

Uncertainty, Productivity and Unemployment in the Great Recession*

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Abstract

The current 2007-2010 recession has witnessed two phenomena that standard search models of the labor market have difficulty reconciling: a large and persistent increase in unemployment and a sharp rise, above pre-recession levels, in labor productivity following a small initial drop. In addition, these observations were accompanied by a significant increase in the dispersion of firm growth rates. In this paper, I develop a tractable dynamic search model of heterogeneous firms with decreasing returns, in which I introduce uncertainty shocks. An increase in the idiosyncratic uncertainty faced by firms leads to higher unemployment, larger measured productivity, and more dispersion in firm growth rates. A combination of aggregate productivity and uncertainty shocks is able to explain many of the patterns observed in the ongoing recession, including the joint dynamics of unemployment and productivity. In addition to these findings, the model performs well at explaining business cycle statistics in ordinary times and is able to reproduce a range of observations at the establishment and cross-sectional levels, such as the employment behavior of establishments.

1 Introduction

The recession that followed the 2007-2008 collapse of the financial markets resulted in one of the deepest downturns in post-war U.S. labor markets. While GDP contracted by up to 6.8% in the fourth quarter of 2008¹, the unemployment rate grew from 5% in January 2008 to 10.1% in October 2009 according to the Bureau of Labor Statistics. At the same time, labor productivity experienced a small drop, but quickly recovered to a level above its pre-recession trend. While one would expect firms to start hiring again at such productivity levels, the unemployment rate has remained persistently high.

To summarize these facts, I construct detrended time-series of output, unemployment and productivity. The trend is computed with an HP-filter with parameter 1600, as is usually done with quarterly data. Figure 1 presents the log of these series, centered at the peak (or trough) preceding the recession. Graph (a) and (b) show the depth of the recession

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¹Source: BEA. Percent change in real GDP from preceding period in annual rate.

in output and unemployment. Figure (c) presents two different measures of productivity: output per person and TFP. The two time-series display an initial drop of about 2 to 3% in the first quarter of 2009, followed by a quick recovery during the rest of the year to a level of 2% above their pre-recession trend in 2010.

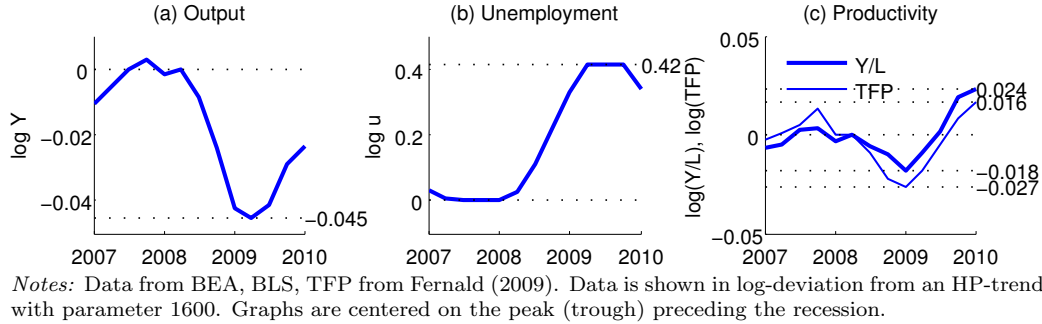


Figure 1: Output, Unemployment and Productivity in the 2007-2010 Recession

Apart from their importance for welfare, these patterns offer a challenge to search-and-matching models of the labor market. First of all, it has been noticed by several authors such as Shimer (2005) and Hall (2005) that standard calibrations of models with matching frictions cannot explain large fluctuations in unemployment. Accounting for the recent increase in the unemployment rate would require a counterfactually large fall of the exogenous component of aggregate productivity, unless one follows the calibration strategy suggested in Hagedorn and Manovskii (2008). But most importantly, a large part of the literature on search and business cycles is based on one underlying shock process in aggregate productivity. In such an environment, recessions are periods of technological regress: it is then difficult to explain the coexistence of high productivity and persistent, high unemployment. To illustrate these results, I present in figure 2 the responses of the Mortensen-Pissarides model in the Shimer (2005) and Hagedorn-Manovskii (2008) calibrations when the aggregate productivity process is calibrated on the empirical time-series of output per person². As mentioned before, the Shimer calibration predicts too little variation in unemployment. The Hagedorn-Manovskii calibration being designed to generate a higher elasticity of unemployment to productivity produces a much larger response (about 50% of the observed increase). However, because of the large recent increase in productivity, the two calibrations predict a counterfactual fall in unemployment from the middle of 2009 to a level between 10% and 20% below its pre-recession trend, as well as a counterfactually large recovery in output.

The coexistence of high productivity and high unemployment can suggest that a different kind of shock may be needed to explain the patterns observed in the current recession. This paper proposes uncertainty shocks as one possible explanation. Bloom et al. (2009) indeed show evidence that different measures of uncertainty have largely increased during the recession. A measure of the increase in uncertainty is the evolution of the dispersion of firm sales growth rates from Compustat over 2006Q1-2009Q3, which I present in Figure 3. A large increase in the growth rate dispersion can be observed at the end of 2008. This rise in dispersion can be observed at different levels of aggregation (establishment, firm and industry levels), and in the establishment-level employment growth rates as well.

²A comparison with Mortensen and Pissarides (1994) with uncertainty shocks can be found in the appendix.

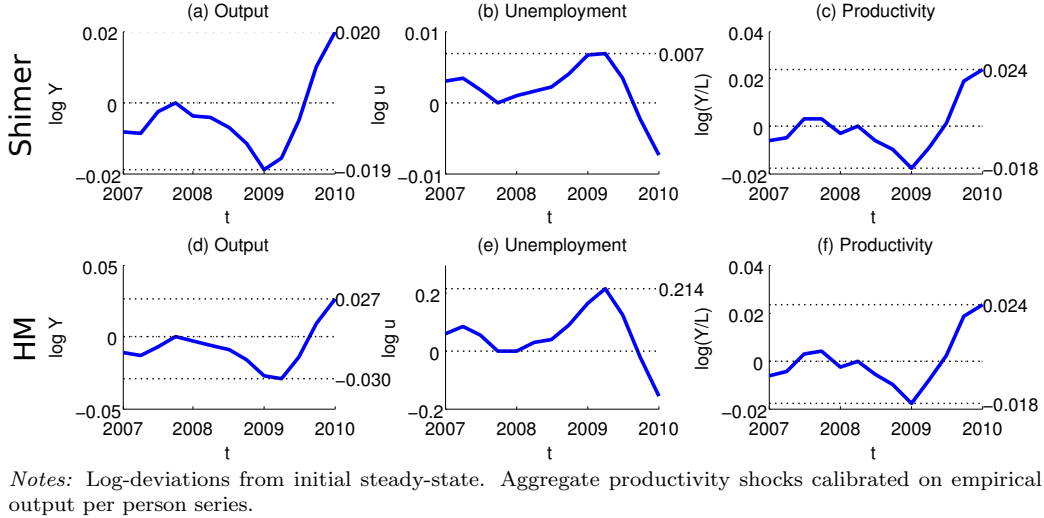
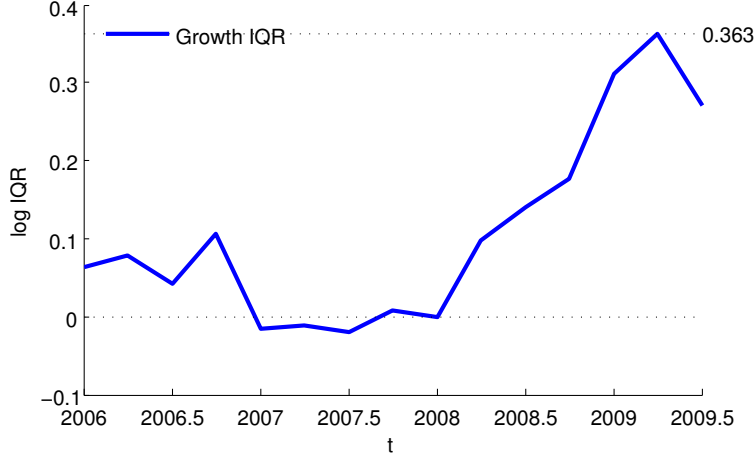


Figure 2: Predicted paths for output and unemployment in the Shimer (05) and Hagedorn-Manovskii (08) calibrations

Uncertainty shocks can have a significant impact on the economy. In an economy with labor market frictions, adjustments in a firm's employment are costly. Therefore, in the face of greater uncertainty about the future state of the economy or future sales prospects, firms may decide to delay or reduce their hirings. Unemployment may rise as a consequence. Greater uncertainty may also imply an increased probability for firms to get a bad shock. In that case, one would expect to observe a rise in the number of firm exits, resulting in more job destructions and a higher inflow of laid off workers into unemployment. Furthermore, it may lead to a rise in productivity as firms reduce the scale of their operation in the presence of decreasing returns in labor. A second channel due to the selection of highly productive firms and the exit of unproductive ones may further increase the aggregate measure of productivity. This mechanism depicts how uncertainty may explain a high level of unemployment along a contraction in output and a high level of productivity.

To investigate whether this mechanism can help models with labor market imperfections to account for the current crisis, I develop a dynamically tractable search model of firms with decreasing returns in which I allow for productivity and idiosyncratic uncertainty shocks. I calibrate a version of the model with aggregate productivity shocks only and show that the model can reproduce comovements of aggregate variables along the business cycle in ordinary times. Next, I compare the responses of the economy to productivity and uncertainty shocks. When idiosyncratic uncertainty increases, the model predicts a substantial rise in unemployment, an increase in output per person and TFP, and a larger dispersion of firm growth rates. This does not suffice, however, to account for the contraction in output and initial drop in productivity. I then turn to show that the combination of a productivity and an uncertainty shock is able to account for many of the patterns observed in output, unemployment, productivity and a range of labor market flows since 2007.

A second contribution of the paper is to develop a model of firms with decreasing returns and search frictions in which dynamics are fully tractable. Dynamic models featuring heterogeneous firms indeed raise a certain number of technical issues that make them difficult to solve, even numerically. When search frictions are added, the task of solving the model becomes even more complex. To address this issue, I use the structure of labor mar-



Notes: Quarterly sales data from Compustat. Interquartile range of sales growth rates presented in log-deviation from trough before recession (2008Q1). Growth rates are computed with $g_{i,t} = \frac{s_{i,t} - s_{i,t-4}}{1/2(s_{i,t} + s_{i,t-4})}$. See data appendix for details.

Figure 3: Sales growth dispersion in the 2007-2010 Recession

kets with directed search developed by Menzio and Shi (2008, 2009) in order to exploit the convenient property of *block recursivity*. I show that the property continues to hold with the introduction of multiworker firms with decreasing returns under some conditions. This property allows me to easily characterize firm and aggregate dynamics in an exact way. The model features realistic firm dynamics and a rich description of labor markets flows. In the model, heterogeneous firms can endogenously expand/contract, enter/exit over the business cycle. Workers search for new job opportunities both on- and off-the-job, which allows me to distinguish quits from layoffs. I show in the last section of the paper, that the model is able to reproduce a range of facts at the establishment and cross-sectional levels. First, it matches a number of features of the micro-level employment policies of establishments and their cross-sectional growth rates as reported in Davis et al. (2006) and Davis et al. (2010). Second, turning to the evolution of the cross-section of firms over the business cycle, I find that the model is able to explain the recent finding that large firms are more cyclically sensitive as reported by Moscarini and Postel-Vinay (2009). Finally, the model has predictions in terms of wages. The presence of on-the-job search leads to a substantial wage dispersion that lies within the range of empirical estimates. The model also produces a realistic size-wage differential.

This paper is related to several strands in the literature. It first relates to the growing literature suggesting uncertainty shocks as a driving force of the business cycles. Bloom (2009) and Bloom et al. (2009) study the effects of time-varying uncertainty in models of firms with non-convex adjustment costs and show that uncertainty can lead to large drops in economic activity. Arellano et al. (2010) study how financial frictions and uncertainty shocks combine to explain a number of facts related to the current crisis and conclude that uncertainty shocks can explain a substantial fraction of fluctuations in output as well as large movements in the labor wedge. In a slightly different framework, Gilchrist et al. (2010) analyse how time-varying uncertainty and frictions in financial markets in a general equilibrium setting can produce aggregate fluctuations observationally equivalent to TFP-driven cycles. Bachmann and Bayer (2009) shows that realistically calibrated uncertainty shocks do not alter significantly the dynamics of an otherwise standard heterogeneous-firm

RBC model. Bachmann et al. (2010) uses business survey data on confidence to analyse the impact of uncertainty on aggregate variables, and finds little evidence of the “wait-and-see” effect highlighted in the literature. From a different perspective, my paper investigates the role of labor market imperfections in explaining the patterns observed in the data, and shows that uncertainty shocks in a model with search frictions can help us account for a large fraction of the recent trends in unemployment, productivity and a range of labor market variables.

This paper also relates to the recent strand in the literature seeking to introduce multiworker firms with decreasing returns to scale in search-and-matching models. Acemoglu and Hawkins (2010) and Elsby and Michaels (2010) extend the Mortensen-Pissarides model to firms with decreasing returns and Stole and Zwiebel bargaining. Acemoglu and Hawkins (2010) emphasizes the time-consuming aspect of matching to generate persistence in unemployment. Elsby and Michaels (2010) shows that the gap between average and marginal products of labor resulting from the decreasing returns allows a reasonable calibration of the model to generate large fluctuations in unemployment and vacancies. However, the dynamics are sometimes intractable and their resolution may require the use of approximation methods. My paper explores another approach in which dynamics are easily solvable. This tractability enables me to enrich the model further by adding job-to-job transitions and endogenous firm entry/exit, which play an important role in business cycles. Kaas and Kircher (2010) develops a model with features similar to the ones presented in this paper. Addressing the question of efficiency of search models with large firms, they show that dynamic contracts in a directed search setting offer a nice alternative to Stole and Zwiebel bargaining as they offer an efficient benchmark. Based on a slightly different market structure, my model allows for job-to-job transitions, which enables me to distinguish between quits and layoffs. As a result, my model is able to reproduce a number of additional observations at the micro-level such as the employment behavior of establishment, as well as a realistic cross-sectional wage dispersion. Efficiency continues to hold in my model even in the presence of on-the-job search without commitment from workers.

2 Model

In order to study the impact of idiosyncratic uncertainty on unemployment and productivity, I build a dynamic search model of heterogeneous firms with decreasing returns. Extending standard search models along this dimension is not trivial and raises a number of difficulties. Allowing firms to differ in size amounts to introducing a new layer of heterogeneity. This significantly complicates the resolution of the model as it raises the dimensionality of the problem and renders the aggregation of the variety of firms’ and workers’ behaviors more difficult. In this paper, I explore an approach initially suggested by Menzio and Shi (2009). Switching the labor market structure from random search as in Mortensen and Pissarides (1994) to directed search can greatly simplify the resolution of the dynamics. Indeed, the combination of a free-entry condition and a directed search structure delivers the property of *block recursivity*, under which an individual firm’s problem can be solved independently from the distribution of employment across other firms. As I explain below, this convenient property keeps the dimensionality of the problem tractable and allows for an exact characterization of the model dynamics. Let me now introduce the model.

2.1 Population and technology

Time is discrete. The economy is populated by a continuum of equally productive workers of mass 1. An unrestricted mass of firms can potentially enter the economy. Firms and workers are risk neutral and share the same discount rate β . Firms all produce an identical homogeneous good. All are subject to the same time-varying aggregate productivity y that takes a finite number of values in $\mathcal{Y} = \{y_{min} < \dots < y_{max}\}$ and follows a discrete Markov process with transition matrix $\pi_y(y_t, y_{t+1})$. In addition, firms differ in their idiosyncratic productivity z , that follows the finite Markov process $\pi_z(z_t, z_{t+1})$ in $\mathcal{Z} = \{z_{min} < \dots < z_{max}\}$ with stationary distribution g_z . A firm with a mass of n workers operates the production technology

$$e^{y+z}F(n)$$

where F is an increasing concave production function with $F(0) = 0$. Upon entry, firms must pay a sunk entry cost k_e . Also, firms must pay an operating cost k_f every period in order to use the production technology.

2.2 Labor market

Search is directed on the worker and firm sides. Firms post dynamic contracts that guarantee a certain utility x to the workers. Posting is costly and there is a cost c per vacancy. The labor markets are organized in a continuum of submarkets indexed by the utility $x \in [\underline{x}, \bar{x}]$ promised by the firm to the workers. Workers can direct their search and choose in which submarket to look for a job. Each submarket is characterized by its tightness $\theta(x) = v(x)/u(x)$, where $v(x)$ stands for the number of vacancies posted on submarket x and $u(x)$ the corresponding number of searching workers. On a submarket with tightness θ , workers find jobs with probability $p(\theta)$, while firms find candidates with probability $q(\theta) = p(\theta)/\theta$. I allow workers to search on-the-job, but at a reduced efficiency. Denoting λ the relative search efficiency of the employed compared to the unemployed, the job finding probability of employed workers is $\lambda p(\theta)$.

Firms are assumed to post a mass of vacancies so that a law of large number applies and there is no uncertainty on the number of worker they recruit. Figure 4 describes the functioning of the labor market. Notice that the market tightness $\theta(x)$ is the equilibrium variable that adjusts to guarantee that markets clear.

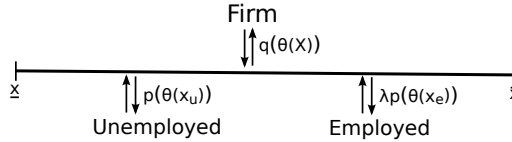


Figure 4: Directed search

2.3 Contracting and timing

The introduction of multiworker firms into search models raises important questions about the wage determination process. Most of the recent literature with large firms has used the multilateral bargaining procedure proposed by Stole and Zwiebel (1996). This wage determination process nicely extends the standard Nash bargaining procedure to firms with multiple workers, but loses its property of efficiency. In this paper, I opt for the less explored alternative of dynamic contracts. On top of lending themselves very naturally to a directed

search setting, dynamic contracts lead to an efficient outcome, which considerably simplifies the resolution of the model as I can focus on a planner's allocation.

To simplify the exposition, assume for now that contracts are complete, and that there is full commitment from both worker and firm sides. In particular, this means that a contract can specify any possible variable or action taken by one party in every possible state of the economy. A contract specifies $\{w_{t+s}, \tau_{t+s}, x_{t+s}, d_{t+s}\}_{s=0}^{\infty}$, where w is a wage, τ a firing probability, x the submarket that the worker visits on-the-job, and d an exit dummy for the firm. Each element is contingent on the current state of the firm and the economy. I maintain the assumptions of completeness and full commitment throughout this section, but will however show in section 3 that completeness and commitment from the worker side can be relaxed, as the optimal allocation can be implemented by a contract that satisfies the worker's participation and incentive constraints.

The timing is illustrated in Figure 5. At date t , the shock to aggregate productivity y is revealed. Potentially entering firms decide whether or not to enter. Immediately after, each firm learns its idiosyncratic productivity z . Firms then decide whether to exit ($d_t = 1$) or stay. Separations (layoffs) at probability τ_t then take place. Search and matching between new and incumbent firms on one side and unemployed/employed workers on the other side follows, before production takes place.

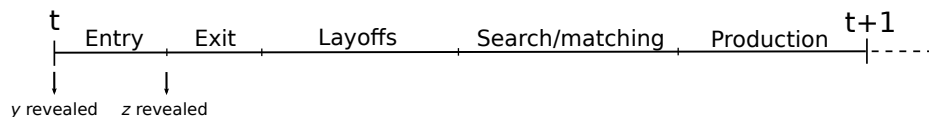


Figure 5: Timing

2.4 Worker's problem

As mentioned earlier, the introduction of firms with decreasing returns in a search model raises the dimensionality of workers' and firms' problems. In principle, the aggregate state variables should include the current aggregate productivity y , as well as the distribution of employment across firms $g(z, n)$. The latter is particularly critical as one must keep track of an infinite-dimensional object in the state space, making the resolution of the dynamics intractable. Fortunately, the structure of the model will give rise to the property of *block recursivity*, in which firms' and workers' problems are independent of distribution g .³ In that case, the aggregate state space reduces to productivity y . I focus the analysis on block-recursive equilibria from now on and will examine in the next section under what conditions such a property arises.

Let me now introduce the worker's problem. In what follows, all value functions are expressed at the time just before production takes place. As a convention, hatted variables denote next period's values.

While unemployed, workers enjoy a utility b from home production or leisure. Unemployed workers choose to visit the submarket $\hat{x}_u$ that offers the best balance between

³Another possibility explored in the literature is to use limited-information techniques as introduced by Krusell and Smith (1998) in which distribution g is approximated by a series of moments. Such methods render the analysis tractable, but are still subject to limitations. First, the number of moments needed to approximate the employment distribution may still be large and require intensive computations. Second, it is sometimes difficult to evaluate the accuracy of the approximation.

the utility they can get and the probability at which they can find a job. They solve the following problem:

$$\mathbf{U}(y) = \max_{\hat{x}_u(\hat{y})} b + \beta \mathbf{E}_{\hat{y}} \left\{ (1 - p(\theta(\hat{x}_u, \hat{y}))) \mathbf{U}(\hat{y}) + p(\theta(\hat{x}_u)) \hat{x}_u \right\}$$

Abusing notation slightly, denote $p(x) \equiv p(\theta(x, y))$. The problem can be rewritten as:

$$\mathbf{U}(y) = b + \beta \mathbf{E}_{\hat{y}} \left\{ \mathbf{U}(\hat{y}) + \max_{\hat{x}_u(\hat{y})} p(\hat{x}_u) (\hat{x}_u - \mathbf{U}(\hat{y})) \right\} \quad (1)$$

Turning to employed workers, we will now write the contracts in their recursive form. In each period, a contract specifies $\{w, \hat{\tau}, \hat{x}, \hat{d}, \hat{W}\}$, where the hatted variables implicitly depend on next period's state $(\hat{y}, \hat{z})$. $\hat{W}$ stands for the promised utility that the firm guarantees to the worker in the next period. Workers valuation is given by:

$$\begin{aligned} \mathbf{W}(y, z; \{w, \hat{\tau}, \hat{x}, \hat{d}, \hat{W}\}) = & w + \beta \mathbf{E}_{\hat{y}, \hat{z}} \left\{ \hat{d} \mathbf{U}(\hat{y}) + (1 - \hat{d}) \left(\hat{\tau} \mathbf{U}(\hat{y}) + (1 - \hat{\tau}) \lambda p(\hat{x}) \hat{x} \right. \right. \\ & \left. \left. + (1 - \hat{\tau})(1 - \lambda p(\hat{x})) \hat{W} \right) \right\} \end{aligned} \quad (2)$$

The first term in the expectation corresponds to the utility of going back to unemployment if the firm exits. The second is the utility he receives when laid-off. The third is the utility he enjoys if he manages to find a job in submarket $\hat{x}$. The last term is the continuation utility offered by the firm.

2.5 Firm's problem

Firms hire a continuum of workers with potentially different contracts. Let $\varphi(W)$ denote the cumulative distribution of promised utilities across workers within a given firm. The total mass of workers is $n = \int d\varphi$. The contracts offered by the firm

$$\{w(W), \hat{\tau}(\hat{y}, \hat{z}, W), \hat{x}(\hat{y}, \hat{z}, W), \hat{d}(\hat{y}, \hat{z}, W), \hat{W}(\hat{y}, \hat{z}, W)\}$$

are functions of the promised utility W . To simplify notation, the dependence of the contracts on $(\hat{y}, \hat{z})$ and W should be considered implicit in what follows.

Firms face the following problem:

$$\begin{aligned} \mathbf{J}(y, z, n, \varphi) = & \max_{\substack{w, \hat{\tau}, \hat{x}, \hat{d}, \\ \hat{W}, \hat{v}, \hat{X}}} e^{y+z} F(n) - k_f - \int w d\varphi + \beta \mathbf{E}_{\hat{y}, \hat{z}} \left\{ (1 - \hat{d})(-c\hat{v} + \mathbf{J}(\hat{y}, \hat{z}, \hat{n}, \hat{\varphi})) \right\} \end{aligned} \quad (3)$$

$$\text{s.t. } \forall W, \quad \mathbf{W}(y, z; \{w, \hat{\tau}, \hat{x}, \hat{d}, \hat{W}\}(W)) \geq W \quad (3.a)$$

$$\hat{n} = \int (1 - \hat{\tau})(1 - \lambda p(\hat{x})) d\varphi + \hat{v} q(\hat{X}) \quad (3.b)$$

$$\forall W, \hat{\varphi}(W) = \int_{W' | \hat{W}(W') \leq W} (1 - \hat{\tau})(1 - \lambda p(\hat{x})) d\varphi(W') + \hat{v} q(\hat{X}) \mathbb{I}(W \geq \hat{X}) \quad (3.c)$$

where $\hat{v}$ is the number of vacancies posted, $\hat{X}$ is the submarket where the firm searches for new hires⁴, and $\mathbb{I}$ is an indicator function. In the current period, the firm earns revenue

⁴I implicitly impose here that a firm can only hire in one single market in order to simplify the notation. We will see later that this is no restriction as firms are indifferent between all active submarkets.

from production minus the fixed cost k_f and wage bill $\int w d\varphi$. Next period, the firm stays ($\hat{d} = 0$), or exits ($\hat{d} = 1$), in which case it receives 0. In that case, it pays the vacancy posting cost $c\hat{v}$ for its newly hired workers and goes to the next period. Constraint (3.a) is the *promise-keeping* constraint, ensuring that the firm gives the promised utility to each of its workers. Constraints (3.b) and (3.c) are the laws of motion for the total mass of workers and the corresponding distribution of promised utilities, including newly employed workers on market $\hat{X}$. Notice that for a contract to be optimal, the promise-keeping constraint (3.a) must bind: it is always possible for the firm to reduce the current wage to increase its own profit until the constraint binds.

2.6 Joint surplus maximization

The firm's problem as stated above is a complicated mathematical problem. It is however possible to greatly simplify it thanks to the contract structure. Indeed, the chosen contracts always maximize the joint surplus of a firm and its workers.

Define the following Bellman equation for the joint surplus of a firm and its current workers by:

$$\begin{aligned} \mathbf{V}(y, z, n, \varphi) = & \max_{w, \hat{\tau}, \hat{x}, \hat{d}, \hat{W}, \hat{v}, \hat{X}} e^{y+z} F(n) - k_f + \beta E_{\hat{y}, \hat{z}} \left\{ n \hat{d} \mathbf{U}(\hat{y}) + (1 - \hat{d}) \left(\mathbf{U}(\hat{y}) \int \hat{\tau} d\varphi \right. \right. \\ & \left. \left. + \int (1 - \hat{\tau}) \lambda p(\hat{x}) \hat{x} d\varphi + \mathbf{V}(\hat{y}, \hat{z}, \hat{n}, \hat{\varphi}) - (c + q(\hat{X}) \hat{X}) \hat{v} \right) \right\} \\ \text{s.t. } & \begin{cases} \hat{n} = \int (1 - \hat{\tau})(1 - \lambda p(\hat{x}) d\varphi + \hat{v} q(\hat{X}) \\ \hat{\varphi}(W) = \int_{W' | \hat{W}(W') \leq W} (1 - \lambda p(\hat{x}))(1 - \hat{\tau}) d\varphi(W') + \hat{v} q(\hat{X}) \mathbb{I}(W \geq \hat{X}). \end{cases} \end{aligned} \quad (4)$$

The first term in the joint surplus V corresponds to the present production minus fixed cost. The expectation term includes tomorrow's utility for workers going back to unemployment in case of exit, the utility of laid-off workers, and the utility of workers leaving for another job. The last two terms are next period's surplus net of the utility promised to new hires and vacancy costs. It should be noted that wages cancel out in this expression as they amount to internal transfers between the firm and its employees. Also, the emphasis being on finding efficient allocations, contracts and the *promise-keeping* constraint have disappeared. The following proposition states the equivalence between the firm's problem and the joint surplus maximization.

Proposition 1. *The firm's problem and joint surplus maximization are equivalent in the following sense:*

- (i) *the joint surplus $\mathbf{V}(y, z, n, \varphi) \equiv \mathbf{J}(y, z, n, \varphi) + \int W d\varphi(W)$ solves Bellman equation (4),*
- (ii) *an optimal policy for the firm $\{w^*, \hat{\tau}^*, \hat{x}^*, \hat{d}^*, \hat{W}^*, \hat{v}^*, \hat{X}^*\}$ also maximizes the joint surplus,*
- (iii) *if $\{\hat{\tau}^S, \hat{x}^S, \hat{d}^S, \hat{W}^S, \hat{v}^S, \hat{X}^S\}$ maximizes the joint surplus, there exists a unique wage contract w^S such that $\{w^S, \hat{\tau}^S, \hat{x}^S, \hat{d}^S, \hat{W}^S, \hat{v}^S, \hat{X}^S\}$ solves the firm's problem.*

Proposition 1 tells us that it is possible to solve first for the real variables ($\hat{\tau}$, $\hat{x}$, $\hat{d}$, $\hat{v}$, $\hat{X}$) by finding the efficient allocation that maximizes the joint surplus of the firm and its workers. As a consequence, there is no need to solve for the particular contracts that implement the allocation if we are only interested in worker and job flows. The promise-keeping constraint can be ignored for that matter. From (iii), it is indeed always possible

to adjust the wage so that the promise-keeping constraint is satisfied and the firm's profits maximized.

In its current formulation, the problem can be further simplified. Notice that since the promise-keeping constraint has disappeared, the joint surplus does not depend on distribution φ . In equation (4), φ only appears when we sum over workers. The exact nature of the distribution has no effect on the surplus. In particular, it is possible to find a solution to Bellman equation (4) that does not depend on φ . I will focus from now on equilibria in which $\mathbf{V}(y, z, n)$ does not depend on φ . I will show in the next section that such equilibria are the only solution when a block-recursive competitive equilibrium with positive entry exists. In that case, the firm's profit is equal to

$$\mathbf{J}(y, z, n, \varphi) = \mathbf{V}(y, z, n) - \int W d\varphi.$$

2.7 Free entry

Every period after the aggregate shock y is realized, new firms are allowed to enter the economy. Firms first decide whether or not to enter. Upon entry, firms must pay an entry cost k_e , which allows them to choose a submarket X to post their vacancies. After entry, an idiosyncratic productivity z is drawn from distribution g_z . Depending of the outcome, firms then decide to exit, in which case they get 0, or to stay and choose a number of vacancies $v(y, z)$ to post. The free-entry condition can be written as: $\forall X$,

$$\begin{aligned} k_e &\geq \max_{v(z)} E_{g_z} \left(\mathbf{J}(y, z, vq(\theta(X, y)), \varphi) - cv \right)^+ \\ \Leftrightarrow k_e &\geq \max_{v(z)} E_{g_z} \left(\mathbf{V}(y, z, vq(\theta(X, y))) - vq(\theta(X, y))X - cv \right)^+ \\ \Leftrightarrow k_e &\geq \max_{n(z)} E_{g_z} \left(\mathbf{V}(y, z, n) - (c/q(\theta(X, y)) + X)n \right)^+ \end{aligned} \quad (5)$$

and the corresponding complementary slackness condition as

$$\forall X, \quad \theta(X) \left[\max_{v(z)} E_{g_z} \left(\mathbf{V}(y, z, n) - (c/q(\theta(X, y)) + X)n \right)^+ - k_e \right] = 0 \quad (6)$$

where $n = vq(\theta(X, y))$ is the number of hired workers, and the $(\cdot)^+$ notation stands for $\max(\cdot, 0)$.

Condition (5) simply states that new firms enter the economy as long as expected profits exceed the entry cost k_e , driving these profits down to k_e . Condition (6) is essential as it pins down the equilibrium market tightness θ for all submarkets. It imposes that if the submarket is active, $\theta > 0$, then expected profits equal k_e . If the submarket is inactive, $\theta = 0$, the entry cost exceeds anticipated profits.

The free-entry condition is crucial to guarantee the existence of a block-recursive equilibrium. Indeed, as was said earlier, the distribution of firms $g_t(n, z)$ with employment n and productivity z could in principle be part of the state-space of agents as they need to forecast the tightness of the different submarkets. This could make the computation of the equilibrium intractable because of high dimensionality. However, the free-entry condition

(6) makes the equilibrium labor market tightness $\theta(x, y)$ a function of the joint surplus. If the joint surplus V does not depend on distribution g , then θ does not either. Similarly, since distribution g only enters the joint surplus maximization problem through its potential effect on θ , it is possible to find a *block-recursive* solution in which neither θ , nor V depend on the distribution of employment across firms. This is what makes positive entry a key condition to satisfy for a block-recursive equilibrium to exist. As Proposition 2 will show in the next section, there always exists a *block-recursive* solution to equations (1)-(6) in which the worker-firm joint surplus does not depend on the employment distribution g .

2.8 Unemployment and firm distribution dynamics

Although the firm behavior does not depend on the aggregate distribution of employment in a block-recursive equilibrium, the unemployment level and distribution $g_t(z, n)$ follow their own history-dependent dynamics that can easily be computed recursively. Their laws of motion are given by

$$u_{t+1} = u_t(1 - p(x_{u,t+1})) + \int n_t(d_{t+1} + (1 - d_{t+1})\tau_{t+1})g_t(z_t, n_t)\pi_z(z_t, z_{t+1})dz_t dz_{t+1}dn_t \quad (7)$$

and

$$g_{t+1}(z_{t+1}, n_{t+1}) = \int_{n_{t+1}=\hat{n}(y_{t+1}, z_{t+1}, n_t)} (1 - d_{t+1})g_t(z_t, n_t)\pi_z(z_t, z_{t+1})dz_t dz_{t+1}dn_t + \underbrace{h(y_{t+1}, z_{t+1}, n_{t+1}, g_t)}_{\text{new entrants}} \quad (8)$$

where the number of new entrants h can be computed from period to period as the residual that makes the equilibrium market tightness θ equal to the ratio of vacancies over searching workers.

3 Properties of the equilibrium

I characterize in this section a few properties of the competitive equilibrium. I first establish the existence of a block-recursive solution to equations (1)-(6) under some weak conditions, which provides us with a well-defined block-recursive competitive equilibrium as long as it implies positive entry in every period. I next show that when they exist, such equilibria coincide with the unique efficient equilibrium of the economy. I then describe some features of firms' optimal behavior such as the existence of a band of inactivity, a characteristic that the model shares with the adjustment cost literature. In the last part, I relax the assumption of contract completeness and commitment from workers to show that the optimal allocation can be decentralized with contracts that satisfy workers' participation and incentive constraints.

3.1 Existence of a block-recursive solution

A block-recursive equilibrium can be defined as follows.

Definition 1. *A block-recursive competitive equilibrium of this economy is a set of value functions $\mathbf{V}(y, z, n)$, $\mathbf{U}(y)$, $\mathbf{J}(y, z, n, \varphi)$, $\mathbf{W}(y, z; \{w, \hat{\tau}, \hat{x}, \hat{d}, \hat{W}\})$, optimal policy functions $\{w^*, \hat{\tau}^*, \hat{x}^*, \hat{d}^*, \hat{W}^*, \hat{v}^*, \hat{X}^*\}$, where w^* depends on (y, z, W) and all the others on $(y, z, \hat{y}, \hat{z}, W)$,*

and an equilibrium labor market tightness $\theta^*(x, y)$ such that equation (1)-(6) are satisfied and equations (7)-(8) imply a positive number of entrants in every period.

To prove the existence of a block-recursive solution, I make the following assumptions:

Assumption 1. F is bi-Lipschitz continuous, i.e there exists $(\underline{\alpha}_F, \bar{\alpha}_F)$ such that

$$\forall(n_1, n_2), \quad \underline{\alpha}_F |n_2 - n_1| \leq |F(n_2) - F(n_1)| \leq \bar{\alpha}_F |n_2 - n_1|$$

Assumption 2. (i) p, q are twice continuously differentiable, (ii) p is strictly increasing, concave; q is strictly decreasing and convex, (iii) $p(0) = 0$, $q(0) = 1$, (iv) $p \circ q^{-1}$ is strictly concave⁵.

To proceed with the proof, I show that there exists a common solution to the joint surplus maximization, free-entry condition and unemployed workers' problem. This establishes the behavior of variables $(\hat{\tau}, \hat{x}, \hat{d}, \hat{v}, \hat{X})$ without the need to describe the set of contracts that implement the efficient allocation. I will show in subsections 3.3 and 3.4 how these contracts can be easily recovered from the firm's optimal behavior. Let us first define the set where our optimal surplus $\mathbf{V}$ lies and introduce our last assumption on parameters. Let n be the firm size and $\bar{n}$ an upper-bound chosen sufficiently large so that it does not constrain the equilibrium.

Definition 2. Let $\mathcal{V}$ be the set of value functions $V : (y, z, n) \in \mathcal{Y} \times \mathcal{Z} \times [0, \bar{n}] \rightarrow \mathbb{R}$ strictly increasing in n , satisfying $\forall y, \mathbb{E}_{g_z} V(y, z, 0)^+ \leq \beta k_e$ ⁶, bounded in $[\underline{V}, \bar{V}]$, and bi-Lipschitz continuous in n such that

$$\forall V \in \mathcal{V}, \forall (y, z), \forall n_2 \geq n_1, \quad \underline{\alpha}_V (n_2 - n_1) \leq V(y, z, n_2) - V(y, z, n_1) \leq \bar{\alpha}_V (n_2 - n_1),$$

with the constants $\underline{\alpha}_V$, $\bar{\alpha}_V$, $\underline{V}$ and $\bar{V}$ defined in the appendix.

Assumption 3. Assume $\bar{n} > \underline{\alpha}_V^{-1}(k_e + k_f)$.

Assumption 3 is a sufficient condition on parameters that guarantees that there is always a solution to the free-entry problem. We can now establish the existence of a solution to the free-entry problem.

Lemma 1. Under Assumptions 1-3, for $V \in \mathcal{V}$, $y \in \mathcal{Y}$, the free-entry problem (5)-(6) admits a solution with equality. The hiring cost per worker $c/q(\theta(X, y)) + X$ is equalized across active submarkets. The solution can be summarized by a unique hiring cost $\kappa^V(y)$ and an optimal level of hiring $n_e^V(y, z)$ such that

- (i) Submarket X is active $\Leftrightarrow \theta^V(X, y) > 0 \Leftrightarrow c/q(\theta(X, y)) + X = \kappa^V(y)$,
- (ii) $k_e = \max_{n(y, z)} \mathbb{E}_{g_z} [V(y, z, n) - \kappa^V n]^+ = \mathbb{E}_{g_z} [V(y, z, n_e^V) - \kappa^V n_e^V]^+$.
- (iii) $\theta^V(X, y) = \begin{cases} q^{-1} \left(\frac{c}{\kappa^V(y) - X} \right), & \text{for } \underline{x} \leq X \leq \kappa^V(y) - c, \\ 0, & \text{for } X \geq \kappa^V(y) - c. \end{cases}$

⁵(iv) is a regularity condition ensuring that workers' problem is well defined and concave.

⁶This condition guarantees the existence of a solution to the free-entry problem.

This lemma tells us that there is a continuum of active submarkets for $x \in [\underline{x}, \kappa^V(y) - c]$ in which firms can enter. It is important to notice that the free-entry problem amounts to an indifference condition from the point of view of firms. This results from the fact that there is a potentially infinite number of entrants that can arbitrarily choose between the different submarkets. As a consequence, the hiring cost per worker is identical across active submarkets. It can be decomposed between the expected utility promised to the worker X and the total vacancy cost to attract one worker $c/q(\theta(X, y))$. Figure 6 displays how the market tightness evolves with X . We see that markets that offer a low utility to the workers have a higher tightness: as it is less costly for them, firms keep entering these markets until the probability to find a worker becomes so low that they would actually prefer to make better offers. Another point worth noticing is that the hiring cost is the same for both incumbent and entering firms: the indifference result applies to all.

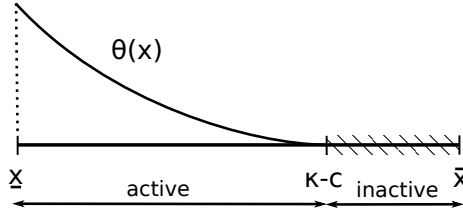


Figure 6: Equilibrium market tightness

We can now turn to the main proposition of this section that establishes the existence of a block-recursive solution.

Proposition 2. *Under Assumptions 1-3, there exists a block-recursive solution to equations (1)-(6), i.e the mapping $T : \mathcal{V} \rightarrow \mathcal{V}$ such that*

$$TV(y, z, n) = \max_{\hat{\tau}, \hat{x}, \hat{d}, \hat{W}, \hat{n}_i, \hat{X}} e^{y+z} F(n) - k_f + \beta E_{\hat{y}, \hat{z}} \left\{ n \hat{d} \mathbf{U}^V(\hat{y}) + (1 - \hat{d}) \left(\mathbf{U}^V(\hat{y}) \int \hat{\tau} dj + \int (1 - \hat{\tau}) \lambda p^V(\hat{x}) \hat{x} dj - \kappa^V(\hat{y}) \hat{n}_i + V(\hat{y}, \hat{z}, \hat{n}) \right) \right\}$$

with $\hat{n} = \int (1 - \hat{\tau})(1 - \lambda p^V(\hat{x})) dj + \hat{n}_i$, (θ^V, κ^V) solution to the free-entry problem (5)-(6) and $\mathbf{U}^V$ solution to (1), admits a fixed point. The existence of valuations $\mathbf{J}^V$, $\mathbf{W}^V$ and optimal policies $\{\hat{\tau}^V, \hat{x}^V, \hat{d}^V, \hat{W}^V, \hat{n}_i^V, \hat{X}^V\}$ follows.

Notice that I have substituted without loss of generality the arbitrary distribution φ with a summation on a uniform distribution j for $j \in [0, n]$ as it does not affect the equilibrium.

Proposition 2 shows the existence of a joint solution to the surplus maximization, free-entry and unemployed workers' problems. To be a well-defined competitive block-recursive equilibrium, one must still check one additional feasibility condition: since we have imposed a free-entry condition, we must check that the required number of entrants is always non-negative. Unfortunately, this condition cannot be put into a block-recursive form as the number of entrants depends on the distribution of employment across firms. We can, however, easily check in the simulations whether this condition is satisfied. This provides us with a constructive way to find candidates for block-recursive equilibria.

3.2 Uniqueness and efficiency

To study efficiency, I now introduce the planning problem of this economy. In particular, I show what conditions guarantee the uniqueness and efficiency of a block-recursive equilibrium. Let $h_t(y^t)$ be the total number of new entrants every period. The planner's objective is to maximize the total welfare in the economy

$$E_y \sum_t \beta^t \left[\sum_{z_{t-1}, z_t, n_{t-1}} g_{t-1}(z_{t-1}, n_{t-1}) \pi_z(z_{t-1}, z_t) (1 - d_t) (F(n_t) - k_f - cv_t) - k_e h_t + u_t b \right] \quad (9)$$

which is the discounted sum of production net of operating cost k_f and vacancy posting cost c over all existing firms, minus total entry costs for new firms h_t every period, plus home production b of unemployed agents.

The following proposition establishes the existence and unicity of the planning solution and characterizes when a competitive equilibrium is efficient.

Proposition 3. *(i) There exists a unique solution to the social planner's problem. (ii) If a block-recursive competitive equilibrium with positive entry exists, it coincides with the unique efficient allocation.*

Although a solution to the planning problem always exists, there is no easy way to characterize or compute the optimal allocation in general because of the high dimensionality of the problem. This is where the block-recursivity of the competitive equilibrium is useful, as it provides us with a constructive and tractable way to solve for it. Proposition 3 highlights the importance of the feasibility condition requiring that there is non-negative gross entry in every period, under which a block-recursive solution to equations (1)-(6) is a well-defined competitive equilibrium. It shows, in addition, that when such an equilibrium exists, it coincides with the unique efficient allocation. In contrast, when the positive entry condition is not satisfied, a solution to the planning problem still exists, but no longer satisfies block-recursivity: the analysis loses its tractability. This raises the natural question of how strong and restrictive such a condition is. Data from the U.S and other developed economies shows that this is true empirically: available datasets usually display a positive number of firm entry even during the deepest recessions. The condition of positive entry is therefore no restriction for any empirical application. It would however start to matter if one were to test counterfactual hypotheses or policies faraway from the calibrated economy. Theoretically, one may also wonder whether a block-recursive equilibrium with positive entry always exists. The answer depends on the parameters, but it is always possible to introduce a minimum amount of exogenous exits in the model to make sure that a positive number of firms always enter.

3.3 Optimal firm behavior

Now that the results of existence and efficiency of the equilibrium have been established, we can characterize a few features of the optimal policy for firms in terms of hirings, layoffs, quits and exit. A few analytical results can be established about symmetry and uniqueness of the layoff probability τ , the market for job-to-job transitions x , and wages, but we rapidly face the problem that the model is too complex to be solved analytically. We can however go further by doing numerical simulations.

In what follows, I simulate the model with some parameters and show in figure 7 how the optimal decision of firms varies in the (z, n) -space in terms of expansion (hirings), contraction (layoffs and quits⁷) and exit. The aggregate productivity y is fixed.

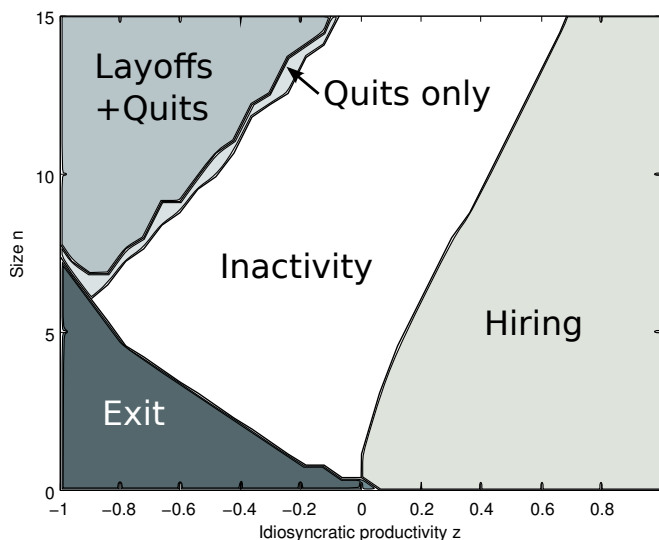


Figure 7: Optimal hirings, quits, layoffs and exit as a function of (z, n)

We see that exits are concentrated at small unproductive firms. Indeed, small firms with low productivity are those for which the current production and expected future surpluses are not enough to cover the operating cost k_f . This is a feature that has also been observed empirically, as evidenced in Evans (1987). Hirings tend to occur at small productive firms, since they have a high marginal product of labor. These firms actually expand until the marginal product of workers is equalized to the hiring cost $\kappa(y)$. In contrast, separations occur mostly at large unproductive firms, because the marginal product in these firms is low. Indeed, they choose to dismiss workers up to the point where their marginal productivity is equal to the utility they could get if unemployed, $U(y)$. Another point to notice is that there is a thin band in which firms do not lay workers off, but let them quit for better jobs. This can be explained by the fact that quits are in general more profitable than layoffs since workers can immediately get a higher utility by avoiding a spell of unemployment. Firms therefore let their workers quit more easily if their marginal productivity is not too high. Finally, we see that there is a large band of inactivity, in which firms experience no change in employment. This result is due to the existence of a gap between the utility of unemployment, $U(y)$, and the marginal productivity of workers at hiring firms, which is also equal to the hiring cost $\kappa(y)$. As a consequence, there is a non-negligible mass of firms with a marginal product of labor comprised between these two values that do not adjust their employment in every period. This fact has been often documented in the data as in Davis et al. (1996).

After this short discussion of the qualitative features of the optimal firm behavior, I now establish a few analytical results about the contracts offered by firms. In the model, firms are allowed to discriminate among workers by offering them different contracts. I examine in proposition 4 how different elements of the contracts (layoff probability τ , market for on-the-job search x , etc.) vary between workers in a given firm.

⁷In the model, all quits are job-to-job transitions.

Proposition 4. *If an equilibrium joint surplus $\mathbf{V} \in \mathcal{V}$ exists with its corresponding optimal contract $\{w_t, \hat{\tau}_{t+1}, \hat{x}_{t+1}, \hat{d}_{t+1}, \hat{W}_{t+1}\}_{t \geq 0}$, then:*

- (i) *If workers can commit, wages are not uniquely determined. In particular, the transformation $\{w + \Delta, \hat{\tau}, \hat{x}, \hat{d}, \hat{W} - a\Delta\}$ leaves the worker and the firm indifferent, with $a = [\beta(1 - \hat{d})(1 - \hat{\tau})(1 - \lambda p(\hat{x}))]^{-1}$ and $\Delta \in \mathbb{R}$.*
- (ii) *$\hat{x}$ is identical for all workers in the same firm.*
- (iii) *only the total number of layoffs $\int \hat{\tau} d\varphi$ is determined; the distribution of layoffs $\hat{\tau}$ over workers is not.*

Proposition 4 shows that there is a variety of contracts that can implement the optimal allocation. First, if workers can commit, as I have assumed so far, then an infinite number of wage profiles can be used by the firm. This result is due to the risk-neutrality of firms and workers and the fact that they value utility in the same way. The market for on-the-job search x is the same for all workers in the firm. This result may seem surprising if we think that firms prefer to separate early from their workers with high promised utility. This does not happen here because firms are committed to their contracts: even if it dismisses one of these highly paid workers, the firm has to fulfill its promise and compensate the worker at the time of the dismissal, instead of later, in a way that leaves it indifferent. The promise-keeping constraint therefore does not distort the firm's decision in favor of some group of workers. Finally, the individual layoff probability τ is not uniquely determined, but only the total number of layoffs at the firm level ($\int \hat{\tau} d\varphi$) is. Indeed, since all agents value utility in the same way, it is always possible to design compensation schemes to sustain any distribution of layoff probabilities among workers to yield the same utility profile to each of them.

3.4 Decentralization with incentive compatible contracts

I have assumed so far that contracts are complete and that firms and workers are able to commit. These assumptions may seem somewhat restrictive. I show in this subsection that they can be relaxed along two dimensions. First, the feature that contracts specify $\hat{x}$, the submarket for on-the-job search, seems unrealistic. I show that it is possible for the firm to write contracts only specifying $\{w, \hat{\tau}, \hat{d}, \hat{W}\}$ that make the worker choose the right submarket $\hat{x}$. Second, commitment from the worker side can be relaxed. It is indeed possible for the firm to write incentive compatible contracts that satisfy workers participation constraint.

Relaxing these two assumptions reduce to the following additional constraints on the equilibrium contract $\{w^*, \hat{\tau}^*, \hat{x}^*, \hat{d}^*, \hat{W}^*\}$. To guarantee that the worker chooses the right submarket, we have the incentive constraint:

$$\begin{aligned} \hat{x}^* &= \operatorname{argmax}_{\hat{x}} \lambda p(\hat{x})\hat{x} + (1 - \lambda p(\hat{x}))\hat{W}^* \\ \Leftrightarrow \hat{x}^* &= \operatorname{argmax}_{\hat{x}} p(\hat{x})(\hat{x} - \hat{W}^*). \end{aligned} \tag{10}$$

Now, to guarantee that the worker never wants to break the contract and leave the company to go back to unemployment, we have the following participation constraint:

$$\hat{W} \geq U(\hat{y}). \tag{11}$$

I now show that for any given contract with $\{\hat{\tau}^*, \hat{x}^*, \hat{d}^*\}$, there is a unique equivalent contract with wage w^{IC} and future utility $\hat{W}^{IC}$ that satisfies the above incentive and participation constraints and delivers the same promised utility to the worker.

Proposition 5. *Given an optimal contract $\{w^*, \hat{\tau}^*, \hat{x}^*, \hat{d}^*, \hat{W}^*\}$, there exists a unique equivalent incentive-compatible contract $\{w^{IC}, \hat{\tau}^{IC}, \hat{d}^{IC}, \hat{W}^{IC}\}$ such that:*

- (i) $\hat{\tau}^{IC} = \hat{\tau}^*$ and $\hat{d}^{IC} = \hat{d}^*$,
- (ii) $\hat{W}^{IC} \geq U(\hat{y})$,
- (iii) $\hat{x}^* = \underset{\hat{x}}{\operatorname{argmax}} p(\hat{x})(\hat{x} - \hat{W}^{IC})$,
- (iv) $\mathbf{W}(y, z; \{w^*, \hat{\tau}^*, \hat{x}^*, \hat{d}^*, \hat{W}^*\}) = \mathbf{W}(y, z; \{w^{IC}, \hat{\tau}^{IC}, \hat{x}^*, \hat{d}^{IC}, \hat{W}^{IC}\})$.

Proposition 5 tells us that the allocation that maximizes the worker-firm joint surplus can be implemented by an incentive-compatible contract. In particular, the layoff and exit probabilities are the same: $\hat{\tau}^{IC} = \hat{\tau}^*$, $\hat{d}^{IC} = \hat{d}^*$, and the submarket $\hat{x}^*$ chosen by the worker coincides with the efficient one. The wage and future utility $(w^{IC}, \hat{W}^{IC})$ are the only elements that adjust to insure that the two additional constraints (10) and (11) are satisfied. In addition to being more realistic than complete contracts with full commitment, these contracts offer the advantage of pinning down wages uniquely. They thus offer an alternative to other bargaining procedures. Moreover, we will see in part 4 that they can reproduce a number of empirical facts as they generate a realistic wage dispersion and size-wage differential.

4 Standard business cycles and cross-sectional properties

I now turn to the calibration of the model in order to evaluate its performance at explaining business cycle fluctuations. Before addressing the 2007-2010 recession, I present the standard business cycle statistics generated by a version of the model with productivity shocks only as it is able to account for many comovements in U.S. time series from ordinary times. To further validate the model, I then discuss some cross-sectional properties of the model in terms of establishment growth and wages.

4.1 Calibration

4.1.1 Functional forms and stochastic processes

Let us first introduce some functional forms. The production function is a power function $F(n) = An^\alpha$ where α governs the amount of decreasing returns in the economy. I normalize A to 1. Since time is discrete, I must choose a job finding probability function bounded between 0 and 1, which is not satisfied with a Cobb-Douglas matching function. Following Menzio and Shi (2009), I pick the CES contact rate functions

$$p(\theta) \equiv \theta(1 + \theta^\gamma)^{-1/\gamma}, \quad q(\theta) \equiv p(\theta)/\theta = (1 + \theta^\gamma)^{-1/\gamma}.$$

In addition to providing a good fit to the data, these functions satisfy all the regularity conditions stated in Assumption 2. To parameterize them, I estimate function p by non-linear least squares using the job finding rate series constructed by Shimer (2007)⁸ and a measure of market tightness θ . I construct the latter using data on vacancies from the Job Openings and Labor Turnover Survey (JOLTS) and the Conference Board's Help Wanted

⁸This data was constructed by Robert Shimer. For additional details, please see Shimer (2007) and his webpage <http://sites.google.com/site/robertshimer/research/flows>.

Index with unemployment data from the Bureau of Labor Statistics (BLS)⁹. The regression yields $\gamma = 1.60$ with a good fit ($R^2 = 0.90$).

The exogenous component of aggregate productivity is chosen to follow an AR(1) process with parameters (ρ_y, σ_y) :

$$y_t = \rho_y y_{t-1} + \varepsilon_{y,t}, \quad \varepsilon_{y,t} \sim \mathcal{N}(0, \sigma_y^2).$$

Similarly, idiosyncratic productivity z follows an AR(1) process with parameters (ρ_z, σ_z) . The two processes are approximated on finite grids using the method described in Tauchen (1986).

4.1.2 Calibration strategy

The calibration follows a method of simulated moments. To simplify the comparison with existing literature in search and business cycles, I follow the calibration strategy in Menzio and Shi (2008) (MS08 hereafter) as closely as possible. Since the model has a number of additional dimensions, I also target other moments from the data with values that have been widely used in the recent search-and-matching literature.

The time period is set to one month. I set the discount rate β to 0.996 so that the annual interest rate is about 5%. The parameters left to calibrate are the following: the decreasing returns coefficient α , the productivity parameters (ρ_y, σ_y) and (ρ_z, σ_z) , home production b , vacancy posting cost c , entry cost k_e , fixed operating cost k_f and the relative search efficiency of employed workers compared to unemployed ones λ . These parameters are set to minimize the numerical distance between the empirical and simulated moments, with weights chosen to yield relative errors of the same amplitude for each moment. The calibration targets the historical averages of the following monthly transition rates: an Unemployment-Employment (UE) rate of 45%, an Employment-Unemployment (EU) rate of 2.6% according to Shimer (2005), and an Employment-Employment (EE) rate of 2.9% following estimates by Nagypál (2007). The ratio of home production b over average marginal productivity is set to match a ratio of 71% in accordance with Hall and Milgrom (2008).

Turning to establishment-related moments, I must now depart from the MS08 calibration slightly. I target a labor share of 0.66, commonly chosen in the RBC literature. I also fit an average establishment size of 15.6 computed from the 2002 Economic Census. Regarding firm entry, I target an average fraction of hires at opening establishments of 21% according to quarterly data from the Business Employment Dynamics (BED) over the period 1992Q3-2009Q3. Given that MS08 does not have transitory idiosyncratic productivity, I follow Fujita and Nakajima (2009) in targeting a quarterly persistence of newly created jobs¹⁰ of 68% taken from Davis et al. (1996) in order to calibrate (ρ_z, σ_z) .

Finally, the aggregate productivity parameters (ρ_y, σ_y) are set to match the autocorrelation and standard deviation of log-detrended output. To do so, I use seasonally-adjusted quarterly real GDP data from the Bureau of Economic Analysis from 1947Q1 to 2010Q1 and detrend it using an HP filter with parameter 10^5 . The values targeted are an autocorrelation of 0.92 and standard deviation of 0.026.

⁹Because the direct vacancy measure by JOLTS is only available since 2001, I use the Conference Board's Help Wanted Index to complete the measure from 1951Q1 to 2000Q4 and adjust its level to match the JOLTS stock of vacancies in 2001Q1. Unemployment comes from the monthly seasonally-adjusted unemployment rate constructed by the BLS. Data is averaged over 3-month periods and detrended using an HP filter with parameter 10^5 as in Shimer (2005).

¹⁰The quarterly persistence of new jobs is the fraction of newly created jobs that survive after one quarter.

Parameter	Value	Description
Pre-calibrated:		
A	1	Technology parameter
β	0.996	Monthly discount factor
γ	1.60	Job-finding probability parameter
Calibrated:		
α	0.75	Decreasing returns to scale coefficient
b	0.33	Home production
c	0.45	Vacancy posting cost
λ	0.2	Relative search efficiency of employees
k_e	2.07	Entry cost
k_f	2.43	Fixed operating cost
ρ_z	0.3	Persistence of idiosyncratic productivity z
σ_z	0.5	Standard deviation of idiosyncratic productivity z
ρ_y	0.985	Persistence of aggregate productivity y
σ_y	0.030	Standard deviation of aggregate productivity y

Table 1: Calibrated parameters

Since all the targeted moments are endogenous in the model, the parameters are calibrated simultaneously. Table 1 summarizes the parameter values that result from the calibration. Table 2 shows the fit of the model with the targeted moments.

4.2 Business cycle statistics

The model with aggregate productivity shocks only can explain a number of comovements in historical U.S. time series. I simulate the model for a large number of periods and compute a selection of moments. Table 3 compares the standard deviations, autocorrelations and correlations with output of a number of historical time series to the ones generated by the model. The model does reasonably well at reproducing the autocorrelations measured in the data. In terms of comovements with output, the model is able to match the pro- and counter-cyclicality of all the variables. Quantitatively, as is common in the search literature, the correlations are sometimes too large compared to their empirical counterparts. This does not come as a surprise since the cyclical fluctuations are driven by a unique shock in the model. It should also be noted that the model is able to generate a significant Beveridge curve: $cov(U, V) = -0.55$ (empirical: -0.78), even though most models with endogenous destruction fail to account for this fact. Turning to the volatilities, we can see that the model suffers from the unemployment volatility puzzle identified by Shimer (2005). Although the model does better than standard search models by accounting for 40-50% of the variation in unemployment, it does not generate enough movements in unemployment and vacancies. A number of solutions have been suggested to solve this puzzle, including the use of a different calibration strategy as in Hagedorn and Manovskii (2008). I however ignore this problem for now, because we will see in the next section that the model does generate large fluctuations in unemployment when uncertainty shocks are introduced. The entry dimension is difficult to match. Entry in the model is almost acyclical and fluctuates too much. A possible way to improve this dimension would be to introduce convex hiring costs: entry would become more persistent and behave less as an acyclical residual.

Moment	Empirical value	Simulated
UE rate	0.45	0.450
EU rate	0.026	0.026
EE rate	0.029	0.029
b / productivity	71%	71.3%
Labor share	0.66	0.659
Average size of firms	15.6	15.62
Entry / Hirings	21%	21.1%
New jobs persistence	68 %	68.2%
$\rho[\log(output)]$	0.92	0.91
$\sigma[\log(output)]$	0.026	0.026

Notes: UE, EU and EE are monthly transition rates. New jobs persistence, autocorrelation and standard deviations of log-detrended output are stated in quarterly terms.

Table 2: Calibrated moments

	Data			Model		
	Std Dev.	Autocor.	cor(Y,x)	Std Dev.	Autocor.	cor(Y,x)
Y	0.026	0.922	1	0.026	0.909	1
Y/L	0.021	0.879	0.679	0.022	0.902	0.998
U	0.206	0.929	-0.855	0.091	0.925	-0.963
V	0.200	0.946	0.744	0.032	0.595	0.708
UE	0.119	0.905	0.801	0.029	0.899	0.985
EU	0.073	0.694	-0.600	0.062	0.888	-0.921
EE	0.118	0.859	0.622	0.063	0.822	0.960
Hirings	0.075	0.871	0.639	0.026	0.536	0.588
Quits	0.131	0.858	0.629	0.065	0.833	0.968
Layoffs	0.083	0.679	-0.542	0.066	0.887	-0.938
Entry	0.049	0.218	0.357	0.183	0.441	0.030
Exit	0.067	0.662	-0.341	0.083	0.813	-0.818
cov(U, V)		-0.78			-0.55	

Notes: Time series are presented in logs. Quarterly time series detrended using an HP filter with parameter 10^5 . Y is output, Y/L output per person, U unemployment, V vacancies. Entry and exit are number of workers in entering/exiting firms. See data appendix for data sources.

Table 3: Business cycle statistics

4.3 Establishment growth

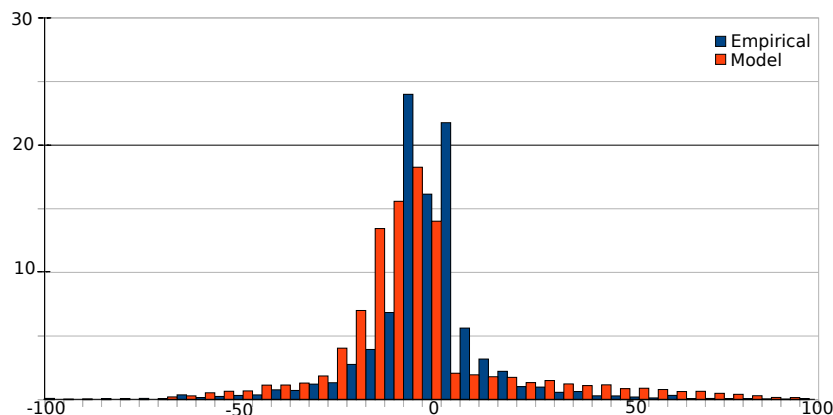
As a validation exercise, I now examine some predictions of the model in terms of establishment growth. It is convenient to introduce the following measure of establishment growth rates as initially used by Davis et al. (1996). Denoting $n_{i,t}$ the total employment of establishment i at date t , define growth rate $g_{i,t}$ as:

$$g_{i,t} = \frac{n_{i,t} - n_{i,t-1}}{\frac{1}{2}(n_{i,t} + n_{i,t-1})}.$$

This measure takes the ratio of net employment growth to the average size of the firm between periods $t - 1$ and t . This measure is helpful in that it can account for entry and exit of firms and treats them in a symmetric fashion. A growth rate of 2 therefore means entry, while -2 stands for exit.

4.3.1 Growth rate distribution

Davis et al. (2010)¹¹ reports the quarterly growth rate distribution of establishments using data from the Business Employment Dynamics dataset in 2008. I compare it with the same distribution from my model calibrated as in part 5. I simulate the model at the aggregate steady state over a 3-month period and compute the growth rate distribution.



Notes: Quarterly data from 2008 tabulated from the BED dataset by Davis, Faberman and Haltiwanger (DFH). Simulated distribution computed over a three-month interval at the aggregate steady state using the same methodology as DFH.

Figure 8: Distribution of quarterly establishment growth rates

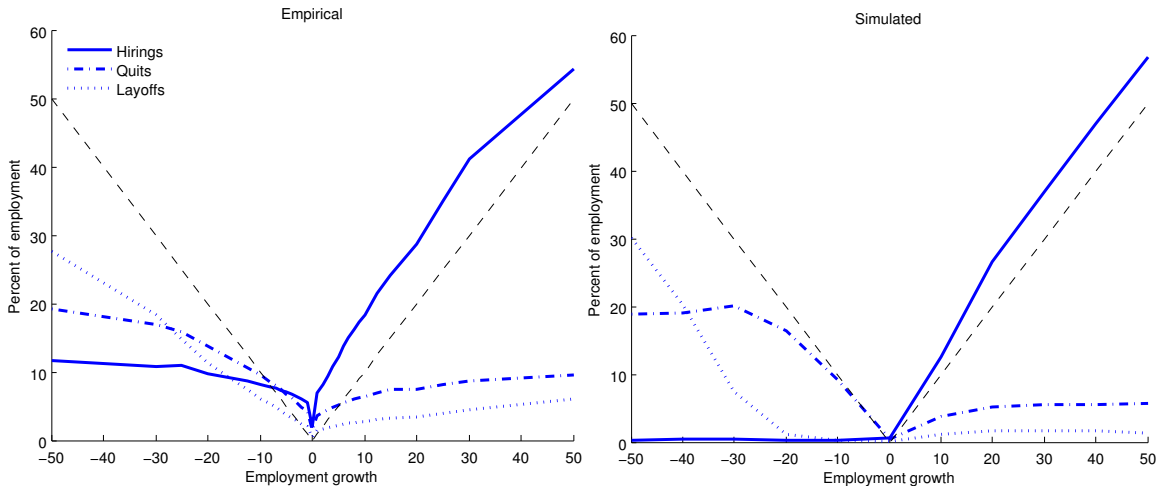
Figure 8 compares the two distributions. The two have similar shapes. The empirical dispersion is 0.22 for a simulated dispersion of 0.24. Both present a large peak at 0 (16% for the empirical one, 14% in the model): a substantial number of establishments do not adjust their employment at all in a quarter. These similarities are especially surprising given that none of the moments targeted in the calibration relate to the growth rate of establishments. On the negative side, the distribution generated by the model is more skewed to the left. This results from the fact that search frictions manifest themselves as

¹¹I would like to thank Steven Davis, Jason Faberman and John Haltiwanger for allowing me to use their tabulations from the Business Employment Dynamics dataset.

linear adjustment costs. Hiring is costly, while separations are not. Therefore, there is a larger tendency for firms to contract by a small amount, while expansions are usually larger. The model-generated distribution has indeed larger tails. This is also confirmed by the fact that the model cannot reproduce the large peak for growth rates between 0 and 5%. A possible way to improve this dimension would be to introduce firing costs in the model in order to restore the symmetry between expansions and contractions.

4.3.2 Employment policy

Empirical evidence shows that firms with different growth rates have different hiring, layoff and quit rates. The composition of hirings against separations and the balance between layoffs and quits present some important non-linearities at the establishment-level. Davis et al. (2010) show with some empirical exercises that these establishment-level regularities are important features that new models should be able to replicate in order to improve their ability to predict the time-series of labor market variables.



Notes: Tabulations from the BED dataset by Davis, Faberman and Haltiwanger. Simulated policy computed by aggregation over a three-month period. The averages are employment weighted. The dashed lines are the -45° and $+45^\circ$ lines to show the minimal level of separations and hirings needed.

Figure 9: Empirical and simulated employment policies as a function of growth

Figure 9 displays the empirical and simulated employment-weighted levels of hirings, quits and layoffs as a function of establishment growth. To produce this graph, I simulate the model over a 3-month period at the aggregate steady-state and compute the corresponding series. The model can reproduce a number of qualitative and quantitative features of the average level of quits and layoffs. In particular, it is able to match the change in their composition for expanding and contracting firms. Establishments that contract by a small amount tend to favor quits over layoffs, as workers can be directly employed without experiencing unemployment. However, the more the firm contracts, the more it will lay off workers. The reason is that in order to dismiss a large number of workers, firms have to direct them to labor markets where the probability to find a job is higher and where wages are lower. At some point, the utility promised to workers on these markets becomes so low that the firm starts using layoffs. Turning to hirings, the model performs reasonably well for expanding firms, but fails for contracting ones. The empirical data shows a non-negligible amount of churning: establishments that grow also separate from some workers, and vice versa. The model is able to generate churning to a certain extent for growing establish-

ments, but very little for contracting ones. The reason lies in the fact that hiring is costly, which breaks the symmetry between expansion versus contraction. For that reason, very little hiring is observed at establishments that contract over a 3-month period.

4.3.3 Differential growth rates over the business cycle

Empirical regularities in the evolution of the cross-sectional distribution of establishments have recently drawn attention from a number of authors. Moscarini and Postel-Vinay (2009) document that large firms are more cyclically sensitive than small ones. They show that large firms shrink faster during recessions, but create more jobs in the later stages of the following expansion. As a result, the differential growth rate between large and small establishments is strongly procyclical. Using different datasets and controls, they report correlations between unemployment and the differential growth rate that range from -0.61 to -0.38 in the U.S. To investigate whether the model can reproduce this different cyclical sensitivity, I compute the differential growth rate between the 50% largest and smallest establishments in the model with aggregate disturbances in productivity. The correlation between the differential and unemployment level is -0.26. This negative correlation remains robust to different definitions of large and small firms. One possible explanation for this differential cyclical sensitivity lies in the presence of decreasing returns. Large firms, being also more productive, produce in a region of their production function with less curvature. Therefore, when a change in hiring cost or unemployment valuation occurs, the triggered change in employment is proportionally larger for more productive firms than for less productive ones that hit their decreasing returns very quickly.

4.4 Wage predictions

The use of optimal dynamic contracts in search models provides an alternative to the standard assumptions of Nash and Stole & Zwiebel bargaining. In particular, since workers can search for jobs while employed, employers have to design contracts that give the right incentives for workers to stay in the firm or apply to the right labor market. Using the incentive-compatible contracts described in part 3.4 yields a unique characterization of wages, that may vary substantially from a worker to the other, even if they belong to the same firm. Due to this rich incentive structure, the model is able to predict an important wage dispersion for observationally equivalent workers, and accounts for a significant fraction of the empirical variation. It also predicts that wages tend to grow with firm size, as seen in the data.

4.4.1 Wage dispersion and elasticity

Hornstein et al. (2007) report that standard calibrations of search-and-matching models without on-the-job search cannot generate much dispersion in wages. In their basic calibration of a standard random search model, they obtain a mean-min ratio for wages of 1.036, while their preferred empirical estimate is about 1.70, and a corresponding coefficient of variation of only 1/12th of the variation in the data. Using wage data from the 1990 Census and using different sets of controls, their estimates for the empirical coefficient of variation for wages range from 0.35 to 0.49. I estimate the same dispersion measure in my model by simulating it at the aggregate steady state and obtain a coefficient of variation of 0.29, which explains between 59% and 83% of the observed dispersion in wages.

Regarding the evolution of wages over business cycles, both the average wage and total wage dispersion appear to be procyclical. After being hit by a negative productivity, wages

tend to fall relatively more in large firms than in smaller ones, resulting in a lower total dispersion. Given the recent attention that wage stickiness has received in the search-and-matching literature, I measure the elasticity of wages with respect to productivity (output per person). My estimated elasticity is of 0.87, above the empirical estimates of about 0.5 reported in the literature. An interesting future development could be to introduce risk-aversion for workers. Combined with the dynamic contracting of the model, this could yield an interesting theory of endogenous stickiness that could significantly improve this dimension.

4.4.2 Size-wage differential

It is a common finding that firm size can explain part of the variation in wages. Brown and Medoff (1989) report that with a variety of data sets and a range of controls chosen to capture much of the differences in labor quality, a substantial size-wage differential remains: an employee working at a firm with $\log(\text{employment})$ one standard deviation above average may expect to earn a wage between 6-15% above the one at a firm with $\log(\text{employment})$ one standard deviation below average. To investigate whether the model can reproduce this finding, I compute the wages in every establishment at the aggregate steady-state. I then run the following regression

$$\log(\text{wage}) = \alpha + \beta \log(\text{employment}) + \varepsilon$$

and evaluate by how much the wage of a worker employed at a firm with $\log(\text{employment})$ one standard deviation above average increases compared to the one with one standard deviation below: I obtain an increase of 6.02% in line with the empirical estimates. Interestingly, this size-wage differential can be explained by a mechanism due to search frictions quite different from more standard explanations based on labor quality or institutions. The mechanism at work in the model is due to the way firms deal with worker incentives. In the model economy, firms that want to expand prefer to retain their current workers in order to save on hirings costs. To do so, it prefers to backload incentives to future periods by promising them a high future utility. Therefore, with all other things kept equal, firms that grow tend to offer lower wages today than firms that shrink, since workers get a higher future compensation. Turning back to firm size, large firms are those that received a high idiosyncratic shock in the recent past, and vice versa for small firms. The idiosyncratic shock being transitory, we can observe a regression to the mean phenomenon. As a result, large firms tend to shrink and offer high wages, while small ones tend to grow and offer low wages. This mechanism underlines the role of search frictions and job-to-job transitions to generate dispersion in wages. It also emphasizes establishment growth as a key determinant for wages.

To go a little further, I simulate the economy after a negative productivity shock and study how the coefficient of dependance of wage with respect to firm size responds. The coefficient β appears to be procyclical, decreasing during a recession, in line with the previous finding that wage dispersion is procyclical.

5 The 2007-2010 Recession

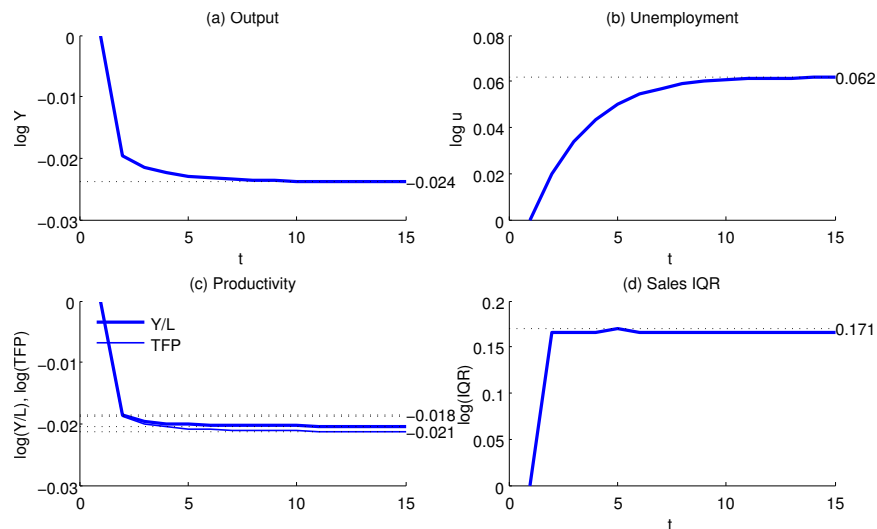
I now turn to the issue of explaining the large downturn in labor markets observed since 2008. I examine in detail the response of the model to productivity shocks, before introducing uncertainty shocks as unanticipated increases in the variance of the innovation to the idiosyncratic productivity process. Such shocks allow the model to explain the coexistence

of high unemployment and high productivity, as observed since 2009. A combination of productivity and uncertainty shocks can account for a large number of the patterns observed in the data during the crisis.

5.1 TFP shocks

In this subsection, I present the response of the economy to a permanent negative productivity shock, and discuss why such a shock fails to explain some characteristic features of the recession.

When an aggregate negative productivity shock hits the economy, all firms and workers in the economy see their surplus decrease. This causes a number of firms at the bottom of the productivity distribution to exit immediately as they can no longer pay the operating cost. At the same time, other firms see the marginal productivity of their workers go below the utility that they could get unemployed. Firms therefore cut hiring and start laying workers off until their marginal productivity is equalized to the unemployment value. All this results in a burst of layoffs and a rapid increase in unemployment. As incumbent and entering firms experience a drop in their profits, job creation goes down with less vacancies posted, causing the equilibrium market tightness to decline. With less job openings, employed and unemployed workers have more difficulty finding new jobs so the the job finding and quit rates drop. As time goes on, the cleansing effect of the recession fades out with a more moderate exit rate. The lower market tightness translates itself into a lower hiring cost, leading to a moderate recovery in entry and hiring as firms start entering again and new jobs are created.



Notes: Responses shown in log deviation from steady-state. The aggregate productivity shock of -1.5% is matched to produce a -1.8% drop in output per person as observed between 2008Q2-2009Q1.

Figure 10: Response to a permanent decrease in aggregate productivity

Figure 10 shows the response of the model to a permanent negative aggregate productivity shock calibrated to produce a 1.8% drop in productivity (output per person) as observed during the 2007-2010 recession. The responses of other labor market variables can be found in figure 17 in the appendix. Figure 16 describes how the optimal firm employment policy evolves in the (z, n) -space in the face of a negative aggregate productivity shock. As explained above, such a shock triggers a drop in output and productivity with a moderate

increase in unemployment. The dispersion of sales growth jumps (17.1%) but remains below the 36% increase observed in the data. We can see that aggregate productivity shocks are not sufficient to account for the patterns observed during the recession. To generate the dramatic increase in unemployment (42%) observed between 2008 and 2009, one would need a counterfactually large drop in aggregate productivity, incompatible with the strong subsequent recovery observed in the data for TFP and output per person.

5.2 Uncertainty shocks

As we have seen above, aggregate productivity shocks cannot account for the patterns in output, productivity and unemployment observed during the 2007-2010 recession. Motivated by the observation that the firm-level dispersion in firm growth rates has increased substantially since 2008, I now study the effect of uncertainty shocks to see whether they can improve the ability of the model to explain the empirical patterns. To keep the analysis tractable, I introduce these shocks parsimoniously as unanticipated increases in the standard deviation σ_z of the innovations to the idiosyncratic process z .

The effect of uncertainty shocks can be decomposed mainly in two categories. There is an effect due to expectations of higher risk in the face of costly adjustments. In the model, matching frictions manifest themselves as hiring costs: there is a gap between the cost of hiring an additional worker and the utility that the firm gets when it lays workers off. Such an increase in risk leads firms to cut or delay hirings.

There is however an opposite effect that Bloom (2009) initially described as “realized volatility”. Higher uncertainty means higher probabilities of getting both good and bad shocks. Since firms have the option to exit when their situation worsens, an uncertainty shock can actually translate on average into higher expectation of future surpluses. If risk aversion is low, an uncertainty shock can therefore be positive on average.

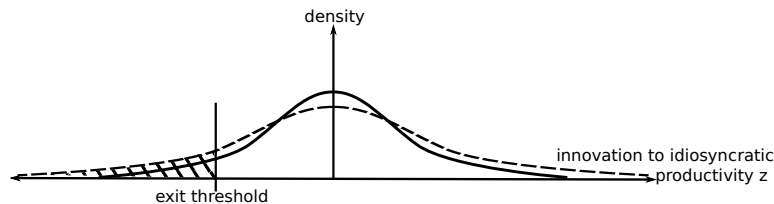
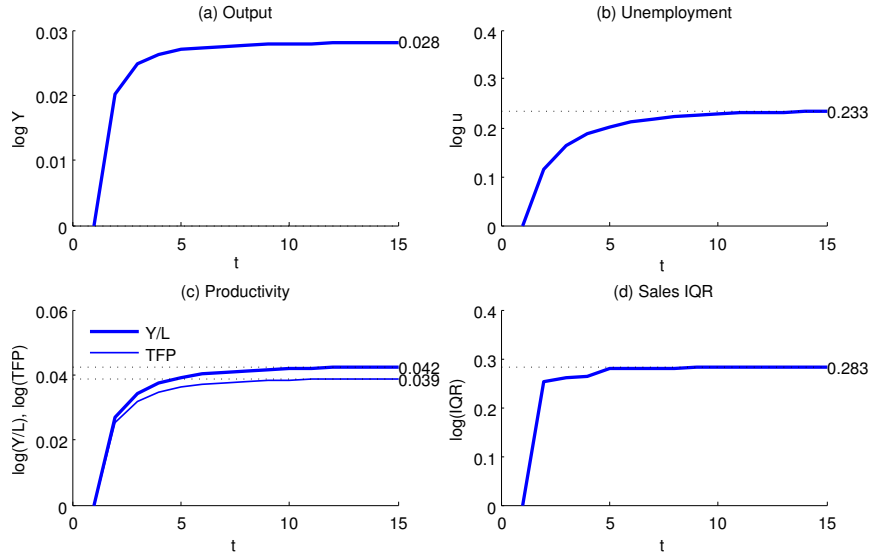


Figure 11: Uncertainty shock

Figure 11 shows how a mean-preserving spread of the distribution of the innovation to idiosyncratic productivity can lead to higher expected surplus. Lower values of productivity z are cut-off since firms can endogenously decide to exit. A mean-preserving spread thus loads proportionally more on the right-hand side of the graph if the exit threshold does not change too much. Of course, the threshold also adjusts when the shocks hits. But since it is endogenously chosen by the firm, the firm can do at least as good as if it kept the threshold constant. So average expected surplus increases when uncertainty rises. This effect is similar to the result that option value increases with volatility. It counterbalances the expectational channel explained earlier and leads to a higher level of entry and hiring. Simultaneously, the mass of firms getting a low shock increases, resulting in more exits and layoffs. On the general equilibrium side, larger expected surpluses lead to a higher market tightness, making the job finding probability larger and therefore higher Unemployment-Employment and Employment-Employment transition rates. Figure 12 illustrates these

effects for the main aggregate variables by presenting the response of the economy when σ_z increases by 5% permanently. The responses of other labor market variables can be found in figure 19 in the appendix. The optimal firm employment policy is shown in figure 18.



Notes: Responses shown in log deviation from steady-state.

Figure 12: Response to a permanent increase of 5% in σ_z

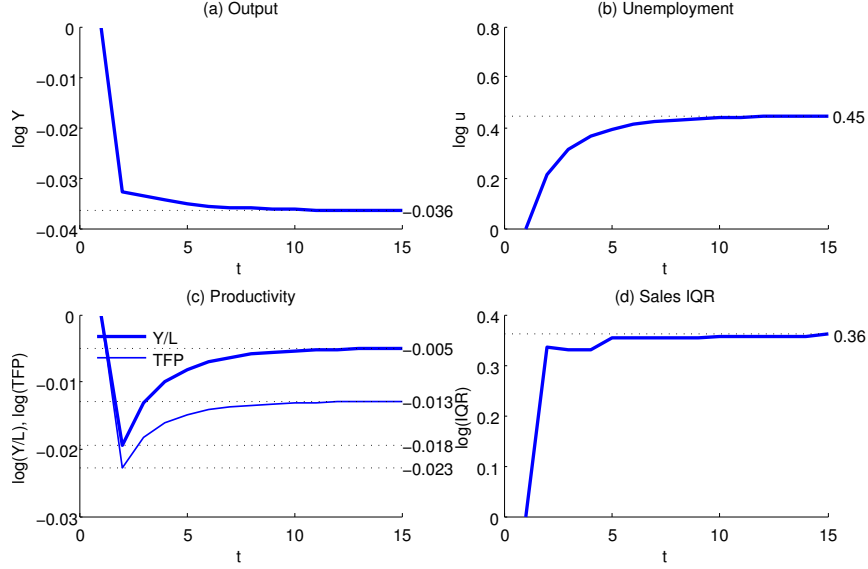
Figure 12 and 19 show the magnitude of the effects described above. It should be noted that the “realized volatility” effects largely dominate. Output, unemployment and productivity jump up, as do the rest of labor market flows. A direct consequence of the uncertainty shock is a substantial increase in the dispersion of sales growth rates. Unemployment goes up because there is a large rise in gross flows: the intensified reallocation of jobs between firms triggered by the rise in volatility leads larger flows of workers to experience a spell of unemployment, as it takes time to create new matches.

5.3 Productivity and uncertainty shocks

The two previous simulations have shown that pure productivity or pure uncertainty shocks cannot reproduce the empirical patterns observed in the 2007-2010 recession. Uncertainty shocks alone cannot account for the contraction in GDP, nor the initial drop in productivity observed in the first quarter of 2009. In particular, the positive “realized volatility” effect needs to be counterbalanced by some negative shock.

I am now going to investigate to what extent a combination of negative aggregate productivity and uncertainty shocks can explain the patterns observed in the data. This combination amounts to a shock that lowers the mean and increases the variance of idiosyncratic productivity. I calibrate the two shocks to match the peak-trough fall in output per person (-1.8%) and jump in the firm-level IQR of sales growth (36%) measured in the Compustat dataset. A combination of a -3.75% drop in the exogenous component of aggregate productivity y and a +4.5% increase in the standard deviation σ_z of the innovation to the idiosyncratic productivity offer the closest match, as shown in figure 13.

As figure 13 shows, the model predicts a drop in output (-3.6%) somewhat smaller than in the data (-4.5%). It offers a good fit for unemployment as it predicts an increase of 45%, close to the one measured in the data. The two measures of productivity display a fall



Notes: Responses shown in log deviation from steady-state. The -3.75% aggregate productivity and +4.5% uncertainty shocks are calibrated to reproduce a drop of -1.8% in output per person and a 36% rise in firm growth dispersion (IQR).

Figure 13: Response to a combination of productivity and uncertainty shocks calibrated on productivity and sales growth dispersion

	Y/L	IQR(g)	Y	U	TFP	Vacancy	Hiring	Quit	Layoff
Data	-0.018	0.36	-0.045	0.42	-0.027	-0.43	-0.12	-0.30	0.28
Model	-0.018	0.36	-0.036	0.45	-0.023	-0.064	-0.056	-0.10	0.42

Notes: The peak-trough variations are computed with the log-deviation from an HP-trend with parameter 1600. Output per person and IQR of sales growth rates are the targeted moments.

Table 4: Peak-trough variations for the 2007-2010 recession and calibrated model with productivity and uncertainty shocks

at the time the shock hits, quickly followed by a strong recovery to levels slightly below their initial trends. The model is able to explain the coexistence of a large and persistent unemployment level with a sharp rise in productivity of labor, as observed during the 2007-2010 recession. The predicted recovery is however weaker than in the data, but we will return to this question later. Turning to the labor market flows reported in figure 20, the model can explain the direction of changes observed in the data. Quantitatively, it also predicts variations broadly consistent with the empirical magnitudes of change in hirings and layoffs. The decline in quits and vacancies is however not as large in the model as in the data. Table 4 summarizes the peak-trough variations in these main variables.

The model is less accurate for the variations in entry and exit. Figure 21 in the appendix compares their fluctuations in the data and in the model. Qualitatively, the model correctly predicts a decline in entry at the time the shock hits, as well as an increase in the number of exits. The model's elasticities along these margins however exceed the empirical one by a factor of slightly less than 4. A possible way to improve this dimension could be to change the type of uncertainty shock and experiment with asymmetric ones. Concerning entry, another potential avenue could be to introduce convex hiring costs in order to increase its persistence and reduce its variability. Solving this issue is however complicated as it would

require us to deepen our understanding of the determinants of firm entry. The available data is however of little help here as entry/exit do not seem to display any strong pattern over the business cycle.

The above exercise has shown the ability of the model to account quantitatively for a large fraction of the empirical peak-trough variations in the 2007-2010 recession, but has largely ignored the dynamic dimension. To conclude this section, I now return to the issue of accounting for the sharp rise in productivity observed since 2009 along with a persistently high unemployment rate. I present the result of a last simulation in which productivity shocks are no longer permanent, but calibrated to reproduce the empirical time-series of output per person. To parameterize the shocks parsimoniously, the uncertainty shock remains permanent and unanticipated, and hits the economy in the fourth quarter of 2008, the date at which uncertainty jumps in the data. Its magnitude remains calibrated as before. Figure 14 displays the results.

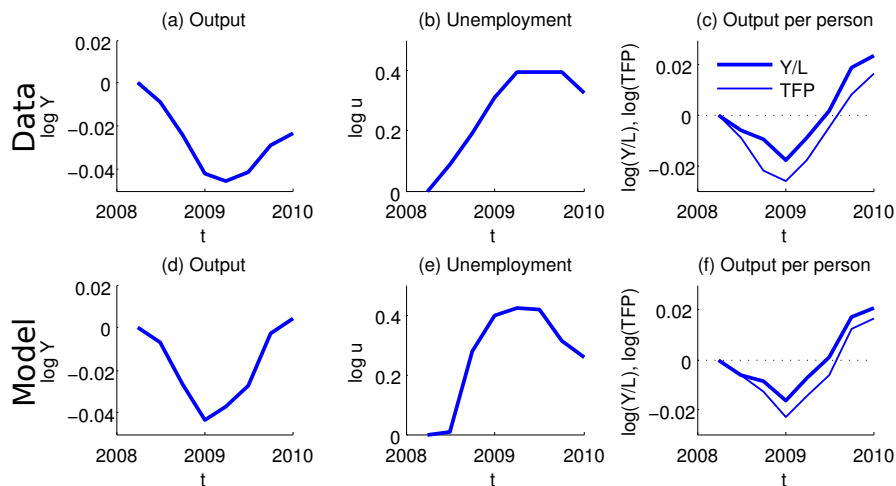


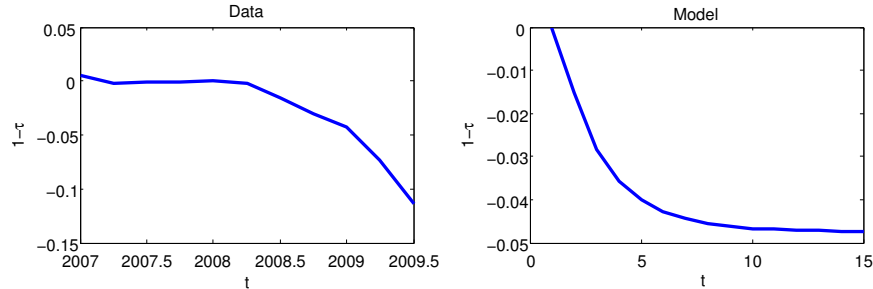
Figure 14: Empirical and simulated time-series when productivity shocks are calibrated to match the output per person series and a permanent uncertainty shock of 4% hits the economy in 2008Q4.

Both qualitatively and quantitatively, the model is able to match a number of features of the time-series of output, TFP and offers a particularly close fit for the joint dynamics of unemployment and productivity. It matches well the magnitudes of the drops in output (-4.4%) and TFP (-2.3%), as well as the jump in unemployment (+42%). It can also account for the persistence in unemployment and its slight decline since the middle of 2009, even though productivity rose well above its initial trend. On the negative side, the model predicts a recovery in output slightly stronger than observed in the data. This results from the fact that negative productivity shocks are receding, while the “realized volatility” effects of uncertainty become more important.

5.4 Labor wedge

Another important aspect of the 2007-2010 recession relates to the labor wedge, the ratio between the marginal rate of substitution of consumption for leisure and the marginal product of labor. Interestingly, the current recession is mostly associated to a large worsening of the labor wedge. Although the model is not meant to address issues related to

this dimension, it is possible to get measures of consumption, output and hours from the simulations to compute a measure of the labor wedge and compare it with the data. This exercise may raise some concerns because of the particular assumptions of the model (risk neutrality, constant interest rate), but is still instructive on the capacity of search models to generate movements in the labor wedge. Figure 15 presents two possible measures from the data and the simulations.



Notes: Data shown in log-deviation from HP-trend with parameter 1600. Labor wedge measured with preferences $u(C, H) = \log C + \xi \frac{H^{1+\nu}}{1+\nu}$ where C is aggregate consumption and H aggregate hours worked. $\nu = 0.5$ for a Frisch elasticity of 2. Consumption is computed with the market clearing condition $C = Y - cv - n_e k_e - m_f k_f$. See appendix for data sources and definitions.

Figure 15: Evolution of the labor wedge in the data and in the model

The data displays a large worsening of the labor wedge. The observed drop of about 11% corresponds to a fall of 5 standard deviations. The model also predicts a substantial decline in the labor wedge, which accounts for about 50% of the empirical one. This is surprising given the recent finding by Shimer (2010) that search models without wage rigidity cannot generate large fluctuations in the labor wedge. We find evidence here that uncertainty shocks can generate important fluctuations in the labor wedge. A large part of this decline is due to the large increase in firm and job reallocation: when uncertainty increases, we observe a large rise in the number of entrants and exiters, as well as in the number of hirings and layoffs. This generates large reallocation costs that substantially reduce the consumption-output ratio that appears in the labor wedge measure.

6 Conclusion

In this paper, I have developed a dynamically tractable search model of firms with decreasing returns in which I have studied the effects of productivity and uncertainty shocks. A combination of productivity and uncertainty shocks significantly improves the ability of search models to explain the patterns observed in the data during the 2007-2010 Recession. In particular, with the two shocks calibrated to match productivity and the dispersion of firm growth rates, the model reproduces closely the joint dynamics of unemployment and productivity and is able to account for the coexistence of the persistently high unemployment level and sharp rise in labor productivity observed since 2009. It correctly predicts the amplitude of fluctuations in a number of other variables such as output, hirings, and layoffs. It is however less accurate on the entry/exit dimension as it tends to generate too large fluctuations along these dimensions.

The model has a range of implications at the establishment and cross-sectional levels. The growth rate distribution and employment behavior of establishments produced by the model replicate a number of features that have been observed in establishment-level data.

Concerning wages, the presence of search frictions and on-the-job search help generate a large wage dispersion, as well as a realistic size-wage differential. It would be interesting for future research to examine these dimensions further in interaction with business cycles.

Because of the high tractability of its dynamics, the model presented in this paper could be used as a framework in a variety of applications in which multiworker firms and search frictions play a role. Concerning the 2007-2010 recession, the entire policy dimension remains to be studied. Could firing costs have attenuated the depth of the recession as suggested by the experience of some European countries? What role did the unemployment benefits extension play in the crisis for the persistence of unemployment? Could it explain the recently observed deviation of unemployment and vacancies from the historical Beveridge curve? Would hiring subsidies or other policies encourage firms to start hiring again? Another interesting extension would be to introduce financial frictions in the model to study the effect of credit tightening on firm employment decisions and see if the negative productivity shocks could arise endogenously from the misallocation of capital across firms. I leave these topics for future research.

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Appendices

A Additional graphs

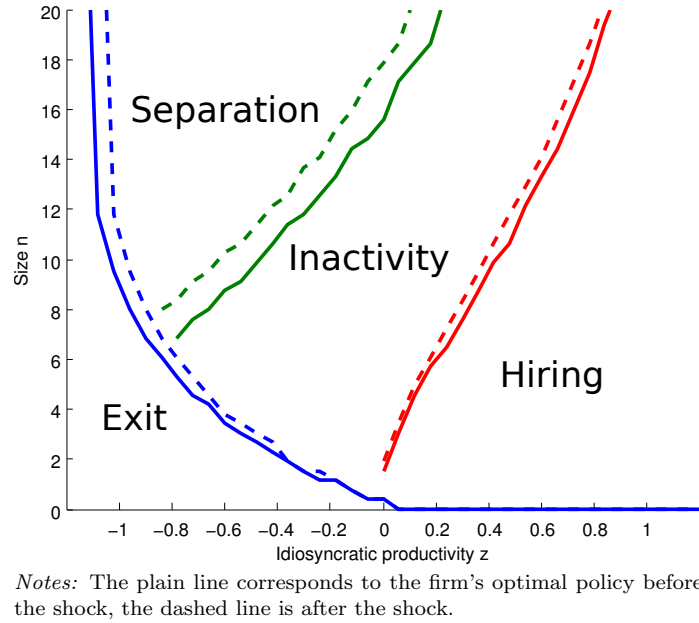


Figure 16: Firm's optimal policy after a negative productivity shock

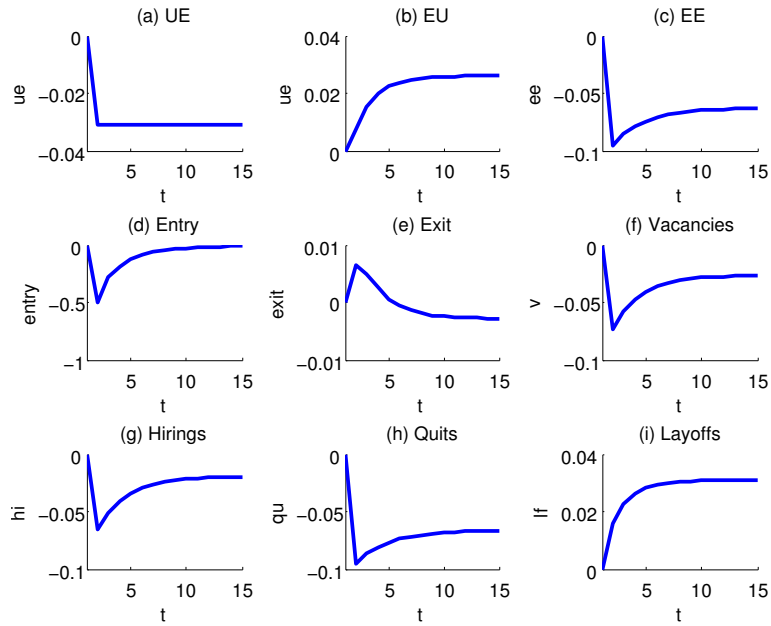


Figure 17: Response to a permanent decrease in aggregate productivity (-1.8% in output per person)

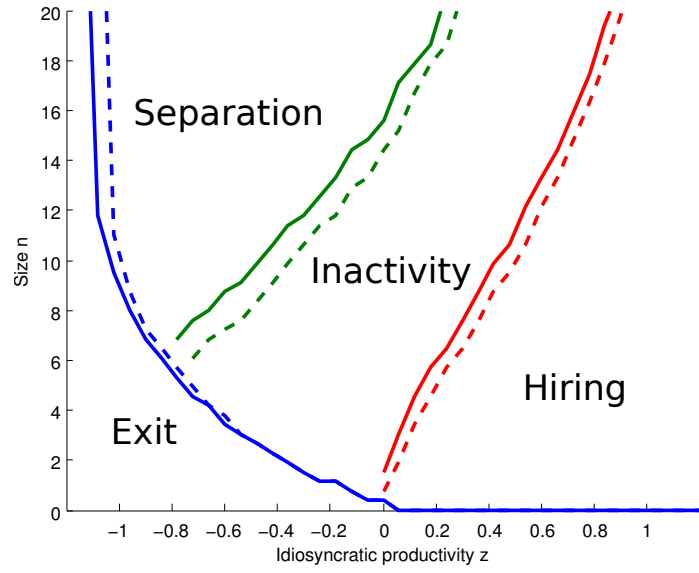


Figure 18: Firm's optimal policy after an uncertainty shock

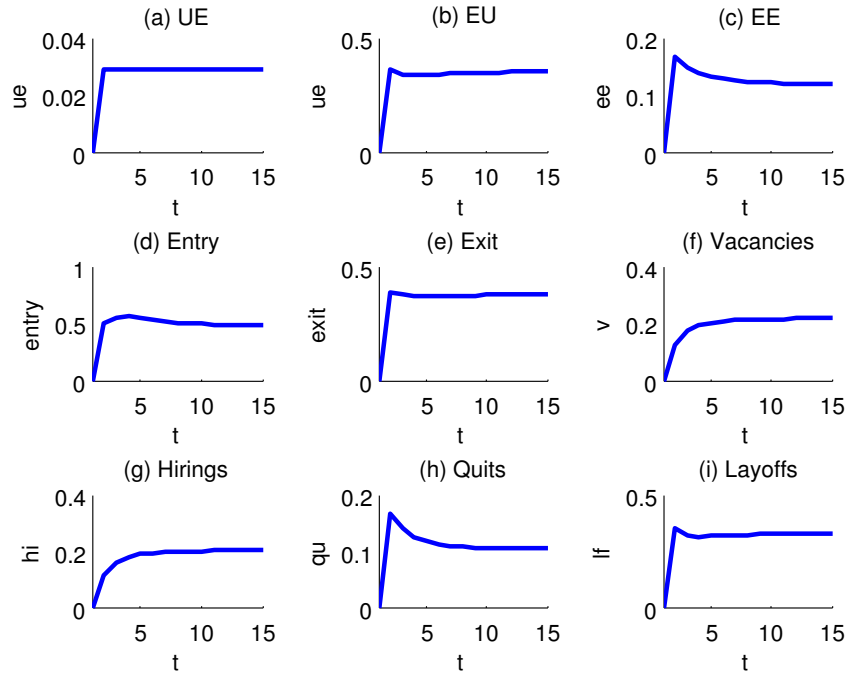


Figure 19: Response to a permanent increase of 5% in σ_z

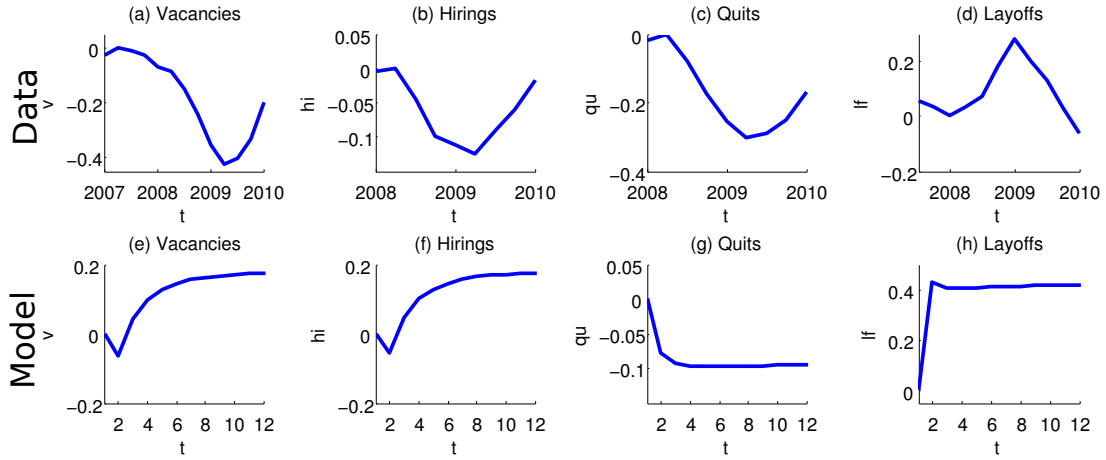


Figure 20: Comparison between labor market flows in the 2007-2010 recession and model response to productivity and uncertainty shock

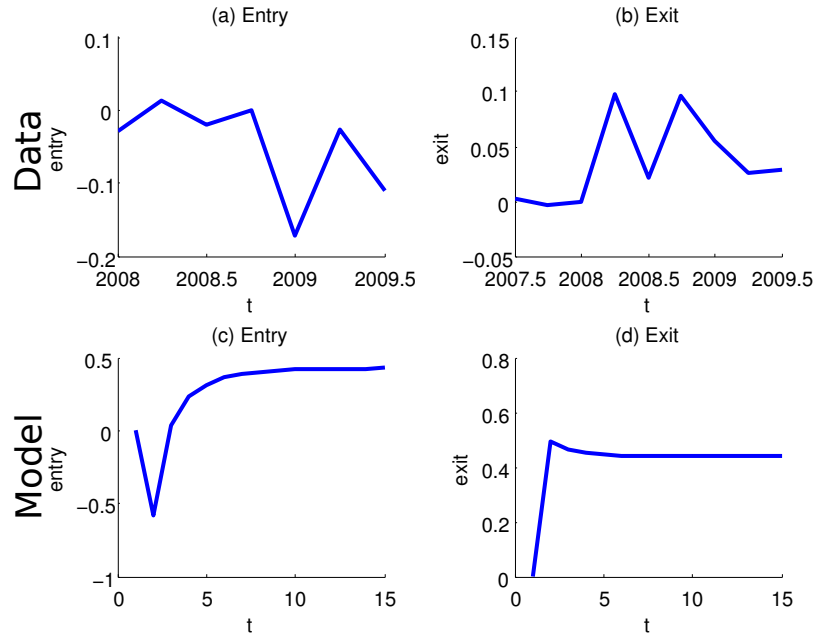


Figure 21: Comparison between entry/exit in the 2007-2010 recession and model response to productivity and uncertainty shock

B Data appendix

This section details the construction and sources of the empirical time series used throughout the paper.

- Output is taken from the NIPA tables constructed by the Bureau of Economic Analysis. I use quarterly GDP in 2005 dollars from 1947Q1 to 2010Q1.
- Productivity Y/L is seasonally adjusted real average output per person in the non-farm sector over the period 1947Q1-2010Q1 from the Bureau of Labor Statistics.
- Unemployment is the seasonally adjusted monthly unemployment rate constructed by the BLS from the Current Population Survey over the period January 1948-June 2010 (for people aged 16 and over). Similarly, I use the total civilian labor force for people aged at least 16 from the BLS over the same period. Data is averaged over quarters.
- Vacancy is quarterly average of the monthly vacancy measure from the Job Openings and Labor Turnover Survey. The measure being only available since 2001, I use the Conference Board's Help Wanted Index to complete the measure from 1951Q1 to 2000Q4 and adjust its level to match the JOLTS stock of vacancies in 2001Q1.
- Historical UE and EU monthly transition rates are taken from Shimer (2007) over the period 1948Q1-2007Q1. For later periods, I use the monthly series on labor force status flows from the Current Population Survey constructed by the BLS over February 1990 to June 2010.
- EE is constructed by taking the ratio of quits from JOLTS over employment $(1 - U)$ from January 2001 to March 2010.
- Entry and exit are quarterly openings and closures in employment terms for total private industries taken from the Business Employment Dynamics over the period 1992Q3-2009Q3.
- Labor market flows for hiring, quits and layoffs are quarterly sums of the JOLTS measures from January 2001 to March 2010. Data is normalized by total labor force.
- The labor wedge measure uses data on hours from Prescott et al. (2009) over the period 1947Q3-2009Q3. Consumption is real personal consumption expenditure on non-durables and services in 2005 dollars from the BEA from 1995Q1-2010Q2.
- Total Factor Productivity is the quarterly unadjusted TFP measure constructed by Fernald (2009) over 1947Q1-2010Q2.
- Firm-level data on sales is taken from Compustat. I use the quarterly sales as well as the annual employment measures from the total US dataset of industrial firms over the period 2001-2009Q3. I only keep firms that report their sales and employment over the entire period 2006Q4-2009Q3. Sales growth is computed with $g_{i,t} = \frac{s_{i,t} - s_{i,t-4}}{1/2(s_{i,t} + s_{i,t-4})}$.

C Comparison with Mortensen-Pissarides (1994)

I briefly compare in this subsection how the Mortensen-Pissarides (1994) model responds to uncertainty shocks. Since this version of the model has an endogenous separation margin, uncertainty shocks are not neutral. I calibrate a discrete time version of the model using the same targets as in part 5 and evaluate the response of the model to a series of aggregate productivity shocks and a permanent uncertainty shock that hits the economy in 2008Q3.

C.1 Model

Let us first introduce the model's equations. Unless stated otherwise, I keep the same notation as in my model. As in the original article, aggregate productivity y follows an AR(1) process with parameters (ρ_y, σ_y) . z is now a match-specific productivity that follows:

$$z_{t+1} = \begin{cases} z_t, & \text{with probability } \rho_z \\ \varepsilon \sim \sigma_z \mathcal{U}(-1, 1), & \text{with probability } 1 - \rho_z \end{cases}$$

The production function is now linear: a firm-worker pair produces Ae^{y+z} . There is a unique labor market with tightness θ where all firms and workers meet. Firms are free to enter in any period. Newly created jobs start at the highest productivity $\bar{z}$.

The value of unemployment is:

$$\mathbf{U}(y) = b + \beta \mathbf{E}_{\hat{y}} \left\{ \mathbf{U}(\hat{y}) + p(\hat{\theta})(\mathbf{W}(\hat{y}, \bar{z}) - \mathbf{U}(\hat{y})) \right\}$$

The value of employment is:

$$\mathbf{W}(y, z) = w(y, z) + \beta \mathbf{E}_{\hat{y}, \hat{z}} \left\{ \hat{\tau} \mathbf{U}(\hat{y}) + (1 - \hat{\tau}) \mathbf{W}(\hat{y}, \hat{z}) \right\}$$

where $w(y, z)$ is the equilibrium wage schedule determined by Nash bargaining, as described below. The firm's problem is given by:

$$\begin{aligned} \mathbf{J}(y, z) &= \max_{\hat{\tau}} Ae^{y+z} - w(y, z) + \beta \mathbf{E}_{\hat{y}, \hat{z}} \left\{ (1 - \hat{\tau}) \mathbf{J}(\hat{y}, \hat{z}) \right\} \\ \Leftrightarrow \mathbf{J}(y, z) &= Ae^{y+z} - w(y, z) + \beta \mathbf{E}_{\hat{y}, \hat{z}} \left\{ \max(\mathbf{J}(\hat{y}, \hat{z}), 0) \right\} \end{aligned}$$

The free-entry condition tells us that firms post vacancies until expected profits exactly equal the vacancy cost c :

$$\forall y, \quad c = q(\theta) \mathbf{J}(y, \bar{z})$$

Wages are determined by Nash bargaining. Denote $S(y, z)$ the joint surplus of the firm and worker. If μ is the worker's bargaining power, we can write:

$$\begin{cases} \mathbf{J}(y, z) = (1 - \mu) S(y, z) \\ \mathbf{W}(y, z) - \mathbf{U}(y) = \mu S(y, z) \end{cases}$$

and

$$S(y, z) = \mathbf{J}(y, z) + \mathbf{W}(y, z) - \mathbf{U}(y).$$

It can be shown that the joint surplus solves the following Bellman equation:

$$S(y, z) = Ae^{y+z} - b + \beta \mathbf{E}_{\hat{y}, \hat{z}} \left\{ \max(\mathbf{S}(\hat{y}, \hat{z}), 0) - \mu p(\hat{\theta}) S(\hat{y}, \bar{z}) \right\}.$$

C.2 Calibration

Let us now calibrate the model. To allow comparison with my model, I use the same targets as in part 5 when they apply (i.e unrelated to multiple-worker firms/decreasing returns): the UE and EU rates, the ratio of home production b over productivity, the quarterly persistence of new jobs, and the autocorrelation/variance of output. Table 5 and 6 summarize the calibration.

Parameter	Value	Description
Pre-calibrated:		
A	1	Technology parameter
β	0.996	Monthly discount factor
γ	1.60	Job-finding probability parameter
Calibrated:		
b	0.79	Home production
c	1.9	Vacancy posting cost
ρ_z	0.96	Persistence of idiosyncratic productivity z
σ_z	0.21	Standard deviation of idiosyncratic productivity z
ρ_y	0.985	Persistence of aggregate productivity y
σ_y	0.048	Standard deviation of aggregate productivity y

Table 5: Calibrated parameters of the MP94 model

Moment	Empirical value	Simulated
UE rate	0.45	0.450
EU rate	0.026	0.026
b / productivity	71%	71%
New jobs persistence	68 %	68%
$\rho[\log(output)]$	0.92	0.92
$\sigma[\log(output)]$	0.026	0.026

Notes: UE and EU are monthly transition rates. New jobs persistence, autocorrelation and standard deviations of log-detrended output are stated in quarterly terms.

Table 6: Calibrated moments of the MP94 model

C.3 Response to aggregate productivity and uncertainty shocks

I now proceed to the same exercise as with my model. I examine the response of the model to a series of aggregate productivity shocks calibrated to reproduce the empirical series of output per person between 2007 and 2010. An additional uncertainty shock hits the economy in 2008Q3. To increase the chance of the model to match the data, the uncertainty shock is calibrated to produce an increase of 42% in unemployment as in the data (in log terms). An increase of 6% in σ_z offers a nice fit. Figure 22 displays the series produced by the model.

The model predicts an initial drop in output of 3.4%, slightly smaller than the 4.5% in the data. Output then experiences a counterfactually large recovery, stronger than the one predicted by my model and the one observed in the data. After an initial peak of about



Figure 22: Response of MP94 to a series of aggregate productivity and uncertainty shocks

42%, unemployment displays little persistence and goes back rapidly to its trend. We can conclude from this exercise that uncertainty shocks can help the model explain some qualitative patterns from the data, but that it fails on some other dimensions quantitatively. An important reason lies in the absence of decreasing returns: without decreasing returns, the rise in productivity following an uncertainty shock is small; the recovery in productivity observed since 2009 is therefore mainly absorbed by the aggregate productivity shock series. As a result, the calibrated productivity shocks are much larger than in the model with decreasing returns: the recovery in output is stronger and unemployment less persistent.

D Proofs of part 2

D.1 Part 2.6

Proof of proposition 1. Denote some firm's policy by $\{w, \hat{\tau}, \hat{x}, \hat{d}, \hat{W}, \hat{v}, \hat{X}\}$ and write the sum of the surpluses:

$$\begin{aligned}
& \tilde{\mathbf{V}}(y, z, n, \varphi, \{w, \hat{\tau}, \hat{x}, \hat{d}, \hat{W}, \hat{v}, \hat{X}\}) \\
& \equiv \mathbf{J}(y, z, n, \varphi, \{w, \hat{\tau}, \hat{x}, \hat{d}, \hat{W}, \hat{v}, \hat{X}\}) + \int \mathbf{W}(y, z; \{w, \hat{\tau}, \hat{x}, \hat{d}, \hat{W}\}) d\varphi \\
& = e^{y+z} F(n) - k_f + \beta E_{\hat{y}, \hat{z}} \left\{ n \hat{d} \mathbf{U}(\hat{y}) + (1 - \hat{d}) \left(\mathbf{U}(\hat{y}) \int \hat{\tau} d\varphi + \int (1 - \hat{\tau}) \lambda p(\hat{x}) \hat{x} d\varphi \right. \right. \\
& \quad \left. \left. - c \hat{v} + \underbrace{\mathbf{J}(\hat{y}, \hat{z}, \hat{n}, \hat{\varphi}) + \int (1 - \hat{\tau})(1 - \lambda p(\hat{x})) \hat{W} d\varphi}_{\mathbf{J}(\hat{y}, \hat{z}, \hat{n}, \hat{\varphi}) + \int \hat{W} d\hat{\varphi} - \hat{v} q(\hat{X}) \hat{X}} \right) \right\}.
\end{aligned}$$

Rewriting the last expression, we obtain:

$$\begin{aligned}
& \tilde{\mathbf{V}}(y, z, n, \varphi, \{w, \hat{\tau}, \hat{x}, \hat{d}, \hat{W}, \hat{v}, \hat{X}\}) \\
& = e^{y+z} F(n) - k_f + \beta E_{\hat{y}, \hat{z}} \left\{ n \hat{d} \mathbf{U}(\hat{y}) + (1 - \hat{d}) \left(\mathbf{U}(\hat{y}) \int \hat{\tau} d\varphi + \int (1 - \hat{\tau}) \lambda p(\hat{x}) \hat{x} d\varphi \right. \right. \\
& \quad \left. \left. - (c + q(\hat{X}) \hat{X}) \hat{v} + \underbrace{\mathbf{J}(\hat{y}, \hat{z}, \hat{n}, \hat{\varphi}) + \int \hat{W} d\hat{\varphi}(\hat{W})}_{\equiv \mathbf{V}(\hat{y}, \hat{z}, \hat{n}, \hat{\varphi})} \right) \right\}. \tag{12}
\end{aligned}$$

Function $\tilde{\mathbf{V}}$ is the joint surplus for some arbitrary contract (not necessarily optimal). To show the result, I will first show that if $\{w^*, \hat{\tau}^*, \hat{x}^*, \hat{d}^*, \hat{W}^*, \hat{v}^*, \hat{X}^*\}$ solves the firm's problem, it must maximize function $\tilde{\mathbf{V}}$. By contradiction, assume that $\{w^*, \hat{\tau}^*, \hat{x}^*, \hat{d}^*, \hat{W}^*, \hat{v}^*, \hat{X}^*\}$ does

not, and assume that there exists another policy $\{\tilde{w}, \tilde{\tau}, \tilde{x}, \tilde{d}, \tilde{W}, \tilde{v}, \tilde{X}\}$ such that:

$$\tilde{\mathbf{V}}(y, z, n, \varphi, \{\tilde{w}, \tilde{\tau}, \tilde{x}, \tilde{d}, \tilde{W}, \tilde{v}, \tilde{X}\}) > \tilde{\mathbf{V}}(y, z, n, \varphi, \{w^*, \hat{\tau}^*, \hat{x}^*, \hat{d}^*, \hat{W}^*, \hat{v}^*, \hat{X}^*\}).$$

If the firm wanted to use the new contract policy $\{\tilde{w}, \tilde{\tau}, \tilde{x}, \tilde{d}, \tilde{W}, \tilde{v}, \tilde{X}\}$, only one problem may arise: it may not satisfy the promise-keeping constraint. However, it is possible for the firm to increase or reduce the wage of its workers so that the promise keeping constraint is exactly satisfied. This transformation does not affect the joint surplus, as it only requires internal transfers between the firm and the workers. Without loss of generality, we can therefore choose wages $\tilde{w}$ such that:

$$\forall W, \quad \mathbf{W}(y, z; \{\tilde{w}(W), \tilde{\tau}(W), \tilde{x}(W), \tilde{d}(W), \tilde{W}(W)\}) = W.$$

The firm's profit under this contract is:

$$\begin{aligned} \mathbf{J}(y, z, n, \varphi, \{\tilde{w}, \tilde{\tau}, \tilde{x}, \tilde{d}, \tilde{W}, \tilde{v}, \tilde{X}\}) &= \tilde{\mathbf{V}}(y, z, n, \varphi, \{\tilde{w}, \tilde{\tau}, \tilde{x}, \tilde{d}, \tilde{W}, \tilde{v}, \tilde{X}\}) - \int W d\varphi \\ &> \tilde{\mathbf{V}}(y, z, n, \varphi, \{w^*, \hat{\tau}^*, \hat{x}^*, \hat{d}^*, \hat{W}^*, \hat{v}^*, \hat{X}^*\}) - \int W d\varphi \end{aligned} \quad (13)$$

Since $\{w^*, \hat{\tau}^*, \hat{x}^*, \hat{d}^*, \hat{W}^*, \hat{v}^*, \hat{X}^*\}$ solves the firm's problem by assumption, the promise-keeping constraint is satisfied:

$$\forall W, \quad \mathbf{W}(y, z; \{w^*(W), \hat{\tau}^*(W), \hat{x}^*(W), \hat{d}^*(W), \hat{W}^*(W)\}) \geq W.$$

Therefore, $\int \mathbf{W}(y, z; \{w^*, \hat{\tau}^*, \hat{x}^*, \hat{d}^*, \hat{W}^*\}) d\varphi \geq \int W d\varphi$ and substituting back in (13), we can conclude:

$$\begin{aligned} &\mathbf{J}(y, z, n, \varphi, \{\tilde{w}, \tilde{\tau}, \tilde{x}, \tilde{d}, \tilde{W}, \tilde{v}, \tilde{X}\}) \\ &> \tilde{\mathbf{V}}(y, z, n, \varphi, \{w^*, \hat{\tau}^*, \hat{x}^*, \hat{d}^*, \hat{W}^*, \hat{v}^*, \hat{X}^*\}) - \int \mathbf{W}(y, z; \{w^*, \hat{\tau}^*, \hat{x}^*, \hat{d}^*, \hat{W}^*\}) d\varphi \\ &> \mathbf{J}(y, z, n, \varphi, \{w^*, \hat{\tau}^*, \hat{x}^*, \hat{d}^*, \hat{W}^*, \hat{v}^*, \hat{X}^*\}) \end{aligned}$$

This is the contradiction we were looking for. We have thus proved result (ii). We can now conclude for (i):

$$\begin{aligned} \mathbf{V}(y, z, n, \varphi) &\equiv \mathbf{J}(y, z, n, \varphi) + \int W d\varphi(W) \\ &= \mathbf{J}(y, z, n, \varphi, \{w^*, \hat{\tau}^*, \hat{x}^*, \hat{d}^*, \hat{W}^*, \hat{v}^*, \hat{X}^*\}) + \int \mathbf{W}(y, z; \{w^*, \hat{\tau}^*, \hat{x}^*, \hat{d}^*, \hat{W}^*\}) d\varphi \\ &= \tilde{\mathbf{V}}(y, z, n, \varphi, \{w^*, \hat{\tau}^*, \hat{x}^*, \hat{d}^*, \hat{W}^*, \hat{v}^*, \hat{X}^*\}) \\ &= \max_{w, \hat{\tau}, \hat{x}, \hat{d}, \hat{W}, \hat{v}, \hat{X}} \tilde{\mathbf{V}}(y, z, n, \varphi, \{w, \hat{\tau}, \hat{x}, \hat{d}, \hat{W}, \hat{v}, \hat{X}\}) \end{aligned}$$

In other words, we have proved (i), i.e $\mathbf{V}$ solves Bellman equation (4).

(iii) This part follows closely the previous proof. Start with a surplus maximizing policy $\{\hat{\tau}^S, \hat{x}^S, \hat{d}^S, \hat{W}^S, \hat{v}^S, \hat{X}^S\}$. Choose the unique w^S such that the promise-keeping constraint binds for all workers. Then, the firm's profit is maximized:

$$\begin{aligned} &\mathbf{J}(y, z, n, \varphi, \{w^S, \hat{\tau}^S, \hat{x}^S, \hat{d}^S, \hat{W}^S, \hat{v}^S, \hat{X}^S\}) \\ &= \tilde{\mathbf{V}}(y, z, n, \varphi, \{w^S, \hat{\tau}^S, \hat{x}^S, \hat{d}^S, \hat{W}^S, \hat{v}^S, \hat{X}^S\}) - \int W d\varphi(W) \\ &= \mathbf{V}(y, z, n, \varphi) - \int W d\varphi(W) = \mathbf{J}(y, z, n, \varphi). \end{aligned}$$

□

E Proofs of part 3

E.1 Part 3.1

Definition 2 (full) Let $\mathcal{V}$ be the set of value functions $V : (y, z, n) \in \mathcal{Y} \times \mathcal{Z} \times [0, \bar{n}] \rightarrow \mathbb{R}$ (i) strictly increasing in n , (ii) satisfying $\forall y, E_{g_z} V(y, z, 0)^+ \leq \beta k_e^{12}$, (iii) bounded in $[\underline{V}, \bar{V}]$, (iv) bi-lipschitz continuous in n such that

$$\forall V \in \mathcal{V}, \forall (y, z), \forall n_2 \geq n_1, \quad \underline{\alpha}_V(n_2 - n_1) \leq V(y, z, n_2) - V(y, z, n_1) \leq \bar{\alpha}_V(n_2 - n_1),$$

with

$$\begin{aligned} \underline{\alpha}_V &= e^{\underline{y} + \underline{z}} \underline{\alpha}_F + \beta(1 - \beta)^{-1} b > 0, \\ \bar{\alpha}_V &= (1 - \beta)^{-1} (e^{\bar{y} + \bar{z}} \bar{\alpha}_F + \beta((1 - \beta)^{-1} (b + \beta \bar{x}) + \lambda \bar{x})), \\ \underline{V} &= -k_f, \\ \bar{V} &= (1 - \beta)^{-1} [e^{\bar{y} + \bar{z}} F(\bar{n}) - k_f + \beta \bar{n} (\lambda \bar{x} + (1 - \beta)^{-1} (b + \beta \bar{x}))]. \end{aligned}$$

Proof of lemma 1. For $V \in \mathcal{V}$, $y \in \mathcal{Y}$, and $\kappa \in \mathbb{R}$, define

$$\psi^V(\kappa) = \max_{0 \leq n_e(z) \leq \bar{n}} E_{g_z}(V(y, z, n_e) - \kappa n_e)^+$$

where $\bar{n}$ is an upper bound on the firm sizes, chosen sufficiently large so that it does not constrain the equilibrium. Since V is continuous in n and z has a finite support, ψ^V is a well-defined function for $\kappa \in \mathbb{R}$. The Theorem of the Maximum tells us that ψ^V is a continuous function of κ . Notice that V being increasing in n , $\psi^V(0) = E_{g_z} V(y, z, \bar{n})^+$. Also, since V is bi-lipschitz continuous with parameters $(\underline{\alpha}_V, \bar{\alpha}_V)$, for $\kappa \geq \bar{\alpha}_V$, $E_{g_z} V(y, z, n_e) - \kappa n_e$ is maximized at $n_e = 0$ and $\psi^V(\kappa) = E_{g_z} V(y, z, 0)^+$. Let us show that ψ^V is a decreasing function of κ . Take $\kappa_1 < \kappa_2$ and the corresponding $n_{i,i=1,2}$ that solve the maximization problem. Denote $\mathcal{Z}_i = \{z \in \mathcal{Z} | V(y, z, n_i) - \kappa_i n_i \geq 0\}$. Then, we have:

$$\begin{aligned} \psi^V(\kappa_1) - \psi^V(\kappa_2) &= E_{g_z}(V(y, z, n_1) - \kappa_1 n_1)^+ - E_{g_z}(V(y, z, n_2) - \kappa_2 n_2)^+ \\ &\geq E_{g_z}[(V(y, z, n_2) - \kappa_1 n_2) \mathbb{I}_{\mathcal{Z}_2}] - E_{g_z}[(V(y, z, n_2) - \kappa_2 n_2) \mathbb{I}_{\mathcal{Z}_2}] \\ &\geq (\kappa_2 - \kappa_1) E_{g_z}[n_2 \mathbb{I}_{\mathcal{Z}_2}] \end{aligned}$$

symmetrically, we can establish that $\psi^V(\kappa_1) - \psi^V(\kappa_2) \leq (\kappa_2 - \kappa_1) E_{g_z}[n_1 \mathbb{I}_{\mathcal{Z}_1}]$. ψ^V is thus decreasing. But this also tells us that if we denote $\bar{\kappa}$ the smallest κ such that $\psi^V(\kappa) = E_{g_z} V(y, z, 0)^+$ (i.e for which $n_e = 0$ is optimal for all z), then we have that ψ^V is strictly decreases on $[0, \bar{\kappa}]$ from $E_{g_z} V(y, z, \bar{n})^+$ to $E_{g_z} V(y, z, 0)^+$ and remains constant thereafter.

If $E_{g_z} V(y, z, 0)^+ < k_e < E_{g_z} V(y, z, \bar{n})^+$, the Intermediate Value Theorem tells us that there exists a unique $\kappa^V(y)$ such that $\psi^V(\kappa^V(y)) = k_e$. This establishes (i):

$$\theta^V(X, y) > 0 \Leftrightarrow c/q(\theta(X, y)) + X = \kappa^V(y).$$

Also, we have (ii): there exists a $n_e^V(y, z) \geq 0$ chosen by entering firms so that

$$k_e = E_{g_z} V(y, z, n_e^V(y, z)) - \kappa^V(y) n_e^V(y, z).$$

For firms that decide not to enter, set $n_e^V(y, z) = 0$.

¹²This condition guarantees the existence of a solution to the free-entry problem.

To conclude, we only need to check that $E_{g_z} V(y, z, 0)^+ < k_e < E_{g_z} V(y, z, \bar{n})^+$. The left-hand side is guaranteed by the fact that $V \in \mathcal{V}$. The right-hand side is guaranteed by assumption 3 as we have $E_{g_z} V(y, z, \bar{n})^+ \geq E_{g_z} V(y, z, 0) + \underline{\alpha}_V \bar{n} > -k_f + \underline{\alpha}_V \bar{n} \geq k_e$.

(iii) The complementary slackness condition (6) implies that either:

$$\theta(X, y) = 0 \quad \text{or} \quad c/q(\theta(X, y)) + X = \kappa^V(y).$$

For $X > \kappa^V(y)$, the second expression admits no solution as the probability q must remain below 1. So θ must be 0 in this region. For $X \leq \kappa^V(y)$, it admits the unique solution $q^{-1}\left(\frac{c}{\kappa^V(y) - X}\right)$. In this region: $c/q(0) + X < \kappa^V$, so $\psi^V(c/q(0) + X) > k_e$. $\theta(X, y)$ cannot be 0 otherwise it would violate the free-entry condition (5). To summarize our results:

$$\theta^V(X, y) = \begin{cases} q^{-1}\left(\frac{c}{\kappa^V(y) - X}\right), & \text{for } \underline{x} \leq X \leq \kappa^V(y) - c, \\ 0, & \text{for } X \geq \kappa^V(y) - c. \end{cases}$$

□

Proof of proposition 2. To prove the existence, I will proceed in 4 steps: (1) establish existence, uniqueness and boundedness of $U^V(y)$ given some $V \in \mathcal{V}$, (2) show that T is a well-defined mapping from $\mathcal{V}$ to $\mathcal{V}$, (3) T is a continuous mapping, (4) $T(\mathcal{V})$ is an equicontinuous family. Since $\mathcal{V}$ is closed, bounded and convex, using Schauder's Fixed Point Theorem as stated in Stokey & Lucas, Theorem 17.4 p.520, this will establish the existence of a solution $\mathbf{V}$ in $\mathcal{V}$ to Bellman equation (4).

Step 1. For $V \in \mathcal{V}$, lemma 1 gives the existence and uniqueness of functions κ^V , n_e^V , θ^V and therefore p^V and q^V . We are going to show that the following mapping M_V that defines U^V is a contraction from the space of functions $U : \mathcal{Y} \rightarrow \mathbb{R}$, bounded between some $\underline{U}$ and $\bar{U}$, to be defined later.

$$M^V U(y) = b + \beta E_{\hat{y}, \hat{z}} \left\{ U(\hat{y}) + \max_{\hat{x}_u(\hat{y})} p^V(\hat{x}_u)(\hat{x}_u - U(\hat{y})) \right\}$$

Applying Blackwell's sufficient conditions for a contraction mapping, check *discounting*: for $a \geq 0$,

$$\begin{aligned} M^V(U + a) &= b + \beta E_{\hat{y}, \hat{z}} \left\{ U(\hat{y}) + a + \max_{\hat{x}_u(\hat{y})} p^V(\hat{x}_u)(\hat{x}_u - U(\hat{y}) - a) \right\} \\ &\leq M^V U + \beta a. \end{aligned}$$

Check *monotonicity*: for $U_1 \leq U_2$, and corresponding optimal choices $\hat{x}_i$, for $i = 1, 2$,

$$\begin{aligned} M^V(U_2) - M^V(U_1) &= \beta E_{\hat{y}, \hat{z}} \left\{ U_2(\hat{y}) - U_1(\hat{y}) + p^V(\hat{x}_2)(\hat{x}_2 - U_2(\hat{y})) - p^V(\hat{x}_1)(\hat{x}_1 - U_1(\hat{y})) \right\} \\ &\geq \beta E_{\hat{y}, \hat{z}} \left\{ U_2(\hat{y}) - U_1(\hat{y}) + p^V(\hat{x}_1)(U_1(\hat{y}) - U_2(\hat{y})) \right\} \geq 0. \end{aligned}$$

It is easy to show now that if $\underline{U} \leq U \leq \bar{U}$, then

$$b + \beta \underline{U} \leq M^V U \leq b + \beta(\bar{x} + \bar{U}).$$

The unique fixed point of M^V is therefore bounded between $\underline{U} = (1 - \beta)^{-1}b$ and $\bar{U} = (1 - \beta)^{-1}(b + \beta\bar{x})$.

Step 2. Let us now check that T is a well defined mapping from $\mathcal{V}$ to $\mathcal{V}$. For what follows, it is useful to denote some policy $\gamma = \{\hat{\tau}, \hat{x}, \hat{d}, \hat{n}_i, \hat{X}\}$, and define

$$\begin{aligned} \Phi^V(y, z, n, \gamma) = & e^{y+z} F(n) - k_f + \beta E_{\hat{y}, \hat{z}} \left\{ n \hat{d} \mathbf{U}^V(\hat{y}) + (1 - \hat{d}) \left(\mathbf{U}^V(\hat{y}) \int \hat{\tau} dj \right. \right. \\ & \left. \left. + \int (1 - \hat{\tau}) \lambda p^V(\hat{x}) \hat{x} dj - \kappa^V(\hat{y}) \hat{n}_i + V(\hat{y}, \hat{z}, \hat{n}) \right) \right\}. \end{aligned}$$

Φ denotes the current joint surplus evaluated at some arbitrary policy γ .

(i) If $V \in \mathcal{V}$, then TV is strictly increasing in n . Take $n_1 < n_2$ and the corresponding optimal policies γ_1 and γ_2 .

$$\begin{aligned} TV(y, z, n_2) - TV(y, z, n_1) &= \Phi(y, z, n_2, \gamma_2) - \Phi(y, z, n_1, \gamma_1) \\ &\geq \Phi(y, z, n_2, \tilde{\gamma}) - \Phi(y, z, n_1, \gamma_1) \end{aligned}$$

with a suboptimal policy $\tilde{\gamma} = \{\tilde{\tau}, \tilde{x}, \tilde{d}, \tilde{n}_i, \tilde{X}\}$ such that $\tilde{x} = \hat{x}_1$, $\tilde{d} = \hat{d}_1$, $\tilde{n}_i = \hat{n}_{i1}$, $\tilde{X} = \hat{X}_{i1}$, and $\tilde{\tau} = \hat{\tau}_1$ for $j \in [0, n_1]$ and 1 for $j \in [n_1, n_2]$. In that case, we have $\tilde{n} = \hat{n}_1$, and many terms cancel to yield the desired result that TV is strictly increasing in n .

$$\begin{aligned} TV(y, z, n_2) - TV(y, z, n_1) &\geq \Phi(y, z, n_2, \tilde{\gamma}) - \Phi(y, z, n_1, \gamma_1) \\ &\geq e^{y+z} (F(n_2) - F(n_1)) + \beta E_{\hat{y}, \hat{z}} \left\{ (n_2 - n_1) \hat{d}_1 \mathbf{U}^V(\hat{y}) + (1 - \hat{d}_1) \mathbf{U}^V(\hat{y}) (n_2 - n_1) \right\} > 0. \end{aligned}$$

(ii) If $V \in \mathcal{V}$, then $\forall y, E_{g_z} TV(y, z, 0)^+ < k_e$. Recall that

$$TV(y, z, 0) = \max_{w, \hat{\tau}, \hat{x}, \hat{d}, \hat{W}, \hat{n}_i, \hat{X}} -k_f + \beta E_{\hat{y}, \hat{z}} \left\{ (1 - \hat{d}) \left(-\kappa^V(\hat{y}) \hat{n}_i + V(\hat{y}, \hat{z}, \hat{n}_i) \right) \right\} \geq -k_f.$$

Thus, integrating over the stationary distribution g_z and using Fubini's theorem:

$$\begin{aligned} E_{g_z} TV(y, z, 0) &= \max_{\hat{d}(\hat{y}, \hat{z}), \hat{n}_i(\hat{y}, \hat{z})} -k_f + \beta E_{g_z} E_{\hat{y}, \hat{z}} \left\{ (1 - \hat{d}) \left(-\kappa^V(\hat{y}) \hat{n}_i + V(\hat{y}, \hat{z}, \hat{n}_i) \right) \right\} \\ &= \max_{\hat{n}_i(\hat{y}, \hat{z})} -k_f + \beta E_{\hat{y}} E_{g_z} [V(\hat{y}, \hat{z}, \hat{n}_i) - \kappa^V(\hat{y}) \hat{n}_i]^+ \\ &= -k_f + \beta k_e \end{aligned}$$

where we have recognized the free-entry problem in the next period. Now, since $TV(y, z, 0) \geq -k_f$,

$$\begin{aligned} E_{g_z} TV(y, z, 0)^+ &= E_{g_z} \max(TV(y, z, 0), 0) \leq E_{g_z} \max(TV(y, z, 0) + k_f, 0) \\ &\leq E_{g_z} TV(y, z, 0) + k_f \leq \beta k_e \end{aligned}$$

(iii) If $V \in \mathcal{V}$, then TV is bounded in $[\underline{V}, \bar{V}]$ with $\underline{V} = -k_f$ and $\bar{V} = (1 - \beta)^{-1} [e^{\bar{y} + \bar{z}} F(\bar{n}) - k_f + \beta \bar{n} (\lambda \bar{x} + (1 - \beta)^{-1} (b + \beta \bar{x}))]$.

$$TV(y, z, n) \leq e^{\bar{y} + \bar{z}} F(\bar{n}) - k_f + \beta [\bar{n} \bar{U} + \bar{n} \lambda \bar{x} + \bar{V}] \leq \bar{V}.$$

Now, for the lower bound:

$$\begin{aligned} TV(y, z, n) &\geq \Phi(y, z, n_2, \tilde{\gamma}) \\ &\geq e^{\underline{y} + \bar{z}} F(n) - k_f + \beta n \underline{U} \geq -k_f = \underline{V} \end{aligned}$$

with suboptimal policy $\tilde{\gamma}$ such that $\tilde{d} = 1$.

(iv) If $V \in \mathcal{V}$, then

$$\forall(y, z), \forall n_2 \geq n_1, \quad \underline{\alpha}_V(n_2 - n_1) \leq TV(y, z, n_2) - TV(y, z, n_1) \leq \bar{\alpha}_V(n_2 - n_1).$$

Take $n_2 \geq n_1$, and corresponding optimal policies $\gamma_i, i = 1, 2$. Choose a suboptimal policy $\tilde{\gamma}$ such that $\tilde{d} = \hat{d}_2$, $\tilde{x} = \hat{\tau}_2$, $\tilde{n}_i = \hat{n}_{i2}$, $\tilde{X}_i = \hat{X}_{i2}$, $\tilde{\tau}(j) = \hat{\tau}_2(j)$ for $j \in [0, n_1]$ and 1 for $j \in [n_1, n_2]$.

$$\begin{aligned} TV(y, z, n_2) - TV(y, z, n_1) &= \Phi(y, z, n_2, \gamma_2) - \Phi(y, z, n_1, \gamma_1) \leq \Phi(y, z, n_2, \gamma_2) - \Phi(y, z, n_1, \tilde{\gamma}) \\ &\leq e^{\underline{y} + \bar{z}} (F(n_2) - F(n_1)) + \beta E_{\hat{y}, \hat{z}} \left\{ (n_2 - n_1) \hat{d}_2 \mathbf{U}^V(\hat{y}) + (1 - \hat{d}_2) \left(\mathbf{U}^V(\hat{y}) \int_{n_1}^{n_2} \hat{\tau}_2 dj \right. \right. \\ &\quad \left. \left. + \int_{n_1}^{n_2} (1 - \hat{\tau}_2) \lambda p^V(\hat{x}_2) \hat{x}_2 dj + V(\hat{y}, \hat{z}, \hat{n}_2) - V(\hat{y}, \hat{z}, \hat{n}_1) \right) \right\} \\ &\leq \left[e^{\bar{y} + \bar{z}} \bar{\alpha}_F + \beta(\bar{U} + \lambda \bar{x} + \bar{\alpha}_V) \right] (n_2 - n_1) = \bar{\alpha}_V(n_2 - n_1) \end{aligned}$$

Proceed similarly for the other side and choose a policy $\tilde{\gamma}$ such that $\tilde{d} = \hat{d}_1$, $\tilde{x} = \hat{\tau}_1$, $\tilde{n}_i = \hat{n}_{i1}$, $\tilde{X} = \hat{X}_1$, $\tilde{\tau}(j) = \hat{\tau}_1(j)$ for $j \in [0, n_1]$ and 1 for $j \in [n_1, n_2]$.

$$\begin{aligned} TV(y, z, n_2) - TV(y, z, n_1) &= \Phi(y, z, n_2, \gamma_2) - \Phi(y, z, n_1, \gamma_1) \geq \Phi(y, z, n_2, \tilde{\gamma}) - \Phi(y, z, n_1, \gamma_1) \\ &\geq e^{\underline{y} + \bar{z}} (F(n_2) - F(n_1)) + \beta E_{\hat{y}, \hat{z}} \left\{ (n_2 - n_1) \hat{d}_2 \mathbf{U}^V(\hat{y}) + (1 - \hat{d}_2) (n_2 - n_1) \mathbf{U}^V(\hat{y}) \right\} \\ &\geq \left[e^{\underline{y} + \bar{z}} \underline{\alpha}_F + \beta \underline{U} \right] (n_2 - n_1) = \underline{\alpha}_V(n_2 - n_1) \end{aligned}$$

Therefore, TV is bi-lipschitz continuous with the desired coefficients.

Step 3. We are now going to show that $T : \mathcal{V} \rightarrow \mathcal{V}$ is a continuous mapping. Denote by $\|\cdot\|$ the infinite norm, i.e $\|V\| = \sup_{(y, z, n) \in \mathcal{Y} \times \mathcal{Z} \times [0, \bar{n}]} V(y, z, n)$. Take $V_1, V_2 \in \mathcal{V}$. For (y, z, n) fixed, denote $\gamma_i, i = 1, 2$, the corresponding optimal policies.

$$\begin{aligned} TV_1(y, z, n) - TV_2(y, z, n) &= \Phi^{V_1}(y, z, n, \gamma_1) - \Phi^{V_2}(y, z, n, \gamma_2) \leq \Phi^{V_1}(y, z, n, \gamma_1) - \Phi^{V_2}(y, z, n, \gamma_1) \\ &\leq \beta E_{\hat{y}, \hat{z}} \left\{ n \hat{d}_1 (\mathbf{U}^{V_1}(\hat{y}) - \mathbf{U}^{V_2}(\hat{y})) + (1 - \hat{d}_1) \left((\mathbf{U}^{V_1}(\hat{y}) - \mathbf{U}^{V_2}(\hat{y})) \int \hat{\tau}_1 dj \right. \right. \\ &\quad \left. \left. + \int (1 - \hat{\tau}_1) \lambda (p^{V_1}(\hat{x}_1) - p^{V_2}(\hat{x}_1)) \hat{x}_1 dj - (\kappa^{V_1}(\hat{y}) - \kappa^{V_2}(\hat{y})) \hat{n}_{i1} + V_1(\hat{y}, \hat{z}, \hat{n}_1) - V_2(\hat{y}, \hat{z}, \hat{n}_1) \right) \right\} \\ &\leq \beta \left[\bar{n} \|\mathbf{U}^{V_1} - \mathbf{U}^{V_2}\| + \bar{n} \lambda \bar{x} \|p^{V_1} - p^{V_2}\| + \bar{n} \|\kappa^{V_1} - \kappa^{V_2}\| + \|V_1 - V_2\| \right] \end{aligned}$$

According to lemma 2 below, we can control each term:

$$TV_1(y, z, n) - TV_2(y, z, n) \leq \beta(\bar{n} \alpha_U + \bar{n} \lambda \bar{x} \alpha_p + \bar{n} \alpha_\kappa + 1) \|V_1 - V_2\|$$

which can be made arbitrarily small as $\|V_1 - V_2\|$ gets smaller. Therefore, T is a continuous mapping.

Lemma 2. *If $V_1, V_2 \in \mathcal{V}$, then*

- (i) $\|\theta^{V_1} - \theta^{V_2}\| \leq \alpha_\theta \|V_1 - V_2\|$, with $\alpha_\theta = \frac{1}{cn_{min}q'(\theta_{max})}$,
- (ii) $\|p^{V_1} - p^{V_2}\| \leq \alpha_p \|V_1 - V_2\|$, with $\alpha_p = p'(0)\alpha_\theta$,
- (iii) $\|\kappa^{V_1} - \kappa^{V_2}\| \leq \alpha_\kappa \|V_1 - V_2\|$, with $\alpha_\kappa = c \frac{|q'(0)|}{q^2(\theta_{max})} \alpha_\theta$,
- (iv) $\|\mathbf{U}^{V_1} - \mathbf{U}^{V_2}\| \leq \alpha_U \|V_1 - V_2\|$, with $\alpha_U = \beta(1 - \beta)^{-1}(\bar{x} + \bar{U})\alpha_p$.

Proof. To prove the lemma, we first need to establish the two following results. Let us prove that there exists $\theta_{max} > 0$ such that

$$\forall V \in \mathcal{V}, \theta^V(\cdot) \leq \theta_{max},$$

and there exists $n_{min} > 0$ such that

$$\forall V \in \mathcal{V}, E_{g_z} n_e^V(y, z) \geq n_{min}.$$

The first result can be established by the fact that $\kappa_V \leq \bar{\alpha}_V$. Then for some $X \in [\underline{x}, \bar{x}]$:

$$c/q(\theta^V(y, X)) + X \leq \bar{\alpha}_V \Rightarrow q(\theta^V(y, X)) \geq c(\bar{\alpha}_V - X)^{-1} \Rightarrow \theta^V(y, X) \leq q^{-1}[c(\bar{\alpha}_V - \underline{x})^{-1}].$$

Setting $\theta_{max} = q^{-1}[c(\bar{\alpha}_V - \underline{x})^{-1}]$ yields the desired result.

Now, for the second result, remember the free-entry condition with positive entry:

$$k_e = E_{g_z} [V(y, z, n_e^V) - \kappa^V n_e^V]^+.$$

Then, using the fact that V is bi-lipschitz:

$$k_e \leq E_{g_z} V(y, z, n_e^V)^+ \leq E_{g_z} [\bar{\alpha}_V n_e^V + V(y, z, 0)]^+ \leq \bar{\alpha}_V E_{g_z} [n_e^V] + E_{g_z} V(y, z, 0)^+$$

Since $E_{g_z} V(y, z, 0)^+ \leq \beta k_e$, we have

$$E_{g_z} n_e^V \geq \bar{\alpha}_V^{-1}(1 - \beta)k_e \equiv n_{min}.$$

(i) For $y \in \mathcal{Y}$, $X \in [\underline{x}, \bar{x}]$, and $n : z \in \mathcal{Z} \rightarrow [0, \bar{n}]$. Define:

$$\Psi^V(y, X, \theta, n) = E_{g_z} [V(y, z, n) - (c/q(\theta) + X)n(z)]^+.$$

Recognize that Ψ represents the expected value of firms considering to enter on submarket X with tightness θ . Fix (y, X) , and denote $(\theta_i, n_i) \equiv (\theta^{V_i}(y, X), n_e^{V_i}(y, z, X))$ the free-entry solutions corresponding the two V 's. By definition we have

$$k_e \geq \Psi^{V_i}(y, X, \theta_i, n_i)$$

with equality if submarket X is open, $\theta_i = 0$ otherwise. Denote $\mathcal{Z}_i = \{z \in \mathcal{Z} | V_i(y, z, n_i) - \kappa^{V_i} n_i \geq 0\}$, $i = 1, 2$. Notice that $\mathbb{P}(z \in \mathcal{Z}_i) > 0$ since there is always strictly positive free-entry. If X is closed for V_1 and V_2 , there is nothing to prove as $\theta_1 = \theta_2 = 0$. Now, let us look at the case where submarket X is open for both V_1 and V_2 . We have:

$$\begin{aligned} 0 &= \Psi^{V_1}(y, X, \theta_1, n_1) - \Psi^{V_2}(y, X, \theta_2, n_2) \geq \Psi^{V_1}(y, X, \theta_1, n_2) - \Psi^{V_2}(y, X, \theta_2, n_2) \\ &\geq E_{g_z} \left[\left(V_1(y, z, n_2) - V_2(y, z, n_2) + n_2 \left(\frac{c}{q(\theta_2)} - \frac{c}{q(\theta_1)} \right) \right) \mathbb{1}_{z \in \mathcal{Z}_2} \right]. \end{aligned}$$

We get

$$\mathbb{E}_{g_z}[n_2]c \frac{q(\theta_1) - q(\theta_2)}{q(\theta_1)q(\theta_2)} \leq \|V_1 - V_2\|.$$

Symmetrically, we can establish that

$$\mathbb{E}_{g_z}[n_1]c \frac{q(\theta_1) - q(\theta_2)}{q(\theta_1)q(\theta_2)} \geq -\|V_1 - V_2\|.$$

By assumption, q is convex. Therefore:

$$q'(\min(\theta_1, \theta_2))(\theta_1 - \theta_2) \leq q(\theta_1) - q(\theta_2) \leq q'(\max(\theta_1, \theta_2))(\theta_1 - \theta_2).$$

Combining with the previous inequalities, we obtain the desired inequality:

$$|\theta_1 - \theta_2| \leq \frac{\|V_1 - V_2\|}{cn_{\min}|q'(\theta_{\max})|}.$$

We are left with the case where market X is closed for one of the V 's but not the other. Assume without loss of generality that X is closed for V_1 , i.e $\theta_1 = 0$. In that case:

$$\forall n_1, \Psi(y, X, 0, n_1) < k_e.$$

We can still derive the above inequality that said

$$\forall z \in \mathcal{I}_2, \quad cn_2 \frac{q(\theta_1) - q(\theta_2)}{q(\theta_1)q(\theta_2)} \leq \|V_1 - V_2\|.$$

In which case we have:

$$0 \leq \theta_2 - \theta_1 \leq \frac{\|V_1 - V_2\|}{cn_2|q'(\theta_2)|}.$$

To summarize, in all cases we have:

$$\|\theta^{V_1} - \theta^{V_2}\| \leq \alpha_\theta \|V_1 - V_2\|$$

with $\alpha_\theta = \frac{1}{cn_{\min}|q'(\theta_{\max})|}$.

(ii) Fix (y, X) . By definition:

$$p^{V_1}(y, X) - p^{V_2}(y, X) = p(\theta^{V_1}(y, X)) - p(\theta^{V_2}(y, X)).$$

Using concavity of p :

$$|p(\theta^{V_1}(y, X)) - p(\theta^{V_2}(y, X))| \leq p'(0)|\theta^{V_1}(y, X) - \theta^{V_2}(y, X)| \leq p'(0)\alpha_\theta \|V_1 - V_2\|.$$

(iii) Fix y and take some submarket X open for both V 's. Using the fact that q is decreasing convex:

$$\begin{aligned} |\kappa^{V_1}(y) - \kappa^{V_2}(y)| &= \left| \frac{c}{q(\theta^{V_1}(y, X))} - \frac{c}{q(\theta^{V_2}(y, X))} \right| \\ &\leq c \frac{|q'(0)|}{q^2(\theta_{\max})} |\theta^{V_1}(y, X) - \theta^{V_2}(y, X)| \leq c \frac{|q'(0)|}{q^2(\theta_{\max})} \alpha_\theta \|V_1 - V_2\|. \end{aligned}$$

(iv) Fix y . Denote $\hat{x}_{u,i}, i = 1, 2$ the corresponding optimal choices for unemployed workers.

$$\begin{aligned}
U^{V_1}(y) - U^{V_2}(y) &= \beta E_{\hat{y}, \hat{z}} \left\{ U^{V_1}(\hat{y}) + p^{V_1}(\hat{x}_{u,1})(\hat{x}_{u,1} - U^{V_1}(\hat{y})) \right. \\
&\quad \left. - U^{V_2}(\hat{y}) - p^{V_2}(\hat{x}_{u,2})(\hat{x}_{u,2} - U^{V_2}(\hat{y})) \right\} \\
&\leq \beta E_{\hat{y}, \hat{z}} \left\{ U^{V_1}(\hat{y}) - U^{V_2}(\hat{y}) + p^{V_1}(\hat{x}_{u,1})(\hat{x}_{u,1} - U^{V_1}(\hat{y})) - p^{V_2}(\hat{x}_{u,1})(\hat{x}_{u,1} - U^{V_2}(\hat{y})) \right\} \\
&\leq \beta E_{\hat{y}, \hat{z}} \left\{ U^{V_1}(\hat{y}) - U^{V_2}(\hat{y}) + \hat{x}_{u,1}(p^{V_1}(\hat{x}_{u,1}) - p^{V_2}(\hat{x}_{u,1})) \right. \\
&\quad \left. - p^{V_1}(\hat{x}_{u,1})U^{V_1}(\hat{y}) + p^{V_2}(\hat{x}_{u,1})U^{V_1}(\hat{y}) - p^{V_2}(\hat{x}_{u,1})U^{V_1}(\hat{y}) + p^{V_2}(\hat{x}_{u,1})U^{V_2}(\hat{y}) \right\} \\
&\leq \beta E_{\hat{y}, \hat{z}} \left\{ (1 - p^{V_2}(\hat{x}_{u,1}))(U^{V_1}(\hat{y}) - U^{V_2}(\hat{y})) + (\hat{x}_{u,1} - U^{V_1}(\hat{y}))(p^{V_1}(\hat{x}_{u,1}) - p^{V_2}(\hat{x}_{u,1})) \right\} \\
&\leq \beta E_{\hat{y}, \hat{z}} \left\{ (1 - p^{V_2}(\hat{x}_{u,1}))\|U^{V_1} - U^{V_2}\| + (\hat{x}_{u,1} - U^{V_1}(\hat{y}))\|p^{V_1} - p^{V_2}\| \right\}.
\end{aligned}$$

We can now conclude:

$$\|U^{V_1} - U^{V_2}\| \leq \beta(1 - \beta)^{-1}(\bar{x} + \bar{U})\alpha_p\|V_1 - V_2\|. \quad \square$$

Step 4. We can now proceed to the last step of the proof of proposition 2. We must show that the family $T(\mathcal{V})$ is equicontinuous, i.e $\forall \varepsilon > 0$, there exists $\delta > 0$ such that for $\xi_i = (y_i, z_i, n_i), i = 1, 2$,

$$\|\xi_1 - \xi_2\| < \delta \Rightarrow |TV(\xi_1) - TV(\xi_2)| < \varepsilon, \forall V \in \mathcal{V}.$$

Fix $\varepsilon > 0$ and denote

$$\begin{cases} \eta_y = \min_{y_1 \neq y_2 \in \mathcal{Y}} |y_1 - y_2| \\ \eta_z = \min_{z_1 \neq z_2 \in \mathcal{Z}} |z_1 - z_2|. \end{cases}$$

Choose $\delta < \min(\eta_y, \eta_z, \varepsilon/\bar{\alpha}_V)$. Take (ξ_1, ξ_2) such that $\|\xi_1 - \xi_2\| < \delta$. Therefore, $y_1 = y_2$ and $z_1 = z_2$. Take $V \in \mathcal{V}$. Using the fact that V is bi-lipschitz:

$$|TV(\xi_1) - TV(\xi_2)| \leq \bar{\alpha}_V |n_1 - n_2| \leq \bar{\alpha}_V \|\xi_1 - \xi_2\| < \varepsilon.$$

Conclusion: $T(\mathcal{V})$ is equicontinuous. Schauder's Fixed Point Theorem applies and tells us that there exists a fixed point $\mathbf{V}$ to the mapping T . All other equilibrium objects $\mathbf{U}, \mathbf{W}, \mathbf{J}, \theta, \kappa$ and optimal policy functions are then well defined. $\square$

E.2 Part 3.2

Proof of prop. 3.

We are now going to define the planner's problem in this economy. There is a first difficulty arising in the way we are going to describe the different labor markets. Since the planner can freely allocate workers between firms without respect to any promised utility, the only relevant information concerning each submarket is its labor market tightness. Let us therefore label each submarket by its tightness θ instead of x . Denote by $(\theta_x, \theta_X, \theta_u)$ the markets chosen respectively by firms for on-the-job search, for hirings, and the one chosen by unemployed workers to search. Second, every workers are identical in the eyes of the planner. Given the strict concavity of the problem and therefore the uniqueness of the maximum, I restrict immediately the allocation to be symmetrical between workers of the same firm for x_t . Similarly, as proposition 4 will make it apparent, τ_t should be understood as the total

fraction of layoffs, which is uniquely determined in equilibrium, whereas the exact distribution of layoffs across workers in the same firm is not. To simplify notation, I will drop the explicit dependence of each variables below on the aggregate state $y^t = (y_0, \dots, y_t)$ and the firm's characteristics at the beginning of the period (z_t, n_{t-1}) . The planning problem is to maximize

$$\mathbb{E}_y \sum_t \beta^t \left[\sum_{z_{t-1}, z_t, n_{t-1}} g_{t-1}(z_{t-1}, n_{t-1}) \pi_z(z_{t-1}, z_t) (1 - d_t) (F(n_t) - k_f - cv_t) - k_e h_t + u_t b \right]$$

subject to: $\forall(t, y^t)$,

$$u_t = u_{t-1}(1 - p(\theta_{u,t})) + \sum_{z_{t-1}, z_t, n_{t-1}} g_{t-1}(z_{t-1}, n_{t-1}) \pi_z(z_{t-1}, z_t) n_{t-1} [d_t + (1 - d_t) \tau_t] \quad (14)$$

$$\forall(z_t, n_{t-1}), \quad n_t = n_{t-1}(1 - \tau_t)(1 - \lambda p(\theta_{xt})) + q(\theta_{Xt}) v_t \quad (15)$$

$$\forall(z, n), g_t(z, n) = \sum_{n_{t-1} | n_t = n} (1 - d_t) g_{t-1}(z_{t-1}, n_{t-1}) \pi_z(z_{t-1}, z_t) + h_t g_z(z) \mathbb{I}(n_{et}(z) = n). \quad (16)$$

The objective function is the discounted sum of production net of operating cost k_f and vacancy posting cost c over all existing firms, minus total entry costs for new firms h_t every period and home production b of unemployed agents. The constraints are the law of motions for the unemployment level u_t , the employment in every firm of type (z_t, n_{t-1}) , and distribution g_t . In addition, the planner is subject to constraints for each labor market specifying that the number of workers finding a job is equal to the number of successful job openings. More precisely, for each labor market θ in every period:

$$\begin{aligned} & \sum_{(z_{t-1}, z_t, n_{t-1}) | \theta_{xt} = \theta} \underbrace{g_{t-1}(z_{t-1}, n_{t-1}) \pi_z(z_{t-1}, z_t) (1 - d_t) n_{t-1} (1 - \tau_t) \lambda p(\theta_{xt})}_{\text{successful job-to-job transitions}} + \underbrace{\mathbb{I}(\theta_{ut} = \theta) u_{t-1} p(\theta_{ut})}_{\text{unemployed finding a job}} \\ &= \sum_{(z_{t-1}, z_t, n_{t-1}) | \theta_{Xt} = \theta} \underbrace{g_{t-1}(z_{t-1}, n_{t-1}) \pi_z(z_{t-1}, z_t) (1 - d_t) v_t q(\theta_{Xt})}_{\text{successful job openings by incumbent and entering firms}} \end{aligned} \quad (17)$$

Let us now write the Lagrangian corresponding to the planner's problem. Write μ_t the Lagrange multiplier on constraint (14) and $\eta_t(\theta)$ the one for each market equilibrium (17). The formal planning problem can be written as follows:

$$\begin{aligned} & \max_{\tau_t, \theta_{xt}, d_t, v_t, \theta_{Xt}, h_t, \theta_{ut}, u_t, n_t, g_t} \mathbb{E} \sum_t \beta^t \left\{ \sum_{z_{t-1}, z_t, n_{t-1}} g_{t-1}(z_{t-1}, n_{t-1}) \pi_z(z_{t-1}, z_t) \left[(1 - d_t) (F(n_t) - k_f - cv_t) \right. \right. \\ & \quad \left. \left. - \eta_t(\theta_{Xt}) q(\theta_{Xt}) v_t + \eta_t(\theta_{xt}) n_{t-1} (1 - \tau_t) \lambda p(\theta_{xt}) + \mu_t n_{t-1} (d_t + (1 - d_t) \tau_t) \right] - k_e h_t \right. \\ & \quad \left. + u_t b - \mu_t (u_t - u_{t-1} (1 - p(\theta_{ut}))) + \eta_t(\theta_{ut}) u_{t-1} p(\theta_{ut}) \right\} \end{aligned}$$

subject to equations (15) and (16). As it is written, the objective function is not concave in all its variables, but if we proceed to the changes of variables $\xi_{xt} = p(\theta_{xt})$, $\xi_{ut} = p(\theta_{ut})$ and $\zeta_{Xt} = q(\theta_{Xt})$, then the problem becomes a well-defined strictly concave maximization problem subject to a convex constraint set. The optimum therefore exists and is unique, and the first-order conditions are sufficient to guarantee optimality.

To complete our proof, we are now going to show that our block-recursive competitive equilibrium with positive entry satisfies the first-order condition and is therefore the optimum. Denote the competitive equilibrium $\{\mathbf{V}, \mathbf{U}, \theta^*(x, y)\}$. Guess the following Lagrange multipliers:

$$\begin{aligned}\mu_t(y^t) &= \mathbf{U}(y_t) \\ \eta_t(y^t, \theta) &= x \text{ s.t. } x = \theta^{*-1}(\theta, y_t).\end{aligned}$$

In particular, notice that the Lagrange multipliers only depend on the current aggregate state of the economy y_t and not on its entire history anymore. One may worry here about the feasibility of inverting the equilibrium function θ^* , but we know thanks to lemma 1 that there always exists a corresponding promised utility x for all values of θ in $[0, \infty)^{13}$. Given this guess, we can now recognize that the planner's objective is to sum the joint-surplus $\mathbf{V}$ of incumbent and entering firms and the utility of unemployed workers $\mathbf{U}$. Each of these problems can be solved independently and we know that the policies obtained in the competitive equilibrium maximize each of them. To see this, let us have a look at the part corresponding to an existing firm given our choice of Lagrange multipliers:

$$\begin{aligned}\max_{\tau_t, \theta_{xt}, d_t, v_t, \theta_{Xt}} E_{y,z} \sum_t \beta^t & \left[(1 - d_t)(F(n_t) - k_f - cv_t - q(\theta_{Xt})v_t X_t(\theta_{Xt}) \right. \\ & \left. + n_{t-1}(1 - \tau_t)\lambda p(\theta_{xt})x_t(\theta_{xt})) + n_{t-1}(d_t + (1 - d_t)\tau_t)\mathbf{U}(y_t) \right]\end{aligned}$$

which corresponds exactly to the surplus maximization problem in the competitive equilibrium. Turning to firms entering at date t :

$$\begin{aligned}\max_{\tau_s, \theta_{xs}, d_s, v_s, \theta_{Xs}, s \geq t} h_t & \left\{ E_{g_z} E_{y,z} \sum_{s \geq t} \beta^{s-t} \left[(1 - d_s)(F(n_s) - k_f - cv_s - q(\theta_{Xs})v_s X_s(\theta_{Xs}) \right. \right. \\ & \left. \left. + n_{s-1}(1 - \tau_s)\lambda p(\theta_{xs})x_s(\theta_{xs})) + n_{s-1}(d_s + (1 - d_s)\tau_s)\mathbf{U}(y_s) \right] - k_e \right\}.\end{aligned}$$

This is exactly the free-entry problem solved in the competitive equilibrium. The planner will increase the number of entrants h_t as long as the expected surplus from entering is equal to the entry cost k_e . Now, comparing the part related to unemployed workers:

$$\max_{\theta_{ut}, u_t} \sum_t \beta^t \left[u_t b - \mathbf{U}(y_t)(u_t - u_{t-1}(1 - p(\theta_{ut}))) + u_{t-1}p(\theta_{ut})x_t(\theta_{ut}) \right].$$

The first-order condition with respect to u_{t+1} and θ_{ut} are equal to

$$\begin{aligned}(u_t) \quad & b - \mathbf{U}(y_t) + \beta E_y[p(\theta_{u,t+1})\mathbf{U}(y_{t+1}) + p(\theta_{u,t+1})x_{t+1}(\theta_{u,t+1})] = 0 \\ (\theta_{ut}) \quad & -\mathbf{U}(y_t)p'(\theta_{ut}) + p'(\theta_{ut})x_t(\theta_{ut}) + p(\theta_{ut})x'(\theta_{ut}) = 0.\end{aligned}$$

We recognize in the first equation the Bellman equation faced by unemployed workers, and in the second equation the first-order of the maximization of $p(\theta_{ut})(x_t(\theta_{ut}) - \mathbf{U}(y_t))$ which is exactly the problem they face. Therefore, the policies obtained from the competitive equilibrium maximize the planner's problem given our choice of Lagrange multipliers. The first-order conditions are thus satisfied. We can now conclude that if a block-recursive equilibrium exists with positive entry, it is also the unique efficient equilibrium. $\square$

¹³The bounds $[\underline{x}, \bar{x}]$ are chosen so that the optimal x lies in the interior, so that we are not constraining the equilibrium.

E.3 Part 3.3

Proof of proposition 4. (i) Let us write the utility of the worker under the new contract $\{w + \Delta, \hat{\tau}, \hat{x}, \hat{d}, \hat{W} - a\Delta\}$:

$$\begin{aligned} \mathbf{W}(y, z; \{w + \Delta, \hat{\tau}, \hat{x}, \hat{d}, \hat{W} - a\Delta\}) &= w + \Delta + \beta E_{\hat{y}, \hat{z}} \left\{ \hat{d} \mathbf{U}(\hat{y}) + (1 - \hat{d}) \left(\hat{\tau} \mathbf{U}(\hat{y}) + (1 - \hat{\tau}) \lambda p(\hat{x}) \hat{x} \right. \right. \\ &\quad \left. \left. + (1 - \hat{\tau})(1 - \lambda p(\hat{x}))(\hat{W} - a\Delta) \right) \right\} \\ &= \mathbf{W}(y, z; \{w, \hat{\tau}, \hat{x}, \hat{d}, \hat{W}\}) + \underbrace{\Delta - \beta(1 - \lambda p(\hat{x}))(1 - \hat{\tau})(1 - \hat{d})a\Delta}_{=0} \end{aligned}$$

The worker's utility is unchanged. His promise-keeping constraint is thus still satisfied. Now, write the firm's expected utility:

$$\begin{aligned} \mathbf{J}(y, z, n, \varphi) &= e^{y+z} F(n) - k_f - \int w d\varphi + \beta E_{\hat{y}, \hat{z}} \left\{ (1 - \hat{d}) \left(-c\hat{v} + \mathbf{J}(\hat{y}, \hat{z}, \hat{n}, \hat{\varphi}) \right) \right\} \\ &= e^{y+z} F(n) - k_f - \int w d\varphi + \beta E_{\hat{y}, \hat{z}} \left\{ (1 - \hat{d}) \left(-c\hat{v} + \mathbf{V}(\hat{y}, \hat{z}, \hat{n}) - \int \hat{W} d\hat{\varphi} \right) \right\} \end{aligned}$$

Under the new contract, since only the contract for workers receiving utility W , the firm's next distribution $\hat{\varphi}_\Delta$ is:

$$\forall W', \hat{\varphi}_\Delta(W') = \hat{\varphi}(W') + (1 - \lambda p(\hat{x}(W'))(1 - \hat{\tau}(W'))(-\mathbb{I}(W' \geq \hat{W}) + \mathbb{I}(W' \geq \hat{W} - a\Delta))$$

Therefore, the firm's utility changes by

$$[-\Delta - \beta(1 - \lambda p(\hat{x}))(1 - \hat{\tau})(1 - \hat{d})(-a\Delta)] d\varphi(W) = 0.$$

The new contract leaves the firm and workers indifferent.

(ii) First, recall that:

$$p(\hat{x}) = p \circ q^{-1} \left(\frac{c}{\kappa - \hat{x}} \right)$$

It is easy to show that under assumption 2, $p(\hat{x})$ is a strictly decreasing, strictly concave function on $[\underline{x}, \kappa(y) - c]$.

Let us first show that for each worker, there is a unique $\hat{x} \in [\underline{x}, \kappa(y) - c]$ that maximizes the surplus. The terms that depend on $\hat{x}$ in the joint surplus are the following:

$$\int (1 - \hat{\tau}) \lambda p(\hat{x}) \hat{x} d\varphi + \mathbf{V} \left(\hat{y}, \hat{z}, \int (1 - \hat{\tau})(1 - \lambda p(\hat{x})) d\varphi + \hat{n}_e \right)$$

Proceed to the change of variable $\hat{\xi} = p(\hat{x})$, which is well defined since p is continuous strictly increasing. The term to maximize as a function of $\hat{\xi}$ is:

$$\int (1 - \hat{\tau}) \lambda \hat{\xi} p^{-1}(\hat{\xi}) d\varphi + \mathbf{V} \left(\hat{y}, \hat{z}, \int (1 - \hat{\tau})(1 - \lambda \hat{\xi}) d\varphi + \hat{n}_e \right) \quad (18)$$

p being decreasing, strictly concave, so p^{-1} is also decreasing, strictly concave. Since $\mathbf{V} \in \mathcal{V}$, i.e concave in n , the whole expression is strictly concave in ξ .

Let us show now that all workers have the same $\hat{\xi}$. Assume by contradiction that two workers have different $\hat{x}_1 \neq \hat{x}_2$, i.e $\hat{\xi}_1 \neq \hat{\xi}_2$. Then, since p^{-1} is strictly concave, the corresponding first term in (18) is:

$$(1 - \hat{\tau}_1) p^{-1}(\hat{\xi}_1) \hat{\xi}_1 + (1 - \hat{\tau}_2) p^{-1}(\hat{\xi}_2) \hat{\xi}_2 < [(1 - \hat{\tau}_1) + (1 - \hat{\tau}_2)] p^{-1}(\hat{\xi}') \hat{\xi}'$$

where $\hat{\xi}' = \frac{(1-\hat{\tau}_1)\hat{\xi}_1 + (1-\hat{\tau}_2)\hat{\xi}_2}{2-\hat{\tau}_1-\hat{\tau}_2}$. The surplus can be strictly higher if we offer a contract $(\hat{x}', \hat{\tau}')$ to both workers with the same $\hat{x}' = p^{-1}(\hat{\xi}')$, and a firing probability $\hat{\tau}'$ such that:

$$(1 - \hat{\tau}_1)(1 - \lambda \hat{\xi}_1) + (1 - \hat{\tau}_2)(1 - \lambda \hat{\xi}_2) = 2 \left(1 - \lambda \hat{\xi}'\right) (1 - \hat{\tau}').$$

Therefore, the rest of the expression in (18) is left equal to the case with the initial contract. The new contract $(\hat{x}', \hat{\tau}')$ that sets the two workers identical strictly increases the total surplus. $\hat{x}$ must therefore be the same across workers.

(iii) $\hat{x}$ being the same for all workers, we can rewrite the only terms in the surplus that depend on $\hat{\tau}$ as:

$$\mathbf{U}(\hat{y}) \int \hat{\tau} d\varphi + \lambda p(\hat{x}) \hat{x} \int (1 - \hat{\tau}) d\varphi + \mathbf{V} \left(\hat{y}, \hat{z}, (1 - \lambda p(\hat{x})) \int (1 - \hat{\tau}) d\varphi + \hat{n}_e \right)$$

This clearly shows that only the total number of layoffs $\int \hat{\tau} d\varphi$ is determined by the maximization. Any combination of $\hat{\tau}$ that keeps this summation constant yields the same joint surplus. $\square$

E.4 Part 3.4

Proof of proposition 5. I will prove the result in two steps. I will first show that if the firm can choose any continuing utility $\hat{W} \in \mathbb{R}$, then there exists a unique $\hat{W}(\hat{x}^*)$ that makes the worker choose $\hat{x}^*$ exactly. We will then show that this continuing utility must satisfy the participation constraint, i.e $\hat{W}(\hat{x}^*) \geq \mathbf{U}(\hat{y})$.

Step 1. Recall that workers solve the problem

$$\hat{x}^* = \operatorname{argmax}_{\hat{x} \in [\underline{x}, \kappa(\hat{y}) - c]} p(\hat{x})(\hat{x} - \hat{W}).$$

Define

$$\tilde{D}(\hat{x}, \hat{W}) = p(\hat{x})(\hat{x} - \hat{W}) \text{ and } \begin{cases} D(\hat{W}) = \max_{\hat{x} \in [\underline{x}, \kappa(\hat{y}) - c]} \tilde{D}(\hat{x}, \hat{W}) \\ C(\hat{W}) = \operatorname{argmax}_{\hat{x} \in [\underline{x}, \kappa(\hat{y}) - c]} \tilde{D}(\hat{x}, \hat{W}) \end{cases}$$

$\tilde{D}$ is a continuous function of $\hat{x}$ and $\hat{W}$. It reaches its maximum in $\hat{x}$ on $[\hat{W}, \kappa(\hat{y}) - c]$. Assumption 2 guarantees that $\tilde{D}$ is strictly concave in $\hat{x}$ on $[\hat{W}, \kappa(\hat{y}) - c]$. The Theorem of the Maximum tells us therefore that $D(\hat{W})$ and $C(\hat{W})$ are continuous functions of $\hat{W}$. p being strictly positive over $[\underline{x}, \kappa(\hat{y}) - c]$, D is strictly decreasing on $[-\infty, \kappa(\hat{y}) - c]$. Therefore, C is strictly increasing on $[-\infty, \kappa(\hat{y}) - c]$, as can be seen from the following: take $\hat{W}_1 < \hat{W}_2 \leq \kappa(\hat{y}) - c$. Denote $\hat{x}_i = C(\hat{W}_i)$, $i = 1, 2$. Then the following is true:

$$p(\hat{x}_2)(\hat{W}_2 - \hat{W}_1) < p(\hat{x}_1)(\hat{x}_1 - \hat{W}_1) - p(\hat{x}_2)(\hat{x}_2 - \hat{W}_2) < p(\hat{x}_1)(\hat{W}_2 - \hat{W}_1)$$

Therefore, $\hat{x}_2 > \hat{x}_1$ and C is strictly increasing.

Now, let us show that C reaches $\underline{x}$ and $\kappa(\hat{y}) - c$. For $\hat{W} = \kappa(\hat{y}) - c$, function $\tilde{D}$ trivially reaches its maximum at $\hat{x} = \hat{W} = \kappa(\hat{y}) - c$. The derivative of $\tilde{D}$ with respect to $\hat{x}$ is

$$p'(\hat{x})(\hat{x} - \hat{W}) + p(\hat{x}).$$

Due to concavity, this is a decreasing function on $\hat{x} \in [\hat{W}, \kappa(\hat{y}) - c]$. For a low value of $\hat{W}$ (ex. $\hat{W} \leq \underline{x} + p(\underline{x})/p'(\underline{x})$), this function can be made negative everywhere. In that

case, the maximum of $\tilde{D}$ is reached at $\hat{x} = \underline{x}$. Therefore, $C(\hat{W})$ is a continuous strictly increasing function that reaches $\underline{x}$ and $\kappa(\hat{y}) - c$. By the Intermediate Value Theorem, for any $\hat{x}^* \in [\underline{x}, \kappa(\hat{y}) - c]$, there exists a unique $\hat{W}^{IC}(\hat{x}^*)$ such that $\max_{\hat{x}} \tilde{D}(\hat{x}, \hat{W}^{IC}(\hat{x}^*))$ is reached at $\hat{x}^*$ exactly. In other words, there exists a unique continuation utility $\hat{W}^{IC} \in [-\infty, \kappa(\hat{y}) - c]$ that makes the worker choose exactly $\hat{x}^*$. To finish this first-step, we must choose the rest of the contract. Set $\hat{\tau}^{IC} = \hat{\tau}^*$ and $\hat{d}^{IC} = \hat{d}^*$. Now, in an optimal allocation, w^{IC} must be chosen so that the promise-keeping constraint is binding. The worker's expected utility is:

$$\mathbf{W}(y, z; \{w, \hat{\tau}, \hat{x}, \hat{d}, \hat{W}\}) = w + \beta E_{\hat{y}, \hat{z}} \left\{ \hat{d} \mathbf{U}(\hat{y}) + (1 - \hat{d}) \left(\hat{\tau} \mathbf{U}(\hat{y}) + (1 - \hat{\tau}) \lambda p(\hat{x}) \hat{x} + (1 - \hat{\tau}) (1 - \lambda p(\hat{x})) \hat{W} \right) \right\}$$

Given $\{\hat{\tau}^{IC}, \hat{d}^{IC}, \hat{W}^{IC}\}$, there exists a unique wage w^{IC} that matches exactly the promised utility. This does not affect the joint surplus, which is maximized by assumption. From proposition 1, the firm's profit is maximized when the level of promised utility is exactly achieved. We have thus found a contract that implements the optimal allocation.

Step 2. I will now proceed to the second step of the proof. Denote $\hat{x}_u(\hat{y}) = C(\mathbf{U}(\hat{y}))$ the submarket chosen by unemployed agents. We will now show that the optimal $\hat{x}^*$ is higher than $\hat{x}_u$. Using the Intermediate Value Theorem for $\hat{W} \in [\mathbf{U}(\hat{y}), \kappa(\hat{y}) - c]$, this will show that $\hat{W}^{IC}(\hat{x}^*) \geq \mathbf{U}(\hat{y})$, i.e that the incentive compatible contract also satisfies the participation constraint.

The terms related to $\hat{x}$ in the joint surplus maximization are

$$\lambda p(\hat{x}) \hat{x} \int (1 - \hat{\tau}) dj + \mathbf{V}(\hat{y}, \hat{z}, (1 - \lambda p(\hat{x})) \int (1 - \hat{\tau}) dj + \hat{n}_i).$$

To simplify the notation, write $n\hat{T} = \int \hat{\tau} dj$, $\hat{T}$ being the total fraction of layoffs. We can rewrite the problem as:

$$n(1 - \hat{T}) \underbrace{\left(\lambda p(\hat{x}) \hat{x} + (1 - \lambda p(\hat{x})) \mathbf{U}(\hat{y}) \right)}_{\text{LHS}} + \underbrace{\mathbf{V}(\hat{y}, \hat{z}, n(1 - \lambda p(\hat{x}))(1 - \hat{T}) + \hat{n}_i) - n(1 - \lambda p(\hat{x}))(1 - \hat{T}) \mathbf{U}(\hat{y})}_{\text{RHS}}$$

The LHS term is the same as the one maximized by unemployed agents. It reaches its maximum in $\hat{x} = \hat{x}_u$. Assume by contradiction that the optimal $\hat{x}^*$ is strictly less than $\hat{x}_u$. As we showed earlier, the LHS term is continuous, strictly concave, the left-hand side is strictly increasing¹⁴ in $\hat{x}$ at this point. That means that the right-hand side term is strictly decreasing in $\hat{x}$. Turning to the right-hand side, since $p(\hat{x})$ is strictly decreasing, it is easy to see that any increase in $\hat{x}$ can be equivalently achieved by a decrease in $\hat{T}$. That means that the RHS term is strictly increasing in $\hat{T}$. At the optimum, this can only be true if $\hat{T} = 1$, i.e all workers are laid-off. However, imposing $\hat{T} = 1$, the maximization problem becomes:

$$n \lambda p(\hat{x}) \hat{x} + n(1 - \lambda p(\hat{x})) \mathbf{U}(\hat{y}) + \mathbf{V}(\hat{y}, \hat{z}, \hat{n}_i)$$

which is the same as the problem solved by unemployed agents, so $\hat{x}^* = \hat{x}_u$. Contradiction. Therefore, $\hat{x}^* \geq \hat{x}_u$, which is implementable with $\hat{W}^{IC} \geq \mathbf{U}(\hat{y})$. $\square$

¹⁴Unless the function is decreasing on $[\underline{x}, \kappa(\hat{y}) - c]$, in which case there is nothing to prove since $\hat{x}_u = \underline{x} \leq \hat{x}^*$.