where

$$
\overline{p}^C_{T-1}=\begin{cases} \max\left\{\underline{p}_{T-1},\min\left\{p^{\max},\overline{p}_{T-1}\right\}\right\} & \text{if } A<0 \\ \overline{p}_{T-1} & \text{if } A>0 \end{cases}.
$$

Since $\underline{p}^U_{T-1}=\overline{p}_{T-1}$ for $x_T=0$, the condition for $\overline{x}^C_{T-1}>\underline{x}^U_{T-1}$ simplifies to

$$
\begin{aligned}
&\beta(1+r)\left[1+\frac{\mu}{1+r-\mu}\right]\left(\bar{c}-E_{T-1}\left(y_T\right)\right) \\
&> \\
&\frac{\beta(1+r)}{a\left[1+r-\mu\right]}\frac{1}{\overline{p}^C_{T-1}} \\
&+\frac{\mu}{1+r}\left[\overline{p}^C_{T-1}-\overline{p}_{T-1}\right] \\
&-\frac{\beta(1+r)}{1+r-\mu}\mu^2\overline{p}^C_{T-1} \\
&-\beta(1+r)\mu\overline{p}_{T-1}.
\end{aligned}
\tag{33}
$$

Since $\overline{p}_{T-1}\geq\overline{p}^C_{T-1}$, the last three terms of the right-hand side of the inequality are certainly negative. Thus, a sufficient condition for multiple equilibria is that the left-hand side equals the first term on the right-hand side of (33). Note that the inequality (33) does not depend on the state variables p_{T-2} and y_{T-1} since they affect x_{T-1} in the same way for constrained and unconstrained choices.

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