

Higher Order Expectations, Illiquidity, and Short-term Trading ^{*}

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Abstract

We propose a theory that jointly accounts for an asset illiquidity and for the asset price potential over-reliance on public information. We argue that, when trading frequencies differ across traders, asset prices reflect investors' Higher Order Expectations (HOEs) about the *two* factors that influence the aggregate demand: fundamentals information *and* liquidity trades. We show that it is precisely when asset prices are driven by investors' HOEs about fundamentals that they over-rely on public information, the market displays high illiquidity, and low volume of informational trading; conversely, when HOEs about fundamentals are subdued, prices under-rely on public information, the market hovers in a high liquidity state, and the volume of informational trading is high. Over-reliance on public information results from investors' under-reaction to their private signals which, in turn, dampens uncertainty reduction over liquidation prices, favoring an increase in price risk and illiquidity. Therefore, a highly illiquid market implies higher expected returns from contrarian strategies. Equivalently, illiquidity arises as a byproduct of the lack of participation of informed investors in their capacity of liquidity suppliers, a feature that appears to capture some aspects of the recent crisis.

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1 Introduction

Liquidity plays an important role in the valuation of financial assets.¹ A relevant aspect of liquidity relates to the provision of *immediacy*, i.e. investors' readiness to hold a position in a risky asset in order to bridge the offsetting trading needs of agents who enter the market at different points in time (Grossman and Miller (1988)). A higher uncertainty over the price at which asset inventories can be unwound – the asset price risk – creates inventory risk, thereby reducing investors' ability to provide immediacy, and curtailing liquidity.² In markets with heterogeneous information, the riskiness of investors' position is exacerbated by the possibility of facing adverse selection risk at interim liquidation dates. When prices are closer to fundamentals, adverse selection risk is less of an issue, as the speculative opportunities offered by private information are scarcer. Price efficiency, however, depends on the informational content of asset prices which, in turn, hinges on investors' responses to their private signals. In equilibrium, both the former and the latter are then endogenously determined. Hence, in these markets, liquidity must proxy for the “amount” of information that, via equilibrium prices, investors transmit to the market.

The impact of private information on asset prices is also at the core of the recent literature that emphasises the role of investors' Higher Order Expectations (HOEs) over asset payoffs in asset price determination (i.e., investors' expectations about other investors' expectations about ... the liquidation value). It is a basic tenet of this literature that if prices are driven by HOEs over the final payoff, they over-rely on public information, systematically departing from fundamentals compared to consensus (Allen, Morris, and Shin (2006)). While this result speaks to the informational properties of asset prices, it is less clear what its implications for market quality are.

In this paper we propose a theory that jointly accounts for an asset illiquidity and for the asset price potential over-reliance on public information. We argue that in general asset prices reflect investors' HOEs about the *two* factors that influence the aggregate demand: fundamentals information *and* liquidity trades. We show that it is precisely when asset prices are driven by investors' HOEs about fundamentals (and therefore over-rely on public information) that the market displays high illiquidity, and the volume of informational trading is low; conversely, when HOEs about fundamentals are subdued, asset prices under-rely on public information, the market hovers in a high liquidity state, and the volume of informational trading is high. Over-reliance on public information results from investors' under-reaction to their private signals which, in turn, dampens uncertainty reduction over liquidation prices, favoring an increase in price risk and illiquidity. Therefore, a highly illiquid market implies higher expected returns from contrarian strategies. Equivalently, illiquidity arises as a byproduct of the lack of participation of informed investors in their capacity of liquidity suppliers, a feature that seems to

¹There is by now a well established literature showing that illiquid assets command a return premium which increases in the asset comovement with the market illiquidity (Pástor and Stambaugh (2003), and Acharya and Pedersen (2005)).

²Several authors document intermediaries' concern over the time-varying patterns of asset illiquidity. See e.g. Comerton-Forde, Hendershott, Jones, Moulton, and Seasholes (2010).

capture some aspects of the recent crisis (see, e.g. Distaso, Fernandes, and Zikes (2010) and Nagel (2010)).

More in detail, we study a market in which overlapping generations of risk-averse, rational investors interact with liquidity traders. The former live for two periods and are informed about a pay-off that will be announced at a future date, which occurs beyond their investment horizon; the latter, instead, are uninformed and submit a random market order, potentially holding their positions during more than one trading round. When young, rational investors absorb liquidity traders' orders, thereby acting as "market-makers." Once old, the former unwind their positions against the reverting portion of the latter demand and take advantage of the market making services provided by the new cohort of rational investors to unload the residual part of their inventory. Thus, as trading frequencies are heterogenous across investors' types, rational investors *provide* liquidity when young and *consume* it when old. This implies that when they determine their holdings, they anticipate the impact that their position unwinding will have on the illiquidity of future markets. We show that this feature of the model generates a number of important implications.

If trading frequencies coincided across agents' types (i.e., if liquidity traders' demand was transient), in every period rational investors would accommodate the demand of liquidity traders, anticipating its full reversion at the end of their investment horizon. In this context, the adverse selection risk related to the unwinding of informed investors' positions would be nil. However, when liquidity traders turn around their positions less often than rational investors, only a fraction of the formers' orders is bound to revert when the latter close out their positions. Hence, each cohort of investors unwinds part of its holdings against the aggregate demand coming from the next period cohort, thereby transferring part of the risk it incorporates but also facing adverse selection risk.

If in a given period investors anticipate that their next period peers will increase their exposure to the risky asset, they face adverse selection risk at the liquidation date which will make the market more illiquid. As a consequence, the price at which they liquidate becomes more difficult to forecast with their private information, and they scale down their response to private information. This, in turn, lowers the informativeness of equilibrium prices, opening more opportunities to speculate on private information to the informed investors who enter the market in the following period, justifying the anticipated increase in illiquidity. If, on the other hand, in a given period investors anticipate that their next period peers lower their exposure to the risky asset, the market in which they liquidate will be more liquid. As a consequence, the liquidation price becomes more easy to forecast with their information, and they step up the use of their private signals. This, in turn, increases the informativeness of equilibrium prices, reducing the opportunities to speculate on private information opened to the next generation of informed investors, justifying the anticipated decrease in illiquidity. We thus show that the presence of such a positive feedback loop *across trading dates* is conducive to multiple equilibria which can be ranked in terms of illiquidity.

As rational investors hold disparate signals, the illiquidity result also has implications for the

information aggregation properties of asset prices. Differently from the existing literature, we show that in our setup the occurrence of over-reliance on public information is *closely related* to the type of equilibrium that arises. Along the equilibrium in which investors anticipate *high* future illiquidity, the risk of adverse price movements at the time of position unwinding causes over-reaction to public information; this generates a price that is poorly related to fundamentals. Conversely, when investors anticipate *low* future illiquidity, the price under-reacts to public information and tracks *more closely* the fundamentals. Thus, while HOEs over fundamentals are *necessary* for over-reaction to public information, they are *not sufficient*.

The existence of multiple equilibria leaves open the question of which equilibrium is more likely to arise. As our intuitive argument has clarified, in our setup multiple equilibria rely on the positive feedback loop that links investors' response to private information in a *given period* to the illiquidity of the *following period* market. Thus, any effect that moderates such a loop is likely to restore uniqueness. With this idea in mind we show that when either liquidity traders' demand becomes transient or when a spike in the residual uncertainty affecting fundamentals hits the market, a unique equilibrium arises. In this equilibrium the market is more illiquid and prices always over-react to public information. Indeed, when agents' liquidity needs are fully synchronized, the cohort of rational investors who enters the market in a given period anticipates that liquidity traders will completely revert their position in the following period, providing the natural counterpart to their position unwinding. This, in turn, reduces the need to anticipate the liquidation price, and enforces an equilibrium in which rational investors over-react to public information.³ Similarly, when fundamentals uncertainty spikes, asset prices increasingly reflect liquidity shocks and rational investors have little use for their signals. In both cases, thus, it is the possibility to focus on servicing the needs of liquidity traders that "distracts" rational investors from forecasting the fundamentals, yielding an equilibrium with higher price risk and higher illiquidity.

Our model can shed light on the impact that heterogeneous trading frequencies have on market liquidity. Indeed, the low illiquidity equilibrium arises precisely because when the trading horizon of liquidity traders differs from the one of informed investors, the latter anticipate a potential adverse selection risk at the liquidation date. Along this equilibrium, liquidity traders' losses are minimized, which leads to conclude that if they had access to the same trading technology available to informed investors, liquidity traders would adopt it. However, in this case as trading frequencies coincide, the adverse selection risk at the liquidation date disappears together with the low illiquidity equilibrium. Thus, an implication of our analysis is that while trading at high frequencies may generate liquidity improvements (as empirically documented, for example, by Hendershott, Jones, and Menkveld (2010)), these improvements are not necessarily exploitable by those who are to gain the most from it, namely liquidity traders. A further implication of our model is that with differential information, the existence of a discrepancy in the speed at which different types of investors can turn around their positions can be responsible for indeterminacy, making liquidity dependent on a coordination problem across different

³In this case our model coincides with Allen, Morris and Shin (2006).

generations of investors, and thereby endogenously creating a source of liquidity risk.

Our model also provides a theoretical underpinning for the empirically documented positive association between illiquidity and asset expected returns.⁴ In fact we show that it is precisely along the high illiquidity equilibrium that the asset expected returns are higher and rational investors concentrate on the supply of immediacy to liquidity traders acting as “contrarians.” If, on the other hand, price risk is low, the market is liquid, liquidity supply is not very profitable (low price risk and thus low risk compensation) and investors act as “trend chasers.” As argued above, whether price risk is high or low depends on a coordination problem and we predict that high price risk occurs when informed investors refrain from using private information exactly because prices are mainly driven by liquidity shocks which are orthogonal to fundamentals. Therefore, our model implies that liquidity provision is profitable when there is little informational trading in the market.

This paper is related to a growing literature that points out the relevance of higher order expectations in influencing asset prices. As our previous discussion suggests, we depart from the main tenet of this literature and point out that price over-reliance on public information is intimately related to the transience of liquidity traders’ demand.⁵ Bacchetta and van Wincoop (2008) study the role of higher order beliefs in asset prices in an infinite horizon model showing that higher order expectations add an additional term to the traditional asset pricing equation, the higher order “wedge,” which captures the discrepancy between the price of the asset and the average expectations of the fundamentals. Kondor (2009), in a model with short-term Bayesian traders, shows that public announcements may increase disagreement, generating high trading volume in equilibrium. Nimark (2007), in the context of Singleton (1987)’s model, shows that under some conditions both the variance and the impact that expectations have on the price decrease as the order of expectations increases. Other authors have analyzed the role of higher order expectations in models where traders hold different initial beliefs about the liquidation value. Biais and Bossaerts (1998) show that departures from the common prior assumption rationalise peculiar trading patterns whereby traders with low private valuations may decide to buy an asset from traders with higher private valuations in the hope to resell it later on during the trading day at an even higher price. Cao and Ou-Yang (2005) study conditions for the existence of bubbles and panics in a model where traders’ opinions about the liquidation value differ.⁶ Banerjee, Kaniel, and Kremer (2009) show that in a model with heterogeneous priors, differences in higher order beliefs may induce price drift.

The paper is also related to the literature that investigates the relationship between the impact of short-term information on prices and investors’ reaction to their private signals (see,

⁴Acharya and Pedersen (2005), estimate that low liquidity stocks command a 4.6% higher expected return over high liquidity stocks. Brennan and Subrahmanyam (1996) find that low liquidity portfolios earn an extra 6.6% yearly return over high liquidity portfolios.

⁵In a related paper, we show that a similar conclusion holds in a model with long term investors (see Cespa and Vives (2009)). In a static model of a binary prediction market where agents hold heterogeneous prior beliefs and are wealth constrained (either exogenously, by the rules of the market, or because of their attitude towards risk), Ottaviani and Sørensen (2009) show that the fully revealing REE price underweights aggregate private information.

⁶Kandel and Pearson (1995) provide empirical evidence supporting the non-common prior assumption.

e.g. Vives (1995), Cespa (2002), and Vives (2008) for a survey). Several authors have argued that when private information is related to an event which occurs beyond the date at which investors liquidate their positions, the latter act on their signals only if they expect them to be reflected in the price at which they liquidate (see, e.g., Dow and Gorton (1994) and Froot, Scharfstein, and Stein (1992)). In our context a similar effect is at work. Note, however, that the main driver in investors' reaction to private signals is the anticipation of a lower price risk which induces lower future illiquidity rather than the anticipation of a *strong* impact of private information on the liquidation price.

Finally, the paper is related to the literature that emphasizes the impact of illiquidity on asset prices. Amihud and Mendelson (1986) argue that the presence of an exogenous transaction cost due to illiquidity leads investors to anticipate higher costs when unwinding their positions, thereby leading to an increase in the discount factor with which they evaluate the asset payoff. Vayanos (1998) in a multi-asset model with proportional transaction costs, shows that in equilibrium a number of counterintuitive results can arise. In our paper illiquidity arises endogenously, as a result of investors' uncertainty over the price at which they unwind their positions. Furthermore, we relate illiquidity to investors' contrarian strategies, and to the role that HOEs about fundamentals have in driving the asset price, showing that in an illiquid market liquidity provision is more profitable, and that prices over-rely on public information.

The rest of the paper is organized as follows: in the next section we spell out the model's assumptions. We then analyse the model in a setting with no private information. This enables us to show that when liquidity traders' demand displays persistence, each cohort of investors shifts part of the risk of its holdings onto the next period cohort, giving rise to illiquidity chains. We thus turn to the market with heterogeneous information, and argue that liquidity trades' persistence in this case also yields multiple equilibria in which a high and a low level of illiquidity arise, and in which anticipated future high illiquidity spawns current high illiquidity. In the following section we relate our illiquidity result to the literature on HOEs. We conclude analyzing extensions and providing a discussion of our results.

2 A three-period market with short term investors

Consider a three-period version of the noisy rational expectations market analyzed by Admati (1985) where a single risky asset with liquidation value $v \sim N(\bar{v}, \tau_v^{-1})$, and a risk-less asset with unitary return are traded by a continuum of risk-averse speculators in the interval $[0, 1]$ together with liquidity traders. At any trading date n , a cohort of risk averse, rational investors in the interval $[0, 1]$ enters the market, loads a position in the risky asset which it unwinds in period $n + 1$. A rational investor i has CARA preferences (denote with γ the common risk-tolerance coefficient) and maximizes the expected utility of his short term profit $\pi_{in} = (p_{n+1} - p_n)x_{in}$.⁷ An investor i who enters the market in period n receives a signal $s_{in} = v + \epsilon_{in}$, where $\epsilon_{in} \sim N(0, \tau_\epsilon^{-1})$, v and ϵ_{in} are independent for all i, n and error terms are also independent both across time

⁷We assume, without loss of generality, that the non-random endowment of investors is zero.

periods and investors. We assume an investor i who enters the market in period n observes the past period prices up to period $n - 1$, denoted by $p^{n-1} \equiv \{p_t\}_{t=1}^{n-1}$, and submits a linear demand schedule (generalized limit order) to the market $X_n(s_{in}, p^{n-1}, p_n) = a_n s_{in} - \varphi_n(p^n)$ indicating the desired position in the risky asset for each realization of the equilibrium price p_n . The constant a_n denotes the private signal responsiveness, while $\varphi_n(\cdot)$ is a linear function of the equilibrium prices p^n .

The stock of liquidity trades is assumed to follow an AR(1) process:

$$\theta_n = \beta \theta_{n-1} + u_n, \quad (2.1)$$

where $u_n \sim N(0, \tau_u^{-1})$ is orthogonal to θ_{n-1} , and $\beta \in [0, 1]$. To interpret, suppose $\beta < 1$, then at any period $n > 1$ four groups of agents are in the market: the $n - 1$ -th and n -th generations of rational investors with demands $x_{n-1} \equiv \int_0^1 x_{in-1} di$, and $x_n \equiv \int_0^1 x_{in} di$, a fraction $1 - \beta$ of the $n - 1$ -th generation of liquidity traders who revert their positions, and the new generation of liquidity traders. Considering the first two trading dates and letting $\Delta x_2 \equiv x_2 - x_1$, $\Delta \theta_2 \equiv \theta_2 - \theta_1 = u_2 + (\beta - 1)\theta_1$, at equilibrium this implies

$$\begin{aligned} x_1 + \theta_1 &= 0 \\ \Delta x_2 + \Delta \theta_2 &= 0 \Leftrightarrow x_2 + \beta \theta_1 + u_2 = 0. \end{aligned}$$

At date 1 the first cohort of rational investors clears the share supply θ_1 . At date 2 a fraction $(1 - \beta)\theta_1$ of the trades initiated by liquidity traders at time 1 reverts. Hence, period 1 rational investors clear the complementary fraction $\beta\theta_1 = -\beta x_1$ against the new aggregate demand: $x_2 + u_2$. In general, the lower is β , the higher is the fraction of period n liquidity traders who revert their positions at time $n + 1$, and the lower is the fraction of rational investors' trades that are cleared against the $n + 1$ -th aggregate demand.⁸

Besides capturing an empirically documented feature of the demand of liquidity traders (see, e.g., Easley, Engle, O'Hara, and Wu (2008)), assuming persistence in liquidity trades allows to model in a parsimonious way the possibility that agents in the market have different horizons: when $\beta = 0$ each generation of rational investors and liquidity traders have the same investment horizon; as β grows, investment horizons become increasingly different. Different investment horizons can be justified on grounds of asynchronous liquidity needs, or of differential access to a trading technology. Accordingly, as we will argue in section 7.1, informed investors can be seen as high frequency traders that are able to turn around their positions at a faster speed compared to liquidity traders. Table 1 displays the evolution of liquidity trades and rational investors' positions in the three periods.⁹

We denote by $E_{in}[Y] = E[Y|s_{in}, p^n]$, $E_n[Y] = E[Y|p^n]$ ($\text{Var}_{in}[Y] = \text{Var}[Y|s_{in}, p^n]$, $\text{Var}_n[Y] = \text{Var}[Y|p^n]$), respectively the expectation (variance) of the random variable Y formed by a trader

⁸The AR(1) assumption for liquidity traders' demand is not new in the literature. For instance, He and Wang (1995) consider a model with long term investors in which liquidity trading is generated by an AR(1) process.

⁹For a sufficiently large number of trading rounds N , the stock of liquidity trades reverts completely. Indeed, according to the table, at any generic time N the fraction of the first period stock of liquidity that has reverted is given by $\theta_1(1 - \beta) \sum_{t=0}^{N-2} \beta^t = (1 - \beta^{N-1})\theta_1 \rightarrow \theta_1$ as N grows large.

Trading Date		1	2	3
Liquidity traders	Position	$\theta_1 = u_1$	$\theta_2 = \beta\theta_1 + u_2$	$\theta_3 = \beta(\beta\theta_1 + u_2) + u_3$
	Reverting	—	$(1 - \beta)\theta_1$	$(1 - \beta)(\beta\theta_1 + u_2)$
	Holding	—	$\beta\theta_1$	$\beta(\beta\theta_1 + u_2)$
	New shock	u_1	u_2	u_3
Rational investors	Position	x_1	x_2	x_3
	Reverting	—	x_1	x_2

Table 1: The evolution of liquidity trades and rational investors' positions in the three periods.

conditioning on the private and public information he has at time n , and that obtained conditioning on public information only. The consensus opinion about the fundamentals at time n is denoted by $\bar{E}_n[v] \equiv \int_0^1 E_{in}[v]di = \int_0^1 E_n[v]di$. Finally, we let $\alpha_{E_n} = \tau_\epsilon/\tau_{in}$, where $\tau_{in} \equiv (\text{Var}_{in}[v])^{-1}$ and make the convention that, given v , at any time n the average signal $\int_0^1 s_{in}di$ equals v almost surely (i.e. errors cancel out in the aggregate: $\int_0^1 \epsilon_{in}di = 0$). Therefore, we have $E_{in}[v] = \alpha_{E_n}s_{in} + (1 - \alpha_{E_n})E_n[v]$, and $\bar{E}_n[v] = \alpha_{E_n}v + (1 - \alpha_{E_n})E_n[v]$.

3 The market without private information

In this section we assume away private information (i.e., we impose $\tau_{\epsilon_n} = 0$). In this case, rational investors can perfectly observe the stock of liquidity trades and always act as market makers, providing immediacy as in Grossman and Miller (1988). It is then possible to show that

Proposition 1. *In the market without private information, there exists a unique equilibrium in linear strategies where for $n = 1, 2, 3$:*

$$X_n(\theta_n) = -\theta_n \quad (3.1)$$

$$p_n = \bar{v} + \Lambda_n\theta_n, \quad (3.2)$$

and for $n = 1, 2$:

$$\Lambda_n = \frac{\text{Var}_n[p_{n+1}]}{\gamma} + \beta\Lambda_{n+1}, \quad (3.3)$$

while $\Lambda_3 = (\gamma\tau_v)^{-1}$.

Proof. See the appendix. □

According to (3.1) and (3.2), rational investors always take the other side of the order flow, buying the asset at a discount when $\theta_n < 0$ and selling it at a premium otherwise. Indeed, as demand shocks are only driven by liquidity trading, rational investors anticipate return reversal

and profit from the supply of immediacy to liquidity traders:

$$\begin{aligned} p_{n+1} - p_n &= \Lambda_{n+1} u_{n+1} + (\beta \Lambda_{n+1} - \Lambda_n) \theta_n \\ &= \Lambda_{n+1} u_{n+1} - \frac{\text{Var}_n[p_{n+1}]}{\gamma} \theta_n. \end{aligned} \quad (3.4)$$

The larger is the stock of liquidity trades θ_n , the larger is the adjustment in the price rational investors require in order to absorb it. Therefore, the price impact of trades Λ_n proxies for the *illiquidity* of the market. A higher price risk increases illiquidity and also investors' expected profits, as one can see by taking the expectation of (3.4):

$$E_n[p_{n+1} - p_n] = -\frac{\text{Var}_n[p_{n+1}]}{\gamma} \theta_n. \quad (3.5)$$

According to (3.3), at any time $n < 3$, market illiquidity captures two effects, both of which are related to the inventory risk due to liquidity traders' demand. On the one hand, due to the randomness of liquidity trades, investors bear the risk related to the price at which they will unwind their position; on the other hand, due to the persistence of liquidity traders' demand, they anticipate the impact that the unwinding of a fraction of their holdings to the next cohort of rational investors will have on the next period price as a result of the *increased* risk exposure this implies for the next cohort of rational investors. The former effect is reflected in the conditional variance of the liquidation price (scaled by risk tolerance) $\text{Var}_n[p_{n+1}]/\gamma$; the latter, is instead captured by the anticipated illiquidity component $\beta \Lambda_{n+1}$.

More persistent liquidity trades imply a stronger effect due to investors' position unwinding. This, in turn, pushes investors to adjust more harshly the price for a given realization of liquidity traders' demand: ¹⁰

Corollary 1. *In the market with homogeneous information $\partial \Lambda_n / \partial \beta > 0$, for $n = 1, 2$.*

4 The market with heterogeneous information

With heterogeneous information, the aggregate demand is driven by liquidity trades and fundamental information shocks, and illiquidity thus proxies for the two risks investors face: inventory and adverse selection. This creates two types of adverse selection problems: On the one hand, when loading their position, investors cannot perfectly observe the stock of noise and need to infer it from the aggregate demand to decide how to position themselves with respect to the market; on the other hand, when trading frequencies differ, when unwinding their position, investors do not know who will take the other side of their orders. These two problems are interrelated since, if in period n investors anticipate that their next period peers will increase their exposure to the risky asset (i.e. that they will unwind in the hands of informed investors), this implies that they will suffer from adverse selection risk at the liquidation date. This,

¹⁰Equivalently, all else equal, a larger β by making the demand of liquidity traders more persistent increases the risk that investors at n shed on the shoulders of their next period peers who in turn require a larger compensation which feeds back into the illiquidity of the period n market.

added to the inventory risk will increase the illiquidity of the market, making the liquidation price harder to forecast, potentially leading period n investors to scale down their reliance on private information. As a consequence, the aggregate demand in period n is less influenced by informed trading, a fact that in turn affects investors' decision on how to position themselves with respect to the market, and opens more speculative opportunities to informed investors in period $n + 1$. If investors in period n anticipate that their next period peers will reduce their exposure to the risky asset, the illiquidity of the market in which the former liquidate is set to decrease (as adverse selection risk now is reduced). This implies that investors in period n can better forecast the liquidation price with their information, leading them to scale up their reliance on their private signal. This can reinforce the impact of informed trading in period n aggregate demand, influencing informed investors' position in the market, and reducing the speculative opportunities of informed investors in period $n + 1$. As we will argue in Section 4.1, this leads to equilibria with self-fulfilling illiquidity (see Figure 1).

We start by giving a general characterisation of the equilibrium. The following proposition characterises equilibrium prices:

Proposition 2. *In the market with heterogeneous information, the equilibrium price is given by*

$$p_n = \alpha_{P_n} \left(v + \frac{\theta_n}{a_n} \right) + (1 - \alpha_{P_n}) E_n[v], \quad (4.1)$$

where $\theta_n = u_n + \beta \theta_{n-1}$, and a_n, α_{P_n} denote respectively the responsiveness to private information at time n displayed by investors and the equilibrium price.

According to (4.1), at period n the equilibrium price is a weighted average of the market expectation about the fundamentals v , and a monotone transformation of the n -th period aggregate demand intercept.¹¹ A straightforward rearrangement of (4.1) yields

$$\begin{aligned} p_n - E_n[v] &= \frac{\alpha_{P_n}}{a_n} E_n[\theta_n] \\ &= \Lambda_n (a_n (v - E_n[v]) + \theta_n), \end{aligned} \quad (4.2)$$

implying that there is a discrepancy between p_n and $E_n[v]$ which, as in (3.2), captures a premium which is proportional to the expected stock of liquidity that is available at time n . Given that via the observation of the aggregate demand, investors infer fundamentals information this premium drives a wedge between the equilibrium price p_n and the semi-strong efficient price $E_n[v]$:

Corollary 2. *At any linear equilibrium of the market, the price incorporates a premium above the semi-strong efficient price:*

$$p_n = E_n[v] + \Lambda_n E_n[\theta_n], \quad (4.3)$$

¹¹This is immediate since in any linear equilibrium $\int_0^1 x_{in} di + \theta_n = a_n v + \theta_n - \varphi_n(p^n)$.

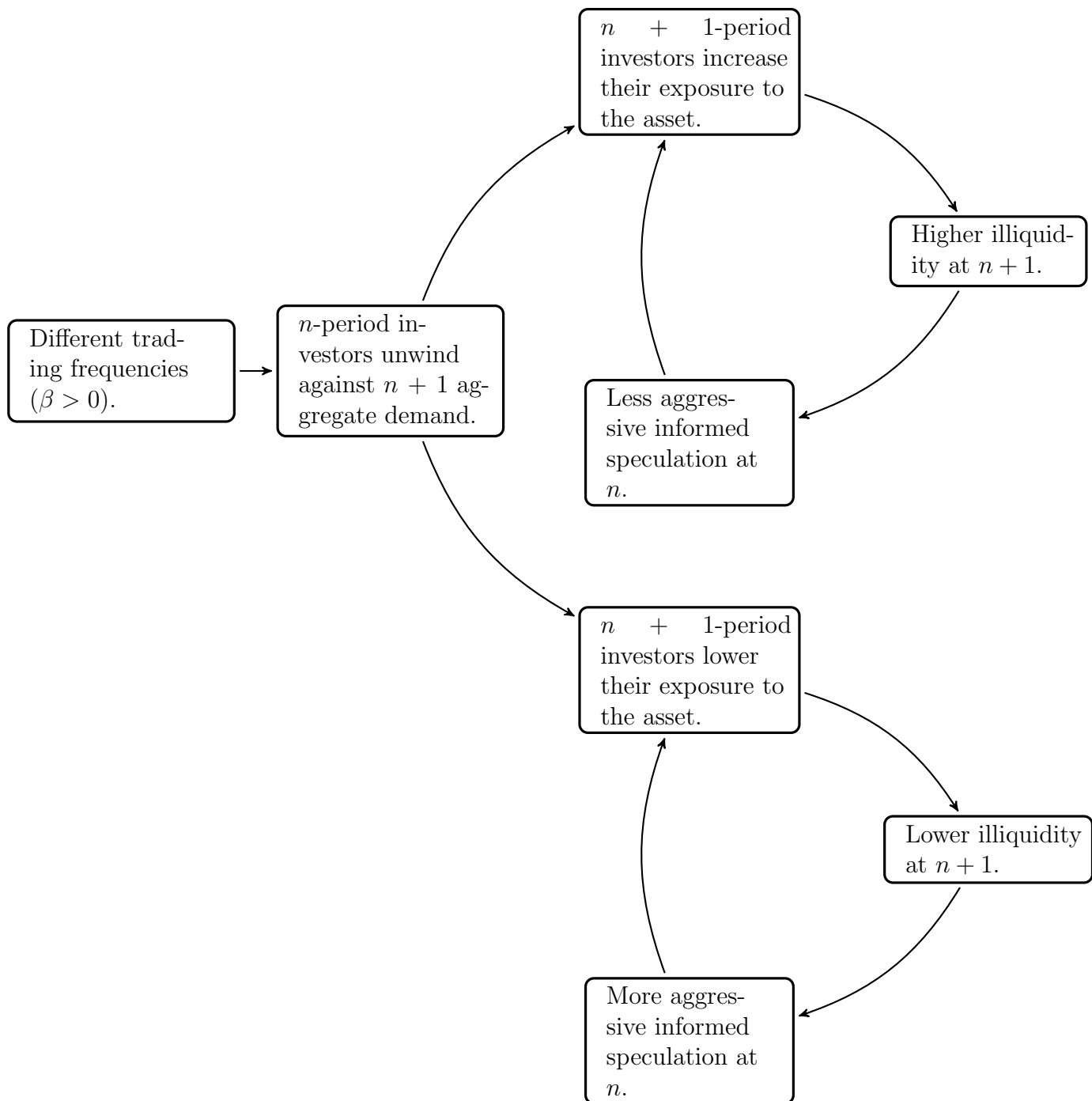


Figure 1: Information driven liquidity spillovers across time.

where for $n = 1, 2$,

$$\Lambda_n = \frac{\text{Var}_{in}[p_{n+1}]}{\gamma} + \beta\Lambda_{n+1}, \quad (4.4)$$

while $\Lambda_3 = 1/(\gamma\tau_{i3})$.

Expressions (4.3) and (4.4) parallel (3.2) and (3.3), in that also in the presence of heterogeneous information rational investors require a compensation to clear the market. However, in this case due to the presence of informed investors, price changes also reflect the arrival of new information about the asset liquidation value, and Λ_n only captures one component of the illiquidity at time n . More formally, denoting by $\Delta a_n = a_n - \beta a_{n-1}$, and by $z_n = \Delta a_n v + u_n$ the “new” information reflected in the n -th period aggregate demand by the change in position of informed investors and the new liquidity shock, we obtain

Corollary 3. *At any linear equilibrium of the market, short term returns are given by*

$$p_n - p_{n-1} = \lambda_n \left(z_n + \Delta a_n \frac{\alpha_{P_{n-1}} - \alpha_{E_{n-1}}}{a_{n-1}} E_{n-1}[\theta_{n-1}] - \Delta a_n p_{n-1} \right), \quad (4.5)$$

where

$$\lambda_n = \Lambda_n + (1 - \Lambda_n a_n) \frac{\Delta a_n \tau_u}{\tau_n}, \quad (4.6)$$

captures the illiquidity of the market at time n .

According to (4.6), illiquidity reflects the “total” price impact of net trades and is given by the sum of two components: the first component (Λ_n) corresponds to the illiquidity measure in the market with no private information, and reflects the inventory risk investors bear when clearing the position of liquidity traders. The second component $((1 - \Lambda_n a_n) \Delta a_n \tau_u / \tau_n)$ reflects the adverse selection risk investors face owing to the presence of heterogeneous information.¹² Note that if $\beta > 0$, adverse selection risk can either magnify or reduce illiquidity, depending on the sign of Δa_n . Intuitively, when $\beta > 0$ informed investors in period $n - 1$ unwind a fraction of their orders against the new cohort of investors who enter the market in the following period. How informed investors in period n decide to react to these orders depends on the speculative opportunities that they envisage to exploit. If, given the information that has been revealed in the previous trading rounds, period n informed investors anticipate the possibility to exploit their private information, they absorb these orders, increasing their exposure to the risky asset. In this case $a_n > \beta a_{n-1}$ and thus $\Delta a_n > 0$ and, as more private information is reflected in the price in period n , adverse selection has a positive effect on illiquidity. However, if in period n informed investors view little speculative opportunities, they choose to lower their exposure to the risky asset. In this case $a_n < \beta a_{n-1}$, $\Delta a_n < 0$, and the adverse selection component has a negative impact on (i.e., it lowers) illiquidity.¹³

An immediate consequence of (4.5) is that, similarly to the market with no private information, a higher price risk commands higher expected returns (see equation (3.5)):

¹²Indeed, $\Delta a_t \tau_u / \tau_t$ denotes the OLS regression coefficient assigned to z_t in the regression of v over $\{z_1, z_2, \dots, z_n\}$.

¹³In this discussion we are taking $\Lambda_n a_n < 1$, which, as we will argue in the next section is always true in the equilibrium of the 2-period market.

Corollary 4. *At any linear equilibrium of the market*

$$E_n[p_{n+1} - p_n] = -\frac{\text{Var}_{in}[p_{n+1}]}{\gamma} E_n[\theta_n]. \quad (4.7)$$

Expression (4.7) shows that short term returns are predictable based on public information and that such predictability depends on the possibility to correctly extrapolate liquidity traders' demand from the aggregate demand. Due to the presence of informed trading, however, estimated positive (negative) liquidity shocks, do not necessarily lead investors to take the other side of the market:

Corollary 5. *At any linear equilibrium of the market, a rational investor's strategy is given by*

$$X_n(s_{in}, z^n) = \frac{a_n}{\alpha_{E_n}} (E_{in}[v] - p_n) + \frac{\alpha_{P_n} - \alpha_{E_n}}{\alpha_{E_n}} E_n[\theta_n], \quad (4.8)$$

where $\alpha_{E_n} = \tau_\epsilon / \tau_{in}$,

$$\alpha_{P_n} = \alpha_{E_n} \left(1 + \gamma \tau_n \sum_{t=n+1}^N \Lambda_t (\beta \rho_{t-1} - \rho_t) \right), \quad n = 1, 2, \quad (4.9)$$

$\alpha_{P_3} = \alpha_{E_3}$, $\rho_n = a_n / (\gamma \tau_\epsilon)$, $a_3 = \gamma \tau_\epsilon$, and for $n = 1, 2$,

$$a_n = \gamma \frac{\lambda_{n+1} \Delta a_{n+1}}{\text{Var}_{in}[p_{n+1}]} \alpha_{E_n}. \quad (4.10)$$

According to (4.8), at any period $n < 3$, a rational investor's strategy is the sum of two components. The first component captures the investor's activity based on his *private* estimation of the difference between the fundamentals and the n -th period equilibrium price. This is akin to “long-term” speculative trading, aimed at taking advantage of the investor's superior information on the liquidation value of the asset. The second component captures the investor's activity based on the extraction of order flow, i.e. *public*, information. This trading is instead aimed at timing the market by exploiting short-run movements in the asset price determined by the evolution of the future aggregate demand. Upon observing this information, and depending on the sign of the difference $\alpha_{P_n} - \alpha_{E_n}$, rational investors engage either in “market making” (when $\alpha_{P_n} - \alpha_{E_n} < 0$, thereby accommodating the aggregate demand) or in “trend chasing” (when $\alpha_{P_n} - \alpha_{E_n} > 0$, thus following the market). To fix ideas, suppose that $E_n[\theta_n] > 0$ (which, given (4.7), implies $E_n[p_{n+1} - p_n] < 0$). Given that

$$E_n[\theta_n] = a_n (v - E_n[v]) + \theta_n,$$

rational investors' reaction to this observation depends on whether they believe it to be driven by liquidity trades or fundamentals information. In the former case, they anticipate that the impact of their position unwinding on the $n + 1$ price will be negative. Hence, if $\alpha_{P_n} < \alpha_{E_n}$, they take the other side of the market, acting as market makers and shorting the asset (at a premium above $E_n[v]$) in the expectation of buying it back at a lower price in period $n + 1$. If, on the other hand, they attribute their estimate to fundamentals information, they instead

anticipate that their position unwinding will have a positive impact on the $n + 1$ price. Hence, if $\alpha_{P_n} > \alpha_{E_n}$, they buy the asset (once again at a premium above $E_n[v]$), expecting to resell it at a price that reflects the positive news, effectively chasing the trend.

Accordingly, the impact that rational investors' estimate of the supply shock has on p_n and on $p_{n+1} - p_n$ changes depending on whether they act as contrarians or trend chasers. To see this it suffices to impose market clearing on (4.8) and solve for the equilibrium price, obtaining:

$$\int_0^1 X_n(s_{in}, z^n) di + \theta_n = 0 \Leftrightarrow p_n = \bar{E}_n[v] + \frac{\alpha_{P_n} - \alpha_{E_n}}{a_n} E_n[\theta_n] + \frac{\alpha_{E_n}}{a_n} \theta_n, \quad (4.11)$$

where $\bar{E}_n[v] \equiv \int_0^1 E_{in}[v] di = \alpha_{E_n} v + (1 - \alpha_{E_n}) E_n[v]$, and $\alpha_{E_n} = \tau_\epsilon / \tau_{in}$. Shifting the time index one period ahead in (4.5), we obtain:

$$p_{n+1} - p_n = \lambda_{n+1} \left(z_{n+1} + \Delta a_{n+1} \frac{\alpha_{P_n} - \alpha_{E_n}}{a_n} E_n[\theta_n] - \Delta a_{n+1} p_n \right). \quad (4.12)$$

According to (4.11) and (4.12) when rational investors act as contrarians ($\alpha_{P_n} < \alpha_{E_n}$), an estimated positive supply shock at time n ($E_n[\theta_n] > 0$) has a negative impact on p_n and on $p_{n+1} - p_n$; conversely, when they chase the market ($\alpha_{P_n} > \alpha_{E_n}$), the same estimate has a positive impact both on p_n and on $p_{n+1} - p_n$. This latter possibility can never occur in a market without private information. Indeed, as noted in the previous section, in such a market prices are not tied to a persistent factor, and thus are *never* expected to trend.

4.1 Multiple equilibria and illiquidity

When $N = 2$, we can prove existence and analytically characterize the set of equilibrium solutions:

Proposition 3. *When $N = 2$, linear equilibria always exist. If $\beta \in (0, 1]$:*

1. *There are two equilibria in which the responsiveness to private signals are $a_2 = \gamma\tau_\epsilon$, and*

$$a_1^* = \frac{1 + \gamma\tau_u a_2(1 + \beta) - \sqrt{(1 + \gamma\tau_u a_2(1 + \beta))^2 - 4\beta(\gamma\tau_u a_2)^2}}{2\beta\gamma\tau_u} \quad (4.13)$$

$$a_1^{**} = \frac{1 + \gamma\tau_u a_2(1 + \beta) + \sqrt{(1 + \gamma\tau_u a_2(1 + \beta))^2 - 4\beta(\gamma\tau_u a_2)^2}}{2\beta\gamma\tau_u}, \quad (4.14)$$

where $a_1^* < a_1^{**}$. When $a_1 = a_1^*$ investors are contrarians ($\alpha_{P_1} < \alpha_{E_1}$), while when $a_1 = a_1^{**}$, they chase the market ($\alpha_{P_1} > \alpha_{E_1}$);

2. *When $a_1 = a_1^*$, $\lambda_2(a_1^*, a_2) > 0$, while when $a_1 = a_1^{**}$, $\lambda_2(a_1^{**}, a_2) < 0$. Furthermore, $|\lambda_2(a_1^{**}, a_2)| < \lambda_2(a_1^*, a_2)$, and prices are more informative along the low illiquidity equilibrium.*

If $\beta = 0$, the equilibrium is unique: $a_2 = \gamma\tau_\epsilon$,

$$a_1 = \frac{\gamma a_2^2 \tau_u}{1 + \gamma a_2 \tau_u}, \quad (4.15)$$

investors are contrarians ($\alpha_{P_1} < \alpha_{E_1}$), and the second period market is more illiquid.

According to the above result, multiple equilibria where investors display different levels of private signal responsiveness a_1 can arise. The intuition is as follows. In the first period, investors use their private signals to forecast the price at which they liquidate their position, p_2 . However, with heterogeneous information, prices are driven by fundamentals information, and liquidity trades. The less illiquid is the second period market (i.e., the lower is λ_2), the weaker is the reaction of p_2 to the information contained in the second period aggregate demand, and the easier it is for informed investors in the first period to predict the second period price with their signals. When $\beta > 0$, if first period investors anticipate that their second period peers will absorb part of their orders, they expect the market in period 2 to be more illiquid. As a consequence, they scale down their response to private information, conveying less information to the price and opening more speculative opportunities to second period investors. In this case, indeed, the second period market is more illiquid. If, on the other hand, first period investors anticipate that investors in the second period reduce their exposure to the risky asset, they expect a more liquid market in period 2. As a consequence, they ramp up their response to their private signals, conveying more information to the price and narrowing the speculative opportunities available to second period investors. In this case, thus, the market in the second period is less illiquid. When $\beta = 0$, first period investors anticipate liquidating their position against the reverting demand of liquidity traders. In this case, the adverse selection component of the second period price impact is always positive, implying that prices react more aggressively to z_2 . As a consequence, first period investors scale back their response to private information.

The existence of a negative (and small) price impact of trades, along the low illiquidity equilibrium, is consistent with Boehmer and Wu (2006) who find a negative association between “uninformed” investors’ imbalances and contemporaneous returns.¹⁴ Saar and Linnainmaa (2010) in their analysis of brokers’ activity in the Helsinki Stock Exchange, also find that price impacts of households (who are arguably uninformed) are negative. In our context, in the low illiquidity equilibrium, the price impact is small and negative exactly because, in view of a very informative first period price, second period informed investors reduce their exposure to the risky asset, implying that liquidity (i.e., uninformed) traders end up having a more relevant role in clearing the market.

Proposition 3 clarifies that the responsiveness to private information *and* the reaction to the estimated demand of liquidity traders (measured by $(\alpha_{P_1} - \alpha_{E_1})/\alpha_{E_1}$, see (4.8)) are closely related. Indeed, the stronger is investors’ reaction to private signals, the more likely that the latter estimate of the liquidity stock is influenced by fundamentals information, implying that investors chase the trend. This is what happens in the equilibrium with low second period illiquidity. Conversely, in the equilibrium along which second period illiquidity is high, investors scale back the responsiveness to private signals, implying that estimates of liquidity trades are more likely to signal non-fundamentals driven orders. This justifies contrarian behavior.

Other authors have argued that when private information is related to an event which occurs beyond the date at which investors liquidate their positions, the latter act on their signals only

¹⁴In Boehmer and Wu (2006), uninformed investors are individuals, market makers, and institutions who adopt program trades.

if they expect them to be reflected in the price at which they liquidate. This effect is responsible for Dow and Gorton (1994)'s arbitrage chains, as well as for Froot, Sharfstein and Stein (1992)'s herding on short term private information. In the present context a similar effect is at work. Note, however, that the main driver in first period investors' reaction to private signals *is not* the anticipation of a *strong* impact of private information on the liquidation price. Indeed, if that happened the second period market would not necessarily be liquid. It is rather the anticipation of a lower informational advantage held by second period investors, which implies a lower illiquidity for first period investors when unwinding their positions that matters.

As the persistence in liquidity trading is reduced, in both equilibria first period informed investors speculate more aggressively on their private information:

Corollary 6. *When $N = 2$, and $\beta \in (0, 1)$, in any equilibrium of the market $\partial a_1 / \partial \beta < 0$.*

Proof. When $N = 2$, rearranging (4.10) yields

$$\phi(a_1) \equiv \lambda_2 \tau_{i2} a_1 - \gamma \Delta a_2 \tau_u \tau_\epsilon = 0.$$

The result follows immediately, since from implicit differentiation of the above with respect to β :

$$\frac{\partial a_1}{\partial \beta} = - \frac{\gamma \tau_u a_1 (a_2 - a_1)}{(1 + \gamma \tau_u \Delta a_2) + \gamma \beta \tau_u (a_2 - a_1)} < 0,$$

independently of the equilibrium that arises; in the high illiquidity equilibrium we have $\beta a_1 < a_1 < a_2 \equiv \gamma \tau_\epsilon$, and in the low illiquidity equilibrium $a_1 > a_2 / \beta > a_2$ and $1 + \gamma \tau_u \Delta a_2 < 0$. $\square$

The intuition for this result is as follows: along the equilibrium with high illiquidity, a higher β implies that a larger fraction of the position informed investors hold in the first period will be cleared by second period informed investors. This, in turn, amplifies both the inventory risk and the adverse selection risk to which first period investors are exposed when liquidating, making p_2 less predictable and leading them to lower their response to private information. Along the low illiquidity equilibrium, more persistent liquidity trading means that more informed investors are escalating their responsiveness to private information, making p_2 more dependent on p_1 . As a consequence, individually each trader scales down his reliance on private information. Note that as $\beta \rightarrow 0$, along the high illiquidity equilibrium a_1 converges to (4.15), while along the low illiquidity equilibrium a_1 diverges. Intuitively, along the low illiquidity equilibrium, the smaller is β , the lower is the number of first period informed investors who cannot count on the reversion of liquidity traders to unwind their positions. As a consequence, the more aggressively each informed investor needs to respond to private information in the first period for second period investors to lower their exposure to the risky asset.

We now show that the expected volume of informational trading is higher along the low illiquidity equilibrium. Indeed, as investors step up their response to their signals, the position change due to private information is higher along such equilibrium. Defining the volume of informational trading as expected traded volume in the market with heterogeneous information

net of the expected volume that obtains in the market with no private information analyzed in Section 3, yields:¹⁵

$$\begin{aligned}
V_2 &\equiv \int_0^1 E[|X_2(s_{i2}, z^2) - X_1(s_{i1}, z_1)|] di - \int_0^1 E[|X_2(\theta_2) - X_1(\theta_1)|] di \\
&= \int_0^1 \sqrt{\frac{2}{\pi} \text{Var}[X_2(s_{i2}, z^2) - X_1(s_{i1}, z_1)]} di - \int_0^1 \sqrt{\frac{2}{\pi} \text{Var}[X_2(\theta_2) - X_1(\theta_1)]} di \\
&= \sqrt{\frac{2}{\pi}} \left(\sqrt{(a_1^2 + a_2^2)\tau_\epsilon^{-1} + (1 + (\beta - 1)^2)\tau_u^{-1}} - \sqrt{(1 + (\beta - 1)^2)\tau_u^{-1}} \right). \tag{4.16}
\end{aligned}$$

Corollary 7. *When $N = 2$, for all $\beta \in (0, 1]$ the expected volume of informational trading is higher along the low illiquidity equilibrium. When $\beta = 0$ only the equilibrium with a low volume of informational trading survives.*

Proof. Rearranging the expressions for investors' strategies obtained in Corollary 5 yields $x_{in} = a_n \epsilon_{in} - \theta_n$, for $n = 1, 2$. Owing to the fact that for a normally distributed random variable Y we have

$$E[|Y|] = \sqrt{\frac{2}{\pi} \text{Var}[Y]},$$

which implies (4.16). Recall that while $a_2 = \gamma\tau_\epsilon$, in the first period the response to private information is higher along the equilibrium with low illiquidity: $a_1^{**} > a_1^*$, and the result follows. Finally, from Proposition 3 when $\beta = 0$,

$$a_1 = \frac{\gamma a_2^2 \tau_u}{1 + \gamma a_2 \tau_u} < a_1^{**}.$$

□

We conclude this section by showing that along the high illiquidity equilibrium, price risk is larger. As a consequence, consistently with Corollary 4, liquidity supply is more profitable.

Corollary 8. *When $N = 2$, for all $\beta \in (0, 1]$ liquidity provision entails higher expected returns along the high illiquidity equilibrium.*

Proof. We need to prove that along the equilibrium with low illiquidity, price risk (captured by $\text{Var}_{i1}[p_2]$) is lower compared to the equilibrium with high illiquidity. To see this, note that according to (4.10)

$$a_1 = \gamma \frac{\lambda_2 \Delta a_2}{\text{Var}_{i1}[p_2]} \alpha_{E_1}.$$

Now, given that $a_2 = \gamma\tau_\epsilon$, by direct comparison one can verify that $\lambda_2(a_1^*, a_2)(a_2 - \beta a_1^*) > \lambda_2(a_1^{**}, a_2)(a_2 - \beta a_1^{**})$. Given that $a_1^{**} > a_1^*$ and $\alpha_{E_1}(a_1^*) > \alpha_{E_1}(a_1^{**})$, this implies that $\text{Var}_{i1}[p_2]$ must be lower along the equilibrium with low illiquidity compared to the other equilibrium. □

¹⁵This is consistent with He and Wang (1995).

5 Illiquidity and over-reliance on public information

In this section we investigate the implications that asynchronous liquidity needs have for over-reliance on public information. We start by obtaining a general expression for the equilibrium price which shows that in general asset prices are driven by investors' HOEs about the two factors that influence the aggregate demand: fundamentals and liquidity trades.

Starting from the third period, and imposing market clearing yields

$$\int_0^1 X_3(s_{i3}, p^3) di + \theta_3 = 0. \quad (5.1)$$

At any linear equilibrium, the price will be a normally distributed random variable, which, owing to the fact that investors' utility display CARA implies that

$$X_3(s_{i3}, p^3) = \gamma \frac{E_{i3}[v] - p_3}{\text{Var}_{i3}[v]}.$$

Replacing the above in (5.1) and solving for the equilibrium price we obtain

$$p_3 = \bar{E}_3[v] + \Lambda_3 \theta_3,$$

where $\Lambda_3 = \text{Var}_{i3}[v]/\gamma$. Similarly, in the second period, imposing market clearing yields:

$$\int_0^1 X_2(s_{i2}, p^2) di + \theta_2 = 0,$$

and solving for the equilibrium price we obtain

$$p_2 = \bar{E}_2[p_3] + \frac{\text{Var}_{i2}[p_3]}{\gamma} \theta_2. \quad (5.2)$$

Substituting the above obtained expression for p_3 in (5.2) yields

$$\begin{aligned} p_2 &= \bar{E}_2 \left[\bar{E}_3[v] + \frac{\text{Var}_{i3}[v]}{\gamma} \theta_3 \right] + \frac{\text{Var}_{i2}[p_3]}{\gamma} \theta_2 \\ &= \bar{E}_2 [\bar{E}_3[v]] + \frac{\text{Var}_{i3}[v]}{\gamma} \beta \bar{E}_2 [\theta_2] + \frac{\text{Var}_{i2}[p_3]}{\gamma} \theta_2. \end{aligned} \quad (5.3)$$

According to (5.3), there are three terms that form the second period price: investors' second order average expectations over the liquidation value ($\bar{E}_2[\bar{E}_3[v]]$), the risk-adjusted impact of the second period stock of liquidity trades (θ_2), and investors' average expectations over second period liquidity trades ($\bar{E}_2[\theta_2]$). As liquidity trades are persistent, rational investors anticipate unwinding a fraction β of their inventory (θ_2) to third period investors, thereby affecting p_3 . Due to heterogeneous information, however, θ_2 cannot be perfectly assessed. Thus, p_2 reflects the second period market consensus over the size of liquidity traders' demand.

In the first period, a similar argument yields

$$p_1 = \bar{E}_1 [\bar{E}_2 [\bar{E}_3 [v]]] + \frac{\text{Var}_{i3}[v]}{\gamma} \beta \bar{E}_1 [\bar{E}_2 [\theta_2]] + \frac{\text{Var}_{i2}[p_3]}{\gamma} \beta \bar{E}_1 [\theta_1] + \frac{\text{Var}_{i1}[p_2]}{\gamma} \theta_1, \quad (5.4)$$

and generalizing (5.4) to an arbitrary number of periods N we obtain

$$\begin{aligned}
p_n = & \bar{E}_n [\bar{E}_{n+1} [\bar{E}_{n+2} [\cdots \bar{E}_N[v] \cdots]]] \\
& + \frac{\beta}{\gamma} \text{Var}_{iN}[p_{N+1}] \bar{E}_n [\bar{E}_{n+1} [\bar{E}_{n+2} [\cdots \bar{E}_{N-1}[\theta_{N-1}] \cdots]]] \\
& + \frac{\beta}{\gamma} \text{Var}_{iN-1}[p_N] \bar{E}_n [\bar{E}_{n+1} [\bar{E}_{n+2} [\cdots \bar{E}_{N-2}[\theta_{N-2}] \cdots]]] \\
& + \cdots \\
& + \frac{\text{Var}_{in}[p_{n+1}]}{\gamma} \theta_n.
\end{aligned} \tag{5.5}$$

The above expression shows that in period n the equilibrium price reflects investors' HOEs over the liquidation value *and* over the liquidity trades in periods $n, n+1, \dots, N-1$. The former factor reflects the findings of Morris and Shin (2002), and Allen et al. (2006) who prove that when investors have heterogeneous information, the law of iterated expectations fails to hold:

$$\bar{E}_n [\bar{E}_{n+1} [\bar{E}_{n+2} [\cdots \bar{E}_N[v] \cdots]]] \neq \bar{E}_n[v],$$

so that when prices are driven by HOEs over the final payoff they are systematically farther away from fundamentals compared with consensus or, equivalently, they over-rely on public information (compared to optimal statistical weights). In our context, the presence of asynchronous liquidity needs implies that an additional factor adds to the weight contributed by HOEs. Computing the expectations (5.3) and (5.4) we obtain

$$\bar{E}_2 [\bar{E}_3[v]] = \bar{\alpha}_{E_2} v + (1 - \bar{\alpha}_{E_2}) E_2[v], \quad \bar{E}_1 [\bar{E}_2 [\bar{E}_3[v]]] = \bar{\alpha}_{E_1} v + (1 - \bar{\alpha}_{E_1}) E_1[v],$$

where

$$\bar{\alpha}_{E_1} = \alpha_{E_1} \left(1 - \frac{\tau_1}{\tau_2} (1 - \bar{\alpha}_{E_2}) \right), \quad \bar{\alpha}_{E_2} = \alpha_{E_2} \left(1 - \frac{\tau_2}{\tau_3} (1 - \alpha_{E_3}) \right), \tag{5.6}$$

denote the weights that HOEs about the final payoff assign to v in the first and second period price, and $\tau_n = 1/\text{Var}[v|p^n]$. Similarly,

$$\begin{aligned}
\bar{E}_n[\theta_n] &= a_n(1 - \alpha_{E_n})(v - E_n[v]) + \theta_n \\
\bar{E}_1[\bar{E}_2[\theta_2]] &= (a_2(\alpha_{E_1} - \bar{\alpha}_{E_1}) + \beta a_1(1 - \alpha_{E_1}))(v - E_1[v]) + \beta \theta_1.
\end{aligned}$$

According to Proposition 2, at any linear equilibrium, the price can be expressed as follows:

$$p_n = \alpha_{P_n} \left(v + \frac{\theta}{a_n} \right) + (1 - \alpha_{P_n}) E_n[v].$$

Hence, we obtain:

Lemma 1. *When $\beta > 0$, the weights the price assigns to the fundamentals in the first and second period are given by*

$$\alpha_{P_1} = \bar{\alpha}_{E_1} + \beta((1 - \alpha_{E_1})\Lambda_2 a_1 + (\alpha_{E_1} - \bar{\alpha}_{E_1})\Lambda_3 a_2) \tag{5.7}$$

$$\alpha_{P_2} = \bar{\alpha}_{E_2} + \beta(1 - \alpha_{E_2})\Lambda_3 a_2. \tag{5.8}$$

Similarly as in Allen et al. (2006), we say that at time n the price is systematically farther away from investors' consensus opinion if the following condition holds true:

$$|E[p_n - v|v]| > |E[\bar{E}_n[v] - v|v]|. \quad (5.9)$$

The above condition then holds if, for any liquidation value, averaging out the impact of noise trades, the discrepancy between the price and the fundamentals is always larger than that between investors' average opinion and the fundamentals.¹⁶ Using the expression obtained in Proposition 2, the following result offers two alternative characterizations of condition (5.9).

Lemma 2. *At any linear equilibrium of the 3-period market the following three conditions are equivalent:*

$$|E[p_n - v|v]| > |E[\bar{E}_n[v] - v|v]| \quad (5.10)$$

$$\alpha_{P_n} < \alpha_{E_n} \quad (5.11)$$

$$\text{Cov}[p_n, v] < \text{Cov}[\bar{E}_n[v], v]. \quad (5.12)$$

Thus, as intuition suggests, the equilibrium price is systematically farther away from fundamentals compared to consensus, whenever the price overweights public information (compared to optimal statistical weights); equivalently, whenever the price scores worse than investors' average opinion in predicting the fundamentals.

When $\beta = 0$, it is easy to see that $\alpha_{P_1} = \bar{\alpha}_{E_1} < \alpha_{E_1}$, so that over-reliance on public information occurs. However, as we argued in Proposition 3, when $\beta > 0$, $\alpha_{P_1} < \alpha_{E_1}$ if and only if $a_1 = a_1^*$. Thus, we can immediately conclude

Proposition 4. *When $N = 2$ and $\beta > 0$, along the equilibrium with high illiquidity the first period price over-relies on public information. Conversely, along the equilibrium with low illiquidity, the price under-relies on public information.*

Thus, the existence of different trading frequencies (due to persistence in liquidity trades) generate an effect that can *offset* the gravitational pull towards over-reliance on public information due to HOEs over fundamentals. This effect works precisely by forcing investors to internalize the negative impact that their position unwinding has on future periods' illiquidity. Thus, short-term investment horizons per-se do not warrant over-reliance on public information. It is rather a matter of how heterogeneous investment horizons are across agents' types. The need to forecast a price that reflects investors' valuations which are also based on a public signal (the equilibrium price), leads to over-reliance on public information. On the other hand, the need to minimize the illiquidity cost borne when unwinding their positions, pushes investors to over-rely on their private signals. Along the equilibrium with high illiquidity, the former effect prevails and the price is driven by HOEs over the final payoff. Conversely, along the equilibrium with low illiquidity, HOEs over the payoff are subdued and prices track fundamentals more efficiently. Our analysis thus portrays a more complex picture of Keynes' Beauty Contest asset pricing allegory.

¹⁶Condition (5.9) can be given the following intuitive interpretation. In the market, two estimators of the fundamentals are available: the equilibrium price, p_n , and the average opinion traders hold about v , $\bar{E}_n[v]$.

Table 2 collects our results stressing the interplay between illiquidity, expected returns, and the impact of HOEs on asset prices.

	$\beta = 0$	$\beta \in (0, 1]$
Over-reliance on public information	$\alpha_{P_1} < \alpha_{E_1}$	$\alpha_{P_1} \gtrless \alpha_{E_1}$
Illiquidity	High	Low/High
Expected returns	High	Low/High
Expected volume of informational trading	Low	High/Low

Table 2: A summary of our results.

6 Extensions

In this section we check the robustness of our conclusions, extending the model to account for the presence of an additional trading round and for the impact of an increase in fundamentals uncertainty.

6.1 An additional trading round

If we add an extra trading round to the market analyzed in Section 4.1, the coordination problem across different cohorts exacerbates, and the cardinality of the equilibrium set increases. Intuitively, in the second period rational investors anticipate either a liquid or an illiquid third period market. Accordingly, two different levels of second period equilibrium signal responsiveness arise. One period before, the same problem is now faced by first period investors *for each* equilibrium on which second period investors coordinate. This gives rise to four equilibrium candidates: first period investors anticipate low second and third period illiquidity; alternatively, they anticipate low illiquidity in the second, followed by high illiquidity in the third period, or high illiquidity in the second followed by low illiquidity in the third period; finally, they may anticipate that illiquidity will stay high in both the second and third periods. Correspondingly, four different levels of first period signal responsiveness can arise.

We illustrate our findings with a numerical simulation. In Figure 2 we plot market illiquidity across the three trading dates (panel (a)) and the difference between α_{P_n} and α_{E_n} in the first two trading periods (panel (b)).¹⁷ There are four equilibria, two “extreme” ones in which illiquidity is persistently high or low across all trading dates and the price over- or under-reacts to public information at dates 1 and 2. However, there are also two “intermediate” equilibria in which illiquidity is high (low) in the first period and low (high) in the second period and the market over- (under-) reacts in the first period and under- (over-) reacts in the second period.

¹⁷In the last trading round we know that $\alpha_{P_3} = \alpha_{E_3}$.

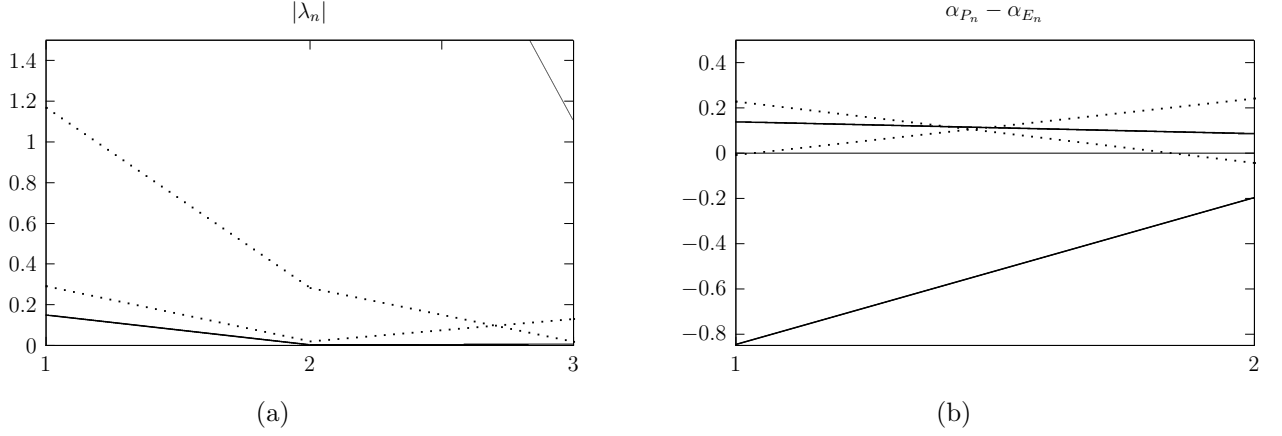


Figure 2: In panel (a) we plot $|\lambda_n|$ for $n = 1, 2, 3$, and in panel (b) $\alpha_{P_n} - \alpha_{E_n}$ for $n = 1, 2$, along the four equilibria. The solid lines in panel (b) correspond to equilibria in which the market over- (under-) reacts to public information in both periods. Correspondingly, the solid lines in panel (a) show that when the market is over- (under-) reacting to public information illiquidity is high (low) in both periods (along the high illiquidity equilibrium $\lambda_1 = 26.9$ and $\lambda_2 = 3.4$). The dotted lines in both panels correspond to equilibria in which the market over- (under-) reacts in the first (second) period and under- (over-) reacts in the second (first) period implying an illiquidity pattern of high (low) illiquidity in the first period, low (high) illiquidity in the second period and finally high (low) illiquidity in the third period. Other parameter values are as follows: $\tau_v = \tau_\epsilon = \tau_u = 1$, $\gamma = .5$, and $\beta = .8$.

6.2 An increase in the residual uncertainty affecting fundamentals

In this section we redefine the final liquidation value as $v + \delta$, with $\delta \sim N(0, \tau_\delta^{-1})$, orthogonal to v and to all the other random variables of the model. As in our baseline model, informed investors' private signals are only informative about v : $s_{in} = v + \epsilon_{in}$. Hence, δ captures a factor that matters for the final liquidation value and about which no investor in the market is informed. The presence of an additional layer of fundamentals uncertainty complicates the analysis but does not affect our main conclusion. In particular, when $N = 2$, we can prove the following result:

Corollary 9. *When $N = 2$, linear equilibria always exist.*

1. If $\beta \in (0, 1]$ and $0 < 1/\tau_\delta < \infty$, there are two equilibria in which the responsiveness to private signals are (a_1^*, a_2^*) and (a_1^{**}, a_2^{**}) , with

$$a_1^* = \frac{1 + \gamma\tau_u(a_2^* + \beta\gamma\tau_\epsilon) - \sqrt{1 + \gamma\tau_u(2(a_2^* + \beta\gamma\tau_\epsilon) + \gamma\tau_u(a_2^* - \beta\gamma\tau_\epsilon)^2)}}{2\beta\gamma\tau_u} \quad (6.1)$$

$$a_1^{**} = \frac{1 + \gamma\tau_u(a_2^{**} + \beta\gamma\tau_\epsilon) + \sqrt{1 + \gamma\tau_u(2(a_2^{**} + \beta\gamma\tau_\epsilon) + \gamma\tau_u(a_2^{**} - \beta\gamma\tau_\epsilon)^2)}}{2\beta\gamma\tau_u}, \quad (6.2)$$

and a_2^* (a_2^{**}) obtains as the unique real solution to the cubic equation $\phi_2(a_1^*, a_2) \equiv a_2(\tau_\delta + \tau_{i2}) - \gamma\tau_\delta\tau_\epsilon = 0$ ($\phi_2(a_1^{**}, a_2) \equiv a_2(\tau_\delta + \tau_{i2}) - \gamma\tau_\delta\tau_\epsilon = 0$). Furthermore, $a_1^* < a_1^{**}$, and $0 < a_2^{**} < a_2^* < \gamma\tau_\epsilon$. When $a_n = a_n^*$ first period investors are contrarians, while when $a_n = a_n^{**}$, they chase the market.

2. If $\beta = 0$ or $1/\tau_\delta \rightarrow \infty$, the equilibrium is unique, $\alpha_{P_1} < \alpha_{E_1}$, and the second period market is more illiquid.

Thus, an increase in the residual uncertainty that affects fundamentals loosens the relationship between prices and fundamentals, crowding out informed trading – thereby eliminating the “trend chasing” equilibrium – and prompting rational investors to act as liquidity suppliers in view of the higher expected profits that this entails.

7 Implications

7.1 Endogenous illiquidity and High Frequency Trading

Our model can shed light on the impact that heterogeneous trading frequencies have on market liquidity. In particular, when $\beta > 0$, we can interpret the informed traders as High Frequency Traders (HFTs) that are able to turn around their positions at a higher speed compared to liquidity traders. More formally, consider the 2-period case analyzed in section 4.1, and assume that, prior to entering the market, first period liquidity traders get to choose the fraction of their positions that will be unwound at the end of the first period.¹⁸ If $\beta = 0$ in the second period there is more liquidity trading as first period investors can unwind against $-\theta_1$. At the other extreme, if $\beta = 1$, the total amount of second period liquidity trading is much smaller, as first and second period investors share the additional shock u_2 . Thus, the case $\beta = 0$ captures the extreme situation in which the technological features of HFT are available to all liquidity traders too, whereas when $\beta = 1$ the technological gap between HFT and liquidity trading is maximal. Formally, the sequence of events and decisions for first period liquidity traders is illustrated in Figure 3.

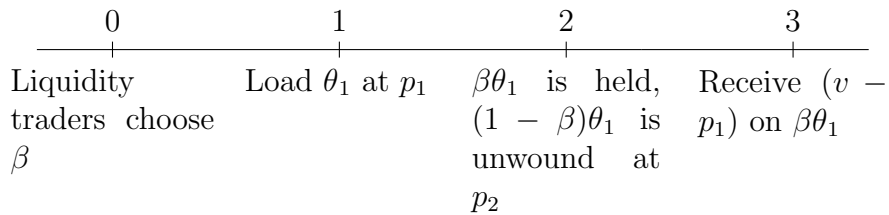


Figure 3: The sequence of events and decisions of the first period liquidity trader.

The expected profit that the first period liquidity traders obtain from this strategy is given by

$$\begin{aligned}\Pi_{\theta_1} &\equiv E[\beta\theta_1(v - p_1) + (1 - \beta)\theta_1(p_2 - p_1)] \\ &= - \left(\beta\lambda_1 + (1 - \beta)\lambda_2\Delta a_2 \left(\frac{\tau_{i1} - \tau_v}{a_1\tau_{i1}} \right) \right) \tau_u^{-1} < 0.\end{aligned}\tag{7.1}$$

¹⁸This is similar to the discretionary liquidity traders’ idea in Admati and Pfleiderer (1988).

It is easy to see that along the low illiquidity equilibrium $\lim_{\beta \rightarrow 0} \Pi_{\theta_1} = 0$, since in this case the first period signal responsiveness diverges at $\beta = 0$ (see Corollary 6). On the contrary, along the equilibrium with high illiquidity

$$\lim_{\beta \rightarrow 0} \Pi_{\theta_1} = \Pi_{\theta_1} \Big|_{a_1 = \frac{\gamma a_2^2 \tau_u}{1 + \gamma a_2 \tau_u}, \beta=0} < 0.$$

In general, plotting (7.1) along the two equilibria that arise with $\beta > 0$ one obtains Figure 4. Depending on parameters values the two plots intersect or not (we need τ_v high for them to

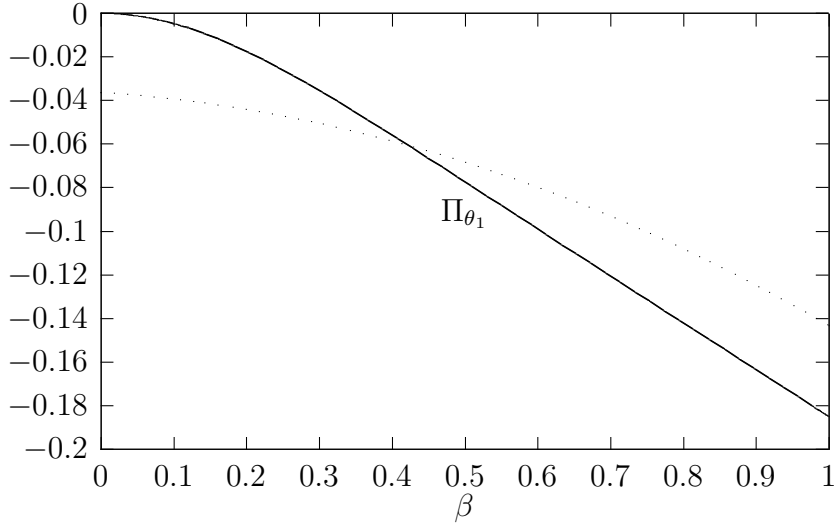


Figure 4: Expected profit along the high illiquidity equilibrium (dotted line) and along the low illiquidity equilibrium (continuous line) as a function of β . Other parameter values are $\tau_v = 10$, $\tau_u = 1$, $\tau_\epsilon = 1$, $\gamma = 1$.

intersect), but the bottomline is that for β small, liquidity traders' expected losses are always smaller along the low illiquidity equilibrium (in the second period the conclusion is immediate given the properties of the equilibrium). This implies that if liquidity traders could choose their trading frequency, they would equate it to the one that is available to informed investors, and set $\beta = 0$. However, at $\beta = 0$, the low illiquidity equilibrium disappears and the only equilibrium that is available features high illiquidity. Indeed, as argued in Corollary 6, when first period informed investors can count on the reversion of first period liquidity traders' positions at the liquidation date, the low illiquidity equilibrium can only obtain if the first period responsiveness is infinite.

Several authors suggest that the introduction of algorithmic trading and high frequency trading have generated a large improvement in market liquidity accompanied by a reduction in adverse selection risk (see, e.g. Hendershott, Jones, and Menkveld (2010)). This finding is consistent with the low illiquidity equilibrium where, short of profitable speculative opportunities, informed investors in the second period lower their exposure to the risky asset (see Proposition 3), adverse selection decreases across the two periods, and the first period price under-relies on public information (and thus over-relies on private information, see Proposition 4).¹⁹ As

¹⁹If we measure adverse selection at time n via the OLS conditional regression coefficient of the liquidation

argued above, our model suggests, however, that the liquidity improvement that is brought by HFT is not necessarily exploitable by those who are to gain the most from it, namely liquidity traders. A further implication of our model is that with differential information, the existence of a discrepancy in the speed at which different types of investors can turn around their positions can be responsible for indeterminacy, making liquidity dependent on a coordination problem across different generations of investors, and thereby endogenously creating a source of liquidity risk.

7.2 Illiquidity and contrarian behavior

Our model predicts that investors are contrarians, acting as dealers, when the level of private information (as measured by the difference $\alpha_{P_n} - \alpha_{E_n}$) affecting the aggregate demand is low (see Corollary 5, and Proposition 3). In this case, indeed, as the trading process does not lead to a considerable reduction in the uncertainty on the liquidation price, the supply of liquidity entails high expected returns (see Corollary 8). This is what occurs in the high illiquidity equilibrium. Conversely, in the low illiquidity equilibrium, investors step up their response to private signals, the aggregate demand is more driven by fundamentals information, and liquidity provision is less profitable. Based on this, we should expect that information on the supply of liquidity represents a better proxy for price changes in the high illiquidity equilibrium, compared to the low illiquidity equilibrium. Using (4.5), we can analyze the covariance between investors' aggregate positions in period 1 ($-\theta_1$) and the subsequent price change (Δp_2) across the two equilibria that arise with $\beta > 0$:

$$\text{Cov}[-\theta_1, \Delta p_2] = \frac{\text{Var}_{i1}[p_2]\text{Var}[\theta_1]}{\gamma} \left(1 + a_1^2 \frac{\text{Var}[\epsilon_{i1}]}{\text{Var}[\theta_1]} \right). \quad (7.2)$$

The above expression captures the idea that for investors to accept clearing the market, prices must move in a way to allow for a risk-based compensation. Several studies (see, e.g. Henderson and Seasholes (2007)) use a measure like (7.2) to document the ability of dealers' inventory positions to forecast future returns at short horizons. While in both equilibria the above covariance is positive, it is easy to see that for "small" values of β , $\text{Cov}[-\theta_1, \Delta p_2]$ is higher along the equilibrium with high illiquidity. Using the result of Proposition 3, and computing the limit of (7.2) when $a_1 = a_1^*$,

$$\lim_{\beta \rightarrow 0} \text{Cov}[-\theta_1, \Delta p_2] = \frac{\text{Var}_{i1}[p_2]\text{Var}[\theta_1]}{\gamma} \left(1 + a_1^2 \frac{\text{Var}[\epsilon_{i1}]}{\text{Var}[\theta_1]} \right) \Big|_{a_1 = \frac{\gamma a_2^2 \tau_u}{1 + \gamma a_2 \tau_u}, \beta=0} > 0,$$

whereas along the low illiquidity equilibrium given that with $\beta \rightarrow 0$, a_1^* diverges to infinity (see Proposition 3), prices become fully revealing and $\text{Cov}[-\theta_1, \Delta p_2]$ tends to 0. Numerical simulations confirm that this result holds also for larger values of β , as shown in Figure 5. The

value over the time n informational addition, we can verify that along the low illiquidity equilibrium adverse selection decreases between periods 1 and 2. To see this, note that along the low illiquidity equilibrium adverse selection at time 1 is $a_1^{**} \tau_u / \tau_1$, whereas at time 2 we have $(a_2 - \beta a_1^{**}) \tau_u / \tau_2$. As in this case $\Delta a_2 < 0$, we look at the absolute value and check that $a_1^{**} / \tau_1 > a_1^{**} / \tau_2 > |\Delta a_2 / \tau_2|$. The first inequality is trivially always satisfied, whereas the second one requires $a_1^{**} > a_2 / (1 + \beta)$, a condition that is always satisfied along the equilibrium with low illiquidity.

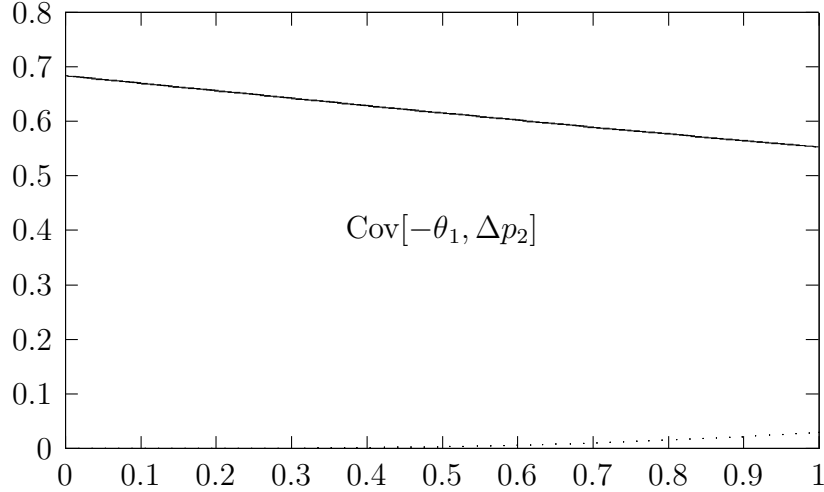


Figure 5: The figure displays the evolution of $\text{Cov}[-\theta_1, \Delta p_2]$ along the high illiquidity equilibrium (continuous line) and the low illiquidity equilibrium (dotted line) as a function of β . Parameters' values are as follows: $\gamma = 1$, $\tau_u = \tau_v = \tau_\epsilon = 1$.

figure displays the evolution of $\text{Cov}[-\theta_1, \Delta p_2]$ along the high illiquidity equilibrium (continuous line) and the low illiquidity equilibrium (dotted line) as a function of β . When β increases, a lower fraction of liquidity traders revert their positions, and thus first period inventories are a worse predictor of the next period price change along the high illiquidity equilibrium. This explains why along the high illiquidity equilibrium $\text{Cov}[-\theta_1, \Delta p_2]$ declines with β .

The above results and Corollaries 4, 7, and 8 suggest the following interpretation for our findings. If price risk is high, the market is illiquid, liquidity supply is thus profitable, and informed investors act in a “market making” fashion, absorbing liquidity traders’ demand with large price concessions (and thereby earning high expected returns from liquidity provision). If, on the other hand, price risk is low, the market is liquid, liquidity supply is not very profitable, and investors act as “trend chasers” (low price risk and thus low risk compensation). Whether price risk is high or low depends on a coordination problem and we predict that high price risk occurs when informed investors refrain from using private information exactly because prices are mainly driven by liquidity shocks which are orthogonal to fundamentals. Conversely, price risk is low when investors use private signals exactly because the main driver of asset prices is fundamentals information. Therefore, liquidity provision is more profitable when there is little informational trading in the market. Conversely, liquidity provision is less profitable when the market is rife with informed trading. As argued in the introduction many authors document the existence of a positive relationship between illiquidity and expected returns. In our model, this effect is due to the lack of participation of rational agents in their capacity of informed investors which prevents the reduction in fundamentals uncertainty due to order flow and private information, thus making holding the asset more risky.

This intuition allows us to put into perspective our findings in Section 6.2. There we argued that a shock to the residual uncertainty affecting fundamentals “crowds out” informed investors

from the market. Intuitively, a sudden increase in residual uncertainty is one of the features of a crisis, as it implies that investors' information is a poor guide to investment decisions based on fundamentals. In this respect, our findings are consistent with the recent literature that analyzes the impact of the crisis started in 2007 on investors' behavior. Nagel (2010), shows that expected returns from contrarian strategies for Nasdaq stocks were highly positively correlated with the level of the VIX index during the recent crisis (2007–2009).²⁰ Distaso, Fernandes, and Zikes (2010), supply evidence that hedge funds of different styles reduced their exposure to the market when the liquidity dry up climaxed. Our model provides a theoretical interpretation for both of these facts. Indeed, our analysis shows that investors reduce their reliance on private information, yielding a lower volume of informational trading, and act as contrarians exactly when due to high price risk, illiquidity and the (expected) returns due to liquidity provision are high, and the risk to face informed investors is instead low. This suggests that part of the spike in illiquidity documented in the literature was favored by an increase in price risk due to a lack of participation of informed investors in their capacity of liquidity suppliers.

8 Conclusions

When a market is populated by short term investors, the persistence of liquidity traders' demand implies that only a fraction of the latter's orders is bound to revert when informed investors close out their positions. Hence, each cohort of investors unwinds part of its holdings against the aggregate demand coming from the next period cohort, thereby shifting part of the risk it incorporates and potentially facing an adverse selection problem at the liquidation date. This makes portfolio decisions more sensitive to the illiquidity of the market in which investors plan to unwind their positions. With heterogeneous information, a more illiquid market in the future, lowers price dependence on fundamentals, reducing investors' reliance on their private information. Conversely, when investors anticipate a more liquid future market they over-rely on their private signals. Thus, with persistent liquidity trades and heterogeneous information, short term horizons deliver multiple equilibria which can be ranked in terms of illiquidity.

We show that along the equilibria in which investors anticipate future higher illiquidity, prices are driven by HOEs about fundamentals, and therefore over-rely on public information. In these equilibria investors scale down their response to private signals yielding an increase in price risk, which implies that returns from liquidity provision are higher. Indeed, it is precisely in these equilibria that contrarian strategies are profitable, as the aggregate demand for the stock features little private information. When instead investors anticipate a more liquid market in the future, they step up their response to private signals, reducing price risk and returns from liquidity provision. In this case, indeed, investors are trend chasers as the aggregate demand for the stock is rife with private information.

Our model provides a theoretical foundation to the empirical regularity that links illiquidity to expected returns. In our story, a more illiquid market is one in which price risk is huge

²⁰The VIX is traditionally interpreted as a measure of financial intermediaries' risk-appetite (this is, e.g., the interpretation in Adrian and Shin (2010)).

exactly because private information is scarce, and investors require a higher compensation (price concession) to hold the inventory of the risky asset. Based on this intuition we show that a spike in the residual uncertainty that affects fundamentals causes the impact of private information on the aggregate demand to wane, singling out as a unique equilibrium the one in which investors are contrarians.

Our model also sheds light on the implications that heterogeneous trading frequencies have for market liquidity. Indeed, while along the low illiquidity equilibrium, liquidity traders' losses are minimized, this equilibrium disappears whenever trading frequencies coincide. Thus, an implication of our analysis is that while trading at high frequencies may generate liquidity improvements, these improvements are not necessarily exploitable by those who are to gain the most from it, namely liquidity traders. A further implication of our model is that with differential information, heterogeneous trading frequencies make liquidity dependent on a coordination problem across different generations of investors, and thereby can endogenously create liquidity risk.

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A Appendix

PROOF OF PROPOSITION 1

Assume that in a linear equilibrium $x_n = -\phi_n(p_n)$, with $\phi_n(\cdot)$ a linear function of p_n . This implies that the market clearing equation at time n reads as follows:

$$x_n + \theta_n = 0 \Leftrightarrow -\phi_n(p_n) + \theta_n = 0.$$

Hence, at equilibrium p_n is observationally equivalent to θ_n , i.e. at time n rational investors *know* the realisation of the noise shock θ_n . To solve for the equilibrium we proceed by backward induction and start from the third trading round, where due to CARA and normality, we have

$$\begin{aligned} X_3(p_3) &= \gamma \frac{E_3[v] - p_3}{\text{Var}_3[v]} \\ &= -\Lambda_3^{-1}(p_3 - \bar{v}), \end{aligned} \tag{A.1}$$

with

$$\Lambda_3 \equiv \frac{1}{\gamma \tau_v}, \tag{A.2}$$

and

$$p_3 = \bar{v} + \Lambda_3 \theta_3. \tag{A.3}$$

In the second period, we have

$$\begin{aligned} X_2(p_2) &= \gamma \frac{E_2[p_3] - p_2}{\text{Var}_2[p_3]} \\ &= -\Lambda_2^{-1}(p_2 - \bar{v}), \end{aligned} \tag{A.4}$$

with

$$\Lambda_2 \equiv \frac{1 + \gamma \beta \Lambda_3^{-1} \tau_u}{\gamma \Lambda_3^{-2} \tau_u}, \tag{A.5}$$

and

$$p_2 = \bar{v} + \Lambda_2 \theta_2. \tag{A.6}$$

Similar calculations show that in the first period $x_1 = -\Lambda_1^{-1}(p_1 - \bar{v})$ and $p_1 = \bar{v} + \Lambda_1 \theta_1$, with

$$\Lambda_1 \equiv \frac{1 + \gamma \beta \Lambda_2^{-1} \tau_u}{\gamma \Lambda_2^{-2} \tau_u}.$$

QED

PROOF OF COROLLARY 1

In the third period $\Lambda_3 = 1/\gamma \tau_v$, implying that $\partial \Lambda_3 / \partial \beta = 0$. In the second period, using (A.5), $\partial \Lambda_2 / \partial \beta = \Lambda_3 > 0$. Finally, in the first period we have

$$\Lambda_1 = \frac{\Lambda_2^2}{\gamma \tau_u} + \beta \Lambda_2,$$

which is increasing in β .

QED

The following lemma establishes that working with the sequence $z^n \equiv \{z_t\}_{t=1}^n$ is equivalent to working with $p^n \equiv \{p_t\}_{t=1}^n$:

Lemma 3. *In any linear equilibrium the sequence of informational additions z^n is observationally equivalent to p^n .*

Proof. Consider a candidate equilibrium in linear strategies $x_{in} = a_n s_{in} - \varphi_n(p^n)$. In the first period imposing market clearing yields $\int_0^1 a_1 s_{i1} - \varphi_1(p_1) di + \theta_1 = a_1 v - \varphi_1(p_1) + \theta_1 = 0$ or, denoting with $z_1 = a_1 v + \theta_1$ the informational content of the first period order-flow, $z_1 = \varphi_1(p_1)$, where $\varphi_1(\cdot)$ is a linear function. Hence, z_1 and p_1 are observationally equivalent. Suppose now that $z^{n-1} = \{z_1, z_2, \dots, z_{n-1}\}$ and $p^{n-1} = \{p_1, p_2, \dots, p_{n-1}\}$ are observationally equivalent and consider the n -th period market clearing condition: $\int_0^1 X_n(s_{in}, p^{n-1}, p_n) di + \theta_n = 0$. Adding and subtracting $\sum_{t=1}^{n-1} \beta^{n-t+1} a_t v$, the latter condition can be rewritten as follows:

$$\sum_{t=1}^n z_t - \varphi_n(p^n) = 0,$$

where $\varphi_n(\cdot)$ is a linear function, $z_t = \Delta a_t v + u_t$ denotes the informational content of the t -th period order-flow, and $\Delta a_t = a_t - \beta a_{t-1}$. As by assumption p^{n-1} and z^{n-1} are observationally equivalent, it follows that observing p_n is equivalent to observing z_n . $\square$

PROOF OF PROPOSITION 2

We provide this proof assuming that residual uncertainty affects the final liquidation value ($\tau_\delta < \infty$). As the reader can easily verify our argument goes through also when we set $\tau_\delta \rightarrow \infty$. To prove our argument, we proceed by backwards induction. In the last trading period traders act as in a static model and owing to CARA and normality we have

$$X_3(s_{i3}, z^3) = \gamma \frac{E_{i3}[v] - p_3}{\text{Var}_{i3}[v + \delta]}, \quad (\text{A.7})$$

and

$$p_3 = \alpha_{P_3} \left(v + \frac{\theta_3}{a_3} \right) + (1 - \alpha_{P_3}) E_3[v], \quad (\text{A.8})$$

where

$$a_3 = \frac{\gamma \tau_\epsilon}{1 + \kappa}, \quad (\text{A.9})$$

$$\alpha_{P_3} = \frac{\tau_\epsilon}{\tau_{i3}}, \quad (\text{A.10})$$

$\kappa = \tau_\delta^{-1} \tau_{i3}$. An alternative way of writing the third period equilibrium price is

$$p_3 = \lambda_3 z_3 + (1 - \lambda_3 \Delta a_3) \hat{p}_2, \quad (\text{A.11})$$

where

$$\lambda_3 = \alpha_{P_3} \frac{1}{a_3} + (1 - \alpha_{P_3}) \frac{\Delta a_3 \tau_u}{\tau_3}, \quad (\text{A.12})$$

captures the price impact of the net informational addition contained in the 3rd period aggregate demand, while

$$\begin{aligned}\hat{p}_2 &= \frac{\alpha_{P_3}\tau_3\beta(\sum_{t=1}^2\beta^{2-t}z_t) + (1 - \alpha_{P_3})a_3\tau_2E_2[v]}{\alpha_{P_3}\tau_3\beta a_2 + (1 - \alpha_{P_3})a_3\tau_2} \\ &= \frac{\gamma\tau_2E_2[v] + \beta(1 + \kappa)(z_2 + \beta z_1)}{\gamma\tau_2 + \beta a_2(1 + \kappa)},\end{aligned}\tag{A.13}$$

$z_n = \Delta a_n v + u_n$, and $\Delta a_n = a_n - \beta a_{n-1}$.

SECOND PERIOD

In the second period owing to CARA and normality, an agent i trades according to

$$X_2(s_{i2}, z^2) = \frac{\gamma(E_{i2}[p_3] - p_2)}{\text{Var}_{i2}[p_3]},\tag{A.14}$$

where

$$E_{i2}[p_3] = \lambda_3\Delta a_3E_{i2}[v] + (1 - \lambda_3\Delta a_3)\hat{p}_2.\tag{A.15}$$

$$\text{Var}_{i2}[p_3] = \lambda_3^2 \left(\frac{\tau_{i3}}{\tau_{i2}\tau_u} \right).\tag{A.16}$$

Replacing (A.15) and (A.16) in (A.14) yields

$$X_2(s_{i2}, z^2) = \frac{\gamma\Delta a_3\tau_{i2}\tau_u}{\lambda_3\tau_{i3}}(E_{i2}[v] - \hat{p}_2) + \frac{\gamma\tau_{i2}\tau_u}{\lambda_3^2\tau_{i3}}(\hat{p}_2 - p_2).$$

Imposing market clearing and identifying equilibrium parameters yields

$$p_2 = \alpha_{P_2} \left(v + \frac{\theta_2}{a_2} \right) + (1 - \alpha_{P_2})E_2[v],\tag{A.17}$$

where

$$\alpha_{P_2} \equiv \alpha_{E_2} \left(1 + \frac{(\beta\rho_2 - 1)\tau_2}{\tau_{i3}} \right)\tag{A.18}$$

$$a_2 = \frac{\gamma\Delta a_3\tau_u\tau_\epsilon}{\lambda_3\tau_{i3}},\tag{A.19}$$

and $\rho_2 \equiv a_2(1 + \kappa)/(\gamma \sum_{t=1}^2 \tau_{\epsilon t})$. Alternatively, in the spirit of what done for the third period analysis

$$p_2 = \lambda_2 z_2 + (1 - \lambda_2\Delta a_2)\hat{p}_1,\tag{A.20}$$

where

$$\lambda_2 = \alpha_{P_2} \frac{1}{a_2} + (1 - \alpha_{P_2}) \frac{\Delta a_2\tau_u}{\tau_2},\tag{A.21}$$

and

$$\hat{p}_1 = \frac{\alpha_{P_2}\tau_2\beta z_1 + (1 - \alpha_{P_2})a_2\tau_1E_1[v]}{\alpha_{P_2}\tau_2\beta a_1 + (1 - \alpha_{P_2})a_2\tau_1}.\tag{A.22}$$

Finally, note that using (A.18) and (A.19) and rearranging the expression for the second period strategy yields

$$X_2(s_{i2}, z^2) = \frac{a_2}{\alpha_{E_2}}(E_{i2}[v] - p_2) + \frac{\alpha_{P_2} - \alpha_{E_2}}{\alpha_{E_2}} \frac{a_2}{\alpha_{P_2}}(p_2 - E_2[v]).$$

FIRST PERIOD

To compute the first period equilibrium we can use the results of the second period analysis. Indeed, $X_1(s_{i1}, z_1) = (\gamma/\text{Var}_{i1}[p_2])(E_{i1}[p_2] - p_1)$ and using (A.20)

$$\begin{aligned} E_{i1}[p_2] &= \lambda_2 \Delta a_2 E_{i1}[v] + (1 - \lambda_2 \Delta a_2) \hat{p}_1 \\ \text{Var}_{i1}[p_2] &= \lambda_2^2 \left(\frac{\tau_{i2}}{\tau_{i1} \tau_u} \right), \end{aligned}$$

The above expressions highlight the recursive structure of the problem and imply

$$X_1(s_{i1}, z_1) = \frac{\gamma \Delta a_2 \tau_{i1} \tau_u}{\lambda_2 \tau_{i2}} (E_{i1}[v] - \hat{p}_1) + \frac{\gamma \tau_{i1} \tau_u}{\lambda_2^2 \tau_{i2}} (\hat{p}_1 - p_1),$$

and

$$p_1 = \alpha_{P_1} \left(v + \frac{\theta_1}{a_1} \right) + (1 - \alpha_{P_1}) E_1[v], \quad (\text{A.23})$$

where

$$\alpha_{P_1} = \alpha_{E_1} \left(1 + \tau_1 \left(\frac{\gamma \alpha_{P_2} (\beta \rho_1 - \rho_2)}{a_2} + \frac{(\beta \rho_2 - 1)}{\tau_{i3}} \right) \right), \quad (\text{A.24})$$

$$a_1 = \frac{\gamma \Delta a_2 \tau_u \tau_\epsilon}{\lambda_2 \tau_{i2}}, \quad (\text{A.25})$$

and $\rho_1 \equiv a_1(1 + \kappa)/(\gamma \tau_{\epsilon_1})$. In the first period too we can express the price as

$$p_1 = \lambda_1 z_1 + (1 - \lambda_1 a_1) \bar{v},$$

with

$$\lambda_1 = \alpha_{P_1} \frac{1}{a_1} + (1 - \alpha_{P_1}) \frac{a_1 \tau_u}{\tau_1},$$

and, using (A.24), obtain that

$$X_1(s_{i1}, z_1) = \frac{a_1}{\alpha_{E_1}} (E_{i1}[v] - p_1) + \frac{\alpha_{P_1} - \alpha_{E_1}}{\alpha_{E_1}} \frac{a_1}{\alpha_{P_1}} (p_1 - E_1[v]). \quad (\text{A.26})$$

QED

PROOF OF COROLLARY 2

The first part of the Corollary is proved in the paper. For the second part, we show how to obtain the expression for Λ_2 , the argument for Λ_1 being similar. Imposing market clearing on (A.7) yields

$$p_3 = \bar{E}_3[v] + \frac{\text{Var}_{i3}[v + \delta]}{\gamma} \theta_3,$$

where $\bar{E}_3[v] \equiv \int_0^1 E_{i3}[v] di$. Similarly, due to short term horizons, the second period price is given by

$$\begin{aligned} p_2 &= \bar{E}_2[p_3] + \frac{\text{Var}_{i2}[p_3]}{\gamma} \theta_2 \\ &= \bar{E}_2[\bar{E}_3[v]] + \frac{\text{Var}_{i3}[v + \delta]}{\gamma} \beta \bar{E}_2[\theta_2] + \frac{\text{Var}_{i2}[p_3]}{\gamma} \theta_2 \\ &= \alpha_{P_2} v + (1 - \alpha_{P_2}) E_2[v] + \left(\beta \frac{\text{Var}_{i3}[v + \delta]}{\gamma} + \frac{\text{Var}_{i2}[p_3]}{\gamma} \right) \theta_2, \end{aligned} \quad (\text{A.27})$$

where

$$\alpha_{P_2} = \bar{\alpha}_{E_2} + \frac{\text{Var}_{i3}[v + \delta]}{\gamma} \beta a_2 (1 - \alpha_{E_2}),$$

and $\bar{\alpha}_{E_2}$ is given by:

$$\bar{\alpha}_{E_2} = \alpha_{E_2} \left(1 - \frac{\tau_2}{\tau_3} (1 - \alpha_{E_3}) \right).$$

Now, we know that at a linear equilibrium

$$p_2 = \alpha_{P_2} \left(v + \frac{\theta_2}{a_2} \right) + (1 - \alpha_{P_2}) E_2[v]. \quad (\text{A.28})$$

Comparing (5.3) and (A.28), we then see that an alternative expression for a_2 is the following:

$$a_2 = \gamma \frac{\alpha_{P_2}}{\text{Var}_{i2}[p_3] + \beta \text{Var}_{i3}[v + \delta]}.$$

Given that we define the reciprocal of market depth in period 2 as $\Lambda_2 \equiv \alpha_{P_2}/a_2$, from the last equation we can conclude that

$$\Lambda_2 = \frac{\text{Var}_{i2}[p_3] + \beta \text{Var}_{i3}[v + \delta]}{\gamma}. \quad (\text{A.29})$$

QED

PROOF OF COROLLARY 3

To obtain equation (4.5) we start from the equilibrium price equation at time n :

$$p_n = \alpha_{P_n} \left(v + \frac{\theta_n}{a_n} \right) + (1 - \alpha_{P_n}) E_n[v],$$

and expanding the expression for θ_n , we obtain

$$p_n = \frac{\alpha_{P_n}}{a_n} (a_n v + u_n + \beta \theta_{n-1}) + (1 - \alpha_{P_n}) E_n[v].$$

Adding and subtracting $(\alpha_{P_n}/a_n)\beta a_{n-1}v$ at the r.h.s. of the above expression and rearranging

$$p_n = \lambda_n z_n + \frac{\beta \alpha_{P_n}}{a_n} (a_{n-1}v + \theta_{n-1}) + (1 - \alpha_{P_n}) \frac{\tau_{n-1}}{\tau_n} E_{n-1}[v], \quad (\text{A.30})$$

where $z_n \equiv \Delta a_n v + u_n = (a_n - \beta a_{n-1})v + u_n$, and

$$\lambda_n = \alpha_{P_n} \frac{1}{a_n} + (1 - \alpha_{P_n}) \frac{\Delta a_n \tau_u}{\tau_n}.$$

Adding and subtracting $(1 - \lambda_n \Delta a_n) p_{n-1}$ to the r.h.s. of (A.30) yields

$$\begin{aligned} p_n &= \lambda_n z_n + (1 - \lambda_n \Delta a_n) p_{n-1} - (1 - \lambda_n \Delta a_n) \times \\ &\quad \left(\frac{\alpha_{P_{n-1}}}{a_{n-1}} (a_{n-1}v + \theta_{n-1}) + (1 - \alpha_{P_{n-1}}) E_{n-1}[v] \right) + \frac{\beta \alpha_{P_n}}{a_n} (a_{n-1}v + \theta_{n-1}) + (1 - \alpha_{P_n}) \frac{\tau_{n-1}}{\tau_n} E_{n-1}[v] \\ &= \lambda_n z_n + (1 - \lambda_n \Delta a_n) p_{n-1} + \left(\frac{\beta \alpha_{P_n}}{a_n} - (1 - \lambda_n \Delta a_n) \frac{\alpha_{P_{n-1}}}{a_{n-1}} \right) E_{n-1}[\theta_{n-1}]. \end{aligned} \quad (\text{A.31})$$

We now prove that

$$\frac{\beta\alpha_{P_n}}{a_n} - (1 - \lambda_n\Delta a_n)\frac{\alpha_{P_{n-1}}}{a_{n-1}} = \lambda_n\Delta a_n\frac{\alpha_{P_{n-1}} - \alpha_{E_{n-1}}}{a_{n-1}}.$$

Starting from the third period price we have

$$\begin{aligned} p_3 &= \alpha_{P_3} \left(v + \frac{\theta_3}{a_3} \right) + (1 - \alpha_{P_3}) E_3[v] \\ &= \lambda_3 z_3 + (1 - \lambda_3\Delta a_3) p_2 + \left(\frac{\beta\alpha_{P_3}}{a_3} - (1 - \lambda_3\Delta a_3)\frac{\alpha_{P_2}}{a_2} \right) E_2[\theta_2]. \end{aligned} \quad (\text{A.32})$$

Now using the definition for α_{P_3} we obtain

$$\begin{aligned} \frac{\beta\alpha_{P_3}}{a_3} - (1 - \lambda_3\Delta a_3)\frac{\alpha_{P_2}}{a_2} &= \frac{\beta(1 + \kappa)}{\gamma\tau_{i3}} - \frac{\gamma\tau_2 + \beta(1 + \kappa)a_2}{\gamma\tau_{i3}} \frac{\alpha_{P_2}}{a_2} \\ &= \frac{\beta(1 + \kappa)a_2 + \gamma\tau_2}{\gamma\tau_{i3}} \left(\frac{\beta(1 + \kappa)}{\beta(1 + \kappa)a_2 + \gamma\tau_2} - \frac{\alpha_{P_2}}{a_2} \right) \\ &= \frac{(1 + \kappa)(\beta(1 + \kappa)a_2 + \gamma\tau_2)}{\gamma\tau_{i3}} \left(\frac{\beta}{\beta(1 + \kappa)a_2 + \gamma\tau_2} - \frac{1}{\gamma\rho_2\tau_{i2}} \left(1 + \frac{(\beta\rho_2 - 1)\tau_2}{\tau_{i3}} \right) \right) \\ &= \frac{1 + \kappa}{\gamma^2\rho_2\tau_{i2}\tau_{i3}} \left(\gamma\tau_1(\beta\rho_2 - 1) - (\beta(1 + \kappa)a_2 + \gamma\tau_2) \frac{(\beta\rho_2 - 1)\tau_2}{\tau_{i3}} \right) \\ &= \frac{(1 + \kappa)(\beta\rho_2 - 1)\tau_2}{\gamma^2\rho_2\tau_{i2}\tau_{i3}^2} (\gamma\tau_{i3} - \beta(1 + \kappa)a_2 - \gamma\tau_2) \\ &= \frac{(1 + \kappa)(\beta\rho_2 - 1)\tau_2}{\gamma\rho_2\tau_{i2}\tau_{i3}} \lambda_3\Delta a_3 \\ &= \lambda_3\Delta a_3 \frac{\alpha_{P_2} - \alpha_{E_2}}{a_2}, \end{aligned}$$

where we use the definition of α_{P_2} to move from the second to the third row of the above expression. Summarizing, using the above result the second period price can be expressed as follows:

$$p_3 = \lambda_3 z_3 + (1 - \lambda_3\Delta a_3) p_2 + \lambda_3\Delta a_3 \frac{\alpha_{P_2} - \alpha_{E_2}}{a_2} E_2[\theta_2]. \quad (\text{A.33})$$

Going back one period, we need to prove that

$$\frac{\beta\alpha_{P_2}}{a_2} - (1 - \lambda_2\Delta a_2)\frac{\alpha_{P_1}}{a_1} = \lambda_2\Delta a_2\frac{\alpha_{P_1} - \alpha_{E_1}}{a_1}. \quad (\text{A.34})$$

To prove this, we start by noting that

$$\frac{\beta\alpha_{P_2}}{a_2} - (1 - \lambda_2\Delta a_2)\frac{\alpha_{P_1}}{a_1} = \frac{1}{a_1} \left(\alpha_{P_1}\lambda_2\Delta a_2 + \frac{\beta\alpha_{P_2}}{a_2}a_1 - \alpha_{P_1} \right),$$

which in turn implies that (A.34) is correct if and only if

$$\alpha_{P_1} = \alpha_{E_1}\lambda_2\Delta a_2 + \frac{\beta\alpha_{P_2}}{a_2}a_1. \quad (\text{A.35})$$

To prove that the above expression is correct, we manipulate the definition of α_{P_1} to obtain:

$$\begin{aligned}
\alpha_{P_1} &= \alpha_{E_1} \left(1 + \frac{\gamma\tau_1}{1+\kappa} \left(\frac{\alpha_{P_2}(\beta\rho_2 - \rho_1)}{a_2} + \frac{\alpha_{P_3}(\beta\rho_2 - 1)}{a_3} \right) \right) \\
&= \alpha_{E_1} \left(1 + \tau_1 \left(\frac{\alpha_{P_2}}{a_2} \left(\frac{\beta a_1}{\tau_{\epsilon_1}} - \frac{a_2}{\sum_{t=1}^2 \tau_{\epsilon_t}} \right) + \frac{\beta\rho_2 - 1}{\tau_{i3}} \right) \right) \\
&= \frac{\alpha_{P_2}}{a_2} \beta a_1 + \alpha_{E_1} \left(1 - \frac{\alpha_{P_2}}{a_2} \beta a_1 + \left(\frac{\beta\rho_1 - 1}{\tau_{i3}} - \frac{\alpha_{P_2}}{\sum_{t=1}^2 \tau_{\epsilon_t}} \right) \right). \tag{A.36}
\end{aligned}$$

Thus, to prove our claim we need to show that

$$1 - \frac{\alpha_{P_2}}{a_2} \beta a_1 + \left(\frac{\beta\rho_1 - 1}{\tau_{i3}} - \frac{\alpha_{P_2}}{\sum_{t=1}^2 \tau_{\epsilon_t}} \right) = \lambda_2 \Delta a_2. \tag{A.37}$$

According to our definition of α_{P_2} we have

$$\alpha_{P_2} = \alpha_{E_2} \left(1 + \frac{(\beta\rho_2 - 1)\tau_2}{\tau_{i3}} \right),$$

which in turn implies that

$$1 - \alpha_{P_2} = (1 - \alpha_{E_2}) \left(1 - \frac{\sum_{t=1}^2 \tau_{\epsilon_t}}{\tau_{i3}} (\beta\rho_2 - 1) \right). \tag{A.38}$$

Rearranging (A.37) yields

$$\begin{aligned}
1 - \frac{\alpha_{P_2}}{a_2} \beta a_1 + \left(\frac{\beta\rho_1 - 1}{\tau_{i3}} - \frac{\alpha_{P_2}}{\sum_{t=1}^2 \tau_{\epsilon_t}} \right) &= \frac{\alpha_{P_2}}{a_2} \Delta a_2 + (1 - \alpha_{P_2}) - \frac{\tau_1}{\sum_{t=1}^2 \tau_{\epsilon_t}} \left(\alpha_{P_2} - \frac{\sum_{t=1}^2 \tau_{\epsilon_t} (\beta\rho_2 - 1)}{\tau_{i3}} \right) \\
&= \frac{\alpha_{P_2}}{a_2} \Delta a_2 + (1 - \alpha_{P_2}) - \frac{\tau_1}{\tau_2} (1 - \alpha_{P_2}) \\
&= \lambda_2 \Delta a_2, \tag{A.39}
\end{aligned}$$

where we use (A.38) to simplify the first row of the above expression. This allows us to write

$$p_2 = \lambda_2 z_2 + (1 - \lambda_2 \Delta a_2) p_1 + \lambda_2 \Delta a_2 \frac{\alpha_{P_1} - \alpha_{E_1}}{a_1} E_1[\theta_1], \tag{A.40}$$

and completes our proof. QED

PROOF OF PROPOSITION 4

Rearranging (4.5) yields

$$\begin{aligned}
p_{n+1} - p_n &= \lambda_{n+1} \left(z_{n+1} + \Delta a_{n+1} \frac{\alpha_{P_n} - \alpha_{E_n}}{a_n} E_n[\theta_n] - \Delta a_{n+1} p_n \right) \\
&= \lambda_{n+1} \Delta a_{n+1} \left((1 - \alpha_{E_n})(v - E_n[v]) - \frac{\alpha_{E_n}}{a_n} \theta_n \right) + \lambda_{n+1} u_{n+1},
\end{aligned}$$

and recalling that

$$a_n = \gamma \frac{\lambda_{n+1} \Delta a_{n+1} \alpha_{E_n}}{\text{Var}_{in}[p_{n+1}]},$$

we can see that from the point of view of period n investors

$$\begin{aligned} E_n[p_{n+1} - p_n] &= -\frac{\lambda_{n+1}\Delta a_{n+1}\alpha_{E_n}}{a_n} E_n[\theta_n] \\ &= -\frac{\text{Var}_{in}[p_{n+1}]}{\gamma} E_n[\theta_n]. \end{aligned} \quad (\text{A.41})$$

QED

PROOF OF COROLLARY 5

The expression for rational investors' strategies are obtained in the proof of Proposition 2, whereas those for α_{P_2} and α_{P_1} follow immediately after rearranging (A.18) and (A.24). The recursive equation defining the couple a_1, a_2 follows from (A.19) and (A.25).

QED

PROOF OF PROPOSITION 3

For any $\beta \in [0, 1]$, in the second period an equilibrium must satisfy $a_2 = \gamma\tau_\epsilon$. In the first period an equilibrium must satisfy

$$\begin{aligned} \phi_1(a_1, a_2) &\equiv a_1\lambda_2(\tau_2 + \tau_\epsilon) - \gamma\tau_\epsilon\Delta a_2\tau_u \\ &= a_1(1 + \gamma\tau_u\Delta a_2) - \gamma^2\tau_\epsilon\Delta a_2\tau_u = 0. \end{aligned} \quad (\text{A.42})$$

The above equation is a quadratic in a_1 which for any $a_2 > 0$ and $\beta > 0$ possesses two positive, real solutions:

$$a_1^* = \frac{1 + \gamma\tau_u(a_2 + \beta\gamma\tau_\epsilon) - \sqrt{1 + \gamma\tau_u(2(a_2 + \beta\gamma\tau_\epsilon) + \gamma\tau_u(a_2 - \beta\gamma\tau_\epsilon)^2)}}{2\beta\gamma\tau_u} \quad (\text{A.43})$$

$$a_1^{**} = \frac{1 + \gamma\tau_u(a_2 + \beta\gamma\tau_\epsilon) + \sqrt{1 + \gamma\tau_u(2(a_2 + \beta\gamma\tau_\epsilon) + \gamma\tau_u(a_2 - \beta\gamma\tau_\epsilon)^2)}}{2\beta\gamma\tau_u}, \quad (\text{A.44})$$

with $a_1^{**} > a_1^*$. This proves that for $\beta > 0$ there are two linear equilibria.

Inspection of the above expressions for a_1 shows that $\beta a_1^* < a_2$, while $\beta a_1^{**} > a_2$. This implies that $\beta\rho_1 > 1$ for $a_1 = a_1^{**}$ and $\beta\rho_1 < 1$ otherwise. Thus, using (4.9), we obtain $\alpha_{P_1}(a_1^*, a_2) < \alpha_{E_1}(a_1^*, a_2)$, and $\alpha_{P_1}(a_1^{**}, a_2) > \alpha_{E_1}(a_1^{**}, a_2)$. The result for second period illiquidity follows from substituting (A.43) and (A.44) in λ_2 . To see that prices are more informative along the low illiquidity equilibrium note that in the first period $\text{Var}[v|z_1]^{-1} = \tau_1 = \tau_v + a_1^2\tau_u$. In the second period, the price along the low illiquidity equilibrium is more informative than along the high illiquidity equilibrium if and only if

$$\frac{(1 + \beta^2 + \gamma a_2\tau_u((1 - \beta^2) + \beta(1 + \beta^2)))\sqrt{(1 + \gamma a_2\tau_u(1 + \beta))^2 - 4\beta(\gamma a_2\tau_u)^2}}{\gamma^2\beta^2\tau_u} > 0,$$

which is always true.

To see that with $\beta = 0$, a unique equilibrium arises, note that

$$\lim_{\beta \rightarrow 0} \frac{1 + \gamma\tau_u(a_2 + \beta\gamma\tau_\epsilon) + \sqrt{1 + \gamma\tau_u(2(a_2 + \beta\gamma\tau_\epsilon) + \gamma\tau_u(a_2 - \beta\gamma\tau_\epsilon)^2)}}{2\beta\gamma\tau_u} = \infty,$$

while, using l'Hospital's rule,

$$\lim_{\beta \rightarrow 0} \frac{1 + \gamma\tau_u(a_2 + \beta\gamma\tau_\epsilon) - \sqrt{1 + \gamma\tau_u(2(a_2 + \beta\gamma\tau_\epsilon) + \gamma\tau_u(a_2 - \beta\gamma\tau_\epsilon)^2)}}{2\beta\gamma\tau_u} = \frac{\gamma a_2^2 \tau_u}{1 + \gamma a_2 \tau_u}.$$

From (4.9) it then follows that in this case $\alpha_{P_1} < \alpha_{E_1}$. Finally, taking the limit of λ_2 as $\beta \rightarrow 0$ when $a_1 = a_1^*$ yields

$$\lim_{\beta \rightarrow 0} \lambda_2(a_1^*, a_2) = \frac{1 + \gamma\tau_u a_2}{\gamma(\tau_v + (a_1^*)^2 \tau_u + a_2^2 \tau_u + \tau_\epsilon)} > 0,$$

whereas $\lim_{\beta \rightarrow 0} \lambda_2(a_1^{**}, a_2) = 0$.

QED

PROOF OF LEMMA 2

Note that at any linear equilibrium $E_n[\theta_n] = a_n(v - E_n[v]) + \theta_n$. This, in turn, allows us to express the equilibrium price as follows:

$$p_n = \bar{E}_n[v] + \frac{\alpha_{P_n} - \alpha_{E_n}}{a_n} E_n[\theta_n] + \frac{\alpha_{E_n}}{a_n} \theta_n.$$

As a consequence

$$p_n - v = \bar{E}_n[v] - v + \frac{\alpha_{P_n} - \alpha_{E_n}}{a_n} E_n[\theta_n] + \frac{\alpha_{E_n}}{a_n} \theta_n.$$

Thus, if $\alpha_{P_n} > \alpha_{E_n}$ the price is closer to the fundamentals compared the consensus opinion, while the opposite occurs whenever $\alpha_{P_n} < \alpha_{E_n}$.

We now prove that at equilibrium (5.11) and (5.12) are equivalent. To see this, note that using (4.1) the covariance between p_n and v is given by

$$\text{Cov}[v, p_n] = \alpha_{P_n} \frac{1}{\tau_v} + (1 - \alpha_{P_n}) \left(\frac{1}{\tau_v} - \frac{1}{\tau_n} \right), \quad (\text{A.45})$$

and carrying out a similar computation for the time n consensus opinion

$$\text{Cov}[\bar{E}_n[v], v] = \alpha_{E_n} \frac{1}{\tau_v} + (1 - \alpha_{E_n}) \left(\frac{1}{\tau_v} - \frac{1}{\tau_n} \right), \quad (\text{A.46})$$

where $\tau_n \equiv \text{Var}[v|p^n] = \tau_v + \tau_u \sum_{t=1}^n \Delta a_t^2$. We can now subtract (A.46) from (A.45) and obtain

$$\text{Cov}[p_n - \bar{E}_n[v], v] = \frac{\alpha_{P_n} - \alpha_{E_n}}{\tau_n}, \quad (\text{A.47})$$

implying that the price at time n over relies on public information if and only if the covariance between the price and the fundamentals falls short of that between the consensus opinion and the fundamentals.

QED

PROOF OF COROLLARY 9

For any $1/\tau_\delta > 0$, $\beta \in [0, 1]$, in the second period an equilibrium must satisfy $a_2 = (1 + \kappa)^{-1}\gamma\tau_\epsilon$. Rearranging the latter condition we obtain the following cubic in a_2 :

$$\phi_2(a_1, a_2) \equiv a_2^3\tau_u - 2a_2^2a_1\beta\tau_u + a_2(\tau_1 + \tau_\delta + (\beta a_1)^2\tau_u + \tau_\epsilon) - \gamma\tau_\delta\tau_\epsilon = 0. \quad (\text{A.48})$$

Differentiating (A.48) with respect to a_2 yields

$$\frac{\partial\phi_2(a_1, a_2)}{\partial a_2} = 3a_2^2\tau_u - 4a_2a_1\beta\tau_u + \tau_1 + \tau_\delta + (\beta a_1)^2\tau_u + \tau_\epsilon,$$

a quadratic whose discriminant is given by

$$\nabla = 4\tau_u(a_1^2\tau_u(\beta^2 - 1) - 3\tau_u(\tau_v + \tau_\delta + \tau_\epsilon) - 2a_1^2\tau_u),$$

which is always negative for all $\beta < 1$. This implies that for any a_1 ,

$$\frac{\partial\phi_2(a_1, a_2)}{\partial a_2} > 0, \quad (\text{A.49})$$

and thus that for any a_1 , there always exists a unique real solution $a_2^* \in (0, \gamma\tau_\epsilon)$ to (A.48) which defines the second period equilibrium signal responsiveness. In the first period an equilibrium must satisfy

$$\begin{aligned} \phi_1(a_1, a_2) &\equiv a_1\lambda_2(\tau_2 + \tau_\epsilon) - \gamma\tau_\epsilon\Delta a_2\tau_u \\ &= a_1(1 + \gamma\tau_u\Delta a_2) - \gamma^2\tau_\epsilon\Delta a_2\tau_u = 0. \end{aligned} \quad (\text{A.50})$$

The above equation is a quadratic in a_1 which for any $a_2 > 0$ and $\beta > 0$ possesses two positive, real solutions:

$$a_1^* = \frac{1 + \gamma\tau_u(a_2 + \beta\gamma\tau_\epsilon) - \sqrt{1 + \gamma\tau_u(2(a_2 + \beta\gamma\tau_\epsilon) + \gamma\tau_u(a_2 - \beta\gamma\tau_\epsilon)^2)}}{2\beta\gamma\tau_u} \quad (\text{A.51})$$

$$a_1^{**} = \frac{1 + \gamma\tau_u(a_2 + \beta\gamma\tau_\epsilon) + \sqrt{1 + \gamma\tau_u(2(a_2 + \beta\gamma\tau_\epsilon) + \gamma\tau_u(a_2 - \beta\gamma\tau_\epsilon)^2)}}{2\beta\gamma\tau_u}, \quad (\text{A.52})$$

with $a_1^{**} > a_1^*$. This proves that for $\beta > 0$ there are two linear equilibria.

From implicit differentiation we have

$$\frac{\partial a_2}{\partial a_1} = \frac{\partial\phi_2(a_1, a_2)/\partial a_2}{\Delta a_2}.$$

According to (A.49), the numerator in the above expression is positive, and since using (A.51) and (A.52) $a_2^* - \beta a_1^* > 0$, while $a_2^{**} - \beta a_1^{**} < 0$, we can conclude that $a_2^{**} < a_2^*$. As argued above $\beta a_1^* < a_2^*$, while $\beta a_1^{**} > a_2^{**}$. This implies that $\beta\rho_1 > 1$ for $a_1 = a_1^{**}$ and $\beta\rho_1 < 1$ otherwise. Thus, using (4.9), we obtain $\alpha_{P_1}(a_1^*, a_2^*) < \alpha_{E_1}(a_1^*, a_2^*)$, and $\alpha_{P_1}(a_1^{**}, a_2^{**}) > \alpha_{E_1}(a_1^{**}, a_2^{**})$.

The result for second period illiquidity follows immediately from the fact that $\Lambda_2 = \text{Var}_{i2}[v + \delta]/\gamma = (\tau_{i2}^{-1} + \tau_\delta^{-1})/\gamma$, where $\tau_{i2} = \tau_v + \tau_u(a_1^2 + (\Delta a_2)^2) + \tau_\epsilon$.

To see that with $\beta = 0$, a unique equilibrium arises, note that

$$\lim_{\beta \rightarrow 0} \frac{1 + \gamma\tau_u(a_2 + \beta\gamma\tau_\epsilon) + \sqrt{1 + \gamma\tau_u(2(a_2 + \beta\gamma\tau_\epsilon) + \gamma\tau_u(a_2 - \beta\gamma\tau_\epsilon)^2)}}{2\beta\gamma\tau_u} = \infty,$$

while, given that $a_2 < \gamma\tau_\epsilon$, and using l'Hospital's rule

$$\lim_{\beta \rightarrow 0} \frac{1 + \gamma\tau_u(a_2 + \beta\gamma\tau_\epsilon) - \sqrt{1 + \gamma\tau_u(2(a_2 + \beta\gamma\tau_\epsilon) + \gamma\tau_u(a_2 - \beta\gamma\tau_\epsilon)^2)}}{2\beta\gamma\tau_u} = \frac{\gamma a_2^2 \tau_u}{1 + \gamma a_2 \tau_u}.$$

From (4.9) it then follows that in this case $\alpha_{P_1} < \alpha_{E_1}$.

We now check what happens to $\rho_1 = a_1(\tau_\delta + \tau_2 + \tau_\epsilon)/(\gamma\tau_\delta\tau_\epsilon)$ as $\tau_\delta \rightarrow 0$. Notice that

$$\begin{aligned} \lim_{\tau_\delta \rightarrow 0} a_1 &= \lim_{\tau_\delta \rightarrow 0} \frac{\gamma^2 \tau_\delta \tau_\epsilon \tau_u \Delta a_2 (\tau_2 + \tau_\epsilon)}{(\tau_2 + \tau_\epsilon) (\tau_\delta (1 + \gamma \Delta a_2 \tau_u) + (\tau_2 + \tau_\epsilon))} \\ &= 0, \end{aligned}$$

and $\lim_{\tau_\delta \rightarrow 0} a_2 = \lim_{\tau_\delta \rightarrow 0} \gamma\tau_\delta\tau_\epsilon/(\tau_\delta + (\tau_2 + \tau_\epsilon)) = 0$. Then,

$$\begin{aligned} \lim_{\tau_\delta \rightarrow 0} \rho_1 &= \lim_{\tau_\delta \rightarrow 0} \frac{\gamma\tau_u \Delta a_2 (\tau_2 + \tau_\epsilon)}{\tau_2 + \tau_\epsilon} \\ &= 0, \end{aligned}$$

since, as argued above, in the limit $a_1 = a_2 = 0$. By continuity of ρ_1 as a function of (a_1, a_2) , there must then exist an interval $(0, \tau_\delta^*)$, such that for all $\tau_\delta \in (0, \tau_\delta^*)$ we have $\rho_1 < 1$. As $a_n \rightarrow 0$, second period illiquidity coincides with the one of the market without private information, and since $\text{Var}[v + \delta | s_{i2}, p^2] \leq \text{Var}[v + \delta]$, the result follows. QED