

Selection, Reallocation and the Shape of Aggregate Fluctuations: A General Equilibrium Analysis

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ABSTRACT

We study the cyclical implications of endogenous firm-level entry and exit decisions in a dynamic, stochastic general equilibrium model wherein firms face persistent shocks to both aggregate and individual productivity. Firms' decisions regarding entry into production and their subsequent continuation are affected not only by their expected productivities, but also by the presence of fixed capital adjustment costs, and thus their existing stocks. Such adjustment frictions distort the allocation of capital away from an efficient one and muddy the optimal selection that would otherwise operate at the firm level, thus reducing the average level of productivity. The more volatile are individual productivity shocks, the greater is the overall distortion.

The model we explore is in the spirit of Hopenhayn (1992) and, more directly, an extension of Clementi and Palazzo (2010). Our interest here is to better understand how selection reshapes the dynamics of macro aggregates in a general equilibrium setting with realistic firm-level investment patterns and life-cycle dynamics. More specifically, in an analysis disciplined by long-run observations on both aggregate and firm-level variables, we examine the extent to which procyclical entry and countercyclical exit lead to amplification, greater propagation and nonlinearities in the movements of aggregate production, employment and investment.

Keywords: entry, exit, heterogeneity, investment frictions, (S,s) policies, reallocation, propagation, business cycles

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1 Introduction

It is well-understood that the dynamics of capital investment have enormous implications for an economy's business cycle fluctuations. When endogenous capital accumulation is introduced into a typical equilibrium business cycle model, the consequences of temporary disturbances are amplified and propagated in quantitatively important ways. Given this observation, one might expect that the dynamics of other forms of investment would also be important in shaping the size and persistence of aggregate fluctuations. When viewed from an aggregate perspective, microeconomic decisions that influence the number and characteristics of an economy's firms have the capacity to generate such alternative investment dynamics.

How do heterogenous firms' entry, growth and exit decisions influence the size and shape of aggregate fluctuations? To explore this question, we design a dynamic stochastic general equilibrium model with endogenous entry and exit, alongside heterogenous capital accumulation. In our model economy, as in actual economies, firms have persistent differences in productivities, they face fixed costs of entering into production, as well as costs of continuing in operation, and the reallocation of capital across them is hindered by microeconomic adjustment frictions.

To be informative about the ways in which firms' entry and exit decisions shape aggregate fluctuations in actual economies, it is essential that our theoretical environment generate firm life-cycle dynamics resembling those actually observed in the data. Beyond merely matching the average extent of entry and exit in the economy, as a setting with exogenous death and birth might easily do, we construct a model capable of reproducing a set of key stylized facts about the characteristics of new firms, incumbent firms in production, and those exiting the economy.

At the core of our setting, we have in essence Hopenhayn's (1992) model of industry dynamics. Here, as there, potential firms receive informative signals about their future productivities and determine whether to pay a fixed cost to enter production, while incumbent firms' productivities are affected by a persistent common component and a persistent idiosyncratic component, with orthogonal innovations, and they select whether to pay fixed costs to remain in operation after observing both components of their current productivity. This set of assumptions immediately implies a selection effect whereby the average productivity and value of surviving members within a cohort rise as the cohort ages. As a result, firms that have recently entered production are, on average, smaller, less productive and more likely to exit than are older firms, as consistent with the observations of Dunne, Roberts and Samuelson (1989). Moreover, all else equal, large

firms (those with high employment) are those that have relatively high productivities, so mean-reversion in productivity immediately delivers the observed unconditional negative correlations between size and growth and between age and growth.

One limitation of the original Hopenhayn framework is its perfect mapping between productivity, firm size and growth. After controlling for size, this leaves no independent negative relationship between firm age and growth, in contrast to evidence presented by Evans (1987) and Hall (1987). As in Clementi and Palazzo (2010), we overcome this problem by including capital in the production function and imposing frictions on capital reallocation, so that idiosyncratic productivity and capital become separately evolving state variables for a firm. Because firms cannot immediately adjust their capital stocks following changes in their productivities, those that are observed to be large in the usual employment-based sense need not all be firms with high productivity. Instead, some large firms will be relatively unproductive, but carrying large capital stocks accumulated in the past.

Consider a group of firms sharing a common size. Given one-period time-to-build in capital, those among them that have the smallest stocks and highest idiosyncratic productivity will exhibit the fastest growth between this period and the next, as they raise their capital, and thus their employment, toward a level consistent with their high current (and hence expected future) productivity. By contrast, those with the largest stocks and lowest productivity will shrink as they shed excess capital. Of course, to be in the latter position, a firm must have experienced a sufficiently long episode of high productivity to have accumulated a large stock. This is far more likely for an old firm than a young one. Thus, among firms of a given size, older firms tend to have lower growth rates. Notice that this negative conditional correlation is reinforced by our inclusion of investment frictions, as these protract firms' adjustments toward their otherwise first-best capital stocks.

Given its success in reproducing the essential aspects of firm life-cycle dynamics, the model of Clementi and Palazzo (2010) serves as our starting point. There, changes in entry and exit over the cycle are seen to not only amplify the unconditional variation of aggregate series such as GDP and employment, but also generate greater persistence in the economy's responses to shocks. We revisit the findings there, extending the environment to general equilibrium by explicit introduction of a representative household supplying labor and savings to firms. One problem we confront in doing so is the fact that aggregate excess demand moves discontinuously

in a search for an equilibrium interest rate path if small changes in prices induce sharp changes in the number of operating firms. We overcome this obstacle by introducing randomness in firms' fixed costs of entry and continuation.

We calibrate the parameters of our model using long-run observations on aggregate and firm-level variables, including the average entry rate and the relative sizes of new firms and exiting firms. Thereafter, we verify that our model is a useful laboratory in which to explore that aggregate implications of selection and reallocation by confirming that its microeconomic predictions are consistent with the above-mentioned empirical observations regarding the average size distribution of firms, the average exit hazard as a function of age, the relationship between firm size and growth controlling for age and that between age and growth controlling for size, as well as a separate set of observations regarding the average distribution of firm-level investment rates. Next, we solve the model using a nonlinear method, thus permitting aggregate nonlinearities that would otherwise be suppressed.

Nonlinearities are absent in representative agent models, which necessarily abstract from extensive margin decisions. However, the setting we explore is sufficiently rich in micro-level decision-making that it has the potential to generate these features. In particular, changes in three distinct sets of threshold decisions offer the possibility of nonlinear movements in aggregate employment, investment and production in response to aggregate shocks. First, there are the entry decisions. When the common exogenous component of productivity is unusually high, a potential firm that might otherwise avoid its fixed entry cost sees its value increased. As a result, the minimum idiosyncratic productivity level potential entrants are willing to accept is counter-cyclical. Thus, at the onset of a boom, the number of new firms in the economy rises, while the average productivity among them falls.

Next, there are extensive margin changes associated with the investment decisions among incumbent firms. In any given period, our model has a family of capital adjustment hazards, one for each idiosyncratic productivity realization. Firms' willingness to suffer capital adjustment costs tends to rise with the difference between their existing stocks and the target capitals they would adopt conditional on undertaking adjustment. As such, each hazard is centered around a capital associated with the target capital corresponding to the given productivity and rising in the distance from that particular capital. Following a positive aggregate productivity shock, firms anticipate high marginal product of capital schedules in periods to come, so the target capitals

associated with each firm productivity level rise. As a result, conditional on an investment, the level of adjustment undertaken from any given capital rises. Moreover, the corresponding rightward shifts in the adjustment hazards imply that these rises along the intensive margin are accompanied by rises in the numbers of firms undertaking an investment.

This brings us to a third rise occurring along the extensive margin, that associated with the numbers of incumbent firms choosing to remain in production. Given the rises in all firms' values at the onset of a boom, the willingness to pay operating costs to remain in production rises at each current capital and idiosyncratic productivity pair, implying reduced exit rates relative to normal. More incumbents remain in production, and they are less selective than usual about accepting relatively low individual productivity levels. On balance, our model delivers procyclical entry and countercyclical exit, alongside procyclical movements in the measures of investing firms, with each of these extensive margin effects serving to amplify the responses in aggregate production and investment following an aggregate shock.

The discrete decisions made in our model can also imply greater persistence in aggregate fluctuations. Following a positive aggregate shock, an unusually large number of young firms are in production. Over subsequent periods, as aggregate productivity begins to revert toward its mean, the typical surviving member of this larger-than-average group of young firms grows in productivity and size. Meantime, incumbent firms of low productivity that had been induced to remain in production at the onset of the boom and continue to have low productivity begin exiting. This implies a reallocation of the economy's resources toward more productive, maturing firms that serves to maintain aggregate productivity at its above-average level despite the declining exogenous component. As a result, the procyclical movements in aggregate employment, investment and production are protracted.

The remainder of the paper is organized as follows. Section 2 presents our theoretical environment. Next, section 3 analyzes the three sets of threshold policy rules that arise therein and derives a series of implications useful in developing a numerical algorithm to solve for competitive equilibrium. Section 4 discusses our model's calibration to moments drawn from postwar U.S. aggregate and firm-level data and thereafter describes the solution method we adopt. Section 5 presents results, first exploring aspects of the steady state of our model economy and next considering its aggregate fluctuations, with an emphasis on how the mechanics of selection and (S,s) investment rules at the firm jointly affect macroeconomic outcomes. Section 6 concludes.

2 Model

Our model economy is a close reproduction of that in Clementi and Palazzo (2010), extending the setting to general equilibrium, with few changes elsewhere.¹ It contains three groups of optimizing agents: households, incumbent firms and potential firms. Households are standard and ultimately own all firms. Potential firms face fixed costs of entering production, while incumbent firms face both fixed operating costs and nonconvex costs of capital adjustment. The latter compound the effects of persistent differences in their total factor productivities to yield substantial heterogeneity in production. We begin our description of the economy with an initial look at the optimization problems facing incumbent and potential firms, then follow with a brief discussion of the economy's identical, infinitely-lived households and a description of equilibrium. Thereafter, in section 3, we use a simple implication of equilibrium to restate incumbent firms' problems and characterize their continuation and investment decisions. Once this is done, entry decisions are easily characterized. With the resulting (S,s) rules for entry, continuation and capital adjustment in hand, it becomes possible to apply a convenient, computationally tractable algorithm to solve for equilibrium allocations in our model, despite its heterogeneity in capital and productivities and evolving measure of firms.

2.1 Incumbent firms

Our economy houses a large, time-varying number of incumbent firms. Conditional on survival, each firm produces a homogenous output using predetermined capital stock k and labor n , via an increasing and concave production function, $y = zsF(k, n)$. Here, z represents exogenous stochastic total factor productivity common across firms, while s is a persistent firm-specific counterpart. For convenience, we assume that s is a Markov chain, $s \in \mathbf{S} \equiv \{s_1, \dots, s_{N_s}\}$, where $\Pr(s' = s_m \mid s = s_l) \equiv \pi_{lm}^s \geq 0$, and $\sum_{m=1}^{N_s} \pi_{lm}^s = 1$ for each $l = 1, \dots, N_s$. Similarly, $z \in \{z_1, \dots, z_{N_z}\}$, where $\Pr(z' = z_j \mid z = z_i) \equiv \pi_{ij} \geq 0$, and $\sum_{j=1}^{N_z} \pi_{ij} = 1$ for each $i = 1, \dots, N_z$.

At the beginning of any period, each firm is defined by its predetermined stock of capital, $k \in \mathcal{K} \subset \mathbf{R}_+$, and by its current idiosyncratic productivity level, $s \in \{s_1, \dots, s_{N_s}\}$. We summarize

¹Beyond our explicit treatment of households, there are only two modifications to the Clementi and Palazzo environment. We eliminate the firm-level convex cost of capital adjustment, as these tend to have a lesser role in general equilibrium settings, and we add an idiosyncratic component to operating costs to help convexify the production-side response to small changes in prices.

the distribution of firms over (k, s) using the probability measure μ defined on the Borel algebra for the product space $\mathcal{K} \times \mathcal{S}$; $\mu : \mathcal{B}(\mathcal{K} \times \mathcal{S}) \rightarrow [0, 1]$. The aggregate state of the economy is fully described by (z, μ) , with the distribution of firms evolving over time according to an equilibrium mapping, Γ , from the current state; $\mu' = \Gamma(z, \mu)$. The evolution of the firm distribution is determined in part by the actions of continuing firms and in part by endogenous entry. As will be described below, each potential entrant draws a productivity s and chooses whether to pay a fixed cost to become a firm. Having no capital initially, an entering firm produces nothing in the current period; however, it does invest in capital for the subsequent date. Thus, any such new firm appears as a part of the incumbent firm distribution at the start of the next date with its chosen capital and an s' influenced by the same π_{lm}^s probabilities as affect incumbent firms.

On entering a period, any given firm (k, s) observes the economy's aggregate state (hence equilibrium prices, though we suppress the arguments of equilibrium price functions until we have described the model further) and also observes the total output-denominated costs it must pay to remain in production. These operating costs are $c_f + \varphi$, where c_f is constant and common across firms, and φ is an individual draw each period from a time-invariant distribution $H(\varphi)$ with bounded support $[\varphi_L, \varphi_U]$. After observing aggregate prices and its current operating costs (but before production), the firm decides whether to continue on as a firm. Its alternative is to avoid the current operating costs by immediately exiting the economy. In the event of exit, the firm forfeits its existing capital and achieves zero value.

If a firm pays its operating costs to continue, it then chooses its current level of employment, n , undertakes production, and pays its wage bill. Thereafter, it observes its current realization of a fixed cost associated with capital adjustment, $\xi \in [\xi_L, \xi_U]$, which is denominated in units of labor and drawn from the time-invariant distribution $G(\xi)$. At that point, the firm chooses its investment in capital for the next period; in doing so, it chooses whether it will pay or avoid its current capital adjustment costs. If it undertakes any nonzero level of investment, the firm's total adjustment costs in units of output will be $c_0 k + \omega \xi$. These costs reflect that, in the event of active capital adjustment, the firm must hire ξ units of labor at equilibrium wage rate ω to manage the activity, and it must also suffer an output-disruption cost of $c_0 k$. The proportional fixed cost coefficient c_0 is assumed to be constant and common across firms. Given the firm's chosen level of investment, i , its capital stock at the start of the next period is determined by a

familiar accumulation equation,

$$k' = (1 - \delta) k + i, \quad (1)$$

where $\delta \in (0, 1)$ is the rate of capital depreciation, and primes indicate one-period-ahead values. In the analysis section to follow, we will show how the fixed costs capital adjustment naturally give rise to two-sided (S, s) investment decision rules. Given increasing returns in the adjustment technology, firms undertaking nonzero investment make larger, more forward-looking, adjustments to their stocks than they would do otherwise. However, the inclusion of a proportional disruption cost over and above the traditional scale-independent fixed cost generates some upward inertia in capital adjustment, since it in effect imposes a linear penalty on shedding capital. This partially offsets the downward inertia implied by depreciation.

We are now in a position to set out the optimization problems solved by each existing firm in our economy. Let $v^1(k, s_l, \varphi; z_i, \mu)$ represent the expected discounted value of a firm that enters the period with k and firm-specific productivity s_l , when the aggregate state of the economy is (z_l, μ) , just after it learns the random component of its current operating costs φ . Let $v^0(k, s_l; z_i, \mu)$ represent its expected value just beforehand;

$$v^0(k, s_l; z_i, \mu) \equiv \int_{\varphi_L}^{\varphi_U} v^1(k, s_l, \varphi; z_i, \mu) H(d\varphi). \quad (2)$$

The first decision the firm faces is whether to operate or exit. Defining its potential current flow profits as

$$\pi(k, s_l; z_i, \mu) \equiv \max_n [zsF(k, n) - \omega(z_i, \mu)n], \quad (3)$$

v^1 solves the following binary maximization problem.

$$v^1(k, s_l, \varphi; z_i, \mu) = \max \left\{ 0, \pi(k, s_l; z_i, \mu) - [c_f + \varphi] + v^2(k, s_l; z_i, \mu) \right\} \quad (4)$$

As the firm cannot observe its current random capital adjustment cost until it has produced, the ex-production continuation value, v^2 , in equation 4 is itself an expectation over the possible realizations of ξ ;

$$v^2(k, s_l; z_i, \mu) = \int_{\xi_L}^{\xi_U} v^3(k, s_l, \xi; z_i, \mu) G(d\xi) \quad (5)$$

The value function v^3 reflects an operating firm's maximized discounted continuation value net of investment and capital adjustment costs. Because adjustment costs are independent of the scale of the adjustment, the firm has a second binary decision to make at the end of the current period as it chooses its investment. Let $d_j(z_i, \mu)$ represent the discount factor applied

by firms to their next-period expected value if aggregate productivity at that time is z_j and the current aggregate state is (z_i, μ) . Taking as given the evolution of s and z according to the transition probabilities specified above, and taking as given the evolution of the firm distribution, $\mu' = \Gamma(z, \mu)$, the firm solves the optimization problem in (6) - (7) to determine its investment.

$$v^3(k, s_l, \xi; z_i, \mu) = \max \left\{ \sum_{j=1}^{N_z} \sum_{m=1}^{N_s} \pi_{ij} \pi_{lm}^s d_j(z_i, \mu) v^0((1 - \delta)k, s_m; z_j, \mu'), \right. \\ \left. -\omega(z_l, \mu) \xi + e(k, s_l; z_i, \mu) \right\}, \quad (6)$$

where

$$e(k, s_l; z_i, \mu) = [1 - \delta - c_0]k + \max_{k' \in \mathcal{K}} \left[-k' + \sum_{j=1}^{N_z} \sum_{m=1}^{N_s} \pi_{ij} \pi_{lm}^s d_j(z_i, \mu) v^0(k', s_m; z_j, \mu') \right] \quad (7)$$

The firm chooses between two options in 6. The option in the first line is to avoid all capital adjustment costs and continue to the next period with the $(1 - \delta)k$ that remains of its current capital after depreciation. The second option is to pay its random adjustment cost ξ (transformed into output units by the wage) and the proportional fixed cost $c_0 k$ and select the k' that maximizes its continuation value net of investment. For expositional convenience below, we have broken the continuation value under adjustment into two terms; $-\omega(z_l, \mu) \xi$ is the random component of adjustment costs, while $e(k, s_l; z_i, \mu)$ represents the firm's maximized continuation value net of investment, $k' - (1 - \delta)k$, and net of the deterministic adjustment costs $c_0 k$.

We simplify the firm's problem to isolate its decision rules in section 3 below. For now, notice that there is no friction associated with the firm's employment choice, since the firm pays its current wage bill after production takes place, and its capital choice for next period also has no implications for current production. Thus, conditional on paying the fixed costs to operate, but irrespective of the capital with which they exit the period, firms sharing in common the same (k, s) combination select the same employment, which we will denote by $N(k, s; z, \mu)$, and hence common production, $y(k, s; z, \mu)$. By contrast, such operating firms make differing intertemporal decisions, given differences in the capital adjustment costs they face. Thus, $K(k, s, \xi; z, \mu)$ represents their choice of next-period capital. With that noted, it is useful to observe that, irrespective of their existing stocks, all operating firms with the same productivity s that do undertake non-zero investment will exit the period with a common capital stock arising from the continuous decision problem in (7), and their ex-production continuation values will differ only by the linear term $\omega(z_l, \mu) \xi$. Further, all operating (k, s) firm that choose zero investment exit

the period with a common $(1 - \delta)k$ stock, and thus achieve a common ex-production continuation value. Stepping back to the beginning of the period, a firm of type (k, s) may or may not choose to remain in operation, depending upon the operating costs it faces. At points below, we represent the continuation decision of an existing firm using the indicator function χ .

$$\chi(k, s, \varphi; z, \mu) = \begin{cases} 1 & \text{if } v^1(k, s, \varphi; z, \mu) > 0 \\ 0 & \text{otherwise} \end{cases} \quad (8)$$

2.2 Potential firms

At any date, there is a fixed measure M of potential new firms on hand. Each observes the current aggregate state and draws a productivity level s from an initial distribution $Q(s)$ that is defined on the same support as the distribution of incumbent firm productivities. Because a potential firm has no capital, it cannot produce in the current period. However, given persistence in productivity draws, the current s draw is informative about its future prospects should it choose to become a firm. Given its s , the potential firm decides whether to pay a (constant and common) fixed entry cost to become a firm, c_e . Conditional on paying this cost, it then chooses its capital for the next period when it will become an incumbent firm, and it invests accordingly.

Conditional on entry, the new firm's next-period capital solves

$$\max_{k' \in \mathcal{K}} \left[-k' + \sum_{j=1}^{N_z} \sum_{m=1}^{N_s} \pi_{ij} \pi_{lm}^s d_j(z_j, \mu) v^0(k', s_m; z_j, \mu') \right]. \quad (9)$$

Referring back to equation 7, notice that the ex-entry-cost value of a new firm is $e(k, s; z_i, \mu)$ with $k = 0$. As such, the potential firm with productivity draw (signal) s becomes an entrant if this value is at least as great as the entry cost. At points below, we represent the entry decision of a potential firm using the indicator function χ^e .

$$\chi^e(s; z, \mu) = 1 \text{ iff } e(0, s; z_i, \mu) - c_e \geq 0 \quad (10)$$

Because new firms encounter the same transition probabilities for productivity, π_{lm}^s , as face existing firms, any entrant with productivity/signal s chooses the same future capital stock as is chosen by every incumbent firm with that same productivity that is currently undertaking a nonzero investment. Given this observation, the capital decision rule for an entering firm is

$$K^e(s; z, \mu) = K(k, s, \xi; z, \mu) \text{ with } \xi = 0 \text{ and } k = 0 \quad (11)$$

2.3 Households

The economy is populated by a unit measure of identical households. Household wealth is held as one-period shares in firms, which we denote using the measure λ .² Given the prices they receive for their current shares, $\rho_0(k, s; z, \mu)$, and the real wage they receive for their labor effort, $\omega(z, \mu)$, households determine their current consumption, c , hours worked, n^h , as well as the numbers of new shares, $\lambda'(k', s')$, to purchase at prices $\rho_1(k', s'; z, \mu)$. The lifetime expected utility maximization problem of the representative household is listed below.

$$W^h(\lambda; z, \mu) = \max_{c, n^h, \lambda'} \left[U(c, 1 - n^h) + \beta \sum_{m=1}^{N_z} \pi_{lm}^z W^h(\lambda'; z_m, \mu') \right] \quad (12)$$

subject to

$$c + \int_{\mathcal{K} \times \mathcal{S}} \rho_1(k', s'; z, \mu) \lambda'(d[k' \times s']) \leq \omega(z, \mu) n^h + \int_{\mathcal{K} \times \mathcal{S}} \rho_0(k, s; z, \mu) \lambda(d[k \times s]).$$

Let $C^h(\lambda; z, \mu)$ describe the household choice of current consumption, and let $N^h(\lambda; z, \mu)$ be the allocation of current available time to working. Finally, let $\Lambda^h(k', s', \lambda; z, \mu)$ be the quantity of shares purchased in firms that will begin the next period with k' units of capital and idiosyncratic productivity s' .

2.4 Recursive equilibrium

A *recursive competitive equilibrium* is a set of functions,

$$\left(\omega, (d_j)_{j=1}^{N_z}, \rho_0, \rho_1, v_1, N, K, \chi, \chi^e, K^e, W^h, C^h, N^h, \Lambda^h \right),$$

that solve firm and household problems and clear the markets for assets, labor and output, as described by the following conditions.

- (i) v_1 solves (4) - (7), given the definitions in (2) and (3), and (χ, N, K) are the associated policy functions for incumbent firms, and χ^e and K^e are the resulting policy functions for potential firms satisfying (10) - (11)
- (ii) W^h solves (12), and (C^h, N^h, Λ^h) are the associated policy functions for households

²Households also have access to a complete set of state-contingent claims. However, as there is no heterogeneity across households, these assets are in zero net supply in equilibrium. Thus, for sake of brevity, we do not explicitly model them here.

(iii) $\Lambda^h(k', s', \mu; z, \mu) = \mu'(k', s'; z, \mu)$, for each $(k', s') \in \mathcal{K} \times \mathcal{S}$

$$\begin{aligned} \text{(iv)} \quad N^h(\mu; z, \mu) &= \int_{\mathcal{K} \times \mathcal{S}} \int_{\varphi_L}^{\varphi_U} \chi(k, s, \varphi; z, \mu) \left[N(k, s; z, \mu) + \int_{\xi_L}^{\xi_U} \xi \mathcal{J}(K(k, s, \xi; z, \mu) \right. \\ &\quad \left. - (1 - \delta)k) G(d\xi) \right] H(d\varphi) \mu(d[k \times s]), \text{ where } \mathcal{J}(x) = 0 \text{ if } x = 0; \mathcal{J}(x) = 1 \text{ otherwise.} \end{aligned}$$

$$\begin{aligned} \text{(v)} \quad C^h(\mu; z, \mu) &= \int_{\mathcal{K} \times \mathcal{S}} \int_{\varphi_L}^{\varphi_U} \chi(k, s, \varphi; z, \mu) \left[z s F(k, N(k, s; z, \mu)) - [c_f + \varphi] - \int_{\xi_L}^{\xi_U} [K(k, s, \xi; z, \mu) \right. \\ &\quad \left. - (1 - \delta)k + c_0 k] \mathcal{J}(K(k, s, \xi; z, \mu) - (1 - \delta)k) G(d\xi) \right] H(d\varphi) \mu(d[k \times s]) \\ &\quad - M \int_{\mathcal{S}} \chi^e(s; z, \mu) [c_e + K^e(s; z, \mu)] Q(ds), \text{ where again } \mathcal{J}(x) = 0 \text{ if } x = 0; \mathcal{J}(x) = 1 \end{aligned}$$

$$\begin{aligned} \text{(vi)} \quad \mu'(D, s_m) &= \int_{\{(k, s_l, \xi) \mid K(k, s_l, \xi; z, \mu) \in D\}} \chi(k, s_l, \varphi; z, \mu) \pi_{lm}^s G(d\xi) H(d\varphi) \mu(d[\varepsilon_l \times k]) \\ &\quad + M \int_{\{s_l \mid K^e(s_l; z, \mu) \in D\}} \chi^e(s_l; z, \mu) \pi_{lm}^s Q(ds_l), \text{ for all } (D, s_m) \in \mathcal{K} \times \mathcal{S}, \text{ defines } \Gamma \end{aligned}$$

Let C and N represent the market-clearing values of household consumption and hours worked satisfying conditions (iv) and (v) above. It is straightforward to show that market-clearing requires that (a) the real wage equal the household marginal rate of substitution between leisure and consumption, $\omega(z, \mu) = D_2 U(C, 1 - N) / D_1 U(C, 1 - N)$, and that (b) firms' (and potential firms') state-contingent discount factors agree with the household marginal rate of substitution between consumption across states. Letting C'_{ij} denote household consumption next period given current state (z_i, μ) and next-period state $(z_j, \mu'(z_i, \mu))$ and with N'_{ij} as the corresponding labor input, the discounting requirement is: $d_m(z_l, \mu) = \beta D_1 U(C'_{ij}, 1 - N'_{ij}) / D_1 U(C, 1 - N)$.

Given the observations above, we may compute equilibrium by solving a single Bellman equation that combines the incumbent firm profit maximization problem with the equilibrium implications of household utility maximization. Here, we effectively subsume households' decisions into the problems faced by firms. Without loss of generality, we assign $p(z, \mu)$ as an output price at which firms value current profits and payments, and we correspondingly assume that firms discount their future values by the household subjective discount factor. Given this alternative means of expressing the discounting, the following two conditions ensure all markets clear in our

economy.

$$p(z, \mu) = D_1 U(C, 1 - N) \quad (13)$$

$$\omega(z, \mu) = D_2 U(C, 1 - N) / p(z, \mu) \quad (14)$$

2.5 Reformulated firm problem

We reformulate (2) - (7) to arrive at an equivalent description of an incumbent firm's dynamic problem where each firm's value is measured in units of marginal utility, rather than output, with no change in the resulting decision rules. Exploiting the fact that the choice of n is independent of the k' choice, suppressing subscripts indexing current aggregate and idiosyncratic productivity for convenience, and defining $V^0(k, s; z, \mu) \equiv \int_{\varphi_L}^{\varphi_U} V^1(k, s, \varphi; z, \mu) H(d\varphi)$, we have the following recursive representation for the start-of-period value of a type (k, s) firm drawing an operating cost φ .

$$V^1(k, s, \varphi; z, \mu) = \max \left\{ 0, [\pi(k, s; z, \mu) - c_f - \varphi] p(z, \mu) + V^2(k, s; z, \mu) \right\} \quad (15)$$

$$V^2(k, s; z, \mu) = \int_{\xi_L}^{\xi_U} V^3(k, s, \xi; z, \mu) G(d\xi) \quad (16)$$

$$V^3(k, s, \xi; z, \mu) = \max \left\{ \beta \sum_{j=1}^{N_z} \sum_{m=1}^{N_s} \pi_{ij} \pi_{lm}^s V^0((1 - \delta)k, s_m; z_j, \mu'), \right. \\ \left. -p(z, \mu) \omega(z, \mu) \xi + E(k, s; z, \mu) \right\}, \quad (17)$$

$$E(k, s; z, \mu) = p(z, \mu) [1 - \delta - c_0] k + \max_{k' \in \mathcal{K}} \left[-p(z, \mu) k' + \beta \sum_{j=1}^{N_z} \sum_{m=1}^{N_s} \pi_{ij} \pi_{lm}^s V^0(k', s_m; z_j, \mu') \right] \quad (18)$$

An incumbent firm of type (k, s) continues operation if it draws an operating cost such that $V^1(k, s_l, \varphi; z_i, \mu) > 0$; in that case, $\chi(k, s_l, \varphi; z_i, \mu) = 1$. Given the above reformulation of the incumbent firm problem, we may restate a potential firm's decision rules as follow. If $E(0, s; z_i, \mu) - p(z, \mu) c_e \geq 0$, then entry occurs and $\chi^e(s; z, \mu) = 1$. Conditional on entry, a new firm adopts the future capital stock solving (18) above.

3 Threshold policy rules and their implications

Toward deriving an reasonably efficient numerical algorithm with which to solve our model economy, it is useful to first characterize optimizing decisions among incumbent and potential

firms. We begin with incumbents. As is typical in cases where there are binary decisions between one action and another, it is convenient to begin with the continuous decisions that are made thereafter and work backwards from there. Throughout this subsection, we suppress the aggregate state arguments of the p and ω functions to shorten the equations, and similarly write $\mu'(z, \mu)$ as simply μ' .

3.1 Capital decisions

Consider first the end-of-period decision made by a continuing firm that has chosen to pay its adjustment cost and undertake a nonzero investment. Any such firm will adopt a *target capital* consistent with its current productivity and the aggregate state, which we denote by $k^*(s; z, \mu)$.

$$k^*(s; z, \mu) \equiv \max_{k' \in \mathcal{K}} \left[-pk' + \beta \sum_{j=1}^{N_z} \sum_{m=1}^{N_s} \pi_{ij} \pi_{lm}^s V^0(k', s_m; z_j, \mu') \right] \quad (19)$$

Notice that a firm's target capital choice is independent of its current capital, since the price of investment goods (p) is unaffected by its level of investment and the current capital adjustment cost draw ξ carries no information about future ones (and thus does not enter V^0). Given this observation, we know that all firms with the same current productivity level that undertake nonzero investment will move into the next period with a common capital stock.

Given the solution for the target capital above, we now turn to the binary capital adjustment decision. For a continuing firm of type (k, s) , the ex-production value of undertaking no adjustment is $\beta \sum \sum \pi_{ij} \pi_{lm}^s V^0((1-\delta)k, s_m; z_j, \mu')$. The value of the alternative option may be written as $-p\omega\xi + E(k, s; z, \mu)$, where:

$$E(k, s; z, \mu) = p[(1-\delta) - c_0]k + E^*(s, z, \mu), \text{ with} \quad (20)$$

$$E^*(s, z, \mu) = -pk^*(s; z, \mu) + \beta \sum \sum \pi_{ij} \pi_{lm}^s V^0(k^*(s; z, \mu), s_m; z_j, \mu'). \quad (21)$$

Notice that E^* is a common gross adjustment value achieved by all concurrently adjusting firms with common productivity s , while E is common across adjusting firms with the same (k, s) pair; both observations can be used to expedite model solution. Obviously, the firm pays its capital adjustment cost only if the net benefit of adjustment is positive, i.e., if:

$$-p\omega\xi + E(k, s; z, \mu) \geq \beta \sum \sum \pi_{ij} \pi_{lm}^s V^0((1-\delta)k, s_m; z_j, \mu').$$

The firm's capital decision rule can be described using a threshold policy. Define $\tilde{\xi}(k, s; z, \mu)$ as the random adjustment cost that leaves the firm indifferent to adjustment, and define $\xi^T(k, s; z, \mu)$

as the resulting threshold cost confined to the support of the cost distribution.

$$\tilde{\xi}(k, s; z, \mu) = \frac{E(k, s; z, \mu) - \beta \sum \sum \pi_{ij} \pi_{lm}^s V^0((1 - \delta)k, s_m; z_j, \mu')}{p\omega} \quad (22)$$

$$\xi^T(k, s; z, \mu) = \max\{\xi_L, \min\{\xi_U, \tilde{\xi}(k, s; z, \mu)\}\} \quad (23)$$

If the firm draws a random adjustment cost at or below its threshold, ξ^T , it pays that cost and adopts the target $k^*(s; z, \mu)$. Otherwise, it undertakes zero investment. The resulting capital decision rule is listed below.

$$K(k, s, \xi; z, \mu) = \begin{cases} k^*(s, z, \mu) & \text{if } \xi \leq \xi^T(k, s; z, \mu) \\ (1 - \delta)k & \text{otherwise} \end{cases} \quad (24)$$

Note that the threshold adjustment cost depends not only on s and the aggregate state through the target, but also on a firm's current capital. All else equal, a firm tends to be more willing to pay adjustment costs when its existing stock is farther away from its target. When this is so, the threshold cost is raised, which in turn implies a higher likelihood that the firm will adopt its k^* . Thus, our model implies (S,s) capital decisions and rising adjustment hazards as in Caballero and Engel (1999), Khan and Thomas (2003, 2007) and other studies involving nonconvex decisions at the microeconomic level.

Observe from (22) that all firms of type (k, s) share in common the same threshold cost ξ^T . This immediately implies that each of them has the same probability of capital adjustment and hence the same expected ex-production continuation value before the individual ξ draws have been realized, since all draw from the same cost distribution. Let $\alpha^k(k, s; z, \mu)$ denote any such firm's probability of capital adjustment, which is simply the probability of drawing $\xi \leq \xi^T$: $\alpha^k(k, s; z, \mu) = G(\xi^T(k, s; z, \mu))$, and let $\Phi^k(k, s; z, \mu)$ denote the conditional expectation of the random cost paid: $\Phi^k(k, s; z, \mu) = \int_{\xi_L}^{\xi^T(k, s; z, \mu)} \xi G(d\xi)$. Recalling the definition of $E(k, s; z, \mu)$ from (20), we can now re-express the type (k, s) firm's mid-period expected continuation value after production, $V^2(k, s; z, \mu) = \int_{\xi_L}^{\xi_U} V^3(k, s, \xi; z, \mu) G(d\xi)$, as follows.

$$\begin{aligned} V^2(k, s; z, \mu) &= [1 - \alpha^k(k, s; z, \mu)] \beta \sum_{j=1}^{N_z} \sum_{m=1}^{N_s} \pi_{ij} \pi_{lm}^s V^0((1 - \delta)k, s_m; z_j, \mu') \\ &\quad + \alpha^k(k, s; z, \mu) E(k, s; z, \mu) - p\omega \Phi^k(k, s; z, \mu) \end{aligned} \quad (25)$$

3.2 Operating decisions

Given the expected ex-production continuation values from above, the start-of-period operating decision of an incumbent firm is easily obtained. If the firm exits the economy, it achieves zero value. If it operates, it achieves the flow profits $\pi(k, s; z, \mu)$ in equation 3 (which it values by p), and an expected continuation value of $V^2(k, s; z, \mu)$ from (25). The firm continues into production if these expected benefits of doing so are not outweighed by value of its current operating costs, i.e., if:

$$p\pi(k, s; z, \mu) + V^2(k, s; z, \mu) \geq p[c_f + \varphi]. \quad (26)$$

At this point, we can summarize the binary operating decision of any (k, s) firm by reference to a second threshold policy, this one involving the random operating cost φ . Define $\tilde{\varphi}(k, s; z, \mu)$ as the cost that leaves the firm indifferent to exit, and define $\varphi^T(k, s; z, \mu)$ as the resulting threshold cost confined to the support of H .

$$\tilde{\varphi}^T = \pi(k, s; z, \mu) - c_f + \frac{V^2(k, s; z, \mu)}{p} \quad (27)$$

$$\varphi^T(k, s; z, \mu) = \max\{\varphi_L, \min\{\tilde{\varphi}^T(k, s; z, \mu), \varphi_U\}\} \quad (28)$$

If the firm realizes a random operating costs at or below the threshold, φ^T , it pays that cost and remains in production; otherwise, it exits. This binary decision rule is summarized below; when it implies continuation ($\chi = 1$), the firm hires labor and produces according to the frictionless decision rules $N(k, s; z, \mu)$ and $y(k, s; z, \mu)$ that maximize current flow profits.

$$\chi(k, s, \xi; z, \mu) = \begin{cases} 1 & \text{if } \varphi \leq \varphi^T(k, s; z, \mu) \\ 0 & \text{otherwise} \end{cases} \quad (29)$$

Before leaving this subsection, note that (27) implies all firms entering the period with the same (k, s) pair shares in common the same threshold operating cost φ^T . This means that each of them has equal probability of survival [$\alpha^c(k, s; z, \mu) = H(\varphi^T(k, s; z, \mu))$] and equal expectation of the level of random operating costs that will be paid [$\Phi^c(k, s; z, \mu) = \int_{\varphi_L}^{\varphi^T(k, s; z, \mu)} \varphi H(d\varphi)$]. Thus we obtain the start-of-period expected value of any incumbent firm just prior to the operating cost draws, $V^0(k, s; z, \mu) \equiv \int_{\varphi_L}^{\varphi_U} V^1(k, s, \varphi; z, \mu) H(d\varphi)$, as:

$$V^0(k, s; z, \mu) = \alpha^c(k, s; z, \mu) \left[p[\pi(k, s; z, \mu) - c_f] + V^2(k, s; z, \mu) \right] - p\Phi^c(k, s; z, \mu) \quad (30)$$

3.3 Entry decisions

We use the target capitals from equation 19, alongside the start-of-period expected value of an incumbent firm from (30), to easily characterize the entry decision rule of any potential firm drawing initial productivity signal s_l . If its expected value of investing to the target capital associated with s_l and beginning next period as an incumbent firm is at least as great as the value of the fixed entry cost, it will enter. This obvious choice is implemented by the rule below.

$$\chi^e(s_l; z, \mu) = \begin{cases} 1 & \text{if } \beta \sum_{j=1}^{N_z} \sum_{m=1}^{N_s} \pi_{ij} \pi_{lm}^s V^0(k^*(s_l; \cdot), s_m; z_j, \mu') - p[k^*(s_l; \cdot) + c_e] \geq 0 \\ 0 & \text{otherwise} \end{cases} \quad (31)$$

3.4 Aggregation

Given the probabilities of continuation and capital adjustment isolated above, the conditional fixed cost expectations, and the accompanying labor, output and capital decision rules, aggregation in our model is entirely straightforward. We cover it here largely to define the aggregate variables. Total output-denominated costs associated with firm entry (Ψ^e), incumbent operations (Ψ^c), and capital adjustment (Ψ_y^k) are

$$\Psi^e = c_e M \sum_{l=1}^{N_s} Q(s_l) \chi^e(s_l; z, \mu) \quad (32)$$

$$\Psi^c = \int_{\mathcal{K} \times \mathcal{S}} \left[\alpha^c(k, s; z, \mu) c_f + \Phi^c(k, s; z, \mu) \right] \mu(d[k \times s]) \quad (33)$$

$$\Psi_y^k = \int_{\mathcal{K} \times \mathcal{S}} \left[\alpha^c(k, s; z, \mu) \alpha^k(k, s; z, \mu) c_0 k \right] \mu(d[k \times s]), \quad (34)$$

while total labor-denominated fixed costs associated with capital adjustment (Ψ_n^k) are:

$$\Psi_n^k = \int_{\mathcal{K} \times \mathcal{S}} \left[\alpha^c(k, s; z, \mu) \Phi^k(k, s; z, \mu) \right] \mu(d[k \times s]).$$

Aggregate production (Y), employment (N), investments by incumbent firms (I^c) and by entering firms (I^e) are listed below, with $\Theta(k, s; z, \mu)$ summarizing firms' start of period probabilities of both operating and adjusting capital; $\Theta(k, s; z, \mu) \equiv \alpha^c(k, s; z, \mu) \alpha^k(k, s; z, \mu)$.

$$Y = \int_{\mathcal{K} \times \mathcal{S}} \left[y(k, s; z, \mu) \alpha^c(k, s; z, \mu) \right] \mu(d[k \times s]) \quad (35)$$

$$N^D = \Psi_n^k + \int_{\mathcal{K} \times \mathcal{S}} \left[N(k, s; z, \mu) \alpha^c(k, s; z, \mu) \right] \mu(d[k \times s]) \quad (36)$$

$$I^c = \int_{\mathcal{K} \times \mathcal{S}} \left[[k^*(s; z, \mu) - (1 - \delta)k] \Theta(k, s; z, \mu) \right] \mu(d[k \times s]) \quad (37)$$

$$I^e = M \sum_{l=1}^{N_s} \pi^q(q_l) \chi^e(q_l; z, \mu) k^*(s_l, z, \mu) \quad (38)$$

Finally, the implied household consumption is:

$$C = Y - (I^c + I^e) - (\Psi^e + \Psi^c + \Psi_y^k). \quad (39)$$

4 Calibration and solution

In the sections to follow, we will consider how the mechanics of our (full) model with entry and exit alongside capital adjustment frictions compare to those in two relevant reference models. The first of these references is a model otherwise identical to ours, but where there are no capital adjustment costs. There, all firms choosing to continue in operation will frictionlessly adopt capital stocks consistent with the expected future productivity implied by their current s , and the selection of which firms continue versus exit is optimal and independent of firms' existing capital. Our second reference economy is a frictionless setting with a fixed unit measure of firms. It has neither capital adjustment frictions, nor any fixed costs associated with operation or waiting potential firms. We will use these two reference models to try to isolate how much the frictions on capital reallocation *interact* with the entry and exit margins in our model economy, and how each set of micro-level imperfections influences our economy's aggregate dynamics. Aside from the values of the frictions noted here, all three models share a common parameter set that is selected in our full model to best match moments drawn from postwar U.S. aggregate and firm-level data. To be clear, we do not re-calibrate the reference models; thus, the average capital/output ratio, hours worked, and other important aspects of these economies are allowed to vary across them.

4.1 Functional forms and common aggregate targets

Across our model economies, we assume that the representative household's period utility is the result of indivisible labor (Rogerson (1988)): $u(c, L) = \log c + \theta L$. Firm-level production is

Cobb-Douglas: $zsF(k, n) = zsk^\alpha n^\nu$. In specifying our exogenous stochastic process for aggregate productivity, we begin by assuming a continuous shock following a mean zero AR(1) process in logs: $\log z' = \rho_z \log z + \eta'_z$ with $\eta'_z \sim N(0, \sigma_{\eta_z}^2)$. Next, we estimate the values of ρ_z and σ_{η_z} from Solow residuals measured using NIPA data on US real GDP and private capital, together with the total employment hours series constructed by Prescott, Ueberfeldt, and Cociuba (2005) from CPS household survey data over 1959-2002. Next, we discretize the productivity process using a grid with 5 shock realizations to obtain (z_i) and (π_{ij}) . We determine the firm-specific productivity shocks (s_l) and the Markov Chain governing their evolution (π_{lm}^s) similarly by discretizing a log-normal process, $\log s' = \rho_s \log s + \eta'_s$ using 15 values, and we assign the initial distribution of productivity signals, $Q(s)$, as a discretized Pareto distribution with curvature parameter 15.

We set the length of a period to correspond to one year, and we determine the values of β , ν , δ , α , and θ using moments from the aggregate data as follows. First, we set the household discount factor, β , to imply an average real interest rate of 4 percent, consistent with recent findings by Gomme, Ravikumar and Rupert (2008). Next, the production parameter ν is set to yield an average labor share of income at 0.64 (Cooley and Prescott (1995)). The depreciation rate, δ , is taken to imply an average investment-to-capital ratio of roughly 0.069, which corresponds to the average value for the private capital stock between 1954 and 2002 in the U.S. Fixed Asset Tables, controlling for growth. Given this value, we determine capital's share, α , so that our model matches the average private capital-to-output ratio over the same period, at 2.3, and we set the parameter governing the preference for leisure, θ , to imply an average of one-third of available time is spent in market work. The parameter set obtained from this common calibration exercise is summarized by the table below.

β	ν	δ	α	θ	ρ_z	σ_{η_z}
0.962	0.640	0.069	0.256	2.40	0.852	0.014

4.2 Establishment-level targets

The remaining parameters, those specific to our full model with capital adjustment costs and endogenous entry and exit, are determined using a series of moments from U.S. establishment-level data. For comparability with previous aggregate findings by Clementi and Palazzo (2010) regarding the same microeconomic phenomena we consider here, we adopt a common set of micro-level targets, which are drawn from two recent studies involving the Longitudinal Research

Database. These include observations from Lee and Mukoyama (2009) on the average annual entry rate across firms (6.2 percent), the average size of a new firm relative to that of a typical incumbent (60 percent) and the average relative size of an exiting firm (49 percent). They also observations on micro-level investment patterns from Cooper and Haltiwanger (2006), including the average mean investment rate (0.122), standard deviation of investment rates (0.337), serial correlation of investment rates (0.058) and fraction of establishments with investment rates less than 1 percent in absolute value (0.081). While our models has life-cycle aspects affecting firms' investments, the Cooper and Haltiwanger (2006) dataset includes only large manufacturing establishments that remain in operation throughout their sample period, Thus, in undertaking this part of our calibration, we must select an appropriate model-generated sample for comparability with their sample. This we do by simulating a large number of firms for 30 years, retaining only those firms that survive throughout, and then restricting the dates over which investment rates are measured to eliminate life-cycle effects.

This second calibration exercise remains to be done. For the moment, we simply explore an *example* economy with the parameterization listed in the table below. In this example, we eliminate the nonrandom costs of operation and capital adjustment, and assume that both random costs φ and ξ are drawn from uniform distributions with lower bound 0. The upper support on the adjustment cost distribution and the persistence and volatility of idiosyncratic productivities are taken from Khan and Thomas (2008). Given these, the fixed entry cost c_e and the upper support on continuation costs are chosen for consistency with the 6 percent entry rate in the data.

c_e	$[\varphi_L, \varphi_U]$	$[\xi_L, \xi_U]$	ρ_ε	σ_η	c_0, c_f
0.004	[0.0, 0.11]	[0.0, 0.008]	0.859	0.022	0.0

4.3 Solution

The distribution μ in the aggregate state vector of our model economy is a large object. In general, discrete choices imply that this distribution is highly non-parametric. For each level of productivity, we store the conditional distribution using a fine grid defined over capital. However, firms' choices of investment are not restricted to conform to this grid. To allow the possibility that nonconvex capital adjustment may interact with endogenous entry and exit over the business cycle in a way that delivers aggregate nonlinearities, we adopt a nonlinear solution method. Given μ , an exact solution is obviously numerically intractable; thus, we use selected moments of μ as a

proxy for the distribution in the aggregate state vector when computing expectations.

Our solution method is adaptation of that in Khan and Thomas (2007), which in turn extends Khan and Thomas (2003) to replace one- with *two*-dimensional heterogeneity among firms. Following the approach developed by Krusell and Smith (1997, 1998), we assume that firms approximate the distribution in the aggregate state vector with a vector of moments, $m = (m_1, \dots, m_I)$, drawn from the true distribution. Because our model implies a discrete distribution over k and over s , conditional means from I equal-sized partitions of the capital distribution work well, implying small forecasting errors.

As in Krusell and Smith (1997), we solve our model by iterating between an inner loop step and an outer loop step until we isolate forecasting rules satisfyingly consistent with equilibrium outcomes. In the inner loop, we take as given a current set of forecasting rules for p and m' and use them to solve incumbent firms' expected value functions V^0 (from equation 30). This we do by combining value function iteration with multivariate piecewise polynomial cubic spline interpolation allowing firms to evaluate and select off-grid options. We next move to the outer loop to simulate the economy for 1000 periods. The current set of m' forecast rules are used in the outer loop, while p is endogenously determined in each date. Each period in the simulation begins with the actual distribution of firms over capital and productivity implied by the decisions of the previous date. Given incumbent firms' value functions from the most recent inner loop, and the aggregation of section 3.4, we determine equilibrium prices and quantities, and thus the subsequent period's distribution. Once the simulation has finished, we use the resulting data to update the forecasting rules, with which we return to the inner loop.

5 Results

5.1 Steady state

In this section, we begin to explore aspects of our model in its steady state and how our setting compares to a control model with the same capital adjustment frictions, but where a constant unit measure of firms produces each period and is exempt from fixed costs of operating. On average, our model economy forfeits roughly 8.9 percent of its GDP to operating costs. However, the average level of consumption is actually about 1.4 percent higher than it is in the control model where there are no such costs. This higher level of consumption is achieved in part by

the fact that households work harder in our economy, roughly 12.5 percent more of their unit time endowment. However, the more direct explanation lies in the distribution of firms over productivity levels, which encourages this higher work effort.

Figure 0 compares the stationary distribution of firms over total factor productivity in our model economy (top panel) to that in the control model without entry and exit (bottom panel). All else equal, firms with relatively low productivities are induced to exit our model economy by the fixed costs they must pay to remain. Furthermore, the presence of fixed entry costs induces those potential entrants that observe relatively low productivity signals to stay out. As such, the typical exiting firm is replaced by a firm with a higher productivity. Given these aspects of selection, the stationary distribution of firms in our model economy has substantially less mass over lower productivity levels and more in the higher regions of productivity than does the control model. Despite the fact that productivities are drawn from the same lognormal distribution, the implication of Figure 0 is that the average firm in our economy has roughly 3 percent higher productivity. This implies a greater return to any given aggregate level of productive inputs, thus encouraging households in our setting to work, produce, and save more.

As mentioned above, our model has all the necessary elements to achieve consistency with the core regularities involving firm-level entry, exit, size and growth. By allowing mean-reverting idiosyncratic productivities alongside fixed costs of entry and fixed operating costs, we obtain the selection-based successes of Hopenhayn's (1992) original model of industry dynamics. In particular, given selection, the average productivity and value of surviving members within a cohort rise as the cohort ages. Thus, firms that have recently entered production are, on average, smaller, less productive and more likely to exit than are older firms, as consistent with the observations of Dunne, Roberts and Samuelson (1989). Moreover, in keeping with observations from the same study, large firms (those with high employment) tend to be those with relatively high productivities, so mean-reversion in productivity immediately delivers the observed unconditional negative correlations between size and growth and between age and growth.

The inclusion of one-period time-to-build capital stocks at the firm breaks the perfect mapping between productivity, firm size and growth inherent in the Hopenhayn model. As a result, our model has the capacity to generate not only the negative correlation between size and growth conditional on age, but also that between age and growth conditional on size, thus rendering it consistent with the observations of Evans (1987) and Hall (1987). In our setting, a firm's

size (employment) is not determined by its current productivity alone, but the history of its productivities reflected in its existing capital stock. Because firms cannot immediately adjust their capital inputs in response to changes in their productivities, those that are observed to be large are not all firms with high productivity. Instead, some are relatively unproductive firms that have high employment simply because they are carrying large capital stocks.

Among firms with a common size, those that are older tend to exhibit lower growth rates than those that are young for a fairly simple reason. Those firms in the size group with the smallest stocks and highest idiosyncratic productivity will have the fastest growth between this period and the next, as they raise their capital, and thus their employment, toward a level consistent with their high current (and hence expected future) productivity. By contrast, those with the largest stocks and lowest productivity will shrink as they shed excess capital. Of course, to be in this situation, a firm must have experienced a sufficiently long episode of high productivity to have acquired a large stock. Such a history is far more likely for an older firm than a younger one.

The negative conditional correlation between age and growth is strengthened in our model by the fact that firms face capital adjustment frictions, since these prolong the transitions toward what would otherwise be their first-best capital stocks. We have chosen to include *nonconvex* capital adjustment costs at the firm in an effort to simultaneously ensure consistency with a core set of observations regarding microeconomic investment patterns presented by Cooper and Haltiwanger (2006), among these, the fractions of establishments undertaking very small and very large stock adjustments in a typical year. The presence of these costs add a further layer of richness to our model's microeconomic decision-making. As clear from the analysis above in section 3, the (S,s) investment policies they imply affect the capital stocks with which operating firms exit any given period, thus influencing their expected continuation values, which in turn feeds back into the exit decisions that are made at the start of a period. Moreover, they increase the likelihood that the stochastic economy will exhibit aggregate nonlinearities, as we will discuss further below.

Figures 1 and 2 display the stationary distributions of firms in our economy at the start of a period and at the time of production, respectively. In each of these figures, population density increases perceptibly as one looks toward the back left corner representing the highest levels of capital and productivity. Comparing the first figure to the second, we see how selection imprints these shapes on the distributions. Among firms with a common capital level at the start of

the period, it is those with the lowest productivities (entering the period at the left edge of the given capital slice) that do not remain to produce. Conversely, among firms with a common productivity level, those in the foreground with the lowest productivities are the ones most likely to exit. These results (which may be seen more directly from the exit hazard displayed in Figure 6) arise as a natural consequence of the fact that firms draw their operating costs from a common time-invariant distribution, while firm value is increasing in both capital and idiosyncratic productivity. Their implications for the stationary distribution of firms are compounded by the fact that exiting firms are typically replaced by firms with higher productivity levels.

This observation is confirmed by the distribution of newly producing firms plotted in Figure 3. Because the productivity signals potential entrants draw are understood to be informative about subsequent productivity draws, and because all face the same fixed entry cost, only those drawing from the top third of the signal distribution will choose to become firms. Thus, when they enter production in the next period, new firms tend to have high levels of productivity. It is worth pointing out that this feature is overly stark in the current formulation of the model, and it has a problematic implication for the level of capital held by new firms relative to that of older incumbents. New firms enter the period with the target capital levels associated with the high productivity signals they accepted, so they are large relative to the incumbents in Figure 2, adopting a capital-based measure of size. With one small change to the environment above, we muddy the signal selection effect to allow an empirically viable degree of dispersion in the productivities and capitals among new firms. In this reformulation, we assume that the fixed entry costs are drawn from an iid distribution. Among potential firms facing a common entry cost, it is still the case that those with higher signals are more likely to enter. However, now, some firms with low signals choose to enter, because they face relatively low entry costs, while some firms with high signals choose not to enter for the opposite reason.

There is another reason why the entrants in Figure 3 appear large relative to incumbents, and it has prompted our undertaking a second minor modification to our original setting. Incumbent firms contemplating exit in light of low productivity realizations tend to delay doing so for at least one period. This explains the hill present in the foreground of the distribution shown in Figure 2, and why the bulk of this hill disappears by Figure 3. In the version of our model that we have presented above, firms choosing to exit at the start of a period are assumed to achieve zero value. Knowing it will receive zero scrap value for its existing stock, a firm considering

exit will do so only if its value net of operating costs is negative (see equation 26). As such, unless it realizes a particularly large operating cost, a firm that would otherwise exit will instead cover its operating costs and produce, thereafter either actively selling off capital (in the case of a relatively low capital adjustment cost realization) or else investing nothing towards maintaining its stock. Only at the start of a subsequent period, when it has little capital left to forfeit, do we actually observe the firm's exit. Thus, while the exit hazard in Figure 6 ultimately rises at lower productivities and the lowest levels of capital as it should, it is constant at zero over a wide range of higher capital stocks. In the more recent version of our model, we eliminate this problematic lingering exitor phenomenon simply by allowing incumbent firms to sell off their capital stocks at the start of any period in which they choose to exit. Any such abrupt shedding of capital at a firm's exit is associated a fixed $\lambda \in (0, 1)$ recoup of the value of the stock, so that the condition under which a firm continues to operate is $p(z, \mu)\pi(k, s; z, \mu) + V^2(k, s; z, \mu) \geq p(z, \mu)[c_f + \varphi - \lambda k]$.

Figure 4 displays the distribution of capital across firms in production, aggregating over all levels of productivity. The disjoint spike at the far left of the distribution is generated by the lingering exitor behavior discussed just above; thus it is eliminated by our reformulation allowing exitors a fraction of the value of their capital. Elsewhere, notice that the distribution has the most mass at a capital level around 2.05, and population density falls fairly steadily as we look leftward toward lower capital levels. The selection involved in entry and exit that we have emphasized above has some part in explaining why this capital distribution looks so different in shape from the underlying lognormal productivity distribution. However, the more direct explanation lies in the (S, s) capital adjustment decisions implemented by continuing firms.

Consider for a moment the steady state of an environment analogous to our model economy, but one where firms do not differ in their individual productivities. There, with nothing else distinguishing them, those firms drawing relatively low fixed costs of capital adjustment at the end of a period will undertake investment to a common steady-state target capital, while those realizing relatively high adjustment costs will defer investment and allow their stocks to decay. With every consecutive period that investment does not occur, a firm loses δ fraction of its capital, so observing it over time, we see discrete drops. Of course, given the common target stock, all firms that last invested at the same date will enter a period with the same capital level. Generally speaking, firms' willingness to suffer adjustment costs grows larger as their capital is eroded farther and farther away from the target by depreciation. Given this, and the fact that all draw from

the same adjustment cost distribution, the fraction of firms paying fixed costs and investing back to the target rises as start-of-period capital falls. Thus, as we look from groups of firms entering the period with capital near the target stock toward those groups with capital further and further away from the target, we see an ever smaller fraction of a given group remaining once investing firms have departed. As a result, we observe a monotone distribution with precisely the shape depicted in Figure 4.

Firms in our model do not have a common target capital level, because they have persistence differences in their productivities. Moreover, following a decline in its relative productivity, unlike the example above, a firm can enter a period with capital exceeding the target stock it will adopt conditional on an adjustment. Nonetheless the reasoning above broadly explains the stationary distribution of capital arising here as well. Below, we merely sketch in a few details.

Figure 5 plots three of the fifteen adjustment hazards relevant in our model's steady state - those associated with current productivity levels s_i , $i = 9, 12, 15$. Each hazard is centered at the current capital level where a firm is unwilling to pay any nonzero fixed cost to adjust its capital. This centering point for each hazard is the current capital from which, conditional on paying the fixed adjustment cost, the firm would optimally select a zero investment. In other words, it the capital from which $i = 0$ delivers the target associated with its current productivity by default, $\frac{k^*(s_i)}{1-\delta}$. Of course, because higher current productivity predicts higher expected productivity in the next period, target capital is higher at higher current s . Thus the productivity specific hazards are centered at higher capital values as we look from the top panel of our figure ($s = s_9$) to the middle panel ($s = s_{12}$) and to the bottom ($s = s_{15}$).

As in the setting with common productivities described above, firms in our model economy are more impatient to adjust capital when their existing stocks are further from their targets, so their individual probabilities of adjustment (taken before the realization of fixed costs) are higher. Thus, each of our fifteen adjustment hazards shows adjustment probabilities (and thus fractions) rising in the distance between existing capital and the relevant target stock. Unlike the example above, however, each of our hazards (except that corresponding to the top productivity level, s_{15}) has some firms entering the period on its right ramp, because firms' relative productivities can fall. This means that not all firms that adjust their capital do so in an upward direction, and the perfect monotonicity in the capital distribution outlined above is muddled to an extent by changes in firms' relative productivities. Nonetheless, given mean-reversion in productivity and

the constant downward pull on capital implied by depreciation, the same basic shape is obtained in the overall distribution of capital as would occur in a setting without productivity differences.

5.2 Aggregate fluctuations

As we have discussed above, the three distinct extensive margin decisions made at the microeconomic level in our model economy have the potential to together deliver larger and more persistent movements in macroeconomic variables in response to shocks than otherwise occur. They also imply a potential for aggregate nonlinearities, for instance, asymmetric changes in GDP over the business cycle, that would further distance our economy's dynamics from the properties of the exogenous disturbances that are fed in to initiate them.

In response to a rise in the exogenous component of aggregate productivity, a potential entrant anticipates higher value than it would in an ordinary date. Thus, the minimum idiosyncratic productivity level accepted falls relative to normal and the number of new firms entering the economy rises. On balance, our economy predicts procyclical entry, which is consistent with observations in the data. While this entry does not raise the level of production directly at the impact of a shock, it is likely to encourage a rise in household labor supply, much as procyclical investment in capital typically does.

Extensive margin changes associated with the capital investment decisions among incumbent firms can compound and propagate the rises implied by investment in new firms mentioned above. As explained in the preceding subsection, our environment implies a family of capital adjustment hazards at any given date, one for each idiosyncratic productivity realization, and these hazards are centered around the productivity-specific target capitals that firms adopt conditional on undertaking capital adjustment. Following a positive aggregate productivity shock, firms anticipate high marginal product of capital schedules over subsequent dates, so the target capitals associated with each firm productivity level rise. As a result, conditional on investment, the level of adjustment undertaken from any given capital rises. This would be the case in any standard representative agent business cycle model. However, in our setting, the resulting rightward shifts in the adjustment hazards imply that these rises along the intensive margin can be compounded by movements along the extensive margin, with rises in the numbers of firms investing.

Finally, the rises in current flow profits, alongside an anticipation of high common productivity over coming periods, generates a third extensive margin shift in our economy, increasing an given

incumbent firm's value of remaining in production. Thus, fewer firms exit than would in an average date, allowing our model to produce empirically desirable countercyclicality in exit rates. While the average productivity level among incumbent firms, as among entrants, is reduced by the fact that they are willing to accept lower idiosyncratic productivity draws than normal, their presence in production nonetheless serves to amplify the initial rise in employment and GDP.

Changes in the stock of young firms in production also have the potential to deliver increased endogenous persistence over the cycle. We noted above that a unusually large number of new firms enter the economy following the realization of a positive aggregate shock. Over subsequent periods, as aggregate productivity begins to revert toward its mean, the typical surviving member of this larger-than-average cohort of young firms will grow in productivity and size. This change in the age distribution of firms, alongside the staggered rise in aggregate capital implied by firm-level capital adjustment, may be sufficient to delay mean-reversion in the aggregate level of productivity beyond what would be implied by its exogenous component, thus protracting the boom in economy-wide employment, investment and production. These observations regarding our economy's aggregate dynamics remain speculative for the present, however, pending convergence in the dynamic solution of the slight reformulation we have outlined in the section above.

6 Concluding remarks

TBA

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FIGURE 0. How selection shapes the productivity distribution

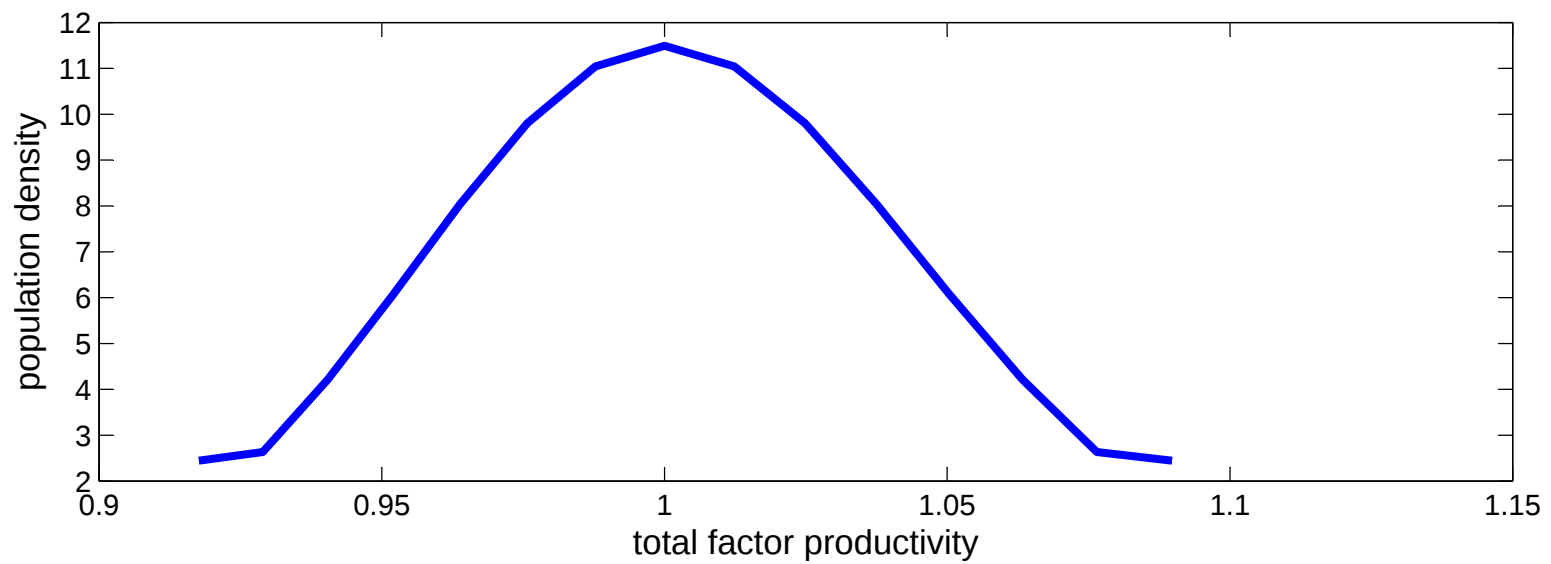
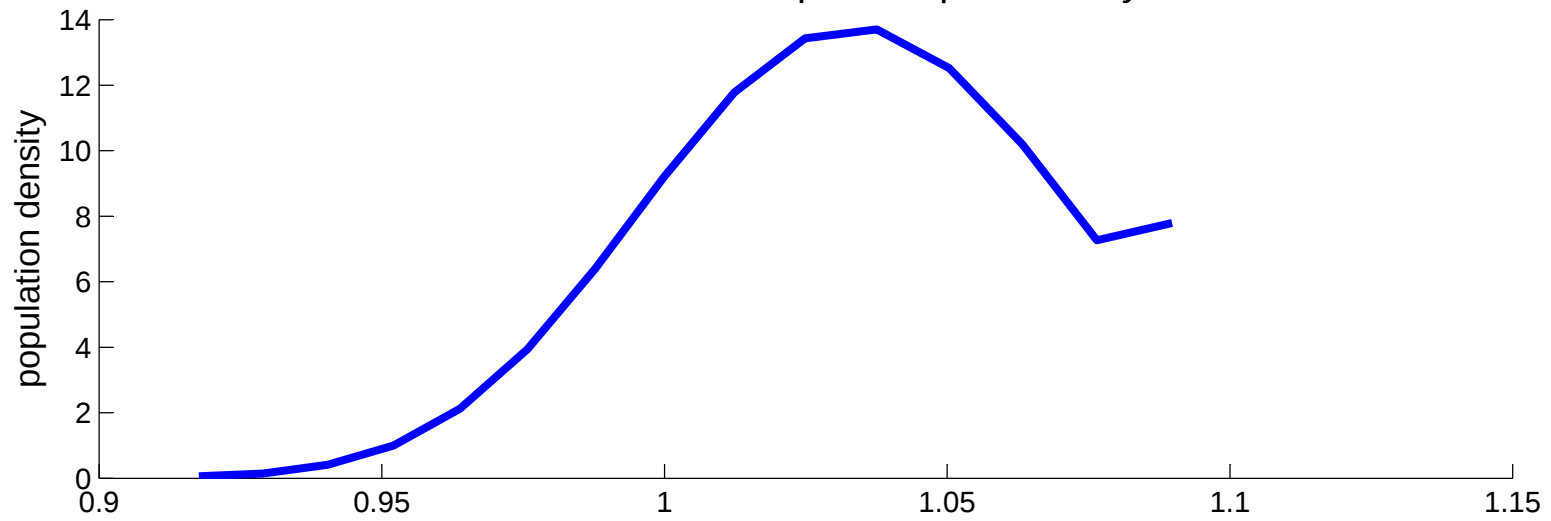


FIGURE 1. Start of period firm distribution

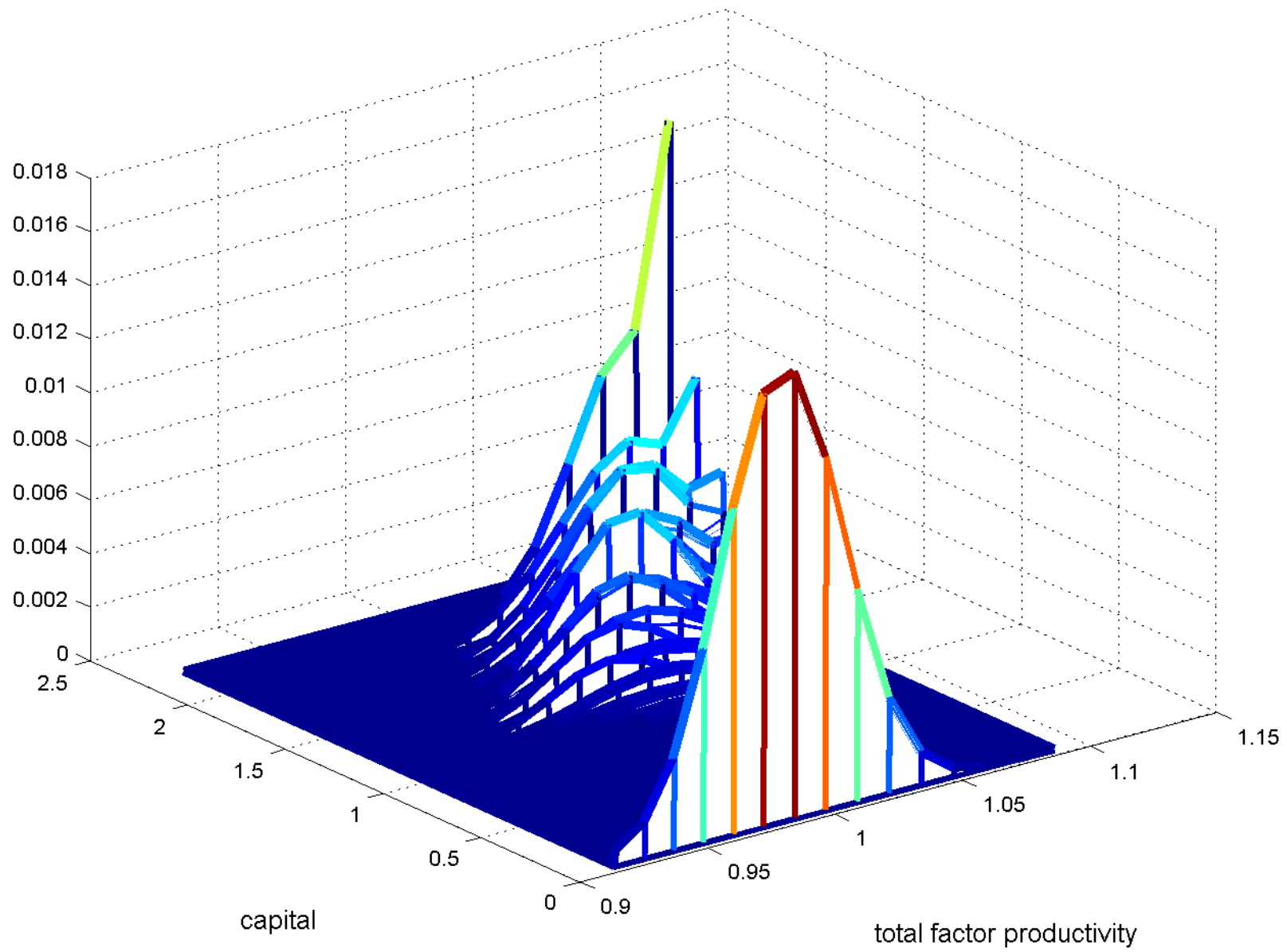


FIGURE 2. Production-time distribution

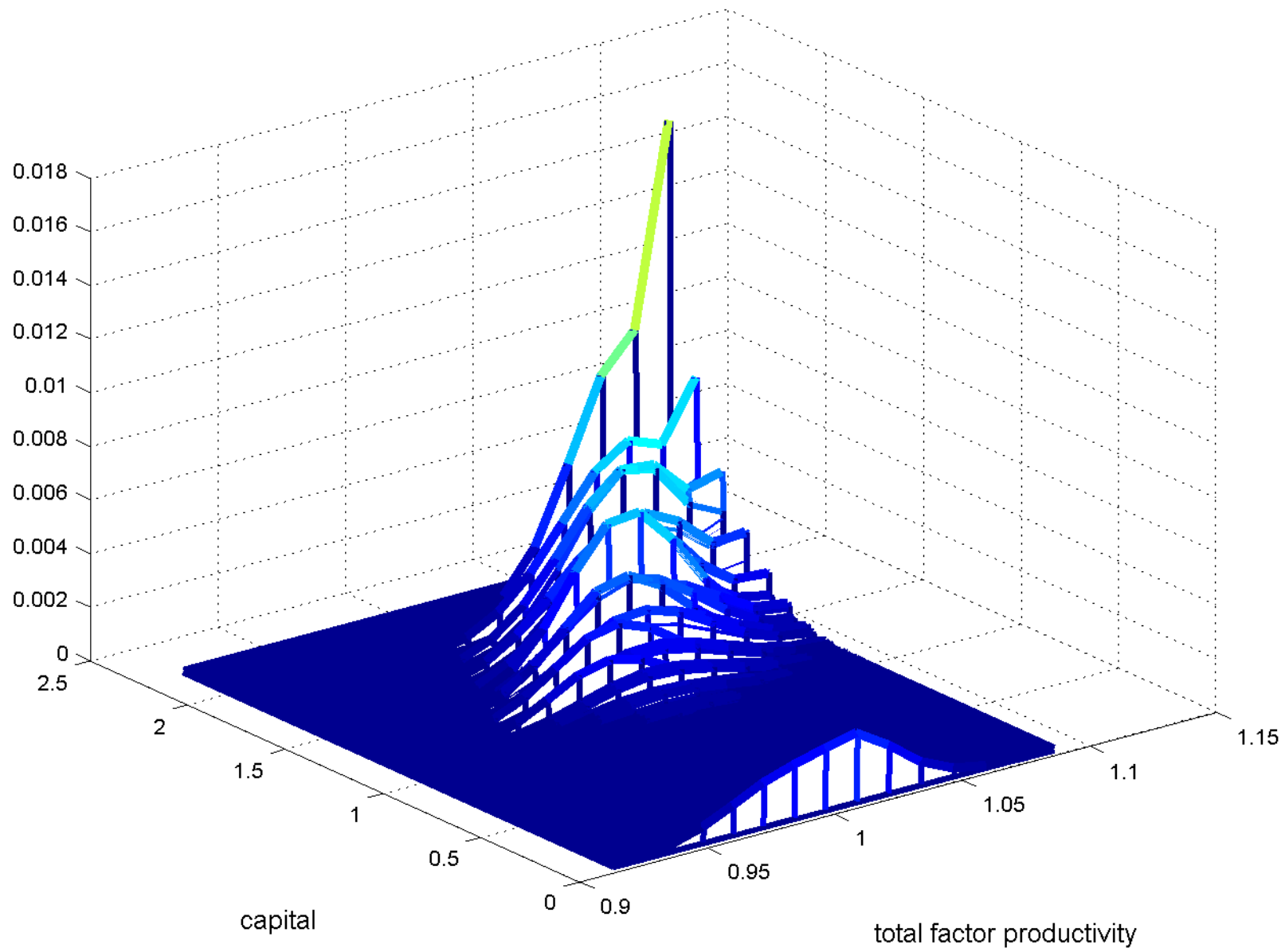


FIGURE 3. Production-time distribution of age 1 firms

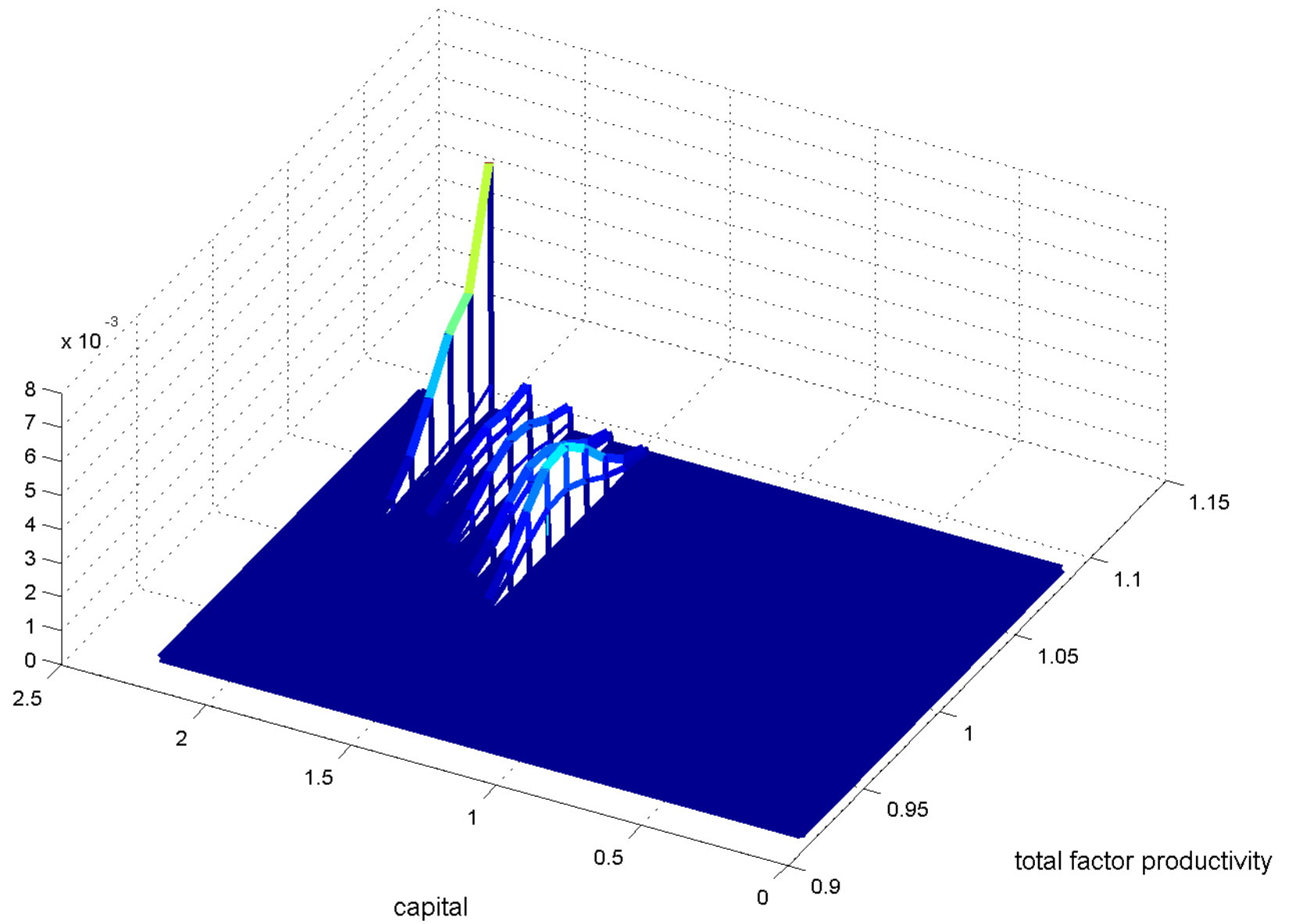


FIGURE 4. Production-time capital distribution

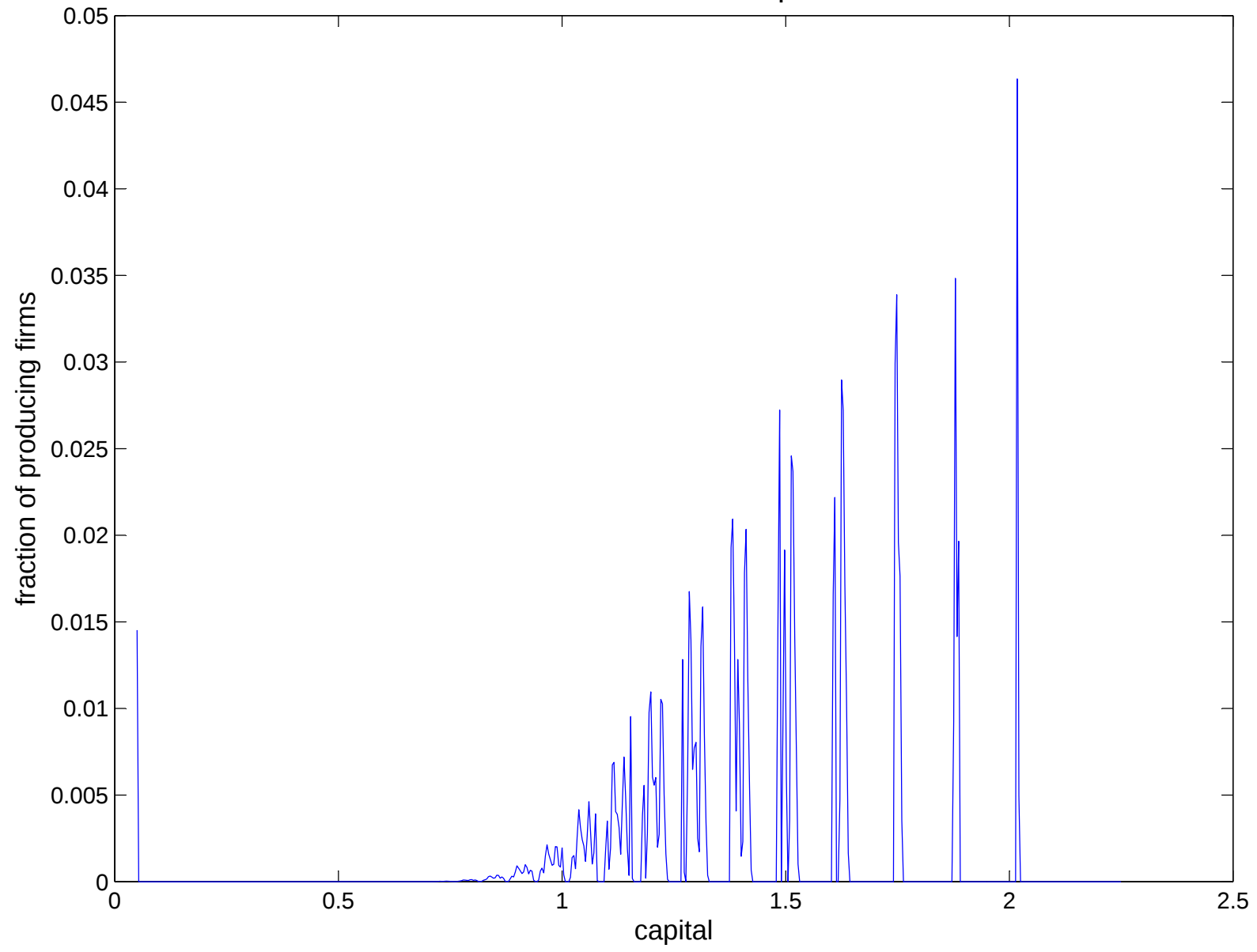


FIGURE 5. Capital adjustment hazards

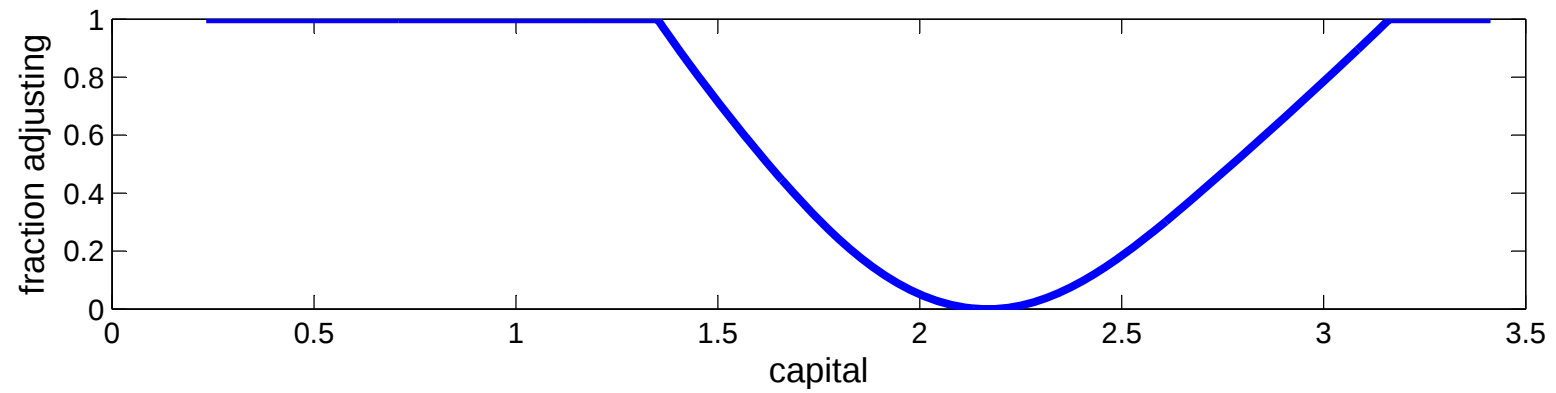
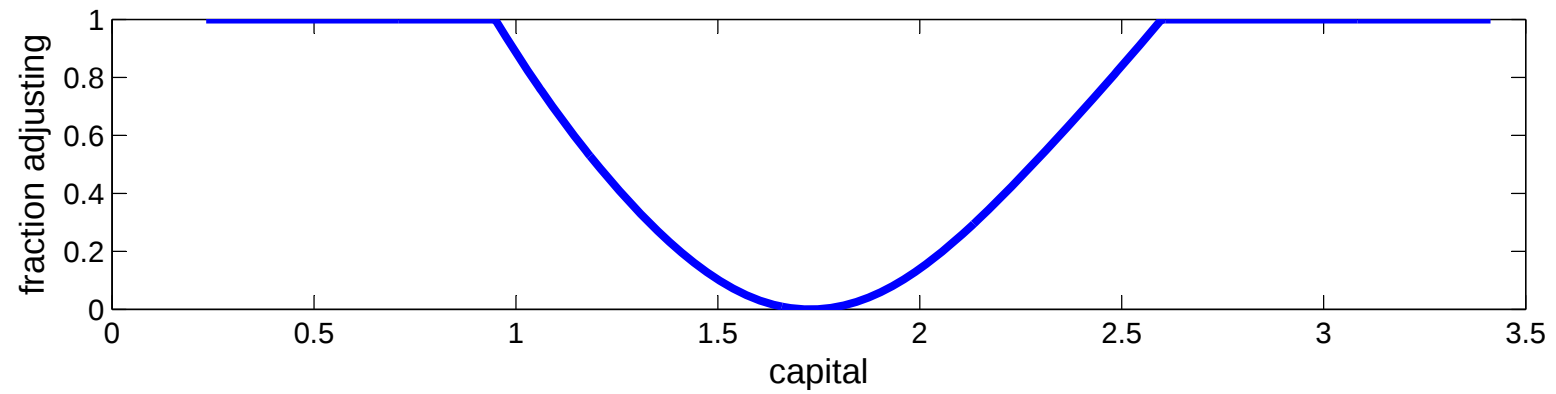
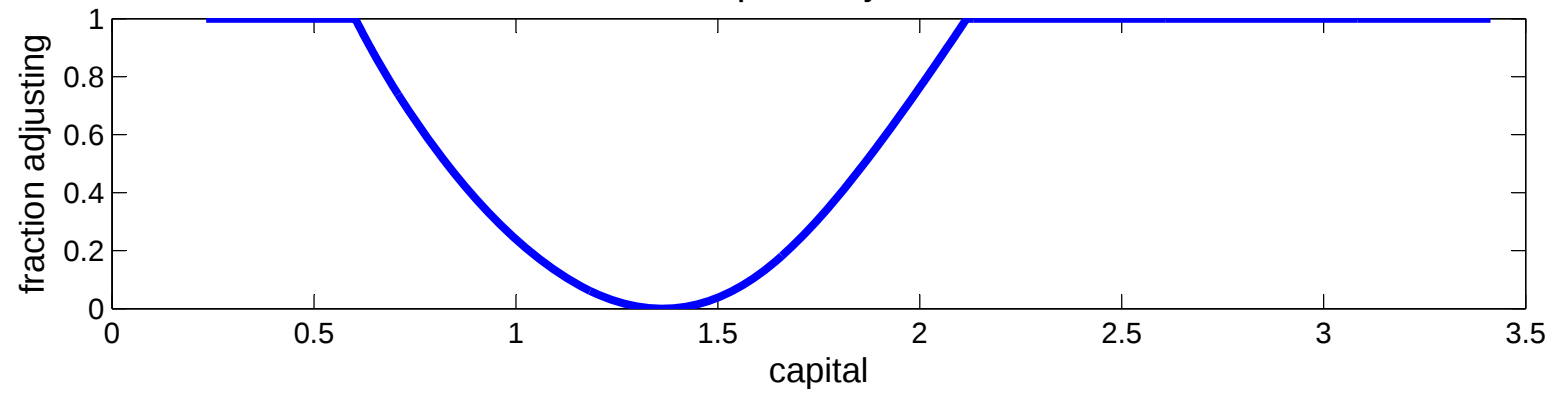


FIGURE 6. Exit probabilities

