Enduring Relationships in an Economy with Capital and Private Information

Aubhik Khan
Ohio State University

Latchezar Popov
University of Virginia

B. Ravikumar[1]
Federal Reserve Bank of St. Louis

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[1]Corresponding author: L. Popov, Department of Economics, University of VIrginia, P.O. Box 400182, Charlottesville, VA 22904-4182; email: lpopov@virginia.edu. We would like to thank Christopher Phelan and Hal Cole and seminar participants at the University of Virginia for their comments. Amanda Kurzendoerfer provided valuable editorial assitance. All errors are ours.

Abstract

We introduce capital accumulation into an economy where individuals have private information with respect to productivity shocks. Efficient, incentive-compatible risk-sharing is achieved by conditioning current and future payoffs on the history of productivity reports. We develop a notion of efficiency, similar to that of Atkeson and Lucas (1992). The capital stock in the economy is endogenous in periods after date zero, so our notion of efficiency is minimizing the initial capital stock that is required to attain an initial distribution of promised utility. Under constant relative risk aversion and linear technology, we find that the planner allocates more capital to agents with a history of high productivity reports relative to agents with a history of low productivity reports. This higher allocation of capital occurs despite the fact that the expected marginal product of capital is the same across all agents. In contrast to the unobservable endowment model, the agent's welfare in the private information economy exceeds the value of autarchy *in every period.* Hence, the contract exhibits voluntary long-term participation by the agent. Risk-sharing does not require net transfers across different wealth groups, so the efficient allocation exhibits scale invariance. The allocation allows a simple decentralization through a sequence of actuarially fair insurance contracts. An alternative, long-run decentralization has implications for the risk-free interest rate in our economy.

1 Introduction

In this paper, we study an economic environment in which agents produce output using capital as the sole factor of production. The production technologies are subject to idiosyncratic productivity shocks that are private information. The capital stock is observable, but output and consumption are not. This creates an incentive for the agent to under-report his income which eliminates the possibility of complete risk-sharing. However, the repeated interaction between the planner and the agent allows that future trades be conditioned on past reports of productivity. We show that this property expands strictly the set of incentive-compatible allocations. The allocation must satisfy a sequence of incentive constraints. Since aggregate capital is endogenous, the planner faces an additional sequence of constraints – the total amount of transfers from all the agents must be greater than or equal to total invested capital in the next period. Thus in the spirit of Atkeson and Lucas (1992), we introduce our efficiency concept – the planner minimizes the capital stock the economy starts with subject to the incentive compatibility constraints and the requirement to provide a certain distribution of promised utility to the agents in the economy. We can consider this to be a "general equilibrium" problem since the planner cannot substitute resources intertemporally except through the production process carried out by the agents.

In our model, investment is observable, so we are able to retain the analytical approach used in endowment economies. In particular, adapting the technique of Green (1987), we ensure that the contract is incentive compatible by imposing his variational equivalent, temporary incentive compatibility, on feasible allocations. We demonstrate these results in Section 2.1 after a formal description of the environment. We relegate all proofs to the Appendix.

In the unobservable endowment models, the individual's pattern of consumption - over time and over histories of endowment shocks - is determined by his utility entitlement, which is distinct from the value of his endowment stream. In our production economy with capital accumulation, capital invested in each agent's production is endogenous and similarly the stochastic process for income is endogenous. We show that under constant relative risk aversion and linear technology, the efficient allocation exhibits a one-to-one mapping from the capital allocated to an individual to his expected lifetime utility. Thus, capital is sufficient to determine an individual's welfare. The one-to-one mapping also gives rise to a scaling result. Having solved the efficient allocation for a given level of expected lifetime utility, the resultant decision rules may be used to recover the optimal allocation associated with any other level of utility. For any agent, allocated capital for production and the consumption transfer are functions of the entire history of productivity reports.

Consider two agents, A and B. The allocation for B will be a scaled version of the allocation for A with the factor of proportionality being the ratio of consumption equivalents of their initial promised utility. This scaling result is similar to that of Atkeson and Lucas (1992), who derived it in a constant endowment economy with unobservable shocks to preferences.

Under more general functional forms for utility and production, the scaling result no longer holds. In particular, the capital and consumption allocation for a particular agent is no longer independent of the entire distribution of promised utilities. If production is linear, then the allocation of capital among agents has no effect on aggregate production (since the law of large numbers holds) and it only has an effect on risk-sharing and incentives. We show that if the utility is in the CRRA class, it is efficient to impose production (and hence consumption) risk proportional to the consumption equivalent of promised utility. These are the assumptions that drive our results.

The presence of production and capital accumulation alters some of the implications of the private information endowment model. In the endowment model, insurance is incomplete and, given convexity of individual preferences, negative income reports imply a reduction in lifetime consumption, while positive income reports are rewarded with increases in lifetime consumption. Consequently, the efficient allocation generates a random walk in lifetime utility and, hence, increasing dispersion in consumption. This increasing dispersion implies that, for almost all individuals, welfare in the private information economy eventually falls below the autarchy value of the risky endowment process. In contrast, the one-to-one mapping in our economy implies that the planner allocates less capital to agents with lower expected lifetime utility. These reductions in capital decrease the value of the autarchy option for the individual. Consequently, at the onset of any period in our economy, agents always prefer to remain in the risk-sharing arrangement since it offers insurance i.e., no individual wants to revert to autarchy.

While our production economy retains the characteristic of increasing dispersion of wealth or utility entitlements, this is also characteristic of the autarchic production economy. In numerical examples, we see that inequality increases at a lower rate in our private information economy relative to that in autarchy. The key to this result is that the contract dampens consumption *and* investment fluctuations relative to autarchy.

The scaling result implies that after each history for every agent the expected net transfer to the planner is equal to expected capital allocation in the subsequent period. There is no cross-subsidization between agents and, after the initial capital stock is provided, the planner performs better than autarchy solely because she is able to provide insurance (subject to some incentive-compatibility constraints). We are able to exploit this property to construct a simple decentralization of the optimal allocation. Agents enter into single-period insurance contracts with private insurance companies. The insurers offer actuarially fair contracts that insure against productivity risk. To ensure incentive compatibility, a maximum investment requirement is added. The allocation can be achieved through short-term contracts because, in the optimal allocation, the agent receives zero expected net transfers after every history. Thus there is no need for either party to commit to long-run transfer of resources: enforcement of the infra-period contract is all that is required.

An alternative approach to decentralization is to study a private market allocation in which a private insurer is in a long-run relationship with the agent, make the investment and consumption decisions subject to incentive and promise-keeping constraints, and trade between themselves deficits and surpluses of resources at different times. This decentralization will replicate the optimum if and only if there is no trading in equilibrium. This allows us to determine the unique interest rate at which the private insurers have no incentive to borrow or to invest. Thus we are able to provide asset-pricing implications from our model. We find that the interest rate is between that in the autarchy and the full information economy.

In allowing for capital accumulation under private information, our work is related to that of Marcet and Marimon (1993) who examine the implications of unobservable investment shocks in a two-agent environment where a risk-neutral investor lends to a risk-averse producer operating a neoclassical production technology.

Another paper that considers the effect of private information on capital accumulation is by Clementi and Hopenhayn (2006). In the paper, the authors consider the relationship between risk-neutral entrepreneurs and financiers, when output can be hidden. Similar to our results they show that under some conditions the long-run optimal allocation can be achieved through a sequence of one period debt contracts in which existing debt is rolled over. This result is depends on the fact that under some parametric restrictions the contract is renegotiation-proof – for all possible histories, the continuation long-run optimal contract is not Pareto dominated by any other contract. Second, it is assumed that the principal's claim over the agent (in the form of debt) will always be costlessly enforced. Our paper differs from theirs in that we solve for the problem in a general equilibrium framework. Second, we are able to provide stricter characterization of the decentralization mechanism: under all possible histories, both parties prefer staying in the relationship. Moreover, there is no intertemporal link (such as enforceable debt) that forces commitment upon both parties – all that is required is that one period contracts be enforced.

The paper most similar to ours is due to Bohacek (2005). In his model, unobservable labor input and a stochastic relationship between labor input and production lead to a moral hazard problem, preventing the full information allocation. He shows that for some interest rate lower than the discount factor, a stationary recursive equilibrium exists with mobility and a nondegenerate distribution of agent's utilities. In our model, the efficiency notion is valid off the steady state. Since capital input is correlated with promised utility then if the production function is strictly concave, equality of promised utility increases total output (equivalently reduces required total capital stock). Thus it appears that the optimal allocation will limit the amount of inequality.

Another paper that considers a similar problem is Mookherjee and Ray (2002). The implications of moral hazard and lack of commitment are explored in this paper. The authors concentrate on the role of bargaining power between principals and agents and show that either growth or a poverty trap can ensue from different bargaining power arrangements. In our model, we study the efficiency problem: find all possible distributions of promised utilities to different agents in the economy, which can be attained with given initial capital stock.

In Section 2.1, we characterize the constraint set and develop notion of efficiency similar to the one in Atkeson and Lucas (1992); the allocation achieves a given lifetime utility with the least amount of capital. We restrict attention to constant-relative-risk-aversion (CRRA) preferences and linear technologies. In Section 3, we characterize the efficient risk-sharing and contrast it with the unobservable endowment economies. A series of numerical examples is presented in section 3.2 which serve to further illustrate the properties of the efficient allocation. Section 4 introduces a decentralization mechanism that implements the optimal allocation. Section 5 briefly concludes.

2 Model

2.1 Preliminaries

In this section, we describe our environment, which involves idiosyncratic risk across agents. We develop an efficiency concept in the spirit of Atkeson and Lucas (1992) for this environment. Next, we develop a standard result on incentive compatibility that we will use to characterize risk-sharing recursively. In Section 2.3, we use our efficiency concept to give a convenient recursive representation of the planner's problem and to prove a welfare theorem. In Section **??**, we illustrate our results through a few examples.

In our economy, there is a continuum of individuals of measure 1. Time is discrete and is indexed by $t = 0, 1, \ldots$. Each individual operates a production technology subject to privately observed stochastic productivity. Capital, which must be allocated prior to the realization of productivity, is the sole factor of production and is accumulable. Let z_t denote an individual's *productivity shock* at time t and K_t denote the individual's capital stock allocated at the beginning of period t. The individual's output in period t is $z_t K_t$. There is no aggregate uncertainty. Preferences of every agent are described by

$$E_0 \sum_{s=0}^{\infty} \beta^s (1-\beta) u\left(C_s\right)$$

where C_s is consumption in period s and u is a utility function of the Constant Relative Risk Aversion class. That is $u(c) = c^\gamma/\gamma$ for $\gamma \neq 0$ or $u(c) = \log(c)$.

Productivity is independently and identically distributed over time and across individuals. Specifically, $z_t \in \mathbf{Z} \equiv \{z_1, z_2\}$. Let $\mathcal{Z} \equiv 2^{\mathbf{Z}}$ be the complete σ-algebra and define $\mu : \mathcal{Z} \to [0,1]$ as a probability measure on $\mathcal{Z}$, summarized by $\mu(z_i) = \mu_i > 0$.

Each agent is initially entitled to (and identified by) an expected utility w_0. Let $\mathcal{V} \subseteq \mathbb{R}$ be the set of promised expected utilities. The distribution of promised utilities is given by a Borel measure ν on $\mathcal{V}$.

We will consider direct revelation mechanisms, in which the agents truthfully report their productivity. Since the agents are treated differently in the future if they have made different reports, it is possible to induce truthtelling in the model.

In each period, the principal will assign each agent capital for current production and, after receiving a report of production, the principal will assign a net transfer b of current consumption good. Since the technology requires that capital be invested prior to the realization of the productivity shock, the capital allocation can be conditioned only on variables determined up to the past period. On the other hand, consumption takes part after production, so the transfer to and from the principal can also be conditioned on variables determined in the current period.

In the rest of the paper, we use the now standard recursive representation of an allocation. In Section **??** we show that a more general formulation yields the same results. The sufficient statistic that describes the constraints that the planner faces is the agent's expected utility. Let the agent be entitled to some promised utility w_t at the beginning of the period. Then informally, a period allocation is an assignment of capital, transfers conditional on reports, and continuation utility conditional on productivity reports. More formally we have:

Definition 1 *An allocation rule* $\mathbf{P}$ *is a sequence of Borel-measurable functions* $\mathbf{P} = \{k_t(w), b_t(w,z), v_{t+1}(w,z)\}_{t=0}^{\infty}$ *that satisfies the following constraints:* $k_t(w) \geq 0$ *for all* t *and* $w \in \mathcal{V}$, $b_t(w,z) \geq -zk(w)$ *for all* t, $w \in \mathcal{V}$, $z \in \mathbf{Z}$ *and* $v_{t+1}(w,z) \in \mathcal{V}$ *for all* t, $w \in \mathcal{V}$, $z \in \mathbf{Z}$.

The constraints on the allocations are intuitive: capital cannot depend on the report of productivity and it must be nonnegative. Secondly, the agent cannot transfer more to the principal than total output and continuation utility must be feasible.

We can apply the allocation rule recursively to generate a mapping between capital and transfers on one hand and histories of productivity reports on the other.

Definition 2 *The sequential allocation generated by the allocation rule is a sequence of functions* $\{\tilde{w}_t : \mathbf{Z}^{t-1} \times \mathcal{V} \to \mathcal{V},\ \tilde{k}_t : \mathbf{Z}^{t-1} \times \mathcal{V} \to \mathbb{R}_+,\ \tilde{b}_t : \mathbf{Z}^t \times \mathcal{V} \to \mathbb{R}\}_{t=0}^{\infty}$ *that is determined as follows:*

1. $\tilde{w}_0(\emptyset, w_0) = w_0$ *and* $\tilde{w}_t(z^{t-1}, w_0) = v_t(\tilde{w}_{t-1}(z^{t-2}, w_0), z_{t-1})$.[1]
2. $\tilde{k}_t(z^{t-1}, w_0) = k_t(\tilde{w}_t(z^{t-1}, w_0))$
3. $\tilde{b}_t(z^t, w_0) = b_t(\tilde{w}_t(z^{t-1}, w_0), z_t)$.

The construction above can be performed for an arbitrary starting point and promised utility. We will denote this, somewhat abusing notation, by $\tilde{k}_t(z^{t-s-1}, w_s)$, $\tilde{b}_t(z^{t-s}, w_s)$, for $s < t$.

It is possible to specify a sequential allocation directly. Not all such allocations can be generated by a recursive allocation rule, however. For ease of exposition, we will make use of allocation rules and allocations generated by them exclusively. In an online appendix, we show that for the efficiency concept developed below it is optimal to use allocations that can be generated by an allocation rule. Whenever we are referring to an individual agent, it will be understood that the allocation rule is conditioned on initial promised utility w_0, so there will be no ambiguity in suppressing it.

Next we turn to the agent. The agent faces an allocation rule (and an implicit sequential allocation generated by it) determined by the planner. For a particular allocation, the stochastic process for productivity and the reporting strategy of the agent induce a stochastic process for investment and consumption. The direct revelation mechanism requires that the agent's expected utility of reporting truthfully dominates any other strategy.

Definition 3 *A reporting strategy is* $\sigma = (\sigma_0, \sigma_1, \ldots)$ *such that* $\sigma_t : \mathbf{Z}^{t+1} \to \mathbf{Z}$ *is the time* t *report of productivity by a particular agent.*

Thus the productivity report can (potentially) depend on the entire history of observed shocks. Note that past reports are not arguments in the function since they can be determined from the history of actual shocks.

[1] Throughout, we use the following notation: for any element $x \in \mathbf{X}$, where $\mathbf{X}$ is a vector space, we define the time t value of x as x_t, the partial history of values through period t as $x^t = (x_0, \ldots, x_t) \in \mathbf{X}^{t+1}$ and a sequence of values as $\mathbf{x} = (x_0, x_1, \ldots) \in \mathbf{X}^{\infty}$. It is understood that $x^{-1} = \{\emptyset\}$. Finally, it is convenient to define the history from t through s, where $t < s$ as ${}_t x^s = (x_t, \ldots, x_s) \in \mathbf{X}^{s-t+1}$.

Denote the truthtelling strategy by σ^*, that is $\sigma_t^*(z^t, .) = z_t, \forall z^t \in \mathbf{Z}^{t+1}$. Let $\mathbf{\Sigma}$ represent the set of all measurable reporting strategies.

Let $\mathbf{\Pi}$ be the set of all allocation rules that satisfy the constraints above and let $\Sigma(\mathbf{P}) \subseteq \mathbf{\Sigma}$ be the set of all measurable reporting strategies which allow for nonnegative consumption given an allocation $\mathbf{P} \in \mathbf{\Pi}$.

Each agent in the economy has an increasing and strictly concave utility function $u : \mathbf{R}_+ \to \mathbf{R}$ and subjective discount factor $\beta \in (0, 1)$. Each agent's *expected utility* from an allocation $\mathbf{P}(w_0) \in \mathbf{\Pi}$ is a function of his reporting strategy, $\sigma \in \Sigma(\mathbf{P}(w_0))$, and denoted $V(\mathbf{P}(w_0), \sigma)$.[2]

$$V(\mathbf{P}(w_0), \sigma) = \sum_{s=0}^{\infty} \beta^s \int_{\mathbf{Z}^{s+1}} (1-\beta) u \left(z_s \tilde{k}_s \left[\sigma^{s-1}(z^{s-1}), w_0 \right] + \tilde{b}_s \left[\sigma^s(z^s), w_0 \right] \right) \mu^{s+1}(dz^s)$$

where $\tilde{k}$ and $\tilde{b}$ are defined above.

An allocation $\mathbf{P}$ is *incentive compatible* if it offers every agent a level of expected lifetime utility, when he reports each period's productivity accurately, that is no less than the level of utility from any other reporting strategy:

$$V(\mathbf{P}(w_0), \sigma^*) \geq V(\mathbf{P}(w_0), \sigma), \ \forall \sigma \in \Sigma(\mathbf{P}(w_0)), \ \forall w_0 \in \mathcal{V}. \tag{1}$$

An allocation satisfies *promise keeping* if each agent's expected utility is greater than or equal to promised initial utility w_0:

$$V(\mathbf{P}(w_0), \sigma^*) = w_0, \ \forall w_0 \in \mathcal{V}. \tag{2}$$

If the allocation rule satisfies the incentive compatibility constraint, then the expected transfer *from all agents to the principal* at date t is given by:

$$-\int_{\mathcal{V}} \left[\int_{\mathbf{Z^{t+1}}} \tilde{b}_t(z^t, w) \mu(dz^t) \right] \nu(dw).$$

Green (1994) shows how to introduce a construction of a continuum of i.i.d. random vector such that each sample has the same distribution as that of the random vectors from which it is drawn. Therefore, with probability one, the empirical distribution of any variable in the model will coincide with that of the random process that generates it. This implies that the distribution of all variables in the model will develop deterministically. Similarly, an

[2]The probability space $(\mathbf{Z}, \mathcal{Z}, \mu)$ generates the finite dimensional product probability spaces $(\mathbf{Z}^t, \mathcal{Z}^t, \mu^t)$, $t = 0, 1, \ldots$, as well as the infinite dimensional probability space of all measurable sequences of technology shocks $(\mathbf{Z}^\infty, \mathcal{Z}^\infty, \mu^\infty)$ in the standard manner.

analog of the strong law of large numbers will hold - sample averages of all realized vectors will be equal to their expectation almost surely.

In a similar fashion, we can derive the total amount of capital that the planner allocates for production at date t:

$$\int_{\mathcal{V}} \left[\int_{\mathbf{Z^t}} \tilde{k}_t(z^{t-1}, w)\mu(dz^{t-1}) \right] \nu(dw).$$

The principal's resources at the end of period t are equal to the net transfers from the agents. Since capital allocated to investment is bounded by the principal's resources, the principal faces the following resource constraint:

$$-\int_{\mathcal{V}} \left[\int_{\mathbf{Z^{t+1}}} \tilde{b}_t(z^t, w)\mu(dz^t) \right] \nu(dw) \geq \int_{\mathcal{V}} \left[\int_{\mathbf{Z^t}} \tilde{k}_t(z^{t-1}, w)\mu(dz^{t-1}) \right] \nu(dw). \tag{3}$$

In the spirit of Atkeson and Lucas (1992), our notion of efficiency is the minimum amount of capital necessary to attain a distribution of promised utility w_0. The planner minimizes date-0 capital.

$$\begin{aligned} \varphi^*(\nu) &= \inf \int_{\mathcal{V}} K_0(w_0) \\ & \quad \text{s. to } (1) - (3). \end{aligned} \tag{4}$$

We will call this mapping between the distribution of promised utility to required date-0 capital the infimum function.

We will show later that if a distribution of promised utilities ν_1 first-order stochastically dominates another distribution ν_2 then $\varphi^*(\nu_1) \geq \varphi^*(\nu_2)$ and the inequality is strict if the domination is strict. Therefore the efficient allocation respects Pareto optimality.

2.2 Constraints

In this section we will consider the structure of the allocation rule and the constraints in more detail. In the section above we constructed an allocation in an explicitly recursive manner - at each date we allow the principal to differentiate the agents along a single dimension (a subset of the real line). Thus each agent, identified with a point from the real line, makes a report and is assigned a new index for the next period. In our discussion we call this index continuation utility. In this section we derive some necessary and sufficient conditions that ensure that this index is indeed the agent's expected discounted utility conditional on all the information at some date. Then we show necessary and sufficient conditions to ensure that incentive compatibility is satisfied.[3]

[3] This approach is very common in the literature, following the pioneering work of Green (1987) and Spear and Srivastava (1987)

This construction imposes the restrictions that two agents with the same expected discounted utility must face an identical continuation allocation. In an online appendix we construct a more general allocation and show that there is no benefit from this additional degree of generality and we can restrict our attention to recursively generated allocations.

In the preceding section we introduced the *the lifetime expected utility* of an agent from a strategy σ. We can generalize and find the expected utility of an agent at date t with promised utility w_t, a strategy σ and a history z^{t-1}.

$$\begin{aligned} V_t(\mathbf{P}, \sigma; w_t, z^{t-1}) &= \sum_{s=t}^{\infty} \beta^{s-t} \int_{\mathbf{Z}^{s-t+1}} (1-\beta) u[z_s \tilde{k}_s(\sigma^{t,s}(z^{s-t}, z^{t-1}, w_t), w_t) + \\ &\quad \tilde{b}_s(\sigma^{t,s}(z^{s-t+1}, z^{t-1}, w_t), w_t)] \mu^{s-t}(dz^{s-t}) \\ &= \int_{\mathbf{Z}} \left[(1-\beta) u(z_t k_t(w_t) + b_t(\sigma_t(z^t), w_t)) + \beta V_{t+1}(\mathbf{P}, \sigma; v_{t+1}(w_t, \sigma_t(z^t)), z^t)\right] \mu(dz_t) \end{aligned} \tag{5}$$

Notice that past history is relevant only in that it conditions the reporting strategy. In the case of truthtelling, the past history is irrelevant, so we may omit it.

We called the index w_t continuation utility. This terminology would be suggestive if an agent following the truthtelling strategy actually gets expected utility w_t, or $w_t = V_t(\mathbf{P}, \sigma^*; w_t)$. Then equation (5) gives us the following necessary condition for promise-keeping:

$$w_t = \int_{\mathbf{Z}} \left[(1-\beta) u(z_t k_t(w_t) + b_t(z_t, w_t)) + \beta v_{t+1}(w_t, z_t)\right] \mu(dz_t), \tag{6}$$

for all t and $w_t \in \mathcal{V}$. However, it is feasible for the principal to offer zero consumption and ever-increasing continuation utility. Thus condition (6) is not sufficient: we need to ensure that promised utility does not explode:

$$\lim_{s \to \infty} \beta^s E_t[w_s | w_t, \sigma^*] = 0 \text{ for all } t \text{ and } w_t \in \mathcal{V}. \tag{7}$$

$E[w_s | w_t, \sigma^*]$ denotes that the expectation is taken conditional on truthtelling by the agent and promised utility at date t being w_t.

These two conditions turn out to be necessary and sufficient.

Lemma 1 *Given that (7) holds, then* $w_t = V_t(\mathbf{P}, \sigma^*; w_t)$ *for all* t *and* $w_t \in \mathcal{V}$ *if and only if* $\mathbf{P}$ *satisfies (6).*

The second step in simplifying the constraints is to consider the incentive compatibility constraint. The set of possible reporting strategies is incredibly complex and verifying that the incentive-compatibility constraint holds is infeasible. We will consider strategies that involve misreporting only in one period. We show that if truthtelling dominates these strategies, then truthtelling dominates all possible reporting strategies, i.e. the allocation rule is incentive-compatible.

Consider an agent who has promised utility w_t at date t. If incentive compatibility holds, then the agent should not prefer misreporting in period t alone and telling the truth afterwards to telling the truth forever. Then this implies that for any $z_i, z_j \in \mathbf{Z}$:

$$(1-\beta)u(z_i k_t(w_t) + b_t(z_i, w_t)) + \beta v_t(w_t, z_i)) \geq (1-\beta)u(z_i k_t(w_t) + b_t(z_j, w_t)) + \beta v_t(w_t, z_j) \quad (8)$$

Here we have used Lemma 1 that ensures that the utility of truthtelling is w_{t+1}. An additional condition is that expected utility does not converge to $-\infty$ too fast:

$$\lim_{s\to\infty} \beta^s \inf[w_s|w_0] = 0 \text{ for all } w_0 \in \mathcal{V}. \quad (9)$$

This condition implies that even with the worst history, *discounted* promised utility does not diverge to negative infinity.

Our final result in this section is to show that this condition and the boundedness of expected continuation utility ensure incentive compatibility:

Lemma 2 *Given that (6), (7) and (9) hold, then an allocation rule* $\mathbf{P}$ *is incentive-compatible if and only if condition (8) holds.*

2.3 Bellman equation

In the preceding section we showed that without loss of generality, the planner's allocation can be represented as a sequence of functions mapping promised utility and reports to capital allocation and transfers. In this subsection, we show how we can represent the optimal allocation as a time-invariant mapping from the agent's promised utility, the agent's report and the planner's state variable into capital and transfers.

Since the planner must incorporate general equilibrium considerations into the optimal allocation (through the accumulation constraint), her state variable is the entire distribution of promised utilities. Suppose that starting from tomorrow the capital required to attain some distribution of utilities ν' is given by $\varphi(\nu')$. Then the capital necessary to attain a distribution of utilities ν today is given by:

$$(T\varphi)(\nu) = \inf_{(k,b,v)} \int_{\mathcal{V}} k(w)\nu(dw) \quad (10)$$

$$\text{s. t. } (1-\beta)u\left(z_i k(w) + b(w, z_i)\right) + \beta v(w, z_i) \geq$$

$$(1-\beta)u\left(z_i k(w) + b(w, z_j)\right) + \beta v(w, z_j),\ \forall w \in \mathcal{V}, z_i, z_j \in \mathbf{Z} \quad (11)$$

$$w = \int_{\mathbf{Z}} \left[(1-\beta)u\left(zk(w) + b(w, z)\right) + \beta v(w, z)\right] \mu(dz) \quad (12)$$

$$-\int_{\mathcal{V}} \left[\int_{\mathbf{Z}} b(w, z)\mu(dz)\right] \nu(dw) \geq \varphi(\nu') \quad (13)$$

where ν' is the distribution of promised utility for the next period, generated by (k, b, v) and ν. For any Borel set A the measure of continuation utilities falling in the set A is given by:

$$\nu'(A) = \int_{\mathcal{V}} \int_{\mathbf{Z}} \chi_A[v(w, z)]\mu(dz)\nu(dw).$$

Equation (11) is the temporary incentive compatibility constraint, (12) is the promise keeping constraint, and (13) is the capital accumulation constraint.

For any allocation rule P that satisfies inequality (13) weakly we can find an alternative allocation P' such that the accumulation constraint binds and the distribution of utilities is attained with less capital. Therefore, without loss of generality we can assume that the accumulation constraint is binding.

Next, we establish the relationship between the sequential problem and the functional equation problem.

Lemma 3 *φ^* solves the Bellman equation, that is $T\varphi^* = \varphi^*$.*

Following Thomas and Worral (1990), it is easy to find economically meaningful bounds on the cost functions. Let φ_L be the initial capital necessary to attain ν with perfect information. Since this is the solution to a relaxed problem, it is a lower bound on φ^*. The autarchy allocation satisfies incentive compatibility and feasibility so that is an upper bound for the cost function φ_H.

Define the autarchy value function as:

$$v_A(k) = \int_{\mathbf{Z}} \left[\max_{k' \in [0, zk]} (1-\beta)u(zk - k') + \beta v_A(k') \right] \mu(dz) \tag{14}$$

Then we can define a feasible upper bound for φ:

$$\varphi_H(\nu) = \int_{\mathcal{V}} v_A^{-1}(w)\nu(dw)$$

The operator T is not a contraction, so standard approaches to finding φ^* are not applicable. One strategy would be to iterate on the two bounds. The following result, by exploiting the monotonicity of the operator T, implies that if the two sequences of functions have a common limit, it is the infimum function.

Lemma 4 *$\varphi_L < \varphi^* \leq \varphi_H$. If $\lim_{n\to\infty} T^n\varphi_L = \lim_{n\to\infty} T^n\varphi_H$ then $\varphi^* = \lim_{n\to\infty} T^n\varphi_H$.*

Finally, we show that the allocation generated recursively by the Bellman equation attains the supremum provided that the promised utility does not grow too fast.

Proposition 1 *Let* $\mathbf{P}$ *be an allocation rule generated recursively by iterating on the Bellman equation above. If*

$$\lim_{s\to\infty} \beta^s \inf[w_s|w_0] = 0 \textit{ for all } w_0 \in \mathcal{V}$$

then $\mathbf{P}$ *attains the infimum in the sequence problem.*

This problem as stated is computationally infeasible. We exploit the functional forms for utility and production to find the analytical solution to the Bellman equation. We shall show that in this particular environment, in expectation agents do not transfer resources between each other. This simplifies the principal's problem dramatically, since the allocation at each date now needs to be conditioned only on the agent's promised utility and not on the whole distribution.

The allocation rule is then a triplet of functions: $\mathbf{k} : \mathcal{V} \to \mathbb{R}_+$ that determines capital to the agent, $\mathbf{b} : \mathcal{V} \times \mathbf{Z} \to \mathbb{R}$ that determines consumption transfers and $\mathbf{v} : \mathcal{V} \times \mathbf{Z} \to \mathcal{V}$ that determines continuation utility.

In order to ensure that the full-information and autarchy allocations are well-defined we make the following assumption.

Assumption 1 *If* $u(c) = c^\gamma/\gamma$, $\gamma \neq 0$, *then* $\beta^{\frac{1}{1-\gamma}}(Ez)^{\frac{\gamma}{1-\gamma}} < 1$ *and* $\beta^{\frac{1}{1-\gamma}}(E\{z^\gamma\})^{\frac{1}{1-\gamma}} < 1$.

The first condition ensures that the full-information case is well-defined; the second is the analogue for the autarchy allocation.

Since the utility functions we work with allow aggregation, φ_L has a particularly simple form. Let $v_{FB}(w)$ be the value function in a representative agent model with deterministic productivity $\bar{z} = Ez$. Then if $u(c) = c^\gamma/\gamma$,

$$\varphi_L(\nu) = v_{FB}^{-1}\left[\left(\int |w|^{\frac{1}{\gamma}}\nu(dw)\right)^\gamma\right].$$

Similarly, if $u(c) = \log(c)$, then:

$$\varphi_L(\nu) = v_{FB}^{-1}\left[\log\left(\int \exp(w)\nu(dw)\right)\right].$$

Since both v_a and v_{FB} are of the form $Bu(k)$, where $B > 0$ is some constant, the upper and lower bounds on the infimum function are of the form $\varphi_i(\nu) = A_i \int_{\mathcal{V}} u^{-1}(w)$, $i = a, FB$. We will show that the infimum function is of the same form. Again, we will show that the Bellman operator maps the space of functions of this form to itself and we will use Lemma 4 to prove that the unique fixed point of the operator in that space is the infimum function.

Since the planner internalizes the general equilibrium concerns, the planner must allocate capital, consumption and continuation utility to all agents. In other words, for every possible distribution of utility, the planner chooses a triplet of functions rather than vectors. To solve this problem we guess and verify that the optimal contract is a scaled version of the optimal contract for a particular promised utility. We define the auxiliary problem of finding the optimal contract at $\operatorname{sgn}(\gamma)$ as follows:

$$\begin{aligned} \phi(A) &= \inf_{k,b,v} k && (15) \\ \text{s. to } (1-\beta)u(zk+b(z))+\beta v(z) &\geq \\ (1-\beta)u\left(zk+b(z')\right)+\beta v(z'), \quad z,z' &\in \mathbf{Z}, z\neq z' && (16) \\ \int_{\mathbf{Z}}\{(1-\beta)u\left(zk+b(z)\right)+\beta v(z)\}\mu(dz) &= \operatorname{sgn}(\gamma) && (17) \\ -\int_{\mathbf{Z}} b(z)\mu(dz) &\geq A\int_{\mathbf{Z}} u^{-1}(v(z))\mu(dz) && (18) \end{aligned}$$

First, we show that if the set function φ has the particular form we conjectured, the solution to the capital minimization problem (10) - (13) can be derived from the solution of the simpler auxiliary problem.

Lemma 5 *Assume that* $\varphi(\nu) = A\int u^{-1}(w)\nu(dw)$ *for some* $A > 0$. *Then the optimal contract is given by* $\mathbf{k}(w) = k|w|^{\frac{1}{\gamma}}$, $\mathbf{b}(w,z) = b(z)|w|^{\frac{1}{\gamma}}$, *and* $\mathbf{v}(w,z) = v(z)|w|$, *if* $\gamma \neq 0$ *and* $\mathbf{k}(w) = k\exp(w)$, $\mathbf{b}(w,z) = b(z)\exp(w)$ *and* $\mathbf{v}(w,z) = v(z)+w$ *for the* log *case, where* (k,b,v) *is the solution to the auxiliary problem (15)-(18).*

Moreover, $T\varphi = \phi(A)\int_{\mathcal{V}} u^{-1}(w)\nu(dw)$.

Since $\phi(A_l) > A_l$, $\phi(A_h) < A_h$ and ϕ is continuous, the auxiliary problem has at least one fixed point. We have not been able to show that the fixed point is unique but numerical examples suggest that this is the case. We will show that the smallest value of the fixed point corresponds to the infimum function and we show that the corresponding allocation rule attains the infimum.

Let $\phi^{(n)}$ denote the usual n-times composition of the function. Then the properties of ϕ ensure that $A^* = \lim_{n\to\infty}\phi^n(A_L)$ exists and is a fixed point for the function ϕ. Moreover, it is the smallest fixed point on the interval $[A_l, A_h]$. If that fixed point is unique then it corresponds to the solution of the infimum problem; otherwise some additional conditions are required to ensure that A^* corresponds to the solution of the infimum problem.

Proposition 2 *Suppose that utility is in the CRRA class. Assume that either* A^* *is the unique fixed point of* ϕ *on* $[A_l, A_h]$ *or that* $\beta^{\frac{1}{1-\gamma}}(E\{z^\gamma\})^{\frac{\gamma}{1-\gamma}}z_L^\gamma < 1$. *Then the infimum function* φ^* *is given by:*

$$\varphi^*(\nu) = A^*\int_{\mathcal{V}}|w|^{\frac{1}{\gamma}}\nu(dw) \tag{19}$$

and the optimal allocation rule is:

$$\mathbf{k}(w) = k|w|^{\frac{1}{\gamma}}, \mathbf{b}(w,z) = b(z)|w|^{\frac{1}{\gamma}}, \mathbf{v}(w,z) = v(z)|w|,$$

where (k, b, v) solve the auxiliary problem for A^*. Moreover, the allocation rule induces an allocation that satisfies all the constraints.

If utility is logarithmic, the fixed point $\phi(A^*) = A^*$ is unique.

Proposition 3 *Suppose that utility is logarithmic. There is a unique $A^* \in [A_l, A_h]$ such that $\phi(A^*) = A^*$.*
The infimum function φ^* is given by:

$$\varphi^*(\nu) = A^* \int_{\mathcal{V}} \exp(w)\nu(dw) \tag{20}$$

and the optimal allocation rule is:

$$\mathbf{k}(w) = k\exp(w), \mathbf{b}(w,z) = b\exp(w), \mathbf{v}(w,z) = v(z) + w,$$

where (k, b, v) solve the auxiliary problem for A^*. Moreover, the allocation rule induces an allocation that satisfies all the constraints.

3 Analysis

In the optimal contract allocated capital and consumption are both proportional to the consumption equivalent of the current period promised utility (the constant flow of consumption that would have the same utility level), which is $|w|^{\frac{1}{\gamma}}$ (or $\exp(w)$ for log utility). Since the consumption equivalent is a geometric random walk, capital and consumption are geometric random walks. This is a consequence of the fact that agents in our economy have constant relative risk aversion. Therefore, it is optimal to make consumption variability exactly proportional to average level of consumption. This is accomplished by making allocated capital proportional to average consumption. On the other hand, since agents prefer smoother consumption profile, the effect of a bad productivity report is to permanently (and proportionally) reduce all future consumptions.

Since allocated capital, output, and consumption are proportional, it follows that there is no net expected transfer of resources (consumption transfers and capital) between agents. However, there is some transfer of resources between states of the world (subject to incentive compatibility constraints). Therefore, the planner's power to improve upon the autarchy allocation (discussed in more detail below) stems from the fact that the resource constraint is satisfied agent by agent, but only in expectation and not state by state. A second condition is that investment is observable, which helps in the dynamic provision of incentives. We exploit this fact to derive a simple decentralization of the optimal allocation in Section 4.

A further implication of the structure of the optimal allocation is *scale invariance*: the growth rates of aggregate capital, output and consumption are independent of either current distribution of promised utility or initial conditions. In the decentralization below, we interpret capital as wealth. Therefore the scale invariance result implies that the *change* in measures of inequality – variance of log wealth or variance of log consumption – is independent of the distribution of promised utilities or current conditions generally.

3.1 A comparison with autarchy

To see whether an individual may eventually be better off without the contract, we briefly consider the autarchic allocation. In autarchy, savings is equal to investment at every point in time and in every state of nature; therefore $-b_t\left(z^t\right) = k_{t+1}\left(z^t\right)$ for all $z^t \in \mathbf{Z}^{t+1}$, $t = 0, 1, \ldots$. Let $v_a(z_i, k)$ denote the expected lifetime utility of an individual with k units of capital, given z_i as the current level of productivity. The functional equation, in the absence of trade, is given by

$$v_a(z_i, k) = \max_{k' \in [0, z_i k]} \left\{ u(z_i k - k') + \beta \int_{\mathcal{V}} v_a(z, k')\mu(dz) \right\}. \tag{21}$$

As is well known, the optimal policy is $k^{'} = \beta z_i k$ for logarithmic utility and $k^{'} = \left[\beta \int_{\mathbf{Z}} z^{\gamma}\mu(dz)\right]^{\frac{1}{1-\gamma}} z_i K$ for iso-elastic utility.

Next we take the expectation of $v_a\left(z_i, k\right)$ over z_i to define $v_a(k) = \int_{\mathbf{Z}} v_a(z, k)\mu(dz)$, which yields expected lifetime utility. While $v_a\left(z_i, k\right)$ represents the value of k units of capital conditional on the current productivity shock, $v_a\left(k\right)$ values the current capital stock prior to the realization of current productivity. This latter function ensures that the value function for the autarchic problem is consistent with the timing of events used to solve the contracting problem.

For brevity we will only consider iso-elastic utility in the rest of this section – the same arguments hold for the log utility case. From the discussion above, it is clear that $v_a(k) = Bk^{\gamma}$, where B is some constant. Then the upper bound on capital that the planner needs to deliver $\text{sgn}(\gamma)$ units of utility is $A_h = |B|^{-\frac{1}{\gamma}}$. Since $A^* < A_h$, and the planner will allocate capital $k = A^*|w|^{\frac{1}{\gamma}}$, the utility the agent will get from this allocation is

$$w = \text{sgn}(\gamma) A^{*-\frac{1}{\gamma}} k^{\gamma} > \text{sgn}(\gamma) A_H^{-\frac{1}{\gamma}} = Bk^{\gamma} = v_a(k)$$

Therefore at each date the agent's ex ante expected utility from the allocation is strictly higher from the autarchic value of the allocated capital. Since there is some transfer of resources between *states of the world*, this result will not necessarily hold if we allow the agent to leave the contract after the productivity shock is realized.

3.2 A numerical example

Further analytical characterization is difficult. However, computation of the contract economy is relatively easy. We do not have to solve for the value function or track the distribution of wealth in order to obtain the equilibrium contract allocation. The scaling property, necessarily absent in the unobservable endowment model, allows us to solve a small number of non-linear equations that describe the typical contract. This yields $(b_i, v_i)_{i=1}^2$. Through numerical examples below we highlight several additional features of our economy that distinguish it from the endowment model.

We set the parameters as follows. We allow productivity to vary around its mean value ξ, symmetrically. In other words, let $\xi - z_1 = z_2 - \xi = \Delta z$ and $\mu_1 = 0.5$. After choosing a plausible value of $\xi = 1.05$, we set Δz to as to imply a reasonably high coefficient of variation, 0.15, for the productivity process. Finally, let $\beta = 0.97$; this implies that the economy with logarithmic preferences, under complete risk-sharing, would have a real interest rate of 5 per cent and a growth rate of roughly 2 per cent per annum. We examine the contract for three different values of γ: 0.5, 0, and -1. It is important to emphasize that while we illustrate these features of our private information economy using specific parameters, these findings hold across all parameterizations that we have examined.

As noted in the introduction, an individual's lifetime wealth in the endowment model is determined by his utility entitlement. Since our purpose here is to contrast the long-term contract under production and capital accumulation with the endowment model, we adopt individual utility entitlements as our metric even though our model has a distinct, and perhaps preferable, measure of wealth, which is capital. To begin with, consider the evolution of lifetime utility under the enduring relationship. Our strategy is to examine an economy with a unit measure of agents who are ex ante identical, each having the same utility entitlement in the initial period. A fraction μ_1 of these agents will experience a capital growth $|v_1|^{\frac{1}{\gamma}}$ while a fraction μ_2 will experience a growth rate $|v_2|^{\frac{1}{\gamma}}$.[4] As a result, the population is no longer homogeneous in period 1. In period 2, each of these two groups are again split into proportions μ_1 and μ_2. Each period we calculate the population mean and standard deviation of expected lifetime utility under the assumption of long-term contracts. We then contrast the results with those under autarchy.

Figure 1 displays the economy-wide mean level of utility for both the contract and autarchic environments for the first 50 periods. In contrast to the endowment economy, we

[4]When $\gamma = 0$, the corresponding growth rates are e^{v_1} and e^{v_2}.

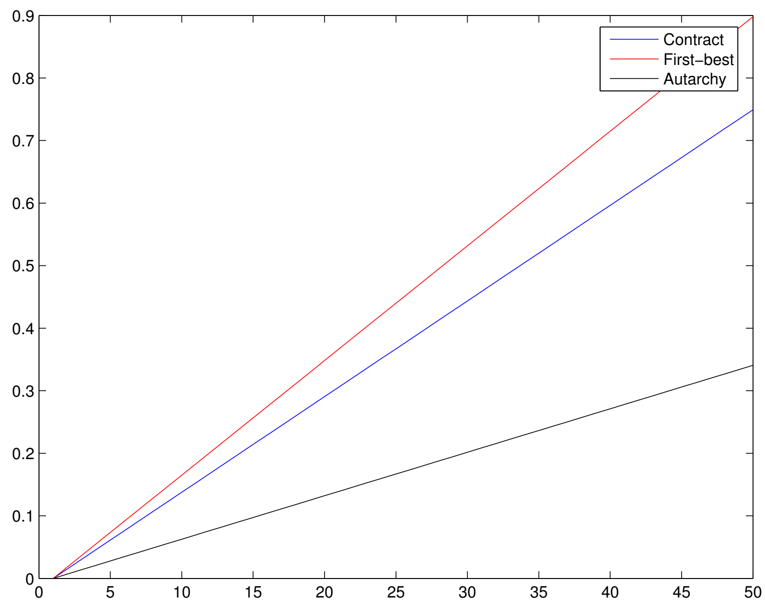

Figure 1: Average expected utility for logarithmic utility

see that expected lifetime utility in the enduring relationship rises above that in autarchy over time. This is despite the fact that in some cases the growth rate of wealth is higher under autarchy.[5] For example, when $\gamma = -1$, which corresponds to a coefficient of relative risk aversion $(1-\gamma)$ of 2, the strong self-insurance motive present under autarchy leads to a doubling of the mean rate of growth of capital, relative to the contract economy. However, the insurance offered through long-term contracts implies reduced variability in investment rates, relative to autarchy. This is sufficient to ensure that the mean level of expected lifetime utility in the contract grows relative to autarchy.

In Figure 2, we examine the standard deviation of lifetime utility. Dispersion in lifetime utility grows in both settings. However, the long-term contract dampens the inequality in lifetime utility relative to autarchy, and the dampening is dramatic when γ is sufficiently low ($\gamma = -1$). The contract provides insurance by dampening variability in both consumption and investment. As a result inequality rises more slowly than in autarchy.

[5]The rates of growth for the contract economies, when $\gamma = -1$, 0 and 0.5, are 0.98, 1.95 and 3.95 per cent, respectively. The corresponding growth rates under autarchy are 2.15, 2.0 and 3.45 percent.

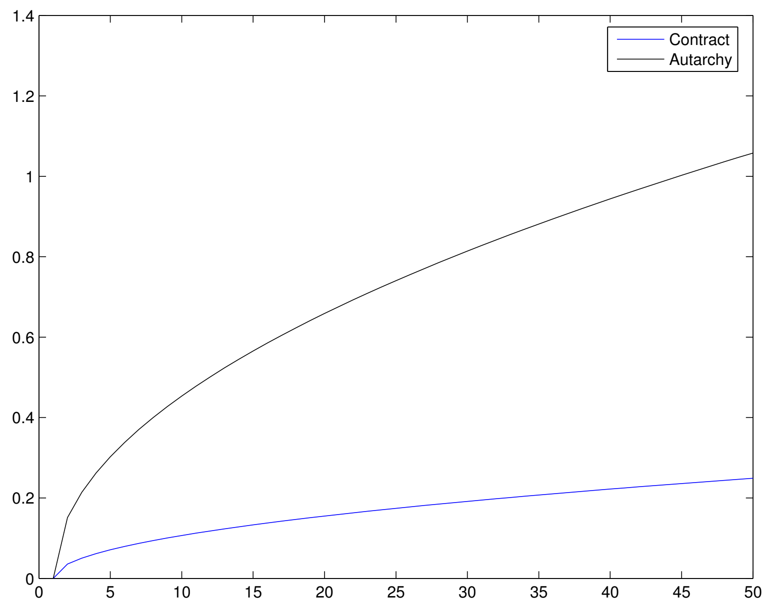

Figure 2: Standard deviation of utility for logarithmic utility

4 Decentralization

In this section, we follow the seminal work of Atkeson and Lucas (1992) to show how the distributions of promised utilities can be achieved through a decentralized market mechanism. We postulate a continuum of risk-neutral insurance companies who can commit and can trade consumption between themselves. The agents enter into long-term relationships with the insurers. The insurers can intertemporally trade with each other, so one of the functions the insurer performs for the agent is to transfer consumption between different states and dates (subject to incentive constraints). Since every one of these insurers is tasked with delivering utility to one agent, we call them, in accordance with the literature, component planners.

At the end of each period, after net transfers have been made, all component planners exchange resources. We can assume that these trades were agreed upon in some date-0 Arrow-Debreu market. There is no aggregate uncertainty and the law of large numbers holds, so there is only one price for each date. The timing of the trading scheme is as follows:

$t-1$	t					$t+1$
	The planner has saved enough capital	Planner gives capital for investment	Shock is realized	Report	Planner makes transfers and saves capital for the next period	
					Trade at price p_t	

Then the planner's cost to deliver w_0 is

$$C(w_0) = \inf \int k + \sum_{t=0}^{\infty} p_t \left[E_0(b_t + k_{t+1})\right] \tag{22}$$

where the allocations are chosen subject to the same constraints.

The following theorem relates the solution of this problem to the planner's problem.

Theorem 1 *Suppose that there exists a sequence of prices $\{p_t\}_{t=0}^{\infty}$, and an allocation that:*

1. *at the given prices it solves (22) subject to PK and IC*
2. *at each date the net trades are zero:*

$$\int_{\mathcal{V}} \int_{\mathbf{Z}^t} \left[b_t(w_0, z^t) + k_{t+1}(w_0, z^t)\right] \mu(dz^t)\nu(dw_0) = 0$$

3. $$1 + \sum_{t=1}^{\infty} p_t \left| \int_{\mathbf{Z^{t+1}}} [b_t(w_0, z^t) + k_{t+1}(w_0, z^t)]\mu(dz^t) \right| < \infty \tag{23}$$

.

Then this allocation attains ν with minimum initial capital.

The theorem above plays the role of the first welfare theorem in our environment. Its proof follows the basic logic of the classic first welfare theorem.

The structure of the optimal allocation allows us to go further and to show that such decentralization exists under some weak conditions. In the planner's allocation, the distribution of promised utility and the date are irrelevant to capital and transfers allocated to a particular agent. Capital and consumption are proportional to the consumption equivalence of the agent's promised utility. This suggests that the same invariance should carry over in the decentralization. So, we conjecture that $p_{t+1}/p_t = q$ for all t. Then if a decentralization with the desired properties exists, the component planner's problem would solve the simple

Bellman equation:

$$\begin{aligned} C(w) &= \min k + qE[b_i + C(w_i)] && (24) \\ E[(1-\beta)u(z_i k + b_i) + \beta w_i] &= w && (25) \\ (1-\beta)u(z_H k + b_H) + \beta w_H &\geq (1-\beta)u(z_H k + b_L) + \beta w_L && (26) \\ k \geq 0, w_i \in \mathcal{V}, z_i k + b_i \geq 0. & && (27) \end{aligned}$$

If the component planners are not subject to incentive constraints, allocating capital to the agent and lending to other component planners are both riskless and they must yield the same return. One bound for the intertemporal price q is then $q^{FB} = [Ez]^{-1}$. On the other hand, if we do not allow the component planner to transfer resources between states, the intertemporal price q that would clear the market is $q^{AU} = Ez^{\gamma-1}/Ez^{\gamma}$. The next theorem states that for some q between these bounds, the allocation chosen by the component planners will satisfy the conditions of theorem 1.

Theorem 2 *Assume that $q^{AU} < 1$. There exists some price $q^* \in [q^{FB}, q^{AU}]$ such that the insurer's optimal allocation coincides with the planner's solution and there is no trading between insurers.*

This result is a counterpart to our results about consumption smoothing - the optimal allocation does not provide perfect insurance, but it reduces consumption variability relative to the autarchy allocation. Similarly, the value of a safe asset is somewhere in between that of full information environment and autarchy. Another way of looking at this result is as an analogue of the second welfare theorem - it is always possible to decentralize the allocation through component planners.

4.1 An alternative decentralization

Notice that at no date is there a transfer of resources between planners. This permits us to find a much simpler decentralization. Since there is no cross-subsidization across agents, the decentralization can be accomplished through a sequence of one-period contracts. For an arbitrary distribution of initial promised utilities, let's endow each agent with the capital that the planner would allocate ($A^*|w|^{\frac{1}{\gamma}}$ or $A^* \exp(w)$). In our decentralized economy, the agents will own the capital stock and carry it from one period to the next. As in the planner's problem, we will assume that invested capital is fully observable.

In addition to productive agents, there is a mass of competitive insurers. We will assume that the law of large numbers holds for the insurers. So, when we consider their relationship with a single agent, they can be considered risk-neutral and their budget constraints need to hold only in expectation.

The insurers and the agents can fully commit to infraperiod contracts. The insurance contracts consist of a net transfer from the insurer to the agent $\tau(z)$ which is conditioned on reported productivity shock and an investment requirement $\kappa(z)$. The ability of the insurer to control (but not own) the agent's investment is crucial to providing the intertemporal incentives. In particular an agent with a high productivity shock might want to report that he got the low productivity shock and collect a positive net payment from the insurer, but the agent will have to reduce his investment and hence future consumption.

The timeline in this economy is as follows:

1. At the beginning of the period the agent has some invested capital k, but the productivity shock is not yet realized.

2. Competitive insurers offer contracts to the agent.

3. A contract is signed.

4. The productivity shock is realized and privately observed.

5. The agent makes a report to the insurer.

6. The net payment is carried out and the agent consumes and (publicly) invests the required amount.

Since we assume that the insurers are risk-neutral and perfectly competitive, the equilibrium contract will maximize the agent's utility subject to incentive-compatibility and break-even constraints. Let $v^D(k)$ be the equilibrium value function for the agent. Then the equilibrium contract for an agent with capital k will solve the following problem:

$$\max_{\tau(z),\kappa(z)} \int_{\mathbf{Z}} (1-\beta)u(zk+\tau(z)-\kappa(z)) + \beta v^D(\kappa(z))\mu(dz) \tag{28}$$

$$\int_{\mathbf{Z}} \tau(z)\mu(dz) \geq 0 \tag{29}$$

$$u(zk+\tau(z)-\kappa(z)) + \beta v^D(\kappa(z)) \geq u(zk+\tau(z')-\kappa(z')) + \beta v^D(\kappa(z')) \tag{30}$$

Let $\tau(k,z), \kappa(k,z)$ be the equilibrium contract for an agent with capital k. Define $w(k) = \operatorname{sgn}(\gamma)k^\gamma/A^{*\gamma}$ for the iso-elastic case and $w(k) = \log(k) - \log(A^*)$ for the log utility case; $w(k)$ is the utility corresponding to some capital level in the planner's problem. Then we have the following proposition:

Proposition 4 *The pair of functions* $\tau(k,z) = \mathbf{b}(w(k),z) + \mathbf{k}(\mathbf{v}(w(k),z))$ *and* $\kappa(k,z) = \mathbf{k}(\mathbf{v}(w(k),z))$ *constitute an equilibrium in the decentralized economy. Moreover,* $v^D(k) = w(k)$*.*

The proposition implies that in order to satisfy the utility entitlements in an optimal manner, the planner can allocate initial capital appropriately and then let the market equilibrium play itself out.

This decentralization

5 Concluding remarks

We have examined the qualitative behavior of an economy with long-term contracts with production and capital accumulation. Insurance is implemented in our environment by exploiting differences in agents' willingness to intertemporally substitute consumption: individuals with low productivity receive a net transfer of resources at the expense of reduced future expected utility. Under the assumption of linear technology and CRRA preferences, we establish a one-to-one mapping between an individual's stock of capital and his expected lifetime utility. In particular, we show that adjustments to lifetime utility are implemented through changes in an individual's capital. This feature provides an interesting contrast between our model and the long-term contracts model with risky endowments.

Both the endowment environment and our environment exhibit increasing dispersion in lifetime utility (or wealth) across agents. For the endowment economy, however, wealth in the typical contract eventually falls below the autarchy value of the endowment process. In the production economy with capital accumulation, the adjustment in individuals' stocks of capital affect the value of autarchy. We find that the contract is able to maintain welfare above autarchy levels for any agent.

We are able to characterize the efficient allocation in a closed economy in which the resource constraint (consumption plus investment equals output) holds in each period. Incentive compatibility prevents the separation of production and consumption decisions, and this in turn prevents aggregation. Thus, the planner's state variable is a distribution of agents' promised utilities. However, the scaling property of the optimal allocation makes the problem tractable: the optimal allocation can be derived for one agent in an auxiliary problem and then "scaled up" for everyone.

The one-to-one mapping, in equilibrium, implies that the expected profit from providing insurance is zero for each firm. Hence, unlike the endowment model our contract exhibits two-sided voluntary participation: provided any agreement is enforceable during the period, it is never rejected by either party at the onset of any period. Thus, our contract may be interpreted as allowing for a two-sided lack of commitment, thereby extending Phelan's (1995) one-sided lack of commitment.

Atkeson and Lucas (1995) and Phelan (1995) have shown that the increasing dispersion of wealth property in the endowment model can be eliminated if the set of possible lifetime utility entitlements is bounded below. Atkeson and Lucas motivate their bound as reflecting the fact that, in a dynasty model, the current generation cannot promise away the utility entitlement of future generations. In our model, lifetime utility is proportional to capital and future utility is determined by current investment. Bounding future utility below is equivalent to forcing a lower bound on the agent's current savings. This is far stronger than the assumption that agents cannot sell the utility of future generations. Phelan assumes that at the onset of any period, the agent is free to recontract with another firm. Again, the one-to-one mapping in our model implies that if agents were free to do so, they would never want to exercise this option. Hence, the methods in Atkeson and Lucas and Phelan do not generate an invariant limiting distribution of wealth in our environment. In related work, Aiyagari and Alvarez (1995) and Bohacek (2001) have shown that the presence of an invariant distribution in the endowment model relies on individuals' consumption possibilities sets being compact. In our model with capital accumulation, the possibility of long run growth makes the assumption of upper bounds on consumption unattractive, while the assumption of a lower bound on consumption is eventually non-binding.

Our approach assumes that capital accumulation is observable, and exploits this assumption to model the firm as allocating capital. In a recent paper, Cole and Kocherlakota (2001) characterize efficient allocations in an economy where individuals have risky, privately observed, endowment shocks and may operate a risk-free, unobservable storage technology. They show that this leads to an elimination of state-contingent payoffs or insurance. While we view the assumption of observable investment as restrictive in certain contexts, we also view the lack of insurance arrangements which arises under the Cole and Kocherlakota model as extreme.

A Proofs

Proof of Lemma 1. Assume (6) holds. Then it implies that for any w_0 and z^{-1} we have:

$$\tilde{w}_t(z^{t-1}, w_0) = \int_{\mathbf{Z}} \Big[(1-\beta)u(z_t\tilde{k}_t(z^{t-1}, w_0) + b_t(z^t, w_0)) + \beta v_{t+1}(z^t, w_0))\Big]\, \mu(dz_t)$$

By repeated application of application of the promise-keeping constraint , we get:

$$\begin{aligned} w_t &= (1-\beta)\sum_{s=t}^{T} \beta^{s-t} \int_{\mathcal{Z}^{s-t+1}} u[z_s\tilde{k}_s(z^{s-t-1}, w_t) + \tilde{b}_s(z^{s-t}, w_t)]\mu(dz^{s-t}) \\ &\quad +\beta^{T-t}\int_{\mathcal{Z}^{T-t+1}} \tilde{w}_{T+1}(z^{T-t}, w_t)\mu(dz^{T-t-}) \\ &= (1-\beta)\sum_{s=t}^{\infty} \beta^{s-t} \int_{\mathcal{Z}^{s-t+1}} u[z_s\tilde{k}_s(z^{s-t-1}, w_t) + \tilde{b}_s(z^{s-t}, w_t)]\mu(dz^{s-t}) \\ &\quad + \lim_{T\to\infty} \beta^{T-t}\int_{\mathcal{Z}^{T-t+1}} \tilde{w}_{T+1}(z^{T-t}, w_t)\mu(dz^{T-t}) \end{aligned}$$

Then adding the boundedness condition on expected utility, (7), we get the sufficiency condition.

Now assume that $V_t(\mathbf{P}, \sigma^*; w_t) = w_t$ for all t and $w_t \in \mathcal{V}$. Then substituting this in (5) we get:

$$w_t = \int_{\mathbf{Z}} \left[(1-\beta)u\left(z_t k_t(w) + b_t(w_t, z_t)\right) + \beta v_{t+1}(w_t, z_t)\right] \mu(dz_t)$$

But this is exactly the recursive condition (6) that we require. ■

Proof of Lemma 2. The first step in the proof is to show that truthtelling σ^* dominates any strategy that involves misreporting in finitely many periods. Define $\Sigma_t = \{\sigma \in \Sigma : \sigma_t(z^t) = z_t \text{ if } t \geq T\}$, that is the set of strategies that involve truthtelling in all periods after (and including) T. Note that $\sigma^* \in \Sigma_0$ (and is in fact the only member of Σ_0). For any $\sigma \in \Sigma_T$:

$$\begin{aligned} V(\mathbf{P}(w_0), \sigma) &= \sum_{s=0}^{T} \beta^s \int_{\mathbf{Z}^{s+1}} (1-\beta)u\left(z_s\tilde{k}_s\left[\sigma^{s-1}(z^{s-1}), w_0\right] + \tilde{b}_s\left[\sigma^s(z^s), w_0\right]\right)\mu^{s+1}(dz^s) + \\ &\quad \beta^{T+1}\int_{\mathbf{Z}^{t+1}} \tilde{w}_{T+1}(\sigma^T(z^T), w_0)\mu(dz^T) \end{aligned}$$

We used the representation (5) and lemma 1.

The proof is by induction on T. σ^* trivially dominates all $\sigma \in \Sigma_0$. Now assume that σ^* dominates all $\sigma \in \Sigma_T$. We will show that this is true for $T+1$. Let $\sigma \in \Sigma_{T+1}$. Integrating the t.i.c. constraint (8) over all possible histories and using the promise-keeping constraint and lemma 1 we get that:

$$\int_{\mathbf{Z}^{T+1}} \left\{(1-\beta)u\left(z_T\tilde{k}_T(\sigma^{t-1}(z^{T-1}), w_0) + \tilde{b}_T(\sigma^T(z^T), w_0)\right) + \beta\tilde{w}_{T+1}(\sigma^T(z^T), w_0)\right\}\mu(dz^T) \leq \int_{\mathbf{Z}^T} w_T(\sigma^{T-1}(z^{T-1}), w_0)\mu(dz^{T-1})$$

Then applying the definition of V and the result above:

$$\begin{aligned} V(\mathbf{P}(w_0),\sigma) \quad &\leq \quad \sum_{s=0}^{T-1} \beta^s \int_{\mathbf{Z^{s+1}}} (1-\beta) u \left(z_s \tilde{k}_s(\sigma^{s-1}(z^{s-1}), w_0) + \tilde{b}_s(\sigma^s(z^s), w_0) \right) + \\ & \quad + \beta^T \int_{\mathbf{Z}^T} w_T(\sigma^{T-1}(z^{T-1}), w_0)\mu(dz^{T-1}) \leq V(\mathbf{P}(w_0),\sigma^*) \end{aligned}$$

So we showed that a strategy involving misreporting in T periods is dominated by a strategy involving misreporting in $T-1$ periods which in turn is dominated by truthtelling. Then by induction, any strategy in which the agent lies for finite periods is dominated by truthtelling.

Now let's consider an arbitrary strategy σ. Assume that it dominates σ^*. Then:

$$\sum_{s=0}^{T-1} \beta^s \int_{\mathbf{Z^{s+1}}} (1-\beta) u \left(z_s \tilde{k}_s(\sigma^{s-1}(z^{s-1}), w_0) + \tilde{b}_s(\sigma^s(z^s), w_0) \right) - w_0 > \epsilon \tag{A.1}$$

for all t sufficiently large $t \geq T$ and some $\epsilon > 0$. Pick some $T' > T$ such that $\beta^{T'} \inf w_{T'} > -\epsilon/2$. This implies that:

$$\beta^{T'} \int_{\mathbf{Z}^{T'}} w'_T(\sigma^{T'-1}(z^{T'-1}), w_0)\mu(dz^{T'-1}) > -\epsilon/2$$

Define σ' to be the strategy identical to σ in all periods up to $T'-1$ and truthtelling afterwards. Then the utility of that strategy is:

$$V(\mathbf{P}(w_0),\sigma') > w_0 + \epsilon/2$$

Thus we get a strategy with finite misreporting that strictly dominates truthtelling. We have reached a contradiction.

Finally, we need to show that (8) is necessary. Consider a strategy that involves misreporting only in period t, given promised utility w_t. It must be dominated by truthtelling and that together with lemma 1 implies the result. ■

Proof of Lemma 3. The proof is standard. Clearly if the measure on promised utlities ν gives probability 1 to utility equal to its lower bound (utility of consuming zero forever), then both $\varphi^*(\nu) = 0$ and $T\varphi^*(\nu) = 0$ and $\varphi^*(\nu) > 0$ and $T\varphi^*(\nu) > 0$ otherwise. So we need to prove the result only for the latter case.

First, we show that $\varphi^* \geq T\varphi^*$. Assume not: $\varphi^*(\nu) < T\varphi^*(\nu)$ for some ν. Then, by definition, there exists an allocation $\mathbf{P}$ that satisfies all the constraints and

$$\int_{\mathcal{V}} \mathbf{k}_0(w)\nu(dw) < T\varphi^*(\nu)$$

This allocation induces a continuation distribution ν'. By the resource constraint,

$$-\int_{\mathcal{V}} \int_{\mathbf{Z}} \mathbf{b}_0(w,z)\mu(dz)\nu(dw) \geq \int_{\mathcal{V}} \mathbf{k}_1(w)\nu'(dw) \geq \varphi^*(\nu')$$

The last inequality follows from the fact that the continuation allocation is attainable and therefore cannot be better than φ^*. Define $\hat{P}$ by $\hat{k}(w) = \mathbf{k}_0(w), \hat{b}(w,z) = \mathbf{b}_0(w,z), \hat{v}(w,z) = \tilde{w}_1(z,w_0)$. Then $\hat{P}$ satisfies all the constraints for the Bellman equation and

$$\int_{\mathcal{V}} \hat{k}(w)\nu(dw) < T\varphi^*(\nu)$$

which is a contradiction. Therefore $\varphi^* \geq T\varphi^*$.

Now we prove that $\varphi^* \leq T\varphi^*$. Assume not: $T\varphi^*(\nu) < \varphi^*(\nu)$ for some ν with nonzero mass of utility entitlements above the lower bound. Let P satisfy all the constraints in the Bellman equation and

$$\int_{\mathcal{V}} k(w)\nu(dw) < T\varphi^*(\nu) + \epsilon < \varphi^*(\nu)$$

for some small enough $\epsilon > 0$. Let ν' be the continuation distribution induced by P.

The incentive constraint implies that if $k > 0$, then w is above the lower bound of utility. Then there exists a set A with positive measure such that w is above its lower bound and $k(w) > 0$. By the Inada conditions, for each $w \in A$, $z_L k(w) + b(w, z_L) > 0$. Then for each w and k, it is possible to increase k and b_H and b_L in such a way to keep all the constraints. Let $\delta(w)$ be the maximum possible increase in k. This is a Borel-measurable function. Redefine the allocation on A by increasing k at each (w) by $\lambda\delta(w, k)$ and the transfers b to keep the promise-keeping and incentive constraints. For any positive λ expected net transfers to the planner have increased strictly. We can find a $\lambda > 0$ small enough such that

$$\int_{\mathcal{V}} k(w; \lambda) d\nu < \varphi^*(\nu)$$

where $k(.;\lambda)$ and $b(.;\lambda)$ is the updated allocation of capital. By construction:

$$-\int_{\mathcal{V}} \int_{\mathbf{Z}} b(w, z; \lambda)\mu(dz)\nu(dw) > \varphi^*(\nu')$$

By definition of φ^* there exists some allocation ($\mathbf{P'}$ that attains ν' with capital less than $-\int_{\mathcal{V}} \int_{\mathbf{Z}} b(w, z; \lambda) d\nu d\mu$. Then the allocation $((\mathbf{k}_0(.;\lambda), \mathbf{B}(.;\lambda), \mathbf{B'}), \mathbf{P'})$ dominates $\varphi^*(\nu)$ which is a contradiction. ∎

Proof of Lemma 4. First, we show that the operator is monotone. Let φ_1 and φ_2 be two set functions, such that $\varphi_1(\nu) \leq \varphi_2(\nu)$ for all ν. Let E_{φ_i} be the set of allocation rules P that satisfy the constraint set of φ_i for some ν. Since constraint (13) is relaxed for φ_1, we have $E_{\varphi_1} \supseteq E_{\varphi_2}$. Therefore:

$$T\varphi_1(\nu) = \inf_{P \in E_{\varphi_1}} \int_{\mathcal{V}} k(w)\nu(dw) \leq \inf_{P \in E_{\varphi_2}} \int_{\mathcal{V}} k(w)\nu(dw) = T\varphi_2(\nu)$$

Consider φ_H - the upper bound on φ^* that is achieved by replicating the optimal autarchy allocation. If (K_a, B_a, V_a) are the functions that attain the autarchy allocation, $(K_a, B_a, V_a) \in E_{\varphi_H}$. Then

$$T\varphi_H(\nu) = \inf_{P \in E_{\varphi_1}} \int_{\mathcal{V}} \int_{\mathbf{M}} kP(dk|w)\nu(dw) \leq \int_{\mathcal{V}} K_a(w)\nu(dw) = \varphi_H(\nu)$$

We showed that T is monotone, therefore by induction $T^n\varphi_H \geq T^{n+1}\varphi_H$.

Let E_{FB} be the set of measures P that satisfy a relaxed constraint - without the incentive compatibility constraint. Clearly, $E_{FB} \supseteq E_{\varphi_L}$. This implies that

$$T\varphi_L(\nu) = \inf_{P \in E_{\varphi_H}} \int_{\mathcal{V}} k(w)\nu(dw) \geq \inf_{P \in E_{\varphi_{FB}}} \int_{\mathcal{V}} k(w)\nu(dw) = \varphi_L(\nu)$$

We showed that T is monotone, therefore by induction $T^{n+1}\varphi_L \geq T^n\varphi_L$. To summarize:

$$\varphi_H \geq T^n\varphi_H \geq T^{n+1}\varphi_H \geq \cdots \geq T^{n+1}\varphi_L \geq T^n\varphi_L \geq \varphi_L$$

This implies that $\lim_{n\to\infty} T^n\varphi_L$ and $\lim_{n\to\infty} T^n\varphi_H$ exist. Now assume that $\lim_{n\to\infty} T^n\varphi_L = \lim_{n\to\infty} T^n\varphi_H$.

Since $\varphi^* \leq \phi_H$, $T^n\varphi^* = \varphi^*$ and the operator T is monotone, $\varphi^* \leq T^n\phi_H$ and therefore $\varphi^* \leq \lim_{n\to\infty} T^n\varphi_H$. Similarly $\varphi^* \geq \lim_{n\to\infty} T^n\varphi_L$. Using the equality of the two limits, $\varphi^* = \lim_{n\to\infty} T^n\varphi_H$. ■

Proof of Proposition 1. Let $\mathbf{P}$ attain the infimum in the Bellman equation. Lemmas 1 and 8 ensure that the allocation $(\mathbf{K}, \mathbf{B})$ satisfies incentive compatibility and promise-keeping. Feasibility is ensured the capital accumulation constraint. Therefore, the allocation satisfies all the constraints.

Assume that $(\mathbf{P'})$ satisfies all the constraints and dominates $\mathbf{P}$. $\mathbf{P'}$ is generated by an the sequence of lotteries P'_t. Clearly P'_0 satisfy the constraints in the Bellman equation. By assumption

$$\int_{\mathcal{V}} k'(w)\nu(dw) < \int_{\mathcal{V}} k(w)\nu(dw)$$

which is a contradiction since P attains the infimum in the Bellman equation. ■

First, we define a (slightly) more general form of the auxiliary problem (AP2). We will characterize the solution to this problem in a sequence of lemmas and then use these results to prove our main results lemma 5, proposition 2 and proposition 3.

$$\hat{k}(w, C; A) = \inf k \tag{A.2}$$

$$\text{s. to } [(1-\beta)u(zk + b(z)) + \beta v(z)] \geq$$

$$[(1-\beta)u(zk + b(z')) + \beta v(z')], \quad z, z' \in \mathbf{Z}, z \neq z' \tag{A.3}$$

$$\int_{\mathbf{Z}} \{(1-\beta)u(zk + b(z)) + \beta v(z)\}\mu(dz) = w \tag{A.4}$$

$$-\int_{\mathbf{Z}} b(z)\mu(dz) \geq C + A\int_{\mathbf{Z}} u^{-1}(v(z))\mu(dz) \tag{A.5}$$

We consider the problem of delivering arbitrary utility level w and a net transfer of resources C. (If $C > 0$, it means that the agent transfers more output than the expected cost of delivering her continuation utility.)

If we set $C = 0$ and $w = sgn(\gamma)$ we get the auxiliary problem from the main body of the text.

Lemma A.1 *In AP2 $b_L \geq b_H$.*

Proof. That is lemma 3 in Thomas and Worrall (1990). ■

In the case of log utility or iso-elastic utility with $\gamma < 0$, the constraint set is always nonempty. If $\gamma > 0$, there are some infeasible combinations (w, C). In all the lemmas below, we assume that (w, C) is feasible.

Lemma A.2 *For every w and A, there exists some $\bar{C} > 0$ such that if $C \leq \bar{C}$, (w, C) is feasible, given A.*

Proof. Clearly if a contract satisfies the constraints for some (w, C), then it will satisfies the constraints for (w, C') if $C' < C$. Then all we need to show that (w, C) is feasible for some $C > 0$. In the case of log utility or iso-elastic utility with $\gamma < 0$, pick some arbitrary $v_i = \bar{v} \in \mathcal{V}$ and set $b_i = -C - Au^{-1}(\bar{v})$. This contract satisfies accumulation constraint and incentive compatibility. What's left is to find a k that will keep feasibility and promise-keeping. Capital then must be in the interval $(C/z_L, \infty)$ and the utility from this allocation is strictly increasing in k, and in the set $(-\infty, \sup(\mathcal{V}))$. Therefore for some k the contract (k, b, v) satisfies all the constraints.

Now assume $\gamma \in (0, 1)$. Define

$$s = \frac{1}{1 + |\beta\gamma A^{-\gamma}|^{\frac{1}{\gamma-1}}}$$

The best autarchy allocation (the capital accumulation constraint holds state by state, not in expectation) is $b_i = -s(z_i k - C)$, $v_i = \operatorname{sgn}(\gamma) A^{-\gamma}(s(z_i k - C))^\gamma$ and k is such to determined to keep the promise-keeping constraint. It is the optimal autarchy allocation, therefore it satisfies incentive compatibility and it satisfies the accumulation constraint by construction. To satisfy the promise-keeping constraint we need:

$$\left[(1-\beta)(1-s)^\gamma/\gamma + \beta \operatorname{sgn}(\gamma) A^{-\gamma} s^\gamma\right] E[(z_i k - C)^\gamma] = w$$

Also $k > C/z_L$. The expression on the left-hand side is strictly increasing on $(C/z_L, \infty)$, so for C small enough it has a solution. This gives us a feasible contract. ■

Lemma A.3 *The maximum in the auxiliary problem is attained.*

Proof. Take any feasible contract $(\tilde{k}, \tilde{b}, \tilde{v})$.

Without loss of generality we can impose the following additional constraint, concentrating on contracts that do at least as well as $(\tilde{k}, \tilde{b}, \tilde{v})$:

$$k \leq \tilde{k}$$

Feasibility implies that

$$b_i \geq -z_i k \geq -z_i \tilde{k}$$

Incentive compatibility requires that $b_H \leq b_L$. That and the fact that

$$-Eb_i \geq AEu^{-1}(v_i) + C \geq C$$

implies that $b_H \leq -C$. The same inequality and the fact that $b_H \geq -z_H k \geq -z_H \tilde{k}$ implies that $b_L \leq [\pi_H z_H \tilde{k} - C]/\pi_L$.

$$(Ez)\tilde{k} \geq -Eb_i \geq AEu^{-1}(v_i) + C \geq \pi_i A f(v_i)$$

This implies that $|v_i| \leq u^{-1}[Ez\tilde{k}/\pi_i]$

All the constraints are continuous, so the constraint set is closed. We showed that without loss of generality, we can constrain the contract to be in a compact subset of $\mathbb{R}^5$. The intersection of a closed set with a compact set is compact, so without loss of generality the constraint set is compact. The objective function is continuous, therefore the minimum is attained. ■

Lemma A.4 *Without loss of generality the downward incentive constraint binds and the upward incentive constraint is slack.*

Proof. The downward incentive constraint and the fact that $b_L \geq b_H$ implies that $v_H \geq v_L$. Take any contract (k, b_H, b_L, v_H, v_L) that satisfies all the constraints. Define v'_H, v'_L by
$\pi_H v'_H + \pi_L v'_L = \pi_H v_H + \pi_L v_L$ and $v'_H - v'_L = \frac{1-\beta}{\beta}[u(z_H k + b_L) - u(z_H k + b_H)]$. The upward incentive constraint is relaxed (so it is slack), the downward incentive constraint binds with equality and the accumulation constraint is relaxed too, since u^{-1} is convex. The new contract does not affect the objective k. ■

Lemma A.4 implies that without loss of generality we can rewrite the auxiliary problem without the upward incentive constraint.

If $C < 0$ the principal receives a net transfer of resources and would therefore need less capital to deliver the promised utility. If the net transfer is large enough, there will be no need for production at all and the accumulation constraint will be slack. The next lemma shows that the accumulation constraint will be binding if C is not too small.

Lemma A.5 *If $C \geq 0$, at the optimum the accumulation constraint binds.*

Proof. Let $C > 0$. The accumulation constraint implies that $-\pi_H b_H - \pi_L b_L \geq C$ and since $b_i \geq -z_i k$ we get that $(Ez)k \geq C > 0$, so $k > 0$. If $C = 0$ and $k = 0$, then the accumulation constraint implies that $b_i = 0$, $u^{-1}(w_i) = 0$, which contradicts the promise-keeping constraint. So it must be that $k > 0$.

As shown above, we can assume that the downward incentive constraint is binding. Assume that the accumulation constraint is slack. Then we can define a new contract (k', b', v') by $k' = k - \epsilon$, $b'_i = b_i + z_i \epsilon$, $v'_i = v_i$. If $\epsilon > 0$ is small enough, the accumulation constraint is still satisfied. Consumption in both states of the world is not affected, so the promise-keeping constraint is satisfied. The original upward incentive constraint is slack, so for ϵ small enough it will still be satisfied. Finally, since $z_H k' + b'_L < z_H k + b_L$

$$(1-\beta)u(z_H k' + b'_L) + \beta v'_L < (1-\beta)u(z_H k + b_L) + \beta v_L$$

Then since the original downward constraint is satisfied, the new contract satisfies the downward incentive constraint too.

We showed that the modified contract satisfies all the constraints and lowers the objective function. This is a contradiction, since we showed that the minimum is attained. ■t

Lemma A.6 *If utility is iso-elastic, $\hat{k}(\lambda w, C\lambda^{\frac{1}{\gamma}}; A) = \lambda^{\frac{1}{\gamma}} k(w, C; A)$ for all $\lambda > 0$. If utility is logarithmic, $\hat{k}(\lambda + w, \exp(\lambda)C; A) = \exp(\lambda)\hat{k}(w, C; A)$ for all λ.*

Proof. Consider the iso-elastic case first. Let (k, b, v) be the optimal contract for $\hat{k}(w, C; A)$. Then $(k\lambda^{\frac{1}{\gamma}}, b_i\lambda^{\frac{1}{\gamma}}, v_i\lambda)$ is a contract that satisfies all the constraints for $(\lambda w, C\lambda^{\frac{1}{\gamma}})$. Thus $\hat{k}(\lambda w, C\lambda^{\frac{1}{\gamma}}; A) \leq \lambda^{\frac{1}{\gamma}} k(w, C; A)$.

Similarly, $\hat{k}(w, C; A) = \hat{k}((1/\lambda)\lambda w, (1/\lambda)^{\frac{1}{\gamma}} C\lambda^{\frac{1}{\gamma}}; A) \leq (1/\lambda)^{\frac{1}{\gamma}} \hat{k}(\lambda w, C\lambda^{\frac{1}{\gamma}}; A)$. Taking the two inequalities implies the result.

Now, consider the case when utility is logarithmic. Let (k, b, v) be the optimal contract for $\hat{k}(w, C; A)$. Then $(\exp(\lambda)k, \exp(\lambda)b_i, v_i + \lambda)$ is a contract that satisfies all the constraints for $(\lambda w, \exp(\lambda)C)$. Thus $k(\lambda + w, \exp(\lambda)C; A) \leq \exp(\lambda)\hat{k}(w, , C; A)$.

Similarly, $\hat{k}(w, C; A) = \hat{k}(-\lambda + (w + \lambda), \exp(-\lambda)(C \exp(\lambda)); A) \leq \exp(-\lambda)\hat{k}(w + \lambda, \exp(\lambda)C; A)$. Taking the two inequalities implies the result. ∎

Lemma A.7 *$\hat{k}(w, C; A)$ is convex in C and strictly convex for $C \geq 0$. The contract that attains the maximum is unique for $C \geq 0$.*

Proof. Let (k', b', v') and (k'', b'', v'') be the minimizers for C' and C'' respectively. Let $\lambda \in (0, 1)$. Define $k''' = \lambda k' + (1-\lambda)k''$ and $v_i''' = \lambda v_i' + (1-\lambda)v_i''$. Define b_H''' and b_L''' implicitly by $u(z_i k''' + b_i''') = \lambda u(z_i k' + b_i') + (1 - \lambda)u(z_i k'' + b_i'')$.

The new contract satisfies promise-keeping by construction. Since $z_i k''' + b_i''' \leq \lambda(z_i k' + b') + (1 - \lambda)(z_i k'' + b'')$ we have that $b_i''' \leq \lambda b_i' + (1 - \lambda)b_i''$. Therefore:

$$\begin{aligned}
-\int_{\mathbf{Z}} b'''(z)\mu(dz) &\geq -\lambda \int_{\mathbf{Z}} b'(z)\mu(dz) - (1 - \lambda) \int_{\mathbf{Z}} b''(z)\mu(dz) \\
&\geq \lambda[C' + A \int_{\mathbf{Z}} u^{-1}(v'(z))\mu(dz)] + \lambda[C'' + A \int_{\mathbf{Z}} u^{-1}(v''(z))\mu(dz)] \\
&\geq C''' + A \int_{\mathbf{Z}} u^{-1}(v'''(z))\mu(dz)
\end{aligned}$$

The last inequality follows from the fact that $u^{-1}(x)$ is convex on the relevant range of x. This shows that the accumulation constraint is satisfied. Finally, we need to show that the incentive constraint is satisfied.

$$\begin{aligned}
(1 - \beta)u(z_H k''' + b_H) + \beta v_H''' &= \lambda[(1 - \beta)u(z_H k' + b_H') + \beta v_H'] + (1 - \lambda)[(1 - \beta)u(z_H k'' + b_H'') + \beta v_H''] \\
&= (1 - \beta)(\lambda u(z_H k' + b_L') + (1 - \lambda)u(z_H k'' + b_L'')) + \beta v_L''' \\
&\geq (1 - \beta)u(z_H k''' + b_L''') + \beta v_L'''
\end{aligned}$$

where the last inequality follows from the fact that u displays decreasing absolute risk aversion. As described in lemma A.4 we can modify the contract further to make sure that the downward incentive constraint binds and the upward incentive constraint is slack. Therefore $\hat{k}(w, C'''; A) \leq k' = \lambda\hat{k}(w, C'; A) + (1 - \lambda)\hat{k}(w, C''; A)$.

Finally, we know that $\hat{k}(., C; A)$ is strictly increasing for $C \geq 0$. So if $C' \leq 0 < C''$, it must be that $k'' > k'$. Then $Eb''' < \lambda Eb' + (1 - \lambda)Eb''$, which implies that the accumulation constraint will be slack. Therefore we can reduce k''' further, which proves that $\hat{k}$ is strictly convex if $C > 0$.

Finally, assume that $C \geq 0$ and that there are two contracts attaining the minimum. Set $\lambda = 1/2$ and perform the variation described above. As shown it will reduce k strictly, which is a contradiction. ∎

Now we are ready to prove the main result.

Proof of Lemma 5. If $\varphi(\nu) = A \int_{\mathcal{V}} u^{-1}(w)\nu(dw)$, then the planner faces the following problem:

$$(T\varphi)(\nu) = \inf_{c(w)} \int_{\mathcal{V}} \hat{k}(w, c(w); A)\nu(dw)$$

$$\text{s. to} \int c(w)\nu(dw) \geq 0$$

$\hat{k}(w, C; A)$ is strictly increasing if $C > 0$. Therefore the accumulation constraint must bind, because otherwise we can reduce $c(w)$ on a set with positive measure, where $c(w) > 0$, reducing total required capital.

Define the probability measure ν' by

$$\nu'(A) = \frac{1}{\int_{\mathcal{V}} u^{-1}(w)\nu(dw)} \int_A u^{-1}(|w)\nu(dw),$$

where $A \subseteq \mathcal{V}$ is an arbitrary Borel set. Then the planner's problem can be rewritten as:

$$\inf_{c(w)} \left[\int_{\mathcal{V}} u^{-1}(w)\nu(dw)\right] \left[\int_{\mathcal{V}} \hat{k}(\text{sgn}(\gamma), c(w)/u^{-1}(w); A)\nu'(dw)\right]$$

$$\text{s. to} \left[\int_{\mathcal{V}} u^{-1}(w)\nu(dw)\right] \left[\int c(w)/u^{-1}(w)\nu'(dw)\right] = 0$$

Here we used the implication of lemma A.6 that $\hat{k}(w, C; A) = u^{-1}(w)\hat{k}(sgn(\gamma), C/u^{-1}(w); A)$. By the fact that $\hat{k}(\text{sgn}(\gamma), x; A)$ is convex in x and strictly convex for $x > 0$, and that ν' is a probability measure, it follows that it is optimal to set $c(w) = 0$ a.e. $[\nu']$. But $\nu' << \nu$, so it's optimal to set $c(w) = 0$ a.e. $[\nu]$.

Therefore, the optimal allocation is a scaled version of $\hat{k}(\text{sgn}(\gamma), 0; A)$ with a factor of proportionality of $u^{-1}(w)$.

So we showed that the optimal contract is $\mathbf{k}(w) = k|w|^{\frac{1}{\gamma}}, \mathbf{b}(w, z) = b(z)|w|^{\frac{1}{\gamma}}$ and $\mathbf{v}(w, z) = v(z)|w|$ for the CRRA case and $\mathbf{k}(w) = k\exp(w), \mathbf{b}(w, z) = b(z)\exp(w)$ and $\mathbf{v}(w, z) = v(z) + w$ for the log case, where (k, b, v) is the optimal (unique) contract in the auxiliary problem. Plugging the optimal contract in the objective function, we immediately get that:

$$T\varphi(\nu) = \phi(A) \int_{\mathcal{V}} u^{-1}(w)\nu(dw)$$

■

Proof of Proposition 2. From the definition of ϕ, A_H and A_L it immediately follows that:

1. ϕ is increasing
2. $\phi(A_L) > A_L$
3. $\phi(A_H) \leq A_H$

Combining these results we get:

$$A_L < \phi(A_L) \leq \phi^{(n)}(A_L) \leq \phi^{(n)}(A_H) \leq A_H$$

Moreover, since ϕ is monotone, $\phi^{n+1}(A_L) \geq \phi^n(A_L)$. Therefore the limit $\lim_{n\to\infty} \phi^{(n)}(A_L)$ exists.

Let $A^* = \lim_{n\to\infty} \phi^{(n)}(A_L)$. Since ϕ is continuous, $\phi(A^*) = A^*$. Then $\varphi(\nu) = A^* \int |w|^{\frac{1}{\gamma}} \nu(dw)$ is a solution to the Bellman equation.

Let (k, b, v) be the optimal contract in the auxiliary problem for A^*. Lemma 1 ensures that the allocation induced by this contract will satisfy promise-keeping if $\beta E|v| < 1$. If $\gamma > 0$, $\beta E v \geq 1$ violates the assumption that $\beta (Ez)^\gamma < 1$. If $\gamma < 0$, $\beta E v \leq -1$ violates the assumption that $\beta E z^\gamma < 1$. Finally, since $A^* > A_L$, the condition $\beta^{\frac{1}{1-\gamma}} (E\{z^\gamma\})^{\frac{-\gamma}{1-\gamma}} z_L^\gamma < 1$ ensures that $\beta|v_L| < 1$. Then lemma 8 ensures that the allocation is incentive compatible. Therefore the allocation is incentive-compatible, delivers the promised utility and it satisfies period by period accumulation constraint by construction. Therefore $A^* \int |w|^{\frac{1}{\gamma}} \nu(dw) \geq \varphi^*(\nu)$ for every feasible ν.

Let $\varphi_L(\nu) \leq \varphi^*(\nu)$ be an arbitrary lower bound on the necessary capital. Since T is monotone, $T\varphi_L(\nu) \leq T\varphi^*(\nu) = \varphi^*(\nu)$ for any feasible ν. But

$$A^* \int |w|^{\frac{1}{\gamma}} \nu(dw) = \lim_{n\to\infty} T^n \left[A_L \int |w|^{\frac{1}{\gamma}} \nu(dw) \right] \leq \varphi^*(\nu)$$

Therefore $A^* \int |w|^{\frac{1}{\gamma}} \nu(dw) = \varphi^*(\nu)$. ■

Proof of Proposition 3.

First, we will show that the equation $\phi(A) = A$ has exactly one solution. To do that we first show that $\phi(A) = \exp(B + \beta \log(A))$, where B is some number that depends on the parameters, but not on A.

Consider the auxiliary problem. Define $b'(z) = b(z)/k$ and $v'(z) = v(z) - \log(k)$. Then we can solve for k from the promise-keeping constraint and rewrite the problem as:

$$\begin{aligned} \phi'(A) = \min_{b', v'} \exp \left(- \int_{\mathbf{Z}} \{(1-\beta)\log(z + b'(z)) + \beta v'(z)\} \mu(dz) \right) \\ \text{s. to } (1-\beta)\log(z + b'(z)) + \beta v'(z) \geq \\ (1-\beta)\log(z + b'(z')) + \beta v'(z'), \quad z, z' \in \mathbf{Z} \\ -\int_{\mathbf{Z}} b'(z)\mu(dz) \geq A \int_{\mathbf{Z}} \exp(v'(z))\mu(dz) \end{aligned}$$

If $f_1(b'_L, b'_H) = \frac{1-\beta}{\beta}[\log(z_H + b'_H) - \log(z_H + b'_L)]$, then $v'_H = v'_L + f_1(b'_L, b'_H)$, since the downward incentive compatibility constraint binds. Since the accumulation constraint binds at the optimum, we can also determine that $v'_L = \log(-\sum_i \mu_i b'_i) - \log[\mu_L + \mu_H f_1(b'_L, b'_H)] - \log(A) = f_2(b'_L, b'_H) - \log(A)$. Then plugging into the objective function, we get

$$\begin{aligned} \phi(A) &= \min_{b'} \exp \left(- \int_{\mathbf{Z}} \{(1-\beta)\log(z + b'(z))\}\mu(dz) - \beta f_2(b'_L, b'_H) - \beta \mu_H f_1(b'_L, b'_H) + \beta \log(A) \right) \\ &= \min_{b'} \exp\left(f_3(b'_L, b'_H) + \beta \log(A)\right) \\ &= \exp\left(\min_{b'} f_3(b'_L, b'_H) + \beta \log(A) \right) \end{aligned}$$

Clearly, the only fixed point of ϕ is $A^* = \exp\left(\min_{b'} f_3(b'_L, b'_H)/(1-\beta)\right)$.

Then by the same arguments as for the CRRA case $\lim_{n\to\infty}\phi^{(n)}(A_l)=\lim_{n\to\infty}\phi^{(n)}(A_h)=A^*$. Then since $(T^n\varphi_i)(\nu)=\phi^{(n)}(A_i)\int_{\mathcal{V}}\exp(w)\nu(dw)$, we get that:

$$\lim_{n\to\infty}(T^n\varphi_l)(\nu)=\lim_{n\to\infty}(T^n\varphi_l)(\nu)=A^*\int_{\mathcal{V}}\exp(w)\nu(dw).$$

Then lemma 4 and proposition 1 imply the result. ■

A.1 Decentralization

We will define the following version of the problem:

$$\begin{aligned} T_qC(w) &= \min k+qE[b_i+C(w_i)] && \text{(A.6)}\\ E[(1-\beta)u(z_ik+b_i)+\beta w_i] &= w && \text{(A.7)}\\ (1-\beta)u(z_Hk+b_H)+\beta w_H &\geq (1-\beta)u(z_Hk+b_L)+\beta w_L && \text{(A.8)}\\ k\leq BC(w), k\geq 0 & && \text{(A.9)} \end{aligned}$$

With $B>1$ some arbitrary constant. We characterize the modified problem in a sequence of lemmas which we will use in order to prove the main result.

Lemma A.8 *For any $q\geq 0$, the problem above attains the minimum.*

Proof. The vector $k=0$, $b_i=u^{-1}(w)$ and $w_i=w$ is feasible and has a cost $M=q(u^{-1}(w)+C(w))$. Then M is an upper bound on the cost. Then without loss of generality we can impose the additional restrictions that $b_i\leq M/(q\pi_i)$, $w_i\leq C^{-1}(M)/(q\pi_i)$. If $\inf\mathcal{V}=0$, then $w_i\geq 0$.

On the other hand, if $\inf\mathcal{V}=-\infty$. Let $f_i(x)$ be the solution of the problem, conditional on $w_i=x$ and no incentive constraint. This is a strictly convex problem and has a well-defined solution. By the maximum theorem $f_i(x)$ is continuous. There exists some $\bar{w}_i$ such that if $x\leq\bar{w}_i$, $f_i(x)\geq M$. So, we can impose the restriction that $w_i\geq\bar{w}_i$ without loss of generality. Then the constraint is finite-dimensional, closed and bounded, therefore compact. The objective function is continuous, therefore a solution exists. ■

Lemma A.9 *At the optimum the incentive constraint is binding, and $z_Hk+b_H>0$.*

Proof. If $k=0$ then it is optimal to set $b_L=b_H=b$ and $w_L=w_H=v$, in which case the incentive constraint binds trivially. Moreover since $u'(0)=\infty$ and $b>0$, proving the second claim.

Assume that $k>0$. Additionally, assume that the incentive constraint does not bind. We proved above that $b_H\leq b_L$ and $z_Hk+b_H\geq z_Lk+b_L$. Then reduce w_H-w_L while keeping Ew_i constant till the constraint binds. This reduces the principal's cost strictly.

Now assume that $z_ik+b_i=0$. Assume that $w_i>\inf\mathcal{V}$. Then increasing b_i and reducing w_i marginally while keeping $(1-\beta)u(z_ik+b_i)+\beta w_i$ constant reduces the principal's cost while keeping all the constraints. Combining the incentive and promise-keeping constraints, it follows that $(1-\beta)u(z_Hk+b_H)+\beta w_H>w$, so $w_H=\inf\mathcal{V}$ is impossible. ■

Proof of Theorem 1. Suppose there exists an optimal allocation $\mathbf{P'}$ that attains the distribution of promised utilities with strictly less initial capital. Define $v_1(w)$ to be the value of this allocation at the prices p_t:

$$v_1(w_0) = k'_0(w_0) + \sum_{t=0}^{\infty} p_t \left[\int_{\mathbf{Z^{t+1}}} [b'_t(w_0, z^{t+1}) k'_{t+1}(w_0, z^{t+1})] \mu(dz^{t+1}) \right] = k'(w_0)$$

We have used the fact that at the optimum, $E_{t-1}[b_t(w_0, z^{t+1}) + k_{t+1}(w_0, z^{t+1})] = 0$.

The market clearing condition and the boundedness condition implies that:

$$0 \leq \int v(w)\nu(dw) = \int k_0(w_0)\nu(dw_0) < \infty$$

Then we know that

$$\int v_1(w)\nu(dw) = \int k'(w_0)\nu(dw_0) < \int k_0\nu(dw_0) = \int v(w)\nu(dw).$$

This implies that some component planner (a positive measure of them) is not optimizing, therefore we have reached a contradiction. ■

Proof of Theorem 2. Let $C(w) = A|w|^{\frac{1}{\gamma}}$ for the iso-elastic utility case and $C(w) = A\exp(w)$ for the log utility case with A the solution of the planner's problem. By arguments analogous to lemma A.3 the operator T_q defined in the Bellman equation attains a maximum if $A > 0$, the minimizing correspondence is u.s.c. and T_qC is continuous; by arguments identical to proposition **??** the minimizing correspondence is single-valued. We will show that for some q^*, $T_{q^*}C = C$ and that $q^* \in [0, 1)$. The rest of the proof proceeds in 6 steps.

1. If $q = q^{FB} = [Ez]^{-1}$, then $E[b_i + C(w_i)] > 0$.

   Perform the minimization without the incentive constraint. Within the set of minimizers, $z_Hk + b_H = z_Lk + b_L$, $w_L = w_H$. Take an arbitrary minimizing vector (k, b_i, w_i), and modify it by setting $k' = 0$, $b'_i = z_ik + b_i$. Since $q = (Ez)^{-1}$, this new vector is still a minimizer, but now it is incentive compatible and $E(b_i + C(w_i)) > 0$. Any incentive compatible contract with $k > 0$ will have $w_H > w_L$, so it will have higher cost for the component planner. Therefore at the optimum $k = 0$ and $E[b_i + C(w_i)] > 0$.

2. Define $\bar{b} \equiv E[b_i + C(w_i)]$. If $k > 0$, then at the optimum $(z_Hk + b_H)/(z_Hk + \bar{b}) \equiv \alpha_H < \alpha_L \equiv (z_Lk + b_L)/(z_Lk + \bar{b})$.

   Assume not: $\alpha_H \geq \alpha_L$. Let $m_i \equiv [z_ik + b_i + C(w_i)]/[z_ik + \bar{b}]$

   Forming the Lagrangian of the problem and taking first-order conditions, we see that $C'(w_L)u'(z_Lk + b_L) < \beta/(1-\beta) = C'(w_H)u'(z_Hk + b_H)$. Then the functional forms of $C()$ and $u()$ imply that $C(w_L)/(z_Lk + b_L) < C(w_H)/(z_Hk + b_H)$. This fact and the assumption that $\alpha_H \geq \alpha_L$ imply that $m_H > m_L$. Now assume that $m_L \geq 1$. This implies that:

   $$(Ez)k + \bar{b} = E[z_ik + b_i + C(w_i)] = E[m_i(z_ik + \bar{b})] > E[z_ik + \bar{b}] = (Ez)k + \bar{b},$$

which is a contradiction, therefore $m_L < 1$. By a similar argument $m_H > 1$. Therefore, if an agent with a high shock misreports, the value of his consumption and continuation utility will be

$$m_L(z_L k + \bar{b}) + (z_H - z_L)k < (z_L k + \bar{b}) + (z_H - z_L)k < m_H(z_H k + \bar{b})$$

But by the dual problem, b_H, w_H maximize the agent's utility subject to a resource constraint:

$$\max(1-\beta)u(z_H k + b_H) + \beta w_H$$

$$z_H k + b_H + C(w_H) \leq m_H(z_H k + \bar{b})$$

Therefore, the incentive constraint is slack, contradicting lemma A.9. Therefore, $\alpha_H < \alpha_L$.

3. If $k < C(w)$, then $z_L k + b_L > 0$.

   If $\bar{b} \leq 0$, the component planner's allocation will satisfy the accumulation constraint and be feasible for the planner, so $k < C(w)$ contradicts the optimality of the planner's solution. From lemma A.9 we know that $\alpha_H > 0$ and from the claim 2 we know that $\alpha_L > \alpha_H > 0$. So $z_L k + b_L = \alpha_L(z_L k + \bar{b}) > 0$.

4. Define $q^{AU} = E(z^{\gamma-1})/E(z^{\gamma})$, with $\gamma = 0$ for the log utility case. If $q = q^{AU}$, then at the optimum, $\bar{b} = E(b_i + C(w_i)) < 0$.

   The planner's decision is always feasible and gives a cost of $C(w)$. If the component planner's $k > C(w)$, it must be that $E(b_i + C(w_i)) < 0$.

   Now assume that $k \leq C(w)$, which implies $\bar{b} \geq 0$. Consider increasing allocated capital and decreasing transfers b_i. Step 3 of the proof shows that this is possible. From the implicit function theorem and the (binding) constraints, it follows that this variation must satisfy:

$$\sum_i \mu_i u'(z_i k + b_i)\left[z_i + \frac{db_i}{dk}\right] = 0$$

$$u'(z_H k + b_H)\left[z_H + \frac{db_H}{dk}\right] \geq u'(z_H k + b_L)\left[z_H + \frac{db_L}{dk}\right]$$

   For ease of exposition we will consider variation in which $db_H/dk = db_L/dk = db/dk$. The second condition implies that incentive compatibility will be satisfied as long as $db/dk > -z_H$. The promise-keeping constraint implies that

$$\frac{db}{dk} = -\frac{Eu'(z_i k + b_i)z_i}{Eu'(z_i k + b_i)}$$

   so $db/dk > -z_H$ and this variation is feasible and incentive-compatible. The marginal change in the planner's cost from this variation is

$$1 - q\frac{E\{(z_i k + b_i)^{\gamma-1} z_i\}}{E\{(z_i k + b_i)^{\gamma-1}\}}$$

   With $\gamma = 0$ for the log utility case. We will show that $-db/dk = \frac{E\{(z_i k+b_i)^{\gamma-1} z_i\}}{E\{(z_i k+b_i)^{\gamma-1}\}} > 1/q$, so the variation reduces the component planner's cost.

$$-\frac{db}{dk} = \frac{E\{(z_i k + b_i)^{\gamma-1} z_i\}}{E\{(z_i k + b_i)^{\gamma-1}\}} = \frac{E\{\alpha_i^{\gamma-1}\left(1 + \frac{\bar{b}}{z_i k}\right)^{\gamma-1} z_i^{\gamma-1} z_i\}}{E\{\alpha_i^{\gamma-1}\left(1 + \frac{\bar{b}}{z_i k}\right)^{\gamma-1} z_i^{\gamma-1}\}}$$

Since $\bar{b} \geq 0$, $\alpha_L > \alpha_H$ and $\gamma - 1 < 0$, it follows that $\frac{\alpha_L^{\gamma-1}\left(1+\frac{\bar{b}}{z_L k}\right)^{\gamma-1} z_L^{\gamma-1}}{E\{\alpha_i^{\gamma-1}\left(1+\frac{\bar{b}}{z_i k}\right)^{\gamma-1} z_i^{\gamma-1}\}} < \frac{z_L^{\gamma-1}}{E\{z_i^{\gamma-1}\}}$ and $\frac{\alpha_H^{\gamma-1}\left(1+\frac{\bar{b}}{z_H k}\right)^{\gamma-1} z_H^{\gamma-1}}{E\{\alpha_i^{\gamma-1}\left(1+\frac{\bar{b}}{z_i k}\right)^{\gamma-1} z_i^{\gamma-1}\}} > \frac{z_H^{\gamma-1}}{E\{z_i^{\gamma-1}\}}$. Then $-db/dk > \frac{E\{z_i^{\gamma-1} z_i\}}{E z_i^{\gamma-1}} = 1/q$, which implies that the variation reduces costs, reaching a contradiction.

5. For some $q^* \in [q^{AU}, q^{FB}]$, $T_q C(w) = C(w)$. By the maximum theorem, $E(b_i + C(w_i))$ is u.s.c. in q. It is also single-valued, therefore it is continuous. Then steps 1 and 4 imply the first result. Clearly $T_q C(w) \leq C(w)$. Since we showed that the optimal contract will satisfy the accumulation constraint, it is feasible for the planner's problem too, so $T_q C(w) < C(w)$ is impossible. This also implies that the component planner will set $k = C(w)$.

6. We showed that for q^*, $T_q C(w) = C(w)$ and $k = C(w) < BC(W)$, thus for the optimal solution the constraint A.9 does not bind, so $C(w)$ is a fixed point for the original problem. Moreover, by assumption $0 < q^* < 1$, so by theorem 1, the allocation is optimal.

■

Proof of Proposition 4. Assume that the equilibrium contract from the next period on is $\tau(k,z), \kappa(k,z)$. From the construction of these functions, it follows that for every agent's reporting strategy the consumption process will be the same as in the planner's problem with a starting utility $w(\kappa(k,z))$. So from the next period on, it would be optimal for the agent to follow truthtelling and expected continuation utility is $w(\kappa(k,z)$.

Then by the way we constructed it, the period contract $\tau(k,z), \kappa(k,z)$ satisfies incentive compatibility from the temporary incentive compatibility constraint of the planner's problem and the break-even condition from the auxiliary problem's accumulation constraint.

Now assume that for an agent with some k there is a contract (τ', κ') that's better for the agent and still satisfies the constraints, that is:

$$\sum_i \mu_i \left\{(1-\beta)u(z_i k + \tau'(k,z_i) - \kappa'(k,z_i)) + \beta w(\kappa'(k,z))\right\} = \tilde{w} > w(k)$$

In the case of log utility, define $k' = k\exp(-\tilde{w}), b'(z) = \exp(-\tilde{w})[\tau'(k,z) + \kappa'(k,z)], v'(z) = \log(\kappa'(k,z)) - \tilde{w} - \log A^*$; if utility is iso-elastic define $k' = k|\tilde{w}|^{-\frac{1}{\gamma}}, b'(z) = [\tau'(k,z) + \kappa'(k,z)]|\tilde{w}|^{-\frac{1}{\gamma}}, v'(z) = \kappa'(k,z)^\gamma/(\tilde{w}A^{*\gamma})$. In both cases (k', b', v') satisfies the constraints in the auxiliary problem and $k' < A^*$ which is a contradiction since A^* is a fixed point for the auxiliary problem. ■

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