Common Values Procurement Auctions with Bidder Solicitation*

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Abstract

A buyer seeks to procure a service and solicits bids from sellers. The cost of the service depends on characteristics that are only known to the buyer ("common values"). The buyer can choose how many bids to solicit and the number of solicited bids cannot be disclosed verifiably. We characterize equilibrium. We show that there are potentially important differences between a standard common values procurement auction with a fixed number of a bidders and a common values auction with endogenous biddership.

Very preliminary and Incomplete. Please do not post publicly.

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1 Introduction

We develop a theory of common values auctions in which the auctioneer solicits bidders. In the standard analysis of common values auctions, the number of bidders is assumed to be exogenous and the number of bidders is commonly known. In many relevant settings, this is a bad assumption. Bidders are often actively solicited by auctioneers, by advertising an auction or by contacting potential bidders directly. Auctioneers such as Sotheby's invest considerable effort in attracting buyers. Moreover, the actual number of relevant bidders is not always known. This project aims to investigate the optimal solicitation strategy and the effects of the endogeneity of the number of bidders on the bidding strategies and on the auctioneer's revenue.

We consider a buyer who seeks to procure a service, where the cost of the service depends on characteristics that are known only to the buyer ("common values"). The buyer solicits sellers to bid in an auction and procures the service from the seller with the lowest bid. Consider the following example of public procurement of construction through an auction. The procuring agency might have private information about the cost of the construction project. The agency solicits bidders for a procurement auction by contacting building companies. The number of contacted builders and solicited bids is unobserved by the bidders at the time of bid submission. If the agency solicits bidders with different effort depending on privately known characteristics, then bidders who are contacted will take this into account when determining their optimal bid.

We are particularly interested in the price that the buyer pays as a function of the true cost of the project when soliciting bidders becomes cheap. Among other things, we use our framework to re-consider the question of information aggregation in auctions posed by Milgrom (1979) and Wilson (1977). Wilson and Milgrom derived conditions on the informativeness of the signals under which the price aggregates information when the number of bidders is large. They consider a scenario where the number of bidders is *exogenous*, *known*, and *independent* of the state.

Rather than letting the number of bidders grow large exogenously, we assume that it becomes cheaper to solicit additional bidders. In principle, lower costs allow the auctioneer to solicit an increasingly large number of bids. This is not the same as an exogenous increase in the number of bidders, however, because the number of solicited bidders may now depend on the state of nature through the solicitation strategy of the buyer. Being contacted and invited to submit a bid contains information. Therefore, a seller who is contacted will update based on being invited to submit a bid. If a buyer who has characteristics that induce high costs on the seller is soliciting many more bids than a buyer who induces low cost, contacted sellers become pessimistic and are reluctant to bid aggressively.

We provide a simple example (on Page 9) in which we demonstrate the effect of a state-dependent number of bidders. In this example, a "high cost buyer" is assumed to contact many more sellers than a "low cost buyer." As demonstrated, the two buyers may end up paying the same expected price in equilibrium—despite the fact that bidders receive informative signals about the true cost of the service. Thus, information aggregation in auctions becomes more difficult when the number of bidders is not exogenously fixed and independent of the state of the world.

The example mentioned before also illustrates that there are some, potentially interesting technical differences to standard common values auction. In standard auctions, it is possible to use revealed preference arguments to rule out atoms in the equilibrium bid distribution. In the example, all bidders submit the same bid, i.e., there is an atom. With a state dependent biddership, we cannot rule out atoms in the equilibrium bid distribution. Therefore, standard existence proofs for discontinuous games cannot be applied (standard existence proofs rely on the absence of discontinuities of expected payoffs in equilibrium). As a consequence, we currently do not provide an existence proof.

We first study partial equilibria in which the auctioneer's solicitation strategy is exogenous. We provide an explicit characterization of the partial equilibrium bid distribution in the limit, given an increasingly large and potentially state dependent number of bidders. An important intermediate step towards this characterization is to show that there can be no atoms in the *limit* distribution of the winning bid when the number of bidders is growing large. (The probability of two bidders being tied at the lowest bid converges to zero.) We use this characterization of the limiting partial equilibrium bid distribution to characterize the incentives of the buyer types to solicit bids when solicitation costs are small. We show that (given a suitable existence assumption) for some sequence of equilibria both types of buyers sample an increasingly large number of bidders, and the high cost type solicits more bids than the low cost type in the limit. Crucially, we show that the high cost type does not sample "too many" more bids, so that the information contained in being sampled does not entirely cancel the information contained in the signal. Consequently, we show that buyers with different costs receive different prices. Specifically, we show that the expected winning bid converges to the true cost of the service along some sequence of equilibria if and only if the most informative signals are close to being fully informative. Thus, we re-affirm the results by Wilson and Milgrom for a setting where the number of bidders is not exogenous.[1]

Our results for the common values auction with endogenous biddership contrast with our results in a companion paper.[2] In that paper, we consider a search environment, where the buyer samples sellers sequentially. We show that, with sequential search, when search costs becomes small, infor-

[1]Our current result is weaker because we cannot rule out the existence of additional equilibria that do not aggregate information in the limit.

[2]Lauermann and Wolinsky, "Search with Adverse Selection," in preparation.

mation aggregation requires stronger assumptions onto the strength of the signals. The existence of unboundedly informative signals alone is not sufficient.

This paper is incomplete. In addition to the missing existence proof, there are a number of potentially interesting questions that are left open. What is the equilibrium bid distribution away from the limit? When are there atoms in the bid distribution? Are there other equilibria in which the number of solicited bidders is not large when solicitation costs are small and in which information is not aggregated? Which type of buyer will sample more bidders for different level of sampling costs? What would happen if the auctioneer could commit to—and publicly announce—a fixed number of bidders (so the number of solicited bidders becomes a signal)? Would it be good to invite few bidders to signal optimism (low cost)?

2 The Setup and some Preliminary Observations

Our setup: We consider a common value first price procurement auction with an endogenous number of bidders. The buyer knows the true cost of the service c_w, $w \in \{L, H\}$. Each state w is equally likely and we often set $c_L = 0$ and $c_H = 1$. The value of the buyer for the good is commonly known to be v, independent of the state. The buyer can invite some number n of bidders at costs s each, with $n \in \left\{2, ..., \bar{N}\right\}$, $\bar{N} >> \frac{v-c_L}{s}$.[3] [4] Sellers do not observe the number of other invitees. If the winning bid is p, payoffs to the buyer are $v - p - ns$, and the payoffs to the winning seller are $p - c_w$, depending on the state. Sellers receive conditionally independent signals $x \in [a, b]$, distributed according to G_L and G_H with continuous and strictly positive densities. Signals are normalized such that a signal x induces a posterior x.

Denote a (mixed) sampling strategy by $\mathcal{N} \in \left[\triangle_L^{[1,\bar{N}]}, \triangle_H^{[1,\bar{N}]}\right]$, with $\mathcal{N}_w(i)$ denoting the probability of inviting i bidders. A bidding strategy is denoted $\beta(\cdot) : [a, b] \to [c_L, v]$. We restrict attention to symmetric pure bidding strategies. We show that equilibrium bidding strategies are monotone non-decreasing using a single-crossing argument. By the same argument, the restriction to pure strategies is without loss of generality, see the discussion below.

Bidders maximize expected payoffs. Consider a price p and denote the probability of winning with a bid equal to p conditional on state w and conditional on a bidding strategy β used by $(n-1)$

[3] Assuming $n \geq 2$ ensures that we do not get a degenerate equilibrium with $n = 1$ and $\beta(x) \equiv v$.

[4] Games with a random number of players are non-standard. We consider therefore the following auction with a fixed number of player instead: For any fixed s, there exists a number $\bar{N}(s)$ of potential bidders, with $\bar{N}(s) >> \frac{v-c_L}{s}$. Each of these bidders is invited to submit a bid with probability $\frac{n}{\bar{N}}$ if the buyer chooses to sample n bidders.

other bidders by $\pi[p|w,\beta,n]$. For example, if the bidding function is strictly increasing[5] we have

$$\pi[p|w,\beta,n] = \left(1 - G_w\left(\beta^{-1}(p)\right)\right)^{n-1}.$$

In most of our analysis, bidding functions are monotone non-decreasing. Monotonicity of the bidding strategy makes the analysis tractable. However, we will have to deal with the possibility of ties. Given a monotone strategy β, the set of sellers which are tied at any price p is the interval

$$[x_l(p),\ x_h(p)] = [\inf\{x|\beta(x) = p\},\ sup\{x|\beta(x) = p\}]$$

and the probability to win is given by

$$\pi[p|w,\beta,n] = \sum_{i=0}^{n} \binom{n}{i} \frac{1}{i+1} \left(1 - G_w(x_h(p))\right)^{n-i-1} \left(G_w(x_h(p)) - G_w(x_l(p))\right)^i.$$

When submitting a bid at p and buyers sample n_H and n_L bidders respectively, payoffs are given by

$$\frac{(1-x)\frac{n_L}{\bar{N}}}{x\frac{n_H}{\bar{N}} + (1-x)\frac{n_L}{\bar{N}}} \pi[p|L,\beta,n_L](p - c_L) + \frac{x\frac{n_H}{\bar{N}}}{x\frac{n_H}{\bar{N}} + (1-x)\frac{n_L}{\bar{N}}} \pi[p|H,\beta,n_H](p - c_H).$$

Cancelling the denominator shows that maximizing the above expression is equivalent to maximizing[6]

$$(1-x)\ n_L\ \pi[p|L,\beta,n_L]\ \ (p - c_L) + x\ n_H\ \pi[p|H,\beta,n_H]\ \ (p - c_H). \tag{1}$$

We say a bidding strategy β constitutes a partial equilibrium given a sampling strategy $\mathcal{N}$ if bidding $p = \beta(x)$ maximizes the payoffs of a seller with signal x, given β and $\mathcal{N}$.

Example for decreasing bidding function. Suppose that $n_L = 1$ and $n_H = 100$ and suppose the support is $[a,b] = [0,1]$. Consider a partial equilibrium bidding strategy β given n_L and n_H. A seller who sees a signal equal to zero is sure to be in the low state with no competitor. Hence, the optimal bid must be $\beta(0) = v$. A seller who sees a signal equal to one is sure to be in the high state with many competitors. Hence, the optimal bid is strictly below v. Hence, $\beta(0) > \beta(1)$, i.e., a more pessimistic buyer with a higher signal submits a lower bid.[7]

In the example, the H buyer samples more bidders than the L buyer. Hence, a bidder with

[5] We use the generalized inverse throughout, $\beta^{-1}(p) = sup\{x|\beta(x) \leq p\}$ and $\beta^{-1}(p) = b$ if $\beta(b) < p$.

[6] If buyers use a mixed sampling strategy, payoffs are equovalent to $x\ \sum_{i\in[1,..,\bar{N}]} i\ \mathcal{N}_H(i)\,\pi[p|H,\beta,i]\ \ (p - c_H) + (1-x)\ \ \sum_{i\in[1,..,\bar{N}]} i\mathcal{N}_L(i)\ \ \pi[p|L,\beta,i]\ \ \ (p - c_L)$.

[7] Note that we have not shown that a partial equilibrium exists given the sampling strategy. The example shows that bidding functions are decreasing, *assuming* existence.

a higher signal expects more competition and bids more aggressively, counteracting the higher expected cost. We will now show that this example of a decreasing bidding strategy relies on the assumption that $n_L = 1$. When $n_w \geq 2$ is imposed for both buyers, equilibrium strategies must be non-decreasing. We now prove monotonicity of any symmetric partial equilibrium bidding strategy β when $n_w \geq 2$. The main step towards proving monotonicity is to observe that in equilibrium sellers must incur losses in the high state, i.e., almost all bids are below the cost c_H.

Lemma 1 *Suppose β is a partial equilibrium bidding strategy given a sampling strategy $\mathcal{N}$ for which $n_w \geq 2$. Then $\beta(x) < c_H$ for almost all $x \in [a, b]$.*

Proof: Step 1. $\beta(x) \leq c_H$ follows from Bertrand logic (it is common knowledge that cost are smaller than c_H).

Step 2. $\beta(x) < c_H$ for almost all x. Suppose not and suppose instead that there is a strictly positive mass of sellers bidding equal to c_H, $\int_{x|\beta(x)=c_H} dG_L(x) > 0$ and (therefore) $\int_{x|\beta(x)=c_H} dG_H(x) > 0$. We now argue that every bidder $x < b$ would have an incentive to undercut the atom slightly: Consider the profit at $p' = c_H - \varepsilon$. For ε small enough, payoffs at p' are (see (1))

$$\begin{aligned}
& x\ n_H\ \pi\left[p'|H, \beta, n_H\right]\ \left(p' - c_H\right) + (1-x)\ n_L\ \pi\left[p'|L, \beta, n_L\right]\ \left(p' - c_L\right) \\
\geq\ & x n_H(-\varepsilon) + (1-x)\ n_L\ \pi\left[p'|L, \beta, n_L\right](c_H - c_L - \varepsilon) \\
>\ & 0 + (1-x)\ n_L\ \pi\left[c_H|L, \beta, n_L\right]\ (c_H - c_L) \\
=\ & x\ n_H\ \pi\left[c_H|H, \beta, n_H\right]\ (c_H - c_H) + (1-x)\ n_L\ \pi\left[c_H|L, \beta, n_L\right]\ (c_H - c_L).
\end{aligned}$$

Hence payoffs at p' are higher than at c_H. We used the fact that the winning probability in the atom is strictly lower than the winning probability when undercutting the atom:

$$\begin{aligned}
\pi\left[p'|L, \beta, n_L\right] \ &\geq\ \left[\int_{x|\beta(x)=c_H} dG_L(x)\right]^{(n_L-1)} \\
&>\ \frac{1}{n_L}\left[\int_{x|\beta(x)=c_H} dG_L(x)\right]^{(n_L-1)} = \pi\left[c_H|L, \beta, n_L\right]
\end{aligned}$$

and we used the fact that winning in the low state is desirable. **QED**

Lemma 2 *Suppose β is a partial equilibrium bidding strategy given a sampling behavior $\mathcal{N}$ for which $n_w \geq 2$. Then $\beta(\cdot)$ is non-decreasing.*

Proof: The proof uses the observation that the strict single crossing condition must hold for all bids strictly below c_H for which the winning probability is positive. First, we establish that expected payoffs at any price are a strictly decreasing linear function of the signal if the winning probability is positive. Second, the winning probability is indeed positive at all equilibrium bids (except possibly $\beta(b)$). This implies the strict single crossing condition.

Step 1. Payoffs are decreasing linear functions. It is useful for this proof to consider ex ante expectations of a seller, before being sampled. We denote the ex ante expectations by hats. Consider a seller who plans to bid p. His expected probability of being sampled and winning in state w is

$$\hat{\pi}[p|w,\beta,\mathcal{N}] = \sum_i \frac{\mathcal{N}_w(i)}{\bar{N}} \pi[p|w,\beta,i].$$

Denote by $\hat{U}^S(p,w,\beta,\mathcal{N})$, the ex ante expected payoffs from bidding p, given state w, sampling strategy $\mathcal{N}$ and bidding strategy β, calculated before the seller knows whether he is sampled and denote interim expected payoffs given signal x as

$$\begin{aligned}
\hat{U}^S(p,x,\beta,\mathcal{N}) &= x\hat{U}^S(p,H,\beta,\mathcal{N}) + (1-x)\hat{U}^S(p,L,\beta,\mathcal{N}) \\
&= x\hat{\pi}[p|H,\beta,\mathcal{N}](p-c_H) + (1-x)\hat{\pi}[p|L,\beta,\mathcal{N}](p-c_L) \\
&= \hat{\pi}[p|L,\beta,\mathcal{N}](p-c_L) - x(\hat{\pi}[p|L,\beta,\mathcal{N}](p-c_L) - \hat{\pi}[p|H,\beta,\mathcal{N}](p-c_H)).
\end{aligned}$$

Thus, for all $p < c_H$, payoffs $\hat{U}^S(p,x,\beta,\mathcal{N})$ are a non-increasing linear of function of the signal x, since $(p-c_H) < 0$ implies that the coefficient of x is positive (we will denote the coefficient by $\Delta(p)$ below). If the trading probability is strictly positive (if $\hat{\pi}[p|L,\beta,\mathcal{N}] > 0$ then $\hat{\pi}[p|H,\beta,\mathcal{N}] > 0$, since the support is state independent), payoffs are a strictly decreasing linear function of x.

Step 2. The equilibrium trading probability $\hat{\pi}[\beta(x)|w,\beta,\mathcal{N}]$ is strictly positive for almost all types: Suppose the probability were zero for some set $X \subset [a,b]$ with positive mass. With positive probability all invited bidders are from the set X and one of the invited bidders has to win - a contradiction to the construction of X. This implies that $\hat{\pi}[\beta(x)|w,\beta,\mathcal{N}]$ is strictly positive for *all* types x with $x < b$: Given any x, choose some $x' \in (x,b)$ such that $\hat{\pi}[\beta(x')|w,\beta,\mathcal{N}] > 0$. Since payoffs at $\beta(x')$ are strictly decreasing in the signal, equilibrium payoffs for x must be strictly positive,

$$\hat{U}^S(\beta(x),x,\beta,\mathcal{N}) \geq \hat{U}^S(\beta(x'),x,\beta,\mathcal{N}) > \hat{U}^S(\beta(x'),x',\beta,\mathcal{N}) \geq 0.$$

A positive equilibrium payoff implies that a positive equilibrium trading probability $\hat{\pi}[\beta(x)|w,\beta,\mathcal{N}]$.

Step 3. We now show that for any two prices p_l and p_h, $p_l < p_h$, for which the winning probabilities are positive, the strict single crossing condition holds,

$$\hat{U}^S(p_h,x_l,\beta,\mathcal{N}) \geq \hat{U}^S(p_l,x_l,\beta,\mathcal{N}) \Rightarrow \hat{U}^S(p_h,x_h,\beta,\mathcal{N}) > \hat{U}^S(p_l,x_h,\beta,\mathcal{N}), \;\; p_h > p_l,\, x_h > x_l.$$

Payoffs in the high state are higher when bidding higher, $\hat{U}^S(p_h, H, \beta, \mathcal{N}) > \hat{U}^S(p_l, H, \beta, \mathcal{N})$ since losses in case of winning are smaller and the probability of winning (and incurring losses) is lower at the higher price. Hence, the conclusion of the single crossing condition is immediate if p_h is better even in the low state since then p_l would be strictly dominated. So assume $\hat{U}^S(p_h, L, \beta, \mathcal{N}) < \hat{U}^S(p_l, L, \beta, \mathcal{N})$. Then, the linear coefficient on x in the payoff function $\hat{U}^S(p, x, \beta, \mathcal{N})$ is smaller at p_h,

$$\begin{aligned}\Delta(p_l) &\equiv \hat{\pi}[p_l|L,\beta,\mathcal{N}](p_l - c_L) - \hat{\pi}[p_l|H,\beta,\mathcal{N}](p_l - c_H) > \hat{\pi}[p_h|L,\beta,\mathcal{N}](p_h - c_L) - \hat{\pi}[p_l|H,\beta,\mathcal{N}](p_l - c_H) \\ &> \hat{\pi}[p_h|L,\beta,\mathcal{N}](p_h - c_L) - \hat{\pi}[p_h|H,\beta,\mathcal{N}](p_h - c_H) \equiv \Delta(p_h)\end{aligned}$$

(where we used $\hat{U}^S(p_l, L, \beta, \mathcal{N}) > \hat{U}^S(p_h, L, \beta, \mathcal{N})$ to derive the first inequality and $\hat{U}^S(p_h, H, \beta, \mathcal{N}) > \hat{U}^S(p_l, H, \beta, \mathcal{N})$ to derive the second. Hence, the single crossing condition holds:

$$\begin{aligned} & \hat{U}^S(p_h, x_l, \beta, \mathcal{N}) - \hat{U}^S(p_l, x_h, \beta, \mathcal{N}) = \hat{U}^S(p_h, L, \beta, \mathcal{N}) - \hat{U}^S(p_l, L, \beta, \mathcal{N}) + x_l(\Delta(p_l) - \Delta(p_h)) \geq 0 \\ \Rightarrow\ & \hat{U}^S(p_h, x_h, \beta, \mathcal{N}) - \hat{U}^S(p_l, x_h, \beta, \mathcal{N}) = \hat{U}^S(p_h, L, \beta, \mathcal{N}) - \hat{U}^S(p_l, L, \beta, \mathcal{N}) + x_h(\Delta(p_l) - \Delta(p_h)) > 0\end{aligned}$$

using $x_h > x_l$ and $(\Delta(p_l) - \Delta(p_h)) > 0$.

The strict single crossing condition implies that any selection $\beta(\cdot)$ from $\arg_p \max \hat{U}^S(p, \cdot, \beta, \mathcal{N})$ is monotone non-decreasing which implies the lemma. QED

Remark. During the proof we have shown that payoffs satisfy the strict single crossing condition on the critical set of prices (those at which payoffs are positive). This implies that the optimal bid is uniquely determined for almost all sellers. Furthermore, the proof generalizes to mixed strategies as follows: Suppose other bidders use some potentially mixed bidding strategy in equilibrium. Again, the strict single crossing condition holds by the same arguments (the proof applies verbatim). Hence, the same conclusion holds. For almost all signals there is a unique optimal bid and therefore almost all bidders use pure strategies. In addition, the proof of the lemmas reveals how it is possible to construct a non-montone equilibrium when $n_L = 1$ in the example. When $n_L = 1$, we can no longer argue that $\beta(x) < c_H$ and in fact, in the example, $\beta(a) = v >> c_H$.

While we can show that bids are monotone non-decreasing, we cannot rule out atoms in the bid distribution (a flat region for b). Usually, atoms can be ruled out by showing that, given a monotone symmetric bidding strategy used by other bidders, winning with a bid that occurs with positive probability is bad news, i.e., a bidder could increase his payoff by undercutting the atom. In our setting, winning at an atom can be good news. The reason is that in the low state the number of bidders is lower, hence, the probability of winning can be higher in the low state relative to the high state so that winning in an atom is negatively correlated with costs. We will have to deal with the problem of atoms in the proof of the main proposition. (We show that, for winning to be good news, most bids in the atom must come from bidders with relatively high signals which is unlikely to happen. Otherwise, winning will be bad news and atoms can be ruled out.)

Example for a partial equilibrium with atoms in the bid distribution. Suppose it is known that buyers sample according to $n_L = 3$ and $n_H = 9$ and the support of the signal distribution is $[a, b] = [0.4,\ 0.6]$. Then it is a partial equilibrium given the fixed sampling strategy $\mathcal{N} = [3, 0, 9, 0]$ for all sellers to bid $\beta(x) \equiv 0.6$. Payoffs at $p = 0.6$ for the most pessimistic seller with $b = 0.6$ are

$$\begin{aligned} & b\ n_H\ \frac{1}{n_H}\ (0.6 - c_H) + (1-b)\ \ n_L \frac{1}{n_L}\,(0.6 - c_L) \\ = \quad & 0.6\,(-0.4) + 0.4\,(0.6) = 0. \end{aligned}$$

And hence, payoffs at $p = 0.6$ are strictly positive for all other signals $x < b$. Consider the deviation to a price p' just below 0.6. Payoffs of the most optimistic seller are bounded from above by

$$\begin{aligned} & a\ n_H\ \ (0.6 - 1) + (1-a)\ \ n_L\ \ (0.6) \\ = \quad & 0.4\,(9)\,(-.4) + 0.6\,(3)\,(0.6) = -0.36. \end{aligned}$$

Hence, no seller has an incentive to undercut the atom.[8]

We now consider the buyer's sampling problem. Denote the first order statistic of the winning bid given n draws from bidding strategy β by $F_w(p|w, \beta, n)$ and denote the expected value of the first order statistic by $E[p|w, \beta, n]$,

$$\begin{aligned} F_w(p|w,\beta,n) &= \int_{x|\beta(x)\le p} n\,(1 - G_w(x))^{n-1}\,dG_w(x)\,, \\ E[p|w,\beta,n] &= \int_x n\beta(x)\,(1 - G_w(x))^{n-1}\,dG_w(x) \\ &= \int_{\beta(a)}^{\beta(b)} p dF_w(p|w,\beta,n)\,. \end{aligned}$$

Buyer w's payoff when sampling n sellers who bid according to β is $v - E[p|w, \beta, n] - ns$. The buyer's problem is equivalent to minimizing the expected price plus sampling cost. We now show that this minimization problem is convex and the optimal number of sampled sellers is either a unique number or there are two adjacent numbers both of which are optimal. Hence, given any

[8] Note that, in this example, the ratio of the sampled bidders is larger than the inverse likelihood ratio of the most informative signal, $\frac{n_H}{n_L} = \frac{9}{3} = 3 > \frac{1-a}{a} = \frac{3}{2}$. This implies that, conditional on being sampled, even the most optimistic seller believes that the probability of facing the high cost seller is higher than the prior probability of $\frac{1}{2}$. The posterior of this seller is $\frac{1}{1+\frac{n_L}{n_H}\frac{1-a}{a}} = \frac{1}{1+\frac{1}{3}\frac{3}{2}} = \frac{2}{3}$. Therefore, undercutting the atom and winning for sure at a price $p \cong 0.6$ will lead to a loss, given that the price is below the expected cost.

symmetric and non-decreasing bidding strategy β, we can restrict attention to sampling strategies in which each type of buyer optimally samples either n_w bidders with probability γ_w or n_w+1 bidders with probability $(1-\gamma_w)$.

Lemma 3 *Given any symmetric non-decreasing bidding function β, the expected price plus the total sampling cost is a convex function of the number of sampled bidders. Hence, there is a number n^* such that*

$$\{n^*, n^*+1\} \supseteq \arg\max v - E[p|w,\beta,n] - ns.$$

Proof: The lemma is an immediate consequence of the fact that the first moment of the first order statistic is a convex function of the number of trials. Hence, the marginal benefit of an additional bidder is decreasing in the number of bidders, see, e.g., Rothschild (1974).

The lemma is immediate if β is degenerate and all sellers bid the same. In this case there is a corner solution at $n^*=2$. So, suppose β is non-degenerate. We want to show that the change of the expected winning bid from one additional bidder is strictly decreasing in n. The proof is complicated by the fact that β might have discontinuities. All discontinuities are upward jumps since β is non-decreasing. Let $I=\{x_1,..,x_i,..\}$ denote the set of jump points with jump range $\beta\left(x_i^+\right)-\beta\left(x_i^-\right)$. Integration by parts implies that[9]

$$\begin{aligned} E[p|w,\beta,n] &= \int_a^b n\beta(x)(1-G(x))^{n-1} g(x)\,dx \\ &= \beta(a) + \int_a^b \beta'(1-G(x))^n\,dx + \sum_{i\in I}(1-G(x_i))^n\left(\beta\left(x_i^+\right)-\beta\left(x_i^-\right)\right), \end{aligned}$$

where we use a simple modification of integration by parts that accounts for the jump points,

$$[\beta(x)(1-G(x))^n]_a^b = \int \frac{\partial}{\partial x}\left(\beta(x)(1-G(x))^n\right) + \sum_{i\in I}(1-G(x_i))^n\left(\beta\left(x_i^+\right)-\beta\left(x_i^-\right)\right).$$

[9] For now, we neglect the problem that might arise with a singular component of β.

The expected price difference from sampling one additional seller is given by

$$\begin{aligned}
\Delta(n) &= E[p|w,\beta,n] - E[p|w,\beta,n+1] \\
&= \int_a^b \beta' (1-G(x))^n \, dx - \int_a^b \beta' (1-G(x))^{n+1} \, dx \\
&\quad + \sum_{i\in\{1,\dots,I\}} (1-G(x))^n \left(\beta\left(x_i^+\right) - \beta\left(x_i^-\right)\right) - \sum_{i\in\{1,\dots,I\}} (1-G(x_i))^{n+1} \left(\beta\left(x_i^+\right) - \beta\left(x_i^-\right)\right) \\
&= \int_a^b \beta' G(x) (1-G(x))^n \, dx + \sum_{i\in\{1,\dots,I\}} \left(\beta\left(x_i^+\right) - \beta\left(x_i^-\right)\right) G(x_i) (1-G(x))^n . \qquad (2)
\end{aligned}$$

This difference is strictly decreasing. For $n' > n$,

$$\begin{aligned}
\Delta(n) - \Delta\left(n'\right) &= \int_a^b \beta' G(x) (1-G(x))^n \left(1 - (1-G(x))^{n'-n}\right) dx \\
&\quad + \sum_{i\in\{1,\dots,I\}} \left(\beta\left(x_i^+\right) - \beta\left(x_i^-\right)\right) G(x) (1-G(x))^n \left(1 - (1-G(x))^{n'-n}\right) \\
&> 0,
\end{aligned}$$

using that $1-(1-G(x))^{n'-n} > 0$ and that either $\beta'(x) > 0$ on some interval or $\left(\beta\left(x_i^+\right) - \beta\left(x_i^-\right)\right) > 0$ for some x_i from the assumption that β is non-degenerate. QED

Equilibrium. An equilibrium profile consists of a symmetric bidding strategy β and a sampling strategy $\mathcal{N} = [n_H, \gamma_H, n_L, \gamma_L]$ such that $\beta(x)$ maximizes payoffs (1) and n_w (and $n_w + 1$ if $\gamma_w > 0$) maximize $v - E[p|w,\beta,n] - ns$.

3 Characterization of the Asymptotic Distribution of the Winning Bid

We want to characterize the limit distribution of the winning bid when the number of bidders grows to infinite. This will distribution will determine the sampling incentives of the buyers. Take a sequence n_k with $n_k \to \infty$ and some sequence of bidding strategies β_k. Conditional on state w, a bidding strategy β_k, and n_k sampled bidders, the number of bids below any price p is binomially distributed with parameters n_k and (success probability) $G_w\left(\beta_k^-(p)\right)$.[10] A helpful observation is that the number of bids weakly below any price becomes Poisson distributed when the number of bidders grows large. In particular, the probability that there is no bid below p is equal to the probability that the winning bid is above p, $1 - F_w(p|w,\beta_k,n_k) = \left(1 - G_w\left(\beta_k^-(p)\right)\right)^{n_k}$. Suppose

[10] Recall that we use the generalized inverse, $\beta^-(p) = sup\{x|\beta(x) \le p\}$.

the probability $1 - F_w(p|w, \beta_k, n_k)$ is converging to some value $q \in (0,1)$,

$$\lim 1 - F_w(p|w, \beta_k, n_k) = \lim_{k\to\infty} \left(1 - G_w\left(\beta_k^-(p)\right) n_k \frac{1}{n_k}\right)^{n_k} = q.$$

This implies $\lim G_w\left(\beta_k^-(p)\right) n_k = -\ln q$ by standard arguments. Recall the

"Law of Rare Events." Suppose the number of total successes X_k is a binomial random variable with parameters n_k and π_k. Suppose $n_k\pi_k \to \lambda > 0$ while $n_k \to \infty$. Then the limit distribution of the number of occurrences is a Poisson distribution with parameter λ. Furthermore, suppose there is some other sequence of binomial random variables X' with parameters n'_k and π'_k such that $\frac{n'_k}{n_k} \to R > 0$ and $\frac{\pi'_k}{\pi_k} \to \frac{a}{1-a} > 0$. Then the limit distribution of X'_k is a Poisson distribution with parameter $\lambda' = \frac{a}{1-a}R\lambda$.

An immediate implication is this: Suppose the probability that the winning bid is above p converges in the low state, $(1 - F_L(p|L, \beta_k, n_{Lk})) \to q$. Then, in the limit, the number of bids below p is Poisson distributed with parameter $\lambda = -\ln q$.[11] Now, suppose bids at prices below p come from sellers with signals close to the lower end of the support, $\beta_k^{-1}(p) \to a$. Suppose the H buyer samples n_{Hk} sellers, with $\frac{n_{Hk}}{n_{Lk}} \to R$. Then, in the high state, the probability that the winning bid is above p is given by

$$\lim\left(1 - F_H(p|H, \beta_k, n_{Hk})\right) = q^{\frac{a}{1-a}R},$$

To see why, note that $\frac{n_{Hk}}{n_{Lk}} \to R$ and $\frac{G_H\left(\beta_k^-(p)\right)}{G_L\left(\beta_k^-(p)\right)} \to \frac{g_H(a)}{g_L(a)} = \frac{a}{1-a}$ and, therefore, the number of bids below p in the high state is Poisson distributed with parameter $\lambda' = \frac{a}{1-a}R\lambda$. In particular, $\lim\left(1 - F_H(p|H, \beta_k, n_{Hk})\right) = f\left(0; \lambda'\right) = e^{-\left(-\frac{a}{1-a}R\ln q\right)} = q^{\frac{a}{1-a}R}$.

Whether or not the expected trading price is different for the two types of buyers depends on the likelihood ratio $\frac{a}{1-a}R$, which determines the belief conditional on signal a and conditional on being sampled. If this ratio is smaller than one, then, intuitively, a seller with a low signal believes that the low state is more likely than the high state. As we will show later, if, for $s_k \to 0$, the ratio does not converge to one, the expected trading price cannot be equal to the ex ante expected costs. However, the next examples shows that it is not possible to conclude that $\frac{a}{1-a}R < 1$ without further knowledge of the bidding strategies β_k. Hence, we will have to characterize the bidding strategies and the distribution of the winning bids in the following two lemmas explicitly.

Example for Search Incentives and $R \to \frac{1-a}{a}$. Let us fix a sequence of bids β_k such that $\beta_k(x) = \frac{1}{2}$ for $x \le x_k$ and $\beta_k(x) = 1$ otherwise.[12] Suppose search costs s_k are such that at the

[11] A quick check: $\lim\left(1 - F_L(p|L, \beta_k, n_{Lk})\right) = f(0; \lambda) = \frac{(-\ln q)^0 e^{-(-\ln q)}}{0!} = q$.

[12] β_k is not a partial equilbrium bidding strategy. However, there is an example in the last section of a construction of an equilibrium which is qualitatively similar to the situation considered here.

optimal number n_{kL} the optimality condition holds exactly, i.e.,

$$\left(1-G_L\left(x_k\right)\right)^{n_{kL}} G_L\left(x_k\right)\left(1-\frac{1}{2}\right)=s_k.$$

Further, suppose x_k is such that the probability of not trading at a price of $\frac{1}{2}$, $\left(1-G_L\left(x_k\right)\right)^{n_{kL}}$, converges to some $q\in(0,1)$. The optimality condition for the H buyer implies (approximately)

$$\left(1-G_H\left(x_k\right)\right)^{n_{kH}-1} G_H\left(x_k\right)\left(1-\frac{1}{2}\right)\geq s_k \quad \text{and} \quad \left(1-G_H\left(x_k\right)\right)^{n_{kH}} G_H\left(x_k\right)\left(1-\frac{1}{2}\right)\leq s_k$$

Taking ratios

$$\frac{\left(1-G_L\left(x_k\right)\right)^{n_{kL}}}{\left(1-G_H\left(x_k\right)\right)^{n_{kH}-1}}\leq\frac{G_H\left(x_k\right)}{G_L\left(x_k\right)} \quad \text{and} \quad \frac{\left(1-G_L\left(x_k\right)\right)^{n_{kL}}}{\left(1-G_H\left(x_k\right)\right)^{n_{kH}}}\geq\frac{G_H\left(x_k\right)}{G_L\left(x_k\right)}$$

From the law of rare events and $x_k\to a$, with $R=\lim\frac{n_{kH}}{n_{kL}}$, we can simplify

$$\frac{q}{q^{\frac{a}{1-a}R}}=\frac{a}{1-a}.$$

Solving for R

$$\begin{aligned}\ln\left(q\frac{1-a}{a}\right) &= \frac{a}{1-a}R\ln q\\ \Rightarrow R &= \frac{\ln\left(q\frac{1-a}{a}\right)}{\ln q^{\frac{a}{1-a}}}.\end{aligned}$$

When the probability of not trading at $p=\frac{1}{2}$ approaches zero, $q\to 0$, the ratio $R\frac{1-a}{a}$ gets arbitrarily close to one, i.e., $R\to\frac{1-a}{a}$.[13] Thus, we cannot put a bound on $R\frac{1-a}{a}$ without further knowledge of the bidding strategy.[14]

For the first lemma, fix some sequence of anticipated sampling behavior $\mathcal{N}_k$ such that the number of sampled bidders grows to infinity, $n_{kH}\to\infty$ and $n_{kL}\to\infty$ and suppose $\lim\frac{n_{kH}}{n_{kL}}=\lim R_k=R$ is the limit ratio (we can neglect the possibility of mixed sampling strategies when the number of bidders grows large). Suppose there exists a partial equilibrium bidding strategy β_k for every $\mathcal{N}_k$. Let $F_L\left(p|L,\beta_k,n_{kw}\right)$ denote the induced distribution of the winning bid when $w=L$. Suppose the

[13] Using L'Hospital, $\lim_{q\to 0}\frac{\ln\left(q\frac{1-a}{a}\right)}{\ln q^{\frac{a}{1-a}}}=\lim_{q\to 0}\frac{\frac{1-a}{a}}{q\frac{1-a}{a}}\frac{q^{\frac{a}{1-a}}}{\frac{a}{1-a}q^{\left(\frac{a}{1-a}-1\right)}}=\frac{1-a}{a}$.

[14] Note that $q\to 0$ corresponds to a situation in which the L buyer is almost sure to find a seller who offers a price $\frac{1}{2}$ when sampling n_{kL} times. Hence, the winning bid distribution is degenerate at $\frac{1}{2}$. This fact makes it difficult to rule out a degenerate winning bid distribution in general, since, when $\frac{a}{1-a}R_k\to 1$ for some sampling behavior β_k, we would indeed expect a degenerate winning bid distribution.

distribution converges pointwise to some function F_L.

Lemma 4 ***No gaps or atoms.*** *Let $\mathcal{N}_k$ be a sequence of sampling strategies with $n_{kw} \to \infty$. Suppose $R_k = \frac{n_{kH}}{n_{kL}} \to [0, \frac{1-a}{a})$ and suppose β_k is a partial equilibrium bidding strategy given sampling strategy $\mathcal{N}_k$. The induced distributions of the winning bid are $F_w(p|w, \beta_k, n_{kw})$. Suppose $F_w(p|w, \beta_k, n_{kw}) \to F_w(p)$ pointwise. If $R > 0$, then the limiting distribution of the winning bid does not contain any atoms or gaps. If $R = 0$, then the limiting distribution is degenerate at $p = c_L$ (when $w = L$) and $p = c_H$ (when $w = H$).*

Proof:

Zero profit. Profits at any price p must become zero in the limit. Suppose for some p and signal x, $\lim sup U^S(p|x, \beta_k, \mathcal{N}_k) > 0$. Payoffs are equicontinuous along the sequence and hence in the limit. This follows from standard reasoning about incentive constraints, implying a bound on the slope of the payoffs with respect to the signal. Hence, $\lim sup U^S(p|x, \beta_k, \mathcal{N}_k) > 0$ implies that for some ε, $\lim sup U^S(p, x', \beta_k, \mathcal{N}_k) > 0$ for all $x' \in [x-\varepsilon, x+\varepsilon]$.[15] The expected number of bidders with such signals is

$$EN_k = \frac{1}{2} n_{kL}\left(G_L(x+\varepsilon) - G_L(x-\varepsilon)\right) + \frac{1}{2} n_{kH}\left(G_H(x+\varepsilon) - G_H(x-\varepsilon)\right) \to \infty.$$

The aggregate expected payoff of all bidders with $x' \in [x-\varepsilon, x+\varepsilon]$ in state w is given by

$$\lim sup EN_k U^S\left(\beta_k(x)|x, \beta_k, N_k\right) \geq \lim sup\ EN_k\ U^S(p|x, \beta_k, N_k) = \infty.$$

This violates feasibility (the aggregate payoff is finite and bounded by $c_H - c_L$ in both states).

Case 1: Suppose $R = \lim \frac{n_{kH}}{n_{kL}} = 0$. We want to show that $F_L(p) = 1$ for all $p > c_L$. Suppose $F_L(p) < 1$ for some $p > c_L$. Pick any $p' \in (c_L, p)$ with $F_L(p') \leq F_L(p)$ by definition. The payoff of a bidder $x \in (a, b)$ who submits a bid p' is equal to

$$\begin{aligned} & \frac{(1-x)\frac{n_{kL}}{N}}{x\frac{n_{kH}}{N} + (1-x)\frac{n_{kL}}{N}} \pi\left[p'|L, \beta_k, n_{kL}\right](p' - c_L) + \frac{x\frac{n_{kH}}{N}}{x\frac{n_{kH}}{N} + (1-x)\frac{n_{kL}}{N}} \pi\left[p'|H, \beta_k, n_{kH}\right](p' - c_H) \\ = \ & \frac{1}{\frac{xn_{kH}}{(1-x)n_{kL}} + 1} \pi\left[p'|L, \beta_k, n_{kL}\right](p' - c_L) + \left(1 - \frac{1}{\frac{xn_{kH}}{(1-x)n_{kL}} + 1}\right) \pi\left[p'|H, \beta_k, n_{kH}\right](p' - c_H) \end{aligned}$$

Observe that $\pi[p'|L, \beta_k, n_{kL}] \geq 1 - F_L(p|L, \beta_k, n_{kL})$ by $p' < p$. Taking limits shows that profits

[15] If x is on the boundaries, $x \in \{a, b\}$, consider $x \in [x, x+\varepsilon]$ and $x \in [x-\varepsilon, x]$, respectively.

become

$$
\begin{aligned}
& \lim \frac{1}{\frac{x n_{kH}}{(1-x) n_{kL}}+1} \pi\left[p' | L, \beta_k, n_{kL}\right]\left(p'-c_L\right)+\left(1-\frac{1}{\frac{x n_{kH}}{(1-x) n_{kL}}+1}\right) \pi\left[p' | H, \beta_k, n_{kH}\right]\left(p'-c_H\right) \\
\geq\ & \frac{1}{0+1}\left(1-F_L(p)\right)\left(p'-c_L\right)+0>0 .
\end{aligned}
$$

This contradicts our observation that profits must be zero.

We now show that F_H is degenerate at c_H. Let $\hat{U}^s\left(\beta_k | \beta_k, \mathcal{N}_k\right)$ be the ex ante expected payoff

$$
\hat{U}^s\left(\beta_k | \beta_k, \mathcal{N}_k\right)=\frac{1}{2} \int_a^b \frac{n_{kL}}{\bar{N}} \pi\left[\beta_k(x) | L, \beta_k, n_{kL}\right]\left(\beta_k(x)-c_L\right) dG_L+\frac{1}{2} \int_a^b \frac{n_{kH}}{\bar{N}} \pi\left[\beta_k(x) | H, \beta_k, n_{kH}\right]\left(\beta_k(x)-c_H\right) dG
$$

Let $\{(x_{kli}, x_{khi})\}_{i \in I}$ be the set of intervals on which $\beta_k(x)$ is constant, with typical element $x_{ki} \in (x_{kli}, x_{khi})$. Let $X^T=\cup_{i \in I}\left[x_{kli}, x_{khi}\right]$ be the union of the intervals (x_{kli}, x_{khi}). Then the expected price paid by the L buyer is

$$
\begin{aligned}
E\left[p | L, \beta, n_{kL}\right] = & \int_{[a, b] \backslash X^T}\left(\beta_k(x) n_{kL}\left(1-G_L(x)\right)^{n_{kL}-1}\right) dG_L(x)+ \\
& \sum_{i \in I}\left[\beta\left(x_{ki}\right)\left(\left(1-G_L\left(x_{kil}\right)\right)^{n_{kL}}-\left(1-G_L\left(x_{kih}\right)\right)^{n_{kL}}\right)\right]
\end{aligned}
$$

and note that we can rewrite the ex ante expected payoffs as follows (we derive the winning probability in a tie below, see (4))

$$
\begin{aligned}
& \int_a^b \beta_k(x) \frac{n_{kL}}{\bar{N}} \pi\left[b(x) | L, \beta_k, n_{kL}\right] dG_L \\
= & \int_{[a, b] \backslash X^T} \beta_k(x) \frac{n_{kL}}{\bar{N}}\left(1-G_L(x)\right)^{n_{kL}-1} dG_L \\
& +\sum_{i \in I} \beta\left(x_{ki}\right) \frac{n_{kL}}{\bar{N}} \frac{1}{n_{kL}\left(G_L\left(x_{kh}\right)-G_L\left(x_{kl}\right)\right)}\left[\left(1-G_L\left(x_{kil}\right)\right)^{n_{kL}}-\left(1-G_L\left(x_{kih}\right)\right)^{n_{kL}}\right]\left(G_L\left(x_{kh}\right)-G_L\left(x_{kl}\right)\right) \\
= & \frac{1}{\bar{N}} E\left[p | L, \beta, n_{kL}\right] .
\end{aligned}
$$

This implies the intuitive identity

$$
\bar{N} U^s\left(\beta_k | \beta_k, \mathcal{N}_k\right)=\frac{1}{2}\left(E\left[p | L, \beta_k, n_{kL}\right]-c_L\right)+\frac{1}{2}\left(E\left[p | H, \beta_k, n_{kH}\right]-c_H\right) .
$$

(The average ex ante expected payoff of the sellers is equal to the average price paid to the buyer minus cost.) Ex ante expected profits are positive in equilibrium,

$$\bar{N}U^s\left(\beta_k|\beta_k,\mathcal{N}_k\right)=\frac{1}{2}\left(E\left[p|L,\beta_k,n_{kL}\right]-c_L\right)+\frac{1}{2}\left(E\left[p|H,\beta_k,n_{kH}\right]-c_H\right)\geq 0.$$

Taking limits and noting that we know from before that $\lim E\left[p|L,\beta,n_{kL}\right]=c_L$ implies

$$\begin{aligned}\lim\left(\frac{1}{2}\left(E\left[p|L,\beta_k,n_{kL}\right]-c_L\right)+\frac{1}{2}\left(E\left[p|H,\beta_k,n_{kH}\right]-c_H\right)\right) &\geq 0\\ =\lim\left(\frac{1}{2}\left(c_L-c_L\right)+\frac{1}{2}\left(E\left[p|H,\beta_k,n_{kH}\right]-c_H\right)\right) &\geq 0\\ \Rightarrow\lim\left(E\left[p|H,\beta_k,n_{kH}\right]-c_H\right) &\geq 0.\end{aligned}$$

Lemma 1 implies that $\beta_k(x)\leq c_H$ and hence $\left(E\left[p|H,\beta_k,n_{kH}\right]-c_H\right)\leq 0$. So we find that indeed $\lim E\left[p|H,\beta_k,n_{kH}\right]=c_H$.

(Intuitively, if there is a price p such that in the low state the bid is above p with probability $q=1-F_L(p)>0$ in the limit, any bidder could submit $c_L<p'<p$ and win with probability at least q. $R=0$ implies in particular that the likelihood ratio $\frac{x}{1-x}R=0$. Therefore, the low state is infinitely more likely than the high state conditional on being sampled. Hence, conditional on winning (which does not lead to an update away from zero, since winning happens with positive probability) the profit of the bidder is $p'-c_L>0$. Since the winning probability q is strictly positive, this implies that payoffs are strictly positive, a contradiction.)

Case 2: Suppose $R=\lim\frac{n_{kH}}{n_{kL}}\in\left(0,\frac{1-a}{a}\right)$. This is the case we consider for the remainder of the proof.

No atoms at $\mathbf{c}_H$. Take any sequence of prices $p_k\to p_0$ such that p_0 is an atom and p_0 is in the support of F_L (so, $F_L\left(p_0^-\right)<F_L\left(p_0^+\right)$.) Suppose $p_0=c_H$ and hence, in particular, $F_L\left(p_0^+\right)=1$, by Lemma 1. Consider some sequence of price $p'_k\to p_0$ such that there is no tie at any p'_k and the winning probability is equal to $1-F_L\left(p_0^-\right)$,

$$\lim\pi\left(p'_k|L,\beta_k,n_{kL}\right)\equiv q_L^-=1-F_L\left(p_0^-\right)>0,$$

choosing a converging subsequence if necessary. Such a sequence exists: For any given k, there is no tie at almost all prices and letting p'_k converge to c_H sufficiently slow will ensure a sufficiently high winning probability. Profits at p'_k are

$$\left[\frac{1}{1+\frac{x}{1-x}\frac{n_{kH}}{n_{kL}}}\right]\left[\pi\left[p'_k|L,\beta_k,n_{kL}\right]\left(p'_k-c_L\right)+\frac{x}{1-x}\frac{n_{kH}}{n_{kL}}\pi\left[p|H,\beta_k,n_{kH}\right]\left(p'_k-c_H\right)\right].\tag{3}$$

For $p_k' \to c_H$ and $x = a$,

$$\left[\frac{1}{1+\frac{a}{1-a}R}\right]\left[q_L^-\left(c_H-c_L\right)+\frac{a}{1-a}R\left(c_H-c_H\right)\right]>0.$$

Hence, bidders with signals close to a must earn strictly positive profits if there is an atom at the top, contradicting our earlier observation.

No ties. We first show that there are no atoms in the support of F_L that arise from ties. Suppose there is an atom at $p_0 < c_H$ and let $p_k \to p_0$. The set of sellers bidding equal to p_k along the sequence is the interval $x \in [x_{kl}, x_{kh}]$ with $x_{kl} = \inf\{x|\beta_k(x_k) = p_k\}$, $x_{kh} = sup\{x|\beta_k(x_k) = p_k\}$. There is a non-vanishing tie at p_k if

$$\liminf\left(1-\left(G_L\left(x_{kh}\right)-G_L\left(x_{kl}\right)\right)\right)^{n_{kL}}<1,$$

and if $0 < \lim\left(1-G_L\left(x_{kl}\right)\right)^{n_{kL}} \equiv q_L^- \leq 1-F_L\left(p_0^-\right)$. (So that the tie actually matters for the formation of the atom and the winning bid is not below the tie with probability one.)

The winning probability at p_k in the low state is given by

$$\sum_{j=0}^{n_{kL}-1}\binom{n_{kL}-1}{j}\left(G_L\left(x_{kh}\right)-G_L\left(x_{kl}\right)\right)^j\left(1-G_L\left(x_{kh}\right)\right)^{n_{kL}-1-j}\left(\frac{1}{j+1}\right),$$

which can be rewritten to be exactly equal to

$$\frac{1}{n_{kL}\left(G_L\left(x_{kh}\right)-G_L\left(x_{kl}\right)\right)}\left[\left(1-G_L\left(x_{kl}\right)\right)^{n_{kL}}-\left(1-G_L\left(x_{kh}\right)\right)^{n_{kL}}\right]. \tag{4}$$

where we use the following transformation:

$$
\begin{aligned}
& \sum_{j=0}^{N-1} \binom{N-1}{j} \left(G_L\left(x_{kh}\right)-G_L\left(x_{kl}\right)\right)^j \left(1-G_L\left(x_{kh}\right)\right)^{N-1-j} \left(\frac{1}{j+1}\right) \\
= & \sum_{j=0}^{N-1} \frac{(N-1)!}{j!\,(j+1)\,(N-1-j)!} \left(G_L\left(x_{kh}\right)-G_L\left(x_{kl}\right)\right)^j \left(1-G_L\left(x_{kh}\right)\right)^{N-1-j} \\
= & \frac{1}{N} \sum_{k=1}^{N} \frac{N!}{k!\,(N-k)!} \left(G_L\left(x_{kh}\right)-G_L\left(x_{kl}\right)\right)^{k-1} \left(1-G_L\left(x_k^r\right)\right)^{N-k} \\
= & \frac{1}{N\left(G_L\left(x_{kh}\right)-G_L\left(x_{kl}\right)\right)} \sum_{k=1}^{N} \frac{N!}{k!\,(N-k)!} \left(G_L\left(x_{kh}\right)-G_L\left(x_{kl}\right)\right)^{k} \left(1-G_L\left(x_k^r\right)\right)^{N-k} \\
= & \frac{1}{N\left(G_L\left(x_{kh}\right)-G_L\left(x_{kl}\right)\right)} \left[\sum_{k=0}^{N} \frac{N!}{k!\,(N-k)!} \left(G_L\left(x_{kh}\right)-G_L\left(x_{kl}\right)\right)^{k} \left(1-G_L\left(x_{kh}\right)\right)^{N-k} - \left(1-G_L\left(x_{kh}\right)\right)^N\right] \\
= & \frac{1}{N\left(G_L\left(x_{kh}\right)-G_L\left(x_{kl}\right)\right)} \left[\left(\left(G_L\left(x_{kh}\right)-G_L\left(x_{kl}\right)\right)+\left(1-G_{kL}\left(x_{kh}\right)\right)\right)^N - \left(1-G_L\left(x_{kh}\right)\right)^N\right] \\
= & \frac{1}{N\left(G_L\left(x_{kh}\right)-G_L\left(x_{kl}\right)\right)} \left[\left(1-G_L\left(x_{kl}\right)\right)^N - \left(1-G_L\left(x_{kh}\right)\right)^N\right]
\end{aligned}
$$

Since $q_L^- = \lim\left(1-G_L\left(x_{kl}\right)\right)^N > 0$, the lowest signal must be $x_{kl} \to a$. Denote $\lim\left(1-G_H\left(x_{kh}\right)\right)^{n_{kH}} = q_L^+$ and suppose $q_L^+ = 0$ as the first case. The sign of the profit of a seller $x = x_{kh}$ at p_k is given by the sign of the second term of the profit (3). Substituting the winning probability at p_k

$$
\begin{aligned}
& \frac{1}{n_{kL}\left(G_L\left(x_{kh}\right)-G_L\left(x_{kl}\right)\right)} \left[\left(1-G_L\left(x_{kl}\right)\right)^{n_{kL}} - \left(1-G_L\left(x_{kh}\right)\right)^{n_{kL}}\right] \left(p_k - c_L\right) + \\
& \frac{x_{kh}}{1-x_{kh}} \frac{n_{kH}}{n_{kL}} \frac{1}{n_{kH}\left(G_H\left(x_{kh}\right)-G_H\left(x_{kl}\right)\right)} \left[\left(1-G_H\left(x_{kl}\right)\right)^{n_{kH}} - \left(1-G_H\left(x_{kh}\right)\right)^{n_{kH}}\right] \left(p_k - c_H\right).
\end{aligned}
$$

Given $q_L^+ = 0$, the sign of the profit in the limit is[16]

$$
\left(q_L^- - 0\right)\left(p_0 - c_L\right) + \lim \frac{x_{kh}}{1-x_{kh}} \frac{G_L\left(x_{kh}\right)-G_L\left(x_{kl}\right)}{G_H\left(x_{kh}\right)-G_H\left(x_{kl}\right)} \left(q_H^- - 0\right)\left(p_0 - c_H\right) \tag{5}
$$

Note that

$$
\lim \frac{x_{kh}}{1-x_{kh}} \frac{G_L\left(x_{kh}\right)-G_L\left(x_{kl}\right)}{G_H\left(x_{kh}\right)-G_H\left(x_{kl}\right)} \geq 1
$$

since either $\lim x_{kh} = \lim x_{kl} = a$ implies $\lim \frac{x_{kh}}{1-x_{kh}} \frac{G_L(x_{kh})-G_L(x_{kl})}{G_H(x_{kh})-G_H(x_{kl})} = \lim \frac{g_L(x_{kh})}{g_H(x_{kh})} = \frac{a}{1-a}\frac{1-a}{a} = 1$ (using L'Hospital's rule) or $\lim x_{kh} > a$. In the latter case, observe that the monotone likelihood

[16] Note that the number of sampled bidders cancel and do not affect profits. Intuitively, a higher ratio $\frac{n_{kH}}{n_{kL}}$ has to offsetting effects: A higher ratio implies that, conditional on being sampled the high state is more likely. At the same time, winning is less likely in the high state.

ratio property implies that $\frac{G_L(x_{kh})}{G_H(x_{kh})} > \frac{1-x_{kh}}{x_{kh}}$ for all x_{kh}.[17] The profit of x_{kh} must be positive and rewriting (5) shows that this requires

$$q_L^- (p_0 - c_L) \geq -q_H^- (p_0 - c_H). \tag{6}$$

Consider a sequence of prices $p'_k \to p_0$ that undercut the tie, $p'_k < p_k$, and such that the probability to win with a bid p'_k converges to the probability that the lowest signal is above x_{kl}

$$\lim \pi \left(p'_k | L, \beta_k, n_{kL}\right) = \lim \left(1 - G_L\left(x_{kl}\right)\right)^{n_{kL}-1}.$$

Such a sequence exist by choosing ε_k for which

$$\lim \left(1 - G_L\left(x_{kl} - \varepsilon_k\right)\right)^{n_{kL}-1} = q_L^-$$

and choosing $p'_k = p_k - \frac{1}{2}\left(\beta_k\left(x_{kl}\right) - \beta_k\left(x_{kl} - \varepsilon_k\right)\right)$.[18] This ensures that $\beta_k\left(x_{kl} - \varepsilon_k\right) < p'_k < p_k$ so that the bidder wins against all $x \geq x_{kl}$ while he looses whenever the lowest signal is $x \leq x_k - \varepsilon_k$. Hence, the winning probability $\pi\left(p'_k | L, \beta_k, n_{kL}\right)$ is sandwiched in between $\left(1 - G_L\left(x_{kl}\right)\right)^{n_{kL}-1}$ and $\left(1 - G_L\left(x_{kl} - \varepsilon_k\right)\right)^{n_{kL}-1}$, both of which converge to q_L^-.

Profits at $p'_k \to p_0$ for a seller with $x = a$ become

$$\begin{aligned} & \lim \left[\frac{1}{1 + \frac{a}{1-a}\frac{n_{kH}}{nL}}\right]\left[\pi\left[p'_k | L, \beta_k, n_{kL}\right]\left(p'_k - c_L\right) + \frac{a}{1-a}\frac{n_{kH}}{n_{kL}}\pi\left[p'_k | H, \beta_k, n_{kH}\right]\left(p'_k - c_H\right)\right] \\ = & \frac{1}{1 + \frac{a}{1-a}R}\left[q_L^-\left(p_0 - c_L\right) + \frac{a}{1-a}R q_H^-\left(p_0 - c_H\right)\right] \end{aligned}$$

and using (6)

$$\begin{aligned} & \frac{1}{1 + \frac{a}{1-a}R}\left[q_L^-\left(p_0 - c_L\right) + \frac{a}{1-a}R q_H^-\left(p_0 - c_H\right)\right] \\ \geq & \frac{1}{1 + \frac{a}{1-a}R}\left[-q_H^-\left(p_0 - c_H\right) + \frac{a}{1-a}R q_H^-\left(p_0 - c_H\right)\right] > 0 \end{aligned}$$

since $\frac{a}{1-a}R < 1$ and $p_0 < 1 = c_H$. Thus, it is strictly profitable to undercut the tie and bid p'_k

[17] The MLRP implies that for $x < x'$, $\frac{g_L(x)}{g_H(x)} = \frac{1-x}{x} > \frac{g_L(x')}{g_H(x')} = \frac{1-x'}{x'}$ and rewriting and integrating

$$\int_a^x g_L(x)\, g_H(x')\, dx' > \int_a^x g_H(x)\, g_L(x')\, dx' \quad \Rightarrow \quad g_L(x)\, G_H(x) > g_H(x)\, G_L(x) \Rightarrow \frac{g_L(x)}{g_H(x)} > \frac{G_L(x)}{G_H(x)}.)$$

[18] By definition of x_{kl}, $\beta_k\left(x_{kl}\right) > \beta\left(x_{kl} - \varepsilon\right)$ for every $\varepsilon > 0$.

instead of p_k for k large enough if there is a non-vanishing tie at p_0 and $q_L^+ = 0$ - a contradiction to bidders earning zero profits.

Now consider the second case: $q_L^+ > 0$, (and so, in particular, $x_{kh} \to a$). It will be convenient to normalize $c_H = 1$ and $c_L = 0$ to reduce the algebra (this normalization is without loss of generality relative to the statement of the lemma)[19]. To make sure that profits at p_k' (defined as before) converge to zero it must be that

$$\begin{aligned} \frac{1}{1+\frac{a}{1-a}R}\left[q_L^-\left(p_0-0\right)+\frac{a}{1-a}Rq_H^-\left(p_0-1\right)\right] &= 0 \\ \Rightarrow \frac{\frac{a}{1-a}Rq_H^-}{q_L^-+\frac{a}{1-a}Rq_H^-} &= p_0. \end{aligned} \tag{7}$$

From before, we know that the profit at the tie p_k is equal to

$$\frac{1}{n_{kL}\left(G_L\left(x_{kh}\right)-G_L\left(x_{kl}\right)\right)}\left[\left(1-G_L\left(x_{kl}\right)\right)^{n_{kL}}-\left(1-G_L\left(x_{kh}\right)\right)^{n_{kL}}\right]\left(p_k-0\right)+$$
$$\frac{a}{1-a}\frac{n_{kH}}{n_{kL}}\frac{1}{n_{kH}\left(G_H\left(x_{kh}\right)-G_H\left(x_{kl}\right)\right)}\left[\left(1-G_H\left(x_{kl}\right)\right)^{n_{kH}}-\left(1-G_H\left(x_{kh}\right)\right)^{n_{kH}}\right]\left(p_k-1\right),$$

scaled up by $\left(1+\frac{x}{1-x}\frac{n_{kL}}{n_{kH}}\right)^{-1}$. Note that $\left(1-G_L\left(x_{kh}\right)\right)^{n_{kL}} \to q_L^+ > 0$ implies that $\lim n_{kl}G_L\left(x_{kh}\right) \to -\ln q_L^+ < \infty$ and, in particular, $n_{kH}\left(G_H\left(x_{kh}\right)-G_H\left(x_{kl}\right)\right) \to \ln\frac{q_L^-}{q_L^+}$ (see the law of rare events). Thus, in the limit, the profit at p_k is equal to

$$\left(q_L^- - q_L^+\right)p_0 + \frac{a}{1-a}\frac{1-a}{a}\left(q_H^- - q_H^+\right)\left(p_0-1\right)$$

scaled up by $\left(1+\frac{x}{1-x}R\right)^{-1}\left(\ln\frac{q_L^-}{q_L^+}\right)^{-1}$, observing that $\frac{G_L(x_{kh})-G_L(x_{kl})}{G_H(x_{kh})-G_H(x_{kl})} \to \frac{1-a}{a}$ from $x_{kh} \to a$ and $x_{kl} \to a$.

[19] Suppose for some c_L and c_H, there exists a sequence of partial equilibrium winning bid distributions with either a gap or an atom. Then there will be a sequence of partial equilibrium winning bid distributions with either a gap or an atom after normalizing costs to $c_L = 0$ and $c_H = 1$.

Substituting the zero profit condition (7)

$$
\begin{aligned}
& \left(q_L^- - q_L^+\right) \frac{\frac{a}{1-a} R q_H^-}{q_L^- + \frac{a}{1-a} R q_H^-} + \left(q_H^- - q_H^+\right) \left(\frac{\frac{a}{1-a} R q_H^-}{q_L^- + \frac{a}{1-a} R q_H^-} - 1 \right) \\
= & \frac{1}{q_L^- + \frac{a}{1-a} R q_H^-} \left[\left(q_L^- - q_L^+\right) \frac{a}{1-a} R q_H^- + \left(q_H^- - q_H^+\right) \left(\frac{a}{1-a} R q_H^- - \left(q_L^- + \frac{a}{1-a} R q_H^- \right) \right) \right] \\
= & \frac{1}{q_L^- + \frac{a}{1-a} R q_H^-} \left[\left(q_L^- - q_L^+\right) r q_H^- + \left(q_H^- - q_H^+\right) \left(r q_H^- - \left(q_L^- + r q_H^- \right) \right) \right] \\
= & \frac{1}{q_L^- + \frac{a}{1-a} R q_H^-} \left[q_L^- r q_H^- - q_L^+ r q_H^- + q_H^- \left(r q_H^- - \left(q_L^- + r q_H^- \right) \right) - q_H^+ \left(r q_H^- - \left(q_L^- + r q_H^- \right) \right) \right] \\
= & \frac{1}{q_L^- + \frac{a}{1-a} R q_H^-} \left[q_L^- r q_H^- - q_L^+ r q_H^- + q_H^- r q_H^- - q_H^- \left(q_L^- + r q_H^- \right) - q_H^+ r q_H^- + q_H^+ \left(q_L^- + r q_H^- \right) \right] \\
= & \frac{1}{q_L^- + \frac{a}{1-a} R q_H^-} \left[q_L^- r q_H^- - q_L^+ r q_H^- + q_H^- r q_H^- - q_H^- q_L^- - q_H^- r q_H^- - q_H^+ r q_H^- + q_H^+ q_L^- + q_H^+ r q_H^- \right] \\
= & \frac{1}{q_L^- + \frac{a}{1-a} R q_H^-} \left[q_L^- r q_H^- - q_L^+ r q_H^- + 0 - q_H^- q_L^- - 0 - 0 + q_H^+ q_L^- + 0 \right] \\
= & \frac{1}{q_L^- + \frac{a}{1-a} R q_H^-} \left[q_H^+ q_L^- - q_H^- q_L^- - \frac{a}{1-a} R q_L^+ q_H^- + \frac{a}{1-a} R q_L^- q_H^- \right] \\
= & \frac{q_H^-}{q_L^- + \frac{a}{1-a} R q_H^-} \left[q_H^+ \frac{q_L^-}{q_H^-} - q_L^- - \frac{a}{1-a} R q_L^+ + \frac{a}{1-a} R q_L^- \right]
\end{aligned}
$$

We want to evaluate the sign of the term in brackets. At $q_L^- = q_L^+$ (and hence $q_H^- = q_H^+$) the term is zero, while for all $q_L^- > q_L^+$ the derivative of the term in brackets with respect to q_L^- is strictly negative: After substituting $q_H^+ = \left(q_L^+\right)^{\frac{a}{1-a}R}$ and $q_H^- = \left(q_L^-\right)^{\frac{a}{1-a}R}$ (from the law of rare events)

$$
\begin{aligned}
& \frac{\partial}{\partial q_L^-} \left[\left(q_L^+\right)^{\frac{a}{1-a}R} \left(q_L^-\right)^{1-\frac{a}{1-a}R} - q_L^- - \frac{a}{1-a} R q_L^+ + \frac{a}{1-a} R q_L^- \right] \\
= & \left[\left(1 - \frac{a}{1-a} R\right) \left(\frac{q_L^+}{q_L^-} \right)^{\frac{a}{1-a}R} - \left(1 - \frac{a}{1-a} R\right) \right] \\
= & \left(1 - \frac{a}{1-a} R\right) \left(\frac{q_L^+}{q_L^-} \right)^{\frac{a}{1-a}R} - \left(1 - \frac{a}{1-a} R\right) < 0
\end{aligned}
$$

and hence, the term is strictly negative whenever $q_L^- > q_L^+$, i.e., whenever the tie is non-vanishing. In the limit, the profit of a seller who bids p_k are given by

$$
\left(1 + \frac{x}{1-x} R\right)^{-1} \left(\ln \frac{q_L^-}{q_L^+}\right)^{-1} \frac{q_H^-}{q_L^- + \frac{a}{1-a} R q_H^-} \left[q_H^+ \frac{q_L^-}{q_H^-} - q_L^- - \frac{a}{1-a} R q_L^+ + \frac{a}{1-a} R q_L^- \right],
$$

which is strictly negative, observing that the last term in square brackets is strictly negative when

$q_L^- > q_L^+$ as shown before and the scaling terms are not vanishing to zero. Hence, bidding p_k is not optimal and there cannot be a tie at p_k, contradicting our starting hypothesis.

No atoms. Suppose $p_0 < c_H$ is an atom of F_L, $F_L\left(p_0^-\right) < F_L\left(p_0^+\right) \leq 1$. Take two sequences of signals x_k^1 and x_k^2 such that the winning probabilities are within the atom,

$$1 - F_L\left(p^-\right) > \lim\left(1 - G_L\left(x_k^1\right)\right)^{n_{kL}-1} > \lim\left(1 - G_L\left(x_k^2\right)\right)^{n_{kL}-1} > 1 - F_L\left(p^+\right).$$

Such a sequence exist, recalling that $\lim \pi\left[\beta_k\left(x_k^i\right)|L, \beta_k, n_{kL}\right] = \lim\left(1 - G_L\left(x_k^i\right)\right)^{n_{kL}}$ since there are no ties. From $\lim\left(1 - G_L\left(x_k^i\right)\right)^{n_{kL}} > 0$, $\lim x_k^2 = \lim x_k^1 = a$. Denote the respective winning probabilities by $\pi_w^i = \lim \pi\left[\beta_k\left(x_k^i\right)|w, \beta_k, n_{kw}\right]$ (restricting attention to a convergent subsequence if necessary.) From the law of rare events, $\pi_H^1 = \left(\pi_L^1\right)^{\frac{a}{1-a}R}$. Profits must become zero in the limit

$$\begin{aligned}
\lim\left(1 - x_k^1\right)\pi\left[\beta_k\left(x_k^1\right)|L, \beta_k, n_{kL}\right]\left(\beta_k\left(x_k^1\right) - c_L\right) + x_k^1\frac{n_{kH}}{n_{kL}}\pi\left[\beta_k\left(x_k^i\right)|H, \beta_k, n_{kH}\right]\left(\beta_k\left(x_k^1\right) - c_H\right) &= 0 \\
\Rightarrow (1-a)\,\pi_L^1\left(p_0 - c_L\right) + aR\pi_H^1\left(p_0 - c_H\right) &= 0 \\
\Leftrightarrow (1-a)\,\pi_L^1\left(p_0 - c_L\right) + aR\left(\pi_L^1\right)^{\frac{a}{1-a}R}\left(p_0 - c_H\right) &= 0 \\
\Leftrightarrow -\frac{a}{1-a}R\frac{\left(p_0 - c_H\right)}{\left(p_0 - c_L\right)} &= \left(\pi_L^1\right)^{\left(1-\frac{a}{1-a}R\right)} \\
\Rightarrow -\frac{a}{1-a}R\frac{\left(p_0 - c_H\right)}{\left(p_0 - c_L\right)} &> \left(\pi_L^2\right)^{\left(1-\frac{a}{1-a}R\right)} \\
\Rightarrow (1-a)\,\pi_L^2\left(p_0 - c_L\right) + aR\pi_H^2\left(p_0 - c_H\right) &< 0
\end{aligned}$$

using that $\left(\pi_L^1\right)^{\left(1-\frac{a}{1-a}R\right)} > \left(\pi_L^2\right)^{\left(1-\frac{a}{1-a}R\right)}$ when $\pi_L^1 > \pi_L^2$.Therefore, profits are negative at $p_k = \beta_k\left(x_k^2\right)$ for k large enough, a contradiction to optimality.

No gaps. Suppose there is some interval $[p_1, p_2]$ such that $0 < F_L\left(p_1\right) = F_L\left(p_2\right) < 1$, $p_2 > p_1$. As observed before, p_1 cannot be an atom. Thus, the probability to win at p_1 is the same as the probability to win at any $p' \in \left(p_1, p_2\right)$ in both states. However, at any p', the payoff conditional on winning is strictly higher. Since bidding $\beta_k(x) \to p_1$ must lead to zero profit, bidding any such p' must lead to strictly positive profits when k is large enough. (Note that we cannot not make this conclusion if p_1 is an atom. If p_1 is an atom and winning conditional on being tied is good news, then it is not necessarily the case that overbidding the atom strictly increases profits.) QED

For the next lemma, fix some sequence of anticipated sampling behavior $\mathcal{N}_k$ and partial equilibrium bidding strategies β_k given $\mathcal{N}_k$. Now, suppose the L buyer samples $B_L n_{kL}$ sellers, rather than the anticipated number n_{kL}. (The bidding strategy is determined as a function of the anticipated sampling behavior while the buyers can sample buyers in different ways.) We can calculate the distribution of the winning bid in state L, $F_{kL}\left(p|L, \beta_k, B_L n_{kL}\right)$ and similarly in state H, $F_{kH}(p|H, \beta_k, B_H n_{kL})$. Given a sequence of anticipated sampling behavior $\mathcal{N}_k$, denote by $R = \lim R_k = \lim \frac{n_{kH}}{n_{kL}}$ the limit ratio (restricting attention to a convergent subsequence if necessary). We assume that the number of sampled bidders grows to infinity, $n_{kH} \to \infty$ and $n_{kL} \to \infty$. We will see that there exists a unique limit distribution of the winning bid which depends only on the ratio R and the actual sampling B_L and B_H, respectively, i.e., given any sequence $\mathcal{N}_k$ and partial equilibrium strategies β_k, there is a distribution $F(p|R, L, B_L)$ such that $F_{kL}(p|L, \beta_k, B_L n_{kL}) \to F(p|R, L, B_L)$ (pointwise) whenever $\mathcal{N}_k$ is such that $R_k \to R$. Let $p_w(1 - F_w|R, B_L)$ denote the inverse of F_w, i.e., given R and B_L, in state w, with probability $(1 - F_w)$ the winning bid will be above p in the limit.

Since the algebra becomes messy otherwise, we normalize costs by setting $c_L = 0$ and $c_H = 1$.

The following lemma shows that the inverse c.d.f. of the winning bid depends only on the anticipated ratio R (which determines the bidding strategy of the sellers) and on the actual bidding behavior determined by B_w (e.g., the number of sampled bidders is $B_L n_{kL}$, i.e. the L buyer samples B_L more sellers than expected). The idea of the lemma is this: If there is some price p' such that the winning probability in the low state is $1 - F_L(p')$, we know that the winning probability in the high state is $(1 - F_H(p'))$. The zero profit condition then determines the price p' as a function of $1 - F_L(p')$:

Lemma 5 *Let $\mathcal{N}_k$ be a sequence of sampling strategies with $n_{kw} \to \infty$. Suppose $R_k = \frac{n_{kH}}{n_{kL}} \to [0, \frac{1-a}{a})$ and suppose β_k is a partial equilibrium bidding strategy given sampling strategy $\mathcal{N}_k$. Denote the induced distributions of the winning bid by $F_{kw}(p|w, \beta_k, B_w n_{kL})$ and suppose $F_{wk} \to F_w$ pointwise. Given $c_L = 0$ and $c_H = 1$, the limit distribution satisfies*

$$p_L(1 - F|R, B_L) = \frac{(1-F)^{\frac{R}{B_L}\frac{a}{1-a}}}{(1-F)^{\frac{R}{B_L}\frac{a}{1-a}} + (1-F)^{\frac{1}{B_L}}\frac{1}{R}\frac{1-a}{a}}.$$

.

$$p_H(1 - F|R, B_H) = \frac{\left((1-F)^{\frac{1-a}{B_H a}}\right)^{R\frac{a}{1-a}}}{\left((1-F)^{\frac{1-a}{B_H a}}\right)^{R\frac{a}{1-a}} + \left((1-F)^{\frac{1-a}{B_H a}}\right)R^{-1}\frac{1-a}{a}}$$

Proof: By $R < \frac{1-a}{a}$, $F_L(p)$ is not degenerate from Lemma 4. Consider some interior price p such that $F_L(p) \in (0, 1)$ and consider some $x_k \to x$ such that $\beta_k(x_k) \to p$ (there must exists such a sequence, since there are no gaps, again by Lemma 4). The winning probability must satisfy

$$1 - F_L(p) = \lim (1 - G_L(x_k))^{n_{kL}-1},$$

since there are no atoms and $x_k \to a$. By the law of rare events,

$$1 - F_H(p) = (1 - F_L(p))^{\frac{a}{1-a}R} > 1 - F_L(p),$$

where the last inequality follows from $\frac{a}{1-a}R < 1$. The profit at $\beta_k(x_k)$ must converge to zero, hence

$$\begin{aligned}
&\lim \left[\frac{1}{1+\frac{x}{1-x}\frac{n_{kH}}{n_{kL}}}\right]\left[\pi\left[\beta_k(x_k)|L,\beta_k,n_{kL}\right](\beta_k(x_k)-0) + \frac{x}{1-x}\frac{n_{kH}}{n_{kL}}\pi\left[\beta_k(x_k)|H,\beta_k,n_{kH}\right](\beta_k(x_k)-1)\right] \\
= &\left[\frac{1}{1+\frac{a}{1-a}R}\right]\left[(1-F_L(p))(p-0) + \frac{a}{1-a}R(1-F_L(p))^{\frac{a}{1-a}R}(p-1)\right] = 0
\end{aligned}$$

Thus,

$$p_L(1-F_L) = \frac{(1-F_L(p))^{\frac{a}{1-a}R}}{(1-F_L(p))^{\frac{a}{1-a}R} + \frac{1}{R}\frac{1-a}{a}(1-F_L(p))}.$$

(Alternatively, conditional on winning the posterior probability of the high state is

$$\frac{\frac{1}{2}(1-F_L(p))^{\frac{a}{1-a}R}}{\frac{1}{2}(1-F_L(p))^{\frac{a}{1-a}R} + \frac{1}{2}\frac{1}{R}\frac{1-a}{a}(1-F_L(p))},$$

and setting the price equal to the expected cost conditional on winning yields the same expression for $p_L(1-F_L)$.)

Let p_H be the price such that in state H the winning bid is above p_H with probability $(1-F_H)$. From the law of rare events, in the low state the winning bid is above p_H with probability $(1-F_H)^{\frac{1-a}{a}\frac{1}{R}}$. Hence,

$$\begin{aligned}
p_H(1-F_H) &= p_L\left((1-F_L(p))^{\frac{1-a}{a}\frac{1}{R}}\right) \\
&= \frac{(1-F_H)}{(1-F_H) + (1-F_H)^{\frac{1-a}{a}\frac{1}{R}}\frac{1}{R}\frac{1-a}{a}}.
\end{aligned}$$

General sampling. Let us derive $F(p|R,L,B_L)$ for general B_L. Note that

$$1 - F(p|R,L,B_L=1) = \lim(1-G_L(x_k))^{n_{kL}-1}$$

implies

$$1 - F(p|R,L,B_L) = \lim(1-G_L(x_k))^{n_{kL}B_L} = (1-F_L(p))^{B_L}$$

and similarly (for all B_H not just $B_H = R$):

$$1 - F(p|R, H, B_H) = (1 - F_H(p))^{\frac{a}{1-a} B_H}.$$

Thus, we can find the inverse c.d.f.s in the general case. Let $p(1 - F_L|R, L, B_L)$ denote the price p such that the winning bid is larger than p with probability $(1 - F_L)$ in state L (given that the bids are determined by the anticipated ratio R and the L buyer actually samples $B_L n_{kL}$ sellers.) Then, for the same p, if the buyer does indeed sample n_{kL} sellers, the probability that the winning bid is larger than p is $(1 - F_L)^{\frac{1}{B_L}}$. We know that the price p which has this property is fixed by the zero profit condition. Thus

$$\begin{aligned} p_L(1 - F_L|R, L, B_L) &= p\left((1 - F_L)^{\frac{1}{B_L}}|R, L, 1\right) = \frac{(1 - F_L)^{\frac{a}{1-a}\frac{R}{B_L}}}{(1 - F_L)^{\frac{a}{1-a}\frac{R}{B_L}} + (1 - F_L)^{\frac{1}{B_L}} \frac{1}{R}\frac{1-a}{a}}, \\ p_H(1 - F_H|R, H, B_H) &= \frac{(1 - F_H)^{\frac{R}{B_H}}}{(1 - F_H)^{\frac{R}{B_H}} + (1 - F_H)^{\frac{1-a}{a}\frac{1}{B_H}} \frac{1}{R}\frac{1-a}{a}}. \end{aligned}$$

QED

The lemma allows us to calculate the expected price $\lim E[p|_k, w, B_w]$ with $R = \lim \frac{n_{kH}}{n_{kL}}$,

$$E[p|R, L, B_L] = \lim E[p|L, \beta_k, B_L n_{kL}] = \int_0^1 \frac{x^{\frac{R}{B_L}\frac{a}{1-a}}}{x^{\frac{R}{B_L}\frac{a}{1-a}} + x^{\frac{1}{B_L}} R^{-1}\frac{1-a}{a}} dx \tag{8}$$

$$E[p|R, H, B_H] = \lim E[p|H, \beta_k, B_H n_{kL}] = \int_0^1 \frac{x^{\frac{R}{B_H}}}{x^{\frac{R}{B_H}} + x^{\frac{1-a}{B_H a}} R^{-1}\frac{1-a}{a}} dx. \tag{9}$$

(Here we used the fact that the expected value of a random variable can be written directly as a function of the c.d.f. $(1 - F)$, with $x(1 - F)$ denoting the inverse of $1 - F(x)$,

$$E[x] = \int_{-\infty}^{+\infty} x f(x)\, dx = \int_0^1 x(1 - F)\, d(1 - F).)$$

Let σ_k be a sequence of equilibria and let the total sampling cost of the L buyer be $\bar{s} = \lim n_{kL} s_k$. Payoffs of the L buyer from from sampling $B_L n_{kL}$ times are

$$v - E[p|R, L, B_L] - B_L \bar{s}.$$

The sampling behavior of the buyers can be characterized by the derivative of these payoffs. As we will show below, for the limit of equilibria it has to hold that

$$\begin{aligned} 1 &\in \arg\max_{B_L} v - E\left[p|R,L,B_L\right] - B_L\bar{s} \\ R &\in \arg\max_{B_H} v - E\left[p|R,H,B_H\right] - B_H\bar{s} \end{aligned}$$

and hence, in the limit, the following first order condition has to hold:

$$-\frac{\partial}{\partial B_H}E\left[p|R,H,B_H\right]|_{B_H=R} = -\frac{\partial}{\partial B_L}E\left[p|R,L,B_L\right]|_{B_L=1} = \bar{s}.$$

Now we prove: If a sequence of sampling strategies $\mathcal{N}_k$ and partial equilibrium bidding strategies β_k is such that $\lim \frac{n_{kH}}{n_{kL}} = R$ is close to $\frac{1-a}{a}$, and if the first order condition holds for the L buyer, then the H buyer would have an incentive to sample less (because the marginal price decrease from sampling is smaller for H):

Lemma 6 *There exists some $\bar{R} < \frac{1-a}{a}$ such that for all $R \in \left[\bar{R}, \frac{1-a}{a}\right]$, the derivative of (8) and (9) is such that*

$$-\frac{\partial}{\partial B_L}E\left[p|R,L,B_L\right]|_{B_L=1} > -\frac{\partial}{\partial B_H}E\left[p|R,H,B_H\right]|_{B_H=R}.$$

Proof: The derivative of the expected price is

$$\begin{aligned} \frac{\partial}{\partial B_L}E &= \frac{\partial}{\partial B_L}\left(\int_0^1 \frac{x^{\frac{R}{B_L}\frac{a}{1-a}}}{x^{\frac{R}{B_L}\frac{a}{1-a}} + x^{\frac{1}{B_L}}R^{-1}\frac{1-a}{a}}dx\right) = \frac{\partial}{\partial B_L}\left(\int_0^1 \frac{1}{1+\frac{x^{\frac{1}{B_L}}}{x^{\frac{R}{B_L}\frac{a}{1-a}}}R^{-1}\frac{1-a}{a}}dx\right) \\ &= \frac{\partial}{\partial B_L}\left(\int_0^1 \left(1+\frac{x^{\frac{1}{B_L}}}{x^{\frac{R}{B_L}\frac{a}{1-a}}}R^{-1}\frac{1-a}{a}\right)^{-1}dx\right) = \frac{\partial}{\partial B_L}\left(\int_0^1 \left(1+x^{\frac{1}{B_L}\left(\frac{1-a-Ra}{1-a}\right)}\frac{1-a}{Ra}\right)^{-1}dx\right) \\ &= -\int_0^1 \left(-\frac{1}{(B_L)^2}\ln x^{\left(\frac{1-a-Ra}{1-a}\right)}x^{\frac{1}{B_L}\left(\frac{1-a-Ra}{1-a}\right)}\frac{1-a}{Ra}\right)\left(1+x^{\frac{1}{B_L}\left(\frac{1-a-Ra}{1-a}\right)}\frac{1-a}{Ra}\right)^{-2}dx \\ &= \int_0^1 \frac{1}{(B_L)^2}\ln x^{\left(\frac{1-a-Ra}{1-a}\right)}x^{\frac{1}{B_L}\left(\frac{1-a-Ra}{1-a}\right)}\left(\frac{1-a}{Ra}\right)\left(1+x^{\frac{1}{B_L}\left(\frac{1-a-Ra}{1-a}\right)}\frac{1-a}{Ra}\right)^{-2}dx \end{aligned}$$

where we used $\frac{x^{\frac{1}{B_L}}}{x^{\frac{R}{B_L}\frac{a}{1-a}}} = x^{\frac{1}{B_L}\left(1-R\frac{a}{1-a}\right)} = x^{\frac{1}{B_L}\left(\frac{1-a-Ra}{1-a}\right)}$ and we used

$$\frac{\partial}{\partial B_L}\left(1+x^{\frac{1}{B_L}\left(\frac{1-a-Ra}{1-a}\right)}\frac{1-a}{Ra}\right) = -\frac{1}{(B_L)^2}\ln x^{\left(\frac{1-a-Ra}{1-a}\right)}x^{\frac{1}{B_L}\left(\frac{1-a-Ra}{1-a}\right)}\frac{1-a}{Ra}$$

given the basic derivative $\frac{\partial}{\partial x}A^{\frac{z}{x}} = \frac{\partial}{\partial x}e^{\ln A^{\frac{z}{x}}} = \frac{\partial}{\partial x}e^{\frac{1}{x}\ln A^{z}} = -\frac{1}{x^2}\ln A^{z}e^{\ln A^{\frac{z}{x}}} = -\left(\ln A^{\frac{z}{x^2}}\right)A^{\frac{z}{x}}$.

Similarly,

$$\begin{aligned}
\frac{\partial}{\partial B_H}E\left[p|R,H,B_H\right] &= \frac{\partial}{\partial B_H}\int_0^1 \frac{x^{\frac{R}{B_H}}}{x^{\frac{R}{B_H}} + x^{\frac{1-a}{B_H a}}R^{-1}\frac{1-a}{a}}dx \\
&= \frac{\partial}{\partial B_H}\int_0^1 \frac{1}{1+\frac{x^{\frac{1-a}{B_H a}}}{x^{\frac{R}{B_H}}}R^{-1}\frac{1-a}{a}}dx \\
&= \frac{\partial}{\partial B_H}\int_0^1 \left(1+x^{\frac{1}{B_H}\left(\frac{1-a-Ra}{a}\right)}R^{-1}\frac{1-a}{a}\right)^{-1}dx \\
&= \int_0^1 \frac{1}{\left(B_H\right)^2}\ln\left(x^{\left(\frac{1-a-Ra}{a}\right)}\right)x^{\frac{1}{B_H}\left(\frac{1-a-Ra}{a}\right)}R^{-1}\frac{1-a}{a}\left(1+x^{\frac{1}{B_H}\left(\frac{1-a-Ra}{a}\right)}R^{-1}\frac{1-a}{a}\right)^{-2}dx
\end{aligned}$$

using $\frac{x^{\frac{1-a}{B_H a}}}{x^{\frac{R}{B_H}}} = x^{\frac{1}{B_H}\left(\frac{1-a}{a}-R\right)} = x^{\frac{1}{B_H}\left(\frac{1-a-Ra}{a}\right)}$ and

$$\frac{\partial}{\partial B}x^{\frac{1}{B_H}\left(\frac{1-a-Ra}{a}\right)} = \frac{1}{\left(B_H\right)^2}\ln\left(x^{\left(\frac{1-a-Ra}{a}\right)}\right)x^{\frac{1}{B_H}\left(\frac{1-a-Ra}{a}\right)}$$

Hence, the derivatives are

$$\begin{aligned}
\frac{\partial}{\partial B_L}E\left[p|R,L,B_L\right] &= \int_0^1 \frac{1}{\left(B_L\right)^2}\left(\ln x^{\left(\frac{1-a-Ra}{1-a}\right)}\right)x^{\frac{1}{B_L}\left(\frac{1-a-Ra}{1-a}\right)}\left(\frac{1-a}{Ra}\right)\left(1+x^{\frac{1}{B_L}\left(\frac{1-a-Ra}{1-a}\right)}\frac{1-a}{Ra}\right)^{-2}dx \\
\frac{\partial}{\partial B_H}E\left[p|R,H,B_H\right] &= \int_0^1 \frac{1}{\left(B_H\right)^2}\left(\ln x^{\left(\frac{1-a-Ra}{a}\right)}\right)x^{\frac{1}{B_H}\left(\frac{1-a-Ra}{a}\right)}\left(\frac{1-a}{Ra}\right)\left(1+x^{\frac{1}{B_H}\left(\frac{1-a-Ra}{a}\right)}R^{-1}\frac{1-a}{a}\right)^{-2}dx
\end{aligned}$$

and at $B_L = 1$ and $B_H = R$

$$\begin{aligned}
\frac{\partial}{\partial B_L}E\left[p|R,L,B_L=1\right] &= \int_0^1 \left(\ln x^{\left(\frac{1-a-Ra}{1-a}\right)}\right)x^{\left(\frac{1-a-Ra}{1-a}\right)}\left(\frac{1-a}{Ra}\right)\left(1+x^{\left(\frac{1-a-Ra}{1-a}\right)}\frac{1-a}{Ra}\right)^{-2}dx \\
\frac{\partial}{\partial B_H}E\left[p|R,H,B_H=R\right] &= \int_0^1 \frac{1}{R^2}\left(\ln x^{\left(\frac{1-a-Ra}{a}\right)}\right)x^{\left(\frac{1-a-Ra}{Ra}\right)}\left(\frac{1-a}{Ra}\right)\left(1+x^{\left(\frac{1-a-Ra}{Ra}\right)}\frac{1-a}{Ra}\right)^{-2}dx
\end{aligned}$$

Let $R = \frac{1-a-\varepsilon}{a}$, so that with $\varepsilon \to 0$, $R \cong \frac{1-a}{a}$. Then we simplify $\left(\frac{1-a-Ra}{Ra}\right) = \frac{\varepsilon}{1-a-\varepsilon}$ and

$\left(\frac{1-a-Ra}{1-a}\right) = \frac{\varepsilon}{1-a}$, $\frac{1-a}{Ra} = \frac{1-a}{1-a-\varepsilon}$. The derivatives become

$$\begin{aligned}
\left(\frac{1-a}{Ra}\right)^{-1} \frac{\partial}{\partial B_L} E &= \int_0^1 \left(\frac{1-a-Ra}{1-a}\right)(\ln x)\, x^{\left(\frac{1-a-Ra}{1-a}\right)} \left(1 + x^{\left(\frac{1-a-Ra}{1-a}\right)} \frac{1-a}{Ra}\right)^{-2} dx \\
&= \int_0^1 \frac{\varepsilon}{1-a} (\ln x)\, x^{\frac{\varepsilon}{1-a}} \left(1 + x^{\frac{\varepsilon}{1-a}} \left(\frac{1-a}{1-a-\varepsilon}\right)\right)^{-2} dx \\
&= \frac{\varepsilon}{1-a} \int_0^1 (\ln x)\, x^{\frac{\varepsilon}{1-a}} \left(1 + x^{\frac{\varepsilon}{1-a}} \left(\frac{1-a}{1-a-\varepsilon}\right)\right)^{-2} dx
\end{aligned}$$

and

$$\begin{aligned}
\left(\frac{1-a}{Ra}\right)^{-1} \frac{\partial}{\partial B_H} E &= \int_0^1 \frac{1}{R^2} \left(\frac{1-a-Ra}{a}\right)(\ln x)\, x^{\left(\frac{1-a-Ra}{Ra}\right)} \left(1 + x^{\left(\frac{1-a-Ra}{Ra}\right)} \frac{1-a}{Ra}\right)^{-2} dx \\
&= \int_0^1 \frac{a^2}{(1-a-\varepsilon)^2} \frac{\varepsilon}{1-a-\varepsilon} (\ln x)\, x^{\left(\frac{\varepsilon}{1-a-\varepsilon}\right)} \left(1 + x^{\frac{\varepsilon}{1-a-\varepsilon}} \frac{1-a}{1-a-\varepsilon}\right)^{-2} dx \\
&= \frac{a^2}{(1-a-\varepsilon)^2} \frac{\varepsilon}{1-a-\varepsilon} \int_0^1 (\ln x)\, x^{\left(\frac{\varepsilon}{1-a-\varepsilon}\right)} \left(1 + x^{\frac{\varepsilon}{1-a-\varepsilon}} \frac{1-a}{1-a-\varepsilon}\right)^{-2} dx
\end{aligned}$$

We can evaluate the integral with $\varepsilon \to 0$,

$$\lim \int_0^1 (\ln x)\, x^{\left(\frac{\varepsilon}{1-a-\varepsilon}\right)} \left(1 + x^{\frac{\varepsilon}{1-a-\varepsilon}} \frac{1-a}{1-a-\varepsilon}\right)^{-2} dx = \lim \frac{1}{4} \int_0^1 (\ln x)\, x^{\left(\frac{\varepsilon}{1-a-\varepsilon}\right)} dx = \frac{1}{4}$$

using integration by parts,

$$\begin{aligned}
\int (FG)' &= \int fG + \int Fg \Rightarrow \int Fg = (FG) - \int fG \\
\lim \int (\ln x)\, x^{\varepsilon} &= \left[\frac{1}{(\varepsilon+1)} x^{\varepsilon+1} \ln x\right]_0^1 - \int_0^1 \frac{1}{x} \frac{1}{(\varepsilon+1)} x^{\varepsilon+1} \\
&= 0 - \lim_{\varepsilon \to 0} \int_0^1 \frac{1}{(\varepsilon+1)} x^{\varepsilon} = -1.
\end{aligned}$$

And thus the ratio of the derivatives at $R = \frac{1-a-\varepsilon}{a}$

$$\lim_{\varepsilon \to 0} \frac{\frac{\partial}{\partial B_H} E\left[p | R = \frac{1-a-\varepsilon}{a}, L, B_L = 1\right]}{\frac{\partial}{\partial B_L} E\left[p | R = \frac{1-a-\varepsilon}{a}, H, B_H = R\right]} = \left(\frac{a}{1-a}\right)^2 < 1. \qquad \text{QED}$$

We consider an **auxiliary game** with parameters $\bar{R} \in \left(0, \frac{1-a}{a}\right)$ and $\hat{N}$ with $\bar{R}\hat{N} \geq 2$ in which the L buyer must sample at least $\hat{N}$ bidders and the H buyer must sample at least $\bar{R}\hat{N}$ bidders.

In addition, if the H buyer chooses to invite $\tilde{n}$ bidders, the number of actually sampled bidders is just $n_H = \min\left\{\bar{R}n_L, \tilde{n}\right\}$. Therefore, the ratio of actually sampled bidders satisfies $\frac{n_H}{n_L} \leq \bar{R}$.[20]

An equilibrium of the auxiliary game is some profile $(\beta, \mathcal{N})$ such that β is a partial equilibrium given $\mathcal{N}$ and the number of sampled bidders is optimal, subject to the constraints on the minimal numbers, $n_L \in \left\{\min \hat{N}, ..., \bar{N}(s)\right\}$ and $n_H \in \left\{\min \bar{R}\hat{N}, ..., \bar{N}(s)\right\}$.

Assumption E: There exists an equilibrium $\sigma = (\mathcal{N}, \beta)$ in the auxiliary game for all parameters $\hat{N}$ and $\bar{R}$ if s is sufficiently small.[21]

The following lemma implies that, when $s_k \to 0$, there exists a sequence of equilibria such that the bound $n_{kw} \geq \hat{N}$ does not bind for any $\hat{N}$. Moreover, given $\hat{N}$ and $\bar{R}$, when $s_k \to 0$, there exists a sequence in which the buyers sample an unbounded number of sellers, $n_{kw} \to \infty$. (During the proof of Proposition 1 we will show furthermore that there is some $\bar{R}$ and a sequence of equilibria such that $n_{kH} < \bar{R}n_{kL}$ for s_k small enough. Together with the current lemma this will imply that there is some sequence of equilibria σ_k of the auxiliary game such that the profiles σ_k are also equilibria of the original game when s_k is small enough.)

Lemma 7 *Let $s_k \to 0$. If Assumption E is true, then for any $\bar{R} < \frac{1-a}{a}$ and any $\hat{N}$ there exists a sequence of equilibria σ_k of the auxiliary game such that both buyers sample an unbounded number of bidders, $n_{kL} \to \infty$ and $n_{kH} \to \infty$.*

Proof: Given $\bar{R} < \frac{1-a}{a}$, choose some $\hat{x}$ and μ such that

$$\bar{R}\frac{\hat{x}}{1-\hat{x}} < 1 - \mu.$$

These parameters exists by definition of $\bar{R}$, since $\bar{R}\frac{a}{1-a} < 1$. Choose some $\hat{N}$ large enough such that

$$\frac{\mu}{2-\mu}\left(\frac{c_H - c_L}{2}\right) > \frac{1}{\hat{N}G_L(\hat{x})}(c_H - c_L).$$

Given $s_k \to 0$ let σ_k be a sequence of equilibria of the auxiliary game with parameters $\bar{R}$ and $\hat{N}$ as defined before.[22] By *Assumption E* such a sequence exists. We consider equilibria in which the sampling strategy of the buyers is pure (the same arguments apply to mixed strategies). With pure strategies, there is no loss of generality in assuming that the H buyer invites at most $\bar{R}n_{kL}$

[20] Note that the bidders choose their bids without knowing the number of invitees. Therefore, the L buyer does not have an incentive to manipulate the sampling behavior of the H buyer.

[21] s small ensures that $\bar{N}(s) \geq \hat{N}$, i.e., $\left\{\min \hat{N}, ..., \bar{N}(s)\right\}$ is not empty.

[22] With s_1 small enough such that $\bar{N}(s) \geq \hat{N}$.

bidders, and we identify the number of chosen invitees $\tilde{n}$ and the number of actually sampled bidders $n_H = \min\left\{\tilde{n}, \bar{R}n_{kL}\right\}$. We want to show that $n_{kw} \to \infty$, which implies the lemma (since $n_{kL} \geq \hat{N}$ will not bind.)

Suppose n_{kL} does not diverge to infinity. By $n_{kH} \leq \bar{R}n_{kL}$ for all k, n_{kH} must stay bounded as well. Let us consider a subsequence such that the number of sampled bidders converge, $n_L = \lim n_{kL}$, $n_H = \lim n_{kH}$, $R = \frac{n_H}{n_L} \in (0, \bar{R}]$, the distribution of the winning bid converges, $F_L(\cdot) = \lim F_L(\cdot|L, \beta_k, n_{kL})$ (pointwise) and the bidding strategies converge, $\beta(\cdot) = \lim \beta_k(\cdot)$ (pointwise).

Step 1. β_k must become degenerate, i.e., for some p_0, $\beta_k(x) \to p_0$ for all $x \in (a, b)$. Suppose not and suppose there are $x_1 \in (a, b)$ and $x_2 \in (a, b)$ such that $\lim \beta_k(x_1) \equiv p_1 < p_2 \equiv \lim \beta_k(x_2)$. By monotonicity, $\beta_k(x) \leq p_1$ for all $x \leq x_1$ and $\beta_k(x) \geq p_2$ for all $x \leq x_2$. The benefit from sampling one more seller is $\Delta(n_{kL}) \leq s_k$ by optimality. However, $a < x_1 < x_2 < b$ implies that $(1 - G_L(x_2))^{n_L}$, i.e., with positive probability all bids are above p_2 while there is a positive probability to sample a bidder with $\beta_k(x) \leq p_1 < p_2$. Hence,

$$\begin{aligned} \lim \Delta(n_{kL}) &\geq \lim (1 - G_L(x_2))^{n_{kL}} G_L(x_1)(\beta_k(x_2) - \beta_k(x_1)) \\ &= (1 - G_L(x_2))^{n_L} G_L(x_1)(p_1 - p_2) > 0 = \lim s_k. \end{aligned}$$

Thus, if biding strategies do not degenerate, then, given any number of sampled bidders, sampling one more bidder is strictly profitable when $s_k \to 0$ and $n_{kL} \to n_L < \infty$ cannot hold, contradicting our starting assumption.

Step 2. The atom p_0 is above expected costs, $p_0 \geq \frac{1}{2}(c_L + c_H)$. Ex ante expected profits (for a definition of ex ante payoffs, see the proof of Lemma 2) are

$$\begin{aligned} & \frac{1}{2}\int \hat{U}^S(\beta_k(x), L, \beta_k, \mathcal{N}_k)\, dG_{Lx} + \frac{1}{2}\int \hat{U}^S(\beta_k(x), H, \beta_k, \mathcal{N}_k)\, dG_{Hx} \\ = & \frac{1}{2}\int \hat{\pi}[\beta_k(x)|L, \beta_k(x), \mathcal{N}_k](\beta_k(x) - c_L)\, dG_{Lx} + \frac{1}{2}\int \hat{\pi}[\beta_k(x)|H, \beta_k(x), \mathcal{N}_k](\beta_k(x) - c_L)\, dG_{Hx} \end{aligned}$$

and taking limits, profits become

$$\frac{1}{2}(p_0 - c_L) + \frac{1}{2}(p_0 - c_L) = p_0 - \frac{1}{2}(c_L + c_H),$$

using that the winning probabilities must integrate to one, $\int \hat{\pi}[\beta_k(x)|w, \beta_k, \mathcal{N}_k]\, dG_{wx} = 1$ for all k. Since ex ante expected payoffs are non-negative, $p_0 - \frac{1}{2}(c_L + c_H) \geq 0$, which implies the claim.

Step 3. Profits are bounded. Interim expected payoffs $\hat{U}^S(\beta_k(x), x, \beta_k, \mathcal{N}_k)$ are bounded from above by the payoffs in the low state $\hat{U}^S(\beta_k(x), L, \beta_k, \mathcal{N}_k)$ (since payoffs in the high state are negative by $\beta_k(x) < c_H$, see Lemma 1). Since the number of bidders is at least $\hat{N}$, feasibility and monotonicity of $\hat{U}^S(\beta_k(x), x, \beta_k, \mathcal{N}_k)$ in x implies that

$$\hat{U}^S(\beta_k(\hat{x}), x, \beta_k, \mathcal{N}_k) \leq \hat{U}^S(\beta_k(\hat{x}), L, \beta_k, \mathcal{N}_k) \leq \frac{1}{G_L(\hat{x})\hat{N}}(c_H - c_L) \text{ for all } k.$$

Step 4. Pick some $\varepsilon > 0$. At any $p' = p_0 - \varepsilon$, profits of a seller with signal $\hat{x}$ are

$$\frac{(1-\hat{x})\frac{n_{kL}}{\hat{N}}}{\hat{x}\frac{n_{kH}}{\hat{N}} + (1-\hat{x})\frac{n_{kL}}{\hat{N}}}\pi\left[p'|L, \beta_k, n_{kL}\right](p' - c_L) + \left(1 - \frac{(1-\hat{x})\frac{n_{kL}}{\hat{N}}}{\hat{x}\frac{n_{kH}}{\hat{N}} + (1-\hat{x})\frac{n_{kL}}{\hat{N}}}\right)\pi\left[p'|H, \beta_k, n_{kH}\right](p' - c_H).$$

Observe that $\pi[p'|L, \beta_k, n_{kL}] \to 1$ since p' is below the atom p_0. Taking limits shows that profits become

$$\begin{aligned}
& \frac{1}{1+\frac{\hat{x}}{1-\hat{x}}R}(p' - c_L) + \left(1 - \frac{1}{1+\frac{\hat{x}}{1-\hat{x}}R}\right)(p' - c_H) \\
> \; & \frac{1}{2-\mu}(p' - c_L) + \left(1 - \frac{1}{2-\mu}\right)(p' - c_H) \\
= \; & \frac{1}{2-\mu}(p' - c_L) + \left(\frac{1-\mu}{2-\mu}\right)(p' - c_H) \\
= \; & \frac{1}{2-\mu}(2p' - (c_L + c_H)) + \frac{\mu}{2-\mu}(c_H - p').
\end{aligned}$$

Where we used that $\frac{\hat{x}}{1-\hat{x}}R < 1 - \mu$ implies that $\frac{1}{1+\frac{\hat{x}}{1-\hat{x}}R} > \frac{1}{2-\mu}$. Substituting $p' = p_0 - \varepsilon$ shows that profits at p' are at least

$$\begin{aligned}
& -2\varepsilon + \frac{\mu}{2-\mu}\left(c_H - \frac{1}{2}(c_L + c_H) - \varepsilon\right) \\
= \; & \frac{\mu}{2-\mu}\left(\frac{1}{2}(c_H - c_L)\right) - \varepsilon\left(\frac{2(2-\mu)+\mu}{2-\mu}\right).
\end{aligned}$$

Equilibrium profits are at least as high as profits when bidding p'. Thus

$$\lim \hat{U}^S(\beta_k(\hat{x}), \hat{x}, \beta_k, \mathcal{N}_k) \geq \frac{\mu}{2-\mu}\left(\frac{1}{2}(c_H - c_L)\right) - \varepsilon\left(\frac{2(2-\mu)+\mu}{2-\mu}\right). \tag{10}$$

However, by Step 3 and by definition of $\hat{N}$, payoffs are bounded from above by

$$\hat{U}^S(\beta_k(\hat{x}), \hat{x}, \beta_k, \mathcal{N}_k) \leq \frac{1}{G_L(\hat{x})\hat{N}}(c_H - c_L) < \frac{\mu}{2-\mu}\left(\frac{1}{2}(c_H - c_L)\right) \quad \forall k. \tag{11}$$

Choosing ε small enough implies that (10) and (11) cannot hold simultaneously. Thus, assuming that the number of sampled bidders is bounded when $s_k \to 0$ leads to a contradiction. QED

Proposition 1 *Consider a sequence of vanishing sampling costs $s_k \to 0$ and suppose $0 < a < \frac{1}{2}$. If Assumption E is true (there exists a sequence of equilibria in the auxiliary game), then there exists a sequence of equilibrium profiles of the original game σ_k such that the expected price paid by the low cost buyer is strictly lower than the price paid by the high cost buyer in the limit,*

$$\lim E\left[p|L, \beta_k, n_{kL}\right] < \lim E\left[p|H, \beta_k, n_{kH}\right].$$

Proof of the Proposition. Choose some $\bar{R}$ as in Lemma 6. By Lemma 7 and Assumption E there exists a sequence of equilibrium profiles $(\beta_k, \mathcal{N}_k)$ of the auxiliary game with $\lim sup \frac{n_{Hk}}{n_{Lk}} \leq \bar{R} < \frac{1-a}{a}$ and $n_{kw} \to \infty$.

Step 1. The proposition is true if $\lim sup R_k = \lim sup \frac{n_{Hk}}{n_{Lk}} < \bar{R}$. In this case, the constraint $n_{kH} \leq \bar{R} n_{kL}$ does not bind by convexity of the seller's minimization problem, see Lemma 3. All we need to check is whether the expected price paid by the H buyer is strictly larger than the expected price paid by the L buyer.

Since the limiting ratio is smaller than $\frac{1-a}{a}$, inspection of the expected price shows that F_L is indeed dominated by F_H, i.e., higher price are more likely in the H state. Consider some probability $x \in (0,1)$ and prices $p_L(x)$ and $p_H(x)$ (so that, in state L, the winning bid is above $p_L(x)$ with probability x):

$$\begin{aligned}
p_L(x|R, B_L = 1) &= \frac{x^{R\frac{a}{1-a}}}{x^{R\frac{a}{1-a}} + xR^{-1}\frac{1-a}{a}} \\
&= \frac{x}{x + x^{\frac{1-a}{Ra}} R^{-1}\frac{1-a}{a}} \frac{x^{R\frac{a}{1-a}}}{x} \frac{x + x^{\frac{1-a}{Ra}} R^{-1}\frac{1-a}{a}}{x^{R\frac{a}{1-a}} + xR^{-1}\frac{1-a}{a}} \\
&= \frac{x}{x + x^{\frac{1-a}{Ra}} R^{-1}\frac{1-a}{a}} \frac{1 + x^{\frac{1-a}{Ra}-1} R^{-1}\frac{1-a}{a}}{1 + x^{1-R\frac{a}{1-a}} R^{-1}\frac{1-a}{a}} \\
&= \frac{x}{x + x^{\frac{1-a}{Ra}} R^{-1}\frac{1-a}{a}} \frac{1 + x^{\frac{1-a-Ra}{Ra}} R^{-1}\frac{1-a}{a}}{1 + x^{\frac{1-a-Ra}{1-a}} R^{-1}\frac{1-a}{a}} \\
&< p_H(x|R, B_H = R)
\end{aligned}$$

since $\frac{Ra}{1-a} < 1$ implies $x < x^{\frac{Ra}{1-a}}$ and therefore (via $1 - a - Ra > 0$) $x^{\frac{1-a-Ra}{Ra}} < x^{\frac{1-a-Ra}{1-a}}$, which implies $1 + x^{\frac{1-a-Ra}{Ra}} R^{-1}\frac{1-a}{a} < 1 + x^{\frac{1-a-Ra}{1-a}} R^{-1}\frac{1-a}{a}$.

Step 2. Let $\bar{s} = \lim n_{kL} s_k$. We want to show that the first order conditions have to hold in the limit

$$-\frac{\partial}{\partial B} E\left[p|\bar{R}, L, B\right]|_{B=1} = \bar{s} \quad \text{and} \quad -\frac{\partial}{\partial B} E\left[p|\bar{R}, H, B\right]|_{B=\bar{R}} \geq \bar{s}. \tag{12}$$

Suppose not and $-\frac{\partial}{\partial B} E\left[p|\bar{R}, H, B\right]|_{B=\bar{R}} < \bar{s}$ (the first order condition for the L buyer follows from symmetric reasoning.) Then, for some $B' < \bar{R}$,

$$v - E\left[p|\bar{R}, H, B'\right] - B'\bar{s} > v - E\left[p|\bar{R}, H, B = \bar{R}\right] - \bar{R}\bar{s}.$$

Let $\lfloor B' n_{kL} \rfloor$ denote the largest natural number smaller than $B' n_{kL}$. Then $E\left[p|L, \beta_k, \lfloor B' n_{kL} \rfloor\right] \to E\left[p|\bar{R}, H, B'\right]$ and hence

$$\begin{aligned}
\lim v - E\left[p|H, \beta_k, R_k n_{kL}\right] - R_k n_{kL} s_k &= v - E\left[p|\bar{R}, H, B = \bar{R}\right] - \bar{R}\bar{s} \\
&< v - E\left[p|\bar{R}, H, B'\right] - B'\bar{s} \\
&= \lim v - E\left[p|H, \beta_k, \lfloor B' n_{kL} \rfloor\right] - \lfloor B' n_{kL} \rfloor s_k,
\end{aligned}$$

a contradiction to optimality of $n_{kH} = R_k n_{kL}$ for all k.

Step 3. We now show that the sequence of equilibrium profiles of the auxiliary game corresponds to a sequence of equilibrium profiles of the original game, i.e., we will show that the restriction $n_{kH} \leq \bar{R} n_{kL}$ does not bind.

Suppose the restriction would bind and $\lim sup \frac{n_{Hk}}{n_{Lk}} = \bar{R}$. The last step implies that $-\frac{\partial}{\partial B} E\left[p|\bar{R}, L, B\right]|_{B=1} = \bar{s}$. By definition of $\bar{R}$,

$$-\frac{\partial}{\partial B} E\left[p|\bar{R}, H, B\right]|_{B=\bar{R}} < -\frac{\partial}{\partial B} E\left[p|\bar{R}, L, B\right]|_{B=1} = \bar{s}$$

Hence, the first order condition (12) of the H buyer fails. Therefore, $\lim sup \frac{n_{Hk}}{n_{Lk}} < \bar{R}$ and hence, the restriction in the auxiliary game does not bind for s_k small enough. This implies that the equilibrium profile $(\beta_k, \mathcal{N}_k)$ of the auxiliary game is an equilibrium profile in the original game. By Step 1, the expected price in the limit is higher for the H buyer as claimed. QED

Computing the derivatives at $R = 1$ for $a = 0.1$ shows that

$$\frac{\partial}{\partial B_L} E\left[p|1, L, B_L\right]|_{B_L=1} > \frac{\partial}{\partial B_H} E\left[p|1, H, B_H\right]|_{B_H=R}.$$

And it might well be that this is true for all a. Hence, the H buyer generally samples more bidders than the L buyer when $s_k \to 0$. This implies that, for given a, an auction with an endogenous number of bidders aggregates less information than an auction with an exogenously given, state independent number:

Conjecture 1 *For all $a > 0$,*

$$\frac{\partial}{\partial B_L} E\left[p|1, L, B_L\right]|_{B_L=1} > \frac{\partial}{\partial B_H} E\left[p|1, H, B_H\right]|_{B_H=R}.$$

Therefore, given $s_k \to 0$, there is a sequence of equilibria σ_k such that $R_k \to R \in \left(1, \frac{1-a}{a}\right)$.

4 Information Aggregation

We have shown that there is some separation of the two types of buyers, even when sampling costs become small. Now we want to know what happens when the signal distribution becomes more informative, i.e., we want to know what happens when the most informative signal becomes almost revealing, $a \to 0$. In particular, do we get information aggregation as in an auction with an exogenously given, state independent number of bidders - or not? As conjectured before, in general, information aggregation is harder to achieve with an endogenously chosen number of bidders, since the high cost buyer samples more bidders than the low cost buyer, thus overturning parts of the informational content of the signal.

Proposition 2 *Take some sequence of signal distributions G_{Lm} with $a_m \to 0$ and $0 < a_m \leq \hat{a}$ for all m and a sequence of sampling costs $s_{m,k}$ such that $\lim_{k\to\infty} s_{m,k} \to 0$ for all m. If Assumption E is true (there exists a sequence of equilibria in the auxiliary game), then there exists a sequence of equilibria $\sigma_{m,k} = \left[\beta_{m,k}, \mathcal{N}_{m,k}\right]$ such that the expected winning bid is revealing,*

$$\begin{aligned}\lim_{m\to\infty}\lim_{k\to\infty} E\left(p|a_m, L, \beta_{m,k}, n_{m,kL}\right) &= c_L,\\ \lim_{m\to\infty}\lim_{k\to\infty} E\left(p|a_m, H, \beta_{m,k}, n_{m,kH}\right) &= c_H.\end{aligned}$$

We prove the proposition through a sequence of lemma. The first lemma strengthens Lemma 6 by finding a uniform bound on R that holds when $a_m \to 0$. In the following, we include the lower bound a in the arguments.

Lemma 8 *There exists some $\hat{a}$ and $\hat{\varepsilon}$ such that for all $a \leq \hat{a}$ and $R = \frac{1-a-\hat{\varepsilon}}{a}$, the derivative of (8) and (9) is such that*

$$-\frac{\partial}{\partial B_L} E\left[p|a, R, L, B_L\right]|_{B_L=1} > -\frac{\partial}{\partial B_H} E\left[p|a, R, H, B_H\right]|_{B_H=R}.$$

Proof: From the proof of Lemma 6 we know that the derivatives are

$$\begin{aligned}
\left(\frac{1-a}{Ra}\right)^{-1}\frac{\partial}{\partial B_L}E &= \frac{\varepsilon}{1-a}\int_0^1 (\ln x)\, x^{\frac{\varepsilon}{1-a}}\left(1+x^{\frac{\varepsilon}{1-a}}\left(\frac{1-a}{1-a-\varepsilon}\right)\right)^{-2}dx \\
\left(\frac{1-a}{Ra}\right)^{-1}\frac{\partial}{\partial B_H}E &= \frac{a^2}{(1-a-\varepsilon)^2}\frac{\varepsilon}{1-a-\varepsilon}\int_0^1 (\ln x)\, x^{\left(\frac{\varepsilon}{1-a-\varepsilon}\right)}\left(1+x^{\frac{\varepsilon}{1-a-\varepsilon}}\frac{1-a}{1-a-\varepsilon}\right)^{-2}dx
\end{aligned}$$

Fix some a. As shown in the proof of Lemma 6, there exists $\mu > 0$ and $\hat{\varepsilon} > 0$ such that

$$\frac{\frac{\partial}{\partial B_H}E\left[p|a, R=\frac{1-a-\hat{\varepsilon}}{a}, L, B_L=1\right]}{\frac{\partial}{\partial B_L}E\left[p|a, R=\frac{1-a-\hat{\varepsilon}}{a}, H, B_H=R\right]} < 1-\mu. \tag{13}$$

We now show that there must be some $\hat{a} > 0$ such that (13) holds for all $a \leq \hat{a}$. Suppose not and let $a_k \to 0$ be the sequence of values for a such that

$$\frac{\frac{\partial}{\partial B_H}E\left[p|a_k, R=\frac{1-a_k-\hat{\varepsilon}}{a_k}, L, B_L=1\right]}{\frac{\partial}{\partial B_L}E\left[p|a_k, R=\frac{1-a_k-\hat{\varepsilon}}{a_k}, H, B_H=R\right]} \geq 1-\mu \quad \forall a_k \tag{14}$$

From before

$$\begin{aligned}
&\lim_{a_k\to 0}\frac{\frac{\partial}{\partial B_H}E\left[p|a_k, R=\frac{1-a_k-\hat{\varepsilon}}{a_k}, L, B_L=1\right]}{\frac{\partial}{\partial B_L}E\left[p|a_k, R=\frac{1-a_k-\hat{\varepsilon}}{a_k}, H, B_H=R\right]} \\
= &\lim_{a_k\to 0}\frac{\frac{a_k^2}{(1-a_k-\hat{\varepsilon})^2}\frac{\hat{\varepsilon}}{1-a_k-\hat{\varepsilon}}\int_0^1 (\ln x)\, x^{\left(\frac{\hat{\varepsilon}}{1-a_k-\hat{\varepsilon}}\right)}\left(1+x^{\frac{\hat{\varepsilon}}{1-a_k-\hat{\varepsilon}}}\frac{1-a_k}{1-a_k-\hat{\varepsilon}}\right)^{-2}dx}{\frac{\hat{\varepsilon}}{1-a_k}\int_0^1 (\ln x)\, x^{\frac{\hat{\varepsilon}}{1-a_k}}\left(1+x^{\frac{\hat{\varepsilon}}{1-a_k}}\left(\frac{1-a_k}{1-a_k-\hat{\varepsilon}}\right)\right)^{-2}dx} \\
= &\lim_{a_k\to 0}\frac{0\frac{\hat{\varepsilon}}{1-\hat{\varepsilon}}\int_0^1 (\ln x)\, x^{\left(\frac{\hat{\varepsilon}}{1-\hat{\varepsilon}}\right)}\left(1+x^{\frac{\hat{\varepsilon}}{1-\hat{\varepsilon}}}\frac{1}{1-\hat{\varepsilon}}\right)^{-2}dx}{\frac{\hat{\varepsilon}}{1}\int_0^1 (\ln x)\, x^{\frac{\hat{\varepsilon}}{1}}\left(1+x^{\frac{\hat{\varepsilon}}{1}}\left(\frac{1}{1-\hat{\varepsilon}}\right)\right)^{-2}dx} = 0.
\end{aligned}$$

This contradicts (14). Hence, there is some $\hat{a}$ such that for all $a \leq \hat{a}$ the bound (13) holds given $\hat{\varepsilon}$. QED

The next lemma states that, if there is a sequence of equilibria such that the equilibrium likelihood ratios $R_m\frac{a_m}{1-a_m}$ are bounded away from one, $R_m \in \left[0, \frac{1-a_m-\hat{\varepsilon}}{a_m}\right]$ for all m, then the limiting likelihood ratio must be zero:

Lemma 9 *Take some sequence of signal distributions G_{Lm} with $a_m \to 0$ and $0 < a_m \leq \hat{a}$ for all m. For each a_m, consider a sequence $s_{m,k}$ such that $\lim_{k\to\infty} s_{m,k} \to 0$. Suppose there exists a corresponding sequence of equilibria $\sigma_{m,k} = [\beta_{m,k}, \mathcal{N}_{m,k}]$ such that $R_{m,k} = \frac{n_{m,k,H}}{n_{m,k,L}}$ converges,*

$\lim_{k\to\infty} R_{m,k} = R_m$ *and the limit is bounded* $R_m \in \left[0, \frac{1-a_m-\hat{\varepsilon}}{a_m}\right]$ *for all* m*. Then,* $\lim_{m\to\infty} R_m \frac{a_m}{1-a_m} = 0$*.*

Proof: Take some convergent subsequence of likelihood ratios, $\lim_{m\to\infty} R_m \frac{a_m}{1-a_m} \to X \in [0, 1-\hat{\varepsilon}]$. If $X = 0$ we are done, so consider $X > 0$. We want to show that the distributions $F_L(p|a_m, R_m, B) = \lim_{k\to\infty} F_L(p|a_m, L, \beta_k, Bn_{kL})$ converge to a fixed limit $F_L(p|X, B)$ with $m \to \infty$. Recall that for all a_m, $F_L(p|a_m, R_m, B)$ is determined via the inverse, $p_L(1-F|a_m, R_m, B)$. Taking limits,

$$\begin{aligned}
\lim_{m\to\infty} p_L(1-F|a_m, R_m, B) &= \lim \frac{(1-F)^{\frac{1}{B}R_m\frac{a_m}{1-a_m}}}{(1-F)^{\frac{1}{B}R_m\frac{a_m}{1-a_m}} + (1-F)^{\frac{1}{B}} \frac{1}{R_m}\frac{1-a_m}{a_m}} \\
&= \frac{(1-F)^{\frac{1}{B}X}}{(1-F)^{\frac{1}{B}X} + (1-F)^{\frac{1}{B}} \frac{1}{X}}.
\end{aligned}$$

Hence, the limiting inverse exists and determines the asymptotic distribution $F_L(p|X, B)$.

Using integration by parts, for every a_m

$$\begin{aligned}
E(p|L, a_m, R_m, B) &= \int_{p_l}^{p_h} pB(1-F_{mL}(p))^{B-1} dF_{mL}(p) \\
&= p_l + \int_{p_l}^{p_h} (1-F_{mL}(p))^B dp,
\end{aligned}$$

and from the law of rare events, $1 - F_{Hm}(p, B) = (1-F_{Lm}(p))^{\frac{a}{1-a}B}$,

$$\begin{aligned}
E(p|H, a_m, R_m, B) &= \int_{p_l}^{p_h} p\frac{a_m}{1-a_m}B(1-F_{mL}(p))^{\frac{a_m}{1-a_m}B-1} dF_{mL}(p) \\
&= p_l + \int_{p_l}^{p_h} (1-F_{mL}(p))^{\frac{a_m}{1-a_m}B} dp.
\end{aligned}$$

For every a_m, the first order conditions have to hold with equality in the limit (see the proof of Proposition 1). Hence, with $\bar{s}_m = \lim_{m\to\infty} n_{m,kL}s_{m,k}$,

$$\begin{aligned}
-\frac{\partial}{\partial B}E(p|L, a_m, R_m, B)|_{B=1} &= -\int_{p_l}^{p_h} (1-F_{mL}(p))^1 (\ln(1-F_{mL})) dp = \bar{s}_m \\
-\frac{\partial}{\partial B}E(p|H, a_m, R_m, B)|_{B=R_m} &= -\int_{p_l}^{p_h} \frac{a_m}{1-a_m}(1-F_{mL}(p))^{\frac{a_m}{1-a_m}R_m} (\ln(1-F_{mL})) dp = \bar{s}_m.
\end{aligned}$$

Since $F_{Lm}(p)$ converges to a non-degenerate distribution $F_L(p)$ (as shown above) the aggregate sampling costs do not vanish,

$$\begin{aligned}\lim_{m\to\infty}\bar{s}_m &= \lim_{m\to\infty} -\int_{p_l}^{p_h}(1-F_{mL}(p))^1(\ln(1-F_{mL}))\,dp \\ &= -\int_{p_l}^{p_h}(1-F_L(p))(\ln(1-F_L))\,dp \equiv \bar{s} > 0.\end{aligned}$$

Taking limits on the first order condition of the H buyer

$$\begin{aligned}\lim_{m\to\infty} -\frac{\partial}{\partial B}E(p|H,a_m,R_m,B)\,|_{B=R_m} &= \lim_{m\to\infty} -\int_{p_l}^{p_h}\frac{a_m}{1-a_m}(1-F_{mL}(p))^{\frac{a_m}{1-a_m}R_m}(\ln(1-F_{mL}))\,dp \\ &= -\int_{p_l}^{p_h}0\,(1-F_L(p))^X(\ln(1-F_L))\,dp = 0 < \bar{s}.\end{aligned}$$

Hence, the first order condition of the H buyer must be violated for some $\sigma_{k,m}$, contradicting our starting assumption that $\sigma_{k,m}$ is a sequence of equilibria with $\frac{n_{m,k,H}}{n_{m,k,L}}\frac{a_{m,k}}{1-a_{m,k}} \to X > 0$. QED

Lemma 10 *Take some sequence of signal distributions G_{Lm} with $a_m \to 0$ and $0 < a_m \leq \hat{a}$ for all m. For each a_m, consider a sequence $s_{m,k}$ such that $\lim_{k\to\infty} s_{m,k} \to 0$. Suppose there exists a corresponding sequence of equilibria $\sigma_{m,k} = \left[\beta_{m,k}, \mathcal{N}_{m,k}\right]$ such that $\lim_{k\to\infty} R_{m,k}\frac{n_{m,k,H}}{n_{m,k,L}} = R_m$, and suppose $R_m\frac{a_m}{1-a_m} \to 0$. Then the expected limit prices are revealing,*

$$\begin{aligned}\lim_{m\to\infty}\lim_{k\to\infty} E\left(p|a_m,L,\beta_{m,k},n_{m,kL}\right) &= c_L, \\ \lim_{m\to\infty}\lim_{k\to\infty} E\left(p|a_m,H,\beta_{m,k},n_{m,kH}\right) &= c_H.\end{aligned}$$

Proof: If $R_m\frac{a_m}{1-a_m} \to 0$, the inverse price $p_L(1-F)$ is equal to $c_L(=0)$ for all $(1-F)$, i.e., the distribution F_L is degenerate at $c_L(=0)$ as claimed,

$$\lim_{m\to\infty} p_L(1-F|a_m,R_m,B) = \lim \frac{(1-F)^{\frac{1}{B}\overbrace{R_m\frac{a_m}{1-a_m}}^{\to 0}}}{(1-F)^{\frac{1}{B}\overbrace{R_m\frac{a_m}{1-a_m}}^{\to 0}} + (1-F)^{\frac{1}{B}}\overbrace{\frac{1}{R_m}\frac{1-a_m}{a_m}}^{\to\infty}} = 0.$$

Similarly, $\lim_{m\to\infty} p_L(1-F|a_m,R_m,B) = 1\,(=c_H)$.

(Alternatively, if $R_m\frac{a_m}{1-a_m} \to 0$, note that sellers with signals close to zero are almost sure to be in the low state. Hence, if F_L is non-degenerate and $F_L(p) > 0$ for some $p > 0$, these seller would make strictly positive profits by bidding $p' < p$ and winning with positive probability.) QED

Proof of the Proposition: Lemma 8 and assumption E, imply that there exists a sequence of equilibria $\sigma_{m,k}$ with $\lim_{k\to\infty} R_{m,k} \in \left[0, \frac{1-a_m-\hat{\varepsilon}}{a_m}\right]$ for all m, using the same reasoning as in Proposition 2. Hence, by Lemma 9, $R_m \frac{a_m}{1-a_m} \to 0$. Lemma 10 then implies the Proposition. QED

5 Conclusion, Discussion, and References

To be added.