Solving the new Keynesian model in continuous time*

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Abstract

We show how to formulate and solve the new Keynesian model in continuous time. In our economy, monopolistic firms engage in infrequent price setting á la Calvo. We introduce shocks for preferences, total factor productivity and government expenditure, and then show how the equilibrium system can be written in terms of 8 state variables. Our nonlinear and global numerical solution technique uses the collocation method based on Chebychev polynomials directly computing the continuous-time Bellman equation.

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1. Our Model

We will work with a standard NKE model. The basic structure of the economy is as follows. A representative household consumes, saves, and supplies labor. The final output is assembled by a final good producer, which uses as inputs a continuum of intermediate goods manufactured by monopolistic competitors. The intermediate good producers rent labor to manufacture their good. Also, these intermediate good producers face the constraint that they can only change prices following a Calvo's pricing rule. Finally, there is a government that fixes the one-period nominal interest rate through open market operations with public debt, taxes, and consumes. We will have four shocks: one to preferences (which can be loosely interpreted as a shock to aggregate demand), one to technology (interpreted as a shock to aggregate supply), one to monetary policy, and one to fiscal policy.

1.1. Households

There is a representative household in the economy that maximizes the following lifetime utility function, which is separable in consumption, c_t and hours worked, l_t:

$$\mathbb{E}_0 \int_0^\infty e^{-\rho t} d_t \left\{ \log c_t - \psi \frac{l_t^{1+\vartheta}}{1+\vartheta} \right\} \mathrm{d}t \tag{1}$$

where ρ is the subjective rate of time preference, ϑ is the inverse of Frisch labor supply elasticity, and d_t is a preference shock with law of motion:

$$\mathrm{d} \log d_t = -\rho_d \log d_t \mathrm{d}t + \sigma_d \mathrm{d}B_{d,t}. \tag{2}$$

The household can trade on Arrow securities (which we will not include to save on notation) and on a nominal government bonds b_t that pays a nominal interest rate of r_t. Then, the household's budget constraint is given by:

$$\mathrm{d}b_t = (r_t b_t - p_t c_t + p_t w_t l_t + p_t T_t + p_t \digamma_t)\mathrm{d}t, \tag{3}$$

where w_t is the real wage, T_t is a lump-sum transfer, and $\digamma_t$ are the profits of the firms in the economy.

Let prices p_t follow the process:

$$\mathrm{d}p_t = \pi_t p_t \mathrm{d}t \tag{4}$$

such that the (realized) rate of inflation is locally non-stochastic. We can interpret $\mathrm{d}p_t/p_t$ as the realized inflation over the period $[t, t + \mathrm{d}t]$ and interpret π_t as the inflation rate.

Letting $a_t \equiv b_t/p_t$ denote real financial wealth, we can write the constraint above as:

$$\mathrm{d}a_t = ((r_t - \pi_t)a_t - c_t + w_t l_t + T_t + \digamma_t)\,\mathrm{d}t \tag{5}$$

1.2. The Final Good Producer

There is one final good is produced using intermediate goods with the following production function:

$$y_t = \left(\int_0^1 y_{it}^{\frac{\varepsilon-1}{\varepsilon}}\,\mathrm{d}i\right)^{\frac{\varepsilon}{\varepsilon-1}} \tag{6}$$

where ε is the elasticity of substitution.

Final good producers are perfectly competitive and maximize profits subject to the production function (6), taking as given all intermediate goods prices p_{it} and the final good price p_t. As a consequence the input demand functions associated with this problem are:

$$y_{it} = \left(\frac{p_{it}}{p_t}\right)^{-\varepsilon} y_t \qquad \forall i,$$

and

$$p_t = \left(\int_0^1 p_{it}^{1-\varepsilon}\mathrm{d}i\right)^{\frac{1}{1-\varepsilon}}. \tag{7}$$

1.3. Intermediate Good Producers

Each intermediate firm produces differentiated goods out of labor using $y_{it} = A_t l_{it}$where l_{it} is the amount of the labor input rented by the firm and where A_t follows the following process:

$$\mathrm{d}\log A_t = -\rho_a \log A_t \mathrm{d}t + \sigma_a \mathrm{d}B_{a,t}. \tag{8}$$
$$\Leftrightarrow \mathrm{d}A_t = -\rho_a A_t \log(A_t)\mathrm{d}t + \sigma_a A_t \mathrm{d}B_{a,t} + \tfrac{1}{2}A_t\sigma_a^2\mathrm{d}t$$

Therefore, the real marginal cost of the intermediate good producer is just:

$$mc_t = \frac{w_t}{A_t}$$

Also, note that this marginal cost is the same for all firms.

The monopolistic firms engage in infrequent price setting á la Calvo. We assume that intermediate good producers reoptimize their prices p_{it} only at the time when a price-change signal is received. The probability (density) of receiving such a signal h periods from today is assumed to be independent of the last time the firm got the signal, and to be given by:

$$\delta e^{-\delta h}, \quad \rho > 0.$$

A number of firms δ will receive the price-change signal per unit of time. All other firms keep their old prices. Therefore, prices are set to solve the problem:

$$\max_{p_{it}} \mathbb{E}_t \int_t^\infty \frac{\lambda_\tau}{\lambda_t} \delta e^{-\delta(\tau-t)} \left(\frac{p_{it}}{p_\tau} y_{i\tau} - mc_\tau y_{i\tau} \right) \mathrm{d}\tau$$
$$\text{s.t. } y_{i\tau} = \left(\frac{p_{it}}{p_\tau} \right)^{-\varepsilon} y_\tau$$

where λ_τ is the time t value of a unit of consumption in period τ to the household. Observe that $e^{-\delta(\tau-t)}$ denotes the probability of not having received a signal during $\tau - t$,

$$1 - \int_t^\tau \delta e^{-\delta(h-t)} dh = 1 - \left(-e^{-\delta(\tau-t)} + 1 \right) = e^{-\delta(\tau-t)}. \tag{9}$$

The first-order condition reads:

$$\mathbb{E}_t \int_t^\infty \frac{\lambda_\tau}{\lambda_t} e^{-\delta(\tau-t)} (1-\varepsilon) \left(\frac{p_{it}}{p_\tau} \right)^{1-\varepsilon} \frac{y_\tau}{p_{it}} \mathrm{d}\tau + \mathbb{E}_t \int_t^\infty \frac{\lambda_\tau}{\lambda_t} e^{-\delta(\tau-t)} mc_\tau \varepsilon \left(\frac{p_{it}}{p_\tau} \right)^{-\varepsilon} \frac{y_\tau}{p_{it}} \mathrm{d}\tau = 0.$$

or

$$\mathbb{E}_t \int_t^\infty \lambda_\tau e^{-\delta(\tau-t)} (1-\varepsilon) \left(\frac{p_t}{p_\tau} \right)^{1-\varepsilon} \frac{y_\tau}{p_{it}} \mathrm{d}\tau + \mathbb{E}_t \int_t^\infty \lambda_\tau e^{-\delta(\tau-t)} mc_\tau \varepsilon \left(\frac{p_t}{p_\tau} \right)^{-\varepsilon} p_t \frac{y_\tau}{p_{it}} \mathrm{d}\tau = 0.$$

We may write the first-order condition as:

$$p_{it} x_{1,t} = \frac{\varepsilon}{\varepsilon - 1} p_t x_{2,t} \Rightarrow \Pi_t^* = \frac{\varepsilon}{\varepsilon - 1} \frac{x_{2,t}}{x_{1,t}} \tag{10}$$

in which $\Pi_t^* \equiv p_{it}/p_t$ is the ratio between the optimal new price (common across all firms that can reset their prices) and the price of the final good and where

$$x_{1,t} \equiv \mathbb{E}_t \int_t^\infty \lambda_\tau e^{-\delta(\tau-t)} \left(\frac{p_t}{p_\tau} \right)^{1-\varepsilon} y_\tau \mathrm{d}\tau,$$
$$x_{2,t} \equiv \mathbb{E}_t \int_t^\infty \lambda_\tau e^{-\delta(\tau-t)} mc_\tau \left(\frac{p_t}{p_\tau} \right)^{-\varepsilon} y_\tau \mathrm{d}\tau,$$

Differentiating $x_{1,t}$ with respect to time gives:

$$\begin{aligned}
\frac{1}{\mathrm{d}t}\mathrm{d}x_{1,t} &= e^{\delta t}p_t^{1-\varepsilon}\frac{1}{\mathrm{d}t}\mathrm{d}\mathbb{E}_t\int_t^{\infty}\lambda_\tau e^{-\delta\tau}\left(\frac{1}{p_\tau}\right)^{1-\varepsilon}y_\tau\mathrm{d}\tau+\mathbb{E}_t\int_t^{\infty}\lambda_\tau e^{-\delta\tau}\left(\frac{1}{p_\tau}\right)^{1-\varepsilon}y_\tau\mathrm{d}\tau\frac{1}{\mathrm{d}t}\mathrm{d}\left(e^{\delta t}p_t^{1-\varepsilon}\right)\\
&= -\lambda_t y_t+\left(\delta e^{\delta t}p_t^{1-\varepsilon}+e^{\delta t}(1-\varepsilon)p_t^{1-\varepsilon}\frac{1}{\mathrm{d}t}\frac{\mathrm{d}p_t}{p_t}\right)\mathbb{E}_t\int_t^{\infty}\lambda_\tau e^{-\delta\tau}\left(\frac{1}{p_\tau}\right)^{1-\varepsilon}y_\tau\mathrm{d}\tau\\
&= -\lambda_t y_t+\left(\delta+(1-\varepsilon)\frac{1}{\mathrm{d}t}\frac{\mathrm{d}p_t}{p_t}\right)\mathbb{E}_t\int_t^{\infty}\lambda_\tau e^{-\delta(\tau-t)}\left(\frac{p_t}{p_\tau}\right)^{1-\varepsilon}y_\tau\mathrm{d}\tau\\
&= -\lambda_t y_t+\left(\delta+(1-\varepsilon)\pi_t\right)x_{1,t}\Leftrightarrow\\
\mathrm{d}x_{1,t} &= \left(\left(\delta-(\varepsilon-1)\,\pi_t\right)x_{1,t}-\lambda_t y_t\right)\mathrm{d}t \qquad (11)
\end{aligned}$$

were we identify the actual rate of inflation π_t over the period $[t, t+\mathrm{d}t]$ with $\mathrm{d}p_t/p_t$. We can also renormalize $\lambda_t = e^{\rho t}m_t$ and get:

$$\mathrm{d}x_{1,t} = \big(\left(\delta-(\varepsilon-1)\,\pi_t\right)x_{1,t}-e^{\rho t}m_t y_t\big)\,\mathrm{d}t$$

A similar procedure delivers:

$$\mathrm{d}x_{2,t} = \big(\left(\delta-\varepsilon\pi_t\right)x_{2,t}-e^{\rho t}m_t mc_t y_t\big)\,\mathrm{d}t \qquad (12)$$

Assuming that the price-change is stochastically independent across firms gives:

$$p_t^{1-\varepsilon} = \int_{-\infty}^{t}\delta e^{-\delta(t-\tau)}p_{i\tau}^{1-\varepsilon}\,\mathrm{d}\tau,$$

making the price level p_t a predetermined variable at time t, its level being given by past price quotations. Differentiating with respect to time gives:

$$\begin{aligned}
\mathrm{d}p_t^{1-\varepsilon} &= \left(\delta p_{it}^{1-\varepsilon}-\delta\int_{-\infty}^{t}\delta e^{-\delta(t-\tau)}p_{i\tau}^{1-\varepsilon}\mathrm{d}\tau\right)\mathrm{d}t\\
&= \left(\delta p_{it}^{1-\varepsilon}-\delta p_t^{1-\varepsilon}\right)\mathrm{d}t
\end{aligned}$$

and

$$\frac{1}{\mathrm{d}t}\mathrm{d}p_t^{1-\varepsilon} = (1-\varepsilon)\,p_t^{-\varepsilon}\frac{\mathrm{d}p_t}{\mathrm{d}t}$$

Then

$$\mathrm{d}p_t = \frac{\delta}{1-\varepsilon}\left(p_{it}^{1-\varepsilon}p_t^{\varepsilon}-p_t\right)\mathrm{d}t \Rightarrow \pi_t = \frac{\delta}{1-\varepsilon}\left(\left(\Pi_t^*\right)^{1-\varepsilon}-1\right). \qquad (13)$$

Differentiating the previous expression, we obtain the inflation dynamics:

$$
\begin{aligned}
\frac{1}{\mathrm{d}t}\mathrm{d}\pi_t &= \delta\left(\Pi_t^*\right)^{-\varepsilon}\frac{1}{\mathrm{d}t}\mathrm{d}\Pi_t^* \\
&= \delta\left(\Pi_t^*\right)^{-\varepsilon}\frac{\varepsilon}{\varepsilon-1}\frac{1}{\mathrm{d}t}\mathrm{d}\left(\frac{x_{2,t}}{x_{1,t}}\right) \\
&= \delta\left(\Pi_t^*\right)^{-\varepsilon}\frac{\varepsilon}{\varepsilon-1}\frac{1}{x_{1,t}}\left(\frac{1}{\mathrm{d}t}\mathrm{d}x_{2,t}-\frac{x_{2,t}}{x_{1,t}}\frac{1}{\mathrm{d}t}\mathrm{d}x_{1,t}\right) \\
&= \delta\left(\Pi_t^*\right)^{1-\varepsilon}\frac{1}{x_{2,t}}\left(\frac{1}{\mathrm{d}t}\mathrm{d}x_{2,t}-\frac{x_{2,t}}{x_{1,t}}\frac{1}{\mathrm{d}t}\mathrm{d}x_{1,t}\right) \\
&= \delta\left(\Pi_t^*\right)^{1-\varepsilon}\left(\frac{1}{x_{2,t}}\frac{1}{\mathrm{d}t}\mathrm{d}x_{2,t}-\frac{1}{x_{1,t}}\frac{1}{\mathrm{d}t}\mathrm{d}x_{1,t}\right) \\
&= \delta\left(\Pi_t^*\right)^{1-\varepsilon}\left(\frac{\left(\left(\delta-\varepsilon\pi_t\right)x_{2,t}-e^{\rho t}m_t mc_t y_t\right)}{x_{2,t}}-\frac{\left(\left(\delta-\left(\varepsilon-1\right)\pi_t\right)x_{1,t}-e^{\rho t}m_t y_t\right)}{x_{1,t}}\right) \\
&= -\delta\left(\Pi_t^*\right)^{1-\varepsilon}\left(\pi_t+\left(\frac{mc_t}{x_{2,t}}-\frac{1}{x_{1,t}}\right)e^{\rho t}m_t y_t\right)
\end{aligned}
$$

1.4. The Government Problem

The government sets the nominal interest rates according to the Taylor rule:

$$
\mathrm{d}R_t = (\theta_0+\theta_1\pi_t-\theta_2 R_t)\mathrm{d}t \tag{14}
$$

through open market operations that are financed with lump-sum transfers T_t.

The lump sum transfers insure that the deficit are equal to zero and finance a stream of government consumption g_t:

$$
\begin{aligned}
g_t &= s_g s_{g,t} y_t = -T_t, \\
\mathrm{d}\log s_{g,t} &= -\rho_g \log s_{g,t}\mathrm{d}t+\sigma_g \mathrm{d}B_{g,t}.
\end{aligned} \tag{15}
$$

1.5. Aggregation

First, we derive an expression for aggregate demand:

$$
y_t = c_t + g_t
$$

With this value, the demand for each intermediate good producer is

$$
y_{it} = (c_t+g_t)\left(\frac{p_{it}}{p_t}\right)^{-\varepsilon} \qquad \forall i, \tag{16}
$$

and using the production function is:

$$A_t l_{it} = (c_t + g_t) \left(\frac{p_{it}}{p_t}\right)^{-\varepsilon}.$$

We can integrate on both sides:

$$A_t \int_0^1 l_{it} \mathrm{d}i = (c_t + g_t) \int_0^1 \left(\frac{p_{it}}{p_t}\right)^{-\varepsilon} \mathrm{d}i$$

and get an expression:

$$c_t + g_t = y_t = \frac{A_t}{v_t} l_t$$

where:

$$v_t = \int_0^1 \left(\frac{p_{it}}{p_t}\right)^{-\varepsilon} \mathrm{d}i \tag{17}$$

is the aggregate loss of efficiency induced by price dispersion of the intermediate goods. By the properties of Calvo pricing v_t is a predetermined variable:

$$v_t = \int_{-\infty}^{t} \delta e^{-\delta(t-\tau)} \left(\frac{p_{i\tau}}{p_t}\right)^{-\varepsilon} \mathrm{d}\tau.$$

Differentiating with respect to time gives:

$$\begin{aligned}
\frac{1}{\mathrm{d}t}\mathrm{d}v_t &= \delta \left(\Pi_t^*\right)^{-\varepsilon} + \int_{-\infty}^{t} \delta \frac{1}{\mathrm{d}t} \mathrm{d}e^{-\delta(t-\tau)} \left(\frac{p_{i\tau}}{p_t}\right)^{-\varepsilon} \mathrm{d}\tau \\
&= \delta \left(\Pi_t^*\right)^{-\varepsilon} - \delta \int_{-\infty}^{t} \delta e^{-\delta(t-\tau)} \left(\frac{p_{i\tau}}{p_t}\right)^{-\varepsilon} \mathrm{d}\tau + \int_{-\infty}^{t} \delta e^{-\delta(t-\tau)} p_{i\tau}^{-\varepsilon} \frac{1}{\mathrm{d}t} \mathrm{d}p_t^{\varepsilon} \mathrm{d}\tau \\
&= \delta \left(\Pi_t^*\right)^{-\varepsilon} - \delta v_t + \int_{-\infty}^{t} \delta e^{-\delta(t-\tau)} p_{i\tau}^{-\varepsilon} \varepsilon p_t^{\varepsilon-1} \frac{1}{\mathrm{d}t} \mathrm{d}p_t \mathrm{d}\tau \\
&= \delta \left(\Pi_t^*\right)^{-\varepsilon} + (\varepsilon \pi_t - \delta) v_t.
\end{aligned} \tag{18}$$

For aggregate profits, we use the demand of intermediate producers in (16):

$$\begin{aligned}
\digamma_t &= \int_0^1 \left(\frac{p_{it}}{p_t} - mc_t\right) y_{it} \mathrm{d}i \\
&= \left(\int_0^1 \left(\frac{p_{it}}{p_t} - mc_t\right) \left(\frac{p_{it}}{p_t}\right)^{-\varepsilon} \mathrm{d}i\right) y_t \\
&= \left(\int_0^1 \left(\frac{p_{it}}{p_t}\right)^{1-\varepsilon} \mathrm{d}i - mc_t v_t\right) y_t \\
&= (1 - mc_t v_t) y_t.
\end{aligned} \tag{19}$$

1.6. The Bellman equation first-order conditions

Given our description of the problem, we define the household's *value function* as:

$$V(a_0, d_0, A_0, s_{g,0}, R_0, v_0, x_{1,0}, x_{2,0}) \equiv \max_{\{(c_t, l_t)\}_{t=0}^{\infty}} \mathbb{E}_0 \int_0^{\infty} e^{-\rho t} d_t \left\{ \log c_t - \psi \frac{l_t^{1+\vartheta}}{1+\vartheta} \right\} \mathrm{d}t \text{ s.t.}$$
$$(5), (2), (8), (15), (14), (18), (11), (12). \tag{20}$$

Define the state vector $\mathbb{Z}_t \equiv (a_t, d_t, A_t, s_{g,t}, R_t, v_t, x_{1,t}, x_{2,t})$. For convenience, the constraints are reproduced below

$$\begin{aligned}
\mathrm{d}a_t &= ((R_t - \pi_t)a_t - c_t + w_t l_t + T_t + F_t)\,\mathrm{d}t, \\
\mathrm{d}\log d_t &= -\rho_d \log d_t \mathrm{d}t + \sigma_d \mathrm{d}B_{d,t}, \\
\mathrm{d}\log A_t &= -\rho_a \log A_t \mathrm{d}t + \sigma_a \mathrm{d}B_{a,t}, \\
\mathrm{d}\log s_{g,t} &= -\rho_g \log s_{g,t} \mathrm{d}t + \sigma_g \mathrm{d}B_{g,t}, \\
\mathrm{d}R_t &= (\theta_0 + \theta_1 \pi_t - \theta_2 R_t)\mathrm{d}t, \\
\mathrm{d}v_t &= \left(\delta\,(\Pi_t^*)^{-\varepsilon} + (\varepsilon\pi_t - \delta)v_t\right)\mathrm{d}t, \\
\mathrm{d}x_{1,t} &= \left((\rho + \delta - (\varepsilon - 1)\pi_t)x_{1,t} - e^{\rho t} m_t y_t\right)\mathrm{d}t, \\
\mathrm{d}x_{2,t} &= \left((\rho + \delta - \varepsilon\pi_t)\, x_{2,t} - e^{\rho t} m_t y_t mc_t\right)\mathrm{d}t.
\end{aligned}$$

By choosing the control $(c_t, l_t) \in \mathbb{R}_+^2$ at time t, the Bellman equation reads:

$$\begin{aligned}
\rho V = \; & \max_{(c_t, l_t)} d_t \left\{ \log c_t - \psi \frac{l_t^{1+\vartheta}}{1+\vartheta} \right\} \\
& + ((R_t - \pi_t)a_t - c_t + w_t l_t + T_t + F_t)\, V_a \\
& -(\rho_d \log d_t - \tfrac{1}{2}\sigma_d^2) d_t V_d + \tfrac{1}{2}\sigma_d^2 d_t^2 V_{dd} \\
& -(\rho_a \log A_t - \tfrac{1}{2}\sigma_a^2) A_t V_A + \tfrac{1}{2}\sigma_a^2 A_t^2 V_{AA} \\
& -(\rho_g \log s_{g,t} - \tfrac{1}{2}\sigma_g^2) s_{g,t} V_s + \tfrac{1}{2}\sigma_g^2 s_{g,t}^2 V_{ss} \\
& +(\theta_0 + \theta_1 \pi_t - \theta_2 R_t) V_R \\
& + \left(\delta\,(\Pi_t^*)^{-\varepsilon} + (\varepsilon\pi_t - \delta)v_t\right) V_v \\
& + \left((\rho + \delta - (\varepsilon - 1)\pi_t)x_{1,t} - e^{\rho t} m_t y_t\right) V_{x_1} \\
& + \left((\rho + \delta - \varepsilon\pi_t)\, x_{2,t} - e^{\rho t} m_t y_t mc_t\right) V_{x_2}.
\end{aligned} \tag{21}$$

A neat result about the formulation of our problem in continuous time is that the Bellman equation is, in effect, a deterministic differential equation.

The first-order conditions with respect to c_t and l_t for any interior solution are:

$$\frac{d_t}{c_t} = V_a \tag{22}$$

$$d_t \psi l_t^{\vartheta} = V_a w_t \tag{23}$$

or, eliminating the costate variable (for $\psi \neq 0$):

$$\psi l_t^{\vartheta} c_t = w_t$$

The first-order conditions (22) and (23) make the optimal policy functions functions of the state variables, $c_t = c(\mathbb{Z}_t)$ and $l_t = l(\mathbb{Z}_t)$. Thus the concentrated Bellman equation reads:

$$\begin{aligned} \rho V = {} & d_t \left\{ \log c(\mathbb{Z}_t) - \psi \frac{l(\mathbb{Z}_t)^{1+\vartheta}}{1+\vartheta} \right\} \\ & + \left((R_t - \pi_t) a_t - c(\mathbb{Z}_t) + w_t l(\mathbb{Z}_t) + T_t + \digamma_t \right) V_a \\ & - (\rho_d \log d_t - \tfrac{1}{2}\sigma_d^2) d_t V_d + \tfrac{1}{2}\sigma_d^2 d_t^2 V_{dd} \\ & - (\rho_a \log A_t - \tfrac{1}{2}\sigma_a^2) A_t V_A + \tfrac{1}{2}\sigma_a^2 A_t^2 V_{AA} \\ & - (\rho_g \log s_{g,t} - \tfrac{1}{2}\sigma_g^2) s_{g,t} V_s + \tfrac{1}{2}\sigma_g^2 s_{g,t}^2 V_{ss} \\ & + (\theta_0 + \theta_1 \pi_t - \theta_2 R_t) V_R \\ & + \left(\delta \left(\Pi_t^* \right)^{-\varepsilon} + (\varepsilon \pi_t - \delta) v_t \right) V_v \\ & + \left((\rho + \delta - (\varepsilon - 1)\pi_t) x_{1,t} - e^{\rho t} m_t y_t \right) V_{x_1} \\ & + \left((\rho + \delta - \varepsilon \pi_t) x_{2,t} - e^{\rho t} m_t y_t m c_t \right) V_{x_2}. \end{aligned} \tag{24}$$

Using the envelope theorem, we obtain the costate variable V_a as:

$$\begin{aligned} \rho V_a = {} & (R_t - \pi_t) V_a + \left((R_t - \pi_t) a_t - c_t + w_t l_t + T_t + \digamma_t \right) V_{aa} \\ & - (\rho_d \log d_t - \tfrac{1}{2}\sigma_d^2) d_t V_{da} + \tfrac{1}{2}\sigma_d^2 d_t^2 V_{dda} \\ & - (\rho_a \log A_t - \tfrac{1}{2}\sigma_a^2) A_t V_{Aa} + \tfrac{1}{2}\sigma_a^2 A_t^2 V_{AAa} \\ & - (\rho_g \log s_{g,t} - \tfrac{1}{2}\sigma_g^2) s_{g,t} V_{sa} + \tfrac{1}{2}\sigma_g^2 s_{g,t}^2 V_{ssa} \\ & + (\theta_0 + \theta_1 \pi_t - \theta_2 R_t) V_{Ra} \\ & + \left(\delta \left(\Pi_t^* \right)^{-\varepsilon} + (\varepsilon \pi_t - \delta) v_t \right) V_{va} \\ & + \left((\rho + \delta - (\varepsilon - 1)\pi_t) x_{1,t} - e^{\rho t} m_t y_t \right) V_{x_1 a} \\ & + \left((\rho + \delta - \varepsilon \pi_t) x_{2,t} - e^{\rho t} m_t y_t m c_t \right) V_{x_2 a}. \end{aligned} \tag{25}$$

where we applied the envelope theorem also to the SDF which -at the equilibrium valuecannot be affected by changes in wealth. An alternative formulation is:

$$
\begin{aligned}
\left(\rho-R_t+\pi_t\right) V_a &= V_{aa}\mathrm{d}a_t \\
&\quad+\left(\mathrm{d}\log d_t-\sigma_d\mathrm{d}B_{d,t}+\tfrac{1}{2}\sigma_d^2\right) d_t V_{da}+\tfrac{1}{2}\sigma_d^2 d_t^2 V_{dda} \\
&\quad+\left(\mathrm{d}\log A_t-\sigma_a\mathrm{d}B_{a,t}-\tfrac{1}{2}\sigma_a^2\right) A_t V_{Aa}+\tfrac{1}{2}\sigma_a^2 A_t^2 V_{AAa} \\
&\quad+\left(\mathrm{d}\log s_{g,t}-\sigma_g\mathrm{d}B_{g,t}+\tfrac{1}{2}\sigma_g^2\right) s_{g,t} V_{sa}+\tfrac{1}{2}\sigma_g^2 s_{g,t}^2 V_{ssa} \\
&\quad+V_{Ra}\mathrm{d}R_t+V_{va}\mathrm{d}v_t+V_{x_1a}\mathrm{d}x_{1,t}+V_{x_2a}\mathrm{d}x_{2,t}
\end{aligned}
$$

or:

$$
\begin{aligned}
&\left(\rho-R_t+\pi_t\right) V_a+\sigma_d d_t V_{da}\mathrm{d}B_{d,t}+\sigma_a A_t V_{Aa}\mathrm{d}B_{a,t}+\sigma_g s_{g,t} V_{sa}\mathrm{d}B_{g,t} \\
=\;& V_{aa}\mathrm{d}a_t+V_{da}\mathrm{d}d_t+\tfrac{1}{2}\sigma_d^2 d_t V_{da}+V_{Aa}\mathrm{d}A_t+\tfrac{1}{2}\sigma_a^2 A_t V_{Aa}+V_{sa}\mathrm{d}s_{g,t}+\tfrac{1}{2}\sigma_g^2 s_{g,t} V_{sa} \\
&+V_{Ra}\mathrm{d}R_t+V_{va}\mathrm{d}v_t+V_{x_1a}\mathrm{d}x_{1,t}+V_{x_2a}\mathrm{d}x_{2,t}
\end{aligned}
$$

Observe that the costate variable in general evolves according to:

$$
\begin{aligned}
\mathrm{d}V_a &= V_{aa}da_t+V_{da}\mathrm{d}d_t+\tfrac{1}{2}\sigma_d^2 d_t^2 V_{dda}\mathrm{d}t+V_{Aa}\mathrm{d}A_t+\tfrac{1}{2}\sigma_a^2 A_t^2 V_{AAa}+V_{sa}\mathrm{d}s_{gt}+\tfrac{1}{2}\sigma_g^2 s_{g,t}^2 V_{ssa} \\
&\quad+V_{Ra}\mathrm{d}R_t+V_{va}\mathrm{d}v_t+V_{\pi a}\mathrm{d}\pi_t+V_{x_1a}\mathrm{d}x_{1,t}+V_{x_2a}\mathrm{d}x_{2,t} \\
&= (\rho-R_t+\pi_t)V_a\mathrm{d}t+\sigma_d d_t V_{da}\mathrm{d}B_{d,t}+\sigma_a A_t V_{Aa}\mathrm{d}B_{a,t}+\sigma_g s_{g,t} V_{sa}\mathrm{d}B_{g,t} \qquad (26)
\end{aligned}
$$

where the last equation inserted (25). Note that (26) determines the stochastic discount factor (SDF), which can be used to price any asset in the economy:

$$
\begin{aligned}
\mathrm{d}\ln V_a &= \frac{1}{V_a}dV_a-\tfrac{1}{2}\frac{V_{da}^2}{V_a^2}\sigma_d^2 d_t^2\mathrm{d}t-\tfrac{1}{2}\frac{V_{Aa}^2}{V_a^2}\sigma_a^2 A_t^2\mathrm{d}t-\tfrac{1}{2}\frac{V_{sa}^2}{V_a^2}\sigma_g^2 s_{g,t}^2\mathrm{d}t \\
&= (\rho-R_t+\pi_t)\mathrm{d}t+\frac{V_{da}}{V_a}\sigma_d d_t\mathrm{d}B_{d,t}+\frac{V_{Aa}}{V_a}\sigma_a A_t\mathrm{d}B_{a,t}+\frac{V_{sa}}{V_a}\sigma_g s_{g,t}\mathrm{d}B_{g,t} \\
&\quad-\tfrac{1}{2}\frac{V_{da}^2}{V_a^2}\sigma_d^2 d_t^2\mathrm{d}t-\tfrac{1}{2}\frac{V_{Aa}^2}{V_a^2}\sigma_a^2 A_t^2\mathrm{d}t-\tfrac{1}{2}\frac{V_{sa}^2}{V_a^2}\sigma_g^2 s_{g,t}^2\mathrm{d}t.
\end{aligned}
$$

For $s>t$, we may write:

$$
e^{-\rho(s-t)}\frac{V_a(\mathbb{Z}_s)}{V_a(\mathbb{Z}_t)}=\exp\left(\begin{array}{c}-\int_t^s(R_u-\pi_u)\mathrm{d}u+\int_t^s\frac{V_{da}}{V_a}\sigma_d d_u\mathrm{d}B_{d,u}+\int_t^s\frac{V_{Aa}}{V_a}\sigma_a A_u\mathrm{d}B_{a,u}+\int_t^s\frac{V_{sa}}{V_a}\sigma_g s_{g,u}\mathrm{d}B_{g,u} \\ -\frac{1}{2}\int_t^s\frac{V_{da}^2}{V_a^2}\sigma_d^2 d_u^2\mathrm{d}u-\frac{1}{2}\int_t^s\frac{V_{Aa}^2}{V_a^2}\sigma_a^2 A_u^2\mathrm{d}u-\frac{1}{2}\int_t^s\frac{V_{sa}^2}{V_a^2}\sigma_g^2 s_{g,u}^2\mathrm{d}u\end{array}\right).
$$

Hence, the implied SDF is (cf. Hansen and Scheinkmann 2009):

$$\frac{m_s}{m_t} = e^{-\rho(s-t)} \frac{V_a(\mathbb{Z}_s)}{V_a(\mathbb{Z}_t)},$$

and we can pin down $m_t = e^{-\rho t} V_a(\mathbb{Z}_t)$.

Using the first-order condition,

$$\frac{d_t}{c_t} = V_a$$

we obtain the implicit Euler equation for consumption:

$$(\rho - R_t + \pi_t)V_a \mathrm{d}t + \sigma_d d_t V_{da} \mathrm{d}B_{d,t} + \sigma_a A_t V_{Aa} \mathrm{d}B_{a,t} + \sigma_g s_{g,t} V_{sa} \mathrm{d}B_{g,t}$$

$$\mathrm{d}\left(\frac{d_t}{c_t}\right) = (\rho - R_t + \pi_t)\left(\frac{d_t}{c_t}\right)\mathrm{d}t - \sigma_d \left(\frac{d_t}{c_t}\right)^2 c_d dB_{d,t} - \sigma_a A_t \frac{d_t}{c_t^2} c_A dB_{a,t} - \sigma_g s_{g,t} \frac{d_t}{c_t^2} c_s dB_{g,t}$$

in which we used $V_{ad} = -d_t/c_t^2 c_d$, $V_{Aa} = -d_t/c_t^2 c_A$, and $V_{sa} = -d_t/c_t^2 c_s$.

Hence, by applying Itô's formula, we obtain the Euler equation for consumption:

$$\begin{aligned}
\mathrm{d}\left(\frac{d_t}{c_t}\right) &= -\left(\frac{d_t}{c_t}\right)^{-1}(\rho - R_t + \pi_t)\,\mathrm{d}t + \left(\sigma_d^2 \frac{d_t}{c_t} c_d^2 + \sigma_a^2 \frac{A_t^2}{c_t} c_A^2 + \sigma_g^2 \frac{s_{g,t}^2}{d_t c_t} c_s^2\right) dt \\
&\quad + \sigma_d c_d dB_{d,t} + \sigma_a A_t d_t^{-1} c_A dB_{a,t} + \sigma_g s_{g,t} d_t^{-1} c_s dB_{g,t} \\
&\Leftrightarrow \ \mathrm{d}c_t = -(\rho - R_t + \pi_t)c_t \mathrm{d}t + \left(\sigma_d^2 \frac{d_t^2}{c_t} c_d^2 + \sigma_a^2 A_t^2 c_A^2 \frac{d_t}{c_t} + \sigma_g^2 \frac{s_{g,t}^2}{c_t} c_s^2\right) dt \\
&\quad + \sigma_d c_d d_t dB_{d,t} + \sigma_a A_t c_A dB_{a,t} + \sigma_g s_{g,t} c_s dB_{g,t} \\
&\quad - c_t \rho_d \log d_t \mathrm{d}t + c_t \sigma_d \mathrm{d}B_{d,t} + \tfrac{1}{2} c_t \sigma_d^2 \mathrm{d}t. \qquad (27)
\end{aligned}$$

2. Equilibrium

The equilibrium is given by the sequence $\{y_t, c_t, l_t, mc_t, x_{1,t}, x_{2,t}, w_t, \pi_t, \Pi_t^*, v_t, R_t, d_t, A_t, g_t, s_{g,t}\}_{t=0}^{\infty}$ determined by the following equations:

- The first order conditions of the household and Euler equation:

$$\mathrm{d}c_t = -(\rho - R_t + \pi_t)c_t \mathrm{d}t + \left(\sigma_d^2 \frac{d_t^2}{c_t} c_d^2 + \sigma_a^2 A_t^2 c_A^2 \frac{d_t}{c_t} + \sigma_g^2 \frac{s_{g,t}^2}{c_t} c_s^2\right) dt - c_t \rho_d \log d_t \mathrm{d}t$$

$$+ \tfrac{1}{2} c_t \sigma_d^2 \mathrm{d}t + \sigma_d c_d d_t dB_{d,t} + \sigma_a A_t c_A dB_{a,t} + \sigma_g s_{g,t} c_s dB_{g,t} + c_t \sigma_d \mathrm{d}B_{d,t}$$

$$\psi l_t^{\vartheta} c_t = w_t$$

- Profit maximization is given by:

$$mc_t = \frac{w_t}{A_t}$$
$$\mathrm{d}x_{1,t} = \big((\rho + \delta - (\varepsilon - 1)\pi_t)x_{1,t} - e^{\rho t} m_t y_t\big)\,\mathrm{d}t$$
$$\mathrm{d}x_{2,t} = \big((\rho + \delta - \varepsilon\pi_t)\,x_{2,t} - e^{\rho t} m_t y_t mc_t\big)\,\mathrm{d}t.$$

- Government policy:

$$\mathrm{d}R_t = (\theta_0 + \theta_1\pi_t - \theta_2 R_t)\mathrm{d}t$$
$$g_t = s_g s_{g,t} y_t$$

- Inflation evolution and price dispersion:

$$\pi_t = \frac{\delta}{1-\varepsilon}\left((\Pi_t^*)^{1-\varepsilon} - 1\right)$$
$$\Pi_t^* = \frac{\varepsilon}{\varepsilon - 1}\frac{x_{2,t}}{x_{1,t}}$$
$$\mathrm{d}v_t = \left(\delta\,(\Pi_t^*)^{-\varepsilon} + (\varepsilon\pi_t - \delta)v_t\right)\mathrm{d}t$$

- Market clearing on goods and labor markets:

$$y_t = c_t + g_t$$
$$y_t = \frac{A_t}{v_t} l_t$$

which implies together with profit maximization $y_t = w_t l_t + \Gamma_t$ (income)

- Stochastic processes follow:

$$\mathrm{d}\log d_t = -\rho_d \log d_t \mathrm{d}t + \sigma_d \mathrm{d}B_{d,t}$$
$$\mathrm{d}\log A_t = -\rho_a \log A_t \mathrm{d}t + \sigma_d \mathrm{d}B_{a,t}$$
$$\mathrm{d}\log s_{g,t} = -\rho_g \log s_{g,t} \mathrm{d}t + \sigma_g \mathrm{d}B_{g,t}$$

3. Numerical solution

In what follows we solve the Bellman equation (24) using collocation method based on the Matlab `CompEcon` toolbox (cf. Miranda and Fackler 2002). As an initial guess we use the solution of the corresponding linear-quadratic deterministic problem (cf. Appendix A.2).

3.1. An economy without staggered price setting

Consider the case of $\delta \to \infty$ (the probability of receiving a signal is *one*, i.e., there are no staggered prices, all firms will receive the price-change signal at any time), $x_{1,t} = 0$, $x_{2,t} = 0$, $\Pi_t^* \equiv 1$. Suppose that $\pi_t = \pi_{ss}$ (considering π_t as a parameter). Moreover, $w_t = A_t(\varepsilon - 1)/\varepsilon$ where $\varepsilon > 1$, which in turn implies $mc_t = (\varepsilon - 1)/\varepsilon$. From (17) we get $v_t = 1$. Finally, lump-sum taxes are given by $T = -g_t$, profits $F_t = y_t/\varepsilon$, and aggregate output $y_t = A_t l_t$. Thus, the maximized Bellman equation (24) simplifies to:

$$\begin{aligned}
\rho V &= d_t \log c_t - d_t \psi \frac{l_t^{1+\vartheta}}{1+\vartheta} + \left((R_t - \pi_{ss})a_t - c_t + w_t l_t - g_t + y_t/\varepsilon\right) V_a \\
&\quad + \tfrac{1}{2}\sigma_d^2 d_t^2 V_{dd} - (\rho_d \log d_t - \tfrac{1}{2}\sigma_d^2) d_t V_d + \tfrac{1}{2}\sigma_a^2 A_t^2 V_{AA} - (\rho_a \log A_t - \tfrac{1}{2}\sigma_a^2) A_t V_A \\
&\quad + \tfrac{1}{2}\sigma_g^2 s_{g,t}^2 V_{ss} - (\rho_g \log s_{g,t} - \tfrac{1}{2}\sigma_g^2) s_{g,t} V_s + (\theta_0 + \theta_1 \pi_t - \theta_2 R_t) V_R
\end{aligned} \tag{28}$$

In order to solve the continuous-time Bellman equation, we need the following measures. First, the first-order conditions (22) and (23) we obtain the *optimal controls* given the state and costate variables:

$$\mathbb{X}_t = \mathbb{X}(\mathbb{Z}_t, V_{\mathbb{Z}}(\mathbb{Z}_t)) \equiv \begin{bmatrix} c(\mathbb{Z}_t, V_{\mathbb{Z}}(\mathbb{Z}_t)) \\ l(\mathbb{Z}_t, V_{\mathbb{Z}}(\mathbb{Z}_t)) \end{bmatrix} = \begin{bmatrix} (V_a(\mathbb{Z}_t))^{-1} d_t \\ (V_a(\mathbb{Z}_t) A_t (\varepsilon - 1)/(d_t \psi \varepsilon))^{1/\vartheta} \end{bmatrix}.$$

Second, the *reward function* given the state and the optimal controls:

$$f(\mathbb{Z}_t, \mathbb{X}_t) \;=\; d_t \log c_t - d_t \psi \frac{l_t^{1+\vartheta}}{1+\vartheta}.$$

Third, the *drift* term of the state transition equations:

$$g(\mathbb{Z}_t, \mathbb{X}_t) \;=\; \begin{bmatrix} (R_t - \pi_{ss})a_t - c_t + (1 - s_g s_{g,t}) A_t l_t \\ \theta_0 + \theta_1 \pi_{ss} - \theta_2 R_t \\ -(\rho_d \log d_t - \frac{1}{2}\sigma_d^2) d_t \\ -(\rho_a \log A_t - \frac{1}{2}\sigma_a^2) A_t \\ -(\rho_g \log s_{g,t} - \frac{1}{2}\sigma_g^2) s_{g,t} \end{bmatrix}$$

Fourth, the *diffusion* term of the state transition equations:

$$\sigma(\mathbb{Z}_t, \mathbb{X}_t) = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \sigma_d d_t & 0 & 0 \\ 0 & 0 & 0 & \sigma_a A_t & 0 \\ 0 & 0 & 0 & 0 & \sigma_g s_{g,t} \end{bmatrix}$$

Steady-state. It is useful to consider the non-stochastic steady-state values for our system. By construction, steady-state values for the shock processes are $A_{ss} = d_{ss} = s_{g,ss} = 1$, and

$$\begin{aligned} 0 &= \theta_0 + \theta_1 \pi_{ss} - \theta_2 R_{ss}, \\ 0 &= (R_{ss} - \pi_{ss}) a_{ss} - c_{ss} + (1 - s_g s_{g,ss}) A_{ss} l_{ss} \quad \Rightarrow \quad (R_{ss} - \pi_{ss}) a_{ss} = c_{ss} - (1 - s_g) l_{ss}. \end{aligned}$$

where $\theta_0 \equiv \theta_2 R_{ss} - \theta_1 \pi_{ss}$ (as π_{ss} is set exogenously by the government). From the Euler equation (27) we obtain the non-stochastic steady-state interest rate as

$$R_{ss} = \rho + \pi_{ss}.$$

From the first-order conditions, for $\psi \neq 0$ we obtain $(\varepsilon - 1) A_{ss} = \varepsilon \psi l_{ss}^{\vartheta} c_{ss}$. Finally, from the market clearing condition we obtain $(1 - s_g s_{g,ss}) A_{ss} l_{ss} = c_{ss}$. Thus, summarizing we get:

$$l_{ss} = ((\varepsilon - 1)/((1 - s_g)\psi\varepsilon))^{1/(1+\vartheta)} \quad \text{for} \quad \psi \neq 0, \quad c_{ss} = (1 - s_g) l_{ss}, \quad a_{ss} = 0.$$

3.1.1. Inelastic labor supply ($\psi = 0$)

Consider the case where $\psi = 0$, then the household optimally supplies $l_t = 1$. We may guess a value function:

$$V = d_t \mathbb{C}_1 \log a_t + f(R_t, d_t, A_t, s_{g,t}) \tag{29}$$

and derive conditions under which this actually is the solution of the dynamic control problem. This candidate value function implies:

$$c_t = \mathbb{C}_1^{-1} a_t d_t, \quad V_a = d_t \mathbb{C}_1 / a_t$$

and thus the maximized Bellman equation (28) reads:

$$\begin{aligned}
\rho d_t \mathbb{C}_1 \log a_t &= d_t \log c_t + \big((R_t - \pi_{ss})a_t - \mathbb{C}_1^{-1} a_t d_t + (1 - s_g s_{g,t})\, A_t l_t\big)\, d_t \mathbb{C}_1 / a_t \\
&+ \tfrac{1}{2}\sigma_d^2 d_t^2 f_{dd} - (\rho_d \log d_t - \tfrac{1}{2}\sigma_d^2) d_t \mathbb{C}_1 \log a_t - (\rho_d \log d_t - \tfrac{1}{2}\sigma_d^2) d_t f_d \\
&+ \tfrac{1}{2}\sigma_a^2 A_t^2 f_{AA} - (\rho_a \log A_t - \tfrac{1}{2}\sigma_a^2) A_t f_A + \tfrac{1}{2}\sigma_g^2 s_{g,t}^2 f_{ss} \\
&- (\rho_g \log s_{g,t} - \tfrac{1}{2}\sigma_g^2) s_{g,t} f_s + (\theta_0 + \theta_1 \pi_{ss} - \theta_2 R_t) f_R - \rho f(R_t, d_t, A_t, s_{g,t})
\end{aligned}$$

Collecting terms, and using the equation

$$\begin{aligned}
\rho f(R_t, d_t, A_t) &= -d_t \log \mathbb{C}_1 + d_t \log d_t + R_t d_t \mathbb{C}_1 - d_t^2 + \tfrac{1}{2}\sigma_d^2 d_t^2 f_{dd} - (\rho_d \log d_t - \tfrac{1}{2}\sigma_d^2) d_t f_d \\
&+ \tfrac{1}{2}\sigma_a^2 A_t^2 f_{AA} - (\rho_a \log A_t - \tfrac{1}{2}\sigma_a^2) A_t f_A + \tfrac{1}{2}\sigma_g^2 s_{g,t}^2 f_{ss} - (\rho_g \log s_{g,t} - \tfrac{1}{2}\sigma_g^2) s_{g,t} f_s \\
&+ (\theta_0 + \theta_1 \pi_{ss} - \theta_2 R_t) f_R,
\end{aligned}$$

we obtain:

$$\rho d_t \mathbb{C}_1 \log a_t = d_t \log a_t + ((1 - s_g s_{g,t})\, A_t l_t)\, d_t \mathbb{C}_1 / a_t - (\rho_d \log d_t - \tfrac{1}{2}\sigma_d^2) d_t \mathbb{C}_1 \log a_t$$

which has a solution for the case where $s_g s_{g,t} = 1$ (consumption only out of financial wealth) and $\mathbb{C}_1^{-1} = \rho - \frac{1}{2}\sigma_d^2$. It requires to shut down the government shock, $\rho_g = \sigma_g = 0$, and the reversion in the preference shock, $\rho_d = 0$. Note, the preference shock now has a unit root.

In summary, the optimal policy functions for interior solutions are

$$c_t = (\rho - \tfrac{1}{2}\sigma_d^2) a_t d_t, \quad l_t = 1. \tag{30}$$

It remains to verify that the transversality condition $\lim_{t\to\infty} E_0 e^{-\rho t} V(\mathbb{Z}_t^*) = 0$ holds.

3.1.2. Elastic labor supply ($\psi \neq 0$)

Consider the case of elastic labor supply and no government shocks $\rho_g = \sigma_g = 0$ (and $s_{g,t} = 1$). We may guess:

$$V = d_t \mathbb{C}_1 \log a_t + f(R_t, d_t, A_t), \quad a_t > 0. \tag{31}$$

The first-order conditions (22) and (23) imply:

$$c_t = \mathbb{C}_1^{-1} a_t d_t, \quad \psi l_t^{\vartheta} = \mathbb{C}_1 w_t / a_t, \quad V_a(\mathbb{Z}_t) = d_t \mathbb{C}_1 / a_t,$$

and thus the maximized Bellman equation (28) reads:

$$
\begin{aligned}
\rho d_t \mathbb{C}_1 \log a_t &= d_t \log c_t - d_t \frac{\mathbb{C}_1 l_t w_t / a_t}{1+\vartheta} + \left(R_t a_t - \mathbb{C}_1^{-1} a_t d_t + (1-s_g) A_t l_t\right) d_t \mathbb{C}_1 / a_t \\
&\quad + \tfrac{1}{2}\sigma_d^2 d_t^2 f_{dd} - (\rho_d \log d_t - \tfrac{1}{2}\sigma_d^2) d_t \mathbb{C}_1 \log a_t - (\rho_d \log d_t - \tfrac{1}{2}\sigma_d^2) d_t f_d \\
&\quad + \tfrac{1}{2}\sigma_a^2 A_t^2 f_{AA} - (\rho_a \log A_t - \tfrac{1}{2}\sigma_a^2) A_t f_A + (\theta_0 - \theta_2 R_t) f_R - \rho f(R_t, d_t, A_t, s_{g,t}).
\end{aligned}
$$

Collecting terms, and using the function:

$$
\begin{aligned}
\rho f(R_t, d_t, A_t) &= -d_t \log \mathbb{C}_1 + d_t \log d_t + R_t d_t \mathbb{C}_1 - d_t^2 + \tfrac{1}{2}\sigma_d^2 d_t^2 f_{dd} - (\rho_d \log d_t - \tfrac{1}{2}\sigma_d^2) d_t f_d \\
&\quad + \tfrac{1}{2}\sigma_a^2 A_t^2 f_{AA} - (\rho_a \log A_t - \tfrac{1}{2}\sigma_a^2) A_t f_A + (\theta_0 - \theta_2 R_t) f_R,
\end{aligned}
$$

we obtain:

$$
(\rho - \mathbb{C}_1^{-1} + \rho_d \log d_t - \tfrac{1}{2}\sigma_d^2)\mathbb{C}_1 d_t \log a_t = (1 + \vartheta\varepsilon - s_g(\varepsilon + \vartheta\varepsilon)) \frac{A_t l_t d_t \mathbb{C}_1}{(1+\vartheta)\varepsilon a_t},
$$

which is a solution for $s_g = (1+\vartheta\varepsilon)/(\varepsilon+\vartheta\varepsilon)$, $\rho_d = 0$, and $\mathbb{C}_1^{-1} = \rho - \frac{1}{2}\sigma_d^2$.

In summary, the optimal policy functions for interior solutions are

$$
\begin{aligned}
& c_t = (\rho - \tfrac{1}{2}\sigma_d^2) a_t d_t, \quad l_t = (c_t \varepsilon \psi / (\varepsilon - 1))^{-1/\vartheta}, \quad \varepsilon > 1, \quad a_t > 0 \\
\Rightarrow \quad & c_t = \max\left\{c_{ss}, (\rho - \tfrac{1}{2}\sigma_d^2) a_t d_t\right\}, \quad l_t = \max\left\{(c_t \varepsilon \psi / (\varepsilon - 1))^{-1/\vartheta}, 1\right\}.
\end{aligned} \tag{32}
$$

It remains to verify that the transversality condition $\lim_{t\to\infty} E_0 e^{-\rho t} V(\mathbb{Z}_t^*) = 0$ holds.

3.2. An economy with staggered price setting

Let us replicate the maximized Bellman equation (24):

$$
\begin{aligned}
\rho V(\mathbb{Z}_t) &= d_t \log c_t - d_t \psi \frac{l_t^{1+\vartheta}}{1+\vartheta} + ((R_t - \pi_t) a_t - c_t + w_t l_t + T_t + F_t) V_a \\
&\quad + \tfrac{1}{2}\sigma_d^2 d_t^2 V_{dd} - (\rho_d \log d_t - \tfrac{1}{2}\sigma_d^2) d_t V_d + \tfrac{1}{2}\sigma_a^2 A_t^2 V_{AA} - (\rho_a \log A_t - \tfrac{1}{2}\sigma_a^2) A_t V_A \\
&\quad + \tfrac{1}{2}\sigma_g^2 s_{g,t}^2 V_{ss} - (\rho_g \log s_{g,t} - \tfrac{1}{2}\sigma_g^2) s_{g,t} V_s + (\theta_0 + \theta_1 \pi_t - \theta_2 R_t) V_R \\
&\quad + \left(\delta \left(\Pi_t^*\right)^{-\varepsilon} + (\varepsilon \pi_t - \delta) v_t\right) V_v + \left((\rho + \delta - (\varepsilon - 1)\pi_t) x_{1,t} - e^{\rho t} m_t y_t\right) V_{x_1} \\
&\quad + \left((\rho + \delta - \varepsilon \pi_t) x_{2,t} - e^{\rho t} m_t y_t mc_t\right) V_{x_2},
\end{aligned}
$$

where the algebraic equations $w_t = \psi l_t^\vartheta c_t$, $mc_t = w_t / A_t$, $T_t = -g_t = -s_g s_{g,t} y_t$, $y_t = A_t l_t / v_t$, $F_t = (1 - mc_t v_t) y_t$, and the SDF is $m_t = e^{-\rho t} V_a = e^{-\rho t} d_t / c_t$ are considered *aggregate* variables.

In order to solve the continuous-time Bellman equation, we need the following measures.

First, the first-order conditions (22) and (23) we obtain the *optimal controls* given the state and costate variables:

$$\mathbb{X}_t = \mathbb{X}(\mathbb{Z}_t, V_{\mathbb{Z}}(\mathbb{Z}_t)) \equiv \left[\begin{array}{c} c(\mathbb{Z}_t, V_{\mathbb{Z}}(\mathbb{Z}_t)) \\ l(\mathbb{Z}_t, V_{\mathbb{Z}}(\mathbb{Z}_t)) \end{array} \right] = \left[\begin{array}{c} (V_a(\mathbb{Z}_t))^{-1} d_t \\ (V_a(\mathbb{Z}_t) A_t(\varepsilon - 1)/(d_t \psi \varepsilon))^{1/\vartheta} \end{array} \right].$$

Second, the *reward function* given the state and the optimal controls:

$$f(\mathbb{Z}_t, \mathbb{X}_t) \;\; = \;\; d_t \log c_t - d_t \psi \frac{l_t^{1+\vartheta}}{1+\vartheta}.$$

Third, the *drift* term of the state transition equations:

$$g(\mathbb{Z}_t, \mathbb{X}_t) \;\; = \;\; \left[\begin{array}{c} (R_t - \pi_t) a_t - c_t + (1 - s_g s_{g,t}) A_t l_t / v_t \\ \theta_0 + \theta_1 \pi_t - \theta_2 R_t \\ \delta \left(\Pi_t^*\right)^{-\varepsilon} + (\varepsilon \pi_t - \delta) v_t \\ (\rho + \delta - (\varepsilon - 1)\pi_t) x_{1,t} - d_t A_t l_t / (v_t c_t) \\ (\rho + \delta - \varepsilon \pi_t)\, x_{2,t} - \psi d_t l_t^{\vartheta+1} / v_t \\ -(\rho_d \log d_t - \frac{1}{2}\sigma_d^2) d_t \\ -(\rho_a \log A_t - \frac{1}{2}\sigma_a^2) A_t \\ -(\rho_g \log s_{g,t} - \frac{1}{2}\sigma_g^2) s_{g,t} \end{array} \right].$$

Fourth, the *diffusion* term of the state transition equations:

$$\sigma(\mathbb{Z}_t, \mathbb{X}_t) \;\; = \;\; \left[\begin{array}{cccccccc} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \sigma_d d_t & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \sigma_a A_t & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \sigma_g s_{g,t} \end{array} \right].$$

Steady-state. It is useful to consider the non-stochastic steady-state values for our system.

By construction, steady-state values for the shock processes are $A_{ss} = d_{ss} = s_{g,ss} = 1$, and

$$
\begin{aligned}
0 &= (R_{ss} - \pi_{ss})a_{ss} - c_{ss} + (1 - s_g)l_{ss}/v_{ss}, \\
0 &= \theta_0 + \theta_1\pi_{ss} - \theta_2 R_{ss}, \\
0 &= \delta\left(\Pi^*_{ss}\right)^{-\varepsilon} + (\varepsilon\pi_{ss} - \delta)v_{ss} \quad \Rightarrow \quad v_{ss} = \delta\left(\Pi^*_{ss}\right)^{-\varepsilon}/(\delta - \varepsilon\pi_{ss}), \\
0 &= (\rho + \delta - (\varepsilon - 1)\pi_{ss})x_{1,ss} - y_{ss}/c_{ss}, \\
0 &= (\rho + \delta - \varepsilon\pi_{ss})\, x_{2,ss} - y_{ss}mc_{ss}/c_{ss},
\end{aligned}
$$

where $\theta_0 \equiv \theta_2 R_{ss} - \theta_1\pi_{ss}$ (as π_{ss} is set exogenously by the government). From the definition of the inflation rate (13) we obtain:

$$\delta + (1 - \varepsilon)\pi_{ss} = \delta\left(\Pi^*_{ss}\right)^{1-\varepsilon}.$$

From the market clearing condition, we obtain $c_{ss} = (1 - s_g)l_{ss}/v_{ss}$. Inserting this into the steady-state equation for $x_{1,t}$ and collecting terms we get for $s_g \neq 1$:

$$x_{1,ss} = \left((1 - s_g)(\rho + \delta - (\varepsilon - 1)\pi_{ss})\right)^{-1}, \quad x_{2,ss} = (\varepsilon - 1)\Pi^*_{ss}x_{1,ss}/\varepsilon.$$

This in turn gives the stationary labor supply from the steady-state equation for $x_{2,t}$:

$$l_{ss} = \left((\rho + \delta - \varepsilon\pi_{ss})\, x_{2,ss}v_{ss}/\psi\right)^{1/(1+\vartheta)}, \quad \text{and} \quad l_{ss} = 1 \quad \text{for} \quad \psi = 0.$$

From the Euler equation (27) we obtain the non-stochastic steady-state interest rate as:

$$R_{ss} = \rho + \pi_{ss}.$$

Finally, the steady-state level for financial wealth is:

$$
\begin{aligned}
0 &= (R_{ss} - \pi_{ss})a_{ss} - c_{ss} + \psi l_{ss}^{1+\vartheta}c_{ss} + T_{ss} + F_{ss} \\
\Leftrightarrow 0 &= (R_{ss} - \pi_{ss})a_{ss} - c_{ss} + \psi l_{ss}^{1+\vartheta}c_{ss} + (1 - s_g)l_{ss}/v_{ss} - \psi l_{ss}^{1+\vartheta}c_{ss} \\
&\Rightarrow \quad a_{ss} = 0.
\end{aligned}
$$

Note that the TVC requires that $\lim_{t\to\infty} e^{-\rho t}E_0 V(\mathbb{Z}^*) = 0$.

To avoid the curse of dimensionality, one may use sparse grids, replacing the tensor operator by the Smolyak operator (cf. Winschel, Krätzig 2010).

4. Results

This section reports numerical results for the parameterization summarized in Table 1. In the reported solution we select Chebychev polynomial basis functions for all 8 state variables at standard Chebychev nodes.

Table 1: Parameterization

ϑ	2	Frisch labor supply elasticity
ρ	0.03	subjective rate of time preference
ψ	6	preference for leisure
δ	0.25	Calvo parameter for probability of firms receiving signal
ε	6	elasticity of substitution intermediate goods
s_g	0.25	share of government consumption
ρ_d	0.9	autoregressive component preference shock
ρ_a	0.9	autoregressive component technology shock
ρ_g	0.9	autoregressive component government shock
σ_d	0.025	variance preference shock
σ_a	0.025	variance technology shock
σ_g	0.025	variance government shock
θ_1	1.05	inflation response Taylor rule
θ_2	0.95	interest rate response Taylor rule
π_{ss}	0.02	steady-state inflation

Figure 1: Numerical solution of the new Keynesian model (consumption)

In this figure we show (from left to right, top to bottom) optimal consumption as a function of financial wealth and the interest rate, optimal consumption as a function of price dispersion and expectation x_1, optimal consumption as a function of expectation x_2 and the preference shock, and optimal consumption as a function of the technology shock and the government shock. For illustration the consumption function has been sliced at non-stochastic steady state values of the remaining state variables.

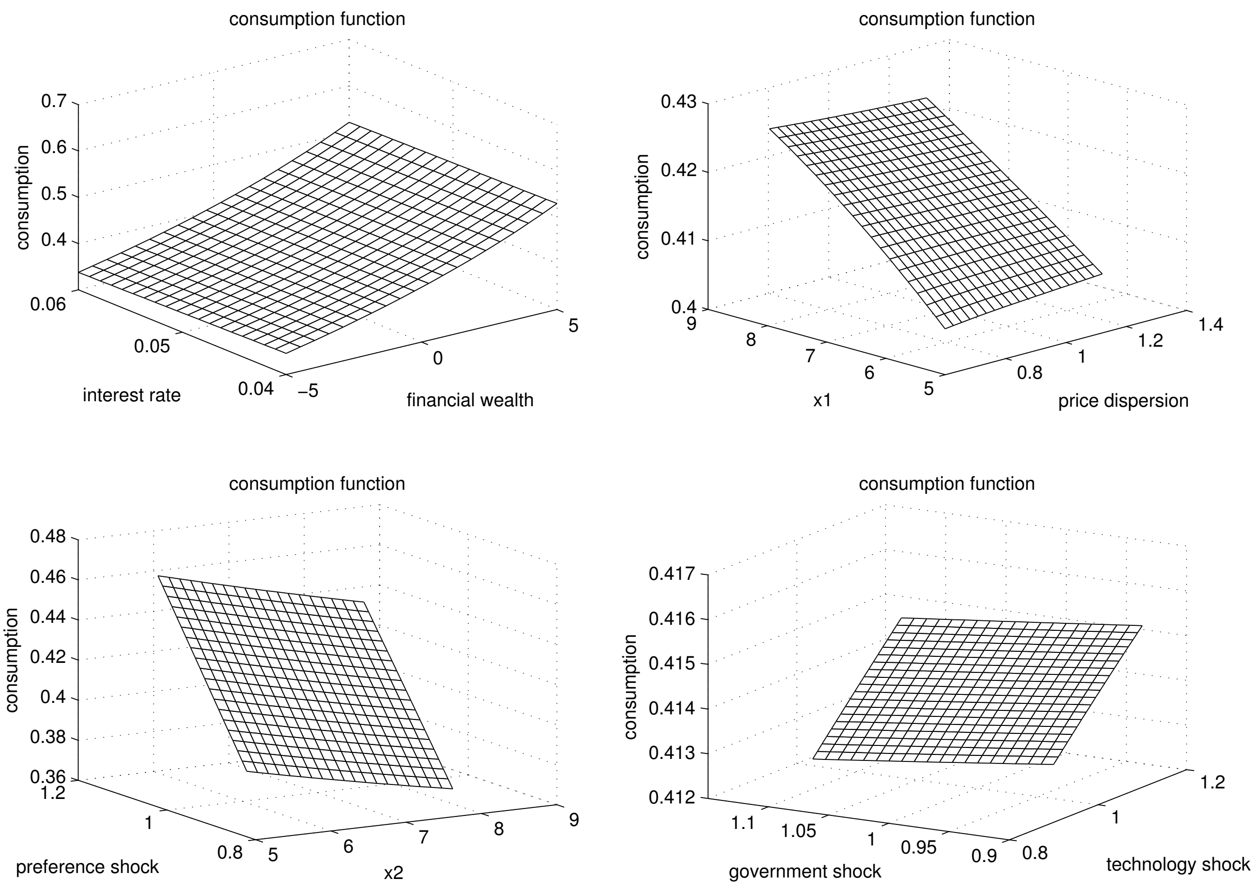


Figure 2: Numerical solution of the new Keynesian model (hours)
In this figure we show (from left to right, top to bottom) optimal hours as a function of financial wealth and the interest rate, optimal hours as a function of price dispersion and expectation x_1, optimal hours as a function of expectation x_2 and the preference shock, and optimal hours as a function of the technology shock and the government shock. For illustration the function for optimal hours has been sliced at non-stochastic steady state values of the remaining state variables.

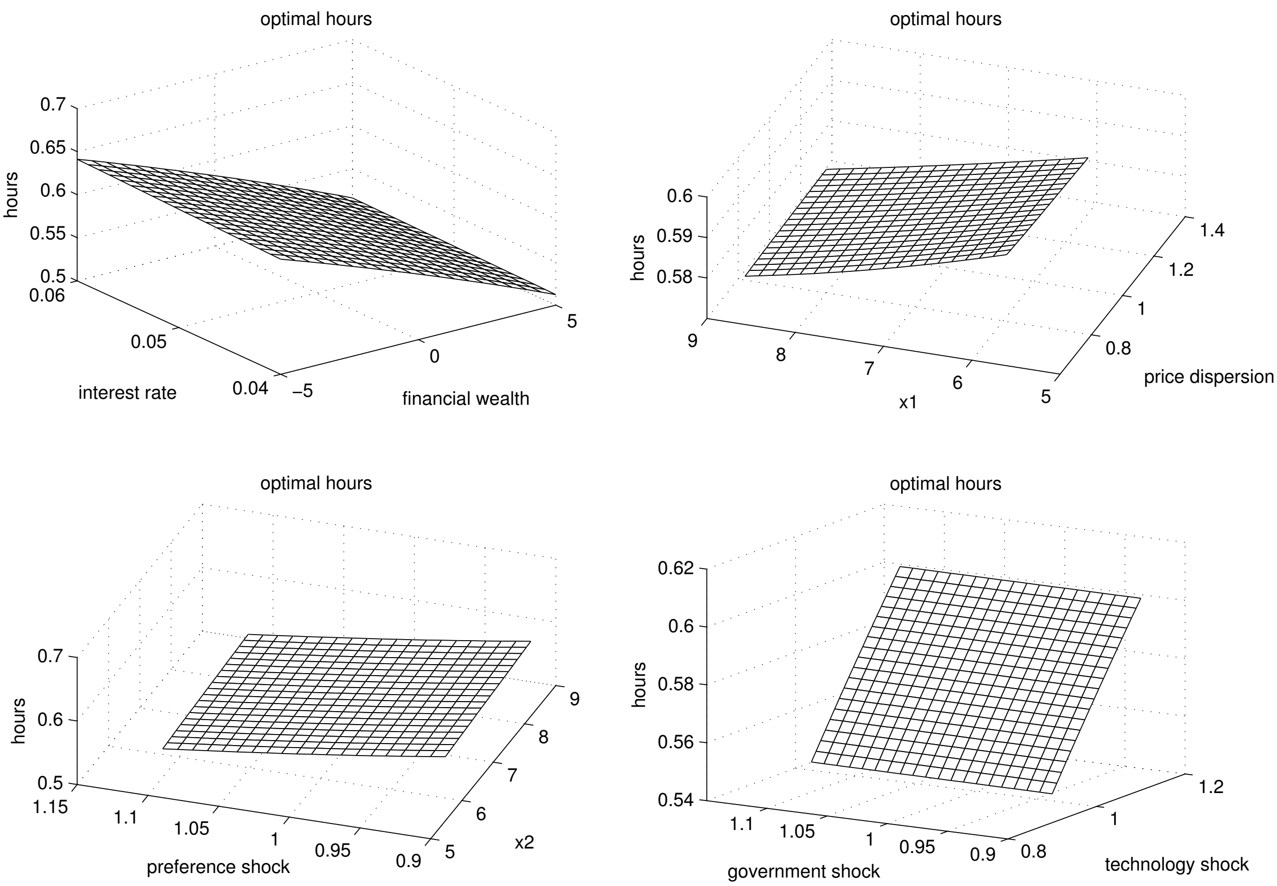
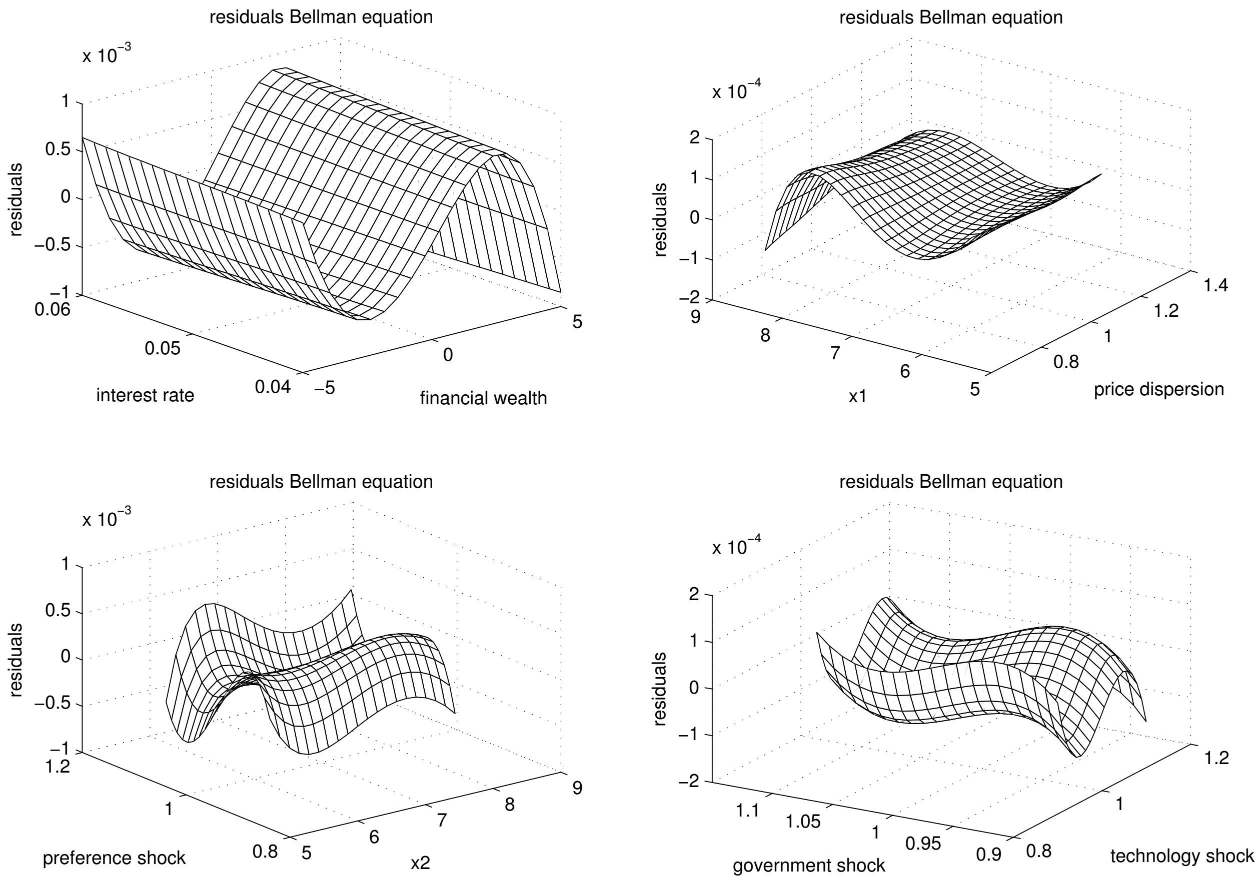


Figure 3: Residuals of the Bellman equation

In this figure we show (from left to right, top to bottom) the residuals of the Bellman equation as a function of financial wealth and the interest rate, the residuals as a function of price dispersion and expectation x_1, the residuals as a function of expectation x_2 and the preference shock, and the residuals as a function of the technology shock and the government shock. For illustration the residual function has been sliced at non-stochastic steady state values of the remaining state variables.



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A. Appendix

A.1. The model

A.1.1. The household's budget constraint for financial wealth (5)

Nominal wealth is given by $p_t a_t = b_t$, where b_t is the individual's nominal bond holdings and p_t is the price of the consumption good:

$$a_t = b_t/p_t. \tag{A.1}$$

Use Itô's formula, the household's real wealth evolves according to:

$$\begin{aligned}
\mathrm{d}a_t &= \mathrm{d}b_t/p_t - b_t/p_t^2\,\mathrm{d}p_t \\
&= (R_t b_t/p_t - c_t + w_t l_t + T_t + \digamma_t)\,\mathrm{d}t - b_t/p_t^2 \pi_t p_t\,\mathrm{d}t \\
&= ((R_t - \pi_t)a_t - c_t + w_t l_t + T_t + \digamma_t)\,\mathrm{d}t,
\end{aligned}$$

which is equation (5) in the text.

A.1.2. Heuristic derivation of the Bellman equation

For illustration, this heuristic derivation obtains the Bellman equation for a value function that depends only on two state variables. A multidimensional version can be obtained along the lines shown below. Splitting up the integral in (20) gives:

$$\mathbb{E}_0 \int_0^{\Delta t} e^{-\rho s} d_s u(c_s, l_s)\,\mathrm{d}s + \mathbb{E}_0 \int_{\Delta t}^{\infty} e^{-\rho s} d_s u(c_s, l_s)\,\mathrm{d}s$$

Following Bellman's idea, the optimal program is:

$$V(a_0, d_0) = \max_{c_0, l_0} \left\{ d_0 u(c_0, l_0)\Delta t + \frac{1}{1+\rho\Delta t}\mathbb{E}_0 V(a_{\Delta t}, d_{\Delta t}) \right\}$$

Multiply by $(1+\rho\Delta t)$ gives:

$$(1+\rho\Delta t)V(a_0, d_0) = \max_{c_0, l_0} \{ d_0 u(c_0, l_0)\Delta t(1+\rho\Delta t) + \mathbb{E}_0 V(a_{\Delta t}, d_{\Delta t}) \}$$

Now divide by Δt and move $\frac{1}{\Delta t}V(a_0, d_0)$ to the right hand side:

$$\rho V(a_0, d_0) \;=\; \max_{c_0, l_0} \left\{ d_0 u(c_0, l_0)(1+\rho\Delta t) + \frac{1}{\Delta t}\left[\mathbb{E}_0 V(a_{\Delta t}, d_{\Delta t}) - V(a_0, d_0)\right] \right\}$$

Letting Δt become infinitesimally small, the problem becomes:

$$\rho V(a_0, d_0) = \max_{c_0, l_0} \left\{ d_0 u(c_0, l_0) + \frac{1}{\mathrm{d}t} \mathbb{E}_0 \, \mathrm{d}V(a_0, d_0) \right\}$$

Observe that using Itô's formula together with (2) and (5) we get:

$$\begin{aligned} \mathrm{d}V(a_t, d_t) &= V_a(a_t, d_t) \, \mathrm{d}a_t + V_d(a_t, d_t) \, \mathrm{d}d_t + \tfrac{1}{2} V_{dd}(a_t, d_t) \sigma_d^2 d_t^2 \, \mathrm{d}t \\ &= V_a(a_t, d_t) \left((R_t - \pi_t) a_t - c_t + w_t l_t + T_t + F_t \right) \mathrm{d}t \\ &\quad + \tfrac{1}{2} V_{dd}(a_t, d_t) \sigma_d^2 d_t^2 \, \mathrm{d}t \\ &\quad + V_d(a_t, d_t) \left(-\rho_d d_t \, \mathrm{d}t + \sigma_d d_t \, \mathrm{d}B_{d,t} + \tfrac{1}{2} d_t \sigma_d^2 \, \mathrm{d}t \right), \end{aligned}$$

using that $B_{d,t}$ and $B_p(t)$ are uncorrelated. Applying the expectation operator gives:

$$\begin{aligned} \mathbb{E}_t \, \mathrm{d}V(a_t, d_t) &= V_a(a_t, d_t) \left((R_t - \pi_t) a_t - c_t + w_t l_t + T_t + F_t \right) \mathrm{d}t \\ &\quad + \tfrac{1}{2} V_{dd}(a_t, d_t) \sigma_d^2 d_t^2 \, \mathrm{d}t \\ &\quad - V_d(a_t, d_t) \rho_d d_t \, \mathrm{d}t + \tfrac{1}{2} V_d(a_t, d_t) d_t \sigma_d^2 \, \mathrm{d}t. \end{aligned}$$

Finally, by inserting we obtain the Bellman equation. A multi-dimensional version gives (21).

A.2. Linear-Quadratic control

Because linear-quadratic solution is relatively easy to obtain, a simple approximate solution to the more general problem is obtained by replacing the state transition function $g(\mathbb{Z}_t, \mathbb{X}_t)$ and objective function $f(\mathbb{Z}_t, \mathbb{X}_t)$ of the general model with linear and quadratic approximants, which is known as the method of linear-quadratic approximation.

Typically, linear and quadratic approximants of g and f are constructed by forming the first- and second Taylor expansion around the steady-state $(\mathbb{Z}^*, \mathbb{X}^*)$,

$$\begin{aligned} f(\mathbb{Z}_t, \mathbb{X}_t) &\approx f^* + f_{\mathbb{Z}}^{*\top} (\mathbb{Z}_t - \mathbb{Z}^*) + f_{\mathbb{X}}^{*\top} (\mathbb{X}_t - \mathbb{X}^*) + \tfrac{1}{2} (\mathbb{Z}_t - \mathbb{Z}^*)^\top f_{\mathbb{Z}\mathbb{Z}}^* (\mathbb{Z}_t - \mathbb{Z}^*) \\ &\quad + (\mathbb{Z}_t - \mathbb{Z}^*)^\top f_{\mathbb{Z}\mathbb{X}}^* (\mathbb{X}_t - \mathbb{X}^*) + \tfrac{1}{2} (\mathbb{X}_t - \mathbb{X}^*)^\top f_{\mathbb{X}\mathbb{X}}^* (\mathbb{X}_t - \mathbb{X}^*), \\ g(\mathbb{Z}_t, \mathbb{X}_t) &\approx g^* + g_{\mathbb{Z}}^* (\mathbb{Z}_t - \mathbb{Z}^*) + g_{\mathbb{X}}^* (\mathbb{X}_t - \mathbb{X}^*), \end{aligned}$$

where $f^*, g^*, f_{\mathbb{Z}}^*, f_{\mathbb{X}}^*, g_{\mathbb{Z}}^*, g_{\mathbb{X}}^*, f_{\mathbb{Z}\mathbb{Z}}^*, f_{\mathbb{Z}\mathbb{X}}^*$, and $f_{\mathbb{X}\mathbb{X}}^*$ are the values and partial derivatives of f and g evaluated at the steady-state. Thus, the approximate solution to the general problem is supposed to capture the local dynamics around that stationary value.

Collecting terms, we may identify the coefficient of the linear-quadratic problem as

$$
\begin{aligned}
\tilde{f}_0 &\equiv f^* - f_{\mathbb{Z}}^{*\top}\mathbb{Z}^* - f_{\mathbb{X}}^{*\top}\mathbb{X}^* + \tfrac{1}{2}\mathbb{Z}^{*\top} f_{\mathbb{Z}\mathbb{Z}}^*\mathbb{Z}^* + \mathbb{Z}^{*\top} f_{\mathbb{Z}\mathbb{X}}^*\mathbb{X}^* + \tfrac{1}{2}\mathbb{X}^{*\top} f_{\mathbb{X}\mathbb{X}}^*\mathbb{X}^*,\\
\tilde{f}_{\mathbb{Z}} &\equiv f_{\mathbb{Z}}^* - f_{\mathbb{Z}\mathbb{Z}}^*\mathbb{Z}^* - f_{\mathbb{Z}\mathbb{X}}^*\mathbb{X}^*,\\
\tilde{f}_{\mathbb{X}} &\equiv f_{\mathbb{X}}^* - f_{\mathbb{Z}\mathbb{X}}^{*\ \top}\mathbb{Z}^* - f_{\mathbb{X}\mathbb{X}}^*\mathbb{X}^*,\\
\tilde{f}_{\mathbb{Z}\mathbb{Z}} &\equiv f_{\mathbb{Z}\mathbb{Z}}^*, \quad \tilde{f}_{\mathbb{Z}\mathbb{X}} \equiv f_{\mathbb{Z}\mathbb{X}}^*, \quad \tilde{f}_{\mathbb{X}\mathbb{X}} \equiv f_{\mathbb{X}\mathbb{X}}^*,\\
\tilde{g}_0 &\equiv g^* - g_{\mathbb{Z}}^*\mathbb{Z}^* - g_{\mathbb{X}}\mathbb{X}^*, \quad \tilde{g}_{\mathbb{Z}} \equiv g_{\mathbb{Z}}^*, \quad \tilde{g}_{\mathbb{X}} \equiv g_{\mathbb{X}}^*.
\end{aligned}
$$

where

$$\tilde{f}(\mathbb{Z}_t, \mathbb{X}_t) = \tilde{f}_0 + \tilde{f}_{\mathbb{Z}}^\top \mathbb{Z}_t + \tilde{f}_{\mathbb{X}}^\top \mathbb{X}_t + \tfrac{1}{2}\mathbb{Z}_t^\top \tilde{f}_{\mathbb{Z}\mathbb{Z}}\mathbb{Z}_t + \mathbb{Z}_t^\top \tilde{f}_{\mathbb{Z}\mathbb{X}}\mathbb{X}_t + \tfrac{1}{2}\mathbb{X}_t^\top \tilde{f}_{\mathbb{X}\mathbb{X}}\mathbb{X}_t, \tag{A.2}$$

and a linear state transition function

$$\tilde{g}(\mathbb{Z}_t, \mathbb{X}_t) = \tilde{g}_0 + \tilde{g}_{\mathbb{Z}}\mathbb{Z}_t + \tilde{g}_{\mathbb{X}}\mathbb{X}_t, \tag{A.3}$$

$\mathbb{Z}_t$ is the $n \times 1$ state vector, $\mathbb{X}_t$ is the $m \times 1$ control vector, and the parameters are $\tilde{f}_0$, a constant; $\tilde{f}_{\mathbb{Z}}$, a $n \times 1$ vector; $\tilde{f}_{\mathbb{X}}$, a $m \times 1$ vector; $\tilde{f}_{\mathbb{Z}\mathbb{Z}}$, an $n \times n$ matrix, $\tilde{f}_{\mathbb{Z}\mathbb{X}}$, an $n \times m$ matrix; $\tilde{f}_{\mathbb{X}\mathbb{X}}$, an $m \times m$ matrix; $\tilde{g}_0$, an $n \times 1$ vector; $\tilde{g}_{\mathbb{Z}}$, an $n \times n$ matrix; and $\tilde{g}_{\mathbb{X}}$, an $n \times m$ matrix.

As from (21), the *Bellman equation* for the (non-stochastic) problem reads

$$\rho V(\mathbb{Z}_t) = \max_{\mathbb{X}_t \in U} \left\{ \tilde{f}(\mathbb{Z}_t, \mathbb{X}_t) + V_{\mathbb{Z}}(\mathbb{Z}_t)^\top \tilde{g}(\mathbb{Z}_t, \mathbb{X}_t) \right\}, \tag{A.4}$$

Observe that we have m *first-order conditions*, $\tilde{f}_{\mathbb{X}} + \tilde{f}_{\mathbb{Z}\mathbb{X}}^\top \mathbb{Z}_t + \tilde{f}_{\mathbb{X}\mathbb{X}}\mathbb{X}_t + \tilde{g}_{\mathbb{X}}^\top V_{\mathbb{Z}}(\mathbb{Z}_t) = 0$, which imply

$$\mathbb{X}_t = -\tilde{f}_{\mathbb{X}\mathbb{X}}^{-1}\left[\tilde{f}_{\mathbb{X}} + \tilde{f}_{\mathbb{Z}\mathbb{X}}^\top \mathbb{Z}_t + \tilde{g}_{\mathbb{X}}^\top V_{\mathbb{Z}}(\mathbb{Z}_t)\right], \tag{A.5}$$

i.e., the controls are linear in the state and the costate variables. We proceed as follows. The solution of the linear-quadratic problem is the value function which satisfies both the maximized Bellman equation and the first-order condition. We may guess a value function and derive conditions under which the guess indeed is the solution to our problem.

An educated guess for the value function is

$$V(\mathbb{Z}_t) = \mathbb{C}_0 + \Lambda_{\mathbb{Z}}^\top \mathbb{Z}_t + \tfrac{1}{2}\mathbb{Z}_t^\top \Lambda_{\mathbb{Z}\mathbb{Z}}\mathbb{Z}_t \tag{A.6}$$

in which the parameters $\mathbb{C}_0$, a constant; $\Lambda_{\mathbb{Z}}$ a $n \times 1$ vector; and $\Lambda_{\mathbb{Z}\mathbb{Z}}$, a $n \times n$ matrix need to be determined. This implies that the costate variable, a $n \times 1$ vector (and thus the controls) is linear in the states

$$V_{\mathbb{Z}}(\mathbb{Z}_t) = \Lambda_{\mathbb{Z}} + \Lambda_{\mathbb{Z}\mathbb{Z}}\mathbb{Z}_t \tag{A.7}$$

Inserting everything into the *maximized Bellman equation* gives

$$\begin{aligned}\rho\mathbb{C}_0 + \rho\Lambda_{\mathbb{Z}}^\top \mathbb{Z}_t + \rho\tfrac{1}{2}\mathbb{Z}_t^\top \Lambda_{\mathbb{Z}\mathbb{Z}}\mathbb{Z}_t &= \tilde{f}_0 + \tilde{f}_{\mathbb{Z}}^\top \mathbb{Z}_t + \tilde{f}_{\mathbb{X}}^\top \mathbb{X}_t + \tfrac{1}{2}\mathbb{Z}_t^\top \tilde{f}_{\mathbb{Z}\mathbb{Z}}\mathbb{Z}_t + \mathbb{Z}_t^\top \tilde{f}_{\mathbb{Z}\mathbb{X}}\mathbb{X}_t + \tfrac{1}{2}\mathbb{X}_t^\top \tilde{f}_{\mathbb{X}\mathbb{X}}\mathbb{X}_t \\ &\quad + \Lambda_{\mathbb{Z}}^\top \left[\tilde{g}_0 + \tilde{g}_{\mathbb{Z}}\mathbb{Z}_t + \tilde{g}_{\mathbb{X}}\mathbb{X}_t\right] + \mathbb{Z}_t^\top \Lambda_{\mathbb{Z}\mathbb{Z}}\left[\tilde{g}_0 + \tilde{g}_{\mathbb{Z}}\mathbb{Z}_t + \tilde{g}_{\mathbb{X}}\mathbb{X}_t\right]\end{aligned}$$

where the vector components are

$$\begin{aligned}\mathbb{X}_t &= -\tilde{f}_{\mathbb{X}\mathbb{X}}^{-1}\left[\tilde{f}_{\mathbb{X}} + \tilde{g}_{\mathbb{X}}^\top \Lambda_{\mathbb{Z}} + \left[\tilde{f}_{\mathbb{Z}\mathbb{X}}^\top + \tilde{g}_{\mathbb{X}}^\top \Lambda_{\mathbb{Z}\mathbb{Z}}\right]\mathbb{Z}_t\right], \\ \mathbb{X}_t^\top \tilde{f}_{\mathbb{X}\mathbb{X}}\mathbb{X}_t &= \left[\tilde{f}_{\mathbb{X}}^\top + \Lambda_{\mathbb{Z}}^\top \tilde{g}_{\mathbb{X}}\right]\tilde{f}_{\mathbb{X}\mathbb{X}}^{-1}\left[\tilde{f}_{\mathbb{X}} + \tilde{g}_{\mathbb{X}}^\top \Lambda_{\mathbb{Z}}\right] + 2\left[\tilde{f}_{\mathbb{X}}^\top + \Lambda_{\mathbb{Z}}^\top \tilde{g}_{\mathbb{X}}\right]\tilde{f}_{\mathbb{X}\mathbb{X}}^{-1}\left[\tilde{f}_{\mathbb{Z}\mathbb{X}}^\top + \tilde{g}_{\mathbb{X}}^\top \Lambda_{\mathbb{Z}\mathbb{Z}}\right]\mathbb{Z}_t \\ &\quad + \mathbb{Z}_t^\top \left[\tilde{f}_{\mathbb{Z}\mathbb{X}_t}^\top + \tilde{g}_{\mathbb{X}}^\top \Lambda_{\mathbb{Z}\mathbb{Z}}\right]^\top \tilde{f}_{\mathbb{X}\mathbb{X}}^{-1}\left[\tilde{f}_{\mathbb{Z}\mathbb{X}}^\top + \tilde{g}_{\mathbb{X}}^\top \Lambda_{\mathbb{Z}\mathbb{Z}}\right]\mathbb{Z}_t\end{aligned}$$

Finally equating terms with equal powers determines our coefficients recursively

$$\begin{aligned}\rho\mathbb{C}_0 &= \tilde{f}_0 - \tilde{f}_{\mathbb{X}}^\top \tilde{f}_{\mathbb{X}\mathbb{X}}^{-1}\left[\tilde{f}_{\mathbb{X}} + \tilde{g}_{\mathbb{X}}^\top \Lambda_{\mathbb{Z}}\right] + \tfrac{1}{2}\left[\tilde{f}_{\mathbb{X}}^\top + \Lambda_{\mathbb{Z}}^\top \tilde{g}_{\mathbb{X}}\right]\tilde{f}_{\mathbb{X}\mathbb{X}}^{-1}\left[\tilde{f}_{\mathbb{X}} + \tilde{g}_{\mathbb{X}}^\top \Lambda_{\mathbb{Z}}\right] \\ &\quad + \Lambda_{\mathbb{Z}}^\top \left[\tilde{g}_0 - \tilde{g}_{\mathbb{X}}\tilde{f}_{\mathbb{X}\mathbb{X}}^{-1}\left[\tilde{f}_{\mathbb{X}} + \tilde{g}_{\mathbb{X}}^\top \Lambda_{\mathbb{Z}}\right]\right], \\ \rho\Lambda_{\mathbb{Z}}^\top &= \tilde{f}_{\mathbb{Z}}^\top - \left[\tilde{f}_{\mathbb{X}} + \tilde{g}_{\mathbb{X}}^\top \Lambda_{\mathbb{Z}}\right]^\top f_{\mathbb{X}\mathbb{X}}^{-1}\tilde{f}_{\mathbb{Z}\mathbb{X}}^\top + \Lambda_{\mathbb{Z}}^\top \tilde{g}_{\mathbb{Z}} + \tilde{g}_0^\top \Lambda_{\mathbb{Z}\mathbb{Z}} - \left[\tilde{f}_{\mathbb{X}} + \tilde{g}_{\mathbb{X}}^\top \Lambda_{\mathbb{Z}}\right]^\top f_{\mathbb{X}\mathbb{X}}^{-1}\tilde{g}_{\mathbb{X}}^\top \Lambda_{\mathbb{Z}\mathbb{Z}}, \\ \rho\tfrac{1}{2}\Lambda_{\mathbb{Z}\mathbb{Z}} &= \tfrac{1}{2}\tilde{f}_{\mathbb{Z}\mathbb{Z}} - \tilde{f}_{\mathbb{Z}\mathbb{X}}\tilde{f}_{\mathbb{X}\mathbb{X}}^{-1}\left[\tilde{f}_{\mathbb{Z}\mathbb{X}}^\top + \tilde{g}_{\mathbb{X}}^\top \Lambda_{\mathbb{Z}\mathbb{Z}}\right] + \tfrac{1}{2}\left[\tilde{f}_{\mathbb{Z}\mathbb{X}}^\top + \tilde{g}_{\mathbb{X}}^\top \Lambda_{\mathbb{Z}\mathbb{Z}}\right]^\top \tilde{f}_{\mathbb{X}\mathbb{X}}^{-1}\left[\tilde{f}_{\mathbb{Z}\mathbb{X}}^\top + \tilde{g}_{\mathbb{X}}^\top \Lambda_{\mathbb{Z}\mathbb{Z}}\right] \\ &\quad + \tfrac{1}{2}\Lambda_{\mathbb{Z}\mathbb{Z}}\tilde{g}_{\mathbb{Z}} + \tfrac{1}{2}\tilde{g}_{\mathbb{Z}}^\top \Lambda_{\mathbb{Z}\mathbb{Z}} - \Lambda_{\mathbb{Z}\mathbb{Z}}\tilde{g}_{\mathbb{X}}\tilde{f}_{\mathbb{X}\mathbb{X}}^{-1}\left[\tilde{f}_{\mathbb{Z}\mathbb{X}}^\top + \tilde{g}_{\mathbb{X}}^\top \Lambda_{\mathbb{Z}\mathbb{Z}}\right].\end{aligned}$$

Rewriting the last condition gives

$$\begin{aligned}0 &= \tilde{f}_{\mathbb{Z}\mathbb{Z}} + \Lambda_{\mathbb{Z}\mathbb{Z}}(\tilde{g}_{\mathbb{Z}} - \tfrac{1}{2}\rho\mathbb{I}_n) + (\tilde{g}_{\mathbb{Z}} - \tfrac{1}{2}\rho\mathbb{I}_n)^\top \Lambda_{\mathbb{Z}\mathbb{Z}} \\ &\quad - \left[\tilde{f}_{\mathbb{Z}\mathbb{X}}\tilde{f}_{\mathbb{X}\mathbb{X}}^{-1}\tilde{f}_{\mathbb{Z}\mathbb{X}}^\top + \tilde{f}_{\mathbb{Z}\mathbb{X}}\tilde{f}_{\mathbb{X}\mathbb{X}}^{-1}\tilde{g}_{\mathbb{X}}^\top \Lambda_{\mathbb{Z}\mathbb{Z}} + \Lambda_{\mathbb{Z}\mathbb{Z}}\tilde{g}_{\mathbb{X}}\tilde{f}_{\mathbb{X}\mathbb{X}}^{-1}\tilde{f}_{\mathbb{Z}\mathbb{X}}^\top + \Lambda_{\mathbb{Z}\mathbb{Z}}\tilde{g}_{\mathbb{X}}\tilde{f}_{\mathbb{X}\mathbb{X}}^{-1}\tilde{g}_{\mathbb{X}}^\top \Lambda_{\mathbb{Z}\mathbb{Z}}\right],\end{aligned}$$

where $\mathbb{I}_n$ is the identity matrix, which gives $\Lambda_{\mathbb{Z}\mathbb{Z}}$ and thus recursively, $\Lambda_{\mathbb{Z}}$ and $\mathbb{C}_0$ as a solution of an algebraic *Ricatti* equation. The vector of coefficients $\Lambda_{\mathbb{Z}}$ is obtained from

$$\Lambda_{\mathbb{Z}} = \left[\rho\mathbb{I}_n + \tilde{f}_{\mathbb{Z}\mathbb{X}}\tilde{f}_{\mathbb{X}\mathbb{X}}^{-1}\tilde{g}_{\mathbb{X}}^\top + \Lambda_{\mathbb{Z}\mathbb{Z}}\tilde{g}_{\mathbb{X}}\tilde{f}_{\mathbb{X}\mathbb{X}}^{-1}\tilde{g}_{\mathbb{X}}^\top - \tilde{g}_{\mathbb{Z}}^\top\right]^{-1}\left[\tilde{f}_{\mathbb{Z}}^\top - \tilde{f}_{\mathbb{X}}^\top \tilde{f}_{\mathbb{X}\mathbb{X}}^{-1}\left[\tilde{f}_{\mathbb{Z}\mathbb{X}}^\top + \tilde{g}_{\mathbb{X}}^\top \Lambda_{\mathbb{Z}\mathbb{Z}}\right] + \tilde{g}_0^\top \Lambda_{\mathbb{Z}\mathbb{Z}}\right]^\top.$$

This closes the proof that the guess indeed is the solution.