Rational Asset Pricing Bubbles Revisited*

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November 2011.

Abstract: Price bubble arises when the price of an asset exceeds the asset's fundamental value, that is, the present value of future dividend payments. The important result of Santos and Woodford (1997) says that price bubbles cannot exist in equilibrium in the standard dynamic asset pricing model with rational agents as long as assets are in strictly positive supply and the present value of total future resources is finite. This paper explores the possibility of asset price bubbles when either one of the sufficient conditions for non-existence of bubbles is violated. We demonstrate that there always exist equilibria with price bubbles on assets in zero supply. Further, we show that endogenous debt constraints generated by limited enforcement of trade are in a certain sense most prone to give rise to equilibrium price bubbles on assets in strictly positive supply and with infinite present value of total resources.

*Preliminary draft.

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1. Introduction

In times of instability in financial markets, news of price bubbles are frequent. The Global Financial Crisis of 2007-2009 provides several examples: real-estate market in the U.S, mortgage-backed securities, stocks of investment banks, etc. Price bubble arises when the price of an asset exceeds the asset's fundamental value. As the notion of fundamental value invites many different interpretations, news of price bubbles can often be ascribed to misjudgement of the fundamental value. Most economists would agree though that the so-called "dot-com" bubble of 1999-2000 was a genuine bubble. Prices of internet stocks reached staggering levels in early 2000. Returns between early 1998 and early 2000 were nearly 1000 %. Then, the market crashed down to virtually no value.

Financial economic theory does not provide a satisfactory answer to the question of whether and how price bubble can arise in asset markets. The theory of rational asset pricing bubbles defines the fundamental value of an asset as the present value of future dividend payments. The main result of the seminal paper by Manuel Santos and Michael Woodford (1997) says that price bubbles cannot exists in equilibrium in the standard dynamic asset pricing model as long as assets are in strictly positive supply and the present value of total future resources over the infinite time is finite. As most assets are believed to be in strictly positive supply and equilibria with infinite present value of aggregate resources are usually considered "exotic," the Santos-Woodford result is interpreted as saying that price bubbles are unlikely to happen.

This paper explores the possibility of asset price bubbles when either one of the sufficient conditions for non-existence of bubbles is violated. We focus mainly on debt/solvency constraints, but the results extend to borrowing and short-sales constraints. We show in Theorems 3 and 4, that there always exist equilibria with price bubbles on infinitely-lived assets that are in zero supply. There is a multiplicity of equilibria, and some equilibrium prices have bubbles.

Equilibria with infinite present value of the aggregate endowment have been known to exist at least since Bewley (1980). The classical example of price bubble on zero-dividend asset (i.e., fiat money) in strictly positive supply due to Bewley (1980) (see also Kocherlakota (1992)) has infinite present value of the aggregate endowment. The recent literature on equilibria with endogenous debt constraints induced by limited enforcement of market transactions (see Alvarez and Jermann (2000), Hellwig and Lorenzoni (2009), Azariadis and Kaas (2008)) provides more examples of equilibria with infinite present value of the aggregate endowment, or "low interest rates.' We show in Theorem 5 that, for endogenous debt constraints, equilibria with low interest rates are associated with price bubbles on some infinitely-lived assets even if those assets' supply is strictly positive.

2. Dynamic Asset Markets with Portfolio Constraints

Time is discrete with infinite horizon and indexed by $t = 0, 1 \ldots$. Uncertainty is described by a set $\mathcal{S}$ of states of the world and an increasing sequence of finite partitions $\{\mathcal{F}_t\}_{t=0}^{\infty}$ of $\mathcal{S}$. The partition $\mathcal{F}_t$ specifies sets of states that can be verified by the information available at date t. An element $s_t \in \mathcal{F}_t$ is called a date-t event. The subset relation $s_\tau \subset s_t$ for $\tau \geq t$ indicates that event s_τ is a successor of s_t We use S^t to denote the set of all successors of event s_t from t to infinity, and S^{t+} the set of all successors of s_t excluding s_t. The set of one-period (date-$(t+1)$) successors of s_t is denoted by s_t^+. The unique one-period predecessor of s_t is denoted by s_t^-.

There is a single consumption good. A consumption plan is a scalar-valued process $c = \{c(s_t)\}_{s_t \in \mathcal{E}}$ adapted to $\{\mathcal{F}_t\}_{t=0}^{\infty}$. The consumption space is the space C of all adapted processes and it can be identified with $\mathcal{R}^\infty$. There are I consumers. Each consumer i has the consumption set C_+ - the cone of nonnegative processes in C - a strictly increasing utility function u^i on C_+, and an initial endowment $w^i \in C_+$. The aggregate endowment $\bar{w} \equiv \sum_i w^i$ is assumed positive, i.e., $\bar{w} \geq 0$.

Security markets consist of J infinitely-lived securities traded at every date. The dividend process x_j of security j is adapted to $\{\mathcal{F}_t\}_{t=0}^{\infty}$ and positive, i.e., $x(s_t) \geq 0$ for every s_t. The ex-dividend price of security j in event s_t is denoted by $p_j(s_t)$. A portfolio of J securities held after trade at s_t is $h(s_t)$. Each agent has an initial portfolio α_0^i at date 0. The supply of securities is $\bar{\alpha}_0 \equiv \sum_i \alpha_0^i$.

Agent i faces the following budget constraints when trading in security markets

$$c(s_0) + p(s_0)h(s_0) = w^i(s_0) + p(s_0)\alpha_0^i, \tag{1}$$

$$c(s_t) + p(s_t)h(s_t) = w^i(s_t) + [p(s_t) + x(s_t)]h(s_t^-) \quad \forall s_t \neq s_0. \tag{2}$$

In addition to these budget constraints some restriction on portfolio holdings has to be imposed for otherwise agents could engage in a Ponzi scheme, that is, borrow any amount of wealth at any date and roll-over the debt forever. We shall at first focus our attention on three types of constraints:

- *debt (or solvency) constraints*

$$[p(s_t) + x(s_t)]h(s_t^-) \geq -D(s_t), \tag{3}$$

- *borrowing constraints*

$$p(s_t)h(s_t) \geq -B(s_t), \tag{4}$$

- *short-sales constraints*

$$h_j(s_t) \geq -b_j(s_t), \quad \forall\ j. \tag{5}$$

The bounds D, B, and b_j are assumed positive.

The set of consumption plans c satisfying budget constraints (1 - 2) and debt constraints (3) is denoted by $B_0^i(p, D^i, \Phi(s_0))$, where $\Phi(s_0)$ denotes date-0 financial wealth which is $p(s_0)\alpha_0^i$, and is called (date-0) budget set under debt constraints.[1] Budget sets under borrowing and short-sales constraints are defined using (4) and (5) and denoted $B_0^i(p, B^i, \Phi(s_0))$ and $B_0^i(p, b^i, \Phi(s_0))$, respectively.

An *equilibrium under debt constraints* is a price process p and consumption-portfolio allocation $\{c^i, h^i\}_{i=1}^I$ such that (i) for each i, consumption plan c^i and portfolio strategy h^i maximize $u^i(c)$ over $(c, h) \in B_0^i(p, D^i, p(s_0)\alpha_0^i)$, and (ii) markets clear, that is

$$\sum_i h^i(s_t) = \bar{\alpha}_0, \qquad \sum_i c^i(s_t) = \bar{w}(s_t) + x(s_t)\bar{\alpha}_0,$$

for all s_t. We restrict our attention throughout to equilibria with positive prices.

Equilibria under borrowing constraints and short-sales constraints are defined in the same way using the respective budget sets. We remark that, for debt and borrowing constraints, the market-clearing (or feasibility) of a consumption allocation implies that there is corresponding market-clearing allocation of portfolios that finance the consumption allocation.

[1] We shall first focus on debt constraints and later, in Section 6, on short-sales constraints. Santos and Woodford (1997) considered borrowing constraints.

3. Arbitrage, Event Prices, and Bubbles.

Portfolio $\hat{h}(s_t)$ is a *one-period arbitrage* in event s_t at t if

$$[p(s_{t+1}) + x(s_{t+1})]\hat{h}(s_t) \geq 0, \qquad \forall s_{t+1} \subset s_t \tag{6}$$

and

$$p(s_t)\hat{h}(s_t) \leq 0, \tag{7}$$

with at least one strict inequality.

A standard result (that follows from the Stiemke's Lemma) is that there is no one-period arbitrage if and only if there exist strictly positive numbers $q(s_t)$ for all s_t such that

$$q(s_t)p_j(s_t) = \sum_{s_{t+1} \subset s_t} q(s_{t+1})[p_j(s_{t+1}) + x_j(s_{t+1})] \tag{8}$$

for every s_t and every j. Strictly positive numbers $q(s_t)$ satisfying (8) and normalized so that $q(s_0) = 1$ are called *event prices.*

In the presence of constraints on portfolio holdings, the question of absence of arbitrage in an equilibrium requires careful consideration. Potentially, there could exist an arbitrage portfolio at equilibrium prices if portfolio constraints were binding for every agent. Clearly, this cannot happen for debt constraints since those constraints cannot be violated by a one-period arbitrage. Thus, we have

Proposition 1: *There is no one-period arbitrage in equilibrium under debt constraints.*

The same holds for borrowing constraints althought the argument is more subtle. If equilibrium prices are positive and non-zero, then a one-period arbitrage with negative price can be turned to an arbitrage with zero price by adding to it a portfolio with positive and non-zero price and positive payoff. An arbitrage portfolio with zero price cannot violate borrowing constraints. Hence. we have

Proposition 2: *If $p(s_t) > 0$ for every s_t, then there is no one-period arbitrage in equilibrium under borrowing constraints.*

The result does not extend to the short-sales constraints. Equilibrium prices may permit one-period arbitrage (see an example in LeRoy and Werner (2001).

If asset prices admit event prices $q >> 0$ satisfying (8), then the present value of a security and the bubble can be defined using any of those event prices. The *present value* of security j in s_t under event prices q is

$$\frac{1}{q(s_t)} \sum_{\tau > t} \sum_{s_\tau \in s_t} q(s_\tau) x_j(s_\tau) \tag{9}$$

This intuitive definition of present value can be given more solid foundations from the view point of the theory of valuation of contingent claims, see Huang (2002). *Price bubble* is the difference between the price and the present value of a security. Price bubble at s_t is

$$\sigma_j(s_t) \equiv p_j(s_t) - \frac{1}{q(s_t)} \sum_{\tau > t} \sum_{s_\tau \in s_t} q(s_\tau) x_j(s_\tau) \tag{10}$$

If there exist multiple event prices, the price bubble may depend on the choice of event prices. Our notation does not reflect that possibility, but we shall keep this mind. Basic properties of price bubbles are stated in the following proposition, the proof of which is standard and therefore omitted.

Proposition 3: *Suppose that p admits strictly positive event prices q. Then*

(i) *If $p(s_t) \geq 0$ for all s_t, then*

$$0 \leq \sigma_j(s_t) \leq p_j(s_t), \quad \forall s_t \; \forall j. \tag{11}$$

(ii) *If security j is of finite maturity (that is, $x_{jt} = 0$ for $t \geq \tau$ for some τ, and that security is not traded after date τ), then $\sigma_j(s_t) = 0$ for all s_t.*

(iii) *It holds*

$$q(s_t)\sigma_j(s_t) = \sum_{s_{t+1} \subset s_t} q(s_{t+1})\sigma_j(s_{t+1}) \tag{12}$$

for every s_t and every j.

Property (12) is referred to as the *discounted martingale property* of bubble σ_j with respect to event prices q. The reason for this terminology is that if a risk-free payoff lies in the one-period asset span for every event s_t, then the discount factor $\rho(s_t)$ can be defined as the product of one-period risk-free returns along the path of events from s_0 to s_t so that event prices rescaled by the discount factor $q(s_t)/\rho(s_t)$ become probabilities.[2] Property (12) says that discounted bubble $\rho(s_t)\sigma_j(s_t)$ is a martingale with respect to these probabilities.

Note that if portfolio strategy $\hat{h}$ satisfies the roll-over condition (**??**), then its value $p(s_t)\hat{h}(s_t)$ is a discounted martingale.

4. No-Bubble Theorems

In this section we present the results establishing sufficient conditions for non-existence of price bubbles in equilibrium. The first result concerns complete markets. Security markets are said to be *complete* at prices p if, for every s_t, the one-period payoff matrix $[p_j(s_t^+) + x_j(s_t^+)]_{j\in J}$ has rank equal to the number of one-period successors of s_t. Of course, if security markets are complete at p and p admits event prices q, then q is unique.

Theorem 1: *Let p be an equilibrium price system under debt constraints such that $p(s_t) > 0$ for every s_t. Suppose that security markets are complete at p, and let q be the (unique) system of strictly positive event prices. If*

$$\sum_{t=1}^{\infty} \sum_{s_t \in F_t} q(s_t)\bar{w}(s_t) < \infty, \tag{13}$$

and

$$\bar{\alpha}_0 >> 0, \tag{14}$$

then price bubbles are zero.

The hypothesis of Theorem 1 remains true for incomplete markets under an additional assumption on agents' utility functions. For a consumption plan c, let $c(-S^t)$ denote the consumption plan for all events other than S^t, and $c(S^{t+})$ the consumption plan for all events in S^{t+}. The assumption is

(A1) For every i, there exists $0 \le \gamma^i < 1$ such that

$$u^i(c^i(-S^t), c^i(s_t) + \hat{w}(s_t), \gamma^i c^i(S^{t+})) > u^i(c^i), \tag{15}$$

for every s_t and every c^i such that $c^i \le \hat{w}$, where $\hat{w}(s_t) = \bar{w}(s_t) + \bar{\alpha}_0 x(s_t)$.

[2] For details, see LeRoy and Werner (2001).

Theorem 2: *Let p be an equilibrium price system under debt constraints. Suppose that A1 holds and securities are in positive supply (14). For every system of strictly positive event prices q such that the present value of the aggregate endowments is finite (13), price bubbles are zero.*

Theorems 1 and 2 are extensions of the main results of Santos and Woodford (1997) from borrowing to debt constraints. In Section 6 we show that a version of Theorem 2 holds for short-sales constraints as well. Whether or not those results hold for collateral constraints or other portfolio constraints merits further investigation. [3]

5. Bubbles on Assets in Zero Supply

We show in this section that there always exist equilibria with price bubbles on infinitely-lived securities that are in zero supply. More precisely, we demonstrate that whenever there exists an equilibrium, there is also an equilibrium with price bubbles. Thus, there is a multiplicity of equilibria.

Let us consider a price system p that admits (possibly multiple) strictly positive event prices q. Let $\{\epsilon_t\}$ by a $\Re^J$-valued process. We say that $\{\epsilon_t\}$ is a *discounted martingale*, if there exists a strictly positive event price process q such that the discounted martingale property (12) holds for $\{\epsilon_t^j\}$ with respect to q for every j. Further, we say that $\{\epsilon_t\}$ is *asset-span preserving* if

$$span\{p_j(s_t^+) + x_j(s_t^+) : j \in J\} = span\{p_j(s_t^+) + \epsilon_j(s_t^+) + x_j(s_t^+) : j \in J\} \quad (16)$$

for every s_t. Property (16) is further discussed in the Appendix, see also Bejan and Bidian (2010). It is important to note that the set of asset-span preserving processes is always non-empty. If markets are complete at p, then almost every $\{\epsilon_t\}$ is asset-span preserving.

The relation $\simeq_c$ between any two sets of consumption-portfolio plans indicates that consumption plans in those sets are the same. We have

Lemma 1: *If $\{\epsilon_t\}$ is a positive asset-span preserving discounted martingale, then*

$$B_0^i(p, D^i, \Phi^i(s_0)) \ \simeq_c B_0^i(p + \epsilon, D^i, \Phi^i(s_0)) \quad (17)$$

[3] See Araujo at al (2009) for results on the existence of bubbles when asset trades are collateralized by durable goods, and Bottazzi et al (2011) for bubbles under repo trading.

for every $\Phi^i(s_0)$.

PROOF: Let $(c,h) \in B_0^i(p, D^i, \Phi^i(s_0))$. Since $\{\epsilon_t\}$ is asset-span preserving, there exists portfolio strategy $\hat{h}$ such that

$$[p(s_{t+1}) + x(s_{t+1})]h(s_t) = [p(s_{t+1}) + \epsilon(s_{t+1}) + x(s_{t+1})]\hat{h}(s_t) \tag{18}$$

for every s_{t+1} and s_t. Further, since $\{\epsilon_t\}$ is a discounted martingale, it follows

$$p(s_t)h(s_t) = [p(s_t) + \epsilon(s_t)]\hat{h}(s_t) \tag{19}$$

It is easy to see now that $(c,\hat{h}) \in B_0^i(p+\epsilon, D^i, \Phi^i(s_0))$. The same argument shows that if $(c,\hat{h}) \in B_0^i(p+\epsilon, D^i, \Phi^i(s_0))$, then $(c,h) \in B_0^i(p, D^i, \Phi^i(s_0))$. This concludes the proof. □

Theorem 3: *Suppose that all infinitely-lived securities are in zero supply. If* p *and* $\{c^i\}$ *are an equilibrium under debt constraints, then, for every positive asset-span preserving discounted martingale* ϵ*,* $p+\epsilon$ *and* $\{c^i\}$ *are an equilibrium, too.*

PROOF: Lemma 1 implies that

$$B_0^i(p, D^i, 0) \ \simeq_c B_0^i(p+\epsilon, D^i, 0)$$

Since $p(s_0)\alpha_0^i = [p(s_0) + \epsilon(s_0)]\alpha_0^i = 0$, it follows that $p+\epsilon$ and $\{c^i\}$ are an equilibrium. □

Theorem 3 holds for an equilibrium under borrowing constraints provided that $p(s_t) > 0$ for every s_t.

Equilibria with price bubbles constructed in Theorem 3 have the same asset span (and the same consumption allocation) as their no-bubble counterparts. There may exist equilibria with price bubbles on securities in zero supply such that the addition of price bubble changes the asset span and leads to different consumption allocation. An example will be given in Section 8.

6. Bubbles when Present Value of the Aggregate Endowment is Infinite

In the section we discuss the possibility of price bubbles on infinitely-lived securities in positive supply when the present value of the aggregate endowment is infinite. We start with the classical example of Bewley (1980) (see Kocherlakota

(1992)) of an equilibrium with price bubble on a zero-dividend security, that is, fiat money with positive price.

Example 2: There is no uncertainty. There are two agents with utility functions

$$u^i(c) = \sum_{t=0}^{\infty} \beta^t \ln(c_t),$$

where $0 < \beta < 1$. Their endowments are $w_t^1 = B$ and $w_t^2 = A$ for even dates $t \geq 2$, and $w_t^1 = A$ and $w_t^2 = B$ for odd dates $t \geq 1$, where $A > B$. Date-0 endowments will be specified later.

There is one security that pays zero dividend at every date, that is, fiat money. Initial security holdings are $\alpha_0^1 = 1$ and $\alpha_0^2 = 0$ so that the total supply is 1. Each agent can short sell at most one unit of the security. Since there is a single security, the short sales constraint can be regarded as a debt constraint or a borrowing constraint.

There exists a stationary equilibrium with consumption plans that depend only on current endowment, positive prices p_t, and debt constraint binding the agent with low endowment at every date t. Such equilibrium has consumption plans $c_t^1 = B+\eta$ and $w_t^2 = A-\eta$ for even dates $t \geq 0$, and $c_t^1 = A-\eta$ and $w_t^2 = B+\eta$ for odd dates $t \geq 1$, security holdings $h_t^1 = -1$ and $h_t^2 = 2$ for even dates and $h_t^1 = 2$ and $h_t^2 = -1$ for odd dates, and constant prices

$$p_t = \frac{1}{3}\eta. \tag{20}$$

The first-order condition for the unconstrained agent,

$$\frac{\beta^t}{c_t^1} p_t = \frac{\beta^{t+1}}{c_{t+1}^1} p_{t+1}, \tag{21}$$

holds provided that $\eta = \frac{\beta A - B}{(1+\beta)}$ and $\beta A > B$. For the constrained agent, the first-order condition requires that the left-hand side in (21) is greater than the right-hand side, and it holds. A transversality condition (see Kocherlakota (1992), Proposition 2) holds, too. If date-0 endowments are $w_0^1 = B + \frac{1}{3}\eta$ and $w_0^2 = A - \frac{1}{3}\eta$, then this is an equilibrium.

Event prices associated with equilibrium prices (20) are $q_t = 1$ for every t. The present value of the aggregate endowment $\sum_{t=0}^{\infty} q_t \bar{w}_t$ is infinite. $\square$

A sufficient condition for the present value of the aggregate endowment being finite is that there exist $\theta \in \Re^J_+$ such that

$$\bar{w}_t \leq \theta x_t, \tag{22}$$

for all $t \geq T$, for some date T. This follows from the fact that the present value of dividend stream x^j_t is finite for every security j, by Proposition 3. Another sufficient condition is that the equilibrium allocation be Pareto optimal and interior. We discuss this in details in Section 8. Needless to say, neither one of those two sufficient conditions holds in Example 2.

7. Endogenous Debt Constraints.

An important class of debt constraints that may lead to an equilibrium with infinite present value of the aggregate endowment and price bubbles on securities in positive supple are endogenous debt constraints induced by limited enforcement of market transactions.

We have assumed thus far that agents are committed to fulfill any obligations implied by their portfolio decisions. We shall relax this assumption now and allow agents to consider defaulting on the payoff of a portfolio at any date. Whether an agent would want to default or not depends on gains and losses that such action would present to him. Once those gains and losses of default are precisely specified, endogenous (or self-enforcing) debt constraints can be defined as a sequence of debt bounds such that the agent is unwilling to default even if his indebtness is at the maximum allowed level.

We proceed now to a formal definition of endogenous debt constraints. We assume throughout this section that agents' utility functions have the time-separable expected utility representation. The continuation utility in event s_t at date t is

$$u^i_{s_t}(c) = \sum_{\tau=t}^{\infty} \beta^{\tau-t} E[v^i(c_\tau)|s_t], \tag{23}$$

where $0 < \beta_i < 1$. Let $B^i_{s_t}(p, D^i, \Phi^i(s_t))$ denote the budget set in event s_t at date t under debt constraints with bounds D^i when initial financial wealth (or debt) at s_t is $\Phi^i(s_t)$. Specifically, this set consists of all consumption plans and portfolio holdings for events in S^{t+} satisfying budget constraints (2) for $s_\tau \in S^{t+}$. Further, let $U^{*i}_{s_t}(p, D^i, \Phi^i(s_t))$ be the maximum continuation utility (23) from event s_t over all consumption plans in the budget set $B^i_{s_t}(p, D^i, \Phi^i(s_t))$.

We shall describe gains and losses of default by a sequence of reservation utility levels that agent i can obtain if she defaults. We denote this sequence by $\bar{V}^i_d = \{\bar{V}^i_d(s_t)\}$ and call it default utilities. We focus on two specifications of default utilities that have been proposed in recent literature. They are

$$\bar{V}^i_d(s_t) = u^i_{s_t}(w^i), \tag{24}$$

and

$$\bar{V}^i_d(s_t) = \hat{U}^{*i}_{s_t}(p, 0, 0). \tag{25}$$

Under the first specification (24), default results in permanent exclusion from the markets so that the agent is forced to consume her endowment from s_t on (see Alvarez and Jermann (2000)). Under the second specification (25), default results in prohibition from taking any debt from s_t on, but the agent continues to participate in the markets under zero-debt constraints (see Hellwig and Lorenzoni (2009)). Note that default utilities (25) depend on security prices p.

Debt bounds D^i are *self-enforcing* for agent i at p given default utilities $\bar{V}^i_d$ if

$$U^{*i}_{s_t}(p, D^i, -D^i(s_t)) \geq \bar{V}^i_d(s_t). \tag{26}$$

Debt bounds D^i are *not too tight* for agent i at p given default option $\bar{V}^i_d$ if (26) holds with equality. Equilibrium with not-too-tight debt constraints is a equilibrium with any debt bounds D^i such that D^i are not too tight at equilibrium price p for every i.

The property of being not too tight does not determine debt constraints in a unique way. To explain this, we introduce another definition. We say that a real-valued process $\{\kappa_t\}$ *lies in the asset span* if

$$\kappa(s_t^+) \in span\{p_j(s_t^+) + x_j(s_t^+) : j \in J\} \tag{27}$$

for every s_t. The spanning condition (27) is closely related to the property asset-span preserving (16). If ϵ_t is asset-span preserving, then ϵ^j_t lies in the asset span at p for every j. The converse holds for almost every ϵ_t. We elaborate on this in the Appendix.

Lemma 2: *If $\{\kappa_t\}$ is a discounted martingale and lies in the asset span, then*

$$B^i_{s_t}(p, D^i, \Phi^i(s_t)) \ \simeq_c B^i_{s_t}(p, D^i + \kappa, \Phi^i(s_t) - \kappa(s_t)) \tag{28}$$

for every s_t.

Proof: Let $(c, h) \in B^i_0(p, D^i, \Phi^i(s_0))$. Since $\{\kappa_t\}$ lies in the asset span, there is a portfolio strategy $\hat{h}$ be such that $[p(s_{t+1}) + x(s_{t+1})]\hat{h}(s_t) = \kappa(s_{t+1})$ for every s_{t+1} and s_t. Since κ is a discounted martingale, it follows that $p(s_t)\hat{h}(s_t) = \kappa(s_t)$. It is easy to see now that $(c, h + \hat{h}) \in B^i_{s_t}(p, D^i + \kappa, \Phi^i(s_t) - \kappa(s_t))$. The same argument shows that if $(c, \tilde{h}) \in B^i_{s_t}(p, D^i + \kappa, \Phi^i(s_t) - \kappa(s_t))$, then $(c, \tilde{h} - \hat{h}) \in B^i_{s_t}(p, D^i, \Phi^i(s_0))$. This concludes the proof. $\Box$

Lemma 2 implies that if debt bounds D^i are not too tight, then $D^i + \kappa$ are not too tight as well, for any discounted martingale κ that lies in the asset span. Indeed, it follows that $U^{*i}_{s_t}(p, D^i, -D^i(s_t)) = U^{*i}_{s_t}(p, D^i + \kappa, -(D^i(s_t) + \kappa(s_t))$.

For the default option (25), zero bounds $D^i \equiv 0$ are not too tight. This implies that $D^i = \kappa$ is not too tight with respect to (25) for every discounted martingale that lies in the asset span. Hellwig and Lorenzoni (2009, Theorem 1, see also Bejan and Bidian (2011)) show that the converse holds, too, if markets are complete: If debt bounds D are not too tight with respect to (25), then D is a discounted martingale. In Example 2, event prices are equal to one and debt bounds are constant, which implies that they are a discounted martingale. Consequently, this equilibrium has not too tight debt constraints with respect to (25).

Lemmas 1 and 2 provide a way of "injecting" (Kocherlakota (2008)) price bubbles on infinitely-lived securities when the supply of securities is positive. Combining those lemmas shows that if an $\Re^J$-valued process $\{\epsilon_t\}$ is an asset span preserving discounted martingale, then

$$B^i_0(p, D^i, p(s_0)\alpha^i_0) \ \simeq_c B^i_0(p + \epsilon, D^i - \alpha^i_0\epsilon, [p(s_0) + \epsilon(s_0)]\alpha^i_0) \tag{29}$$

A critical issue concerning budget identity (29) is whether the adjusted debt bounds $D^i - \alpha^i_0\epsilon$ are positive or not. Debt constraints with negative bounds force agents to hold minimum savings. We excluded negative debt bounds from consideration.

We have the following

Theorem 4: *Let p and $\{c^i\}$ be an equilibrium with not-too-tight debt constraints $\{D^i\}$. For every positive asset-span preserving discounted martingale $\{\epsilon_t\}$ such that $\alpha_0^i \epsilon_t \leq D_t^i$, price process $p+\epsilon$ and consumption allocation $\{c^i\}$ are an equilibrium with not-too-tight debt constraints, too.*

PROOF: It follows from (29) that $p+\epsilon$ and $\{c^i\}$ are an equilibrium under debt constraints with positive bounds $D^i - \alpha_0^i \epsilon$. Debt bounds $D^i - \alpha_0^i \epsilon$ are not too tight by Lemma 2. Further, default utilities (24) and (25) remain the same at prices $p+\epsilon$. For (25), this is so because $\hat{U}_{s_t}^{*i}(p,0,0) = \hat{U}_{s_t}^{*i}(p+\epsilon,0,0)$ by Lemma 1. □

Related results can be found in Kocherlakota (2008) for complete markets, and in Bejan and Bidian (2010). Of course, if the hypothesis of Theorem 4 holds, the present value of the aggregate endowment must be infinite.

A sufficient condition guaranteeing that there exists ϵ_t such that $\alpha_0^i \epsilon_t \leq D_t^i$ for every i is that the discounted value of debt bounds, i.e., $\rho_t D_t^i$ is bounded away from zero for every agent i whose initial portfolio α_0^i is non-zero, and the risk-free payoff lies in the one-period asset span for every s_t. Bejan and Bidian (2011) provide further sufficient conditions.

Hellwig and Lorenzoni (2009, Example 1) presented an example of an equilibrium with not-too-tight debt constraints for default utilities (25) such that debt limits are bounded away from zero and present value of the aggregate endowment is infinite. Theorem 4 implies that there are equilibria with price bubbles in that example. We demonstrate such equilibria in Example 3.

Example 3: Uncertainty is described by a binomial event-tree with the two successor events of every s_t indicated by *up* and *down*. The (Markov) transition probabilities are

$$\text{Prob}(up|s_t) = 1-\alpha, \quad \text{Prob}(down|s_t) = \alpha, \tag{30}$$

whenever $s_t = (s_{t-1}, up)$, and

$$\text{Prob}(up|s_t) = \alpha, \quad \text{Prob}(down|s_t) = 1-\alpha, \tag{31}$$

for $s_t = (s_{t-1}, down)$, where $0 < \alpha < 1$. Initial event is $s_0 = up$.

There are two consumers with utility functions (23) with the same logarithmic function v and discount factor $0 < \beta < 1$. Endowments depend only on the current state and are given by $w^1(up) = A,\ \ w^2(up) = B$, and $w^1(down) = B,\ \ w^2(down) = A$. It is assumed that $A > B > 0$. Note that there is no aggregate risk.

The market structure consists of one-period Arrow securities at every date-event and an infinitely-lived security with zero dividends (fiat money). Arrow securities are in zero supply. Fiat money is in positive supply with each agent holding one share at date 0.

We first find an equilibrium with zero price of the fiat money. There exists a stationary Markov equilibrium such that, at every event, the debt constraint is binding for the agent who receives high endowment. The equilibrium is as follows: Prices of Arrow securities are

$$p_c(s_t) = 1 - \beta(1-\alpha), \quad p_{nc}(s_t) = \beta(1-\alpha), \tag{32}$$

where subscript c stands for "change" of state (for example, from up to down) and nc for "no change." Consumption allocation is

$$c^1(s_t) = \bar{c}, \quad c^2(s_t) = \underline{c} \tag{33}$$

whenever $s_t = (s_{t-1}, up)$, and

$$c^1(s_t) = \underline{c}, \quad c^2(s_t) = \bar{c} \tag{34}$$

whenever $s_t = (s_{t-1}, down)$. Consumption plans $\bar{c}$ and $\underline{c}$ are such that $\bar{c} + \underline{c} = A + B$ and

$$1 - \beta(1-\alpha) = \beta\alpha\frac{\bar{c}}{\underline{c}}. \tag{35}$$

The solution must satisfy $\bar{c} < A$, which is a restriction on parameters α and β. Debt bounds are

$$D(s_t) = \bar{D} = \frac{A - \bar{c}}{2[1 - \beta(1-\alpha)]}, \tag{36}$$

constant over time and across events.

Equilibrium holdings of Arrow securities are such that agent 1 always holds the maximum possible short position $-\bar{D}$ in the Arrow security that pays in the up-event (where she gets high endowment A) and long position $\bar{D}$ in the Arrow security that pays in the down-event. The opposite holds for agent 2. Transversality condition holds. Date-0 endowments must be $w^1(s_0) = A - \bar{D}$ and $w^2(s_0) = B + \bar{D}$.

Event prices are products of Arrow securities prices (32). They have the property that, at any date, the sum of event prices for all date-t events equals one. In other words, there is no discounting (i.e., $\rho(s_t) = 1$). Consequently, the present value of the aggregate endowment is infinite. Further, constant debt bounds $\bar{D}$ are a martingale, and therefore they are not too tight.

Since debt bounds (36) are strictly positive, we can take any positive martingale $\{\epsilon_t\}$ such that $\epsilon_t \leq \bar{D}$ and "inject" it as a bubble on fiat money as described in Theorem 4. For simplicity, we take a deterministic process $\epsilon_t = \bar{\epsilon}$ for $\bar{\epsilon} \leq \bar{D}$. Price process for fiat money given by $p_t = \bar{\epsilon}$ together with consumption allocation $\{c^i\}$ given by (35) and debt bounds $\bar{D} - \bar{\epsilon}$ for each agent constitute an equilibrium with non-too-tight debt constraints.

8. Efficient Equilibria.

If equilibrium allocation under debt or borrowing constraints is Pareto optimal and markets are complete at equilibrium prices, then typically the present value of the aggregate endowment (under unique event prices) is finite.

Example 4: Consider the economy of Example 2. The Arrow-Debreu equilibrium in this economy has time-independent consumption plans $c_t^i = \bar{c}^i$ for $i = 1, 2$ and event prices equal to the discount factor, that is,

$$q_t = \beta^t, \tag{37}$$

for every t. Consumption plans $\bar{c}^i$ are given by

$$\bar{c}^i = \sum_{t=0}^{\infty} \beta^t w_t^i, \tag{38}$$

that is $\bar{c}^1 = \frac{1}{1-\beta^2}[A\beta + B] + \eta$ and $\bar{c}^2 = \frac{1}{1-\beta^2}[A + B\beta] - \eta$.

The Arrow-Debreu equilibrium allocation can be implemented by trading fiat money in zero supply under the natural debt constraints. The natural debt constraints have bounds equal to the present value of current and future agent's endowments, that is, $D_t^i = \frac{1}{q_t} \sum_{\tau=t}^{\infty} q_\tau w_\tau^i$. The price of fiat money equals bubble σ_t

that must satisfy the discounted-martingale property (12), that is

$$q_t\sigma_t = q_{t+1}\sigma_{t+1}. \tag{39}$$

We take $\sigma_t = \beta^{-t}$ so that the price of fiat money is

$$p_t = \beta^{-t}. \tag{40}$$

The present value of agent 1 future endowments at event prices (37) is $\frac{1}{1-\beta^2}[A\beta + B]$ for even dates, and $\frac{1}{1-\beta^2}[A + B\beta]$ for odd dates. The reverse holds for agent 2. Debt bounds are equal to those present values.

9. Short Sales Constraints

In this section we discuss the case of short-sales constraints (5). The major difference between short-sales constraints and borrowing or debt constraints is that the absence of one-period arbitrage cannot be guaranteed in equilibrium, and therefore there may not exist event prices satisfying (8). Yet, there is a way of defining the fundamental value and the bubble similar to what has been done in Section 4. For every security j and every event s_t, there exists at least one agent whose holding of security j is unconstrained at s_t. Assuming that this agent's consumption is interior and her utility function is differentiable, this agent's marginal utilities of consumption in s_t and its successor events s_t^+ can be used as event prices in those events. I follows from the first-order conditions of optimal portfolio choice that (8) will hold for security j with so defined event prices while there will be an inequality in (8) for securities other than j. In the Appendix we present an argument based on the absence of arbitrage that delivers the same result without relying on differentiability of utility functions or interiority of consumption. Summing up, in an equilibrium under short sales constraints, for every security j, there exist strictly positive numbers $q_j(s_t)$ for all s_t such that

$$q_j(s_t)p_j(s_t) = \sum_{s_{t+1}\subset s_t} q_j(s_{t+1})[p_j(s_{t+1}) + x_j(s_{t+1})] \tag{41}$$

and

$$q_j(s_t)p_k(s_t) \geq \sum_{s_{t+1}\subset s_t} q_j(s_{t+1})[p_k(s_{t+1}) + x_k(s_{t+1})] \tag{42}$$

for every $k \neq j$,

We can define the present value of security j and the bubble using security-specific event prices q^j. It is easy to see that Proposition 3 continues to hold. In particular, there cannot be a price bubble on a security that has finite maturity. More interestingly, an inspection of the proof of Theorem 2 reveals that that result holds for an equilibrium with short sales constraints, as well. Thus, if the present value of the aggregate endowment is finite, i.e., $\sum_t q_j(s_t)\bar{w}(s_t) < \infty$ and security j is in positive supply, $\bar{\alpha}_0^j > 0$, then there cannot be a price bubble in equilibrium.

A version of Theorem 3 holds for short sales constraints as well. It is the following

Theorem 5: *Suppose that all infinitely-lived securities are in zero supply. If p and $\{c^i\}$ are an equilibrium under short sales constraints, then there exists positive asset-span preserving discounted martingale ϵ such that $p + \epsilon$ and $\{c^i\}$ are an equilibrium, too.*

PROOF: See Appendix.

Appendix

Proof of Theorem 2. Let $(p, \{c^i, h^i\}$ be the equilibrium. First we observe that

$$p_0 = \sum_{t=1}^{\infty} \sum_{s_t \in F_t} q(s_t)x(s_t) + \lim_{T\to\infty} \sum_{s_T \in F_T} q(s_T)p(s_T). \tag{43}$$

It follows that $\sigma_0 = 0$ if and oly if

$$\lim_{T\to\infty} \sum_{s_T \in F_T} q(s_T)p(s_T) = 0. \tag{44}$$

Let γ^i be as implied by (A1) with $0 \leq \gamma^i < 1$. We claim that

$$(1-\gamma^i)p(s_t)h^i(s_t) \leq \hat{w}(s_t), \tag{45}$$

for every s_t, every i.

To prove (45), suppose that there exists s_t such that

$$(1-\gamma^i)p(s_t)h^i(s_t) > \hat{w}(s_t), \tag{46}$$

for some i. Consider consumption plan

$$\tilde{c}^i = (c^i(-S^t), c^i(s_t) + (1-\gamma^i)p(s_t)h^i(s_t), \gamma^i c^i(S^{t+})). \tag{47}$$

Note that portfolio $\tilde{h}^i = (h^i(-S^t), \gamma^i h^i(S^t))$ finances $\tilde{c}^i$ and satisfies debt constraints. By assumption (A1), $u^i(\tilde{c}^i) > u^i(c^i)$ which is a contradiction. This proves (45).

From (45) it follows that

$$(1-\bar{\gamma})p(s_t)\alpha_0 \leq I\hat{w}(s_t), \tag{48}$$

for every s_t, where $\bar{\gamma} = \max \gamma^i$.

From (48), we obtain

$$\sum_{s_T \in F_T} q(s_T)p(s_T)\alpha_0 \leq \frac{I}{(1-\bar{\gamma})} \sum_{s_T \in F_T} q(s_T)\hat{w}(s_T). \tag{49}$$

From the assumptions of the theorem, it follows that

$$\lim_{T\to\infty} \sum_{s_T \in F_T} q(s_T)\hat{w}(s_T) = 0. \tag{50}$$

Therefore,

$$\sigma_0 \alpha_0 = 0, \tag{51}$$

and consequently $\sigma_0 = 0$. $\square$

Asset span preserving $\{\epsilon_t\}$.

Here, we discuss the properties of asset span preservation introduced in Section 5 and lying in the asset span of Section 7. Let $\{\epsilon_t\}$ be a $\Re^J$-valued process. First, it is easy to see that if $\{\epsilon_t\}$ is asset span preserving at p, then $\{\epsilon_t^j\}$ lies in the asset span at p for every j. The converse holds under a minor rank condition which we explain next.

Suppose that $\{\epsilon_t^j\}$ lies in the asset span at p. For any s_t, let $h^j(s_t)$ be a portfolio such that $\epsilon(s_t^+) = [p(s_t^+) + x(s_t^+)]h^j(s_t)$. It follows that $p(s_t^+) + \epsilon(s_t^+) + x(s_t^+) = [p(s_t^+) + x(s_t^+)](h^j(s_t) + I^j)$, where I^j denotes a portfolio consisting of one share of security j. Let $H(s_t)$ be the $J \times J$ matrix with rows $h^j(s_t)$ for all j and I be the $J \times J$ identity matrix. If matrix $H(s_t) + I$ is invertible, then $\{\epsilon_t\}$ is asset span preserving at p.

Summing up, if $\{\epsilon_t^j\}$ lies in the asset span at p for every j. and $H(s_t) + I$ is invertible for every s_t, then $\{\epsilon_t\}$ is asset span preserving at p.

Equilibrium in Example 4: We use $c^i(0,1)$ to denote equilibrium consumption of agent i if the current dividend is 1 and the dividend at the preceding date was 0. Equilibrium consumption plans are

$$c^1(0,0) = A,\ c^1(0,1) = A + 1 + \frac{27}{13},\ c^1(1,0) = A - \frac{24}{13}, c^1(1,1) = A \tag{52}$$

for agent 1,

$$c^2(0,0) = B,\ c^2(0,1) = B - \frac{27}{13},\ c^2(1,0) = B + \frac{24}{13}, c^2(1,1) = B \tag{53}$$

for agent 2.

Proof of Theorem 5.

For each security j, consider a strategy that starts with buying 1 share of security j at date 0 and consists of rolling over the payoff at every event after date 0. This may by called a reverse Ponzi scheme on security j. Formally it is a real-valued process γ_j defined by

$$p_j(s_t)\gamma_j(s_t) = [p_j(s_t) + x_j(s_t)]\gamma_j(s_t^-)$$

for every s_t, and $\gamma_j(s_0) = 1$. Note that γ_j is positive.

Let ϵ_t be an $\Re^J$-valued process defined by

$$\epsilon_j(s_t) = [p_j(s_t) + x_j(s_t)]\gamma_j(s_t^-)$$

Since

$$p(s_t) + \epsilon(s_t) + x(s_t) = [p(s_t) + x(s_t)](I + \gamma(s_t^-))$$

it follows that ϵ_t is asset-span preserving. Further, it is a discounted martingale.

Let h^i be agent's i portfolio strategy in equilibrium with prices p. Define portfolio strategy $\hat{h}^i$ by

$$\hat{h}^i(s_t) = [I + \Gamma(s_t^-))]h^i(s_t)$$

Clearly $\hat{h}^i$ finances c^i at price $p + \epsilon$ and satisfies short sales constraints.

References

Alvarez, F. and U. Jermann, "Efficiency, Equilibrium and Asset Prices with Risk of Default," *Econometrica*, 2000.

Araujo A., M. R. Pascoa, and J.P. Torres-Martnez (2009), "Long-lived collateralized assets and bubbles," working paper.

Bejan C. and F. Bidian (2010), "Payout Policy and Bubbles," Working Paper.

Bejan C. and F. Bidian (2011), "Martingale Properties of Self-Enforcing Debt." Working Paper,

Gilles, C. and S. LeRoy, *Asset price bubbles*, in The New Palgrave Dictionary of Money and Finance, 1992

Gilles, C. and S. LeRoy, "Bubbles and charges," International Economic Review, 33, 323 - 339, 1992.

Hellwig, Ch and G. Lorenzoni, "Bubbles and Self-Enforcing Debt," *Econometrica*, vol 77, pg 1137-64, 2009.

Harrison, M. and D. Kreps, "Speculative Investor Behavior in a Stock Market with Heterogeneous Expectations," *Quarterly Journal of Economics*, *42*, (1978), 323–36.

Huang K. and J. Werner, "Asset Price Bubbles in Arrow-Debreu and Sequential Equilibrium," *Economic Theory*, *15*, (2000), 253–278.

Kocherlakota, N. "Bubbles and Constraints on Debt Accumulation," *Journal of Economic Theory*, **57** 1992.

Kocherlakota, N. "Injecting rational bubbles," *Journal of Economic Theory*, 2008.

LeRoy, S. and J. Werner, 2001, *Principles of Financial Economics,* Cambridge University Press.

Magill, M. and Quinzii, M, "Incomplete markets over an infinite horizon: Long-lived securities and speculative bubbles," *Journal of Mathematical Economics,* 26(1), pg 133-170, 1996

V. F. Martins-da-Rocha and Y. Vailakis (2011), "Refining Not-Too-Tight Solvency Constraints," Working Paper.

Ofek, E. and M. Richardson, "DotCom Mania: The Rise and Fall of Internet Stock Prices," *The Journal of Finance*, **43**, (2003), 1113–1137.

Santos, M., and M. Woodford, "Rational Asset Pricing Bubbles," *Econometrica*, **65**, (1997), 19–57.