Sorting and Factor Intensity: Production and Unemployment across Skills*

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Abstract

When firms choose the allocation of workers, they can adjust not only the type of worker, the extensive margin, but also the intensive margin, how many of those worker to employ. We propose a tractable matching model with such factor intensity. Positive sorting arises if a simple cross-margin-complementarity condition holds: within-complementarities in extensive and intensive margin have to exceed the between-complementarities across intensive and extensive margin. We characterize the equilibrium allocation, wages and factor intensities. In an immediate extension that allows for frictional hiring we also analyze the presence of unemployment across types. Unemployment is decreasing in skills, and we find conditions under which firm size is increasing in skill.

Keywords. Factor intensity. Unemployment. Competitive Search Equilibrium. Two-Sided Matching. Complementarity.

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1 Introduction

We propose a theory of labor market matching where firms can adjust the factor intensity. When aggregate conditions change, for example due to business cycles, international competition or changes in factor endowments, firm react not only by adjusting the extensive margin (the skill of a worker that is right for the job), but also the intensive margin (how many of those workers to employ). The intensive margin indicates how labor productivity changes. For example, during a recession there can be both increased unemployment *and* growing labor productivity. Firms adjust capacity by laying off workers, which in the presence of a concave production technology raises the marginal product of the workers.

Key in understanding the interplay of productivity and the intensive margin is heterogeneity in skills. When workers differ in skills and ability, and firms employ technologies with different levels of productivity. In standard models of sorting the relevant choice is on the extensive margin by choosing the optimal allocation over different types of skills. Here, firms or technologies simultaneously operate two margins: not only do they optimally choose the skill of a worker, they also choose how many of those skilled workers to employ. The ensuing capital-labor ratio will as a result vary across skills and technologies. The combination of two-sided heterogeneity and endogenous factor intensity pins down the equilibrium outcome and determines wage formation, wage inequality and unemployment.

Unlike the standard Beckerian matching framework, firms are large and we can analyze changes in the firm size and composition. In the presence of an aggregate productivity shock for example, firms are able to adjust the skill quality as well as the number of those skilled workers. Even though our model is static, in a dynamic setting, the latter may turn out to be more easily adjusted since the adjustment is for marginal types only, rather than the entire pool.

Our setting also permits us to analyze equilibrium unemployment. In a directed search environment, firms choose their skill composition and size taking into account frictions in the hiring process. This permits us to analyze how unemployment changes across skills and firm size. Interestingly, in our framework long term unemployment arises naturally and it is determined characteristics of the production technology. In particular, firms might compete for higher skilled workers rather than substituting them with low skilled ones, thus leaving the low skilled permanently unemployed. Interestingly, aggregate productivity shocks will lead to the low skilled to move in and out the state of unemployability, while simultaneously equilibrium unemployment adjusts for higher-skilled workers.

Our main findings. First, we show a surprisingly simple condition for positive sorting. As in the one-to-one matching model, it requires a strong enough degree of complementarity between the type of

firm and that of the worker. Now the strength also depends on the marginal change of output in labor and technology intensity and how that varies across worker and firm types respectively. The relevant condition for sorting therefore has to involve changes in the quality dimensions (extensive margin) and in the quantity of factor inputs (intensive margin). Positive assortative matching arises if the the total degree of intensive and extensive complementarity exceed the cross-complementarities between the intensive and extensive margin. It might be interesting to note the latter become unimportant when the intensive margin is Lientieff, in which case standard supermodularity prevails. Second, even under assortative matching one still has to determine the exact allocation and the associated factor intensities. We derive these expressions, including conditions for them to be increasing. Finally, we integrate labor market frictions into the model and show how unemployment varies accross worker types.

Related literature. The idea of modeling firms as many-to-one matchings has extensively been analyzed by Kelso and Crawford (1982). They propose a general, discrete agent framework. While it is well-known that the stable equilibrium of many-to-one matchings may not exist, Kelso and Crawford derive a sufficient condition for existence, that of gross substitutes. This condition basically means that adding one worker will not dramatically increase the productivity of all the other ones. This condition is satisfied in our setting since the intensive margin is concave: an additional worker is strictly less productive. While Kelso and Crawford allow arbitrary type-dependent production processes only restricted by the cross-substitute condition and show existence, characterization results under intensive margins are essentially missing. Our work is limited to the case where production depends only the type of capital and the capital intensity that is assigned to each worker. For our continuum economy this allows us to capture the intensive margin element while retaining a level of tractability that proves extremely useful in application.

The main difference of our approach to the standrad matching models that feature in many recent applications originating from Kantorovich (1942), Koopmans and Beckmann (1957), Shapley and Shubik (1971), Becker (1973), and Sattinger (1975), is that these earlier models restrict attention to settings where each agent on one side can only be matched with one agent on the other side. Using a continuum economy, we allow any ratio of agents from one side relative to agents on the other side. Restricting the output to Liontief retrievs the standard supermodularity condition, but in general it reveals a richer condition that takes into account the extensive margin in the assignment.

While some settings such as the marriage market, Examples of extensive margins abound. Our leading example is the assignment of workers to firms, where the number of workers per firm does

not need to be unity, but can be freely chosen. Another example is an extension to the classical transportation problem going back to Kantorovich (1942). He considered goods that a produced at various plants and are stored at various locations, and unit costs of transportation that depend on the production site and the storage location. In his setting, each good needs to be stored in exactly one unit of storage, while our setup allows for the case where more goods can be cramped into each unit of storage but damage increases with storage intensity. We discuss this example as well as other applications such as matching between teams of workers at the end, but will focus on our leading example in the main body.

Our model differs from settings such as the Roy (1951) model and its recent variants in e.g. Heckman and Honore (1990) where each firm (or sector) can absorb unbounded numbers of agents. In our setup marginal product decreases as any particular firm gets extensively used.

The extension to search frictions is linked to recent developments on sorting in search markets by Shimer and Smith (2000), Shi (2001), Shimer (2005), Atakan (2006), and Eeckhout and Kircher (2010), but the real novelty of our approach to is to provide general conditions for large firms that can aquire labor either competitively or through a competitive search channel.

2 The Model

Players. The economy consists of heterogeneous firms and workers. Workers are indexed their skill $x \in X = [\underline{x}, \overline{x}]$, and $H_w(x)$ denotes the measure of workers with skills below x, with continuous non-zero density h_w. Also firms are heterogeneous in terms of some propriatory input into production that is exclusive to the firm, such as scarce managerial talent or particular propriatory capital goods. Firms are indexed by their productivity type $y \in Y = [\underline{y}, \overline{y}]$, where $H_f(y)$ denote the measure of firms with type below y, with non-zero continuous density h_f. It will be useful to think of the number of firms as small relative to the number of workers, which will allow each firm to hire a continuum of workers.[1]

Preferences and Production: Firms and workers are risk-neutral expected utility maximizers. If a firm of type y hires an amount of labor l_x of type x, it has to choose a fraction of its propriatory resources r_x that it dedicates to this worker type. This allows the firm to produce output

$$F(x, y, l_x, r_x)$$

[1] Although the set of individuals has the same cardinality as the set of firms, it is helpful to think of the set of firms as a closed interval in $[\underline{y}, \overline{y}] \subseteq \mathbb{R}$, and the set of workers as a two–dimensional subset $[\underline{x}, \overline{x}] \times [0, 1] \subseteq \mathbb{R}^2$. When both sets are endowed with the Lebesgue measure, an active firm employs a continuum of workers, albeit of mass zero.

with this worker type. Note that firms of different type may have different amounts of propratory resources to allocate, yet despite the fact that we only specify the proportion which which they allocate their resources this difference is accounted for in the dependence of the output on the firm type.[2] In the production function the first two arguments (x, y) are quality variables regarding firm productivity and worker skill, while the latter two arguments (l_x, r_x) are quantity variables describing the level of inputs. The output is assumed to be increasing and twice differentiable in all arguments, and strictly concave in each of the quantity variables. For most analysis we also assume that output is constant returns to scale in the quantity variables, which turns it into a theory where the factor intensity r_x/l_x with which each worker is utilized becomes important. Other non-differentiated inputs into production can easily be accomodated, but are suppressed here to focus on the pricing and matching between inputs with high levels of heterogeneity.[3]

The total output of a firm is the sum of the outputs accross all its worker types. A firm that produces with only one worker type produces

$$f(x, y, l) := F(x, y, l, 1).$$

This introduces an intensive margin into models that have traditionally focussed on pair-wise matching (see most literature following Becker (1973)), which is similar to imposing a one-unit labor force for all worker-firm pairs in this model. Since $f(x, y, l)$ is strictly decreasing in l, this theory provides a bridge to the literature on large firms with decreasing returns (see e.g. the literature following Stole and Zwiebel (1996)) that usually analyzes homogeneous workers and firms. The tractability arises because quality complementarities are concentrated on the worker-firm interaction rather than on intra-worker skill-complementarities.

Competitive Market Equilibrium : In equilibrium, workers of type x obtain some expected utility $w(x)$ that coincides with the expected wage that they are paid. Firms take this hedonic schedule as given when they make their hiring decision. We will first fix ideas by outlining the definitions of an intensive-margin hedonic pricing equilibrium without search frictions, and then handle the important case with search frictions and associated unemployment in an extension along the lines of the competitve search literature.

[2] If the true production is given by function $\tilde{F}(x, y, l, t)$ and the firm can allocate up to $T(y)$ units of its resource to this worker type, we can rewrite this in terms of fractions by specifying $F(x, y, l, r) = \tilde{F}(x, y, l, rT(y))$.

[3] Consider for example a small open economy where some non-differentiated commodity "capital" is available at price i, and the production function is $\hat{F}(x, y, l, r, k)$, then we can work with the implied production function after capital is adjusted to the optimal level as $F(x, y, l, r) = \max_k \hat{F}(x, y, l, r, k) - ik$.

Firm optimality in a frictionless competitive market means that a firm of type y maximizes its output minus wage costs as follows:

$$\max_{l_x, r_x} \int [F(x, y, l_x, r_x) - w(x) l_x] dx \tag{1}$$

where r_x can be any probability density function over x. Factoring out r_x from the square bracket reveals that the interior depends only on the factor intensity $\theta = l_x / r_x$, which can be freely chosen at any level in $\Theta = \mathbb{R}_+$ by adjusting the labor input appropriately. Optimality requires that the firm places positive resources only on combinations of $x \in X$ and $\theta \in \Theta$ that solve[4]

$$\max_{x, \theta} f(x, y, \theta) - \theta w(x). \tag{2}$$

If there is only one such combination that solves this maximization problem, then the firm will hire only one worker type, allocate all resources to this type, and hire an amount of labor $l = \theta$.

Feasibility of the allocation implies that firms attempt to hire no more workers than there are in the population. Denote by $\mathcal{R}(x, y, \theta)$ the resource allocation in the economy, which describes the amount of resources that firms with a type below y devote to workers of a type below x that are employed with a factor intensity $l_x / r_x \leq \theta$. Let $\mathcal{R}(y|X, \Theta)$ denote the marginal over y when the other two variables can take any value in their type space. It denotes the amount of resources used by firms with type below y. Since the resources of each firm are normalized to one, this has to equal the amount of firms in the population, so feasibility requires $\mathcal{R}(y|X, \Theta) = H^f(y)$ for all y. Moreover, let $\mathcal{R}^\theta(\theta|x, Y)$ denote the marginal with respect to θ of the distribution conditional on a particular worker type x. It denotes the resources spent by all firms on workers of with type x employed with intensity less than θ. Feasibility requires $\int \theta d\mathcal{R}(\theta|x, Y) \leq h_w(x)$. It states that the amount of workers of type x demanded across all firm types cannot exceed the number of such workers in the population (where the labor demand is the factor intensity times the amount of resources allocated at this factor intensity).

Definition 1 *An intensive-margin hedonic pricing equilibrium is a tuple (w,$\mathcal{R}$) consisting of a non-negative hedonic wage schedule $w(\cdot)$ and a resource allocation $\mathcal{R}$ such that*

1. Optimality: $(x, y, \theta) \in \text{supp}\mathcal{R}$ only if it satisfies (2).

2. Market Clearing: $\int \theta d\mathcal{R}(\theta|x, Y) \leq h_w(x)$, with equality if $w(x) > 0$.

[4] Problem (1) is equivalent to $\max_{r_{(\cdot)}} \int (r_x \max_{\theta_x} [F(x, y, \theta_x, 1) - w(x)\theta_x]) dx$, where $\theta_x = l_x / r_x$ can be adjusted through appropriate hiring of workers. Clearly, resources are only devoted to combinations of x and θ that maximize (2).

Markets with Search Frictions: With a minor modification, the following formulation slightly extends the setup to allow for involuntary unemployment due of search frictions. It generalizes the previous exposition to point out that once we allow for an intensive margin in production, it also enables us to directly consider hiring decisions in a market with search frictions. The presentation follows the competitive search literature (e.g., Peters 1991; Acemoglu and Shimer 1999; Burdett, Shi and Wright 2001; Shi 2001; Shimer 2005; Eeckhout and Kircher 2009; Guerrieri, Shimer and Wright 2010) and its extensions to the analysis of multiworker firms (Menzio and Moen 2010; Garibaldi and Moen forthcoming; Kaas and Kircher 2011).

Each firm decides for each worker type x that it wants to hire how many vacancies v_x to post at cost cv_x. It also decides to post wage ω_x for this worker type. Observing all vacancy postings, workers decide where to search for a job. Let q_x denote the "queue" of workers searching for a particular wage offer, defined as the number of workers per vacancy The probability of filling a vacancy is a function of the number of workers queueing per vacancy, denoted by $m(q_x)$, which is assumed to be strictly increasing and strictly concave.[5] Since there are q_x workers queueing per vacancy with this wage the workers' job-finding rate for these workers is $m(q_x)/q_x$. Firms can attract workers to their vacancies as long as these workers get in expectation their equilibrium utility, meaning that q_x adjusts depending on ω_x to satisfy: $\omega_x m(q_x)/q_x = w(x)$. Note the difference between the wage ω_x which is paid when a worker is actually hired, and the expected wage $w(x)$ of a queueing worker who does not yet know whether he will be hired or not. In equilibrium the firm takes the latter as given because this is the utility that workers can ensure themselves by searching for a job at other firms, while the former is the firm's choice variable with which it can affect how many workers will queue for its jobs. Therefore, a firm maximizes instead of (1) the new problem

$$\max_{r_x,\omega_x,v_x} \int [F(x,y,l_x,r_x) - l_x\omega_x - v_x c]\, dx$$
$$\text{s.t. } l_x = v_x m(q_x); \quad \text{and} \quad \omega_x m(q_x)/q_x = w(x).$$

The first line simply takes into account that the firm has to pay the vacancy-creation cost, and that the number of hires depends on the amount of hiring per vacancy which is in turn related to the wage

[5] Careful elaborations how this queueing problem in a finite economy translates into matching probabilities as the population is exanded is given e.g. in Peters (1991) and Burdett, Shi and Wright (2001). It is based on the idea that workers approach vacancies unevenly due to coordination problems, which leads to excess applicants at some vacancies and to few vacancies at others.

that it offers. Substitution of the constraints allows us to rewrite this problem equivalently as

$$\max_{s_{(.)},r_{(.)}} \int [G(x,y,s_x,r_x) - w(x)s_x]dx \tag{3}$$

where $G(x,y,s_x,r_x) = \max_{v_x} [F(x,y,v_x m(s_x/v_x),r_x) - v_x c]$. Problem (3) has exactly the same structure as the maximization problem (1) in the frictionless case, and we can apply all insights as long as we use production function $G(x,y,s_x,r_x)$. This production function takes as input the quality types of workers and firms, but now the amount of labor is given by the "searchers" s_x that queue in order to get a job at this firm. The particular attraction is that we can now additionally consider questions regarding the unemployment accross different workers. Note that the frictional production function G coincides with the frictionless production function F if the vacancy creation costs are zero.[6]

Assortative Matching: Let $\mathcal{R}(x,y|\Theta)$ be the marginal distribution of $\mathcal{R}$ over the firm and worker types at any level of intensity. It denotes the amount of resources devoted by firms with type below y to workers of skill below x. Matching is assortative if there exists a monotone function $\mu(x)$ and such that the support of $\mathcal{R}(x,y|\Theta)$ only includes points $(x,\mu(x))$. We call μ the assignment function. Matching is (strictly) positive assortative if the assignment function is (strictly) increasing, and it is (strictly) negative assortative if the assignment function is (strictly) decreasing. In the following we will restrict attention to assortative matching that can be supported by a differentiable assignment function.

3 Assortative Matching

In this section we derive necessary and sufficient conditions for assortative matching. Assume that the equilibrium is assortative, supported by some differentiable assignment function $\mu(x)$. Consider any $(x,\mu(x),\theta)$ in the support of the equilibrium allocation with $\theta > 0$, i.e., with positive amount of hiring. By (2) this means that (x,θ) are maximizers of the following problem for a firm of type $y = \mu(x)$:

$$\max_{x,\theta} f(x,y,\theta) - \theta w(x).$$

Assortative matching means that each firm only hires one type, and this problem can be understood as the problem of a firm that could choose any other worker type at any other quantity. The first order

[6] For this result we use the standard assumption that workers always find a job if there are very few workers per vacancy, i.e., $\lim_{\theta\to 0} m(q)/q = 1$. In this case every worker who searches for this firm also gets hired, leading to the setup in the previous section. (This would also arise if $m(\theta) = \theta$.)

conditions for optimality are

$$\begin{aligned} f_\theta(x, \mu(x), \theta(x)) - w(x) &= 0 \quad (4) \\ f_x(x, \mu(x), \theta(x)) - \theta(x) w'(x) &= 0, \quad (5) \end{aligned}$$

where $\mu(x)$ and $l(x)$ are the equilibrium values. The second order condition of this problem requires the Hessian to be positive definite:

$$Hess = \begin{pmatrix} f_{\theta\theta} & f_{x\theta} - w'(x) \\ f_{x\theta} - w'(x) & f_{xx} - \theta w''(x) \end{pmatrix}.$$

This requires $f_{\theta\theta}$ to be negative and the determinant $|Hess|$ to be positive, or

$$f_{\theta\theta}[f_{xx} - \theta w''(x)] - (f_{x\theta} - w'(x))^2 \geq 0. \quad (6)$$

We can differentiate (4) and (5) with respect to the worker type to get

$$\begin{aligned} f_{x\theta} - w'(x) &= -\mu'(x) f_{y\theta} - \theta'(x) f_{ll} \quad (7) \\ f_{xx} - \theta(x) w''(x) &= -\mu'(x) f_{xy} - \theta'(x) \left[f_{x\theta} - w'(x) \right]. \quad (8) \end{aligned}$$

In the following three lines we successively substitute (7), (9) and then (5) into optimality condition (6):

$$\begin{aligned} -\mu'(x) f_{\theta\theta} f_{xy} - \left[\theta'(x) f_{\theta\theta} + f_{x\theta} - w'(x) \right] \left[f_{x\theta} - w'(x) \right] &\geq 0 \\ -\mu'(x) f_{\theta\theta} f_{xy} + \mu'(x) f_{y\theta} \left[f_{x\theta} - w'(x) \right] &\geq 0 \\ -\mu'(x) [f_{\theta\theta} f_{xy} - f_{y\theta} f_{x\theta} + f_{y\theta} f_x / \theta] &\geq 0 \end{aligned}$$

For strictly positive assortative matching ($\mu'(x) > 0$) it has to hold the the term in square brackets is negative, for strictly negative assortative matching the term in square brackets need to be positive. Focussing on positive assortative matching, and using the relationsship in (5), we obtain the condition:

$$f_{\theta\theta} f_{xy} - f_{y\theta} f_{x\theta} + f_{y\theta} f_x / \theta \leq 0. \quad (9)$$

It turns out that this condition can more conveniently be summarized in terms of the original

function $F(x, y, r, s)$, for which we know that $F(x, y, \theta, 1) = f(x, y, \theta)$. The following relationsships will also prove useful. Homogeneity of F implies that $-F_{34} = \theta F_{33}$. Since F is constant returns, so is F_1.[7] A standard implication of constant returns it then $F_1(x, y, \theta, 1) = \theta F_{13} + F_{14}$. We can now rewrite (9) in terms of $F(x, y, \theta, 1)$ and rearrange to obtain the following cross-margin-complementarity condition:

$$\begin{aligned} & F_{33}F_{12} - F_{23}\left[F_{13} - F_1/\theta\right] \leq 0 \\ \Leftrightarrow \quad & F_{33}F_{12} + F_{23}F_{14}/\theta \leq 0 \\ \Leftrightarrow \quad & F_{12}F_{34} \geq F_{23}F_{14} \end{aligned}$$

So the condition depends on the cross-partials in each dimension, relative to the cross-partials across the two dimensions. Only if the within-complementarities in extensive and intensive deminsion on the left hand side exceed the between-complementarities from extensive to intensive margin on the right hand side does positive assortative matching arise. We sum this finding up in the following proposition:

Proposition 1 *A necessary cross-margin-complementarity condition for positive assortative matching is*

$$F_{12}F_{34} \geq F_{23}F_{14} \tag{10}$$

along the equilibrium path. The opposite inequality is necessary for negative assortative matching. If this condition holds at all (x, y, l, r) it is also sufficient to ensure existence of an asssorted equilibrium.

Proof. The proof of necessity is provided above. Since the condition ensures local concavity around the equilibrium path, it is sufficient to ensure that local deviations are not profitable. The proof for global sufficiency is yet to be done. ■

This proposition allows for arbitrary interactions between the quality and quantity dimensions of the problem. That this condition also reflects the efficiency considerations in this market follows form the standard welfare properties of competitive markets. In special cases the condition can be further simplified, revealing further a certain symmetry in the quantity and quality dimensions.

Corollary 1 *In the case of multiplicative separability of the quality and quantity dimensions, i.e., $F(x, y, l, r) = A(x, y)B(l, r)$, then condition (10) for positive assortative matching can be written as $[AA_{12}/(A_1A_2)][BB_{12}/(B_1B_2)] \geq 1$.*

[7] It holds that $F(x, y, r, s) = sF(x, y, r/s, 1)$, so differentiation implies that $F_1(x, y, r, s) = sF_1(x, y, r/s, 1)$.

The sorting conditions of the previous section establishes a positive relation between the productivity of the firm and the skills of the workers that it hires. It does not directly establish how many workers a given firm hires, and therefore it does not directly determine even under assortative matching who matches with whom. This is different to models where workers and firms get matched into pairs, as positive assortative matching in that case directly implies who gets matched to whom. Due to the intensive margin in this model, any given firm type might absorb more or less workers, and the following differential equations describe the equilibrium assignment.

Proposition 2 *If (10) holds, then the equilibrium assignment and factor intensity are determined by the system of differential equations:*

$$\mu'(x) = \frac{h_w(x)}{\theta(x)h_f(x)} \tag{11}$$

$$\theta'(x) = \frac{1}{f_{\theta\theta}}\left[\frac{1}{\theta}f_x - \frac{h_w}{\theta h_f}f_{y\theta} - f_{x\theta}\right] \tag{12}$$

Proof. The market clearing condition implies:

$$H_w(\overline{x}) - H_w(x) = \int_{\mu(x)}^{\overline{y}} \theta(\tilde{x})h_f(\tilde{x})dx$$

Differentiating with respect to x gives a differential equation (11), with initial condition $\mu(\overline{x}) = \overline{y}$ in the case of PAM.

From the first-order condition in equation (4) we know that $f_\theta(x, \mu(x), \theta(x)) = w(x)$ and from (5) we know that $w' = f_x/\theta$. Then from equation (7), after substituting for w' and μ' we obtain:

$$\frac{f_x}{\theta} = f_{x\theta} + \frac{h_w}{\theta h_f}f_{ys} + \theta' f_{\theta\theta},$$

which is equivalent to (12). The initial condition for this differential equation obtains from running down the allocation from the top to the bottom and where the boundary condition holds either when the lowest type is attained or when the number of searchers goes to zero. An equilibrium allocation simultaneously solves the differential equation for μ' and θ' with the respective boundary conditions. ■

Notice that the an increasing firm size means that the interaction between quality and quantity is at least sometimes positive. Otherwise it is better to concentrate one's resources on fewer and fewer workers as the workers become more productive, which can be observed in some specialized industries where highly specialized groups of workers get equipped with much captial.

Corollary 2 *For a general technology, a necessary condition for the firm size to be increasing is that at least one of f_{xs} or f_{ys} be positive.*

Proof. It is immediate from inspection of the expression of $s'(x)$ that if both f_{xs} and f_{ys} are negative, s' is negative as well since $f_{ss} < 0$ and $f_x > 0$. ■

3.1 An Illustration

To illustrate the model further, it might be instructive to consider a specific example of the setting above. Assume that the type are uniformly distributed on the unit interval, and that production is multiplicatively separable between the quality and quantity dimension, with CES form on the latter.

$$\text{Example: } F(x, y, l, r) = A(x, y)(\alpha l^{\gamma} + (1 - \alpha)r^{\gamma})^{1/\gamma}, \tag{13}$$

with parameters $\gamma \in (-\infty, 1)$ to ensure our concavity properties on the production function, and $\alpha \in (0, 1)$. In this case our condition for positive assortative matching reduces by Corollary 1 to

$$\frac{A_{12}A}{A_1 A_2} \geq \frac{1}{1-\gamma}, \tag{14}$$

the strength of which depends on γ. This illustrates the importance of the subsitutability of production factors on the intensive margin. When γ is close to 1 so that production factors are very substitutable, this condition is very difficult to fulfill. The simplicity of this expression also shows the limitations of this example, since it limits the interaction between the quality and quantity dimension of the problem in contrast to the full condition (10) that is relevant whenever quality and quantity dimensions interact in more intricate ways. If the inequality above is fulfilled, we can apply (**??**) and (12) and obtain after some re-organization the assignment equations

$$\theta'(x) = \frac{(1-\alpha)A_2(x, \mu(x)) - \alpha A_1(x, \mu(x))\theta^{1-\gamma}}{A(x, \mu(x))[1 + \theta^{\gamma}][1 - \gamma]} \quad ; \quad \mu'(x) = \frac{1}{\theta(x)}. \tag{15}$$

Symmetry is the first observation that follows from inspection of these equations. Note that under $\alpha = 1/2$ and $A(x, y) = A(y, x)$ the problem is exactly symmetric in firms and workers. Inspecting (15) reveals that $\theta(x) = 1$ and $\mu(x) = x$ solves this system, and in fact is also fulfills the boundary conditions that neither workers nor firms remain unmatched while their neighbors still earn positive returns (since workers and firms exactly are matching one-on-one).

Asymmetries arise when for example the resources of firms become more important than those of workers. This is the case when $\alpha < 1/2$. It is easy to see that for the benchmark assignment $\mu(x) = x$ is not longer sustainable; rather there is a shift of resources towards the more productive firms ($\theta' > 0$). The opposite arises under $\alpha > 1/2$ where workers are more important and better worker types are endowed with more of the firm's resources, leading to reductions in firms size to equip every workers with enough endowment.

The limiting case of one-on-one matching can be envisioned in this setting as follows. Consider a process $\tilde{F}(x, y, l, r)$ but every worker needs exactly one unit of resource and any resources that are not employed are not matched, so that true production is

$$\begin{aligned} F(x, y, l, r) &= \tilde{F}(x, y, \min\{l, r\}, \min\{r, l\}) \\ &= \min\{l, r\}\tilde{F}(x, y, 1, 1). \end{aligned}$$

This "one-on-one" matching case does not fulfill our differentiability assumptions, but we know that it is the limiting case of our production function in (13) when $A(x, y) = \tilde{F}(x, y, 1, 1)$ and $\gamma \to -\infty$, because in this case the CES function converges to the Liontieff function. Inspection of (14) and (15) for this limit case reveals

$$\text{PAM if } \tilde{F}_{12} > 0, \text{ and in this case } \mu(x) = x,$$

where the latter follows from $\theta'(x) = 0$ which implies $\theta(x) = 1$. This captures exactly the conditions of the one-on-one matching literature going back to Becker (1973).

The importance of the more general conditions in (10), (11) and (12) is exactly to highlight the relevant sorting and assignment conditions when substitution between firm and worker inputs is not impossible.

Frictions and Unemployment

Consider a firm that already decided on the worker type x and the number of hires l. Such a firm wants to obtain this number of hires at the lowest cost by posting v vacancies. Its objective function is

$$\begin{aligned} &\min_{v,q} cv + qvw(x) \\ &\text{s.t. } r = vm(q). \end{aligned}$$

Considering the Lagrangian of this problem with multiplier μ gives the following optimality conditions wrt v and R :

$$\begin{aligned} c + \mu[m(q) - qm'(q)] &= 0 \\ w(x) + \mu m'(q) &= 0 \end{aligned}$$

Eliminating μ and writing the elasticity of the matching probability $\eta(q) = qm'(q)/m(q)$ we obtain

$$w(x)q = \frac{\eta(q)}{1-\eta(q)}c \tag{16}$$

This has the following noteworthy implications: First, the optimal q depends only on the utility that this worker obtains in equilibrium, and apart from this it is independent of the type of the worker or the firm or from the number of workers that the firm wants to hire. Second, implicitly differentiating (16) reveals that workers that obtain a higher utility $w(x)$ have a lower market tightness and therefore find it easier to obtain a job. This arises because it is costly to have these workers queuing for a job but not getting hired. Since we know that better workers obtain a higher utility, we can imediately conclude that matching rates for better workers are higher.

A particularly transpart result arises when we consider matching functions with constant elasticity, in which case the utility that the workers obain in equilibrium is inverse proportional to the market tightness that they face. Using $g(x, y, \theta) = \max_v G(x, y, vm(s/v), 1) - cv$ in the frictionless analysis exactly delivers $w(x)$, and we can infer the resulting tightness and matching rate through condition (16).

Characterizing the equilibrium with Unemployment

Recall the the amount of workers $q(x)$ that queue for any particular vacancy is determined by (16) and is decreasing in skill. The amount of labor that is actually hired, $l(x)$, relates to the actual number of searchers and their queue per vacancy as:

$$l(x) = s(x)\frac{m(q(x))}{q(x)}$$

The change of labor amounts across skills can then be computed as:

$$l'(x) = s'\frac{m}{q} + s\frac{m'q - m}{q^2}q'$$

If we are after the question whether more productive firms not only hire more workers, but also more workers, the interest is in whether this function is increasing. Note that s' is determined by (11) under appropriate change of variables (θ and f replaced by s and g). Even if the number of workers that queue for at better firms is not increasing, the number of hires might still be increasing because the second term is strictly positive. So in a frictional labor market it is more likely that more productive firms are larger, since they expand more resources on recruitment of their workers than firms whose workerforce is of lower quality.

Additional Examples

Our main example was the assignment of workers to firms, which can be viewed as the assignment of resources by firms to particular worker types. We expanded this leading example to the case of unemployment. The following gives other interpretation of our setup.

Example 1: Optimal transport. Assume it costs $-r * c(x, y)$ to move a r units of waist from production site x into destination storage y, and if one attempts to move more units r into any given amount l of storage then there is some probability of damage $d(r/l)$ that each unit that is stored gets destroyed. This leads to function $F(x, y, l, r) = -rc(x, y) - \alpha r d(r/l)$, where α represents the lost revenue because of destruction.

Example 3: Matching of men and women. Assume that there are different locations where men and women can meet. These are located in different distinct locations. If r men of type x search for s women of type y, then $M(r, s) \leq \min\{r, s\}$ matches are created, where M is a standard constant

returns to scale matching function. Each match is worth $A(x, y)$ for the pair. Then output is given by $F(x, y, r, s) = M(r, s)A(x, y)$. This is essentially the setup in Eeckhout and Kircher (2010). Under the standard assumption that the search literature makes on M, they find a necessarily and sufficient condition for positive assortative matching is that A is root-supermodular. In a general production environment it is not possible to reduce the analysis to conditions on the quality dimension only, and a more general look at the cross-margin-complementarity is required.

Example 3: Matching of two teams of workers. Rather than thinking about complementarities between the worker and the firm, one can equally well consider complementarities between teams of workers. One way to think about such complementarities may be the following. There is only one quality dimension Q and some distribution $H^Q(q)$ of worker skills. But production needs two teams. So if one team has quality $x \in Q$ and l team members while the other team has quality $y \in Q$ and r team members, then output is $F(x, y, l, r)$. Since the teams are drawn from the same base population, the natural requirement is symmetry of the form $F(x, y, l, r) = F(y, x, r, l)$, at least in settings where the cooperation is symmetric. Then $X = Y = Q$, and each side gets half the number of available workers $H^f(x) = H^w(x) = H^Q(x)/2$. This provides a stylized way to analyze patterns of teams that work together.

4 Concluding Remarks

We have proposed a matching model that incorporates factor intensity and unemployment. We derive a simple condition for assortative matching and characterize the equilibrium firm size, unemployment level and unemployment by skills.

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