# Optimal Government Spending at the Zero Bound: Nonlinear and Non-Ricardian Analysis

Taisuke Nakata*

New York University

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**Abstract**

This paper characterizes optimal government spending when monetary policy is constrained by the zero lower bound under a variety of assumptions about a set of fiscal instruments available to finance government spending. The private sector of the model is given by a standard New Keynesian model. In response to a large and persistent time-preference shock, government chooses a sequence of nominal interest rate and government spending, which can be financed by either lump-sum tax, a mix of labor income tax and debt, or a mix of consumption tax and debt.

There are four main findings. First, optimal government spending policy is characterized by an initial expansion followed by a sharp reduction during the period of zero nominal interest rates. Second, optimal dynamics of debt and primary balance depend on the available distortionary tax and the initial level of debt. Third, welfare gain of having government spending as an additional policy instrument depends importantly on the available distortionary tax, but is generally much smaller than welfare gain of having debt instrument or distortionary tax. Finally, welfare gains of various fiscal instruments are larger in the economy with larger initial debt.

*taisuke.nakata@nyu.edu

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# 1 Introduction

The recent recession led governments across the world to provide large fiscal stimulus. This has generated a renewed interest in the effect of fiscal policy among economists and policymakers. In particular, some authors have examined the effect of government spending, and have noted that government spending is effective in raising output and consumption when nominal interest rate is constrained at the zero lower bound[2]. While these studies consider the effect of exogenous government spending shock, other studies have investigated optimal government spending policy at the zero nominal interest rate and found that, not only can government spending increase output and consumption, it can also increase welfare at the zero bound. However, normative analyses on government spending have thus far focused on the economy in which lump-sum tax is available to finance government spending[3]. It would be useful to consider other financing schemes involving distortionary taxes and debt.

Accordingly, this paper characterizes optimal government spending at the zero nominal interest rate under a variety of assumptions about a set of fiscal instruments available to finance government spending. Private sector of the model is given by a standard New Keynesian model. In response to a discount factor shock, government chooses and commits itself to a sequence of nominal interest rate and government spending, which is financed by either lump-sum tax, labor income tax/debt, and consumption tax/debt. While this paper studies the model in which government can commit, my ongoing research studies the model without commitment.

Following Levin et al. (2010), I consider a large and persistent discount factor shock intended to capture the severity of the Great Recession. In the same New Keynesian model in which government can commit, but nominal interest rate is the only available policy instrument, they find a large decline in output at the zero bound when the discount factor shock is large and persistent, and suggest potential roles for additional policy instruments. This paper studies the potential role of government spending under the same environment. I document how government can improve allocations if it can optimally choose government spending in addition to nominal interest rate, and evaluate welfare gain from having government spending as an additional policy instrument using the representative household's utility as the welfare measure.

Since the focus of the paper is on fiscal policy, I do not assume that government provides lump-sum-tax-financed labor-income subsidy to the representative household to offset the distortion arising from monopolistic competition, which is often assumed in the analysis of optimal fiscal and monetary policy using the New Keynesian model. Thus, the steady-state of the economy is not efficient. For this reason, and also because policy functions are not differentiable due to the zero-lower bound constraint, accurate evaluation of optimal policy and welfare requires nonlinear solution method. I apply a modified Newton method described in Julliard et al. (1998) to solve the model in its original nonlinear form. This turns out to matter quantitatively in the welfare

[2]Christiano, Eichenbaum, and Rebelo (2010), Erceg and Linde (2010), Eggertsson (2010a), and Uhlig (2011) to name a few.

[3]Examples are Eggertsson (2001), Christiano et al. (2010), and Woodford (2010). Eggertsson (2010b) introduces a quadratic adjustment cost of collecting lump-sum taxes to break the Ricardian-Equivalence.

evaluation.

There are four main findings. First, optimal government spending policy is characterized by an initial expansion followed by a sharp reduction to the level below the steady-state. This reduction occurs while the nominal interest rate is still bounded at zero. Government spending eventually rises to a new steady-state level as the zero nominal interest rate ceases to bind. An increase in government spending is desirable as it leads to the increased demand for labor by the firms, which results in an increase in the real wage and inflation. While this is true in all models, the variation in government spending is much smaller in the model with consumption tax and debt than in the other two models, and reflects a concern to align marginal utility of government spending with those of consumption and leisure.

Second, optimal dynamics of debt and primary balance[4] depends on the available distortionary tax and the initial level of debt. In the model with labor income tax and debt, primary balance decreases by a small amount at time one, and spends a few quarters at a level below its terminal steady state level. Nevertheless, due to lower interest expenses, government can reduce the real value of debt. In the model with consumption tax and debt, primary balance drops by a large amount at time one, and gradually increases thereafter. Accordingly, the real value of debt increases at time one and stays above the initial level for several quarters. In the long-run, debt converges to a level smaller than the initial level if the initial debt is positive, regardless of which distortionary tax is available. If government initially has asset (i.e., negative debt), then the asset converges to a smaller level. This long-run dynamics can be explained by the fact that the nominal interest rate stays below its steady-state level for a very long period after the discount factor shock hits.

Third, welfare gain of having government spending as an additional policy instrument is small in absolute terms, and is also small in relative terms compared to welfare gains of distortionary tax and debt instruments. Welfare gain from any of these fiscal instruments is small relative to welfare gain of moving from an economy in which all policy variables are set according to simple rules to an economy in which government can optimally choose nominal interest rate, but cannot optimally choose fiscal instruments. That is, given that nominal interest is chosen optimally, there are not much more government spending can do to improve allocations. However, I should note that this result is specific to the commitment case, and a preliminary analysis in my ongoing research suggests that this result is overturned in the model without commitment.

Finally, welfare gains of various fiscal instruments are larger in the economy with higher initial debt or asset. For example, in the model with labor income tax and debt, welfare gains of government spending, labor income tax, and debt instruments are 0.2, 0.5, and 0.7 percent in the economy with initial debt-to-annualized output ratio of 50 percent, but they are 0.5, 1.7, and 2.9 percent in the economy with initial debt-to-annualized output ratio of 100 percent, and 0.2, 1.8, and 2.5 percent in the economy with initial asset-to-annualized output ratio of 100 percent. Similar results obtain in the model with consumption tax and debt. The reason is as follows. Regardless of the initial debt level, optimal monetary policy is to set nominal interest rate below its steady-state level for a long period. With positive initial debt or asset, such changes in nominal interest

[4]Primary balance is defined as tax-revenue minus government spending.

rates requires adjustment in fiscal instruments in order to keep government budget constraint in equality. Thus, fiscal instruments have a dual role of mitigating the effect of discount factor shock and maintaining government budget constraint in response to sudden changes in nominal interest rates. This second task is large in the economy with larger initial debt and asset because the same change in nominal interest rates leads to larger changes in the interest expenses or receipts. Under such environment, restricting one of fiscal instruments increases the burden for the rest of fiscal instruments, and results in a large welfare loss.

### 1.1 Relation to the Literature

There are several papers closely related. Let me summarize them, and explain how this paper differs from them.

**Relation to Eggertsson and Woodford (2004) and Eggertsson (2001, 2006, 2010a, 2010b):** Eggertsson and Woodford (2004) study optimal fiscal and monetary policy with commitment at the zero nominal interest rate. They study optimal path of labor income tax and debt, and also consumption tax and debt, but they do so while holding government spending constant. Eggertsson (2001) studies optimal government spending financed by lump-sum tax, and my results on the model with lump-sum tax essentially reproduce his findings. Eggertsson (2006 and 2010b) solves for optimal debt, government spending, and nominal interest rate. He breaks Ricardian equivalence by introducing quadratic tax-collection costs to the lump-sum tax, and does not consider distortionary taxes. Eggertsson (2010a) studies the effect of government spending, labor income tax, and consumption tax at the zero bound, one at a time, assuming that there is lump-sum tax to maintain budget balance and that monetary policy is not optimal. Overall, two papers by Eggertsson (2006 and 2010b) are the closest to this paper. My paper differs from them in two ways. First, it breaks Ricardian Equivalence by introducing distortionary taxes, instead of tax-collection cost on lump-sum tax. Second, it focuses on the model with commitment. The focus of Eggertsson (2006 and 2010b) is on the government's ability to use debt to overcome the lack of commitment.

**Relation to Woodford (2010) and Christiano et al. (2010):** While their analyses are mainly on the government spending multiplier, they do consider whether increasing government spending during the zero-bound period enhances welfare. They both analytically derive optimal increase in government spending during the zero bound period. However, they do so by assuming that (i) central bank follows a truncated Taylor rule, (ii) government spending is set to the steady-state level soon after the zero-bound period, and (iii) government spending is financed through lump-sum tax. I will not assume any of them. In particular, nominal interest rate is chosen optimally jointly with fiscal instruments in this paper.

**Relation to Correia et al. (2010):** Correia et al. (2010) show that, once we allow for the simultaneous use of labor income tax, consumption tax, and lump-sum tax, the efficient allocation can be attained even at the zero bound, and suggest that there is no role for government spending. In arriving this result, it is crucial that the government have access to all of the three taxes. This paper finds that, even if the government does not have access to all of the three taxes, welfare gain of using the government spending tends to be small.

### 1.2 Organization of the Paper

Section 2 describes the model and defines the equilibrium. Section 3 formulates government's problem, discusses the steady-state of the economy, and defines the welfare measures. Section 4 discusses calibration and solution method. Section 5 presents main results. Section 6 discusses additional results. Section 7 concludes. There are three appendices. Appendix A describes non-linear solution method. Appendix B describes piecewise linear solution method. Appendix C lists first-order necessary conditions of the Lagrangian problem associated with government's problem. Tables and figures follow.

## 2 Model

The first few subsection describes the private sector of the model, which is given by the New Keynesian model. The economy starts at t=1. The model has one exogenous variable, discount factor shock, but is deterministic. The price-setting environment is given by the Calvo model. Since the model is widely known, the presentation is kept to minimum. The model differs from the standard model because government spending enters into utility function of the household. I do this in order to have the steady-state level of government spending share to be positive[5].

I will consider three versions of the model presented below. The first version allows only for lump-sum tax, the second version allows for labor income tax and debt, and the third version allows for consumption tax and debt. I will describe the model with all of these fiscal instruments, but it should be understood that a subset of them is set to zero in any version of the model.

### 2.1 Household

There is a representative household who maximizes the utility function.

$$\sum_{t=1}^{\infty}\beta^{t-1}\prod_{s=0}^{t-1}\delta_s\Big[\frac{C_t^{1-\chi_c}}{1-\chi_c}-\chi_{n,0}\frac{N_t^{1+\chi_{n,1}}}{1+\chi_{n,1}}+\chi_{g,0}\frac{G_t^{1-\chi_{g,1}}}{1-\chi_{g,1}}\Big]$$

subject to

$$(1+\tau_{c,t})P_tC_t+R_t^{-1}B_t\leq(1-\tau_{n,t})W_tN_t+B_{t-1}-P_tT_t+\Gamma_t$$

and $B_0$ given. $C_t$ is consumption of final goods, $N_t$ is labor supply, $P_t$ is price of consumption goods, and $W_t$ is nominal wage. $T_t$ is lump-sum tax, $B_t$ is the quantity of risk-free one-period bonds carried over from period t, and paying one unit of money in period t+1. $R_t$ denote the gross nominal return on bonds purchased in period t, and $\Gamma_t$ is profits from the intermediate-good producing firms. $\tau_{n,t}$ is labor income tax, and $\tau_{c,t}$ is consumption tax.

[5] Otherwise, I would need to include an inequality constraint on government consumption.

$\{\delta_t\}_{t=0}^{\infty}$ is the discount factor shock that affects how the representative household values the period utility at time t+1 relative to the period utility at time t. $\{\delta_t\}_{t=1}^{\infty}$ is exogenously given, and satisfies

$$\begin{aligned}
\delta_0 &= 1 \\
\delta_1 &= 1 + \epsilon_{\delta,1} \\
\delta_t &= 1 + \rho_\delta(\delta_{t-1} - 1) \text{ for } t \geq 2
\end{aligned}$$

$\epsilon_{\delta,1}$ is revealed at the beginning of t=1 before agents in the model make decisions. $\delta_0$ multiplies all period utilities, and is normalized to one.

The Lagrangian problem is given by

$$\begin{aligned}
L(B_0) &= \sum_{t=1}^{\infty} \beta^{t-1} \prod_{s=0}^{t-1} \delta_s \Big[\frac{C_t^{1-\chi_c}}{1-\chi_c} - \chi_{n,0}\frac{N_t^{1+\chi_{n,1}}}{1+\chi_{n,1}} + \chi_{g,0}\frac{G_t^{1-\chi_{g,1}}}{1-\chi_{g,1}}\Big] \\
&\quad - \sum_{t=1}^{\infty} \beta^{t-1} \prod_{s=0}^{t-1} \delta_s \lambda_t \big[(1+\tau_{c,t})P_t C_t + R_t^{-1} B_t - ((1-\tau_{n,t})W_t N_t + B_{t-1} - P_t T_t + \Gamma_t\big]
\end{aligned}$$

FONCs with respect to $C_t$, $N_t$, and $B_t$ are given by

$$\begin{aligned}
(1+\tau_{c,t})P_t\lambda_t &= \frac{1}{C_t^{\chi_c}} \\
(1-\tau_{n,t})W_t\lambda_t &= \chi_{n,0} N_t^{\chi_{n,0}} \\
\lambda_t &= \beta R_t E_t \delta_t \lambda_{t+1}
\end{aligned}$$

Now, by taking the ratio of the FONCs with respect to consumption and FOC with respect to labor,

$$\frac{(1-\tau_{n,t})W_t}{(1+\tau_{c,t})P_t} = \chi_{n,0} N_t^{\chi_{n,1}} C_t^{\chi_c}$$

Let $w_t \equiv \frac{W_t}{P_t}$. Then we can write

$$\frac{1-\tau_{n,t}}{1+\tau_{c,t}} w_t = \chi_{n,0} N_t^{\chi_{n,1}} C_t^{\chi_c}$$

Using the FONC with respect to consumption to eliminate $\lambda_t$ from the FONC with respect to bond holding, we obtain

$$\begin{aligned} \frac{1}{(1+\tau_{c,t})P_t C_t^{\chi_c}} &= \beta R_t E_t \frac{1}{(1+\tau_{c,t+1})P_{t+1}C_{t+1}^{\chi_c}}\delta_t \\ \Leftrightarrow \frac{1+\tau_{c,t+1}}{1+\tau_{c,t}} C_t^{-\chi_c} &= \beta R_t E_t \delta_t C_{t+1}^{-\chi_c}\Pi_{t+1}^{-1} \end{aligned}$$

## 2.2 Final-Goods Producing Firms

There are a continuum of monopolistically competitive firms producing differentiated intermediate goods indexed by $i \in [0,1]$. These intermediate goods, $Y_t(i)$, are used as inputs by a representative perfectly competitive firm to produce a final good, $Y_t$.

Production technology of the final-goods producing firm is given by

$$Y_t = \left[\int_0^1 Y_t(i)^{\frac{\theta-1}{\theta}} di\right]^{\frac{\theta}{\theta-1}}$$

where $Y_t(i)$ is the quantity of intermediate good j used as an input. Profit maximization of the final goods firm, taking the final good price $P_t$ and the prices for intermediate goods $P_t(i)$ as given, yields the set of demand functions

$$Y_t(i) = \left[\frac{P_t(i)}{P_t}\right]^{-\theta} Y_t$$

and the zero profit condition

$$P_t = \left[\int_0^1 P_t(i)^{1-\theta} di\right]^{\frac{1}{1-\theta}}$$

## 2.3 Intermediate-Goods Producing Firms

An intermediate-goods producing firm sets its price and produces the output demanded by the final-goods producing firms at that price. Labor is chosen to to meet the output demand in a cost minimizing way.

The production function is given by

$$Y_t(i) = N_t(i)$$

### 2.3.1 Cost minimization problem

The intermediate-goods producing firms choose labor to produce $Y_t(i)$ in a cost minimizing way.

$$\begin{aligned} \min \quad & W_t N_t(i) \\ s.t. \quad & Y_t(i) = N_t(i) \end{aligned}$$

The solution is

$$N_t(i)^* = Y_t(i)$$

Thus,

$$Cost(Y_t(i)) = W_t Y_t(i)$$

and

$$MC_t = \frac{\partial Cost(Y_t(i))}{\partial Y_t(i)} = W_t$$

### 2.3.2 Optimal Price

Intermediate goods firms are assumed to set nominal prices in a staggered fashion, as in Calvo (1983). Each firm optimizes its price with probability $\zeta_p$ each period, independently of the time elapsed since the last adjustment.

Optimizing-firms maximize the discounted sum of the future profits where the discount factor comes from the stochastic discount factor in the household's Euler equation.

$$\max_{P_t^*(i)} E_t \sum_{s=0}^{\infty} \zeta_p^s D_{t,t+s} \Big[ P_t^*(i) - W_{t+s} \Big] Y_{t+s}(i)$$

subject to the sequence of demand function

$$Y_{t+s}(i) = \left[ \frac{P_t^*(i)}{P_{t+s}} \right]^{-\theta} Y_{t+s}$$

$D_{t,t+s}$ is given by

$$D_{t,t+s} = \frac{\beta^{t+s} \lambda_{t+s} \prod_{k=0}^{t+s-1} \delta_k}{\beta^t \lambda_t \prod_{k=0}^{t-1} \delta_k} = \beta^s \delta_t \delta_{t+1} ... \delta_{t+s-1} \frac{\lambda_{t+s}}{\lambda_t}$$

## 2.4 Government

The set of potential policy instruments is $[R_t,G_t,T_t,\tau_{n,t},\tau_{c,t},B_t]$. I will sometimes refer to $R_t$ as monetary instrument, and the rest as fiscal instruments. I take the zero-lower bound on the nominal interest rate explicitly into account.

$$R_t \geq 1$$

As mentioned previously, I consider three versions of the model, each associated with a unique set of fiscal instruments. First version is when government has access to lump-sum tax. Government budget constraint is given by

$$P_t G_t = P_t T_t$$

Dividing both sides by $P_t$, we have

$$G_t = T_t$$

Second version is when there are labor income tax and debt. In this case, government budget constraint is given by

$$B_{t-1} + P_t G_t = \tau_{n,t} W_t N_t + R_t^{-1} B_t$$

Dividing both sides by $P_t$, we obtain

$$b_{t-1}\Pi_t^{-1} + G_t = \tau_{n,t} w_t N_t + R_t^{-1} b_t$$

where $\Pi_t = \frac{P_t}{P_{t-1}}$ and $b_t \equiv \frac{B_t}{P_t}$.

Finally, the third version is when there are consumption tax and debt. In this case, government budget constraint is given by

$$B_{t-1} + P_t G_t = \tau_{c,t} P_t C_t + R_t^{-1} B_t$$

Dividing both sides by $P_t$, we obtain

$$b_{t-1}\Pi_t^{-1} + G_t = \tau_{c,t} C_t + R_t^{-1} b_t$$

### 2.5 Market Clearing

The labor market clearing condition is given by

$$N_t = \int N_t(i) di$$

The resource constraint is given by

$$C_t + G_t = Y_t$$

The bond market clearing condition is embedded in the notation already as I use the same notation, $B_t$, in the representative household's budget constraint and government budget constraint.

### 2.6 An Equilibrium

Given an initial level of debt $B_0$, the distribution of initial prices $P_{i,0}$ for all $i \in [0, 1]$, and a sequence of discount factor shocks $\{\delta_t\}_{t=1}^{\infty}$, an equilibrium of this economy consists of

$\{C_t, N_t, Y_t, P_{i,t}, G_t, R_t, T_t, \tau_{n,t}, \tau_{c,t}, B_t\}_{t=1}^{\infty}$ such that

1. $\{C_t, N_t, B_t\}_{t=1}^{\infty}$ solves the household problem.
2. $\{P_{i,t}\}_{t=1}^{\infty}$ solves the firms' problem.
3. Government budget constraint is satisfied.
4. Markets clear.

It is straightforward to show that an equilibrium is characterized by

$$\{C_t, N_t, Y_t, \Pi_t, p_t^*, s_t, C_{d,t}, C_{n,t}, R_t, G_t, T_t, \tau_{n,t}, \tau_{c,t}, b_t\}_{t=1}^{\infty}$$

satisfying the following set of equations:

$$\begin{aligned}
\frac{1+\tau_{c,t+1}}{1+\tau_{c,t}} C_t^{-\chi_c} &= \beta R_t \delta_t C_{t+1}^{-\chi_c} \Pi_{t+1}^{-1} \\
\frac{1-\tau_{n,t}}{1+\tau_{c,t}} w_t &= \chi_{n,0} N_t^{\chi_{n,1}} C_t^{\chi_c} \\
s_t &= (1-\zeta_p)\left[p_t^*\right]^{-\theta} + \zeta_p \Pi_t^{\theta} s_{t-1} \\
1 &= (1-\zeta_p)\left[p_t^*\right]^{1-\theta} + \zeta_p \Pi_t^{\theta-1} \\
p_t^* &= \frac{\theta}{\theta-1} \frac{C_{n,t}}{C_{d,t}} \\
C_{n,t} &= \frac{1}{1+\tau_{c,t}} Y_t w_t C_t^{-\chi_c} + \zeta_p \beta \delta_t \Pi_{t+1}^{\theta} C_{n,t+1} \\
C_{d,t} &= \frac{1}{1+\tau_{c,t}} Y_t C_t^{-\chi_c} + \zeta_p \beta \delta_t \Pi_{t+1}^{\theta-1} C_{d,t+1} \\
Y_t s_t &= N_t \\
Y_t &= C_t + G_t \\
GBC_t &= 0 \\
R_t &\geq 1
\end{aligned}$$

where

$$GBC_t = G_t - T_t$$

when lump-sum tax is used to finance government spending,

$$GBC_t = b_{t-1}\Pi_t^{-1} + G_t - \tau_{n,t} w_t N_t - R_t^{-1} b_t$$

when labor income tax and debt are used to finance government spending,

$$GBC_t = b_{t-1}\Pi_t^{-1} + G_t - \tau_{c,t} C_t - R_t^{-1} b_t$$

when consumption tax and debt are used to finance government spending. $C_{n,t}$ and $C_{n,t}$ are auxiliary variables introduced to describe the equilibrium conditions recursively, $p_t^*$ is the price set by the optimizing firms normalized by the aggregate price, and $s_t$ is cross-sectional price dispersion. Notice that an equilibrium depends only on the real debt $b_0$ and the price dispersion $s_0$.

My main results will be based on nonlinear solution method. However, log-linearized equations are useful in describing the main mechanism behind the results. I will put them here for later reference.

$$\begin{aligned}
\hat{Y}_t &= \hat{Y}_{t+1} + \Gamma_R(\hat{\Pi}_{t+1} - \hat{R}_t - \hat{\delta}_t) + \Gamma_G(\hat{G}_t - \hat{G}_{t+1}) - \Gamma_{\tau,c}(\hat{\tau}_{c,t} - \hat{\tau}_{c,t+1}) \\
\hat{\Pi}_t &= \Omega_Y \hat{Y}_t - \Omega_G \hat{G}_t + \Omega_{\tau,c}\hat{\tau}_{c,t} + \Omega_{\tau,n}\hat{\tau}_{n,t} + \beta\hat{\Pi}_{t+1}
\end{aligned}$$

where the coefficients are shown in the footnote[6].

## 3 Government's Problem

The government's problem is to select an equilibrium that generates the highest utility for the household. That is,

$$\max_{\{u_t\}_{t=1}^T} \sum_{t=1}^{\infty} \beta^{t-1} \prod_{s=0}^{t-1} \delta_s \Big[\frac{C_t^{1-\chi_c}}{1-\chi_c} - \chi_{n,0}\frac{N_t^{1+\chi_{n,1}}}{1+\chi_{n,1}} + \chi_{g,0}\frac{G_t^{1-\chi_{g,1}}}{1-\chi_{g,1}}\Big]$$

where

$$u_t = [C_t, N_t, Y_t, w_t, p_t^*, \Pi_t, C_{d,t}, C_{n,t}, R_t, G_t, \text{either of } (T_t, \{\tau_{c,t}, b_t\}, \{\tau_{n,t}, b_t\})]$$

subject to the set of equations described above, and $b_0$ and $s_0$ are given.

[6] The coefficients of the log-linearized equations as functions of structural parameters are:

$$\begin{aligned}
\Gamma_R &= \frac{1}{\chi_c}\frac{C_{ss}}{Y_{ss}}, \Gamma_G = \frac{G_{ss}}{Y_{ss}}, \Gamma_{\tau,C} = \frac{1}{\chi_c}\frac{C_{ss}}{Y_{ss}}\frac{\tau_{c,ss}}{1+\tau_{c,ss}} \\
\Omega_Y &= \kappa(\chi_{N,1} + \chi_c\frac{Y_{ss}}{C_{ss}}), \Omega_G = \kappa\chi_c\frac{G_{ss}}{C_{ss}}, \Omega_{\tau,C} = \kappa\frac{\tau_{c,ss}}{1+\tau_{c,ss}}, \Omega_{\tau,N} = \kappa\frac{\tau_{n,ss}}{1-\tau_{n,ss}}
\end{aligned}$$

and $\kappa = \frac{(1-\zeta_p)(1-\beta\zeta_p)}{\zeta_p}$.

The associated Lagrangian problem is

$$
\begin{aligned}
L(b_0, s_0) &= \\
\min_{\{\omega_t\}_{t=1}^{\infty}} \max_{\{d_t, u_t\}_{t=1}^{\infty}} \quad & \sum_{t=1}^{\infty} \beta^{t-1} \prod_{s=0}^{t-1} \delta_s \Big[ \frac{C_t^{1-\chi_c}}{1-\chi_c} - \frac{\chi_{n,0}}{1+\chi_{n,1}} N_t^{1+\chi_{n,1}} + \chi_{g,0} \frac{G_t^{1-\chi_{g,1}}}{1-\chi_{g,1}} \Big] \\
&+ \sum_{t=1}^{\infty} \beta^{t-1} \prod_{s=0}^{t-1} \delta_s \Bigg[ \omega_{1,t} [\frac{1+\tau_{c,t+1}}{1+\tau_{c,t}} C_t^{-\chi_c} - \beta \delta_t R_t C_{t+1}^{-\chi_c} \Pi_{t+1}^{-1}] \\
&+ \omega_{2,t} [\frac{1-\tau_{n,t}}{1+\tau_{c,t}} w_t - \chi_{n,0} N_t^{\chi_{n,1}} C_t^{\chi_c}] \\
&+ \omega_{3,t} [s_t - (1-\zeta_p) \left[p_t^*\right]^{-\theta} - \zeta_p \Pi_t^{\theta} s_{t-1}] \\
&+ \omega_{4,t} [1 - (1-\zeta_p) \left[p_t^*\right]^{1-\theta} - \zeta_p \Pi_t^{\theta-1}] \\
&+ \omega_{5,t} [p_t^* - \frac{\theta}{\theta-1} \frac{C_{n,t}}{C_{d,t}}] \\
&+ \omega_{6,t} \big[ C_{n,t} - \frac{1}{1+\tau_{c,t}} Y_t w_t C_t^{-\chi_c} - \zeta_p \beta \delta_t \Pi_{t+1}^{\theta} C_{n,t+1} \big] \\
&+ \omega_{7,t} \big[ C_{d,t} - \frac{1}{1+\tau_{c,t}} Y_t C_t^{-\chi_c} - \zeta_p \beta \delta_t \Pi_{t+1}^{\theta-1} C_{d,t+1} \big] \\
&+ \omega_{8,t} [Y_t s_t - N_t] \\
&+ \omega_{9,t} [Y_t - C_t - G_t] \\
&+ \omega_{10,t} GBC_t \\
&+ \omega_{11,t} [R_t - 1] \Bigg]
\end{aligned}
$$

and $\omega_{11,t} = 0$ if $R_t > 1$ and $\omega_{11,t} > 0$ if $R_t = 1$. First order necessary conditions for this Lagrangian problem at $t \geq 2$ are given in Appendix C.

## 3.1 The modified government's problem

As in many Ramsey problems, the first order necessary conditions (FONCs) at $t = 1$ differ from the FONCs at $t \geq 2$. Therefore, even without any discount factor shock, the solution exhibits a peculiar initial dynamics before it converges to a steady state[7] (I will define the Ramsey steady-state and discuss its properties in the following subsection). In order to focus our attention on the economy's response to the discount factor shock, I will modify the government's problem so as to eliminate the initial dynamics that would exist in the absence of any shock. Specifically, following Kahn, King and Wolman (2003), the government's objective function is modified to include penalty terms involving a hypothetical time-zero Lagrange multipliers. The thought experiment behind this modified government's problem is as follows. Suppose that government solved the Ramsey problem a long time ago, and that the economy is at its Ramsey steady-state at t=0. Now, if government

[7]This is what makes the Ramsey allocation time-inconsistent. In this model, government will reduce the level of debt at time one by levying high labor income or consumption tax.

were to re-optimize again at t=1, government would use this opportunity to improve the welfare from that point on by deviating from what was promised at t=0. The terms involving the time-0 Lagrangian multiplier that appear in the modified objective function penalizes such deviation so that, in the absence of any unanticipated shocks, government continues to choose the same allocation and policy as those chosen at t=0[8].

The modified objective function of the government is given by

$$\sum_{t=1}^{\infty}\beta^{t-1}\prod_{s=0}^{t-1}\delta_s\Big[\frac{C_t^{1-\chi_c}}{1-\chi_c}-\chi_{n,0}\frac{N_t^{1+\chi_{n,1}}}{1+\chi_{n,1}}+\chi_{g,0}\frac{G_t^{1-\chi_{g,1}}}{1-\chi_{g,1}}\Big]-\omega_{1,0}R_0C_1^{-\chi_c}\Pi_1^{-1}-\omega_{6,0}\zeta_p\Pi_1^{\theta}C_{n,1}-\omega_{7,0}\zeta_p\Pi_1^{\theta-1}C_{d,1}$$

and the modified Lagrangian problem is given by

$$\begin{aligned}
L(b_0,s_0,R_0,\omega_{1,0},\omega_{6,0},\omega_{7,0}) \quad = \quad & \\
\min_{\{\omega_t\}_{t=1}^{\infty}}\max_{\{u_t\}_{t=1}^{\infty}} \quad & \sum_{t=1}^{\infty}\beta^{t-1}\prod_{s=0}^{t-1}\delta_s\Big[\frac{C_t^{1-\chi_c}}{1-\chi_c}-\frac{\chi_{n,0}}{1+\chi_{n,1}}N_t^{1+\chi_{n,1}}+\chi_{g,0}\frac{G_t^{1-\chi_{g,1}}}{1-\chi_{g,1}}\Big] \\
- \quad & \omega_{1,0}R_0C_1^{-\chi_c}\Pi_1^{-1}-\omega_{6,0}\zeta_p\Pi_1^{\theta}C_{n,1}-\omega_{7,0}\zeta_p\Pi_1^{\theta-1}C_{d,1} \\
+ \quad & \sum_{t=1}^{\infty}\beta^{t-1}\prod_{s=0}^{t-1}\delta_s\Bigg[\omega_{1,t}[\frac{1+\tau_{c,t+1}}{1+\tau_{c,t}}C_t^{-\chi_c}-\beta\delta_tR_tC_{t+1}^{-\chi_c}\Pi_{t+1}^{-1}] \\
+ \quad & \omega_{2,t}[\frac{1-\tau_{n,t}}{1+\tau_{c,t}}w_t-\chi_{n,0}N_t^{\chi_{n,1}}C_t^{\chi_c}] \\
+ \quad & \omega_{3,t}[s_t-(1-\zeta_p)\big[p_t^*\big]^{-\theta}-\zeta_p\Pi_t^{\theta}s_{t-1}] \\
+ \quad & \omega_{4,t}[1-(1-\zeta_p)\big[p_t^*\big]^{1-\theta}-\zeta_p\Pi_t^{\theta-1}] \\
+ \quad & \omega_{5,t}[p_t^*-\frac{\theta}{\theta-1}\frac{C_{n,t}}{C_{d,t}}] \\
+ \quad & \omega_{6,t}\big[C_{n,t}-\frac{1-\nu_t}{1+\tau_{c,t}}Y_tw_tC_t^{-\chi_c}-\zeta_p\beta\delta_t\Pi_{t+1}^{\theta}C_{n,t+1}\big] \\
+ \quad & \omega_{7,t}\big[C_{d,t}-\frac{1}{1+\tau_{c,t}}Y_tC_t^{-\chi_c}-\zeta_p\beta\delta_t\Pi_{t+1}^{\theta-1}C_{d,t+1}\big] \\
+ \quad & \omega_{8,t}[Y_ts_t-N_t] \\
+ \quad & \omega_{9,t}[Y_t-C_t-G_t] \\
+ \quad & \omega_{10,t}GBC_t \\
+ \quad & \omega_{11,t}[R_t-1]\Bigg]
\end{aligned}$$

where the problem is now indexed by $\omega_{1,0}$, $\omega_{6,0}$, $\omega_{7,0}$, and $R_0$ in addition to $b_0$ and $s_0$. FONCs of this modified Lagrangian problem are the same as the FONCs of the original Lagrangian problem at $t\geq 2$, and are given in Appendix C.

[8] Modifying the government's objective function in this way is often said to be taking "timeless perspective."

### 3.2 The constrained government's problem

In order to document the consequence of various fiscal instruments, I consider various constrained Ramsey problems in which government is constrained to keep one of its fiscal instrument constant. Such constrained problems have additional terms each period in the Lagrangian problem above. For example, the Lagrangian problem in which government is constrained to keep its government spending constant has $\omega_{12,t}[G_t - \bar{G}]$ added.

### 3.3 The Ramsey equilibrium and the Ramsey steady-state

Given the initial values ($b_0$, $s_0$, $R_0$, $\omega_{1,0}$, $\omega_{6,0}$, and $\omega_{7,0}$), the Ramsey equilibrium consists of $\{u_t\}_{t=1}^{\infty}$ and $\{\omega_{j,t}\}_{t=1}^{\infty}$ satisfying the FONCs of the Lagrangian problem above. I define the Ramsey steady-state as a set of values,

$$\{C_{ss}, N_{ss}, Y_{ss}, w_{ss}, p^*_{ss}, \Pi_{ss}, C_{d,ss}, C_{n,ss}, R_{ss}, G_{ss}, b_{ss}, \tau_{n,ss}, \tau_{c,ss}, \omega_{1,ss}, \omega_{2,ss}, ..., \omega_{11,ss}\}$$

satisfying the FONCs of the modified Lagrangian problem. The Ramsey equilibrium and the Ramsey steady-state in the constrained economies are defined in the same way.

Let me make several remarks about the Ramsey steady-state. First, while there is a unique Ramsey-steady state in the model with lump-sum tax, there are infinitely many Ramsey steady-states in the model with debt, each indexed by the rate of distortionary tax[9]. **Figure 1** plots the period utility and debt level associated with the Ramsey steady-states with different tax rates in the model with labor income tax and debt, and **Figure 2** does the same for the model with consumption tax and debt.

First, notice from the right panels that, for the range of tax rates shown in the figures, there is one-to-one mapping between the tax rate and the debt level[10]. Second, the left panels show that that different Ramsey steady-states yield different period utilities for the household. The Ramsey steady-state with highest period utility is the one in which government has a large asset, large enough to pay for labor income subsidy (or consumption subsidy) that eliminates the distortion from monopolistic competition among intermediate-goods firms. In this economy, government's incentive to inflate away nominal debt to reduce the real value of debt, or alternatively the incentive to deflate so that real value of the asset reaches to this best level, is countered by the production inefficiency associated with price dispersion caused by inflation or deflation. Constrained by the given initial debt level, government cannot move to this best Ramsey steady-state. Finally, in any Ramsey steady-state, $\Pi_{ss} = s_{ss} = p^*_{ss} = 1$ and $R_{ss} = \frac{1}{\beta}$. This is because price dispersion leads to production inefficiency ($Y_t s_t = N_t$).

[9]This can be proved by showing that the steady-state value of the Lagrangian multiplier on government budget constraint is indeterminate. Adam (2011) makes the same observation.

[10]The debt level associated with a very high tax rate beyond ones shown in the figure are also consistent with the debt level associated with a lower tax rate. This is due to the Laffer-curve effect.

### 3.4 Welfare Measures

I define **the welfare gain of government spending instrument**, denoted by $WG_G$, as the *one time* consumption transfer I need to make to the household *at time one* in the economy in which government is constrained to keep government spending at its initial level in order to equate the welfare of the constrained economy to the welfare of the unconstrained economy. Specifically, $WG_G$ is defined through

$$
\begin{aligned}
& u(C_{C,1} + \frac{WG_G}{100} C_{ss}, N_{C,1}, \bar{G}) + \sum_{t=2}^{\infty} \beta^{t-1} \prod_{s=1}^{t-1} \delta_s u(C_{C,t}, N_{C,t}, \bar{G}) + \text{penalty term} \\
= & \sum_{t=1}^{\infty} \beta^{t-1} \prod_{s=1}^{t-1} \delta_s u(C_{UC,t}, N_{UC,t}, G_{UC,t}) + \text{penalty term}
\end{aligned}
$$

where the left hand side is welfare in the constrained economy after the consumption transfer and the right hand side is welfare in the unconstrained economy. $X_{C,t}$ refers to the variable X at time t in the constrained economy where the government is constrained to set $G_t$ at $\bar{G}$, and $X_{UC,t}$ refers to the variable X at time t in the unconstrained economy. Notice that welfare of an economy is defined to be the objective function of the government, which is the sum of the household's utility and the penalty term punishing government from deviating from the initial Ramsey steady-state. $C_{ss}$ is the initial Ramsey steady-state consumption (which are the same in both economies). Since we are comparing alternative policy responses to one-time shock, this definition is more natural than that based on the consumption variation applied to all time periods[11]. Similarly, I define **the welfare gain of debt instrument**, denoted by $WG_b$, as the consumption transfer required to equate the welfare of the economy where government is constrained to keep its initial debt level with the welfare of the unconstrained economy. Finally, I define **the welfare gain of distortionary tax instrument**, denoted by either $WG_{\tau,n}$ or $WG_{\tau,c}$, as the consumption transfer required to equate the welfare of the economy where government is constrained to keep both debt and government spending at their initial levels with the welfare in the benchmark economy. Since these welfare gains can be interpreted as measuring the welfare losses from losing the ability to vary fiscal instruments, I will sometimes refer to them as the welfare loss when it is more natural to do so.

In order to put these welfare measures into perspective, I compute welfare in the model in which government sets its policy instruments according to some simple rules. In this "non-optimizing government" regime, nominal interest rate is set according to a truncated Taylor rule augmented with price level target:

$$R_t = \max[1, [\frac{\Pi_t}{\bar{\Pi}}]^{\rho_\Pi} [\frac{P_t}{\bar{P}}]^{\rho_P}]$$

I include the price level gap in the policy rule because standard Taylor rules leads to large deflation and output decline with small $\chi_c$ considered in this paper, and even leads to non-existence of the

[11] To compare the number here with welfare gain number based on perpetual transfer (as in Lucas' welfare calculation), you need to divide the number here by 100 ($=\frac{1}{1-\beta}$), or multiply the Lucas' number by 100.

equilibrium under certain parameter values. Letting nominal interest rate responds to the price level gap improves the outcome considerably[12]. In the model with lump-sum tax, non-optimizing government sets

$$G_t = G_{ss}$$

and sets $T_t$ to satisfy government budget constraint. In the model with labor income tax and debt, non-optimizing government sets

$$\begin{aligned} \frac{G_t}{N_t} &= \frac{G_{ss}}{N_{ss}} \\ \tau_{n,t} &= \tau_{n,ss} + \alpha_n (b_{t-1} - b_0) \end{aligned}$$

and sets $b_t$ to satisfy government budget constraint. $Gss$, $Nss$, and $\tau_{n,ss}$ are the Ramsey-steady-state values of $G_t$, $N_t$, and $\tau_{n,t}$ associated with $b_0$. Finally, in the model with consumption tax and debt, non-optimizing government sets

$$\begin{aligned} \frac{G_t}{N_t} &= \frac{G_{ss}}{N_{ss}} \\ \tau_{c,t} &= \tau_{c,ss} + \alpha_c (b_{t-1} - b_0) \end{aligned}$$

and sets $b_t$ to satisfy government budget constraint. $\tau_{c,ss}$ is the Ramsey-steady-state value of $\tau_{c,t}$ associated with $b_0$. I impose that, in the model with distortionary taxes, non-optimizing government chooses government spending so as to keep its ratio to output constant, whereas non-optimizing government sets government spending constant in the model with lump-sum tax. This imposition is necessary because there turns out to be no equilibrium consistent with constant level of government spending in this "simple-rule regime" model under certain parameter values. Parameter values for the policy rules are listed in Table 1.

# 4 Calibration and Solution Method

## 4.1 Calibration

**Table 1** lists parameter values considered. When multiple values are listed, the bold number represents the benchmark calibration. Since this paper aims to contribute to the literature on the zero nominal interest rate, I calibrate the parameter so that they are in the range of the values considered in this literature. Here, I discuss some key parameters.

Government spending enters into the household's utility function. I choose $\chi_{G,1}$, the inverse of the intertemporal elasticity of substitution for government spending, to be unity, and $\chi_{G,0}$, the weight on the utility from government spending relative to the utility from private consumption,

[12] In the model without fiscal policy instruments, Eggertsson and Woodford (2003) show that a version of price level targeting can replicate the Ramsey outcome.

to be 0.2 so that the steady-state level of government spending to output ratio is roughly about 1/6. Alternative values of $\chi_{G,0} = 0.5$ and $\chi_{G,0} = 1.0$ are also considered to study the effect of a large government sector.

I consider three alternative values for $\chi_c$, the inverse of the intertemporal elasticity of substitution for private consumption. The benchmark is $\chi_c = \frac{1}{6}$ from Jung et al. (2003) and Levin et al. (2010). I also consider $\chi_c = 1$ which corresponds to log-utility function, and $\chi_c = 2$ which is the value used in Eggertsson's work. For the inverse labor supply elasticity, I use $\chi_{n,1} = 1$ as the benchmark, and consider alternative values of $\chi_{n,1} = 0.5$ and $\chi_{n,1} = 2.0$.

For the shock process, I use $\epsilon_{\delta,1} = 0.02$, which makes the annualized nominal interest that would neutralize this shock -4 percent at time one. This is roughly the same magnitude of shock considered by Levin et al. (2010) in their experiment intended to capture the severity of the Great Recession, and is slightly larger than the value considered in Eggertsson and Woodford (2004). For the persistence of the shock, I set $\rho_\delta = 0.9$, which is slightly larger than the value considered in Levin et al. (2010). Eggertsson and Woodford (2004) consider a stochastic two-state Markov process for the discount factor shock, and thus the persistence of their shock cannot directly into the deterministic geometric path of $\delta$ considered in this paper. However, their process implies the average duration of the natural rate being negative to be about 10 quarters, which translates into $\rho_\delta \cong 0.93$ in my setting. I also consider larger shocks ($\epsilon_{\delta,0} = 0.0225$ and $\epsilon_{\delta,0} = 0.025$) and more persistent shocks ($\rho_\delta = 0.925$ and $\rho_\delta = 0.95$).

As discussed above, the model with nominal debt possesses an infinite number of Ramsey steady states, and the initial level of debt determines the the terminal Ramsey steady-state to which the economy converges. Thus, I need to calibrate the initial level of debt. In the benchmark calibration, I choose $b_0$ so that the ratio of debt to annualized output at time zero is 0.5, which is slightly above the ratio of publicly-held debt to GDP in the U.S. during the fiscal year 2007, a period just before the Great Recession. The ratio was 0.362 in the U.S., and some other developed countries who are now constrained by the zero-lower bound had higher debt-to-GDP ratios. I consider alternative values of the initial debt $b_0 = [-2, -1, 0, 1, 2]$, and study how the initial level of debt affects optimal policy and welfare.

## 4.2 Solution Method

Except in the Ramsey steady-state associated with a large government asset, the allocations are inefficient. For this reason, and also because policy functions are not differentiable due to the zero lower bound constraint on nominal interest rate, we need to solve the model in its original nonlinear form in order to accurately evaluate optimal policy and welfare.

Appendix A describes nonlinear solution method. The method is a further modification of a modified Newton algorithm by Julliard et al. (1998). They modify a standard Newton algorithm so as to take advantage of the recursive structure of the problem, and I embed their algorithm in a shooting algorithm where the terminal Ramsey steady-state is searched.

# 5 Results

## 5.1 Optimal Policy with Lump-Sum Tax

**Figure 3** shows two impulse response functions for the model with lump-sum tax. Solid black lines are the impulse response functions with unconstrained government and dashed red lines are the impulse response functions with government constrained to keep its spending at the initial Ramsey steady-state level. Dashed red lines reproduce the result from Jung et al. (2005) and Levin et al. (2010). Optimal monetary policy is characterized by the extended period of holding the nominal interest rate at zero. For $t > 6$, the nominal interest rate that would neutralize the discount factor shock is above 1, and the government could achieve the steady-state allocations from that period on. However, keeping nominal interest rate at the zero for a longer period increases expected inflation, which prevents a large drop in inflation and output at the beginning of the recession. Nevertheless, as emphasized in Levin et al. (2010), a combination of large and persistent shock with high intertemporal elasticity of substitution results in a sharp decline in output and consumption. Notice that introducing government spending policy does not affect this feature of optimal interest rate.

In the unconstrained Ramsey allocation, government spending initially jumps and declines gradually during most of the binding zero-bound period. Government spending declines below its steady-state level around t=5, and reaches its bottom at t=7. After that, it starts rising and comes back to the steady-state level as the zero bound ceases to bind. The intuition for this dynamics can be illustrated with the following log-linearized version of the private section equilibrium.

$$\begin{aligned}
\hat{Y}_t &= \hat{Y}_{t+1} + \Gamma_R(\hat{\Pi}_{t+1} - \hat{R}_t - \hat{\delta}_t) \\
\hat{\Pi}_t &= \Omega_Y \hat{Y}_t - \Omega_G \hat{G}_t + \beta \hat{\Pi}_{t+1}
\end{aligned}$$

where coefficients are functions of structural parameters (see Section 2.5). First equation is the Euler equation, and second equation is the forward-looking Phillips curve. An expected decline in government spending (an increase in $(\hat{G}_t - \hat{G}_{t+1})$) raises output today according to the Euler equation. An increased output demand will increase the demand for labor by the intermediate good firms, which increases the real wage and inflation today, which is captured by the first term in the Phillips Curve. Today's increase in inflation reduces the real interest rate facing the household in the previous period, which works to offset the effect of discount factor shock. Therefore, expected reduction in government spending has the effect of mitigating deflation and output collapse. Although increasing today's government spending lowers today's inflation through $\Omega_G$ terms, this effect is quantitatively small. Since government wants to keep its spending closer to the steady-state level, it is optimal to moderately increase government spending initially, and let it decline below its steady-state, rather than increasing it by a large amount and reducing it straight to the steady-state level.

Variations in government spending have limited effects on allocations. The initial increase in government spending is about 5 percent of its initial steady state level, which is less than 1 percent of initial steady state output. Output multiplier is roughly about one, and government spending does not have any significant effect on consumption. One important difference between the constrained and unconstrained economies is in the response of price dispersion. Price dispersion is always smaller in the unconstrained economy than in the constrained economy. This is a reflection of slightly more subdued inflation under the unconstrained economy during the period of zero nominal interest rates.

In order to further highlight the limited effects of government spending, **Figure 4** shows the impulse response functions associated with government following a truncated Taylor rule and setting government spending constant described in Section 3.4. Dashed blue lines are the allocations associated with such non-optimizing government, and solid black line and dashed red lines are the same as in Figure 3. Under non-optimizing government, nominal interest rate starts rising at t=6. Without government committing itself to an extend period of zero nominal interest rate, there is large output collapse and deflation. It is clear from this figure that, compared to the effects of optimally choosing nominal interest rates, the additional effect of optimally choosing government spending are small. We will see this point again later in the welfare calculation.

## 5.2 Optimal Policy with Labor Income Tax and Debt

**Figure 5** shows two impulse response functions for the model with labor income tax and debt. Solid black lines are the impulse response functions with unconstrained government, and dashed red lines are the impulse response functions with government constrained to keep initial debt level. As in the model with lump-sum tax, optimal monetary policy involves an extended period of the zero nominal interest rate, and optimal government spending is characterized by an initial expansion followed by a sharp reversal.

Labor income tax follows dynamics similar to government spending. It increases at time one, declines below the initial steady-state rate, and rises back to its new steady-state rate as the zero bound ceases to bind. We can again obtain an intuition for such path of labor income tax through the log-linearized private sector equilibrium conditions.

$$
\begin{aligned}
\hat{Y}_t &= \hat{Y}_{t+1} + \Gamma_R(\hat{\Pi}_{t+1} - \hat{R}_t - \hat{\delta}_t) + \Gamma_G(\hat{G}_t - \hat{G}_{t+1}) \\
\hat{\Pi}_t &= \Omega_Y \hat{Y}_t - \Omega_G \hat{G}_t + \Omega_{\tau,n}\hat{\tau}_{n,t} + \beta\hat{\Pi}_{t+1}
\end{aligned}
$$

According to the Phillips curve, an increase in labor income tax leads to an increase in today's inflation. This is because higher labor income tax reduces the household's willingness to work, which increases the real wage and thus inflation. As discussed earlier, such increase in today's inflation reduces the real interest rate facing the household in the previous period, which works to mitigate the effect of the discount factor shock[13].

[13] Eggertsson has discussed this expansionary effect of the labor income tax increase in various papers. See

The initial jump in labor income tax does not generate revenues large enough to cover the initial increase in government spending. As a result, primary balance ($= \tau_{n,t} w_t N_t - G_t$) decreases. Nevertheless, debt declines slightly at time one because the decline in nominal interest rate reduces the interest expense of the government. Thereafter, debt continues to decline until it converges to a new steady-state level, which is lower than the initial debt level. The debt is smaller in the terminal Ramsey steady-state because the interest expense on debt paid by government is lower than the level associated with the initial Ramsey steady-state for all periods after the shock due to below-trend nominal interest rates.

Again, variations on government spending have limited effects on allocation. As before, the initial increase in government spending is around 5 percent of its initial steady-state level, which is less than 1 percent of initial steady-state output. The initial drop in the primary balance is smaller and thus the initial drop in debt is larger in the constrained economy than in the unconstrained economy. As in the model with lump-sum tax, price dispersion is smaller and inflation is slightly more contained at the peak in the unconstrained economy.

**Figure 6** shows the impulse response functions for three different levels of initial debt. Dashed red lines are for the economy with high initial debt ($b_0/(4Y_0) = 2$), solid black lines are for the economy with no initial debt, and the dashed red lines are for the economy with negative initial debt($b_0/(4Y_0) = -2$). For most variables, the impulse response functions exhibit similar dynamics. However, there are several quantitatively important differences. First, the initial increase in government spending is much larger, and the initial increase in labor income tax is much lower, in the economy with large initial debt. In the large-debt economy, government spending jumps by 10 percent while it jumps only by 5 percent in the large-asset economy or no-debt economy. Labor income tax does not fluctuate much in the large-debt economy, but it increases by about 7 percent in the large-asset economy. The differences in the dynamics of government spending and labor income tax lead to a stark difference in the dynamics of primary balance. Primary balance decreases in the large-debt economy, whereas it increases in the large-asset economy.

In the long-run, the high-debt economy converges to a terminal Ramsey steady-state with lower debt, and the high-asset economy converges to a terminal Ramsey steady-state with lower asset. As explained earlier, the reason for the economy with positive initial debt to converge to a terminal steady-state with lower debt is the below-trend nominal interest rate. This below-trend nominal interest rate also leads the high-asset economy to settle in a terminal Ramsey steady-state with lower asset. After the shock, government receives the interest payments from the household that are smaller than the level associated with the initial Ramsey steady-state because of the lower nominal interest rates, and government's asset decreases as a result.

### 5.3 Optimal Policy with Consumption Tax and Debt

**Figure 7** shows the impulse response functions for the model with consumption tax and debt. Solid black lines are the impulse response functions with unconstrained government, and dashed red lines are the impulse response functions with constrained government. As in the previous two

Eggertsson and Woodford (2004), Eggertsson (2010a and 2010c) models, it is optimal for government to set nominal interest rates to zero for an extended period of time.

Dynamics of consumption tax is characterized by an initial drop followed by the gradual increase. After reaching its peak, it gradually declines to its terminal steady state level. The initial drop in consumption tax is about 4 percent. An intuition for such consumption tax dynamics can be again illustrated by the log-linearized version of the private sector equilibrium conditions.

$$\begin{aligned}
\hat{Y}_t &= \hat{Y}_{t+1} + \Gamma_R(\hat{\Pi}_{t+1} - \hat{R}_t - \hat{\delta}_t) + \Gamma_G(\hat{G}_t - \hat{G}_{t+1}) - \Gamma_{\tau,c}(\hat{\tau}_{c,t} - \hat{\tau}_{c,t+1}) \\
\hat{\Pi}_t &= \Omega_Y \hat{Y}_t - \Omega_G \hat{G}_t + \Omega_{\tau,c}\hat{\tau}_{c,t} + \beta\hat{\Pi}_{t+1}
\end{aligned}$$

According to the Euler equation, an increase in tomorrow's consumption tax rate relative to today's consumption tax rate (an increase in $(\hat{\tau}_{c,t+1} - \hat{\tau}_{c,t})$) increases output today. This is because an expected increase in consumption tax induces the household to save less and consume more today, which neutralizes the effect of discount factor shock. Such increase in output demand by the household leads the firms to demand more labor, which pushes up the real wage and thus inflation, and this effect is captured $\Omega_Y$ term in the Phillips Curve. A reduction in consumption tax today can lead to lower inflation today through the $\Omega_{\tau,c}$ term, but this term is small, and the incentive to reduce consumption tax today dominates.

In the Euler equation, the expected increase in consumption tax and the expected reduction in government spending acts in a similar way to mitigate the discount factor shock, holding everything else equal. This explains the reduced role of government spending when consumption tax is available. Government spending path in this model is qualitatively similar in the previous two models, but the variation is much smaller in size. The initial increase is less than 0.5 percent while it was around 5 percent in the previous two models. Government chooses to use variation in consumption tax to mitigate the discount factor shock and vary government spending in a way to maintain the equality of its marginal utility with the marginal utility of private consumption. Consumption at time one actually increases by a small amount due to a combination of a large reduction in today's consumption tax and an expected increase in consumption tax in the future. Since the variation in government spending is very small, the differences between constrained and unconstrained economies are very small.

An expected increase in consumption tax and a reduction in nominal interest rate are close substitutes to government because they both affect the household's intertemporal decision in the same way without perturbing the resource constraint. Thus, in the model with consumption tax, the zero lower bound constraint is less of a problem for government than in the models without it. This is reflected in much smaller values of the Lagrangian multiplier ($\omega_{11,t}$) on the zero lower bound constraint. This Lagrangian multiplier captures the shadow value of further decreasing the nominal interest rate from zero. In the previous two models, this Lagrangian multiplier jumps at time one to 30, and reaches to 50 at the peak (see Figure 3 and 5). In the model with consumption tax and debt, it jumps by less than 1, and reaches to around 3 at the peak. This is because the value of relaxing this inequality is smaller as the variation in consumption tax can accomplish what the further reduction in nominal interest rate could accomplish.

A combination of a very small increase in government spending and a large drop in consumption tax means that primary balance drops at time one. As a result, debt initially increases and stays above the initial level during the first several quarters. As consumption tax rises above its initial level, the debt starts to decline and slowly converges to a new steady-state level. As we saw in the model with labor income tax, the debt in the terminal Ramsey steady-state is below its initial level as government's reduced interest rate expenses allows government to reduce debt level due to the below-trend nominal interest rates.

**Figure 8** shows the impulse response functions for three different levels of initial debt. Dashed red lines are for the economy with high initial debt ($b_0/(4Y_0) = 2$), solid black lines are for the economy with no initial debt, and the dashed red lines are for the economy with negative initial debt, or equivalently positive initial asset ($b_0/(4Y_0) = -2$). Both in the large-debt and large asset economies, it is optimal to increase the nominal interest rate at time one. From time two on, the dynamics of nominal interest rate is the same as in the benchmark model with moderate amount of initial debt or in the model with no initial debt. As discussed earlier, a reduction in nominal interest rate and an expected increase in consumption tax are close substitutes to government. In the model with large initial debt or asset, government chooses to combine a large increase in nominal interest rate and a large expected increase in consumption tax at time one. Government achieves this large expected increase in consumption tax at time one by reducing the consumption tax rate at time one without changing time-two consumption tax rate much. In both high-debt and high asset economies, the initial decrease in consumption tax is more than 10 percent.

Finally, as in the model with labor income tax and debt, a long period of below-trend nominal interest rates leads the economy with positive initial debt and the economy with positive initial asset to converge to terminal Ramsey steady-states with lower debt and asset, respectively.

### 5.4 Welfare

**Table 2, 3 and 4** report the welfare gains for, respectively, the model with lump-sum tax, the model with with labor income tax and debt, and the model with consumption tax and debt. For the models with debt, I compute welfare gains for alternative initial debt levels.

In the model with lump-sum tax, welfare gain of government spending instrument is 0.11. This is substantially smaller than the welfare gain of moving from an economy with non-optimizing government to an economy in which government chooses nominal interest rate optimally, which is consistent with what we saw earlier in Figure 4. Thus, in this economy, there is not much additional improvement government spending can make if nominal interest rate is chosen optimally. In the model with labor income tax and debt, welfare gain of government spending instrument is 0.17, slightly larger than in the model with lump-sum tax. Welfare gain of government spending is smaller than those of labor income tax and debt. As before, conditional on optimally chosen nominal interest rates, additional welfare gain from any of the fiscal instruments is small. Finally, in the model with consumption tax and debt, welfare gain from government spending is even smaller. This is what we would expect from the impulse response functions above; Optimal variations in government spending in this model are much smaller than in the previous two models. Finally, welfare gain of consumption tax is larger than welfare gain of debt in this model.

Welfare gains depend on the initial level of debt or asset. According to the second panels of Table 3 and 4, the larger the initial debt or asset is, the larger the welfare gains of fiscal instruments are, regardless of the available distortionary tax. Why are the welfare gains of various fiscal instruments larger in the economy with large initial debt or asset? The intuition is as follows. Regardless of the level of initial debt or asset, we have seen that optimal monetary policy involves an extended period of zero nominal interest rate followed by the gradual reversal towards its steady-state level. In the model with debt, such change in the nominal interest rate perturbs government budget constraint as it affects the interest expenses or receipts. If the economy starts with a positive initial debt, either debt has to decrease, government spending has to increase, or labor income tax has to decrease, in order to maintain government budget constraint. Therefore, fiscal instruments have a dual role of mitigating the effect of the discount factor shock and maintaining government budget constraint. When the initial debt or asset level is small, changes in the nominal interest rate does not affect the government budget constraint much. However, when the initial level of debt or asset is high, the variation in fiscal instruments needed to balance government budget constraint is large. Under such circumstance, constraining a fiscal instrument forces other fiscal instruments to adjust more to maintain government budget constraint, which can conflict with their role of mitigating the discount factor shock and lead to larger welfare losses.

# 6 Discussion

This section summarizes additional results.

## 6.1 Sensitivity Analysis

This subsection describes how optimal allocation and welfare change with alternative parameter values. The qualitative results are unchanged, but alternative parameter values lead to quantitatively different impulse response functions. Instead of presenting a large number of impulse response functions, I will present three summary statistics (the initial increase in government spending as a percentage of its initial steady-state level, the initial increase in distortionary taxes, and the last period at which nominal interest rate is zero) and welfare gains, for each parameter configurations and for each model. I focus on the sensitivity of the results on the following six parameters; Inverse IES for consumption good ($\chi_{n,1}$), Calvo parameter ($\zeta_p$), inverse labor supply elasticity ($\chi_{n,1}$), the utility weight on government spending ($\chi_{g,0}$), the magnitude of the discount factor shock ($\epsilon_{\delta,1}$), and persistence of the shock ($\rho_\delta$).

**Model with lump-sum tax**

**Table 5** shows the results for the model with lump-sum tax. Notice first that, for all parameter values, welfare gain of government spending is smaller than welfare gain of conducting optimal policy. Thus, the result that additional welfare gain of government spending is small given that monetary policy is chosen optimally do not depend on specific parameter values. Next, the initial increase in government spending and the welfare gain are larger in the economy with parameter values associated with a larger output collapse. High elasticity of substitution for consumption (low $\chi_c$), large price friction (high $\zeta_p$), inelastic labor supply (high $\chi_{n,1}$), a large shock (high $\epsilon_{\delta,1}$), and persistent shock (high $\rho_\delta$) would lead to larger output decline without fiscal policy, and the optimal response of government spending is larger with these parameter values. The response of government spending as a percentage of its steady-state level is smaller if government spending receives more weight in the household's utility function (larger $\chi_{g,0}$). However, since government spending is a larger share of total output when $\chi_{g,0}$ is larger, optimal increase in government spending as a a ratio of total output is actually larger with $\chi_{g,0} = 0.5$ and $\chi_{g,0} = 1.0$, and the welfare gains are also larger. Finally, and not surprisingly, the initial increase in government spending and the welfare gain are larger when the discount factor shock is larger or more persistent (see the bottom two panels of Table 5).

**Model with labor income tax and debt**

**Table 6** shows the results for the model with labor income tax and debt. For most parameter values, the relative importance of three fiscal instruments are the same as in the benchmark case. That is, welfare gain of government spending policy is smaller than welfare gain of labor income tax policy, which in turn is smaller than welfare gain of debt policy. This is no longer true in the economy with large government sector (high $\chi_{g,0}$). There, welfare gain of government spending is larger than those of other two instruments. As before, the optimal increase in government spending and the welfare gain are generally larger in the economy with a larger output collapse (low value of $\chi_c$ and high values of $\zeta_p$, $\chi_{n,1}$, $\epsilon_{\delta,1}$, $\rho_\delta$). One exception is that even though the increase in government spending is larger with more inelastic labor supply (high $\chi_{n,1}$) than the benchmark, the welfare gains are lower.

Variations in the welfare gains of labor income tax instrument across alternative values of $\chi_c$, $\zeta_p$, and $\chi_{n,1}$ are small, being around 0.5 percent most of the times. Welfare gain of labor income tax is smaller in the economy with large government sectors, and is larger when shocks are larger and more persistent. Finally, welfare gain of debt instrument tends to vary with parameter values in a way similar to welfare gain of government spending.

**Model with consumption tax and debt**

**Table 7** shows the results for the model with consumption tax and debt. For most parameter values, the relative importance of alternative fiscal instruments are the same as in the benchmark case. That is, the welfare gain of government spending is smaller than the welfare gain of debt instrument, which in turn is smaller than the welfare gain of consumption tax. One exception is when government sector is large ($\chi_{g,0} = 1.0$). In this case, welfare gain of government spending exceeds welfare gain of debt. Except in this case, welfare gain of government spending is smaller than 0.2 percent, and is between 0.01 and 0.05 percent in most paramater configurations. And, the initial increase in government spending does not deviate much from 0.5 percent of the benchmark case most of the times.

Similar to the labor income tax in the previous model, the response and welfare gains of consumption tax are generally larger in the economies with parameter values implying larger output collapse. One exception occurs when the initial decline of consumption tax increases with $\chi_c$. The welfare gain of debt instrument behaves similarly to that of consumption tax, except when the welfare gain of debt decreases with larger $\chi_{n,1}$.

**Sensitivity of monetary policy responses**

For all parameter configurations considered here, optimal monetary policy is characterized by holding nominal interest rate at zero for an extended period. With the benchmark shock process with $\epsilon_{\delta,1} = 0.02$ and $\rho_\delta = 0.9$, the zero nominal interest rate lasts for between 8 and 11 quarters in the model with lump-sum tax and in the model with labor income tax and debt, and it lasts for exactly 11 quarters in the model with consumption tax and debt. In the two models without consumption tax, the zero-bound periods are larger in the economies with larger $\chi_c$ and $\zeta_p$. The zero-bound period does not vary across different values of labor supply elasticity and government sector size in the model with labor income tax and debt, whereas the zero-bound period is one-period shorter with $\chi_{n,1} = 2.0$ and one period longer with $\chi_{g,0} = 1.0$ in the model with lump-sum tax.

As the size of the shock ($\epsilon_{\delta,1}$) increases, government responds by holding the nominal interest rate at zero for longer periods (see the second-to-the-bottom panels of Table 5-7). The marginal increases in the zero-bound period are about the same in the model with lump-sum tax and in the model with labor income tax and debt, and government set nominal interest rates to zero for 11 quarters with $\epsilon_{\delta,1} = 0.025$. In the model with consumption tax and debt, the marginal increases are larger. With $\epsilon_{\delta,1} = 0.025$, governments keeps nominal interest rate at zero for 15 quarters. A similar observation applies as the persistence of the shock ($\rho_\delta$) increases (see the bottom panels of Table 5-7). Government keeps nominal interest rate at zero for 4 years in the two models without consumption tax, while it keeps the rate at zero for more than 5 years in the model with consumption tax.

## 6.2 Piecewise Linear vs. Nonlinear Solutions

Throughout the paper, models were solved in its original nonlinear form. Do we obtain the similar results based on the piecewise first-order approximation method commonly used to solve the model with zero bound constraints? This subsection compares the results based on nonlinear solution method with those based on a linear method[14]. For the sake of brevity, I will focus on the model with labor income tax and debt.

There are many ways in which one can resort to linearization method. One commonly used approach starts by approximating the household's utility function by a quadratic function of inflation gap and output gap, derives log-linear approximation to the equations characterizing the equilibrium of the model, and solves the government's problem to obtain the set of equations characterizing the Ramsey-equilibrium. The resulting set of equations are linear except for the zero bound, and one can apply the piecewise first-order method described in Eggertsson and Woodford (2003), Christiano (2010), or Appendix B of this paper to obtain the solution. The zero bound literature focusing on optimal policies has relied on this approach. Another approach is to derive the set of nonlinear equations characterizing the Ramsey equilibrium, and linearize those equations. Kahn, King, and Wolman (2003) takes this approach, and I take this second approach here as well[15]. After solving for the first-order accurate law of motions for the endogenous variables, I use the same welfare function described above to measure the welfare (instead of quadratic loss function). These two different linear approaches would generate different results, but a comprehensive assessment of alternative linear methods is outside the scope of this paper[16].

**Figure 8** compares the impulse response functions based on nonlinear and linear solution methods. Solid black line is based on nonlinear method (the same one as in Figure 3), and dashed red line is based on linear method. They are similar qualitatively and quantitatively, but there are a few important differences. First, the linear method ignores price dispersion. Second, inflation is higher during the zero-bound period in the linear solution than in the nonlinear solution. Third, the linear method understates the initial increase in labor tax rate. Finally, debt converges to a lower level in the linear solution than in the nonlinear solution. Overall similarity between two solution methods reflects the fact that fluctuations in many variables are limited when government can optimize and commit. In the model with non-optimizing government, the initial drop in consumption and output is about 20 percent, and there are large differences between two methods (not reported). Nakata (2011) finds that there are large differences between two methods in the model without government's commitment.

**Table 8** compares the welfare gains based on nonlinear and linear solution methods. It shows that the use of linear method can distort welfare gain calculation. While the linear method correctly captures the relative importance of alternative fiscal instruments, it can either overstate or understate the actual welfare gains by a large amount. In particular, welfare gains of debt instrument is substantially larger in the linear method. Also, we see spurious welfare loss from government spending policy if we rely on the linear method.

Overall, the comparison here suggests some caution in using the piecewise linear method for solving the model involving the zero bound. Even when fluctuations in the model's variables are

[14] Braun and Waki (2011) find large differences between government spending multipliers at the zero bound based on piecewise linear and nonlinear methods.

[15] Levin et al. (2010) shows the connection between these two different approaches in the model with labor income subsidy financed by lump-sum tax.

[16] See Kim and Kim (2003) for a systematic comparison of alternative linear-quadratic approaches in the model of international risk-sharing.

small, there are some quantitatively nontrivial differences in the impulse response functions, and welfare calculation can be misleading.

## 7 Conclusion

The paper's findings are summarized in the introduction. Here, I suggest several venues for future research building on this paper.

First, the paper finds that it is optimal to increase labor income taxes during the recession and that the welfare gain from doing so is large, at least compared to the welfare gain of government spending policy. This is at odd with conventional wisdom in which an increase in labor income tax is contractionary, and makes us wonder what features can be introduced to this simplest New Keynesian model in order to reverse this uncomfortable policy recommendation. Various existing work introducing additional frictions in labor markets to the New Keynesian model may be useful in answering this question. Such features include nominal wage rigidities, introduction of borrowing-constrained workers, and introduction of unemployment. Since it is difficult to analyze optimal policy and welfare with these additional features using the linear-quadratic framework, nonlinear analysis conducted in this paper should be a useful benchmark in pursuing such investigations.

Second, the paper abstracts from any political economy concerns. While there are many directions one can take to understand how political aspects of fiscal policy affects allocations and welfare, one useful exercise is to introduce implementation lags. Implementation lags can be introduced as an additional constraint on government's problem, and it would be useful to study their implications for optimal policy and welfare.

Finally, the paper focused on the model with government's commitment. While it is useful to characterize the allocations with commitment as the Ramsey allocation represents the best outcome government can achieve in the market-based economy, the assumption of commitment is clearly unrealistic. My ongoing research analyzes the role of government spending under various sets of financing schemes in the model without government's commitment.

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# A Nonlinear Solution Method

## A.1 Problem

Let $z_t$ be the vector of all the time-t variables in the model. Given $z_0$, the goal is to find $\{z_t\}_{t=1}^{\infty}$ and two integers $T_f$ and $T_l$ such that

$$
\begin{aligned}
&\dots \\
f(z_{T_l+3}, z_{T_l+2}, z_{T_l+1}) &= 0 \text{ and } R_{T_l+1} > 1 \\
f(z_{T_l+2}, z_{T_l+1}, z_{T_l}) &= 0 \text{ and } R_{T_l+1} > 1 \\
g(z_{T_l+1}, z_{T_l}, z_{T_l-1}) &= 0 \text{ and } \omega_{11,T_l} > 1 \\
&\dots \\
g(z_{T_f+2}, z_{T_f+1}, z_{T_f}) &= 0 \text{ and } \omega_{11,T_f+1} > 1 \\
g(z_{T_f+1}, z_{T_f}, z_{T_f-1}) &= 0 \text{ and } \omega_{11,T_f} > 1 \\
f(z_{T_f}, z_{T_f-1}, z_{T_f-2}) &= 0 \text{ and } R_{T_f-1} > 1 \\
&\dots \\
f(z_3, z_2, z_1) &= 0 \text{ and } R_2 > 1 \\
f(z_2, z_1, z_0) &= 0 \text{ and } R_1 > 1
\end{aligned}
$$

where where f and g are respectively the vectors of functions containing the FONCs of the Lagrangean problem when $R_t > 1$ and $R_t = 1$. f and g are identical except for the last element (the last elements of f and g are respectively $\omega_{11,t} = 0$ and $R_t = 1$). $T_f$ and $T_l$ are the first and last period during which $R_t = 1$. In the Lagrangian problem without, f at time 1 is different from f from other periods.

In order to reduce the problem into a finite problem, we assume that the economy converges to a Ramsey steady-state $z_{tss}$ after t=S for some large integer S. Thus, our goal now becomes finding $\{z_t\}_{t=1}^{S}$, two integers $T_f$ and $T_l$, and the terminal Ramsey steady state $z_{tss}$ satisfying

$$
\begin{aligned}
f(z_{tss}, z_{tss}, z_S) &= 0 \text{ and } R_{S+1} > 1 \\
f(z_{tss}, z_S, z_{S-1}) &= 0 \text{ and } R_S > 1 \\
f(z_S, z_{S-1}, z_{S-2}) &= 0 \text{ and } R_{S-1} > 1 \\
&... \\
f(z_{T_l+2}, z_{T_l+1}, z_{T_l}) &= 0 \text{ and } R_{T_l+1} > 1 \\
g(z_{T_l+1}, z_{T_l}, z_{T_l-1}) &= 0 \text{ and } \omega_{11,T_l} > 1 \\
&... \\
g(z_{T_f+2}, z_{T_f+1}, z_{T_f}) &= 0 \text{ and } \omega_{11,T_f+1} > 1 \\
g(z_{T_f+1}, z_{T_f}, z_{T_f-1}) &= 0 \text{ and } \omega_{11,T_f} > 1 \\
f(z_{T_f}, z_{T_f-1}, z_{T_f-2}) &= 0 \text{ and } R_{T_f-1} > 1 \\
&... \\
f(z_3, z_2, z_1) &= 0 \text{ and } R_2 > 1 \\
f(z_2, z_1, z_0) &= 0 \text{ and } R_1 > 1
\end{aligned}
$$

## A.2 Solution Method: A Big Picture

I use "Newton within Shooting" algorithm to solve the problem just described. The algorithm proceeds as follows.

- Guess $T_f$ and $T_l$

  Step 1: Guess $z_{tss}$, the terminal Ramsey steady state.

  Step 2: Given $z_{tss}$, use the modified Newton algorithm to solve for $\{z_t\}_{t=1}^S$.

  Step 3: Check the FONCs at $t = T + 1$ is satisfied. If not, adjust the terminal Ramsey steady-state.

- Check if $\omega_{11,t} \geq 0$ and $R_t \geq 1$ for all t. If not, adjust $T_f$ and $T_l$.

## A.3 A Modified Newton Method

This section elaborates the modified Newton method of Julliard et al. (1998) in Step 2. Given $T_f$, $T_l$, and $z_{tss}$, the goal of the modified Newton algorithm is to find $\{z_t\}_{t=1}^S$ satisfying the equilibrium conditions from t=1 to t=S. The number of equations is the same as the number of variables. While I have not been able to prove the uniqueness of the solution, this Newton algorithm returns the unique solution regardless of the starting values used to initiate the algorithm.

By stacking the $z_t$ into a vector $Y = [z_1', z_2', ..., z_{S-1}', z_S']$, we can express the problem as finding Y such that

$$F(Y) = 0$$

where F is a function stacking f and g from t=1 to t=T. As in any Newton algorithm, we start by an initial guess of Y, $Y^{(1)}$. There are two steps:

**Step 2.1:** Given a previous guess $Y^{(k)}$, we compute an adjustment factor $\Delta Y$ such that

$$\left[\frac{\partial F(Y)}{\partial Y}\right]_{Y^{(k)}} \Delta Y = -F(Y^{(k)})$$

**Step 2.2:** If $||\Delta Y|| > \epsilon_{tol}$, we set our next guess as

$$Y^{(k+1)} = Y^{(k)} + \mu \Delta Y$$

and go back to the first step. Otherwise, stop[17].

When the system is small we can find the $\Delta Y$ in Step 2 by inverting $\left[\frac{\partial F(Y)}{\partial Y}\right]_{Y^{(k)}}$. However, when the system is large, inverting this matrix is either computationally costly or infeasible. Julliard et al. (1998) propose to find $\Delta Y$ without inverting $\left[\frac{\partial F(Y)}{\partial Y}\right]_{Y^{(k)}}$. Their method makes use the sparseness of the matrix, which comes from a recursive nature of the problem.

## A.4 Details on Step 2.1

We want to find $\Delta Y$ such that

$$\left[\frac{\partial F(Y)}{\partial Y}\right]_{Y^{(k)}} \Delta Y = -F(Y^{(k)})$$

We will take advantage of the peculiar structure in $\left[\frac{\partial F(Y)}{\partial Y}\right]_{Y^{(k)}}$

$$\begin{bmatrix} \bar{J}_S^{(k)} & M_{S-1}^{(k)} & 0 & 0 & 0 & \dots & \dots & & 0 & 0 \\ L_S^{(k)} & \bar{J}_{S-1}^{(k)} & M_{S-2}^{(k)} & 0 & 0 & \dots & \dots & & 0 & 0 \\ & & & & & & & & & 0 \\ & & & & & & & & & 0 \\ 0 & 0 & 0 & L_{T+1}^{(k)} & J_T^{(k)} & M_{T-1}^{(k)} & \dots & & \dots & 0 \\ 0 & 0 & 0 & 0 & L_T^{(k)} & J_{T-1}^{(k)} & M_{T-2}^{(k)} & & \dots & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & L_3^{(k)} & J_2^{(k)} & M_1^{(k)} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & & L_2^{(k)} & J_1^{(k)} \end{bmatrix} \begin{bmatrix} \Delta z_S^{(k+1)} \\ \Delta z_{S-1}^{(k+1)} \\ \dots \\ \Delta z_{T+1}^{(k+1)} \\ \Delta z_T^{(k+1)} \\ \Delta z_{T-1}^{(k+1)} \\ \dots \\ \Delta z_3^{(k+1)} \\ \Delta z_2^{(k+1)} \\ \Delta z_1^{(k+1)} \end{bmatrix} = \begin{bmatrix} f(z_{ss}, z_S^{(k)}, z_{S-1}^{(k)}) \\ f(z_S^{(k)}, z_{S-1}^{(k)}, z_{S-2}^{(k)}) \\ \dots \\ f(z_{T+1}^{(k)}, z_T^{(k)}, z_{T-1}^{(k)}) \\ g(z_T^{(k)}, z_{T-1}^{(k)}, z_{T-2}^{(k)}) \\ g(z_{T-1}^{(k)}, z_{T-2}^{(k)}, z_{T-3}^{(k)}) \\ \dots \\ g(z_3^{(k)}, z_2^{(k)}, z_1^{(k)}) \\ g(z_2^{(k)}, z_1^{(k)}, z_{ss}) \end{bmatrix}$$

where

$$\begin{aligned} L_t^{(k)} &\equiv \frac{\partial f(z_t^{(k)})}{\partial z_{t+1}} (\equiv \frac{\partial g(z_t^{(k)})}{\partial z_{t+1}}) \\ \bar{J}_t^{(k)} &\equiv \frac{\partial f(z_t^{(k)})}{\partial z_t} \\ J_t^{(k)} &\equiv \frac{\partial g(z_t^{(k)})}{\partial z_t} \\ M_t^{(k)} &\equiv \frac{\partial f(z_t^{(k)})}{\partial z_{t-1}} (\equiv \frac{\partial g(z_t^{(k)})}{\partial z_{t-1}}) \end{aligned}$$

[17] I set $\mu = 0.1$ and $\epsilon_{tol} = 10e-15$. For most parameter values considered, the Newton algorithm converges to satisfy this criteria within 500 iterations.

Notice that the difference between the zero-bound regime and normal regime show up only in the derivative with respect to $z_t$. Write this as follows:

$$\begin{bmatrix} \bar{J}_S^{(k)} & M_{S-1}^{(k)} & 0 & 0 & 0 & ... & ... & & 0 & 0 \\ L_S^{(k)} & \bar{J}_{S-1}^{(k)} & M_{S-2}^{(k)} & 0 & 0 & ... & ... & & 0 & 0 \\ & & & & & & & & & 0 \\ & & & & & & & & & 0 \\ 0 & 0 & 0 & L_{T+1}^{(k)} & J_T^{(k)} & M_{T-1}^{(k)} & ... & & ... & 0 \\ 0 & 0 & 0 & 0 & L_T^{(k)} & J_{T-1}^{(k)} & M_{T-2}^{(k)} & & ... & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & L_3^{(k)} & J_2^{(k)} & M_1^{(k)} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & & L_2^{(k)} & J_1^{(k)} \end{bmatrix} \begin{bmatrix} \Delta z_S^{(k+1)} \\ \Delta z_{S-1}^{(k+1)} \\ ... \\ \Delta z_{T+1}^{(k+1)} \\ \Delta z_T^{(k+1)} \\ \Delta z_{T-1}^{(k+1)} \\ ... \\ \Delta z_3^{(k+1)} \\ \Delta z_2^{(k+1)} \\ \Delta z_1^{(k+1)} \end{bmatrix} = \begin{bmatrix} d_S^{(k)} \\ d_{S-1}^{(k)} \\ ... \\ d_T^{(k)} \\ d_{T-1}^{(k)} \\ d_{T-2}^{(k)} \\ ... \\ d_2^{(k)} \\ d_1^{(k)} \end{bmatrix}$$

which can also be written as follows:

$$\begin{aligned} \bar{J}_S^{(k)}\Delta z_S^{(k+1)} + M_{S-1}^{(k)}\Delta z_{S-1}^{(k+1)} &= d_S^{(k)} \\ L_S^{(k)}\Delta z_S^{(k+1)} + \bar{J}_{S-1}^{(k)}\Delta z_{S-1}^{(k+1)} + M_{S-2}^{(k)}\Delta z_{S-2}^{(k+1)} &= d_{S-1}^{(k)} \\ &... \\ L_{T+2}^{(k)}\Delta z_{T+2}^{(k+1)} + \bar{J}_{T+1}^{(k)}\Delta z_{T+1}^{(k+1)} + M_T^{(k)}\Delta z_{T+1}^{(k+1)} &= d_{T+1}^{(k)} \\ L_{T+1}^{(k)}\Delta z_{T+1}^{(k+1)} + J_T^{(k)}\Delta z_T^{(k+1)} + M_{T-1}^{(k)}\Delta z_T^{(k+1)} &= d_T^{(k)} \\ L_T^{(k)}\Delta z_T^{(k+1)} + J_{T-1}^{(k)}\Delta z_{T-1}^{(k+1)} + M_{T-2}^{(k)}\Delta z_{T-2}^{(k+1)} &= d_{T-1}^{(k)} \\ &... \\ L_3^{(k)}\Delta z_3^{(k+1)} + J_2^{(k)}\Delta z_2^{(k+1)} + M_1^{(k)}\Delta z_1^{(k+1)} &= d_2^{(k)} \\ L_2^{(k)}\Delta z_2^{(k+1)} + J_1^{(k)}\Delta z_1^{(k+1)} &= d_1^{(k)} \end{aligned}$$

Then,

$$\begin{aligned} \Delta z_S^{(k+1)} &= \left[\bar{J}_S^{(k)}\right]^{-1} d_S^{(k)} - \left[\bar{J}_S^{(k)}\right]^{-1} M_{S-1}^{(k)}\Delta z_{S-1}^{(k+1)} \\ &= \Lambda_S^{(k)} + \Phi_S^{(k)}\Delta z_{S-1}^{(k+1)} \end{aligned}$$

Plug this into the second equation to obtain

$$L_S^{(k)}[\Lambda_S^{(k)} + \Phi_S^{(k)}\Delta z_{S-1}^{(k+1)}] + \bar{J}_{S-1}^{(k)}\Delta z_{S-1}^{(k+1)} + M_{S-2}^{(k)}\Delta z_{S-2}^{(k+1)} = d_{S-1}^{(k)}$$

$$\Rightarrow$$

$$\begin{aligned} \Delta z_{S-1}^{(k+1)} &= \left[L_S^{(k)}\Phi_S^{(k)} + \bar{J}_{S-1}^{(k)}\right]^{-1}\left[d_{S-1}^{(k)} - L_S^{(k)}\Lambda_S^{(k)}\right] - \left[L_S^{(k)}\Phi_S^{(k)} + \bar{J}_{S-1}^{(k)}\right]^{-1} M_{S-2}^{(k)}\Delta z_{S-2}^{(k+1)} \\ &= \Lambda_{S-1}^{(k)} + \Phi_{S-1}^{(k)}\Delta z_{S-2}^{(k+1)} \end{aligned}$$

and so on. The outcome of this process is a sequence of time-varying coefficients for the law of motion.

$$
\begin{aligned}
\Delta z_S^{(k+1)} &= \Lambda_S^{(k)} + \Phi_S^{(k)} \Delta z_{S-1}^{(k+1)} \\
\Delta z_{S-1}^{(k+1)} &= \Lambda_{S-1}^{(k)} + \Phi_{S-1}^{(k)} \Delta z_{S-2}^{(k+1)} \\
&\dots \\
\Delta z_{T+1}^{(k+1)} &= \Lambda_{T+1}^{(k)} + \Phi_{T+1}^{(k)} \Delta z_T^{(k+1)} \\
\Delta z_T^{(k+1)} &= \Lambda_T^{(k)} + \Phi_T^{(k)} \Delta z_{T-1}^{(k+1)} \\
&\dots \\
\Delta z_2^{(k+1)} &= \Lambda_2^{(k)} + \Phi_2^{(k)} \Delta z_1^{(k+1)} \\
\Delta z_1^{(k+1)} &= \Lambda_1^{(k)}
\end{aligned}
$$

where the coefficients are computed recursively as follows:

$$
\begin{aligned}
\Lambda_S^{(k)} &= \left[\bar{J}_S^{(k)}\right]^{-1} d_S^{(k)} \\
\Phi_S^{(k)} &= -\left[\bar{J}_S^{(k)}\right]^{-1} M_{S-1}^{(k)} \\
\Lambda_{S-1}^{(k)} &= \left[L_S^{(k)} \Phi_S^{(k)} + \bar{J}_{S-1}^{(k)}\right]^{-1} \left[d_{S-1}^{(k)} - L_S^{(k)} \Lambda_S^{(k)}\right] \\
\Phi_{S-1}^{(k)} &= -\left[L_S^{(k)} \Phi_S^{(k)} + \bar{J}_{S-1}^{(k)}\right]^{-1} M_{S-2}^{(k)} \\
&\dots \\
\Lambda_{T+1}^{(k)} &= \left[L_{T+1}^{(k)} \Phi_{T+2}^{(k)} + \bar{J}_T^{(k)}\right]^{-1} \left[d_T^{(k)} - L_{T+1}^{(k)} \Lambda_{T+2}^{(k)}\right] \\
\Phi_{T+1}^{(k)} &= -\left[L_{T+1}^{(k)} \Phi_{T+2}^{(k)} + \bar{J}_{S-1}^{(k)}\right]^{-1} M_{T-1}^{(k)} \\
\Lambda_T^{(k)} &= \left[L_S^{(k)} \Phi_S^{(k)} + \bar{J}_{S-1}^{(k)}\right]^{-1} \left[d_{S-1}^{(k)} - L_S^{(k)} \Lambda_{T+1}^{(k)}\right] \\
\Phi_T^{(k)} &= -\left[L_S^{(k)} \Phi_S^{(k)} + \bar{J}_{S-1}^{(k)}\right]^{-1} M_{S-2}^{(k)} \\
&\dots
\end{aligned}
$$

and so on.

# B Piecewise First-Order Solution

My algorithm is an extension of Eggertsson and Woodford (2003), and a variation of Christiano (2010). As in Christiano et al., and unlike EW, my algorithm uses QZ decomposition to be able to handle models with lagged variables. As in EW, and unlike Christiano et al., my algorithm derives time-varying coefficients for the law of motion describing the evolution of the model's variables.

## B.1 Problem

We want to find $z_1, z_2, ..., z_T, z_{T+1}$ such that

$$
\begin{aligned}
z_{T+1} &= Fz_T + d_{T+1} \\
J_0 z_{T+1} + J_1 z_T + J_2 z_{T-1} &= d_T \\
J_0 z_T + J_1 z_{T-1} + J_2 z_{T-2} &= d_{T-1} \\
&\dots \\
J_0 z_{S+2} + J_1 z_{S+1} + J_2 z_S &= d_{S+1} \\
\bar{J}_0 z_{S+1} + \bar{J}_1 z_S + \bar{J}_2 z_{S-1} &= d_S \\
\bar{J}_0 z_S + \bar{J}_1 z_{S-1} + \bar{J}_2 z_{S-2} &= d_{S-1} \\
&\dots \\
\bar{J}_0 z_3 + \bar{J}_1 z_2 + \bar{J}_2 z_1 &= d_2 \\
\bar{J}_0 z_2 + \bar{J}_1 z_1 + \bar{J}_2 z_0 &= d_1
\end{aligned}
$$

where $z_0$ and $d_1, ..., d_{T+1}$ is given, and $F$, $J_0$, $J_1$, $J_2$, $\bar{J}_0$, $\bar{J}_1$, and $\bar{J}_2$ are known. $J_0 \neq 0$ and $\bar{J}_0 \neq 0$.

## B.2 Solution

First, find Q, Z, $\bar{Q}$, and $\bar{Z}$ such that

$$
\begin{aligned}
QJ_0 &= HZ' \\
QJ_1 &= GZ' \\
\bar{Q}\bar{J}_0 &= \bar{H}\bar{Z}' \\
\bar{Q}\bar{J}_1 &= \bar{G}\bar{Z}'
\end{aligned}
$$

where the last `nbottom` rows of H are zero. Multiplying some of the equations above by Q,

$$
\begin{aligned}
QJ_0 z_{T+1} + QJ_1 z_T &= Qd_T - QJ_2 z_{T-1} \\
QJ_0 z_T + QJ_1 z_{T-1} &= Qd_{T-1} - QJ_2 z_{T-2} \\
&\dots \\
QJ_0 z_{S+2} + QJ_1 z_{S+1} &= d_{S+1} - QJ_2 z_S
\end{aligned}
$$

Using the QZ decomposition,

$$
\begin{aligned}
HZ' z_{T+1} + GZ' z_T &= Qd_T - QJ_2 ZZ' z_{T-1} \\
HZ' z_T + GZ' z_{T-1} &= Qd_{T-1} - QJ_2 ZZ' z_{T-1} \\
&\dots \\
HZ' z_{S+2} + GZ' z_{S+1} &= Qd_{S+1} - QJ_2 ZZ' z_S
\end{aligned}
$$

and

$$\begin{aligned}
H\gamma_{T+1} + G\gamma_T &= Qd_T - L\gamma_{T-1} \\
H\gamma_T + G\gamma_{T-1} &= Qd_{T-1} - L\gamma_{T-2} \\
&\dots \\
H\gamma_{S+2} + G\gamma_{S+1} &= d_{S+1} - LZ\gamma_S
\end{aligned}$$

and

$$\begin{aligned}
H_{11}\gamma_{T+1}^{top} + H_{12}\gamma_{T+1}^{bottom} + G_{11}\gamma_T^{top} + G_{12}\gamma_T^{bottom} &= Qd_T^{top} - L_{11}\gamma_{T-1}^{top} - L_{12}\gamma_{T-1}^{bottom} \\
G_{22}\gamma_T^{bottom} &= Qd_T^{bottom} - L_{21}\gamma_{T-1}^{top} - L_{22}\gamma_{T-1}^{bottom} \\
H_{11}\gamma_T^{top} + H_{12}\gamma_T^{bottom} + G_{11}\gamma_{T-1}^{top} + G_{12}\gamma_{T-1}^{bottom} &= Qd_{T-1}^{top} - L_{11}\gamma_{T-2}^{top} - L_{12}\gamma_{T-2}^{bottom} \\
G_{22}\gamma_{T-1}^{bottom} &= Qd_{T-1}^{bottom} - L_{21}\gamma_{T-2}^{top} - L_{22}\gamma_{T-2}^{bottom} \\
&\dots \\
H_{11}\gamma_{S+2}^{top} + H_{12}\gamma_{S+2}^{bottom} + G_{11}\gamma_{S+1}^{top} + G_{12}\gamma_{S+1}^{bottom} &= Qd_{S+1}^{top} - L_{11}\gamma_S^{top} - L_{12}\gamma_S^{bottom} \\
G_{22}\gamma_{S+1}^{bottom} &= Qd_{S+1}^{bottom} - L_{21}\gamma_S^{top} - L_{22}\gamma_S^{bottom}
\end{aligned}$$

Notice that

$$\begin{aligned}
z_{T+1} &= Fz_T + d_{T+1} \\
\Leftrightarrow Z'z_{T+1} &= Z'FZZ'z_T + Z'd_{T+1} \\
\Leftrightarrow \gamma_{T+1} &= Z'FZ\gamma_T + Z'd_{T+1} \\
\Leftrightarrow \gamma_{T+1} &= K\gamma_T + D_{T+1} \\
\Leftrightarrow & \\
\gamma_{T+1}^{top} &= K_{11,T+1}\gamma_T^{top} + K_{12,T+1}\gamma_T^{bottom} + D_{T+1}^{top} \\
\gamma_{T+1}^{bottom} &= K_{21,T+1}\gamma_T^{top} + K_{22,T+1}\gamma_T^{bottom} + D_{T+1}^{top}
\end{aligned}$$

Using the same recursion formula, we obtain

$$\begin{aligned}
\gamma_T^{top} &= K_{11,T}\gamma_{T-1}^{top} + K_{12,T}\gamma_{T-1}^{bottom} + D_T^{top} \\
\gamma_T^{bottom} &= K_{21,T}\gamma_{T-1}^{top} + K_{22,T}\gamma_{T-1}^{bottom} + D_T^{botom} \\
\gamma_{T-1}^{top} &= K_{11,T-1}\gamma_{T-2}^{top} + K_{12,T-1}\gamma_{T-2}^{bottom} + D_{T-1}^{top} \\
\gamma_{T-1}^{bottom} &= K_{21,T-1}\gamma_{T-2}^{top} + K_{22,T-1}\gamma_{T-2}^{bottom} + D_{T-1}^{botom} \\
&\dots \\
\gamma_{S+1}^{top} &= K_{11,S+1}\gamma_S^{top} + K_{12,S+1}\gamma_S^{bottom} + D_{S+1}^{top} \\
\gamma_{S+1}^{bottom} &= K_{21,S+1}\gamma_S^{top} + K_{22,S+1}\gamma_S^{bottom} + D_{S+1}^{botom}
\end{aligned}$$

Notice that

$$
\begin{aligned}
\gamma_{S+1}^{top} &= K_{11,S+1}\gamma_S^{top} + K_{12,S+1}\gamma_S^{bottom} + D_{S+1}^{top} \\
\gamma_{S+1}^{bottom} &= K_{21,S+1}\gamma_S^{top} + K_{22,S+1}\gamma_S^{bottom} + D_{S+1}^{botom} \\
&\Leftrightarrow \\
\gamma_{S+1} &= K_{S+1}\gamma_S + D_{S+1} \\
\Leftrightarrow Z\gamma_{S+1} &= ZK_{S+1}Z'Z\gamma_S + ZD_{S+1} \\
\Leftrightarrow z_{S+1} &= ZK_{S+1}z_S + ZD_{S+1} \\
\Leftrightarrow \bar{Z}'z_{S+1} &= \bar{Z}'ZK_{S+1}\bar{Z}\bar{Z}'z_S + \bar{Z}'ZD_{S+1} \\
\Leftrightarrow \bar{\gamma}_{S+1} &= \bar{Z}'ZK_{S+1}\bar{Z}\bar{\gamma}_S + \bar{Z}'ZD_{S+1} \\
\Leftrightarrow \bar{\gamma}_{S+1} &= \bar{K}_{S+1}\bar{\gamma}_S + \bar{D}_{S+1} \\
\Leftrightarrow \bar{\gamma}_{S+1}^{top} &= \bar{K}_{11,S+1}\bar{\gamma}_S^{top} + \bar{K}_{12,S+1}\bar{\gamma}_S^{bottom} + \bar{D}_{S+1}^{top} \\
\Leftrightarrow \bar{\gamma}_{S+1}^{bottom} &= \bar{K}_{21,S+1}\bar{\gamma}_S^{top} + \bar{K}_{22,S+1}\bar{\gamma}_S^{bottom} + \bar{D}_{S+1}^{bottom}
\end{aligned}
$$

where

$$\bar{K}_{S+1} = \bar{Z}'ZK_{S+1}\bar{Z}$$

Now, we will use

$$
\begin{aligned}
\bar{J}_0 z_{S+1} + \bar{J}_1 z_S &= d_S - \bar{J}_2 z_{S-1} \\
\bar{J}_0 z_S + \bar{J}_1 z_{S-1} &= d_{S-1} - \bar{J}_2 z_{S-2} \\
&\dots \\
\bar{J}_0 z_3 + \bar{J}_1 z_2 &= d_2 - \bar{J}_2 z_1 \\
\bar{J}_0 z_2 + \bar{J}_1 z_1 &= d_1 - \bar{J}_2 z_0
\end{aligned}
$$

$$
\begin{aligned}
\bar{Q}\bar{J}_0 z_{S+1} + \bar{Q}\bar{J}_1 z_S &= \bar{Q}d_S - \bar{Q}\bar{J}_2 z_{S-1} \\
\bar{Q}\bar{J}_0 z_S + \bar{Q}\bar{J}_1 z_{S-1} &= \bar{Q}d_{S-1} - \bar{Q}\bar{J}_2 z_{S-2} \\
&\dots \\
\bar{Q}\bar{J}_0 z_3 + \bar{Q}\bar{J}_1 z_2 &= \bar{Q}d_2 - \bar{Q}\bar{J}_2 z_1 \\
\bar{Q}\bar{J}_0 z_2 + \bar{Q}\bar{J}_1 z_1 &= \bar{Q}d_1 - \bar{Q}\bar{J}_2 z_0
\end{aligned}
$$

Using the QZ decomposition,

$$
\begin{aligned}
\bar{H}\bar{Z}'z_{S+1} + \bar{G}\bar{Z}'z_S &= \bar{Q}d_S - \bar{Q}\bar{J}_2\bar{Z}\bar{Z}'z_{S-1} \\
\bar{H}\bar{Z}'z_S + \bar{G}\bar{Z}'z_{S-1} &= \bar{Q}d_{S-1} - \bar{Q}\bar{J}_2\bar{Z}\bar{Z}'z_{S-2} \\
&\;\;... \\
\bar{H}\bar{Z}'z_2 + \bar{G}\bar{Z}'z_1 &= \bar{Q}d_1 - \bar{Q}\bar{J}_2\bar{Z}\bar{Z}'z_0
\end{aligned}
$$

$$
\begin{aligned}
\bar{H}\bar{\gamma}_{S+1} + \bar{G}\bar{\gamma}_S &= \bar{Q}d_S - \bar{L}\bar{\gamma}_{S-1} \\
\bar{H}\bar{\gamma}_S + \bar{G}\bar{\gamma}_{S-1} &= \bar{Q}d_{S-1} - \bar{L}\bar{\gamma}_{S-2} \\
&\;\;... \\
\bar{H}\bar{\gamma}_2 + \bar{G}\bar{\gamma}_1 &= \bar{Q}d_1 - \bar{L}\bar{\gamma}_0
\end{aligned}
$$

Using the same recursion formula, we obtain

$$
\begin{aligned}
\bar{\gamma}_S^{top} &= K_{11,S}\bar{\gamma}_{S-1}^{top} + K_{12,S}\bar{\gamma}_{S-1}^{bottom} + D_S^{top} \\
\bar{\gamma}_S^{bottom} &= K_{21,S}\bar{\gamma}_{S-1}^{top} + K_{22,S}\bar{\gamma}_{S-1}^{bottom} + D_S^{botom} \\
\bar{\gamma}_{S-1}^{top} &= K_{11,S-1}\bar{\gamma}_{S-2}^{top} + K_{12,S-1}\bar{\gamma}_{S-2}^{bottom} + D_{S-1}^{top} \\
\bar{\gamma}_{S-1}^{bottom} &= K_{21,S-1}\bar{\gamma}_{S-2}^{top} + K_{22,S-1}\bar{\gamma}_{S-2}^{bottom} + D_{S-1}^{botom} \\
&\;\;... \\
\bar{\gamma}_1^{top} &= K_{11,1}\bar{\gamma}_0^{top} + K_{12,1}\bar{\gamma}_0^{bottom} + D_1^{top} \\
\bar{\gamma}_1^{bottom} &= K_{21,1}\bar{\gamma}_0^{top} + K_{22,1}\bar{\gamma}_0^{bottom} + D_1^{botom}
\end{aligned}
$$

where

$$
\begin{aligned}
K_{11,S} &\equiv \bar{\Phi}_{1,S+1}^{-1}\bar{\Phi}_{2,S+1}\bar{G}_{22}^{-1}\bar{L}_{21} - \Phi_{1,S+1}^{-1}\bar{L}_{11} \\
K_{12,S} &\equiv \bar{\Phi}_{1,S+1}^{-1}\bar{\Phi}_{2,S+1}\bar{G}_{22}^{-1}\bar{L}_{21} - \Phi_{1,S+1}^{-1}\bar{L}_{12} \\
D_S^{top} &\equiv \bar{\Phi}_{1,S+1}^{-1}(Qd_S^{top} - \bar{H}_{11}D_{S+1}^{top} - \bar{H}_{12}D_{S+1}^{bottom}) - \bar{\Phi}_{1,S+1}^{-1}\bar{\Phi}_{2,S+1}G_{22}^{-1}Qd_1^{bottom} \\
K_{21,S} &\equiv -\bar{G}_{22}^{-1}\bar{L}_{21} \\
K_{22,S} &\equiv -\bar{G}_{22}^{-1}\bar{L}_{22} \\
D_S^{botom} &\equiv \bar{G}_{22}^{-1}Qd_1^{bottom}
\end{aligned}
$$

and so on.

# C List of First-Order Necessary Conditions for Government's Problem

$$
\begin{aligned}
C_t &: \quad C_t^{-\chi_c} - \omega_{1,t}\chi_c \frac{1}{1+\tau_{c,t}} C_t^{-\chi_c-1} + \omega_{1,t-1}\chi_c R_{t-1} \frac{1}{1+\tau_{c,t}} C_t^{-\chi_c-1}\Pi_t^{-1} - \omega_{2,t}\chi_c\chi_{n,0}N_t^{\chi_{n,1}}C_t^{\chi_c-1} \\
&\quad +\omega_{6,t}\chi_c \frac{Y_t w_t}{1+\tau_{c,t}} C_t^{-\chi_c-1} + \omega_{7,t}\chi_c \frac{1}{1+\tau_{c,t}} Y_t C_t^{-\chi_c-1} - \omega_{9,t} - \omega_{10,t}\tau_{c,t} = 0 \\
Y_t &: \quad -\omega_{6,t}\frac{1}{1+\tau_{c,t}} w_t C_t^{-\chi_c} - \omega_{7,t}\frac{1}{1+\tau_{c,t}} C_t^{-\chi_c} + \omega_{8,t}s_t + \omega_{9,t} = 0 \\
N_t &: \quad -\chi_{n,0}N_t^{\chi_{n,1}} - \omega_{2,t}\chi_{n,1}\chi_{n,0}N_t^{\chi_{n,1}-1}C_t^{\chi_c} - \omega_{8,t} - \omega_{10,t}\tau_{n,t}w_t = 0 \\
w_t &: \quad \omega_{2,t}\frac{1-\tau_{n,t}}{1+\tau_{c,t}} - \omega_{6,t}\frac{1-\nu_t}{1+\tau_{c,t}} Y_t C_t^{-\chi_c} - \omega_{10,t}\tau_{n,t}N_t = 0 \\
\Pi_t &: \quad \omega_{1,t-1}R_{t-1}C_t^{-\chi_c}\Pi_t^{-2} - \omega_{3,t}\theta\zeta_p\Pi_t^{\theta-1}s_{t-1} - \omega_{4,t}(\theta-1)\zeta_p\Pi_t^{\theta-2} \\
&\quad -\omega_{6,t-1}\theta\zeta_p\Pi_t^{\theta-1}C_{n,t} - \omega_{7,t-1}(\theta-1)\zeta_p\Pi_t^{\theta-2}C_{d,t} = 0 \\
s_t &: \quad \omega_{3,t} - \omega_{3,t+1}\beta\delta_t\zeta_p\Pi_{t+1}^{\theta} + \omega_{8,t}Y_t - \omega_{10,t}b_{t-1}\Pi_t^{-2} = 0 \\
p_t^* &: \quad \omega_{3,t}\theta(1-\zeta_p)\left[p_t^*\right]^{-\theta-1} + \omega_{4,t}(\theta-1)(1-\zeta_p)\left[p_t^*\right]^{-\theta} + \omega_{5,t} = 0 \\
C_{n,t} &: \quad -\omega_{5,t}\frac{\theta}{\theta-1}\frac{1}{C_{d,t}} + \omega_{6,t} - \omega_{6,t-1}\zeta_p\Pi_t^{\theta} = 0 \\
C_{d,t} &: \quad \omega_{5,t}\frac{\theta}{\theta-1}C_{n,t}C_{d,t}^{-2} + \omega_{7,t} - \omega_{7,t-1}\zeta_p\Pi_t^{\theta-1} = 0 \\
R_t &: \quad -\omega_{1,t}\beta\delta_t C_{t+1}^{-\chi_c}\Pi_{t+1}^{-1} + \omega_{10,t}R_t^{-2}b_t + \omega_{11,t} = 0 \\
&\quad \omega_{11,t} = 0 \text{ if } R_t > 1, \omega_{11,t} > 0 \text{ if } R_t = 1 \\
G_t &: \quad \chi_{g,0}G_t^{-\chi_{g,1}} - \omega_{9,t} + \omega_{10,t} = 0 \\
\tau_{c,t} &: \quad -\omega_{1,t}(1+\tau_{c,t})^{-2}C_t^{-\chi_c} + \omega_{1,t-1}(1+\tau_{c,t})^{-2}R_{t-1}C_{t-1}^{-\chi_c} \\
&\quad -\omega_{2,t}(1+\tau_{c,t})^{-2}w_t + \omega_{6,t}(1+\tau_{c,t})^{-2}Y_t w_t C_t^{-\chi_c} \\
&\quad +\omega_{7,t}(1+\tau_{c,t})^{-2}Y_t C_t^{-\chi_c} - \omega_{10,t}C_t = 0 \\
\tau_{n,t} &: \quad -\omega_{2,t}\frac{1}{1+\tau_{c,t}}w_t - \omega_{10,t}w_t N_t = 0 \\
b_t &: \quad -\omega_{10,t}R_t^{-1} + \omega_{10,t+1}\beta\delta_t\Pi_{t+1}^{-1} = 0 \\
T_t &: \quad -\omega_{10,t} = 0
\end{aligned}
$$

as well as the equations characterizing the equilibrium of the model shown in Section 2.6.

Table 1: Calibration

| Parameter | Description | Calibrated Value |
|---|---|---|
| $\beta$ | Discount factor | $\frac{1}{1+0.01}$ |
| $\chi_c$ | Inverse intertemporal elasticity of substitution for $C_t$ | [**1/6**, 0.5, 1.0, 2.0] |
| $\chi_{n,0}$ | Labor supply disutility | 1.00 |
| $\chi_{n,1}$ | Inverse labor supply elasticity | [0.5, **1.0**, 2.0] |
| $\chi_{g,0}$ | Utility weight on $G_t$ | [**0.2**, 0.5,1.0] |
| $\chi_{g,1}$ | Intertemporal elasticity of substitution for $G_t$ | 1.0 |
| $\theta$ | Elasticity of substitution among intermediate goods | 10 |
| $\zeta_p$ | Calvo parameter | [0.6, **0.75**, 0.85] |
| $b_0/4Y_0$ | Initial debt-to-output ratio | [-2, -1, 0, **0.5**, 1, 2] |
| $\epsilon_{\delta,1}$ | The size of the discount factor shock | [**0.02**, 0.00225, 0.0025] |
| $\rho_\delta$ | AR(1) coefficient for discount factor shock | [**0.9**, 0.925, 0.95] |
| $\rho_\Pi$ | coefficient on inflation in the Taylor-rule | 2.0 |
| $\rho_P$ | coefficient on price level in the Taylor-rule | 0.5 |
| $\alpha_n$ | coefficient on lagged debt in the tax-rate rule | 0.2 |
| $\alpha_c$ | coefficient on lagged debt in the tax-rate rule | 0.05 |

Table 2: Welfare Gain From Government Spending Policy: Model with Lump-Sum Tax

| | Welfare Loss | Welfare |
|---|---|---|
| $\{R_t, G_t, T_t\}$ chosen optimally | | 11.4111 |
| $\{R_t\}$ chosen optimally, $\{G_t, T_t\}$ fixed | 0.11 | 11.4102 |
| Simple policy rules | 6.14 | 11.3636 |

*Note that the welfare gain/loss is computed as the one-time transfer of consumption to the household. To compare the number here with welfare gain number based on perpetual transfer (as in Lucas' welfare calculation), you need to divide the number here by 100 ($=\frac{1}{1-\beta}$), or multiply the Lucas' number by 100.

Table 3: Welfare Effects From Various Fiscal Policy Instruments: Model with Labor Income Tax and Debt

| | Welfare Loss | Welfare |
|---|---|---|
| $\{R_t, G_t, \tau_{n,t}, b_t\}$ chosen optimally | | 15.6270 |
| $\{R_t, \tau_{n,t}, b_t\}$ chosen optimally, $G_t$ fixed | 0.17 | 15.6259 |
| $\{R_t, G_t, b_t\}$ chosen optimally, $\tau_{n,t}$ fixed | 0.46 | 15.6238 |
| $\{R_t, G_t, \tau_{n,t}\}$ chosen optimally, $b_t$ fixed | 0.70 | 15.6222 |
| Simple policy rules | 8.58 | 15.5676 |

With Alternative Initial Debt Levels

| Initial $b/4Y$ | $WG_G(WG_{\tau,n}, WG_b, WG_{opt})$ |
|---|---|
| -2.0 | 0.76 (7.8,9.0,15.1) |
| -1.0 | 0.20 (1.8,2.5,11.5) |
| 0.0 | 0.07 (0.1,1.2,9.1) |
| 1.0 | 0.45 (1.7,2.9,8.6) |
| 2.0 | 2.12 (6.0,11.7,10.5) |

*Note that the welfare gain/loss is computed as the one-time transfer of consumption to the household. To compare the number here with welfare gain number based on perpetual transfer (as in Lucas' welfare calculation), you need to divide the number here by 100 ($=\frac{1}{1-\beta}$), or multiply the Lucas' number by 100.
**$WG_{opt}$ is the welfare gain of moving from the economy with non-optimizing government following simple policy rules to the economy with optimizing governments with all instruments available.

Table 4: Welfare Effects From Various Fiscal Policy Instruments: Model with Consumption Tax and Debt

| | Welfare Loss | Welfare |
|---|---|---|
| $\{R_t, G_t, \tau_{c,t}, b_t\}$ chosen optimally | | 16.3244 |
| $\{R_t, \tau_{c,t}, b_t\}$ chosen optimally, $G_t$ fixed | 0.02 | 16.3243 |
| $\{R_t, G_t, b_t\}$ chosen optimally, $\tau_{c,t}$ fixed | 1.30 | 16.3154 |
| $\{R_t, G_t, \tau_{c,t}\}$ chosen optimally, $b_t$ fixed | 0.68 | 16.3198 |
| Simple policy rules | 11.0 | 16.2475 |

With Alternative Initial Debt Levels

| Initial $b/4Y$ | $WG_G(WG_{\tau,c}, WG_b, WG_{opt})$ |
|---|---|
| -2.0 | 0.18 (2.5,2.4,6.0) |
| -1.0 | 0.03 (1.9,0.82,7.8) |
| 0.0 | 0.00 (1.0,0.35,9.8) |
| 1.0 | 0.13 (2.1,1.3,12.2) |
| 2.0 | 0.21 (5.6,3.1,13.7) |

*See the footnote in Table 3.

Table 5: Sensitivity Analysis: Model with Lump-Sum Tax

With Alternative IES

| $\chi_c$ | $\Delta G_1/G_{ss}$ | Last t with $R_t = 1$ | $WG_G(WG_{opt})$ |
|---|---|---|---|
| 1/6 | 6.4 | 9 | 0.08 (6.0) |
| 0.5 | 4.4 | 10 | 0.06 (2.8) |
| 1.0 | 3.4 | 10 | 0.04 (1.8) |
| 2.0 | 2.5 | 11 | 0.02 (1.4) |

With Alternative Price Stickiness

| $\zeta_p$ | $\Delta G_1/G_{ss}$ | Last t with $R_t = 1$ | $WG_G(WG_{opt})$ |
|---|---|---|---|
| 0.6 | 3.8 | 8 | 0.03 (3.6) |
| 0.75 | 6.4 | 9 | 0.08 (6.0) |
| 0.85 | 8.8 | 10 | 0.19 (8.3) |

With Alternative Labor Supply Elasticity

| $\chi_{n,1}$ | $\Delta G_1/G_{ss}$ | Last t with $R_t = 1$ | $WG_G(WG_{opt})$ |
|---|---|---|---|
| 0.5 | 4.3 | 9 | 0.04 (4.4) |
| 1.0 | 6.4 | 9 | 0.08 (6.0) |
| 2.0 | 9.0 | 8 | 0.17 (8.2) |

With Alternative Sizes of Government Sector

| $\chi_{g,0}$ | $\Delta G_1/G_{ss}$ | Last t with $R_t = 1$ | $WG_G(WG_{opt})$ |
|---|---|---|---|
| 0.2 | 6.4 | 9 | 0.08 (6.0) |
| 0.5 | 4.7 | 9 | 0.22 (6.7) |
| 1.0 | 2.7 | 10 | 0.45 (8.2) |

With Alternative Magnitudes of Shocks

| $\epsilon_{1,\delta}$ | $\Delta G_1/G_{ss}$ | Last t with $R_t = 1$ | $WG_G(WG_{opt})$ |
|---|---|---|---|
| 0.02 | 6.4 | 9 | 0.08 (6.0) |
| 0.0225 | 8.6 | 10 | 0.16 (8.5) |
| 0.025 | 11.1 | 11 | 0.26 (11.3) |

With Alternative Persistence of Shocks

| $\rho_\delta$ | $\Delta G_1/G_{ss}$ | Last t with $R_t = 1$ | $WG_G(WG_{opt})$ |
|---|---|---|---|
| 0.9 | 6.1 | 9 | 0.08 (6.0) |
| 0.925 | 7.1 | 11 | 0.11 (7.9) |
| 0.95 | 8.2 | 16 | 0.16 (11.8) |

*See the footnote in Table 3.

Table 6: Sensitivity Analysis: Model with Labor Income Tax and Debt

With Alternative IES

| $\chi_c$ | $\Delta G_1/G_{ss}$ | $\Delta\tau_{n,1}$ | Last t with $R_t = 1$ | $WG_G(WG_{\tau,n}, WG_b, WG_{opt})$ |
|---|---|---|---|---|
| 1/6 | 6.1 | 2.0 | 9 | 0.17 (0.46,0.70,8.6) |
| 0.5 | 3.3 | 2.7 | 10 | 0.08 (0.53,0.40,2.80) |
| 1.0 | 1.9 | 2.4 | 11 | 0.01 (0.52,0.26,1.48) |
| 2.0 | 1.0 | 2.1 | 11 | 0.01 (0.44,0.15,8.0) |

With Alternative Price Stickiness

| $\zeta_p$ | $\Delta G_1/G_{ss}$ | $\Delta\tau_{n,1}$ | Last t with $R_t = 1$ | $WG_G(WG_{\tau,n}, WG_b, WG_{opt})$ |
|---|---|---|---|---|
| 0.6 | 3.8 | 1.2 | 8 | 0.04 (0.43,0.79,9.1) |
| 0.75 | 6.1 | 2.0 | 9 | 0.17 (0.46,0.70,8.6) |
| 0.85 | 8.4 | 2.1 | 10 | 0.28 (0.50,0.61,8.0) |

With Alternative Labor Supply Elasticity

| $\chi_{n,1}$ | $\Delta G_1/G_{ss}$ | $\Delta\tau_{n,1}$ | Last t with $R_t = 1$ | $WG_G(WG_{\tau,n}, WG_b, WG_{opt})$ |
|---|---|---|---|---|
| 0.5 | 4.4 | 0.90 | 9 | 0.52 (0.60,1.6,8.1) |
| 1.0 | 6.1 | 2.0 | 9 | 0.17 (0.46,0.70,8.6) |
| 2.0 | 8.5 | 3.1 | 9 | 0.18 (0.48,0.37,9.9) |

With Alternative Sizes of Government Sector

| $\chi_{g,0}$ | $\Delta G_1/G_{ss}$ | $\Delta\tau_{n,1}$ | Last t with $R_t = 1$ | $WG_G(WG_{\tau,n}, WG_b, WG_{opt})$ |
|---|---|---|---|---|
| 0.2 | 6.1 | 2.0 | 9 | 0.17 (0.46,0.70,8.6) |
| 0.5 | 5.2 | 1.6 | 9 | 1.0 (0.21,1.8,16.6) |
| 1.0 | 4.5 | 1.4 | 9 | 8.0 (0.21,4.6,34.5) |

With Alternative Magnitudes of Shocks

| $\epsilon_{1,\delta}$ | $\Delta G_1/G_{ss}$ | $\Delta\tau_{n,1}$ | Last t with $R_t = 1$ | $WG_G(WG_{\tau,n}, WG_b, WG_{opt})$ |
|---|---|---|---|---|
| 0.02 | 6.1 | 2.0 | 9 | 0.17 (0.46,0.70,8.6) |
| 0.0225 | 8.2 | 2.9 | 10 | 0.15 (0.54,0.82,12.1) |
| 0.025 | 10.4 | 3.9 | 11 | 0.33 (0.77,1.0,16.2) |

With Alternative Persistence of Shocks

| $\rho_\delta$ | $\Delta G_1/G_{ss}$ | $\Delta\tau_{n,1}$ | Last t with $R_t = 1$ | $WG_G(WG_{\tau,n}, WG_b, WG_{opt})$ |
|---|---|---|---|---|
| 0.9 | 6.1 | 2.0 | 9 | 0.17 (0.46,0.70,8.6) |
| 0.925 | 7.0 | 2.1 | 11 | 0.18 (0.69,0.91,11.7) |
| 0.95 | 8.2 | 2.1 | 16 | 0.36 (1.4,1.3,16.0) |

*See the footnote in Table 3.

Table 7: Sensitivity Analysis: Model with Consumption Tax and Debt

With Alternative IES

| $\chi_c$ | $\Delta G_1/G_{ss}$ | $\Delta\tau_{c,1}$ | Last t with $R_t = 1$ | $WG_G(WG_{\tau,c}, WG_b, WG_{opt})$ |
|---|---|---|---|---|
| 1/6 | 0.5 | -3.6 | 11 | 0.02 (1.3,0.7,11) |
| 0.5 | 0.6 | -3.8 | 11 | 0.02 (0.64,0.65,4.2) |
| 1.0 | 0.8 | -4.0 | 11 | 0.00 (0.44,0.36,2.4) |
| 2.0 | 0.9 | -4.3 | 11 | 0.01 (0.38,0.17,1.5) |

With Alternative Price Stickiness

| $\zeta_p$ | $\Delta G_1/G_{ss}$ | $\Delta\tau_{c,1}$ | Last t with $R_t = 1$ | $WG_G(WG_{\tau,c}, WG_b, WG_{opt})$ |
|---|---|---|---|---|
| 0.6 | 0.6 | -3.3 | 11 | 0.05 (0.8,0.6,6) |
| 0.75 | 0.5 | -3.6 | 11 | 0.02 (1.3,0.7,11) |
| 0.85 | 0.4 | -3.7 | 11 | 0.03 (1.2,0.7,16) |

With Alternative Labor Supply Elasticity

| $\chi_{n,1}$ | $\Delta G_1/G_{ss}$ | $\Delta\tau_{c,1}$ | Last t with $R_t = 1$ | $WG_G(WG_{\tau,c}, WG_b, WG_{opt})$ |
|---|---|---|---|---|
| 0.5 | 0.7 | -3.5 | 11 | 0.12 (0.3,1.2,9.2) |
| 1.0 | 0.5 | -3.6 | 11 | 0.02 (1.3,0.7,11) |
| 2.0 | 0.3 | -3.6 | 11 | 0.00 (1.4,0.4,13) |

With Alternative Sizes of Government Sector

| $\chi_{g,0}$ | $\Delta G_1/G_{ss}$ | $\Delta\tau_{c,1}$ | Last t with $R_t = 1$ | $WG_G(WG_{\tau,c}, WG_b, WG_{opt})$ |
|---|---|---|---|---|
| 0.2 | 0.5 | -3.6 | 11 | 0.02 (1.3,0.7,11) |
| 0.5 | 0.5 | -4.5 | 11 | 0.14 (1.4,0.7,17) |
| 1.0 | 0.5 | -5.2 | 11 | 1.1 (2.2,1.0,27) |

With Alternative Magnitudes of Shocks

| $\epsilon_{1,\delta}$ | $\Delta G_1/G_{ss}$ | $\Delta\tau_{c,1}$ | Last t with $R_t = 1$ | $WG_G(WG_{\tau,c}, WG_b, WG_{opt})$ |
|---|---|---|---|---|
| 0.02 | 0.5 | -3.6 | 11 | 0.02 (1.3,0.7,11) |
| 0.0225 | 0.6 | -4.8 | 13 | 0.04 (2.3,1.1,16) |
| 0.025 | 0.7 | -6.1 | 15 | 0.01 (3.7,1.8,22) |

With Alternative Persistence of Shocks

| $\rho_\delta$ | $\Delta G_1/G_{ss}$ | $\Delta\tau_{c,1}$ | Last t with $R_t = 1$ | $WG_G(WG_{\tau,c}, WG_b, WG_{opt})$ |
|---|---|---|---|---|
| 0.9 | 0.5 | -3.6 | 11 | 0.02 (1.3,0.7,11) |
| 0.925 | 0.6 | -4.5 | 14 | 0.04 (2.0,1.1,15) |
| 0.95 | 0.8 | -6.1 | 21 | 0.01 (3.6,1.9,21) |

*See the footnote in Table 3.

Table 8: Piecewise Linear vs Nonlinear Methods

Welfare gains in the model with labor income tax and debt

| Solution Method | $WG_G$ | $WG_{\tau,n}$ | $WG_b$ | $WG_{opt}$ |
|---|---|---|---|---|
| Piecewise Linear | -0.05 | 0.59 | 2.16 | 7.8 |
| Nonlinear | 0.17 | 0.46 | 0.70 | 8.6 |

*See the footnote in Table 3.

Figure 1: The Ramsey steady-state in the model with labor income tax and debt

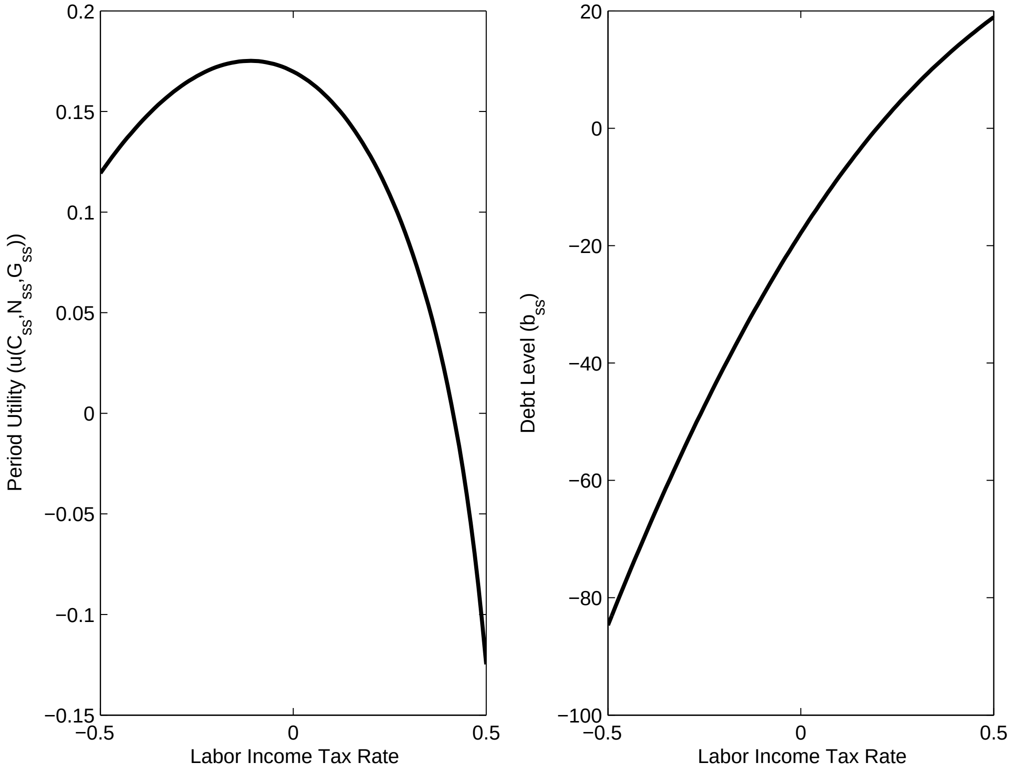



Figure 2: The Ramsey steady-state in the model with consumption tax and debt

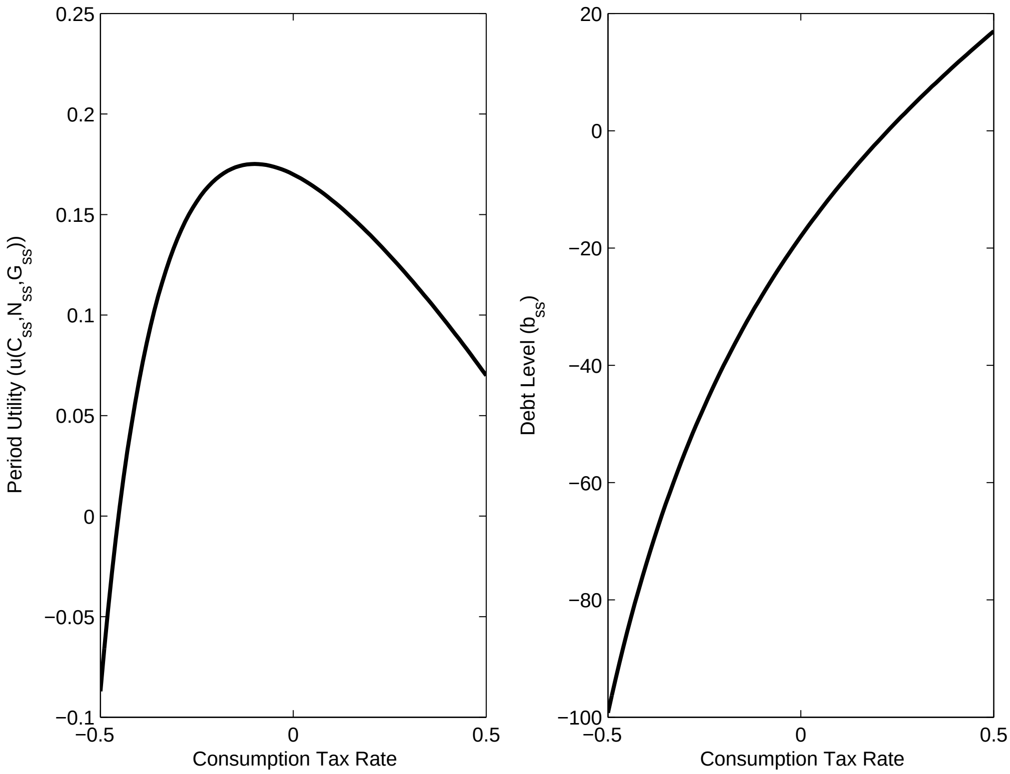
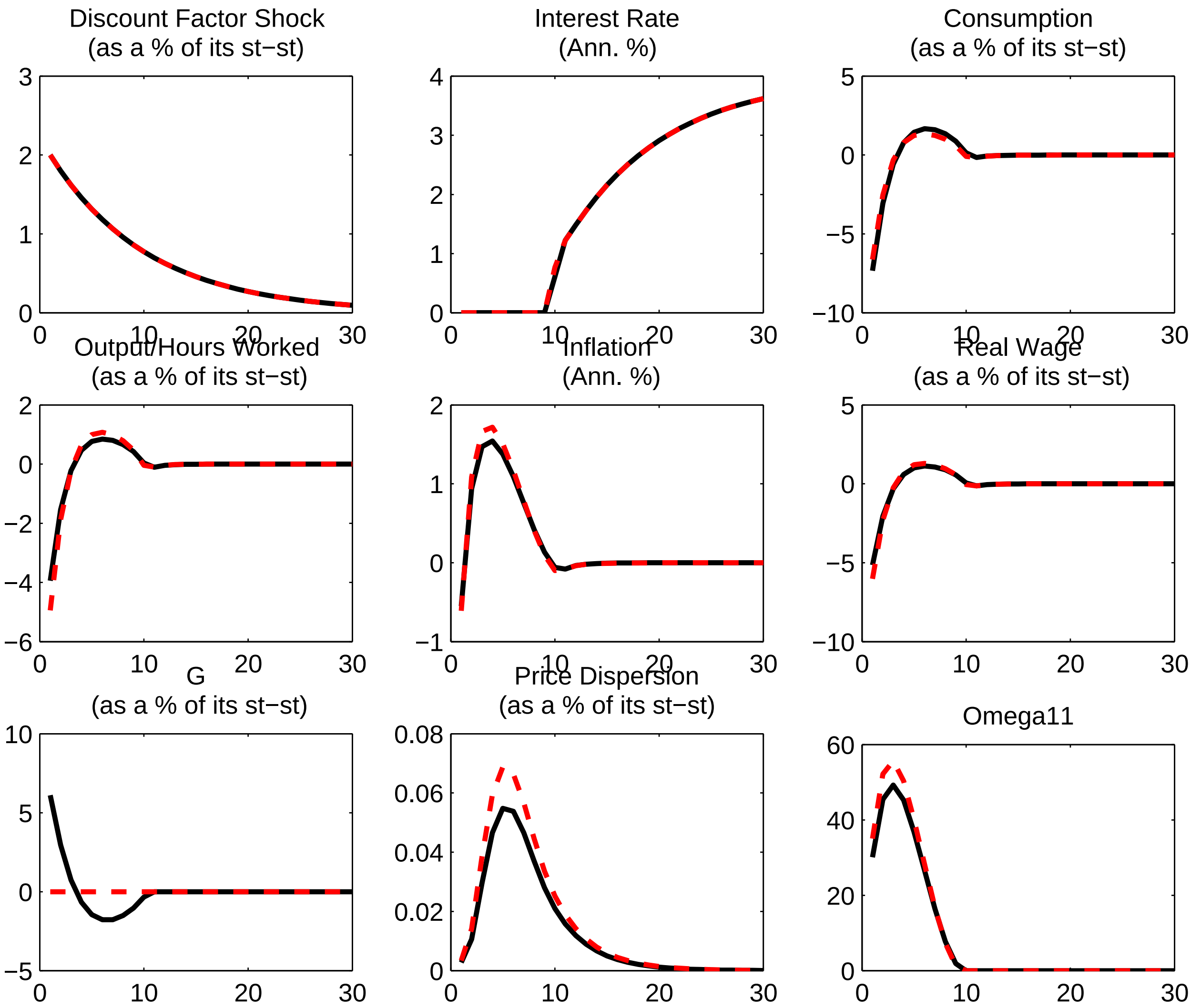
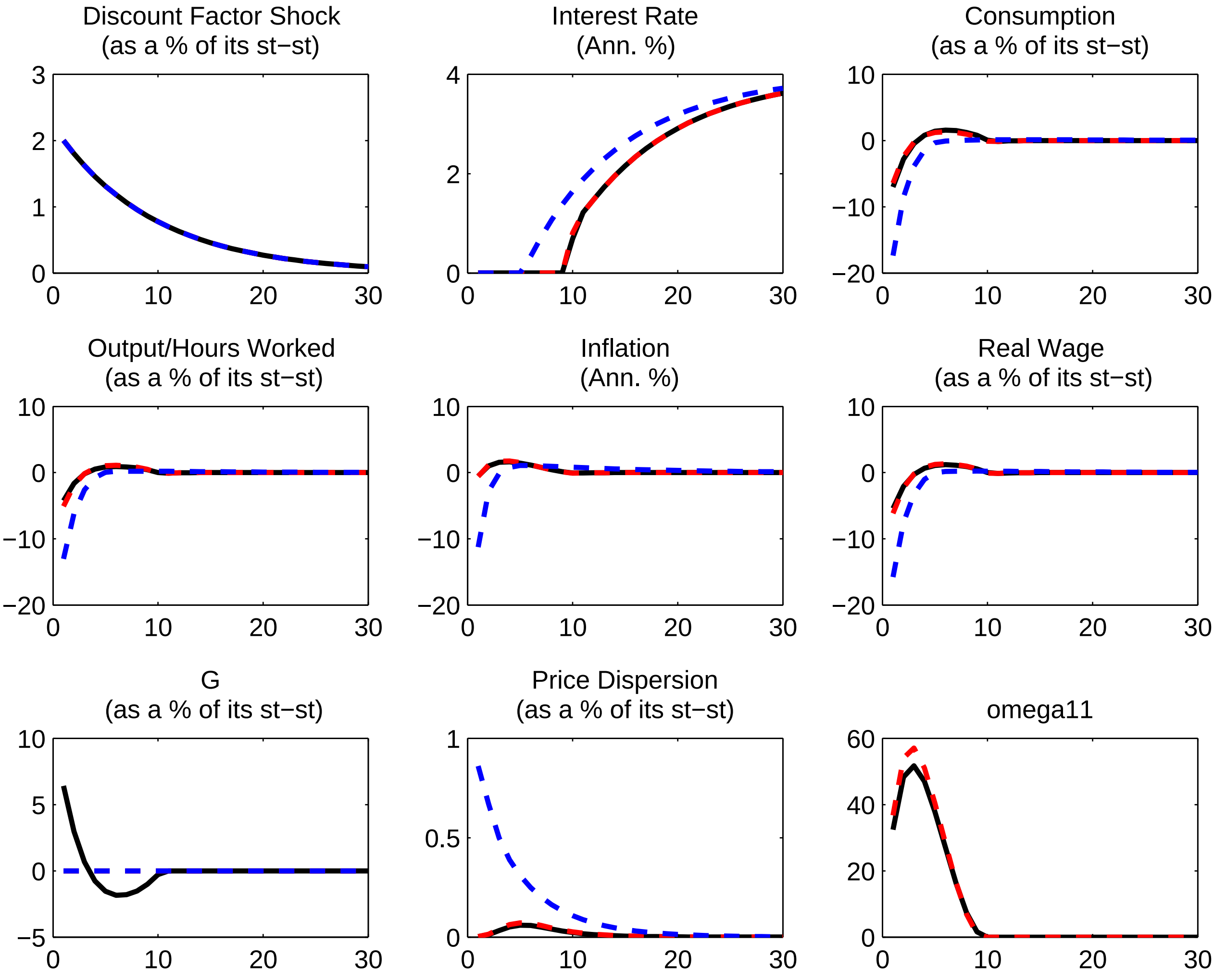


Figure 3: Model with Lump-Sum Tax: Comparing with "no variation in G" case



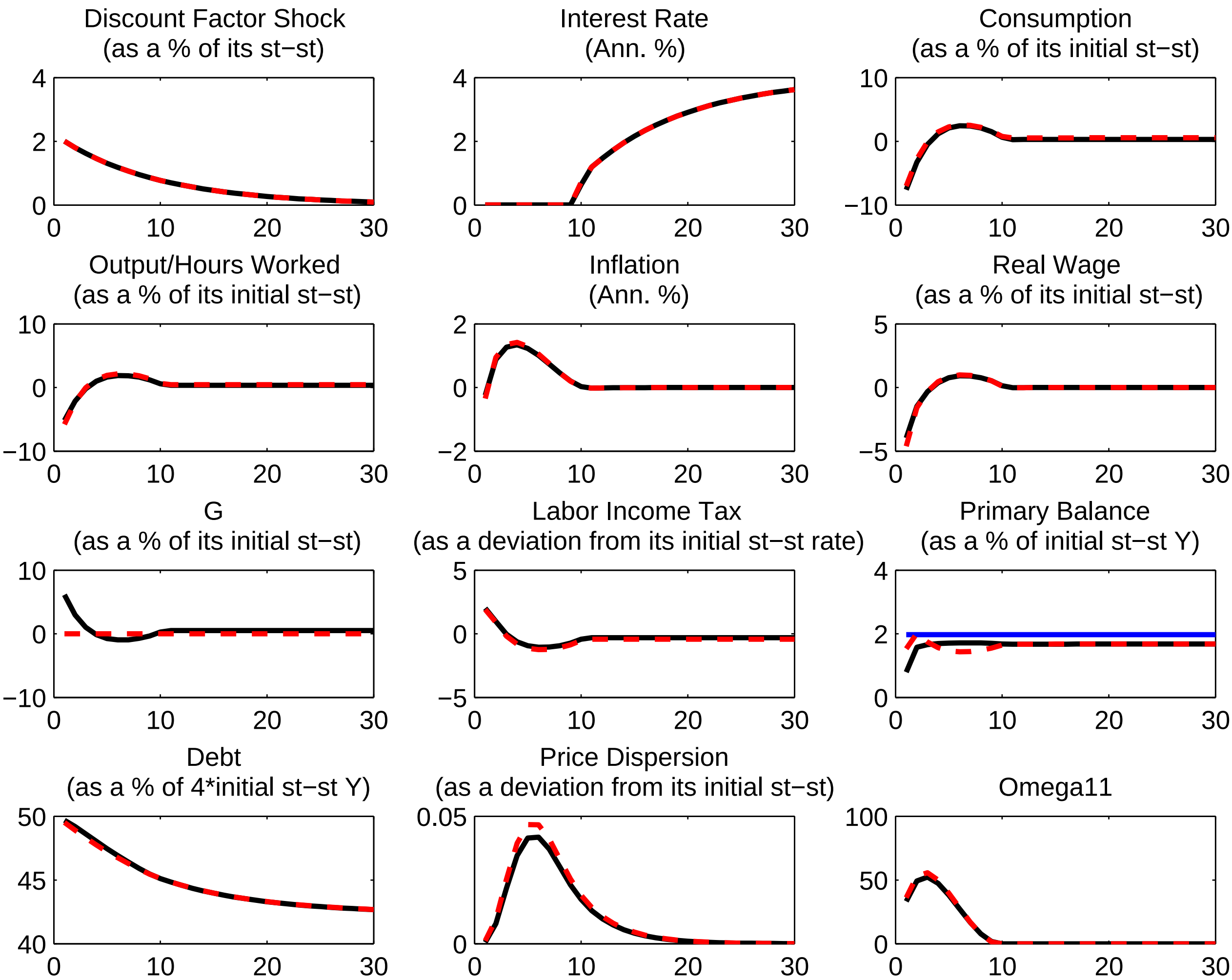

Solid black line: G and R optimally chosen, Dashed red line: Holding G constant.
Omega11 is the Lagrange Multiplier on the inequality constraint on nominal interest rate.

Figure 4: Model with Lump-Sum Tax: Allocations with non-optimizing government



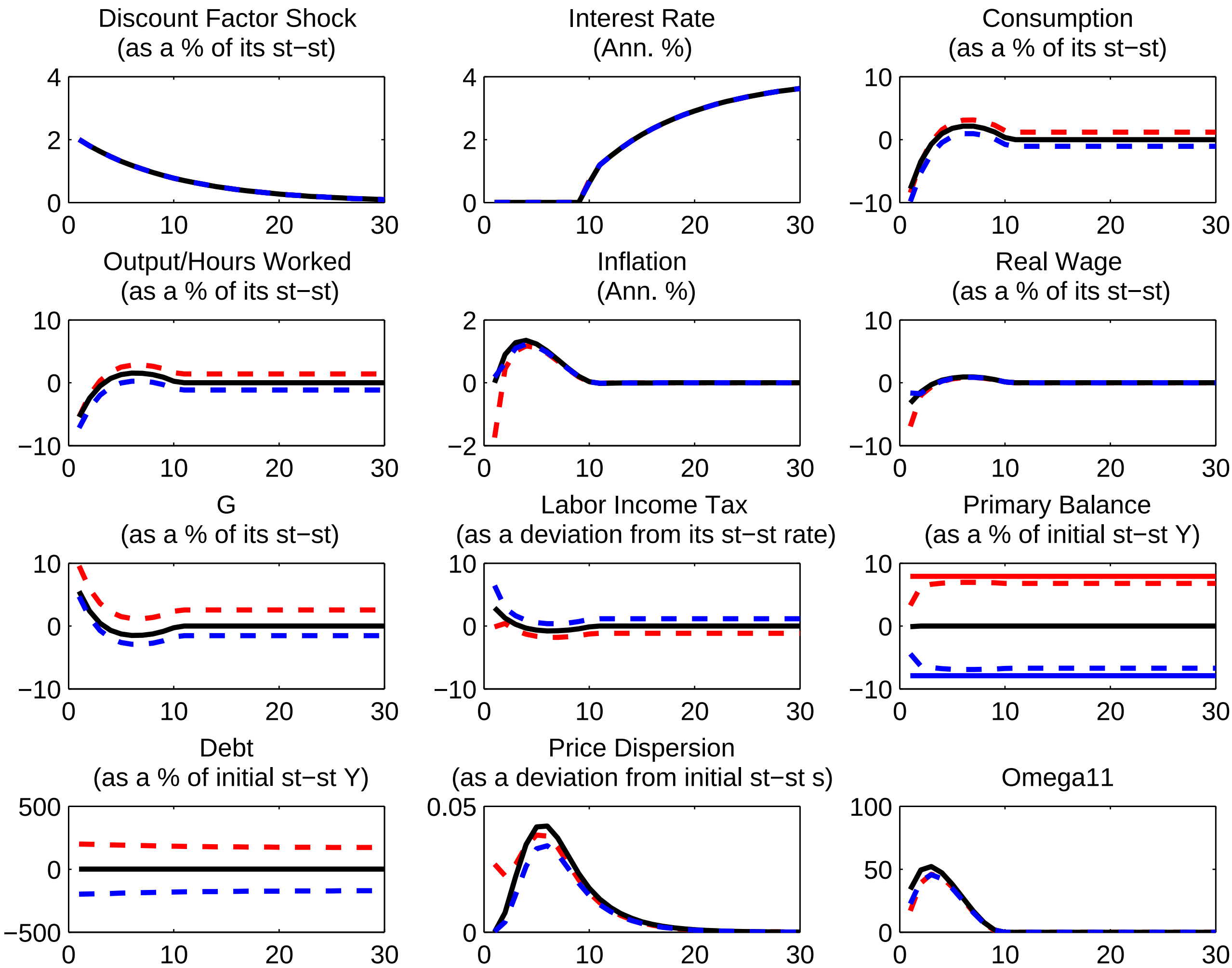

Solid black line: G and R optimally chosen, Dashed red line: Holding G constant.
Dashed red line: Government following simply policy rules.
Omega11 is the Lagrange Multiplier on the inequality constraint on nominal interest rate.

Figure 5: Model with Labor Income Tax and Debt: Comparing with "no variation in G" case



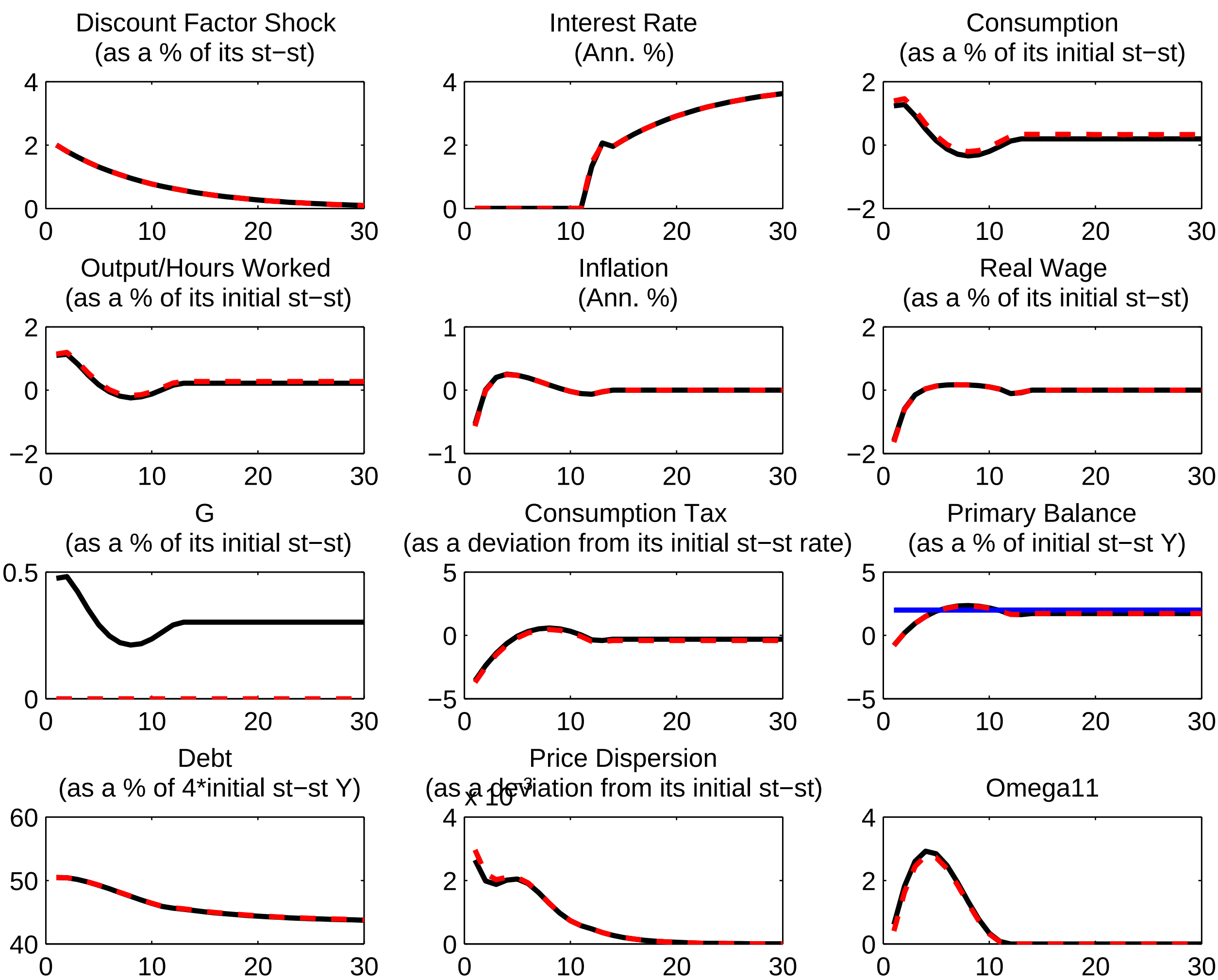

Solid black line: G and R optimally chosen, Dotted red line: Holding G constant.
Solid blue line in the primary balance is the primary balance associated with the initial Ramsey-steady-state.
Omega11 is the Lagrange Multiplier on the inequality constraint on nominal interest rate.

Figure 6: Model with Labor Income Tax and Debt: Different initial debt levels.



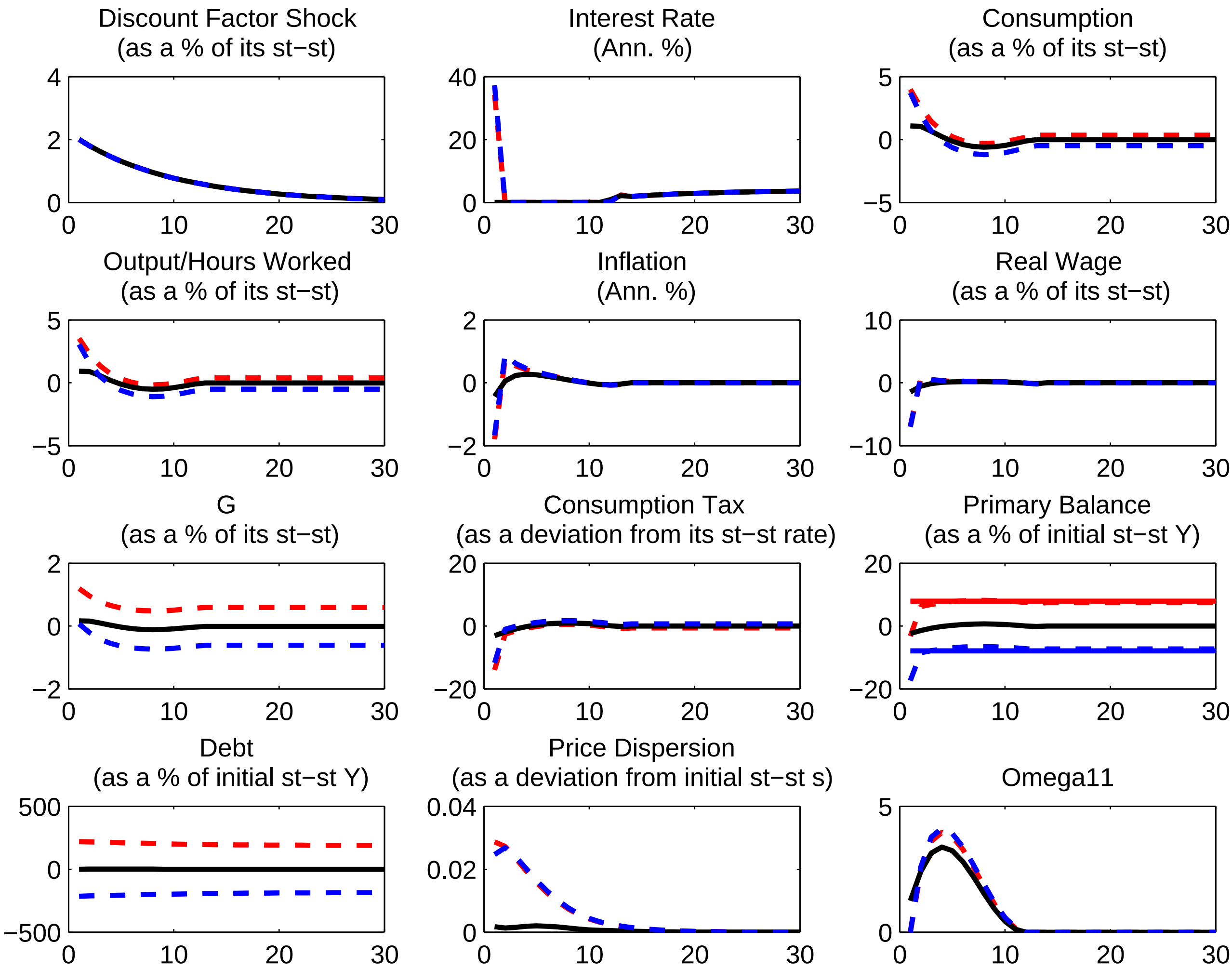

Solid black line: $b_0/(4Y_0) = 0$, Dashed red line: $b_0/(4Y_0) = 2$, Dotted blue line: $b_0/(4Y_0) = -2$
Solid red and blue lines in the primary balance are the primary balances associated with the initial Ramsey-steady-state with $b_0/(4Y_0) = 2$ and $b_0/(4Y_0) = -2$.
Omega11 is the Lagrange Multiplier on the inequality constraint on nominal interest rate.

Figure 7: Model with Consumption Tax and Debt: Comparing with “no variation in G” case



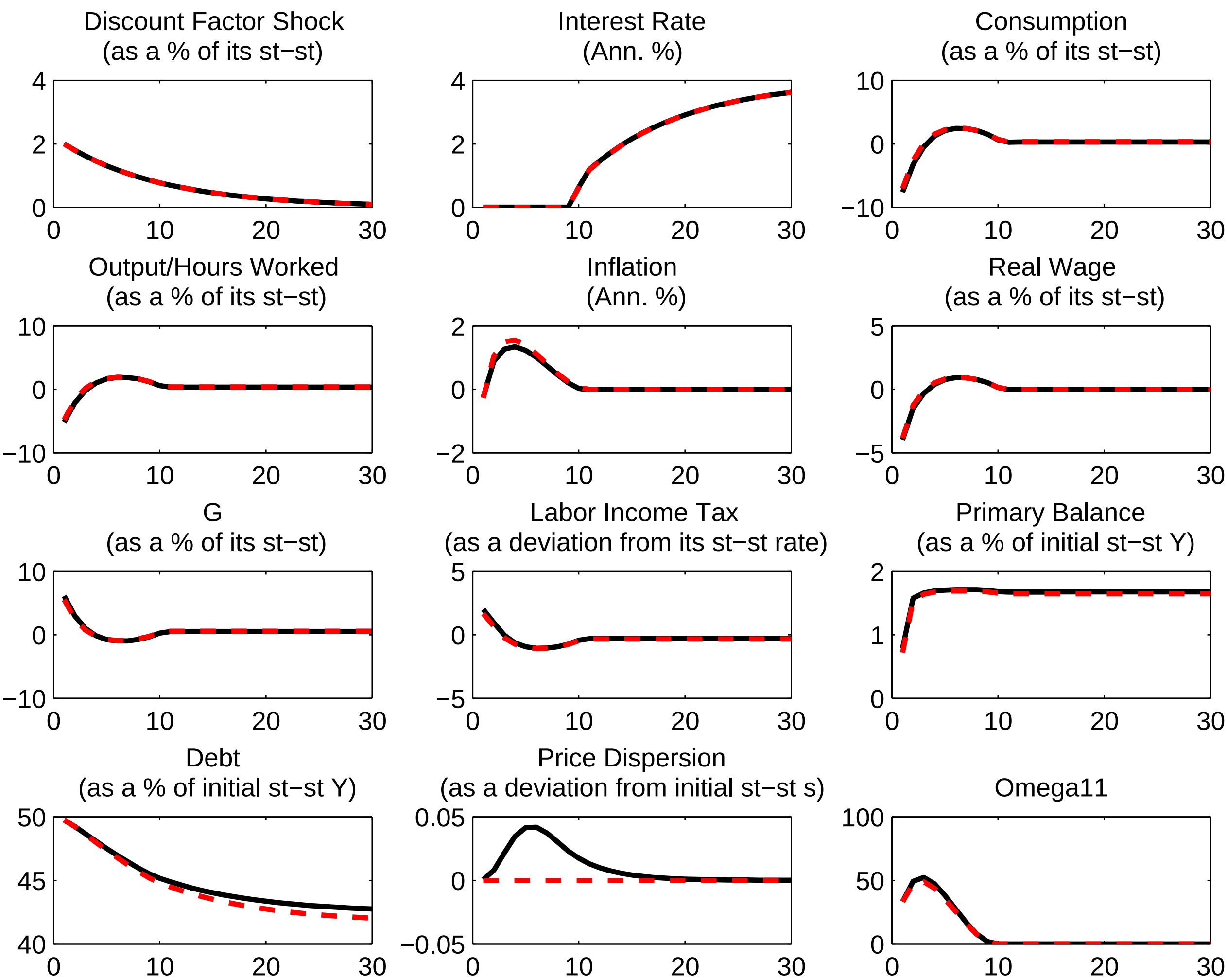

Solid black line: G and R optimally chosen, Dashed red line: Holding G constant.
Solid blue line in the primary balance is the primary balance associated with the initial Ramsey-steady-state.
Omega11 is the Lagrange Multiplier on the inequality constraint on nominal interest rate.

Figure 8: Model with Consumption Tax and Debt: Different initial debt levels.




Solid black line: $b_0/(4Y_0) = 0$, Dashed red line: $b_0/(4Y_0) = 2$, Dotted blue line: $b_0/(4Y_0) = -2$
Solid red and blue lines in the primary balance are the primary balances associated with the initial Ramsey-steady-state with $b_0/(4Y_0) = 2$ and $b_0/(4Y_0) = -2$.
Omega11 is the Lagrange Multiplier on the inequality constraint on nominal interest rate.

Figure 9: Piecewise Linear vs. Nonlinear Methods: Model with Labor Income Tax and Debt




Solid black line: Nonlinear Method

Dashed red line: Piecewise Linear Method

Omega11 is the Lagrange Multiplier on the inequality constraint on nominal interest rate.