

Default Risk and Aggregate Fluctuations in an Economy with Production Heterogeneity

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ABSTRACT

We study aggregate fluctuations in an economy where firms' have persistent differences in their total factor productivities and investment, at the firm level, is funded by retained earnings and non-contingent loans. Loans involve default risk and, in equilibrium, the unit cost of borrowing for any firm rises in its level of debt but falls with the value of its collateral. As the risk of default increases with the level of debt, our model endogenously determines debt limits that vary in firm characteristics, as well as risk premia. Both evolve with the aggregate state of the economy.

Larger firms, with more collateral, have higher levels of investment than smaller, more productive firms with less collateral. This leads to a misallocation of capital across firms. When compared to a model without borrowing constraints, aggregate capital falls, as does GDP and total factor productivity. Quantitative examples suggest that these long-run effects are substantial.

We study business cycle driven by exogenous changes in total factor productivity and credit shocks. The latter involve changes to the fraction of assets that lenders may seize in the event of default. Following changes in aggregate total factor productivity, we find that our non-contingent loan contracts drive countercyclical default risk. When firms face fixed costs of operation, this leads to a worsening of the allocation of capital in recessions that amplifies the effects of technology shocks. Following credit shocks, we see a reduction in economic activity that is qualitatively different from that following a technology shock. The response in investment is more severe, while the responses in consumption, employment and output are more gradual. In contrast to existing analysis of credit shocks with exogenous collateral constraints, our environment does not predict a slow recovery when borrowing conditions return to normal.

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1 Introduction

Over the past three years, events in the real and financial sectors of the U.S. and other large, developed economies have become increasingly difficult to disentangle. In this paper, we develop a quantitative, dynamic, stochastic general equilibrium model to explore how real and financial shocks interact in determining the size and frequency of aggregate fluctuations. In our model, firms experience persistent shocks to both aggregate and individual productivity, while credit market frictions yield persistent disruptions to the efficient allocation of capital across them, and thus persistent reductions in endogenous aggregate productivity. Calibrating our model to aggregate and firm-level data, we use it as a laboratory with which to ask questions such as the following: Can a large shock to an economy's financial sector produce a large and lasting recession? Can it amplify and propagate the effects of a real shock?

Financial frictions are introduced, into our economy, by assuming that borrowing is restricted to one-period non-contingent debt and there is a risk of default. In equilibrium, this leads to loan rates which vary with firm-level, as well as aggregate, conditions. When a firm defaults, it loses its assets and exits production. Firms may default because they prefer to not repay their debt, or because it is infeasible to do so. The latter occurs when current earnings and nondepreciated capital cannot cover debt repayment given the constraints on future borrowing. We assume that, in the event of default, lenders are able to confiscate a fraction of a firm's assets, the sum of its current earnings and capital stock. In general, firms with higher expected future earnings or capital have a lower risk of default. This allows them to borrow more, and at lower cost.

In general, the risk of default rises with the level of debt. This endogenously limits firms' ability to borrow, and these borrowing limits vary across firms as functions of their individual state. Importantly a firm with the same current level of total factor productivity, but higher capital stock, will be able to borrow more than a corresponding firm with the same level of current total factor productivity but less capital. Thus capital serves as collateral.

We show that in the long-run, borrowing and lending with non-contingent debt subject to default risk can lead to a severe misallocation of resources relative to a model where firms do not face borrowing constraints. Larger firms, with more collateral, have higher levels of investment than smaller, more productive firms with less collateral. This reduces average aggregate total factor productivity, as well as GDP and capital. We illustrate this mechanism using a plausibly parameterized example where constraints on borrowing, arising through the risk of default, reduce

aggregate TFP by 2.5 percent, the aggregate capital stock by almost 11 percent, and GDP by 7 percent. These effects of borrowing limits are greater than those found in Khan and Thomas (2010) where firms faced exogenous collateral constraints. This finding follows from the risk of default in the present model. Firms respond to the associated rising loan rate schedule by restraining their borrowing. This leads to a more severe misallocation of capital than that in a model with exogenous borrowing limits without default risk.

Our primary question in this study is whether a temporary crisis in financial markets can generate a large and persistent drop in aggregate productivity by disrupting the distribution of capital away from that implied by firms' relative productivities, and thereby distorting the distribution of production. We model a credit shock as caused by an exogenous change in the fraction of assets that lenders may confiscate when a firm defaults. In our model, such shocks have real effects only when there is a nontrivial risk of default in a large fraction of loans. To generate such a distribution of risk, we assume that firms face random costs of continuation that can only be avoided by exiting from production. Given such continuation costs, we find that credit shocks can, by themselves, generate a large and protracted recession. Moreover, the aggregate response to a credit shock is qualitatively different from that following a real shock. Unlike the response following a persistent reduction productivity, the greatest declines in output, employment and investment do not occur at the onset of a credit crisis, but gradually over time. This is the result of the allocative effects of a credit shock. It affects expected future aggregate total factor productivity, through a worsening in the allocation of resources across firms, but has no direct effect on current production. The shock propagates over time as an increase in the dispersion of firms' net worth leads to larger subsequent distortions in the allocation of capital.

In our model, competitive lenders choose loan rate schedules that provide them with an expected return, for each loan, that is equal to the real return on risk-free debt. As a firm's probability of default changes with its expected future productivity, capital stock and debt, its cost of borrowing is a function of its choice of debt and capital. By assuming that default deprives a firm of its assets and requires it to exit, we derive several results which allow us to characterize borrowing and lending in general equilibrium. First, by abstracting from capital adjustment costs, we show that idiosyncratic productivity and net worth are sufficient to describe a firm's state. Second, we show that firm's values are continuous and weakly increasing in net worth. This implies the existence of threshold levels of net worth that completely describe any firms decision

to default. Specifically, we derive threshold default levels of net worth, which vary with idiosyncratic productivity and the aggregate state, such that any firm will repay its loan and continue to operate only when its own net worth exceeds the threshold level. We exploit these results to numerically characterize borrowing and lending in a model where the aggregate state includes a distribution of firms over idiosyncratic productivity, net worth and capital.

To date, there has been little quantitative research examining the channels through which changes in the availability of credit influence macroeconomic series like business investment, employment and production in a fully specified dynamic stochastic general equilibrium environment. There is a large related literature exploring how financial frictions influence the aggregate response to non-financial shocks. Most notably for our purposes, Kiyotaki and Moore (1997) develop a model of credit cycles and show that collateral constraints can have a large role in amplifying and propagating shocks to the value of collateral.¹

We model financial frictions as arising from non-contingent loans subject to the risk of default. This type of loan contract was first characterized by Eaton and Gersovitz (1981) in their study of international lending. Recent quantitative analysis of sovereign debt are Aguiar and Gopinath (2008) and Arellano (2008). Chatterjee et al (2008) study an environment with unsecured lending to households. While we assume that firms borrow using non-contingent loans, there are well known alternatives. Cooley, Marimon and Quadrini (2004) study constrained-optimal dynamic contracts under limited enforceability. Elsewhere, a large literature examines agency costs as the source of financial frictions.² However, these papers do not consider financial shocks as such.

Other recent studies have begun exploring how financial shocks affect aggregate fluctuations. A leading example is Jermann and Quadrini (2009a), which examines a representative firm model wherein investment is financed using both debt and equity, while costs of adjusting dividends prevent the avoidance of financial frictions. These frictions stem from limited enforceability of intra-temporal debt contracts, which gives rise to endogenous borrowing limits. Specifically, the firm retains its working capital under default, but the lender is able to recover a fraction of the firm's future value. Shocks to the fraction that the lender can confiscate alter the severity of borrowing limits. Measuring these credit shocks, Jermann and Quadrini find that they have been an important source of business cycles.³

¹Cordoba and Ripoll (2004) and Kocherlakota (2000) argue that these effects are quantitatively minor in calibrated versions of the model.

²See Bernanke and Gertler (1989), Carlstrom and Fuerst (1997), and Bernanke, Gertler and Gilchrist (1999).

³Jermann and Quadrini (2009b) adapt this model to address the evolving variability of real and financial variables

Our emphasis on firm-level productivity dispersion is shared by Arellano, Bai and Kehoe (2010), who examine the role of uncertainty shocks in a model with non-contingent debt and equilibrium default. Gomes and Schmid (2009) also develop a model with endogenous default, where firms vary with respect to their leverage, and study the implication for credit spreads.⁴ In contrast to these papers, we study ongoing firm-level investment and the aggregate response to credit shocks. We find that credit shocks can generate recessions through changes in aggregate TFP that, in turn, have sharp implications for investment and employment. In emphasizing the endogenous TFP channel, our study is also related to Buera and Shin (2007), who examine the effect of collateral constraints on economic development and show that these frictions can greatly protract the transition to the balanced growth path if capital is initially misallocated.

2 Model

In our model, non-contingent lending compounds the effect of persistent differences in firm-level total factor productivity to yield substantial heterogeneity in production. We begin our description of the economy with an initial look at the optimization problem facing each firm. Thereafter we focus on the determination of borrowing costs, essentially interest rate schedules for debt, by lenders that seek to earn an expected return that is equal to the risk-free real interest rate. We show that the problem of computing interest rate schedules is vastly simplified because firm values are weakly increasing functions of net worth. This implies that there is a unique default threshold level of net worth. Firms with net worth less than this threshold, which varies with their productivity and the aggregate state, default. Lastly we include a brief description of households and equilibrium.

2.1 Production, credit and capital adjustment

We assume a large number of firms, each producing a homogenous output using predetermined capital stock k and labor n , via an increasing and concave production function, $y = z\varepsilon k^\alpha n^\nu - \phi$, with $\alpha > 0$, $\nu > 0$ and $\alpha + \nu < 1$. The fixed cost of production is $\phi \geq 0$. Here, z represents the technology level in the past 25 years. In a related setting, Christiano, Motto and Rostagno (2009) study a New Keynesian model with lending subject to agency costs; they too find that financial shocks are an important source of economic fluctuations.

⁴Credit spreads are also a focus of Gertler and Kiyotaki (2010), who study a model where such spreads are driven by agency problems arising with financial intermediaries.

resents exogenous stochastic total factor productivity common across firms, while ε is a firm-specific counterpart. We assume that ε is a Markov chain, $\varepsilon \in \mathbf{E} \equiv \{\varepsilon_1, \dots, \varepsilon_{N_\varepsilon}\}$, where $\Pr(\varepsilon' = \varepsilon_j \mid \varepsilon = \varepsilon_i) \equiv \pi_{ij} \geq 0$, and $\sum_{j=1}^{N_\varepsilon} \pi_{ij} = 1$ for each $i = 1, \dots, N_\varepsilon$. Similarly, $z \in \{z_1, \dots, z_{N_z}\}$, where $\Pr(z' = z_m \mid z = z_l) \equiv \pi_{lm}^z \geq 0$, and $\sum_{m=1}^{N_z} \pi_{lm}^z = 1$ for each $l = 1, \dots, N_z$.

Because our interest is in understanding how borrowing constraints shape the investment decisions taken by firms in our economy, we must prevent firms growing so large that none will ever again face costs of borrowing. To ensure this does not occur, we impose exit and entry in the model. In particular, we assume that each firm faces a fixed probability, $\pi_d \in (0, 1)$, that it will be forced to exit the economy following production in any given period. Within a period, prior to investment, each firm learns whether it will continue into the next period. Firms forced to exit are replaced by an equal number of new firms as described below.

At the beginning of each period, a firm is defined by its predetermined stock of capital, $k \in \mathbf{K} \subset \mathbf{R}_+$, by the level of debt it has incurred, $b \in \mathbf{B} \subset \mathbf{R}$, and by its current idiosyncratic productivity level, $\varepsilon \in \{\varepsilon_1, \dots, \varepsilon_{N_\varepsilon}\}$. The aggregate state of the economy is then described by (z, μ) where μ involves a distribution of firms and evolves over time according to a mapping, Γ , $\mu' = \Gamma(z, \mu)$. Given their individual state, and having observed the current aggregate state, each firm takes a series of actions to maximize the expected discounted value of the current and future dividends.

First, it chooses its current level of employment, undertakes production, and pays its wage bill. Thereafter, if it is feasible to do so, it chooses whether to repay its existing debt. Firms that default must exit the economy and lose all profits and any capital. Firms that do not default learn, after the repayment of debt, whether they will survive into the next period. Continuing firms choose their investment, i , current dividends, and the level of debt with which they enter into the next period, b' .⁵ For each unit of debt it incurs for the next period, any firm receives q units of output that it can use toward paying current dividends or investing in its future capital. Thus a loan of qb' involves a debt of b' to be repaid next period.

In contrast to models with exogenous collateral constraints, the risk of default implies that the loan discount factor, q , is a function of not only aggregate state variables, but also a borrower's level of debt and capital. Given a level of debt, a firm's capital choice for next period determines its expected earnings and thus the probability it will repay.

⁵We have adopted this timing to ensure there is no default, in our model, caused by the exogenous exit shock. All firms borrowing this period will be able to produce next period.

If firm undertakes any nonnegative level of investment, then its capital stock at the start of the next period is determined by a familiar accumulation equation,

$$k' = (1 - \delta) k + i \quad \text{for } i \geq 0,$$

where $\delta \in (0, 1)$ is the rate of capital depreciation, and primes indicate one-period-ahead values.

Before characterizing the schedule determining loan rates, q , we examine its choices of employment and production given capital. At the start of any period, every firm chooses employment to solve the following problem,

$$\pi(\varepsilon, z, k, \mu) = \max_n \left[z \varepsilon k^\alpha n^\nu - w(z, \mu) n - \phi \right].$$

The optimal choice of labour is independent of the level of debt, b , and given by

$$n(\varepsilon, z, k, \mu) = \left(\frac{\nu z \varepsilon k^\alpha}{w(z, \mu)} \right)^{\frac{1}{1-\nu}}.$$

This, in turn, implies production,

$$y(\varepsilon, z, k, \mu) = (\varepsilon z)^{\frac{1}{1-\nu}} \left(\frac{\nu}{w(z, \mu)} \right)^{\frac{\nu}{1-\nu}} k^{\frac{\alpha}{1-\nu}} - \phi.$$

Earnings net of labour costs are

$$\pi(\varepsilon, z, k, \mu) = (1 - \nu) y(\varepsilon, z, k, \mu) - \phi.$$

This is common to all firms that, given their choices of (k, b) last period, and observing ε , continue into production.

If a firm defaults, it retains no assets and exits. The lender recovers a fraction, θ , of profits and capital. Let $\gamma \in \{0, 1\}$ be 0 when a firm defaults, and 1 otherwise. When a firm with current idiosyncratic productivity ε chooses a debt of b' , alongside a capital stock of k' , he is able to borrow $q(\varepsilon, z, k', b', \mu) b'$. In other words, a loan of $q(\varepsilon, z, k', b', \mu) b'$ involves a repayment of b' . We refer to $q(\varepsilon, z, k', b', \mu)$ as the loan discount factor. It determines the expected return for lenders.

Competitive lending will equate lenders' expected return, per unit lent, to the risk-free real interest rate. We define the latter. Let $\pi_{ij} d_m(z_l, \mu)$ represent the price of an Arrow security which pays off if $z' = z_m$, then the risk-free real interest rate is $\frac{1}{q_0} - 1$ with

$$q_0(z, \mu) = \sum_{m=1}^{N(z)} \pi_{lm} d_m(z_l, \mu). \quad (1)$$

Every loan discount factor is bounded above by $q(z, \mu)$ and below by 0. If $\gamma = 0$ for some (ε', z') given the choice of (k', b') and $\mu' = \Gamma(z, \mu)$, then $q(\varepsilon, z, k', b', \mu) < q(z, \mu)$.

Let x represent a firm's net worth, its profits, including the value of its nondepreciated capital, after loan repayment,

$$x \equiv \pi(\varepsilon, z, k, \mu) - b + (1 - \delta)k.$$

Importantly, the individual levels of k and b do not separately determine any firm choices beyond their effect in x . To see this, note that a firm choosing (k', b') has a resource constraint,

$$x + q(\varepsilon, z, k', b', \mu) b' - k' \geq 0.$$

Let Φ represent the feasible set of choices,

$$\Phi(\varepsilon, z, x, \mu) = \{(k', b') \in \mathbf{R}_+ \times \mathbf{B} \mid x + q(\varepsilon, z, k', b', \mu) b' - k' \geq 0\}. \quad (2)$$

The result that a firm's choice set, and hence its value, is a function of only x , and does not depend separately upon k and b , is a result of our abstraction from any costs of capital adjustment. This is an important assumption that allows us to reduce a firm's state vector, and the dimension of the value and default functions that characterize any competitive equilibrium.

Let $V(\varepsilon_i, z_l, x, \mu)$ represent the beginning of period value of a firm. If $\Phi(\varepsilon, z, x, \mu) \neq \{\emptyset\}$, and the continuation value function, $V_0^c(\varepsilon_i, z_l, x, \mu)$, is non-negative in

$$V(\varepsilon_i, z_l, x, \mu) = \max\{V_0^c(\varepsilon_i, z_l, x, \mu), 0\}, \quad (3)$$

then the firm chooses to continue and $\gamma(\varepsilon, z, x, \mu) = 1$. Notice that, as mentioned above, default leaves the firm with no income or assets, it exits.

The continuation value of a firm, V_0^c is an expectation over production in the next period, and the probability of continuing into the next period is $1 - \pi_d$. A firm that does not continue into the next period pays its net worth as dividend,

$$V_0^c(\varepsilon_i, z_l, x, \mu) = \pi_d x + (1 - \pi_d) V^c(\varepsilon_i, z_l, x, \mu) \quad (4)$$

and the value of a firm that continues into production next period is V^c .

Firms that continue solve the following problem,

$$\begin{aligned}
V^c(\varepsilon_i, z_l, x, \mu) &= \max_{(k', b')} \left(x + \left(q(\varepsilon, z, k', b', \mu) b' - k' \right. \right. \\
&\quad \left. \left. + \sum_{j=1}^{N(\varepsilon)} \pi_{ij}^\varepsilon \sum_{m=1}^{N(z)} \pi_{lm} d_m(z_l, \mu) V(\varepsilon_j, z_m, x'_{jm}, \mu') \right) \right) \\
&\text{subject to}
\end{aligned} \tag{5}$$

$$(k', b') \in \Phi(\varepsilon_i, z_l, x, \mu), \tag{6}$$

$$x'_{jm} = \pi(\varepsilon_j, z_m, k', \mu') - b' + (1 - \delta) k', \tag{7}$$

$$\mu' = \Gamma(z_l, \mu).$$

Given a default policy, and choices (k', b') , competitive lending by any financial intermediary implies a loan discount factor that solves

$$\begin{aligned}
q(\varepsilon, z, k', b', \mu) b' &= \sum_{j=1}^{N(\varepsilon)} \pi_{ij}^\varepsilon \sum_{m=1}^{N(z)} \pi_{lm} d_m(z_l, \mu) \left(\gamma(\varepsilon_j, z_m, x'_{jm}, \mu') b' \right. \\
&\quad \left. + \left[1 - \gamma(\varepsilon_j, z_m, x'_{jm}, \mu') \right] \theta \left(\pi(\varepsilon_j, z_m, k', \mu') + (1 - \delta) k' \right) \right).
\end{aligned} \tag{8}$$

Equation 8 provides the risk-neutral lender with the same expected return as that associated with risk-free real discount factor, $q(z, \mu)$. If a firm has no probability of default, then $\gamma(\varepsilon_j, z_m, x'_{jm}, s') = 0$ for every (ε_j, z_m) with $\pi_{ij}^\varepsilon > 0$ and $\pi_{lm} > 0$. In this case,

$$q(\varepsilon, z, k', b', \mu) b' = \sum_{j=1}^{N(\varepsilon)} \pi_{ij}^\varepsilon \sum_{m=1}^{N(z)} \pi_{lm} d_m(z_l, \mu) b' = \sum_{m=1}^{N(z)} \pi_{lm} d_m(z_l, \mu) b',$$

and $q(\varepsilon, z, k', b', \mu) = q_0(z, \mu)$.

The economy is populated by a unit measure of identical households. Household wealth is held as one-period shares in firms, which we denote using the measure λ .⁶ Given the prices they receive for their current shares, $\rho_0(x, \varepsilon; z, \mu)$, and the real wage they receive for their labor effort, $w(z, \mu)$, households determine their current consumption, c , hours worked, n^h , as well as the numbers of new shares, $\lambda'(x', \varepsilon')$, to purchase at prices $\rho_1(x', \varepsilon'; z, \mu)$. The lifetime expected

⁶Households also have access to a complete set of state-contingent claims. However, as there is no heterogeneity across households, these assets are in zero net supply in equilibrium. Thus, for sake of brevity, we do not explicitly model them here.

utility maximization problem of the representative household is listed below.

$$V^h(\lambda; z, \mu) = \max_{c, n^h, \lambda'} \left[U(c, 1 - n^h) + \beta \sum_{m=1}^{N_z} \pi_{lm}^z V^h(\lambda'; z_m, \mu') \right] \quad (9)$$

subject to

$$c + \int_{\mathbf{S}} \rho_1(x', \varepsilon'; z, \mu) \lambda'(d[x' \times \varepsilon']) \leq w(z, \mu) n^h + \int_{\mathbf{S}} \rho_0(x, \varepsilon; z, \mu) \lambda(d[x \times \varepsilon]).$$

Let $C^h(\lambda; z, \mu)$ describe the household choice of current consumption, and let $N^h(\lambda; z, \mu)$ be the allocation of current available time to working. Finally, let $\Lambda^h(x', \varepsilon', \lambda; z, \mu)$ be the quantity of shares purchased in firms that will begin the next period with x' units of debt, and idiosyncratic productivity ε' .

A *recursive competitive equilibrium* has firm's solving the problem described by (3) - (7), households solving the problem described in (9), and loans priced according to (8). These problems imply individual decision rules for firms and households that are consistent with the aggregate law of motion, Γ .

Using C and N to describe the market-clearing values of household consumption and hours worked, it is straightforward to show that market-clearing requires that (a) the real wage equal the household marginal rate of substitution between leisure and consumption, $w(z, \mu) = D_2 U(C, 1 - N) / D_1 U(C, 1 - N)$, that (b) the bond price, q_0^{-1} , equal the expected gross real interest rate, $q_0(z, \mu) = \beta \sum_{m=1}^{N_z} \pi_{lm}^z D_1 U(C'_m, 1 - N'_m) / D_1 U(C, 1 - N)$, and that (c) firms' state-contingent discount factors agree with the household discounted marginal utility of consumption across states $d_j(z, \mu) = \beta D_1 U(C'_j, 1 - N'_j) / D_1 U(C, 1 - N)$. Given these results, we may compute equilibrium by solving a single Bellman equation that combines the firm-level profit maximization problem with these equilibrium implications of household utility maximization, effectively subsuming households' decisions into the problems faced by firms.

3 Analysis

Before we summarize our preliminary results, we briefly discuss computation of the model. Any firm, given a state (ε, x) chooses (k', b') from the feasible set $\Phi(\varepsilon_i, z_l, x, \mu)$ defined in (2). This is the set of possible investment and debt levels that are feasible given lenders' zero profit condition. Any choice (k', b') involves a loan of $q(\varepsilon, z, k', b', \mu) b'$ and a debt, to be repaid next period, of b' . The difference between the loan discount factor, $q(\varepsilon, z, k', b', \mu)$, and the risk-free discount factor,

$q_0(z, \mu)$ arises through the possibility of default. This risk makes $\Phi(\varepsilon_i, z_l, x, \mu)$ a potentially complicated object. Most importantly, there is no assurance that it is a convex-valued, continuous correspondence. Consequently we solve firms' choice problems using a grid based method.

Our analysis is considerably simplified by the following observation. An examination of (3) - (7) shows that the value function $V(\varepsilon_i, z_l, x, \mu)$ is continuous and weakly increasing in net worth. This follows from the observation that the period return or earnings is linear in net worth. This property of the model leads to the result that if $x_1 \geq x_0$ then $\Phi(\varepsilon_i, z_l, x_0, \mu) \subseteq \Phi(\varepsilon_i, z_l, x_1, \mu)$.

A firm will default, $\gamma(\varepsilon, z, x, \mu) = 0$, whenever the beginning of period expected continuation value $V_0^c(\varepsilon_i, z_l, x, \mu) \leq 0$. Importantly, (4) and (5) combined with monotonicity of $V(\varepsilon_i, z_l, x, \mu)$ imply strict monotonicity of $V_0^c(\varepsilon_i, z_l, x, \mu)$. This proves that if $\gamma(\varepsilon, z, x_1, \mu) = 0$ for some $x_1 \in \mathbf{X}$, then $\gamma(\varepsilon, z, x_0, \mu) = 0$ for all $x_0 \leq x_1$. Monotonicity of firms' value functions imply that there are threshold levels of net worth,

$$x^d(\varepsilon, z, \mu) = \sup \{x \in \mathbf{X} \mid V_0^c(\varepsilon_i, z_l, x, \mu) \leq 0\}, \quad (10)$$

such that any firm with ε will default if $x \leq x^d(\varepsilon, z, \mu)$. As a result,

$$\begin{aligned} \gamma(\varepsilon, z, x, \mu) &= 0 \text{ if } x \leq x^d(\varepsilon, z, \mu), \\ &= 1 \text{ otherwise.} \end{aligned}$$

Importantly, (10) provides us with a threshold level of net worth for default. Define the indicator function, $\chi_{x^d(\varepsilon_j, z_m, \mu')}(x') = 1$ if $x' \geq x^d(\varepsilon_j, z_m, \mu')$. The loan discount factor may now be rewritten as

$$\begin{aligned} q(\varepsilon, z, k', b', \mu) b' &= \sum_{j=1}^{N(\varepsilon)} \pi_{ij}^\varepsilon \sum_{m=1}^{N(z)} \pi_{lm} d_m(z_l, \mu) \left(\chi_{x^d(\varepsilon_j, z_m, \mu')}(x'_{jm}) b' \right. \\ &\quad \left. + \left[1 - \chi_{x^d(\varepsilon_j, z_m, \mu')}(x'_{jm}) \right] \theta \left(\pi(\varepsilon_j, z_m, k', \mu') + (1 - \delta) k' \right) \right). \end{aligned} \quad (11)$$

These result lead to a dramatic simplification of our computational algorithm. Given the evolution of the aggregate state, Γ , we solve the model by guessing at a vector of $x^d(\varepsilon_j, z_m, \mu')$ for each ε_j , at every value of the aggregate state, (z, μ) at a grid of points that are the bivariate knots used in spline approximation of $V_0^c(\varepsilon_i, z_l, x, \mu)$, for each (ε_i, x) . For any equilibrium discount factor, $q_0(z, \mu)$, this allows us to solve for loan discount factors, $q(\varepsilon, z, k', b', \mu)$ and thus

determine firm's feasible sets, $\Phi(\varepsilon_i, z_l, x, \mu)$. Using these sets, we solve the firm's value function, $V(\varepsilon_i, z_l, x, \mu)$; this also determines the continuation value function, $V_0^c(\varepsilon_i, z_l, x, \mu)$. The latter allows us to update the default thresholds using (10). In determining firm's choices of (k', b') , levels of future net worth that are not on a grid used to solve the value function are recovered using linear interpolation.

The model has a high dimensional aggregate state vector. In order to tractably solve the model, we follow the method of Krusell and Smith (1998) and approximate the distribution μ with moments of the distribution of net worth across firms, m , and the aggregate capital stock, K . This requires replacing the law of motion, $\Gamma(z, \mu)$ with $\Gamma_m(z, m)$.

4 Results

4.1 Parameters

In the following, we will consider how the mechanics of our model with borrowing constraints compare to those in a reference model - one without borrowing constraints. This benchmark model, essentially an equilibrium business cycle model without real or financial frictions, helps us isolate the effects of borrowing constraints on our economy's aggregate dynamics. Aside from the nature of borrowing and lending, the benchmark model and the constrained borrowing model share a common parameter set that is selected to best match moments drawn from postwar U.S. aggregate and firm-level data.

Across our model economies, we assume that the representative household's period utility is the result of indivisible labor (Rogerson (1988)): $u(c, L) = \log c + \varphi L$. In specifying our exogenous stochastic process for aggregate productivity, we begin by assuming a continuous shock following a mean zero AR(1) process in logs: $\log z' = \rho_z \log z + \eta'_z$ with $\eta'_z \sim N(0, \sigma_{\eta_z}^2)$. Next, we estimate the values of ρ_z and σ_{η_z} from Solow residuals measured using NIPA data on US real GDP and private capital, together with the total employment hours series constructed by Prescott, Ueberfeldt, and Cociuba (2005) from CPS household survey data, over the years 1959-2002, and we discretize the resulting productivity process using a grid with 5 shock realizations ($N(z) = 5$) to obtain (z_l) and (π_{lm}^z) . We determine the firm-specific productivity shocks (ε_i) and the Markov Chain governing their evolution (π_{ij}) similarly by discretizing a log-normal process, $\log \varepsilon' = \rho_\varepsilon \log \varepsilon + \eta'$ using 5 values ($N(\varepsilon) = 5$).

We set the length of a period to correspond to one year, and we determine the values of β , ν , δ , α , φ and θ_b using moments from the aggregate data as follows. First, we set the household discount factor, β , to imply an average real interest rate of 4 percent, consistent with recent findings by Gomme, Ravikumar and Rupert (2008). Next, the production parameter ν is set to yield an average labor share of income at 0.60 (Cooley and Prescott (1995)). The depreciation rate, δ , is taken to imply an average investment-to-capital ratio of roughly 0.069, which corresponds to the average value for the private capital stock between 1954 and 2002 in the U.S. Fixed Asset Tables, controlling for growth. Given this value, we determine capital's share, α , so that our model matches the average private capital-to-output ratio over the same period, at 2.3, and we set the parameter governing the preference for leisure, φ , to imply an average of one-third of available time is spent in market work.

The parameters we determine using moments drawn from firm-level data are the exit rate, π_d , and the persistence and variability of the firm-specific productivity shocks, ρ_ε and σ_η . We set the exit rate at 0.10, so that 10 percent of firms enter and exit the economy each year. Our choice for the idiosyncratic shock process is based on the work of Khan and Thomas (2008). We choose the same persistence as in that paper, but a lower variability given the absence of capital adjustment costs in the current model. The result is a standard deviation of firm-level investment rates of 0.2 and a standard deviation of firm-level output, in the benchmark model, of 0.24. These values are comparable to those found in establishment level data. See, for example, Cooper and Haltiwanger (2006).

The table below lists the parameter set obtained from our calibration.

β	ν	δ	α	φ	ρ_z	σ_{η_z}	π_d	ρ_ε	σ_η
0.96	0.60	0.065	0.27	2.15	0.852	0.014	0.10	0.653	0.034

4.2 Steady state of a model without constrained borrowing

We begin by examining the effect of firm-level heterogeneity in an environment without constrained borrowing. This is an Arrow-Debreu environment where firms are fully funded by shareholders and do not need to borrow. Thus the risk of default is irrelevant and has no bearing on firms' investment decisions. Importantly, in any given period, dividends may be negative.

Firms in the benchmark economy solve an optimization problem described by (3), (4) and the following problem:

$$V^c(\varepsilon_i, z_l, x, \mu) = \max_{k' \geq 0} \left(x - k' + \sum_{j=1}^{N(\varepsilon)} \pi_{ij}^\varepsilon \sum_{m=1}^{N(z)} \pi_{lm} d_m(z_l, \mu) V(\varepsilon_j, z_m, x'_{jm}, \mu') \right) \quad (12)$$

subject to

$$\begin{aligned} x'_{jm} &= \pi(\varepsilon_j, z_m, k', \mu') + (1 - \delta) k', \\ \mu' &= \Gamma(z_l, \mu). \end{aligned} \quad (13)$$

It is straightforward to derive the result that firm's decision rules take the form, $k' = g(\varepsilon; z, \mu)$. Since dividends may be negative, net worth has no effect on investment for firms that continue in production. Given our assumption that $N(\varepsilon) = 5$, there are 5 levels of capital chosen in any period, and thus 25 possible pairs of (ε, k) in the economy. This is the extent of heterogeneity, and a simple aggregation result exists whereby this benchmark model may be described using a representative firm with a constant, time-invariant, adjustment to aggregate total factor productivity. The details of this result are in Khan and Thomas (2008), Appendix B.

Figure 1 illustrates the independence of capital choices from firms' net worth. It shows the 25 combinations of x and ε which arise through $g(\varepsilon; \bar{z}, \bar{\mu})$, where $(\bar{z}, \bar{\mu})$ describes the steady state level of aggregate total factor productivity and the distribution of firms. The levels of net worth on the left horizontal axis arise through (13) given the choices described by $g(\varepsilon_i; \bar{z}, \bar{\mu})$, $i = 1, \dots, N(\varepsilon)$. The support of ε (0.915, 0.956, 1.0, 1.0456, 1.0932) leads to a vector of capital (0.928, 1.137, 1.424, 1.777, 2.182). This support for the distribution of capital in the stationary state of the benchmark model varies only as a function of ε and is independent of net worth. Figure 2 illustrates the resultant distribution of capital at each level of firm-level total factor productivity. The top panel is the lowest level of ε and the bottom panel is the highest level of ε . In general, the invariant distribution of firm-level productivity implies that higher levels of productivity are associated with higher average capital. Nonetheless, the variability of idiosyncratic shocks and one-period time-to-build for capital leads to nontrivial dispersion in capital across firms, even when one controls for firm-level differences in TFP.

As we begin to contrast the constrained borrowing model with this frictionless benchmark, it is useful to note that the steady state of the benchmark economy is characterized by aggregate output equal to 0.576 and a capital stock of 1.458. The real wage is 1.0349 and total hours worked is 0.334.

4.3 Model with constrained borrowing

We now turn to the model with non-contingent debt and default. In the stationary state, the firm value function, $V(\varepsilon, x, \bar{z}, \bar{\mu})$ is shown in Figure 1. As discussed above, the value function is continuous and monotonically increasing in net worth. However the value of default, which leaves the firm with a net value of 0 that is independent of (ε, x) , implies that it is not a smooth function. This property of the model requires that we solve for firm-level decisions using piecewise linear interpolation on a grid of values defined upon the set $\mathbf{E} \times \mathbf{X}$. Returning to the issue of default, firms' threshold levels of net worth, $x^d(\varepsilon, \bar{z}, \bar{\mu})$, may be inferred from Figure 1. Recall that these functions identify the maximum level net worth, as a function of firm-level idiosyncratic productivity, associated with default. These thresholds are falling in ε . Firms with higher total factor productivity are less willing to exit production since their expected discounted future continuation value is higher than that of firms with lower levels of ε . In this sense, more productive firms are safer borrowers. Their opportunity cost of default is higher. At the same time, more productive firm, all else equal, would like to borrow more. However higher levels of debt increase the probability of default. As we will see below, in the stationary equilibrium this results in a misallocation of investment that is particularly severe for more productive firms.

Figure 4 shows firms' decisions for future capital and debt. In the top panel, we see firms' choices of k' as a function of their idiosyncratic productivity, ε , and their net worth, x . The second panel illustrates their choices of b' . In contrast to the frictionless model, both now clearly vary with x . Starting with capital, firms with net worth below the default threshold, which ranges from -0.56 at ε_1 to -0.69 for ε_5 , simply choose to exit. This is indicated by a 0 value for future capital. At levels of x above these thresholds, firms choose positive levels of capital. The efficient level of capital that a firm that is unconstrained by its net worth would choose ranges from 1.176, at ε_1 , to 2.6, for ε_5 . Notice that, compared to the frictionless economy, the possibility of default tends to increase levels of investment.

Figure 4a shows that, at levels of net worth close to the default threshold, firms with higher total factor productivity cannot attain their efficient level of capital. Their debt, shown in Figure 4b, is insufficient given their internal funds, x , to finance sufficient investment. These firms must produce with inefficiently low levels of capital, and higher levels of the marginal product of capital, than other firms. Over time, given persistence in firm-level total factor productivity, they tend to accumulate net worth. Eventually this allows them to adopt the efficient level of capital. Notice

that firms with relatively high levels of productivity and high net worth, in the top left corner of Figure 4b, actually begin to save. Their levels of debt, b , are negative.

Figure 5 shows the actual loan discount rates at which firms borrow in the stationary state. For almost all firms in this example, where the fixed cost ϕ is 0, the choice of future capital and debt is constrained to not imply a risk premium on their loans. As we've seen in the discussion above, the risk of default constrains the size of loans that firms choose. In the current example, credit rationing is so severe, that the risk of default is very low or 0 for most firms in the economy. However at levels of net worth close to default thresholds, the optimal loan discount set by lenders falls to 0. In this region firms are simply unable to borrow at any risk premium. Firms that, over time, find their net worth having fallen into this region simply exit. In equilibrium this reduces the number of firms in production by 11 percent relative to the frictionless model.

Figure 6 illustrates the aggregate implications of constrained borrowing in the stationary state. The top panel shows the distribution of firms, in equilibrium, over net worth and total factor productivity. Entrants start with 0 net worth. They finance their initial production using a loan which, if their productivity is lower than expected next period, reduces their net worth below 0. In the stationary distribution of firms, less productive firms tend to have lower levels of net worth and, across the 5 levels of ε , the conditional mean level of net worth is increasing. Figure 7a shows that mean net worth conditional on ε_1 is -0.254 , and rises to 0.356 for firms with a productivity of ε_5 . These conditional means hide considerable dispersion across firms. Figure 7b shows the cumulative distribution of net worth across all levels of firm-specific productivity. The lowest levels of net worth are near the default thresholds, and roughly half the firms in the economy operate with negative levels of net worth. Note that our measure of net worth is not firm value, but simply current earnings net of one-period debt and augmented by the value of nondepreciated capital. Thus all firms that produce, whether with positive or negative net worth, have value bounded below by 0. The mean level of net worth, across all firms that produce, is 0.0534 . The levels of net worth across firms in the steady state range from below -0.5 to above 2.5 .

Net worth allows firms to internally fund investment, or to partly fund investment thereby reducing their loan for any level of future capital. When a firm defaults, the lender is able to confiscate a fraction of its profits, including the value of its nondepreciated capital stock. These serve as collateral and firms with higher levels of net worth tend to have higher levels of collateral.

Dispersion in firms' net worth is associated with dispersion in their levels of collateral. The effect of this dispersion in collateral, and internal funds, is illustrated in Figure 6b which shows the actual levels of capital adopted in the steady state. When examined alongside Figure 6a, it's evident that many firms operate with inefficiently low levels of capital. Most importantly, a firm with low productivity but high levels of collateral will have a higher level of capital than another firm that has higher productivity but less collateral. In equilibrium this reduces aggregate total factor productivity by misallocating the distribution of capital across production units.

There are substantial aggregate effects of this misallocation. In this model where debt limits arise endogenously, their effect appears to be stronger than that found in models where they are exogenous, for example Khan and Thomas (2010). Starting the employment and the real wage, in the steady state, the constrained borrowing model has a real wage of 0.989 and aggregate employment of 0.325. These are 4.5 and 2.7 percent lower, respectively, than the corresponding values for the benchmark model without borrowing constraints. The largest difference between these two economies is in the aggregate capital stock which falls by 10.6 percent. Aggregate GDP falls by 7 percent and aggregate TFP by 2.6 percent.

The effects of financial frictions arise exclusively through the dependence of loan discount rates on firms' choices of capital and debt. In particular, higher levels of debt, at any level of capital, leads to sharp discount or high loan rate as lenders require an expected return equal to the risk-free rate. This, in turn, leads to an allocation of capital where relatively wealthy borrowers, firm's with high levels of net worth, are able fully fund their investment using retained earning or debt, while poorer but equally productive borrowers find themselves operating with low levels of capital. In equilibrium, firm's capital is correlated with its internal funds, which is inefficient. The resultant differences in firms' marginal product of capital reduces aggregate TFP by almost 3 percent. This leads to a reduction in the real wage, employment, and GDP.

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Figure 1: levels of capital without borrowing constraints

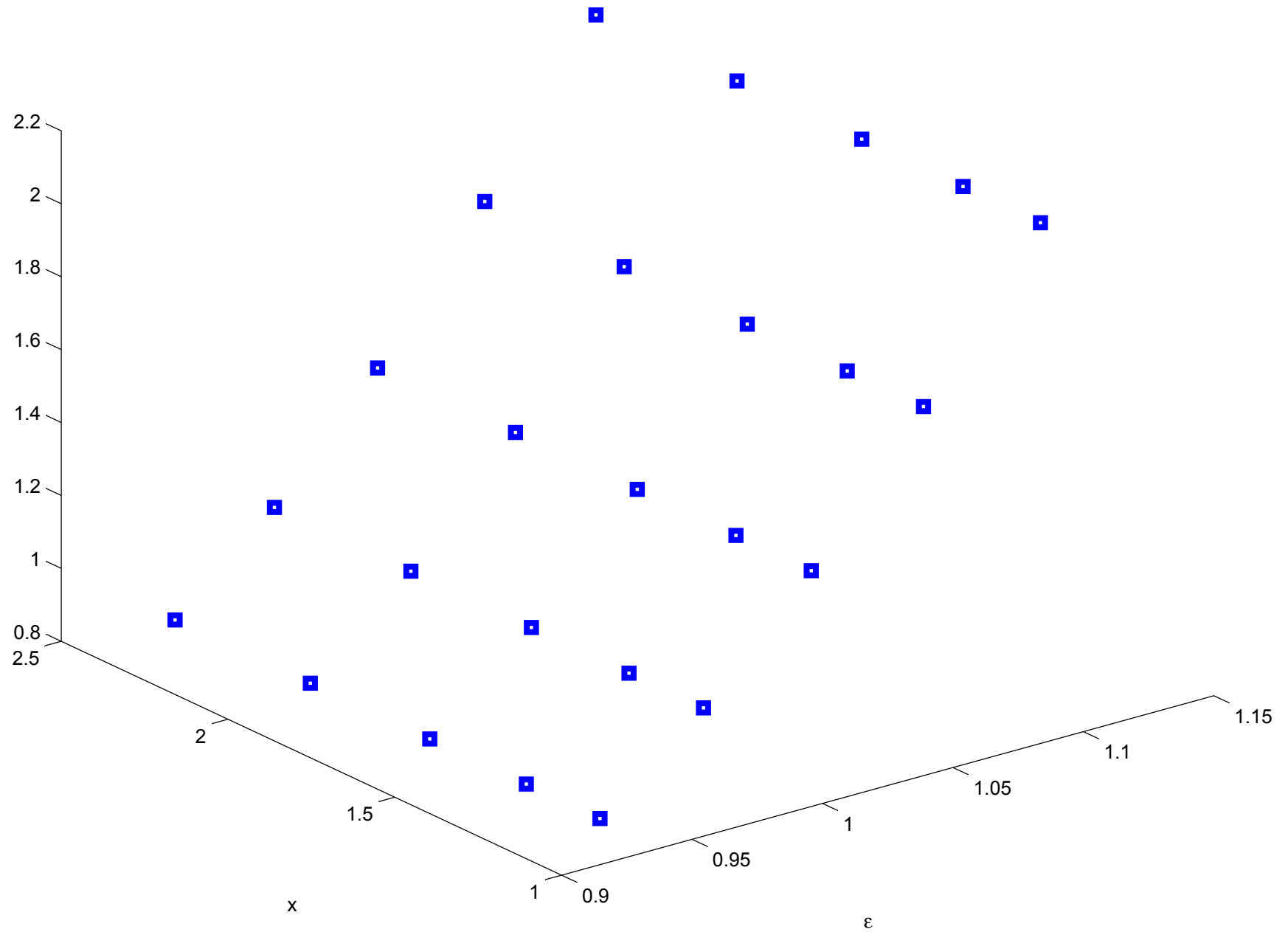


Figure 2: Distribution of capital without borrowing constraints

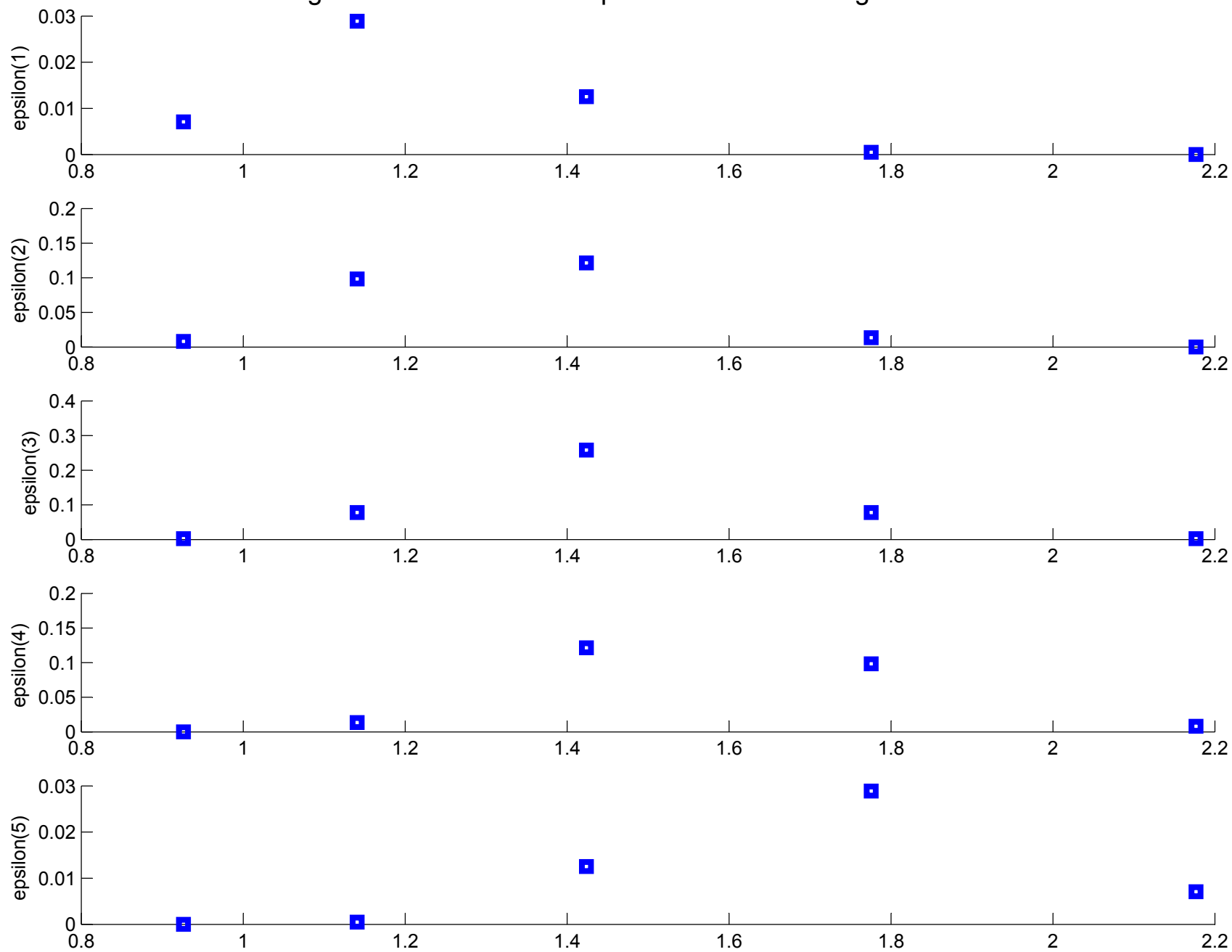


Figure 3: the value function $v(\varepsilon, x)$

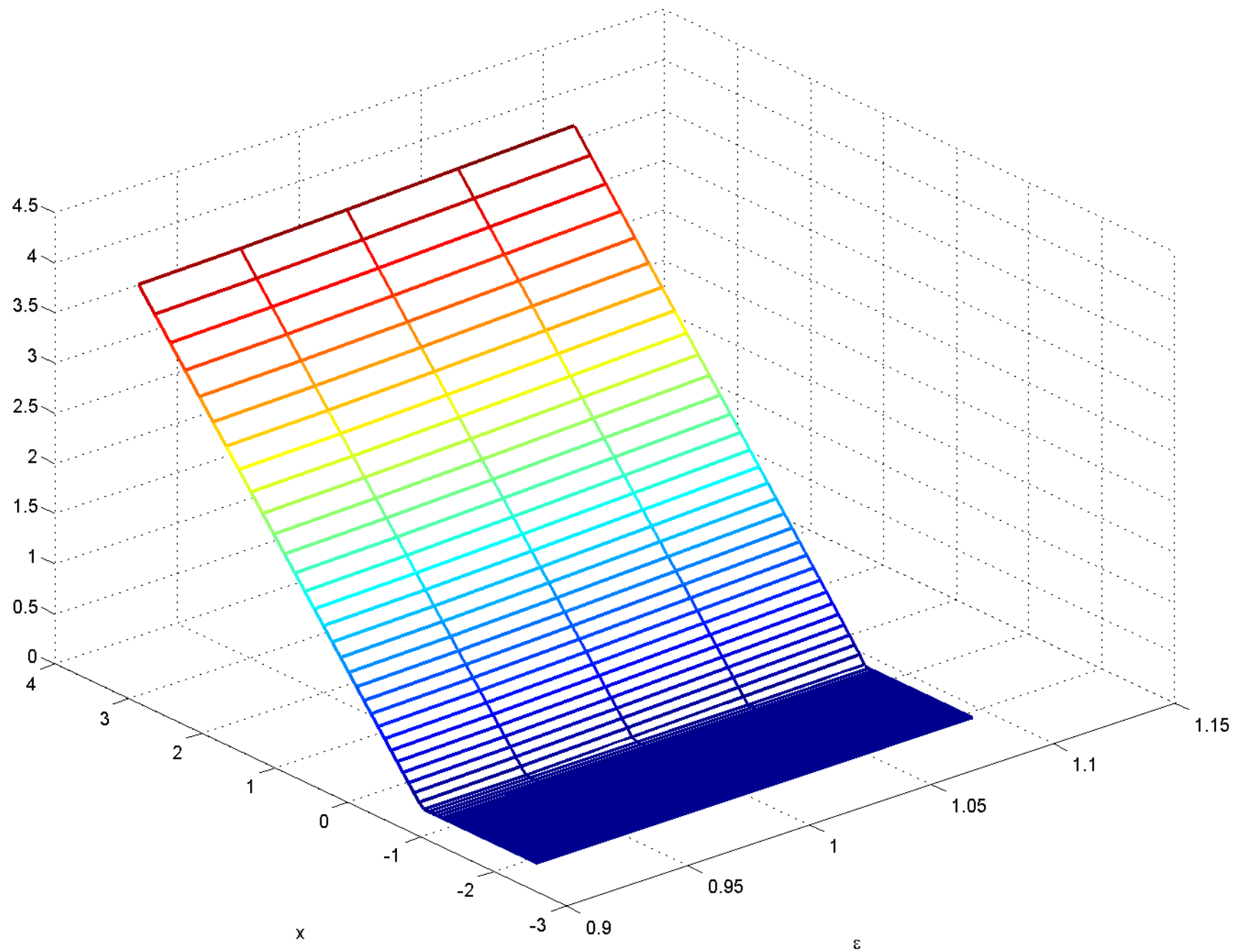


Figure 4a: optimal choice of capital $g(\varepsilon, x)$

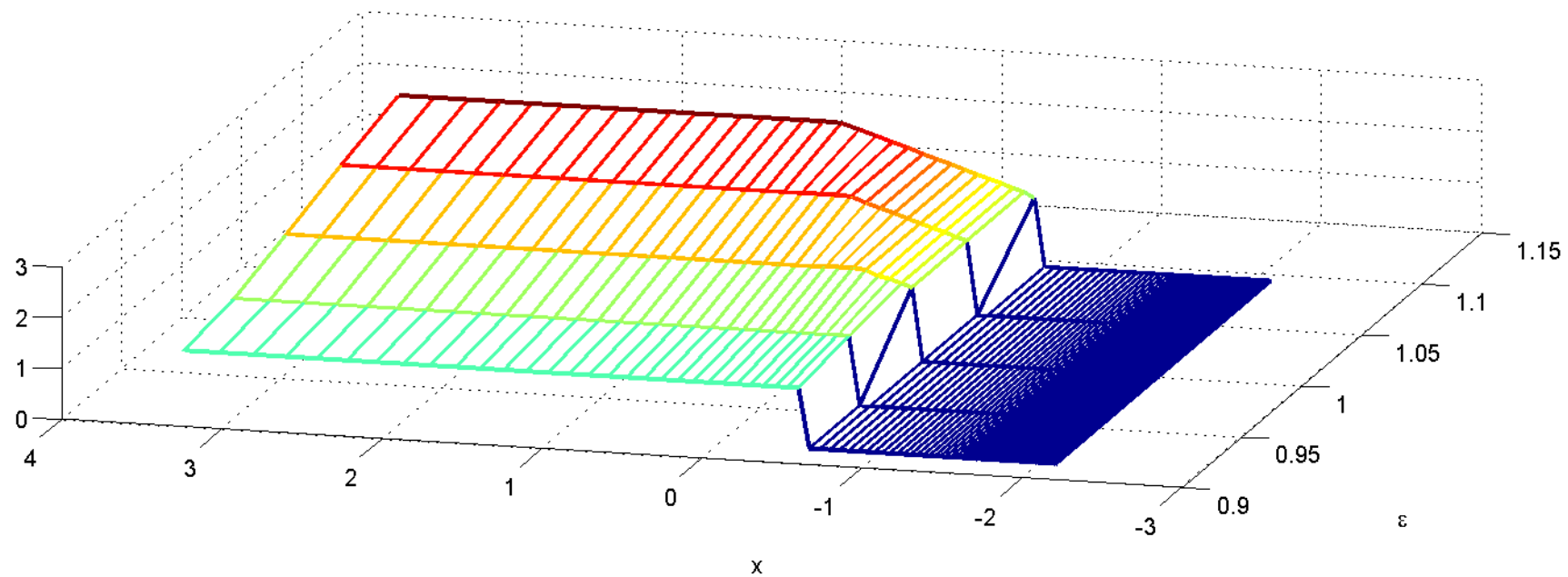


Figure 4b: optimal debt $b(\varepsilon, x)$

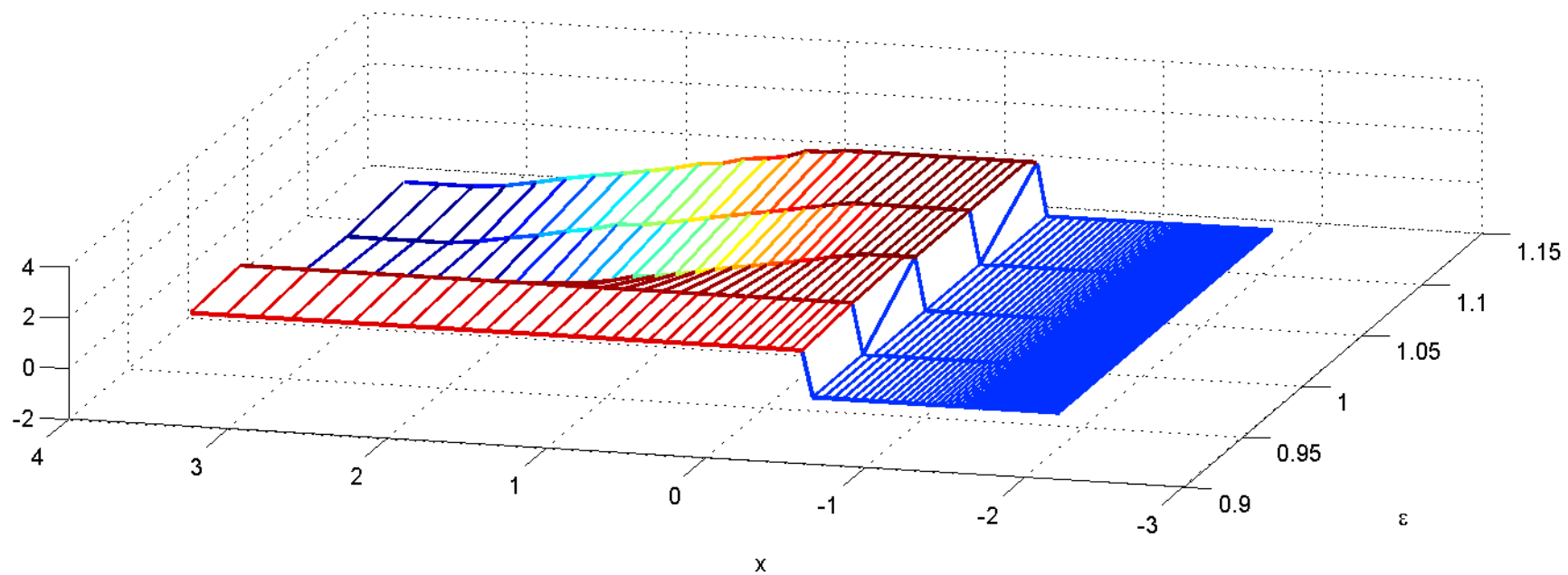


Figure 5: optimal loan rate $q(\varepsilon, x)$

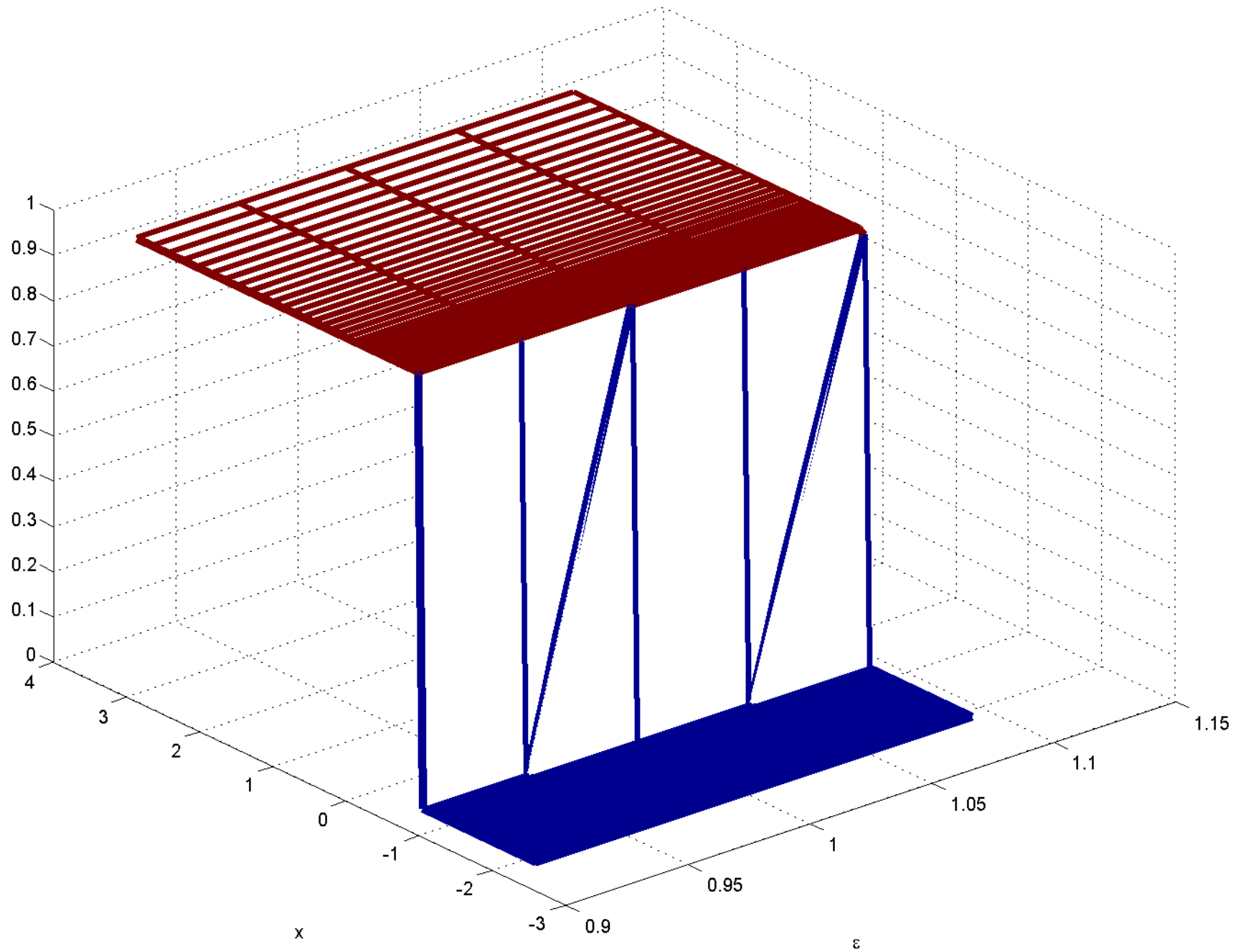


Figure 6a: equilibrium distribution of firms $\mu(\varepsilon, x)$

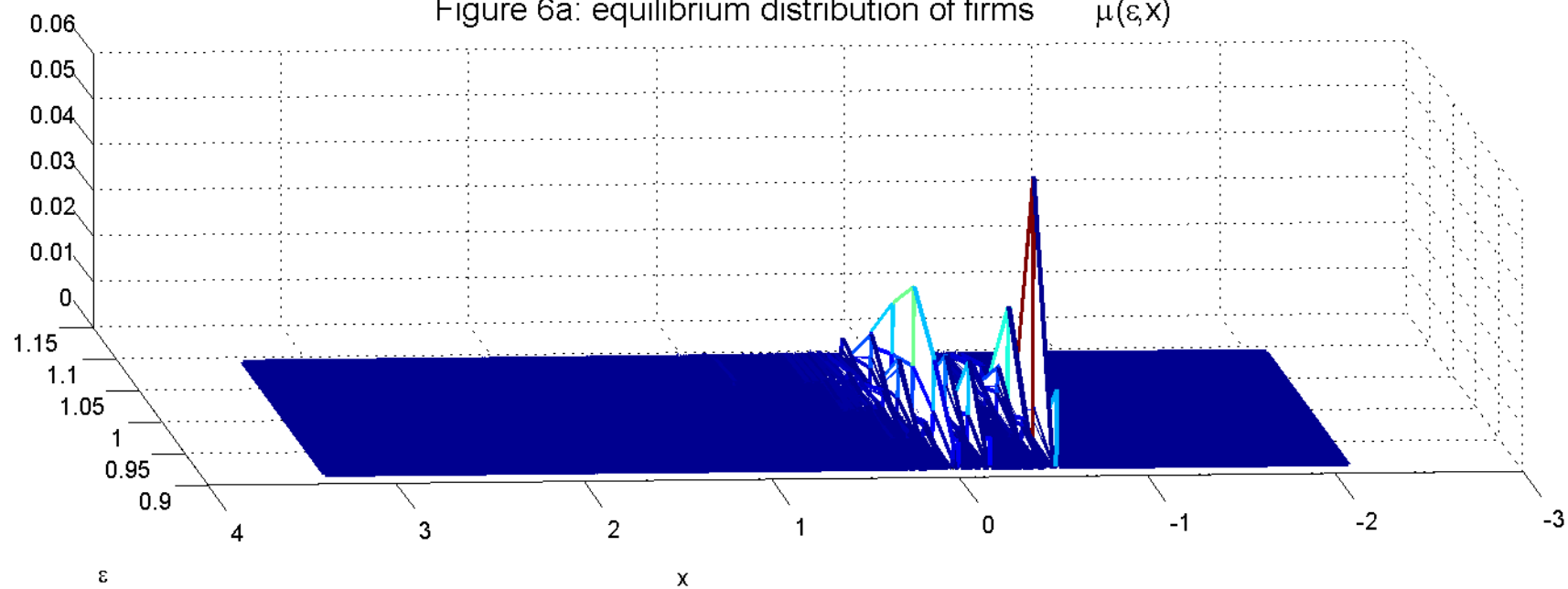


Figure 6b: firm capital choices in equilibrium

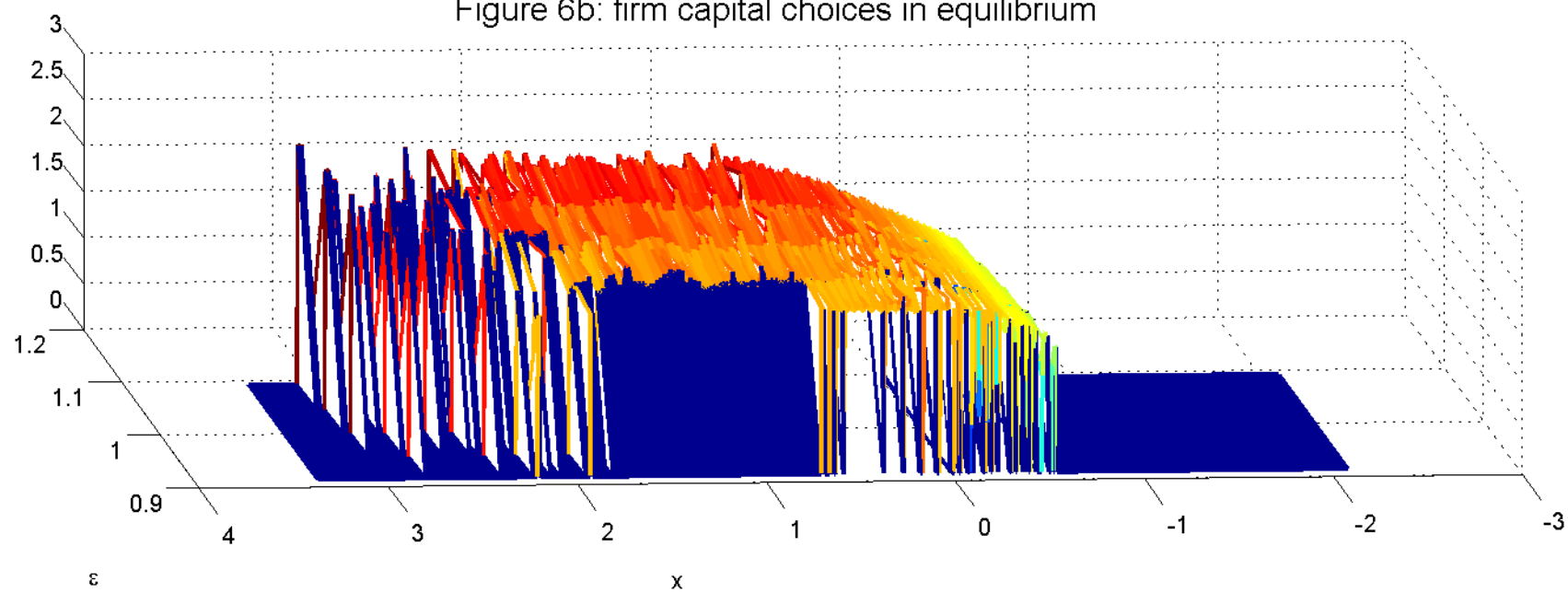


Figure 7a: average net worth over idiosyncratic productivity

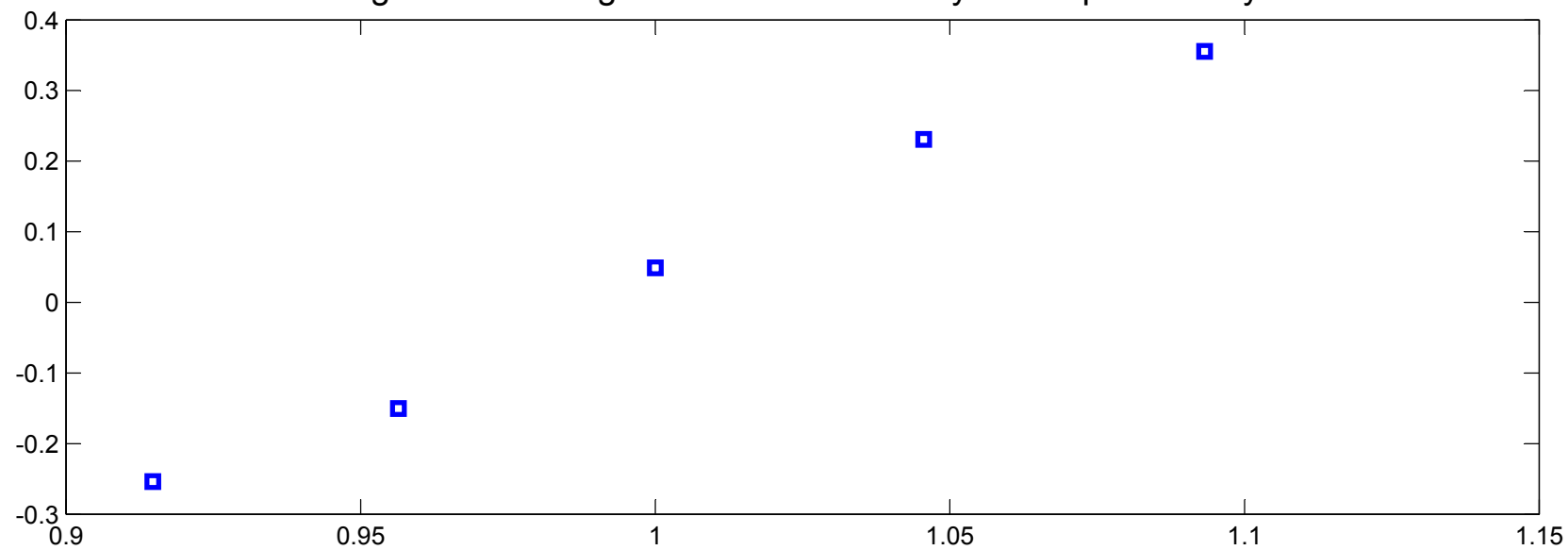


Figure 7b: unconditional distribution of net worth

