

On the costs of disability insurance

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Abstract

The costs of social insurance come from two sources: first, the social insurance changes the behavior of individuals, and second, taxes that are levied to finance these programs create further losses. We extend the standard Ramsey model by a precautionary saving motive and examine the disability insurance program in the United States. A baseline calibration implies that the program lowers per capita consumption by 2.5%: 1/3 of this burden is caused by higher taxes and 2/3 comes from the change in economic behavior. However, precautionary savings are inefficient at insuring people against permanent disability: therefore, social insurance increases welfare. But, a perfect *private* insurance program would provide a 3.5-7% higher per capita consumption than the current disability insurance program.

JEL Codes: E13, E61, H55

Keywords: Incomplete markets, Social insurance, The Ramsey model

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1 Introduction

The disability insurance program is one of the largest social insurance programs in the United States: in 2007, the program covered almost 9 million individuals, its costs were \$99 billion and it constituted 17% of social security benefits. The size of the program is about three times larger than that of the unemployment insurance program and its size far surpasses that of any other similar program (SSA, 2009). Given these facts, it is important to study the cost of the disability insurance program for the U.S. economy.

It is well known that the costs of any social insurance program come from two sources: first, the social insurance program has substantial undesirable effects on incentives and, therefore, on economic performance. Second, these programs are often financed by a proportional tax rate, which causes a further distortion for the economy.¹ However, there no estimate – as far as we know – on the importance of these sources in generating the costs. Moreover, one could ask: What would happen if this program was shut down? Would aggregate output and consumption increase or decrease and would these effects result from the change in economic behavior or the change in the tax rate? Further, could households insure themselves against the risk of disability? Or, is the program extremely important for disabled households? These are the questions that this paper aims to discuss.

The model introduced in this paper gives a new extension for the standard textbook Ramsey model:² we extend the Ramsey model by including a precautionary saving motive for households. That is, there is a risk for the permanent loss of job, which captures the uncertainty associated with disability. Moreover, in the model the government provides the social insurance which affects households' economic behavior by removing the self-insurance (precautionary saving) motive. This program is financed by a proportional tax rate, which reduces the incentives for households to accumulate capital and supply labor. Hence, this model provides a tool to measure the cost of the disability insurance program within the framework of the standard Ramsey model.

The types of models which enable incomplete markets – i.e. where there is a lack of insurance

¹For a more detailed discussion, see Feldstein and Liebman (2002) and Feldstein (2005).

²We refer here, for example, to Barro and Sala-i-Martin (2004, chap. 2).

against idiosyncratic uncertainty – are standard in a general equilibrium framework. Aiyagari (1994) and Huggett (1997) were the first to introduce idiosyncratic labor endowment risk with incomplete markets into dynamic stochastic general equilibrium (DSGE) models. The result from these models is that households save more than they would if there was no uncertainty and this additional saving can be seen as a measure for aggregate precautionary savings (Huggett and Ospina, 2001). However, an important point is made by Marcet, Obiols-Homs, and Weil (2007), who claim: a higher capital stock does not lead to higher output when labor supply is elastic, since a higher capital stock implies a wealth effect which reduces the supply of labor.³ The model presented in this paper captures these aspects within the standard Ramsey model. The origin of the model is taken from Toche (2005) where he gives a tractable model of precautionary saving.⁴ We expand this partial equilibrium model to a general equilibrium. A distinction between this model and the DSGE models cited above is that, whereas we consider the effects of uncertainty associated with rare events, generally the DSGE models are focused more on moderate fluctuations of uncertain labor income. Therefore, this model is a new one and fills a gap in the existing literature.

The second branch of literature associated with the paper is the economics of disability and the analysis of disability insurance targeted for these people.⁵ Furthermore, works by, for example, Diamond and Sheshinski (1995), Autor and Duggan (2003) and Chandra and Samwick (2009), have modeled the economic effects of the risk of disability within a partial equilibrium setting. Moreover, many studies have taken an information approach to this subject and have analyzed optimal planning problems in which some key information is only privately observed. There, the focus is on implementing an optimal mechanism that minimizes the distortions caused by these insurance programs. See, for example, Golosov and Tsyvinski (2006) and references therein. This paper contributes to this literature by using a growth model and studying the effects of disability within a general equilibrium framework. Moreover, here we try to capture the magnitude of

³It is assumed that a household can only adjust its labor supply if it is given the (stochastic) opportunity to work. Pijoan-Mas (2006) uses a model where idiosyncratic shocks are associated with individuals productivity. Then, precautionary saving can be replaced by longer working hours. However, we assume here, as it is traditionally assumed, that stochasticity is associated with the opportunity to work.

⁴See also the recent papers by Carroll and Toche (2009) and Carroll and Jeanne (2009).

⁵A review of this literature is given by Bound and Burkhauser (1999).

the welfare question related to the disability insurance program rather than provide an optimal mechanism for this program. Note that we do not consider any type of strategic behavior when the paper gives a lower limit for the costs of the disability program.⁶

Our model implies that closing the current disability insurance program would increase per capita consumption by 2.5%. One-third of this burden is caused by higher tax rates and 2/3 comes from the change in economic behavior, i.e. from the removed precautionary saving motive. It is surprising that the labor supply decision does not depend on the level of disability insurance, even if taxes are increased when the level of insurance is increased. Rather, the supply of labor is almost constant. However, self-insurance works poorly against permanent shocks, and therefore, the social insurance program increases welfare by providing a higher level of consumption for disabled households. But the real problem is the incompleteness of the private insurance markets: if the perfect insurance against permanent disability is provided by private insurance companies rather than the current social insurance program which is financed by proportional tax rate, per capita consumption would increase by 3.5-7% depending on the Frisch elasticity of labor supply. The result implies that optimizing the tax financed social insurance systems is not the best way to improve welfare. Instead, completing the markets by removing impediments to the private provision of insurance would generate a much higher increase in welfare. Another interpretation of the result is that the cost generated by problems associated with imperfect information – which prevents market-based solutions – is indeed very large.

The rest of the paper is organized as follows: Section 2 introduces the model and Section 3 gives the aggregate variables, derives the steady state and analyzes its stability; Section 4 discusses the calibration of the model and reports the numerical results of simulations; finally, Section 5 concludes the paper.

⁶Strategic behavior created by the disability insurance program causes additional costs to the ones considered here. See, for example, Rust and Phelan (1997) for a more detailed discussion.

2 The model

The model introduced in this paper is a standard neoclassical growth model or the Ramsey model in continuous time with an endogenous labor supply,⁷ but there are two significant exception compared to the baseline model. First, members of households face the constant probability of losing their jobs permanently throughout their lives. Then, the model captures the uncertainty associated with rare and permanent income losses of households, i.e. the model captures the uncertainty associated with a disability. These types of events could, for instance, be a severe injury or compulsory retirement. Second, the government only provides partial insurance against this uncertainty, and this disability insurance is financed by a proportional income tax. Since insurance against uncertainty is only partial, households have a precautionary saving motive and households can self-insure by holding a single asset – physical capital.

2.1 Demographics

There are two types of households: workers and disabled workers, and the latter do not work. At every moment working households get nL_0e^{nt} new members to their households, where L_0 is normalized to 1. The size of the population at time t is $P_t = \int_{-\infty}^t ne^{nv}dv = e^{nt}$, since no one dies. However, every instant there is the constant instantaneous probability μ of permanent disability for each working member of household. The transition from the employed state to the disability state follows a Poisson process with arrival rate μ , which implies that the expected time in the employed state is $\frac{1}{\mu}$. Since μ is also a fraction of households which face permanent disability at each instant, the size of working households is given by $L_t = \int_{-\infty}^t ne^{nv}e^{-\mu(t-v)}dv = \theta e^{nt}$, where $\theta = \frac{n}{n+\mu}$. A fraction θP_t of the entire population are workers, which implies that the size of disabled households is $Z_t = P_t - L_t = (1 - \theta)e^{nt} = \lambda e^{nt}$, where $\lambda = \frac{\mu}{n+\mu}$.

⁷The classical references for the Ramsey model are Ramsey (1928), Cass (1965) and Koopmans (1965).

2.2 Production and factor prices

The Cobb-Douglas production function is assumed and there exists a perfect competition among firms. The economy's production per efficient capita Y_t is given by

$$Y_t = \Theta K_t^\alpha l_t^{1-\alpha}, \quad (1)$$

where $\Theta = \theta^{1-\alpha}$. It is a scaling factor, since production is in the terms of per efficient capita, but only a fraction θ of the entire population is working.⁸ Moreover, K_t is the capital stock, α is the capital share and l_t is labor.

Profit maximization of the representative firm gives following factor prices:

$$r_t = \alpha \Theta K_t^{\alpha-1} l_t^{1-\alpha} - \delta \quad (2)$$

$$w_t = (1 - \alpha) \theta^{-\alpha} K_t^\alpha l_t^{-\alpha}, \quad (3)$$

where δ is the depreciation rate of capital stock. If $\mu = 0$, then the production side of the economy falls to the baseline neoclassical growth model. However, if $\mu > 0$, it decreases the number of workers, which lowers the interest rate and increases the wage rate when everything else is kept constant.

2.3 The behavior of the government

The government offers social insurance for disabled households. Every disabled household receives a social insurance benefit b_t , which is a lump sum transfer, for every period. To finance this program, the government must lay an income tax for capital income ($r_t A_t$) and labor income ($\theta w_t l_t$), which are taxed at the rate τ_t . Moreover, the government is running a balanced budget for every period, or a pay-as-you-go social insurance program, which implies the following budget

⁸Production per efficient capita means that we have divided the actual production by the term $P_t T_t$, where T_t describes technological progress, for which the growth rate is g . Moreover, our Cobb-Douglas production function is in a Harrold-neutral form where effective labor is defined by $T_t L_t l_t$.

constraint:

$$\lambda b_t = \tau_t r_t A_t + \tau_t \theta w_t l_t. \quad (4)$$

The left-hand side of the equation shows the aggregate transfer for disabled households (in per efficient capita form). It is assumed that the government sets a rate η , which is the replacement ratio of after-tax labor income when $0 \leq \eta \leq 1$. Therefore, the level of social insurance is defined as $b_t = \eta(1 - \tau_t)w_t l_t$. The right-hand side of the equation (4) shows the net income of the government, which is composed of the taxed labor and capital incomes of households. Note that the aggregate assets are given by $A_t = A_t^e + A_t^d$, where $A_t^e \equiv \theta a_t^e$ and $A_t^d \equiv \lambda a_t^d$, where superscript indexes the individual's state: e stands for the employed state and d stands for the disabled state. Thus, the level of asset holdings of the representative (or average) agent in both states is weighted according to the population share associated with the state in question.

Rewrite the government budget constraint (4), which now defines the tax rate τ_t by

$$\tau_t = \frac{\lambda \eta w_t l_t}{r_t A_t + (\eta \lambda + \theta) w_t l_t}. \quad (5)$$

Hence, the income tax rate is an endogenous variable, and only the replacement ratio η is decided by the government. That is, the government decides how generous insurance it wants to provide for the disabled households.

2.4 The problem of households

Assume that households start their living within this economy at time $t = 0$. The representative households maximize their own and their prospective descendants' expected utility by making consumption and labor supply decisions. The problem of the representative household is given

by

$$\max_{c_t, l_t} E_0 U = E_0 \int_0^\infty e^{(n-\rho)t} U(c_t, l_t) dt \quad (6)$$

$$\text{s.t. } \dot{a}_t = \begin{cases} [(1-\tau_t)r_t - n - g]a_t - c_t + (1-\tau_t)w_t l_t, & \text{if } \epsilon = e \\ [(1-\tau_t)r_t - n - g]a_t - c_t + b_t, & \text{if } \epsilon = d \end{cases} \quad (7)$$

$$a_t \geq 0 \quad \forall t \quad (8)$$

$$\text{and } \lim_{t \rightarrow \infty} \left[a_t e^{-\int_0^t (1-\tau_s)r_s - g - nds} \right] = 0. \quad (9)$$

E_0 is the conditional expectation operator and household get utility from consumption per adult person c_t and the supply of labor l_t produce disutility for household. However, in the disabled state $l_t = 0 \quad \forall t$ by the definition of disability. n is the growth rate of population in the economy, ρ is the rate of time preference, g is the growth rate of productivity and variables are scaled per efficient capita. Household faces a budget constraint where r_t is the interest rate and w_t is the wage rate. When household is employed ($\epsilon = e$), it receives capital income from assets holdings a_t and labor income from working. Both incomes are taxed with a rate τ_t by the government. When household is disabled ($\epsilon = d$), it receives a social insurance benefit b_t which are tax free but its capital income is still taxed. Moreover, household face a liquidity constraint $a_t \geq 0$ and the last constraint ensures that budget constraint must finally hold with equality.

We can solve the problem by using the familiar tools of optimal control theory, for which Toche (2005) provided the following key insight: The only source of uncertainty in this problem is the timing of transition from the employment state to the disability state. Thus, uncertainty is restricted to a single transition, and after the transition, the problem is deterministic, i.e. disabled households do not face any kind of uncertainty. The key assumption is the persistence of this transition. Therefore, it is possible to solve the full problem by using “backward induction”: first, solve the deterministic problem of disabled households; second, use that solution when solving the problem of employed household.⁹

We assume that new members within the economy were born into working households and that they are loved. Thus, the newborn members enter into the economy with an amount of assets

⁹For a more detailed discussion, see Toche (2005), Carroll and Kimball (2007) and Carroll and Toche (2009).

which equal to the asset holdings of those currently working in the economy. The result of these assumptions is that the liquidity constraint is not binding for employed households, but it will be binding at some point of time for disabled households since, in the disabled state, households start to dis-save. Without the liquidity constraint, the assets of disabled households would approach minus infinity. To prevent that, we assume that disabled households cannot go into debt.

According to backward induction, we must first solve the problem of the disabled household, which is deterministic. But the household faces a liquidity constraint. We can provide a closed-form solution for this problem: first, we ignore the state variable inequality constraint, and next, take into account the effects of liquidity constraint.¹⁰ Hence, we start by ignoring the liquidity constraint and then solve the problem of the representative household given by equations (6), (7) and (9).

2.4.1 Solving the problem without the liquidity constraint

Assume that households have the following utility function:

$$U(c_t, l_t) = \log c_t - \gamma \frac{l_t^{1+\phi}}{1+\phi}, \quad (10)$$

where the Frisch elasticity is equal to $\frac{1}{\phi}$.¹¹ Appendix A provides a detailed derivation for the Euler equations. The Euler equations for consumption are given by:

$$\frac{\dot{c}_t^e}{c_t^e} = (1 - \tau_t)r_t - \rho - g + \mu \left(\frac{c_t^e}{c_t^d} - 1 \right) \quad (11)$$

$$\frac{\dot{c}_t^d}{c_t^d} = (1 - \tau_t)r_t - \rho - g. \quad (12)$$

where $\dot{x}_t = \frac{dx_t}{dt}$. In the disabled state, households' Euler equations are in the standard form, but in the employed state there is a precautionary saving motive, which is given by the last term in

¹⁰This method is given by Kamien and Schwartz (1981, chap. 17) and also utilized by Park (2006).

¹¹As shown by King, Plosser, and Rebelo (1988), the form of utility function given by equation (10) is required for the balanced growth path. Moreover, log-utility for consumption implies that utility is separable with respect to consumption and labor. We assume log-utility for consumption for two reasons: first, we want to reduce the amount of calibrated parameters; second, we try to avoid overestimating the precautionary saving motive, which could be done by assuming a lower intertemporal elasticity of substitution for consumption.

equation (11). Note that the precautionary saving motive disappears when $\mu = 0$ or $c_t^e = c_t^d$. That is, if the probability of a permanent transition to the disability state is zero, or if the levels of consumption are the same in both states when there is a perfect insurance against uncertainty, the precautionary saving motive disappears.¹² The higher the μ or the greater the difference in the levels of consumption between the states (i.e. the lower the level of insurance), the higher the precautionary saving motive.

An employed household must also decide upon its labor supply. The first order condition for labor supply is given by

$$\gamma l_t^\phi = \frac{(1 - \tau_t)w_t}{c_t^e}. \quad (13)$$

Thus, the marginal disutility from working must be equal to the proportion of after-tax wage to consumption, which is the standard result. Now we have derived Euler equations for the problem of households, but we must still define the value of c_t^d . We can do this by solving the problem of disabled households.

2.4.2 The problem of disabled households

Consider a representative household which was disabled at time v . The budget constraint can be written for this household as follows:

$$\begin{aligned} \dot{a}_{t,v}^d &= [(1 - \tau_t)r_t - n - g] a_{t,v}^d - c_{t,v}^d + b_t \\ a_{t=v,v}^d &= a_v^e \\ a_{t,v}^d &\geq 0 \quad \forall t. \end{aligned} \quad (14)$$

Now the modification is that the assets and consumption are function of time v . At moment v the household was disabled and its assets were at the level a_v^e in the employed state at that moment. Moreover, now we must also consider the effects of the liquidity constraint. First, we solve the households' problem without the effects of liquidity constraint. Second, we show that

¹²It is obvious that $c_t^e \geq c_t^d$.

the liquidity constraint will be binding in the future. Third, we show the optimal consumption plan for disabled household.

The first part has already been done previously. The Euler equation for a household that is disabled, but not liquidity constrained, is given by equation (12) and straightforward integration yields

$$c_{t,v}^d = c_v^d e^{[(1-\bar{\tau}_{t,v})\bar{r}_{t,v}-\rho-g](t-v)}, \quad (15)$$

where $(1-\bar{\tau}_{t,v})\bar{r}_{t,v} = \frac{1}{t-v} \int_v^t (1-\tau_s)r_s ds$. Euler equation (12) gives the optimal time path for consumption when the liquidity constraint is not binding and equation (15) gives the level of consumption along that path. However, we still must solve c_v^d in order to define the optimal level of consumption when the liquidity constraint is not binding. Plug equation (15) into household intertemporal budget constraint (14) which then gives

$$c_v^d = (\rho - n) \left[a_v^e + \tilde{b}_v \right], \quad (16)$$

where $\tilde{b}_v = \int_v^\infty b_t e^{-[(1-\bar{\tau}_{t,v})\bar{r}_{t,v}-n-g](t-v)} dt$, i.e. $\tilde{b}_v$ is the present value for social insurance income.

Equation (16) now defines the level of consumption at time v , i.e. at the time when the household was disabled. Moreover, from the viewpoint of *still employed* household, we can replace v with t since the point v is not realized for it, i.e. we get c_t^d . Since the risk for disability is continuous and does not depend on time, equation (16) shows the level of consumption for employed household if it would fall into disability at the present moment. Thus, equation (16) defines – with c_t^e – the magnitude of the precautionary saving motive in equation (11).¹³

The second task was to show that the liquidity constraint will be binding in the future, which can be done by following the evaluation for the assets. The evaluation of assets over time can be

¹³From the viewpoint of the employed household, we can write equation (16) as

$$c_t^d = (\rho - n) \left[a_t^e + \tilde{b}_t \right], \quad \text{where} \\ \tilde{b}_t = \int_t^\infty b_t e^{-[(1-\bar{\tau}_t)\bar{r}_t-n-g]t} dt.$$

derived from equation (14) and details of this derivation are given in Appendix B. This yields

$$a_{t,v}^d = \left[a_v^e + \tilde{b}_v \right] e^{[(1-\bar{\tau}_{t,v})\bar{r}_{t,v}-\rho-g](t-v)} - \tilde{b}_v. \quad (17)$$

It is assumed that most people work and that only a small number of people are disabled. Further, working households are richer than disabled households, and therefore, more influential when the aggregate economy is considered. Hence, the interest rate and the capital stock are mainly set by the decisions of the workers. The Euler equation of workers (11) includes the term $\mu \left(\frac{c_t^e}{c_t^d} - 1 \right)$, which gives the precautionary saving motive and lowers the level of r_t . However, disabled households face the same interest rate r_t in their Euler equation, which implies that $(1 - \tau_t)r_t < \rho + g$ and this causes the dis-saving of disabled households (or a descending path of consumption). Moreover, the cause of continuous dis-saving is that the liquidity constraint will be binding. This can be shown by using equation (17): when $t = v$, equation (17) implies that $a_v^d = a_v^e$. When $t > v$ then $a_{t,v}^d < a_v^e$, since the term $e^{[(1-\bar{\tau}_{t,v})\bar{r}_{t,v}-\rho-g](t-v)}$ has values ranging from one to zero as t increases, which results from the fact that $(1 - \bar{\tau}_t)\bar{r}_t < \rho + g$. Thus, $a_{t,v}^d$ approaches $-\tilde{b}_v$ as times goes on, and hence, the liquidity constraint is binding when $a_{t,v}^d = 0$.

Note that the dis-saving of disabled households and the saving of employed households are results of the general equilibrium setup of the model. We could say that disabled households are “impatient”: the after-tax interest rate is smaller than their efficient discount rate, which causes a descending time path for consumption. The key assumption here is that the per capita social insurance benefit grows at the rate of g . Thus, disabled households also enjoy the benefits of growing productivity. However, “impatience” does not originate from the higher subjective discount factor compared to employed households, but, rather, from the fact that disabled households do not have a precautionary saving motive. Therefore, the prevailing interest rate in the economy – which is set by households that have a precautionary saving motive – causes the dis-saving of disabled households or their “impatience”.

Now we can give the optimal path for consumption. Assume now that the liquidity constraint is binding at time T for a household which is disabled at time v . Then the budget constraint (equation (14)) implies that consumption is $c_{T,v}^d = b_t$, since, by definition, $\dot{a}_{T,v}^d = a_{T,v}^d = 0$. The

moment when the liquidity constraint is binding, T , is still unknown, but it can be solved from equation (17). Furthermore, we can write the solution as $T = \Psi_t + v$, where Ψ_t is a function of the variables given in equation (17).¹⁴

It is now possible to give the optimal consumption plan over t for a household which became a disabled household at time v . It is as follows:

$$c_{t,v}^d = \begin{cases} c_v^d e^{[(1-\bar{r}_t)\bar{r}_t - \rho - g](t-v)} & \text{if } t \in [v, T) \\ b_t & \text{if } t \in [T, \infty). \end{cases} \quad (18)$$

That is, before the liquidity constraint becomes binding, consumption follows the optimal time path implied by Euler equation (12). From the moment when the liquidity constraint becomes binding to infinity, the household consumes its social insurance benefits. This consumption plan maximizes the utility of the representative disabled household. Note that, when $b_t = 0$ (i.e. $\eta = 0$), the liquidity constraint is not bind – or is binding only asymptotically – and assets asymptotically converge to the lower limit: hence, $T = \infty$.

3 Aggregation and the steady state

3.1 Aggregate consumption for the disability state

We start the aggregation from the disability state. According to equation (18), we can separate the consumption of households in the disability state into two parts depending on the asset holdings of the household. If the liquidity constraint is binding, the household's consumption is given by $c_{t,v}^d = b_t$ for all t , and this does not depend on v . However, the level of consumption for the disabled household which has assets depends on v , i.e. consumption depends on the moment when the household became disabled.

Assume now that the economy is living at moment t . Hence, households that have arrived at the

¹⁴Now we have assumed that Ψ_t does not depend on v . We will come back to this assumption in section 3.4, and actually, this assumption holds only approximately. Anyhow, this approximation is feasible.

disability state between times t and $t - \Psi_t$ are not liquidity constrained.¹⁵ In order to capture consumption at every point on that optimal path, we must integrate from t to $t - \Psi_t$. Note that we add the levels of consumption between times t and $t - \Psi_t$, thus we must divide the sum by Ψ_t in order to get the average level of consumption at time t . Hence, by integrating equation (18) from t to $t - \Psi_t$ we can give the average consumption of disabled, *but unconstrained*, households as

$$c_t^{d,u} = \frac{1}{\Psi_t} \int_{t-\Psi_t}^t c_{t,v}^d dv = \frac{1}{\Psi_t} \int_{t-\Psi_t}^t c_v^d e^{[(1-\bar{\tau}_t)\bar{r}_t - \rho - g](t-v)} dv \quad (19)$$

where the second u in the superscript represents unconstrained households.

Now we must find the fraction of liquidity constrained households and the fraction of households that have assets. The number of disabled households at time t is Z_t and it is increasing at rate n . Thus, we can divide Z_t as follows:

$$\begin{aligned} Z_t = \int_{-\infty}^t Z e^{-n(t-v)} dv &= \int_{t-\Psi_t}^t Z e^{-n(t-v)} dv + \int_{-\infty}^{t-\Psi_t} Z e^{-n(t-v)} dv \\ &= (1 - e^{-n\Psi_t}) + e^{-n\Psi_t} = (1 - \kappa_t) + \kappa_t, \end{aligned} \quad (20)$$

where we have now normalized $\frac{Z}{n} = 1$. Since L_0 was normalized to unity, we may normalize Z again without a loss of generality. Moreover, we have defined $\kappa_t \equiv e^{-n\Psi_t}$, when the fraction κ_t from the *disabled households* are liquidity constrained and the fraction $(1 - \kappa_t)$ is not.

Now we can give the aggregate consumption of disabled households as a weighted sum:

$$c_t^{d,a} = (1 - \kappa_t) c_t^{d,u} + \kappa_t b_t, \quad (21)$$

where the weights are just shares of disabled households, as defined by equation (20). The fraction $1 - \kappa_t$ does have assets and they consume, on average, $c_t^{d,u}$ (see equation (19)). The fraction κ_t

¹⁵The liquidity constraint is binding at the point of time $t = \Psi_t + v$ for a household that reached the disability state at time v . So, we can write $v = t - \Psi_t$ which tells the point of time when the household, for which the liquidity constraint for the first time became binding at t , became disabled. If $v < t - \Psi_t$, the liquidity constraint was already binding for the household and the household is consuming just b_t . Thus, the liquidity constraint is not binding for households which reached the disability state between $t \geq v \geq t - \Psi_t$.

does not have assets when the level of consumption is b_t .

3.2 Aggregate consumption and the Euler equation

The aggregate behavior of employed households can be described by the behavior of the representative household, which was defined in Section 2.4. When variables are defined in per efficient capita form, we only have to multiply these variables by the share of the population which is employed. That is, we can write $\theta c_t^e \equiv C_t^e$. Moreover, equation (21) defines the aggregate consumption of disabled households in terms of efficient capita and we can define $\lambda c_t^{d,a} \equiv C_t^d$. Hence, the aggregate consumption in per efficient capita form C_t can be defined by the following equation:

$$C_t = C_t^e + C_t^d = \theta c_t^e + \lambda c_t^{d,a}. \quad (22)$$

In order to derive the aggregate Euler equation for consumption, which tells the optimal time path for aggregate consumption, we must differentiate equation (22) with respect to t . This yields

$$\dot{C}_t = \dot{C}_t^e + \dot{C}_t^d = \theta \dot{c}_t^e + \lambda \dot{c}_t^{d,a}. \quad (23)$$

When equation (11) is multiplied by θc_t^e , it defines $\theta \dot{c}_t^e$. However, the definition of $\lambda \dot{c}_t^{d,a}$ is more complex.

Differentiating (21) with respect to time gives

$$\dot{c}_t^{d,a} = (1 - \kappa_t) \dot{c}_t^{d,u} + \kappa_t \dot{b}_t - \dot{\kappa}_t (c_t^{d,u} - b_t). \quad (24)$$

Disabled households who do not have assets consume b_t for every t and, thus, for the fraction κ_t the Euler equation for consumption is $\dot{b}_t$. Further, the optimal time path of consumption for disabled households which have assets (a fraction $1 - \kappa_t$) is given by Euler equation (12) and the average level of consumption associated with that path is $c_t^{d,u}$ (see equation (19)). Thus, Euler equation (12) defines $\dot{c}_t^{d,u}$. The last term in equation (24) captures the transitions of κ_t in time.

Hence, the aggregate Euler equation (23) for consumption can be rewritten as

$$\begin{aligned}\dot{C}_t = & \left[(1 - \tau_t)r_t - \rho - g + \mu \left(\frac{c_t^e}{c_t^d} - 1 \right) \right] \theta c_t^e \\ & + [(1 - \tau_t)r_t - \rho - g] \lambda (1 - \kappa_t) c_t^{d,u} + \lambda \kappa_t \dot{b}_t - \lambda \dot{\kappa}_t (c_t^{d,u} - b_t).\end{aligned}\quad (25)$$

3.3 The capital stock and its evolution

Since the economy is closed, aggregate assets must equal aggregate capital. Thus, we can write $A_t = K_t = K_t^e + K_t^d$, where $K_t^e \equiv \theta k_t^e$ and $K_t^d \equiv \lambda k_t^d$. The asset holdings of households in the employed state are easy to aggregate: we only have to multiply the asset holdings of the representative agent by population share θ . However, the aggregate asset holdings of disabled households are more complicated to define.

Liquidity constrained disabled households do not have assets, which means that the fraction $1 - \kappa_t$ represents households in the disability state which own capital. Therefore, $k_t^d = (1 - \kappa_t) k_t^{d,u}$, where $k_t^{d,u}$ represents the average asset holdings of unconstrained households in the disability state at time t . We can define $k_t^{d,u}$ by using equation (17) and we apply the same aggregation method that we used for consumption in Section 3.1. Doing this, we get

$$k_t^{d,u} = \frac{1}{\Psi_t} \int_{t-\Psi_t}^t a_{t,v}^d dv = \frac{1}{\Psi_t} \int_{t-\Psi_t}^t \left[k_v^e + \tilde{b}_v \right] e^{[(1-\bar{\tau}_t)\bar{r}_t - \rho - g](t-v)} - \tilde{b}_v dv. \quad (26)$$

Now $k_t^{d,u}$ defines the average capital holdings of an unconstrained household in the disabled state at time t . Thus, the aggregate capital holdings of disabled households is given by $K_t^d \equiv \lambda k_t^d = \lambda (1 - \kappa_t) k_t^{d,u}$, where the average capital holding of disabled households ($k_t^{d,u}$) is just multiplied by the share of disabled households which have assets ($1 - \kappa_t$). This gives the amount of capital for the representative (or average) disabled household k_t^d . Finally, we must multiply it by the population weight of disabled households (λ).

Now we have defined the level of capital. The next task is to derive an equation which gives its

evolution in time. We can easily aggregate equation (7), which gives

$$\dot{K}_t \equiv \theta \dot{a}_t^e + \lambda \dot{a}_t^d = [(1 - \tau_t)r_t - g - n] K_t - C_t + \theta w_t l_t + \lambda b_t,$$

where we have used definition (22). The government was running a balanced budget, or a pay-as-you-go social insurance system, and all of output of was exhausted to households. That is, by using equations (1), (2), (3) and (4) we can write the capital evolution equation in the following way:

$$\dot{K}_t = Y_t - C_t - (\delta + g + n)K_t. \quad (27)$$

Thus, the evolution equation of aggregate capital stock is the standard one.

3.4 The steady state

The dynamics of the economy can now be described by the system of three differential equations, which are the aggregate Euler equation for consumption (25), the evolution equation for the aggregate capital stock (27) and the evolution equation for the assets of disabled households (7). Since the aggregate Euler equation depends on K_t and k_t^d , we also need the evolution equation for k_t^d in addition to the two standard equations. Moreover, we must take into account the effects of the endogenous labor supply so that these equations can be rewritten as

$$\begin{aligned} \dot{\hat{C}}_t &= \left[(1 - \tau_t)r_t - \rho - g + \mu \left(\frac{\hat{c}_t^e}{\hat{c}_t^d} - 1 \right) \right] \theta \hat{c}_t^e \\ &\quad + [(1 - \tau_t)r_t - \rho - g] \lambda (1 - \kappa_t) \hat{c}_t^{d,u} \\ &\quad + \lambda \kappa_t \dot{\hat{b}}_t - \lambda \dot{\kappa}_t (\hat{c}_t^{d,u} - \hat{b}_t) - \frac{\dot{l}_t}{l_t} \hat{C}_t \end{aligned} \quad (28)$$

$$\dot{\hat{K}}_t = \hat{Y}_t - \hat{C}_t - (\delta + g + n)\hat{K}_t - \frac{\dot{l}_t}{l_t} \hat{K}_t \quad (29)$$

$$\dot{\hat{k}}_t^d = [(1 - \tau_t)r_t - n - g] \hat{k}_t^d - (1 - \kappa_t) \left(\hat{c}_t^{d,u} - \hat{b}_t \right) - \frac{\dot{l}_t}{l_t} \hat{k}_t^d \quad (30)$$

where the hat indicates that variables are divided by the term $T_t P_t l_t$.¹⁶ The steady state implies that $\dot{l}_t = \dot{C}_t = \dot{K}_t = \dot{b}_t = \dot{\kappa}_t = 0$. Because of this, the steady state capital stock is given by equation (28) and steady state consumption is given by equation (29). Moreover, the steady state value of $\hat{k}_t^d$ is given by equation (26).

The steady state values are labeled with a star. The steady state capital stock is now defined by three equations. First, we need the definition of aggregate capital stock: $\hat{K}_\star = \hat{K}_\star^e + \hat{K}_\star^d$. Second, we use the aggregate Euler equation (28). Third, we have defined $\hat{k}_\star^d = (1 - \kappa_\star) \hat{k}_\star^{d,u}$, and $\hat{k}_\star^{d,u}$ is given by equation (26). Moreover, we have used the approximation that $\hat{k}_{v,\star}^e$ is a constant, i.e. $\hat{k}_{v,\star}^e \approx \hat{k}_\star^e$.¹⁷ Now we have three equations which are as follows:

$$\hat{K}_\star = \theta \hat{k}_\star^e + \lambda \hat{k}_\star^d \quad (31)$$

$$0 = \left[(1 - \tau_\star) r_\star - \rho - g + \mu \left(\frac{\hat{c}_\star^e}{\hat{c}_\star^d} - 1 \right) \right] \theta \hat{c}_\star^e + [(1 - \tau_\star) r_\star - \rho - g] \lambda (1 - \kappa_\star) \hat{c}_\star^{d,u} \quad (32)$$

$$\hat{k}_\star^d = (1 - \kappa_\star) \left\{ \frac{\hat{k}_\star^e + \frac{\hat{b}_\star}{(1 - \tau_\star) r_\star - n - g}}{\Psi_\star [\rho + g - (1 - \tau_\star) r_\star]} \left[1 - e^{-[\rho + g - (1 - \tau_\star) r_\star] \Psi_\star} \right] - \frac{\hat{b}_\star}{(1 - \tau_\star) r_\star - n - g} \right\}. \quad (33)$$

Now we substitute $\tau_\star$, $r_\star$, $\hat{c}_\star^d$, $\kappa_\star$, $\hat{c}_\star^{d,u}$, $\Psi_\star$ and $\hat{b}_\star$ into equations (32) and (33). Then use equation (22), which gives $\hat{c}_\star^e = \frac{1}{\theta} \left(\hat{C}_\star - \lambda \hat{c}_\star^{d,a} \right)$, where $\hat{C}_\star$ is defined by equation (29) and $\hat{c}_\star^{d,a}$ by equation (21).¹⁸ Then there are three unknowns, $\hat{K}_\star$, $\hat{k}_\star^e$ and $\hat{k}_\star^u$, and three equations. Finally, we use equation (31), which implies that $\hat{k}_\star^e = \frac{1}{\theta} \left(\hat{K}_\star - \lambda \hat{k}_\star^u \right)$, and substitute $\hat{k}_\star^e$ into equations (32) and

¹⁶We also need the equation for the growth of labor supply, which can be derived by using equations (5), (11), (13) and (27). More details are given in Appendix C.

¹⁷Actually, $\hat{k}_{v,\star}^e$ is not exactly constant. Since disabled households are decreasing their asset levels (dis-saving), then employed households must increase their asset levels in order to keep aggregate capital stock constant. Thus, the level of $\hat{k}_{v,\star}^e$ depends on v , i.e. a point of time when the household arrived at the disability state. However, the growth rate of the level of assets (per efficient capita) is very close to zero. Our baseline parameters imply that the growth rate is 0.00004. Thus, we may well approximate that $\hat{k}_{v,\star}^e$ is a constant. This approximation also implied that $\Psi_\star$ was only a function of t , but not a function of v as well.

¹⁸The rest of the steady state variables are given by the following equations: $r_\star$ is given by (2), $\tau_\star$ is given by (5), $\kappa_\star$ is given by (20), $\hat{c}_\star^{d,u}$ is given by (19), $\hat{b}_\star = \eta(1 - \tau_\star) \hat{w}_\star$ and $\Psi_\star$ can be solved using equation (17). Finally, we use equations (1) and (3) to substitute $\hat{w}_\star$ and $\hat{Y}_\star$ into previous definitions. Appendix D gives a detailed derivation of these steady state measures.

(33). Hence, these two equations jointly determine $\hat{K}_\star$ and $\hat{k}_\star^d$ and we can numerically solve these equations. Figure 1 shows the solution.¹⁹

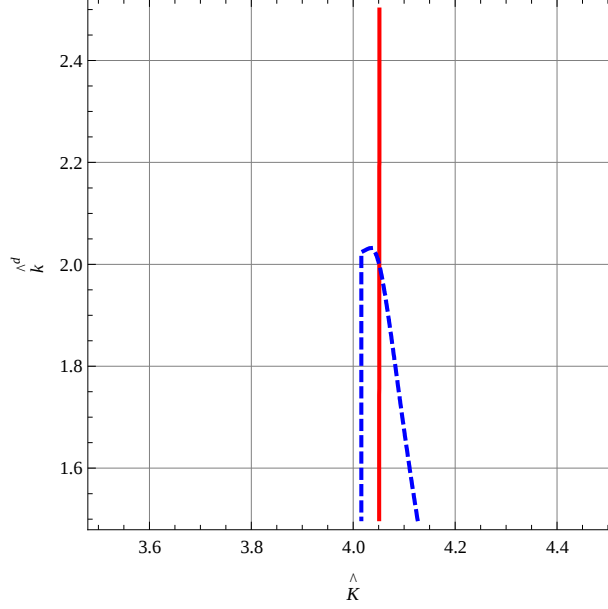


Figure 1: The solution for the steady state in $\hat{K}$ - $\hat{k}^d$ -space. The solid line shows the values of $\hat{K}$ and $\hat{k}^d$ when the aggregate Euler equation (32) holds. The dashed line shows the values of $\hat{K}$ and $\hat{k}^d$ when equation (33) holds. The intersection of the solid and the dashed line defines the steady state values of $\hat{K}_\star$ and $\hat{k}_\star^d$.

The solid line in Figure 1 gives the solution for aggregate Euler equation (32) with respect to the different values of $\hat{K}$ and $\hat{k}^d$. The line is (almost) vertical, which means that the steady state value of $\hat{K}$ does not depend significantly on $\hat{k}^d$. That is, the value of aggregate capital stock $\hat{K}$ can be solved very precisely even if the value of $\hat{k}^d$ is unknown. Since disabled households are in a minority and they are, on average, poorer than working households, they do not matter significantly in the determination of aggregate variables. However, the intersection of the solid and the dashed line determines the steady state values of aggregate capital stock and the capital holdings of disabled households.

¹⁹The parameter values used in Figure 1 are given in Table 1.

3.5 The stability of the steady state and the reduced system

Let us now define the system of equations (28), (29) and (30) as

$$\dot{\mathbf{x}}_t = F(\mathbf{x}_t), \quad (34)$$

where $\mathbf{x} = (\hat{C}, \hat{K}, \hat{k}^d)$ denotes coordinates in the phase space and F represent a smooth map that describes the evolution of the dynamic system. The stability of the system can be locally analyzed by taking a first-order Taylor approximation around the steady state when the real part of eigenvalues ν of the Jacobian matrix determine the nature of the steady state. By applying this procedure to equation (34), we can write our differential equations as

$$\dot{\mathbf{x}}_t = \mathbf{A}(\mathbf{x}_t - \mathbf{x}_\star), \quad (35)$$

where $\mathbf{A} = DF(\mathbf{x}_\star)$ is the Jacobian matrix and the eigenvalues of the Jacobian are $\nu_1 > 0$, $\nu_2 < 0$ and $\nu_3 > 0$. Thus, the Grobman-Hartman Theorem implies that we have a locally saddle path stable steady state. Moreover, the eigenvector associated with the negative eigenvalue corresponds to the stable arm (or manifold) of the linearized system and eigenvectors associated with positive eigenvalues correspond to the unstable arms (or manifolds). The uniqueness of the steady state can be shown in the usual way by using the transversality condition.

We can use this system, but we can also reduce its dimension, and still get an illustration for the dynamics of the system. Gomis-Porqueras and Haro (2009) show that we may find *preserved quantities* that help us simplify the study of the evolution of the system, but the computation of preserved quantities is very model dependent. In this model, we can find a phase space which determines the evolution of aggregate variables. The evolution of aggregate variables $\hat{C}$ and $\hat{K}$ does not depend on equation (30), as suggested by Figure 1. Thus, the dynamic of the model is captured by equations (28) and (29), i.e. the dynamics of the model boil down to the dynamics of the Ramsey model. Moreover, this system has one stable and one unstable manifold (or arm) which implies a saddle path stability. Appendix E provides a more formal discussion of the topics covered here.

4 Simulations

We set most of the parameters on the values that are traditionally used in the existing literature, and those values are as follows: $n = 0.01$, $\alpha = \frac{1}{3}$, $\rho = 0.05$, $g = 0.02$, $\delta = 0.05$. Parameters that control the supply of labor are ϕ and γ , where the Frisch elasticity is given by $\frac{1}{\phi}$, and we assume that labor supply is elastic when $\phi = 0.5$. Finally, γ controls the level of labor supply. It is set so that in the baseline Ramsey model where markets are perfect (i.e, there is a perfect insurance against uncertainty), $l_\star = 0.33$ when $\gamma = 5$.

Now we are left with parameters that are essential for the paper: the arrival rate μ and the replacement ratio η . Note that it is almost impossible to calibrate μ from the data. However, we can observe the value of $\lambda = \frac{\mu}{n+\mu}$, and when $n = 0.01$, we get $\mu = \frac{\lambda n}{1-\lambda}$. λ is the fraction of disabled households (or individuals) out of the working-age population. Since our model does not take into account any aspects of strategic behavior, we should find out the number of “genuine” disabled individuals. But, when people behave strategically, this is obviously difficult. Moreover, it must be noticed that the definition of disability is not very exact. Anyhow, we use self-reported disability status as an estimate for λ .²⁰ Bound and Burkhauser (1999, Table 2) summarize evidence from five different data sets. For working-age males, the percentage of the population with disabilities varies from 8.1% to 11.7%. For women, the corresponding numbers are 7.8% and 11.6%. Given these numbers, we set $\lambda = 0.1$, which seems to be an appropriate number. This then implies that $\mu = 0.0011 \dots$ or that the expected working life is $\frac{1}{\mu} = 900$ years. Hence, permanent disability is a rare event!

The second important parameter was η , which gives the replacement ratio, i.e. how many per cents is the disability insurance benefits of the current after tax labor income. Autor and Duggan (2003, Table 1) report the replacement ratio (including in-kind Medicare benefits) for non-elderly males at various percentiles of the wage distribution and ages for the year 1999. The replacement ratio varies significantly depending on age and earnings: the highest value is 104% and the lowest is 22%. We use a very harsh method to deal with this heterogeneity and just use a mean of that

²⁰These numbers need not reflect the true status of the individuals, but we assume that there is a much weaker reason to misreport the status in an anonymous survey. The same strategy is used by Golosov and Tsyvinski (2006) and Benitez-Silva, Buchinsky, Chan, Cheidvasser, and Rust (2004).

sample which is 50%. That is, we set $\eta = 0.5$, which could be thought of as a replacement ratio for a representative agent. Table 1 summarize our baseline parameter values.

Table 1: The baseline parameter values

Parameter	Value	Explanation
n	0.01	Growth rate of population
α	$\frac{1}{3}$	Capital share
ρ	0.05	Subjective time preference
g	0.02	Growth rate of productivity
δ	0.05	Depreciation rate of capital
ϕ	0.5	Frisch elasticity is $\frac{1}{\phi} = 2$
γ	5	Defines the level of labor supply
λ	0.1	Percent of population with disabilities
θ	0.9	Percent of population without disabilities
η	0.5	Replacement ratio

4.1 The results

Now we consider the numerical values at the steady state implied by the calibrated model. Table 2 reports the results.²¹ To measure the effects of market imperfection, we also report values from the Ramsey model, where we have only scaled output by Θ . This is the complete market case (CM) and the values are reported in column (1). There, we assume that households distribute consumption equally to everyone, i.e. there is a perfect insurance against permanent disability.²²

The case of incomplete markets (IM) is a case in which only the government provides social insurance by deciding the replacement ratio. We give the values $\eta = 0$, $\eta = 0.5$ and $\eta = 1$ to capture the cost and benefits of social insurance in this model, and columns (2), (3) and (4)

²¹Appendix F shows the results when $\lambda = 0.3$. The results are parallel to these results only the magnitudes are much higher due to a degree of higher uncertainty and the higher number of disabled households. Second, we also considered changes in the Frisch elasticity. We set the Frisch elasticity lower and higher than in the baseline case, i.e. we set $\phi = 1$ and $\phi = \frac{1}{3}$. The lower Frisch elasticity implies that the disability insurance program generates a one percentage point smaller costs. The higher Frisch elasticity, in turn, gives 1 percentage point higher costs. Finally, we set $\phi = 2$, which gives a lower bound for welfare costs. Then, replacing the current system with a perfect private one would increase aggregate consumption by 3.5%.

²²Obviously, this kind of arrangement could be based on altruism or some other mechanism which could provide a perfect insurance against disability, or there are private competitive insurance companies which provide insurance at an actuarially fair price.

report these values. However, the incentive-compatible constraint is such that $U^e \geq U^d$. Hence, the utility of working households must be higher than the utility of disabled households, and this implies that the maximum value of $\eta = 0.565$. Hence, the higher values of η are only hypothetical. When $\eta = 0$ there is no social insurance and people must live on their assets in the disability state. In the case of $\eta = 1$, the government provides a perfect insurance when we are back in the Ramsey model, but there is a positive tax rate and $\eta = 0.5$ gives the result from our baseline calibration.

In column (5) the reported values are generated by a model where the tax rate, τ , is not endogenously determined, but it is set instead at the same level as in the case of $\eta = 0.5$. However, here we set $\eta = 0$ when we can separate the sources of distortion caused by the disability insurance program. We can now measure how much the higher tax rate, which is needed to finance the disability insurance program, affects the performance of the economy under the baseline calibration.

By comparing columns (2) and (3), we can measure the burden generated by the social insurance program for the economy. By closing the social insurance program, the government could increase per capita consumption by 2.5%. Moreover, by comparing columns (2) and (5) we see the distortion originating from the increase in the tax rate. Obviously, higher tax rates reduce incentives to save, which lowers the level of capital stock. By comparing columns (3) and (5), we can measure a distortion which results from the reduced precautionary saving motive of households, since in column (5), we have controlled the effects of taxes. The higher level of social insurance reduces the self-insurance (or precautionary saving) motive of working households. Since c_t^u rises, it reduces the value of term $\frac{c_t^e}{c_t^u}$.²³ Thus, we can summarize these comparisons by concluding that 1/3 of the distortion (2.5% lower per capita consumption) is caused by a higher tax rate and 2/3 comes from the change in economic behavior, i.e. from the reduced precautionary saving motive.

Second, the magnitude of precautionary wealth can be calculated by comparing columns (1)

²³This result is generally well known. See, for example, Hubbard and Judd (1987), Hubbard, Skinner, and Zeldes (1995) and Engen and Gruber (2001).

Table 2: The steady state allocations in the complete markets setting (CM) and under incomplete markets (IM).

	(1)	(2)	(3)	(4)	(5)
Market setting	CM	IM	IM	IM	IM
Replacement ratio		$\eta = 0$	$\eta = 0.5$	$\eta = 1$	$\eta = 0$
Tax rate		$\tau = \text{endo.}$	$\tau = \text{endo.}$	$\tau = \text{endo.}$	$\tau = 0.041$
Output, $Y_\star$	0.497	0.480	0.461	0.455	0.475
Agg. consumption, $C_\star$	0.386	0.369	0.360	0.359	0.366 [†]
Capital, $K_\star$	1.379	1.406	1.257	1.203	1.356
Tax rate, $\tau_\star$	0	0	0.041	0.079	0.041
After tax int., $(1 - \tau_\star)r_\star$	0.070	0.064	0.069	0.070	0.064
After tax wage, $(1 - \tau_\star)w_\star$	1.111	1.141	1.056	0.999	1.080
Saving rate, $1 - \frac{C_\star}{Y_\star}$	0.222	0.234	0.218	0.212	0.228
Labor supply, $l_\star$	0.3311	0.3118	0.3101	0.3107	0.3121
Cons. for $\epsilon = e$, $c_\star^e$	0.386	0.409	0.379	0.359	0.387
Cons. for $\epsilon = d$, $c_\star^{d,a}$	0.386	0.001	0.191	0.359	0.001
Utility $U = \theta U^e + \lambda U^d$	-1.523	-1.973	-1.556	-1.545	-2.132

[†] Here aggregate consumption is composed of private and public consumption.

and (2). When the markets are perfect, representative working household capital holdings are $k_\star^e = 1.379$. In the case of imperfect markets, when disabled households capital holdings are approximately zero, we get $k_\star^e = \frac{1}{\theta}K_\star = 1.562$. This additional wealth can be seen as a result of the precautionary saving motive. Hence, the precautionary saving motive increases the asset holdings of working households' by 13%. Chandra and Samwick (2009) gives similar estimates for the magnitude of precautionary savings which are attributable to disability risk. However, we can conclude that social insurance income is important for disabled households, and hence,

the precautionary savings do not insure households against the risk of disability. By increasing the level of social insurance, η , we can raise the consumption of disabled households and welfare. This result is consistent with Deaton (1991), who shows that the effectiveness of savings as an insurance mechanism against shocks to labor earnings declines as the persistence of these shocks rises.

Third, by comparing columns (1) and (4) we can see the difference between two perfect insurance systems. The values in column (1) give the steady state values under perfect markets when there is perfect private insurance. Column (4) shows a case where the government provides the perfect insurance against permanent disability and the program is financed by proportional taxes. It is evident that the public system works much worse than the private one. Actually, per capita consumption is about 7% higher when insurance is provided by private competitive insurance companies. Moreover, the same conclusion can be made when the current system (see column (3)) and complete markets are compared against one another. However, this result depends on the Frisch elasticity of labor supply and when it is set at 0.5 the increase in the amount of aggregate consumption would be 3.5% (see Appendix F). In any case, the results imply that optimizing the tax-financed social insurance systems is not the best way to improve welfare, but that completing the markets by removing impediments to the private provision of insurance would generate a much higher increase in welfare.²⁴ For example, Golosov and Tsyvinski (2006) showed that moving from the current system in the U.S. to an asset-tested disability insurance system increases consumption by about 0.5%. Another interpretation for the result is that the costs generated by problems associated with imperfect information – which prevents market-based solutions – are indeed very large. Further, one can see that this as an estimate (probably an upper limit) for a consumption loss which is generated by government institutions since these institutions crowd out privately provided insurance against permanent disability.

Moreover, imperfect markets cause a significant loss of welfare. If we consider columns (1) and (2), we see that completing the markets would increase output by 3.5% and that aggregate consumption would rise by about 4.5%. But why does market incompleteness decrease welfare? When

²⁴The similar policy recommendation was also expressed by Atkeson, Chari, and Kehoe (1999) when they examined capital taxation.

there is no social insurance, workers do not have to share their consumption with disabled households, which increases their level of consumption by 6%. This “wealth effect” reduces their labor supply by almost 6%, which then reduces aggregate output, even if the level of capital increases in the economy. Thus, completing markets would reduce workers’ consumption, but it would increase aggregate consumption and output. As emphasized by Marcet, Obiols-Homs, and Weil (2007), incomplete markets increase the level of capital stock and then lead to higher output (the so-called Aiygari-Huggett effect) *if labor supply is inelastic*. But, when labor supply is elastic, the ex-post wealth effect dominates the Aiygari-Huggett effect. Thus, economies with less developed financial markets also have a lower level of output and welfare.

Finally, we can conclude by noting a surprising result: higher tax rates, which are levied to finance social insurance programs, do not cause any distortion to the supply of labor. Labor supply is constant when tax rates get higher, even if labor supply is very elastic (see, Table 1). A higher level of social insurance reduces the precautionary saving motive of working households, which lowers their consumption, since capital income is now at a lower level. Hence, the supply of labor is constant even though the after-tax wage has decreased. The decreased level of the after tax-wage rate and consumption balance each other out, which implies that the supply of labor does not depend on the level of social insurance. Hence, the social insurance program causes the distortion to the economy only by reducing the level of capital stock.

5 Conclusions

The model in this paper extends the Ramsey model by using a precautionary saving motive. Households face the constant probability of permanently losing their jobs throughout their lives and this uncertainty is only partially insured by a social insurance policy that is provided by the government. Thus, the model captures the uncertainty associated with the rare and permanent income losses of households. In this paper, we focused on the disability insurance in the United States.

With the calibrated model, we can measure the costs of social insurance that come from two

sources: the extent to which the social insurance changes individuals' behavior and the extent to which the taxes that are levied to finance this program. The disability insurance program lowers per capita consumption by 2.5%. One-third of this burden is caused by higher taxes and 2/3 comes from the change in economic behavior. That is, the social insurance removes households' precautionary saving motive, which lowers the level of capital stock, and the higher tax rate further lowers the level of capital stock. However, the supply of labor does not significantly depend on the level of disability insurance, even if we take into account the effects of higher taxes that are needed to finance the disability insurance program. That is, the extent to which the disability insurance program distorts the economy mainly comes from the lower level of capital stock.

Finally, it is easy to conclude that the leading way to improve welfare is to figure out how impediments that prevent the private provision of insurance against disability can be removed. The private provision of perfect insurance against permanent disability would lead to (depending on the Frisch elasticity of labor supply) 3.5-7% higher per capita consumption than the current disability insurance program. Another interpretation for the result is that the costs generated by problems associated with imperfect information – which prevents market-based solutions – are indeed very large. Yet another alternative interpretation for the result is that the government institutions (i.e. social security) crowd out private solutions at insurance markets. In the past, when financial markets were underdeveloped, government social security institutions provided insurance. However, nowadays financial markets are developed and there could be privately provided insurance, but the government institutions crowd out these solutions. Given the estimates in this paper, the crowding out may result in a large welfare loss.

Appendices

A Solving the problem of representative household without liquidity constraint

In this section we ignore the effects of liquidity constraint, hence the problem of representative household is given by equations (6), (7) and (9). The problem can be solved by using the standard tools of optimal control theory. Good sources of solution methods for these types of problems are given by Dixit and Pindyck (1994, chap. 3 and 4), Bertsekas (2005, chap. 3) and Acemoglu (2009, chap. 6).

First, we suppose that every period of time is length of Δt , when we can write Bellman equation for the problem by

$$V(a_t, t, \epsilon) = \max_{c_t, l_t} \{U(c_t, l_t)\Delta t + (1 + (\rho - n)\Delta t)^{-1} E_t V(a_{t+\Delta t}, t + \Delta t, \epsilon_{t+\Delta t})\} \quad (\text{A1})$$

Assume that V is continuous and differentiable when we can write

$$\begin{aligned} E_t V(a_{t+\Delta t}, t + \Delta t, \epsilon_{t+\Delta t}) &= \frac{\partial V(a_t, t, \epsilon)}{\partial a_t} \dot{a}_t \Delta t + \frac{\partial V(a_t, t, \epsilon)}{\partial t} \Delta t \\ &\quad + E_t [V(a_t, t, \epsilon_t) \Delta t]. \end{aligned} \quad (\text{A2})$$

Remember that uncertainty associated to this problem is given by a Poisson process which implies that

$$E_t [V(a_t, t, \epsilon_t) \Delta t] = \begin{cases} (1 - \mu \Delta t) V^e(a_t, t) + \mu \Delta t V^d(a_t, t), & \text{if } \epsilon = e \\ V^d(a_t, t), & \text{if } \epsilon = d \end{cases} \quad (\text{A3})$$

Now we can write the Bellman equation for both states by substituting equations (A2) and (A3)

into (A1). This gives

$$\begin{aligned}
(1 + (\rho - n)\Delta t)V^e(a_t, t) &= \max_{c_t^e, l_t} \left\{ U(c_t^e, l_t)(1 + (\rho - n)\Delta t)\Delta t + \frac{\partial V^e(a_t, t)}{\partial a_t} \dot{a}_t^e \Delta t \right\} \\
&\quad + \frac{\partial V^e(a_t, t)}{\partial t} \Delta t \\
&\quad + (1 - \mu\Delta t)V^e(a_t, t) + \mu\Delta t V^d(a_t, t)
\end{aligned} \tag{A4}$$

$$\begin{aligned}
(1 + (\rho - n)\Delta t)V^d(a_t, t) &= \max_{c_t^d} \left\{ U(c_t^d, 0)(1 + (\rho - n)\Delta t)\Delta t + \frac{\partial V^d(a_t, t)}{\partial a_t} \dot{a}_t^d \Delta t \right\} \\
&\quad + \frac{\partial V^d(a_t, t)}{\partial t} \Delta t + V^d(a_t, t),
\end{aligned} \tag{A5}$$

where $\dot{a}_t^e = [(1 - \tau_t)r_t - n - g]a_t - c_t^e + (1 - \tau_t)w_t l_t$ and $\dot{a}_t^d = [(1 - \tau)r_t - n - g]a_t - c_t^d + b_t$.

Simplify and divide both equations by Δt . After this let $\Delta t \rightarrow 0$. Then we get the standard Hamilton-Jacobi-Bellman equations for both states:

$$\begin{aligned}
0 &= \max_{c_t^e, l_t} \left\{ U(c_t^e, l_t) + \frac{\partial V^e(a_t, t)}{\partial a_t} \dot{a}_t^e \right\} \\
&\quad + \frac{\partial V^e(a_t, t)}{\partial t} + \mu [V^d(a_t, t) - V^e(a_t, t)] - (\rho - n)V^e(a_t, t)
\end{aligned} \tag{A6}$$

$$\begin{aligned}
0 &= \max_{c_t^d} \left\{ U(c_t^d) + \frac{\partial V^d(a_t, t)}{\partial a_t} \dot{a}_t^d \right\} \\
&\quad + \frac{\partial V^d(a_t, t)}{\partial t} - (\rho - n)V^d(a_t, t).
\end{aligned} \tag{A7}$$

Note that equation (A7) does not depend on equation (A6) which is the key to the tractable solution of this problem. This feature is generated by the assumption that the transition between the states occur only once and the only source of uncertainty is the timing of that transition. Moreover, the term $\mu [V^d(a_t, t) - V^e(a_t, t)]$ in equation (A7) gives the expected reduction in the value function due to income loss from permanent disability.

The maximization of equations (A6) and (A7) yields the envelopment relations. For the employed

state we get

$$U'_{c_t^e} = -\frac{\partial V^e(a_t, t)}{\partial a_t} \quad (\text{A8})$$

$$-U'_{l_t} = (1 - \tau_t)w_t \frac{\partial V^e(a_t, t)}{\partial a_t}, \quad (\text{A9})$$

and for the disability state we get

$$U'_{c_t^d} = -\frac{\partial V^d(a_t, t)}{\partial a_t} \quad (\text{A10})$$

where $U_x^n = \frac{\partial^n U}{\partial x^n}$. Differentiating envelopment conditions, which are associated with consumption, respect to time and assuming that $U''_{c_t^e, l_t} = 0$ (see equation (10)) we get:

$$U''_{c_t^e} \dot{c}_t^e = -\frac{\partial^2 V^e(a_t, t)}{\partial a_t^2} \dot{a}_t^e \quad (\text{A11})$$

$$U''_{c_t^d} \dot{c}_t^d = -\frac{\partial^2 V^d(a_t, t)}{\partial a_t^2} \dot{a}_t^d. \quad (\text{A12})$$

Given sufficient time the value function converges its stationary form implying $\frac{\partial V(a_t, t)}{\partial t} = 0$.

Differentiating equations (A6) and (A7) respect to a_t yields

$$\begin{aligned} 0 &= \frac{\partial^2 V^e(a_t, t)}{\partial a_t^2} \dot{a}_t^e + \frac{\partial V^e(a_t, t)}{\partial a_t} ((1 - \tau_t)r_t - n - g) - (\rho - n) \frac{\partial V^e(a_t, t)}{\partial a_t} \\ &\quad + \mu \left(\frac{\partial V^d(a_t, t)}{\partial a_t} - \frac{\partial V^e(a_t, t)}{\partial a_t} \right) \end{aligned} \quad (\text{A13})$$

$$\begin{aligned} 0 &= \frac{\partial^2 V^d(a_t, t)}{\partial a_t^2} \dot{a}_t^d + \frac{\partial V^d(a_t, t)}{\partial a_t} ((1 - \tau_t)r_t - n - g) \\ &\quad - (\rho - n) \frac{\partial V^d(a_t, t)}{\partial a_t}. \end{aligned} \quad (\text{A14})$$

Substitute equations (A8), (A10), (A11) and (A12) into equations (A13) and (A14). This yields

Euler equation for both states:

$$\frac{\dot{c}_t^e}{c_t^e} = -\frac{U'_{c_t^e}}{c_t^e U''_{c_t^e}} \left[(1 - \tau_t) r_t - \rho - g + \mu \left(\frac{U'_{c_t^d}}{U'_{c_t^e}} - 1 \right) \right] \quad (\text{A15})$$

$$\frac{\dot{c}_t^d}{c_t^d} = -\frac{U'_{c_t^d}}{c_t^d U''_{c_t^d}} [(1 - \tau_t) r_t - \rho - g] \quad (\text{A16})$$

Finally, we substitute equation (A8) into equation (A9) which gives

$$\frac{U'_{l_t}}{U'_{c_t^e}} = (1 - \tau_t) w_t. \quad (\text{A17})$$

Using the utility function $U(c_t, l_t) = \log c_t - \gamma \frac{l_t^{1+\phi}}{1+\phi}$ equations (A15), (A16) and (A17) imply equations (11), (12) and (13) in the text.

B The derivation of asset evaluation in the disabled state

Start by substituting equation (15) and (16) into equation (14). This gives

$$\dot{a}_{t,v}^d = [(1 - \tau_t) r_t - n - g] a_t^d - (\rho - n) [a_v^e + \tilde{b}_v] e^{[(1 - \bar{\tau}_{t,v}) \bar{r}_{t,v} - \rho - g](t-v)} + b_t$$

Bring the term $[(1 - \tau_t) r_t - n - g] a_t^d$ on the left hand side of the equation and multiply both sides by term $e^{-[(1 - \bar{\tau}_{t,v}) \bar{r}_{t,v} - n - g](t-v)}$. Use Leibniz's rule and take an integral over t which then gives

$$\begin{aligned} \int \frac{d}{dt} a_{t,v}^d e^{-[(1 - \bar{\tau}_{t,v}) \bar{r}_{t,v} - n - g](t-v)} dt &= - \int (\rho - n) [a_v^e + \tilde{b}_v] e^{-(\rho - n)(t-v)} dt \\ &\quad + \int b_t e^{-[(1 - \bar{\tau}_{t,v}) \bar{r}_{t,v} - n - g](t-v)} dt. \end{aligned} \quad (\text{A18})$$

Do the integration and multiply both sides with $e^{[(1-\bar{\tau}_{t,v})\bar{r}_{t,v}-n-g](t-v)}$ which gives then equation

$$\begin{aligned} a_{t,v}^d &= \left[a_v^e + \tilde{b}_v \right] e^{[(1-\bar{\tau}_{t,v})\bar{r}_{t,v}-\rho-g](t-v)} \\ &\quad + e^{[(1-\bar{\tau}_{t,v})\bar{r}_{t,v}-n-g](t-v)} \int b_t e^{-[(1-\bar{\tau}_{t,v})\bar{r}_{t,v}-n-g](t-v)} dt. \end{aligned} \quad (\text{A19})$$

Now the term $e^{[(1-\bar{\tau}_{t,v})\bar{r}_{t,v}-n-g](t-v)} \int b_t e^{-[(1-\bar{\tau}_{t,v})\bar{r}_{t,v}-n-g](t-v)} dt$ is unknown but it can be shown that it equals to $-\tilde{b}_v$. Hence, it gives the present value of social insurance transfers. This can be verified by assuming $t = v$ for terms $a_{t,v}^d$ and $\left[a_v^e + \tilde{b}_v \right] e^{[(1-\bar{\tau}_{t,v})\bar{r}_{t,v}-\rho-g](t-v)}$ when these terms can be rewritten as a_v^e and $\left[a_v^e + \tilde{b}_v \right]$. Since we are dealing with a budget constraint, our unknown term $e^{[(1-\bar{\tau}_{t,v})\bar{r}_{t,v}-n-g](t-v)} \int b_t e^{-[(1-\bar{\tau}_{t,v})\bar{r}_{t,v}-n-g](t-v)} dt = -\tilde{b}_v$, otherwise the budget constraint would not hold. Since budget constraint must hold for every t , it implies that previous statement must hold for every t as well. When this substitution is done, we get equation (17) in the text.

C The growth rate of labor supply

Equation (13) defined the optimal labor supply, which now can be rewritten as

$$l_t = \left(\frac{(1-\tau_t)\hat{w}_t}{\gamma\hat{c}_t^e} \right)^{\frac{1}{1+\phi}}, \quad (\text{A20})$$

where $\hat{w}_t = (1-\alpha)\theta^{-\alpha}\hat{K}_t^\alpha$. This equation implies that the growth rate of l is given by

$$\frac{\dot{l}_t}{l_t} = \frac{1}{1+\phi} \left[\alpha \frac{\dot{\hat{K}}_t}{\hat{K}_t} - \frac{\dot{\hat{\tau}}_t}{1-\hat{\tau}_t} - \frac{\dot{\hat{c}}_t^e}{\hat{c}_t^e} \right]. \quad (\text{A21})$$

We can derive the term $\frac{\dot{\hat{\tau}}_t}{1-\hat{\tau}_t}$ from equation (5), which gives

$$\frac{\dot{\hat{\tau}}_t}{1-\hat{\tau}_t} = \hat{\Omega}_t \frac{\dot{\hat{K}}_t}{\hat{K}_t}, \text{ where} \quad (\text{A22})$$

$$\hat{\Omega}_t = \frac{\lambda\eta\delta\theta^\alpha\hat{K}_t^{1-\alpha}}{\left(\lambda\eta + \theta + \frac{\alpha\theta}{1-\alpha} - \frac{\delta\theta^\alpha}{1-\alpha}\hat{K}_t^{1-\alpha} \right) \left(\theta + \frac{\alpha\theta}{1-\alpha} - \frac{\delta\theta^\alpha}{1-\alpha}\hat{K}_t^{1-\alpha} \right)}. \quad (\text{A23})$$

Now rewrite equation (A21)

$$\frac{\dot{l}_t}{l_t} = \frac{1}{1 + \phi} \left[\frac{\dot{\hat{K}}_t}{\hat{K}_t} (\alpha - \hat{\Omega}_t) - \frac{\dot{\hat{c}}_t^e}{\hat{c}_t^e} \right]. \quad (\text{A24})$$

The term $\frac{\dot{\hat{K}}_t}{\hat{K}_t}$ can be defined by using equation (27) and $\frac{\dot{\hat{c}}_t^e}{\hat{c}_t^e}$ can be derived by using equation (11). Then the equation (A24) can be rewritten as

$$\begin{aligned} \frac{\dot{l}_t}{l_t} = & \frac{\alpha - \hat{\Omega}_t}{\phi + \alpha - \hat{\Omega}_t} \left[\frac{\hat{Y}_t}{\hat{K}_t} - \frac{\hat{C}_t}{\hat{K}_t} - (\delta + g + n) \right] \\ & - \frac{1}{\phi + \alpha - \hat{\Omega}_t} \left[(1 - \tau_t)r_t - \rho - g + \mu \left(\frac{\hat{c}_t^e}{\hat{c}_t^d} - 1 \right) \right], \end{aligned} \quad (\text{A25})$$

which now gives the growth rate of labor supply. When equation (A25) is plugged in equations (28), (29) and (30) we have defined our system.

D A detailed derivation of the steady state measures

In the steady state $\dot{K}_t = \dot{l}_t = 0$, hence $K_t = K_\star$ and $l_t = l_\star$ are constants, which implies that $r_\star$, $w_\star$, $\tau_\star$ are constants. This can easily be verified from equations (2), (3) and (5). Let us continue by defining the steady state values for the disability state.

The value of $\tilde{b}_t$ in the steady state is given by

$$\tilde{b}_\star = \int_v^\infty b_\star e^{-\int_v^t (1 - \tau_\star)r_\star - g - nds} dt = \frac{b_\star}{(1 - \tau_\star)r_\star - g - n},$$

where $b_\star = \eta(1 - \tau_t)w_t l_t = \eta(1 - \tau_\star)w_\star l_\star$, which is constant at the steady state. Next we focus on equation (17) or equation (A19) and substitute $\tilde{b}_t = \tilde{b}_\star$ into equation which yields

$$a_{v,\star}^d = \left[a_{v,\star}^e + \frac{b_\star}{(1 - \tau_\star)r_\star - g - n} \right] e^{\int_v^t (1 - \tau_\star)r_\star - \rho - gds} - \frac{b_\star}{(1 - \tau_\star)r_\star - g - n}.$$

Moreover, the liquidity constraint was binding at time T for a household who was disabled at

time v , when we get in the steady state

$$a_{T,v}^d = \left[a_{v,\star}^e + \frac{b_\star}{(1-\tau_\star)r_\star - g - n} \right] e^{\int_v^T (1-\tau_\star)r_\star - \rho - g ds} - \frac{b_\star}{(1-\tau_\star)r_\star - g - n}.$$

Since the liquidity constraint ($a_{t,v}^d \geq 0$) is binding when $t = T$, it implies that $a_{T,v}^d = 0$. Thus, we can easily solve T :

$$\begin{aligned} T &= \Psi_\star + v \quad \text{where} \\ \Psi_\star &\equiv \frac{\log \left[\frac{\frac{b_\star}{(1-\tau_\star)r_\star - n - g}}{a_\star^e + \frac{b_\star}{(1-\tau_\star)r_\star - n - g}} \right]}{(1-\tau_\star)r_\star - \rho - g}. \end{aligned} \quad (\text{A26})$$

Note that we have now assumed that in the steady state $a_{v,\star}^e = a_\star^e$, i.e. $\dot{c}_t^e = \dot{a}_t^e = 0$ in the steady state. This holds only approximately and actually our baseline parameter values imply that in the steady state $\dot{a}_t^e = 0.00004 \approx 0$. Hence, this approximation is quite correct. When λ get larger values this approximation is not so good anymore. However, without this approximation the model is significantly more complicated to solve. Finally,

$$c_\star^d = (\rho - n) \left[a_\star^e + \frac{b_\star}{(1-\tau_\star)r_\star - g - n} \right] \quad (\text{A27})$$

is given by equation (16).

Now we can give the steady state values of our aggregate measures. Let us start from equation (20) which gives the steady state value for $\kappa_\star = e^{-n\Psi_\star}$ where $\Psi_\star$ is defined by (A26). The consumption of disabled but unconstrained households for every t was given by equation (19), which in the steady state gives

$$\begin{aligned} c_\star^{d,u} &= \frac{1}{\Psi_\star} \int_{t-\Psi_\star}^t c_\star^d e^{\int_v^t (1-\tau_\star)r_\star - g - \rho ds} dv \\ &= \frac{c_\star^d}{\Psi_\star [\rho + g - (1-\tau_\star)r_\star]} \left(1 - e^{-[\rho + g - (1-\tau_\star)r_\star]\Psi_\star} \right), \end{aligned} \quad (\text{A28})$$

where equations (A26) and (A27) defines $\Psi_\star$ and $c_\star^d$. The capital holdings of disabled households

are given by equation (26) which yields

$$\begin{aligned} k_{\star}^{d,u} &= \frac{1}{\Psi_{\star}} \int_{t-\Psi_{\star}}^t \left[k_{\star}^e + \frac{b_{\star}}{(1-\tau_{\star})r_{\star} - n - g} \right] e^{\int_v^t (1-\tau_{\star})r_{\star} - g - \rho ds} - \frac{b_{\star}}{(1-\tau_{\star})r_{\star} - n - g} dv \\ &= \frac{k_{\star}^e + \frac{b_{\star}}{(1-\tau_{\star})r_{\star} - n - g}}{\Psi_{\star} [\rho + g - (1-\tau_{\star})r_{\star}]} \left(1 - e^{-[\rho + g - (1-\tau_{\star})r_{\star}] \Psi_{\star}} \right) - \frac{b_{\star}}{(1-\tau_{\star})r_{\star} - n - g}. \end{aligned}$$

Finally, we can define $l_{\star}$ from equation (13) or (A20) which gives the steady state value of the supply of labor as follows

$$l_{\star} = \left(\frac{(1-\tau_{\star})w_{\star}}{\gamma c_{\star}^e} \right)^{\frac{1}{1+\phi}},$$

where equation (22) defines $c_{\star}^e = \frac{1}{\theta} (C_{\star} - \lambda c_{\star}^{d,a})$. We can get the value of $C_{\star} = Y_{\star} - (n+g+\delta)K_{\star}$, from equation (29), where $Y_{\star}$ is defined by equation (1). Moreover, $c_{\star}^{d,a}$ is given by (21) which gives

$$c_{\star}^{d,a} = (1 - \kappa_{\star})c_{\star}^{d,u} + \kappa_{\star}b_{\star},$$

where $c_{\star}^{d,u}$ is defined by equation (A28).

E The preserved quantities and the reduced system

In this model, we can show that given baseline parameter values the stable and unstable manifolds (or arms), which are associated with aggregate variables, are restricted to the level set spanned by a vector $(\hat{C}, \hat{K})$. Thus, our system has 2-dimensional invariant and unique manifolds which are the level sets of the preserved quantity and this subset of phase space defines the flow of our aggregate variables. Hence, we can ignore the effects of $\hat{k}^d$ when we are interested in aggregate behavior of the economy.²⁵

Let us consider a general case where a dynamic system is given $\dot{\mathbf{x}}_t = F(\mathbf{x}_t)$ where the phase

²⁵The following discussion is heavily based on Gomis-Porqueras and Haro (2009). Generally, a good source for the analysis of dynamic systems is Guckenheimer and Holmes (1983).

space is $\mathcal{B} \subset \mathbb{R}^n$ and $F : \mathcal{B} \subset \mathbb{R}^n \rightarrow \mathbb{R}^n$. To find preserved quantities, we can consider manifolds that are defined implicitly by a set of equations. To define these manifolds, given a smooth map $H : \mathcal{B} \subset \mathbb{R}^n \rightarrow \mathbb{R}^{n-m}$, and given by $H_0 \in \mathbb{R}^{n-m}$, the level set

$$\Omega_{H_0} = \{\mathbf{x} \in \mathcal{B} | H(\mathbf{x}) = H_0\}$$

is an $n - m$ -dimensional manifold if for all each point $\mathbf{x} \in \Omega_{H_0}$ the rank of the differential $DH(\mathbf{x})$ is $n - m$. Then, note that the level set Ω_{H_0} is invariant under the dynamic system (34), i.e. $\dot{\mathbf{x}}_t = F(\mathbf{x}_t)$, if and only if

$$H(\mathbf{x}) = H_0 \Rightarrow H(F(\mathbf{x})) = H_0.$$

Hence, the map H is a preserved quantity for the dynamic system $\dot{\mathbf{x}}_t = F(\mathbf{x}_t)$ if and only if for all $\mathbf{x} \in \mathcal{B}$, $H(F(\mathbf{x})) = H(\mathbf{x})$. Then, given an initial state $\mathbf{x}_0$ of the dynamic system, its motion $\mathbf{x}_t = F^t(\mathbf{x}_0)$ can be restricted to the level set of $H_0 = H(\mathbf{x}_0)$. Moreover, the existence of preserved quantity restricts the level set where invariant manifolds can exist, which is a result of following proposition.

Proposition A1. (*Gomis-Porqueras and Haro (2009, Proposition 2)*). *Let the preserved quantity $H : \mathcal{B} \rightarrow \mathbb{R}^{n-m}$ be a smooth map, preserved by a dynamic system $\dot{\mathbf{x}}_t = F(\mathbf{x}_t)$, where $F : \mathcal{B} \subset \mathbb{R}^n \rightarrow \mathbb{R}^n$. Let $\mathbf{x}_\star \in \mathcal{B}$ be a steady state of $\dot{\mathbf{x}}_t = F(\mathbf{x}_t)$. Then, for all $\mathbf{x} \in \mathcal{B}$ s.t. $\lim_{t \rightarrow +\infty} F^t(\mathbf{x}) = \mathbf{x}_\star$ or $\lim_{t \rightarrow -\infty} F^t(\mathbf{x}) = \mathbf{x}_\star$, then $\mathbf{x} \in \Omega_{H_\star}$ where $H_\star = H(\mathbf{x}_\star)$.*

Proof. (Following Gomis-Porqueras and Haro (2009)). Proof is based on a simply continuity argument. Assume that the preserved quantity $H : \mathcal{B} \rightarrow \mathbb{R}^{n-m}$ is a continuous function. Moreover, assume that $\mathbf{x} \in \mathcal{B}$ is such its orbits $\mathbf{x}_t = F^t(\mathbf{x})$ converges to the steady state $\mathbf{x}_\star \in \mathcal{B}$ in the future. That is, assume that $\lim_{t \rightarrow +\infty} F^t(\mathbf{x}) = \mathbf{x}_\star$. Since $H_0 = H(\mathbf{x}) = H(\mathbf{x}_t) \quad \forall t$, then $H_0 = \lim_{t \rightarrow +\infty} H(\mathbf{x}_t) = H(\mathbf{x}_\star)$. The study of $\lim_{t \rightarrow -\infty} F^t(\mathbf{x}) = \mathbf{x}_\star$ is analogous. $\square$

The result is that all the points in set H_0 , that convergence in the steady state, belong the same level surface of such a steady state.

Let us now turn back to our model. Since linearized system (35) has distinct real eigenvalues ν_1 , ν_2 and ν_3 , we have the unique solution to (35). Moreover, we can find a 2-dimensional preserved quantity given by H for the flow of linearized $\dot{\mathbf{x}}_t = F(\mathbf{x}_t)$ and with H we can describe the flow of $\hat{K}, \hat{C}$. The eigenvectors associated with eigenvalues $\nu_{\hat{C}}$ and $\nu_{\hat{K}}$ span the stable E^s and unstable E^u subspaces and in this space there is a subspace $\Omega_{H_0} \in \mathcal{B}$. In this subspace the stable and unstable manifolds (W^s and W^u) exist since the eigenvector associated with $\nu_{\hat{k}^d}$ span only one dimensional space. That is, it can be shown that the eigenvalue associated with $\hat{k}^d$ do not matter for the dynamics of $\hat{K}$ and $\hat{C}$. Hence, as stated by proposition A1, we have a preserved quantity H and this describes the flow of $\hat{K}, \hat{C}$.

To demonstrate previous discussion we consider a numerical example generated by our baseline parameter values which are $n = 0.01$, $\alpha = \frac{1}{3}$, $\rho = 0.05$, $g = 0.02$, $\delta = 0.05$, $\lambda = 0.1$, $\theta = 0.9$, $\eta = 0.5$, $\gamma = 5$, $\phi = 0.5$. Now we may write the solution of the linearized system as follows

$$\begin{aligned} \mathbf{x}_t &= \sum_{j=1}^3 c_j e^{\nu_j t} \mathbf{v}_{\nu_j} + \mathbf{x}_0 \\ \mathbf{x}_t &= c_1 e^{-0.13t} \mathbf{v}_{\nu_1} + c_2 e^{0.34t} \mathbf{v}_{\nu_2} + c_3 \mathbf{v}_{\nu_3} e^{0.04t} + \mathbf{x}_0, \quad \text{where} \\ \mathbf{v}_{\nu_1} &= \begin{bmatrix} -0.80 \\ -0.08 \\ -0.60 \end{bmatrix}, \quad \mathbf{v}_{\nu_2} = \begin{bmatrix} 0.64 \\ -0.45 \\ -0.62 \end{bmatrix} \quad \text{and} \quad \mathbf{v}_{\nu_3} = \begin{bmatrix} -0.0001 \\ 0.0001 \\ 1 \end{bmatrix} \end{aligned}$$

Thus, when we use approximation $-0.0001 \approx 0.0001 \approx 0$, we can write

$$\begin{aligned} \hat{K}_t &= -0.80e^{-0.13t} c_1 + 0.64e^{0.34t} c_2 + \hat{K}_0 \\ \hat{C}_t &= -0.08e^{-0.13t} c_1 - 0.45e^{0.34t} c_2 + \hat{C}_0 \\ \hat{k}_t^d &= -0.60e^{-0.13t} c_1 - 0.62e^{0.34t} c_2 + e^{0.04t} c_3 + \hat{k}_0^d, \end{aligned}$$

where c_1 , c_2 and c_3 must be solved by using the transversality conditions and $\mathbf{x}_0$. Hence, when we are only interested in the evolution of aggregate variables, i.e. $\mathbf{x}^r = (\hat{K}, \hat{C})$, we can reduce the

dimension of our initial system. Thus, we can give our reduced system as

$$\dot{\mathbf{x}}_t^r = H(F(\mathbf{x}_t)) = H(\mathbf{x}_t).$$

Thus, when we have found our steady state, we may analyze the dynamics of the aggregate variables only by focusing on 2-dimensional system which determines the dynamics of aggregate consumption and capital stock. This is the reduced system. In other words, our reduced system corresponds equations (28) and (29). Moreover, when we linearize our reduced system $H(\mathbf{x}_t)$, given $\hat{k}_\star^d$, the eigenvalues (ν_1 and ν_2) are the same as in the case of original system and eigenvectors are linear combinations of $\mathbf{v}_{\nu_1}$ and $\mathbf{v}_{\nu_2}$.

Finally note that, when λ get unfeasible large values like $\lambda = 0.3$ we cannot reduce the dimension of our system. That is obvious, since then the behavior of disabled households also matters for the behavior of the aggregate variables, due to the fact that a large share of population in the economy are disabled when they matter also for the behavior of the aggregate variables.

F The robustness of the results

We only change the value of $\lambda = 0.3$ when $\mu = 0.0043$ or $\frac{1}{\mu} = 233$ and calibrate $\gamma = 6.45$, when under complete markets $l_\star = 0.33$. Table A1 report the results.

Second, we consider *lower* Frisch elasticity and we set $\phi = 2$. But, we still keep $l_\star = 0.33$, which then implies that $\gamma = 26$. Results are given by Table A2.

Third, we consider a Frisch elasticity equal to 1 when $\phi = 1$. But, we still keep $l_\star = 0.33$, which then implies that $\gamma = 8.5$. Results are given by Table A3.

Fourth, we consider a more *higher* Frisch elasticity and we set $\phi = \frac{1}{3}$. But, we still keep $l_\star = 0.33$, which then implies that $\gamma = 4.15$. Results are given by Table A4.

Table A1: The steady state allocations in the complete markets setting (CM) and under incomplete markets (IM).

	(1)	(2)	(3)	(4)	(5)
	CM	IM	IM	IM	IM
		$\eta = 0$ $\tau = \text{endo.}$	$\eta = 0.5$ $\tau = \text{endo.}$	$\eta = 1$ $\tau = \text{endo.}$	$\eta = 0$ $\tau = 0.141$
Output, $Y_\star$	0.385	0.345	0.298	0.285	0.339
Agg. consumption, $C_\star$	0.300	0.253	0.236	0.231	0.258 [†]
Capital, $K_\star$	1.070	1.144	0.772	0.665	1.014
Tax rate, $\tau_\star$	0	0	0.141	0.244	0.141
After tax int., $(1 - \tau_\star)r_\star$	0.070	0.050	0.068	0.070	0.053
After tax wage, $(1 - \tau_\star)w_\star$	1.11	1.214	0.922	0.770	0.990
Saving rate, $1 - \frac{C_\star}{Y_\star}$	0.222	0.265	0.207	0.187	0.239
Labor supply, $l_\star$	0.3303	0.2706	0.2645	0.2660	0.280
Cons. for $\epsilon = e$, $c_\star^e$	0.300	0.362	0.278	0.231	0.290
Cons. for $\epsilon = d$, $c_\star^{d,a}$	0.300	0.001	0.139	0.231	0.001
Utility $U = \theta U^e + \lambda U^d$	-1.776	-3.205	-1.897	-1.876	-3.585

[†] Here aggregate consumption is defined as private+public consumption.

Table A2: The steady state allocations in the complete markets setting (CM) and under incomplete markets (IM).

	(1)	(2)	(3)	(4)	(5)
	CM	IM	IM	IM	IM
		$\eta = 0$	$\eta = 0.5$	$\eta = 1$	$\eta = 0$
		$\tau = \text{endo.}$	$\tau = \text{endo.}$	$\tau = \text{endo.}$	$\tau = 0.141$
Output, $Y_\star$	0.498	0.496	0.478	0.471	0.490
Agg. consumption, $C_\star$	0.387	0.380	0.374	0.371	0.378 [†]
Capital, $K_\star$	1.384	1.454	1.303	1.246	1.401
Tax rate, $\tau_\star$	0	0	0.041	0.079	0.041
After tax int., $(1 - \tau_\star)r_\star$	0.070	0.064	0.069	0.070	0.064
After tax wage, $(1 - \tau_\star)w_\star$	1.111	1.141	1.056	0.999	1.080
Saving rate, $1 - \frac{C_\star}{Y_\star}$	0.222	0.234	0.218	0.212	0.228
Labor supply, $l_\star$	0.3321	0.3223	0.3214	0.3217	0.3225
Cons. for $\epsilon = e$, $c_\star^e$	0.387	0.422	0.393	0.371	0.400
Cons. for $\epsilon = d$, $c_\star^{d,a}$	0.387	0.001	0.198	0.371	0.001
Utility $U = \theta U^e + \lambda U^d$	-1.234	-1.721	-1.261	-1.251	-1.838

[†] Here aggregate consumption is defined as private+public consumption.

Table A3: The steady state allocations in the complete markets setting (CM) and under incomplete markets (IM).

	(1) CM	(2) IM $\eta = 0$ $\tau = \text{endo.}$	(3) IM $\eta = 0.5$ $\tau = \text{endo.}$	(4) IM $\eta = 1$ $\tau = \text{endo.}$	(5) IM $\eta = 0$ $\tau = 0.141$
Output, $Y_\star$	0.502	0.493	0.473	0.467	0.487
Agg. consumption, $C_\star$	0.391	0.377	0.370	0.368	0.376 [†]
Capital, $K_\star$	1.395	1.444	1.292	1.236	1.391
Tax rate, $\tau_\star$	0	0	0.041	0.079	0.041
After tax int., $(1 - \tau_\star)r_\star$	0.070	0.064	0.069	0.070	0.064
After tax wage, $(1 - \tau_\star)w_\star$	1.111	1.141	1.056	0.999	1.080
Saving rate, $1 - \frac{C_\star}{Y_\star}$	0.222	0.234	0.218	0.212	0.228
Labor supply, $l_\star$	0.335	0.320	0.319	0.319	0.320
Cons. for $\epsilon = e$, $c_\star^e$	0.391	0.419	0.390	0.368	0.397
Cons. for $\epsilon = d$, $c_\star^{d,a}$	0.391	0.001	0.226	0.368	0.001
Utility $U = \theta U^e + \lambda U^d$	-1.369	-1.850	-1.399	-1.388	-1.975

[†] Here aggregate consumption is defined as private+public consumption.

Table A4: The steady state allocations in the complete markets setting (CM) and under incomplete markets (IM).

	(1)	(2)	(3)	(4)	(5)
	CM	IM	IM	IM	IM
		$\eta = 0$	$\eta = 0.5$	$\eta = 1$	$\eta = 0$
		$\tau = \text{endo.}$	$\tau = \text{endo.}$	$\tau = \text{endo.}$	$\tau = 0.141$
Output, $Y_\star$	0.497	0.477	0.458	0.451	0.472
Agg. consumption, $C_\star$	0.387	0.366	0.358	0.356	0.364 [†]
Capital, $K_\star$	1.382	1.398	1.249	1.196	1.348
Tax rate, $\tau_\star$	0	0	0.041	0.079	0.041
After tax int., $(1 - \tau_\star)r_\star$	0.070	0.064	0.069	0.07	0.064
After tax wage, $(1 - \tau_\star)w_\star$	1.111	1.141	1.056	0.999	1.080
Saving rate, $1 - \frac{C_\star}{Y_\star}$	0.222	0.234	0.218	0.212	0.228
Labor supply, $l_\star$	0.3316	0.3100	0.3081	0.3087	0.3103
Cons. for $\epsilon = e$, $c_\star^e$	0.387	0.406	0.377	0.356	0.385
Cons. for $\epsilon = d$, $c_\star^{d,a}$	0.387	0.001	0.190	0.356	0.001
Utility $U = \theta U^e + \lambda U^d$	-1.593	-2.097	-1.628	-1.617	-2.203

[†] Here aggregate consumption is defined as private+public consumption.

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