Directed search with endogenous capacity

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Abstract

Preliminary Draft

We consider a large market with frictions. Sellers are capacity constrained, post prices, and buyers contact sellers taking into account price and probability of service. We endogenize the sellers' capacities. To this end we formulate a three stage game, where the sellers choose their capacity in the first stage, in the second stage sellers choose their price given the capacities, and in the third stage buyers choose which firm to visit. We analyse price formation when sellers have different capacities. A non-existence result of an equilibrium is introduced for linear costs. We also characterize equilibrium with free entry and increasing and convex costs. In addition, we derive the matching function and the social welfare function and show that the capacities chosen both under free entry and with a fixed number of sellers are efficient.

Keywords: Directed search, firm size

1 Introduction

What is the equilibrium size of firms in a market with frictions? How many matches are formed and how efficient are markets when firm size is endogenously determined? In the directed search literature these are still unanswered questions. We address them in this article.

We analyze the capacity choice of firms. One limitation of this type of models so far has been that the number of goods a firm has to offer is usually restricted to one. In the cases when larger capacity is allowed for, it is usually either exogenously given or restricted to one or two. As shown by Burdett, Shi and Wright (2001), the size of the firms affects the frictions and number of matches in the economy. An economy with a fixed number of buyers and goods behaves very differently depending on how many sellers are active in the economy. If there were only one large seller the market would be frictionless. The frictions arise from the inability of the buyers to coordinate their actions (which seller to visit). This leads to uncertainty of demand that each seller faces. If the number of buyers visiting a seller exceeds the seller's capacity the good gets rationed so that some of the buyers will be left without the good, while if the number of buyers visiting a seller is less than the capacity of the seller then some of the goods will be left unsold. In our setting sellers are free to choose any capacity. While our model describes an economy with buyers and sellers it could be easily rewritten to describe job market search. The buyers would then be the job searchers and the sellers would be firms posting vacancies and wages.

In section 2 we introduce the model and the timeline of the game. Section 2.1 describes a situation where all sellers have chosen the same capacity in period one. We look at the buyers' and sellers' expected utilities and expected prices given the capacities and proceed to derive the sellers' symmetric equilibrium price. In section 2.2 we derive equilibrium prices when sellers have chosen two different capacities in period one. In section 2.3 we analyze the capacity choice of the sellers in period one. A non-existence result is introduced for linear costs. Section 2.4 characterizes equilibrium with free entry when costs are increasing and convex. In section 2.5 the matching function and the social welfare functions are introduced. We find that the capacities chosen under free entry are constrained efficient. In the extension we show that equilibrium expected prices and expected utilities are equivalent when firms post prices, as in our model, and when prices are determined by auction (with reserve price zero) at the locations of the different sellers. In a second extension (still under construction) we show that equilibrium exists when the measure of sellers is fixed. The capacities (and prices) chosen by the sellers are also in this setting constrained efficient. Preliminary analysis indicates that in equilibrium there are usually sellers of two consecrative capacities and thus price dispersion occurs.

1.0.1 Related literature

There is quite a large literature on directed search. The most relevant to our work are the following. Burdett, Shi and Wright (2001) studies equilibrium price posting. Capacity choice when firms' capacity is either one or two is also studied, but not in equilibrium. The paper finds that firm size affects matching frictions and thus both the equilibrium wages and unemployment levels. This finding is used to partly explain observed shifts in the Beveridge curve. Lester (2010) endogenizes capacity in a directed search model of the labour market. The paper considers capacities of one and two and finds a unique equilibrium in pure or mixed strategies. Our model allows for any capacity. Geromichalos (2008) studies more general mechanisms in the product market. The main difference to our paper is that in Geromichalos (2008) production takes place after the locational demand is realized while in our model the production costs are sunk at the time of trading. Hawkins (2010) considers a rich set of wage contracts. In his model firms can create any number of vacancies after they have paid an entry cost. The timing in both Hawkins (2010) and Geromichalos (2008) is thus different from the timing in our model. Our model describes a product market where a seller has to produce the goods before he can sell them. This assumption seems natural in many settings in the product market.

2 The model

The environment consists of a continuum of size 1 of buyers and a large continuum of potential sellers of which θ^{-1} are active in the market. The ratio of buyers to active sellers is θ. The sellers choose their capacity in units of the indivisible good and post binding prices. Both the capacity and the price of each seller are observable. Each buyer has unit demand and can visit only one seller. Buyers value the good at one, sellers at zero. The sellers choose their capacity and price so as to maximize their expected revenue minus cost. The revenue to the sellers is the price times the number of trades. The buyers maximize their expected utility defined as their utility from the good minus its price times the probability by which they end up with a good.

The timeline is as follows: In stage 1 the active sellers choose capacities $\kappa \in \mathbb{N}$. In stage 2 the sellers choose prices, knowing the measure of active sellers and their capacities. In stage 3 the buyers choose which seller to visit knowing the prices and capacities as well as the measure of active sellers.

As is customary in directed search we focus on symmetric strategies for the buyers. Following Kircher (2009) we assume that the strategies of both the sellers and the buyers are anonymous. Loosely stated anonymity here means that sellers with the same capacity and the same price are treated identically by the buyers and all buyers are treated identically by the sellers.

A strategy of a seller i consists of his choice of capacity κ_i and his posted price p_i given his capacity and the distribution H of capacities chosen by sellers. The pricing strategy p_i is a map p_i: $\{H \cup \{\kappa_i\}\} \to [0, 1]$.

When there are different capacity-price pairs the buyers adjust their behavior in equilibrium so that they are indifferent between the different types of sellers and expect the market utility M from them all. This adjustment of behavior leads to different queue lengths $\beta_{\kappa,p}(\theta, M)$ for the different types of sellers.

2.1 Sellers' and buyers' utility when sellers have identical capacity and price

In order to gain insight in how the sellers' profit and buyers' utility is determined when capacity is higher than one we first look at a situation where all sellers have identical capacities and identical price. After this we solve for price given the distribution of capacities and finally for the equilibrium capacities.

2.1.1 Sellers' profit

When all sellers have capacity k and price q the probability that exactly j buyers visit a certain location is $P\left[x = j\right] = e^{-\theta}\frac{\theta^j}{j!}$. The probability that fewer than k buyers visit a seller is $F_\theta(k-1) = \sum_{i=0}^{k-1} e^{-\theta}\frac{\theta^i}{i!}$. A seller's profit is thus $\sum_{i=0}^{k-1} e^{-\theta}\frac{\theta^i}{i!} iq$ if fewer than k buyers visit him. If k or more buyers visit him he sells all k units. The following expression thus gives a sellers expected profit

$$\pi^s = \sum_{i=0}^{k-1} e^{-\theta} \frac{\theta^i}{i!} iq + \sum_{i=k}^{\infty} e^{-\theta} \frac{\theta^i}{i!} kq.$$

The sellers' expected profit is thus given by[1]

$$\begin{aligned} \pi^s &= q\left[\theta F_\theta(k-2) + k\left(1 - F_\theta\left(k-1\right)\right)\right] \\ &= q\left[\theta F_\theta(k-1) + k\left(1 - F_\theta\left(k\right)\right)\right]. \end{aligned} \tag{1}$$

2.1.2 Buyers' utility

A buyer visiting a seller with k goods expects the following utility:

$$\begin{aligned} u^b &= \sum_{i=0}^{k-1} e^{-\theta} \frac{\theta^i}{i!}\left(1-q\right) + \sum_{i=k}^{\infty} e^{-\theta} \frac{\theta^i}{i!} \frac{k}{i+1}\left(1-q\right) \\ &= \left(1-q\right)\left[F_\theta(k-1) + \frac{k}{\theta} \sum_{i=k}^{\infty} e^{-\theta} \frac{\theta^{i+1}}{(i+1)!}\right] \\ &= \left(1-q\right)\left[F_\theta(k-1) + \frac{k}{\theta}\left(1 - F_\theta(k)\right)\right] \end{aligned} \tag{2}$$

where $(1-q)$ is the utility of the buyer if he gets the good. Given total demand, the probability that there are fewer than k other buyers at the same location is $F_\theta(k-1)$, in which case the buyer acquires the good for sure. If there are k or more other buyers at the same location the good gets rationed, the buyers probability of acquiring the good is then $\frac{k}{\theta}\left(1 - F_\theta(k)\right)$.

[1] The intermediate steps are

$$\sum_{i=k}^{\infty} e^{-\theta} \frac{\theta^i}{i!} = 1 - F_\theta(k-1),$$

and

$$\sum_{i=0}^{k-1} e^{-\theta} \frac{\theta^i}{i!} iq = \sum_{i=1}^{k-1} e^{-\theta} \frac{\theta^{i-1}}{(i-1)!} \theta q = \theta q \sum_{i=0}^{k-2} e^{-\theta} \frac{\theta^i}{i!} = q\theta F_\theta(k-2).$$

3 Price

In this subsection we assume that all firms have chosen capacity k in period one and are competing in prices. We derive the equilibrium price q. We analyze a large economy where the ratio of buyers to active sellers is θ and approximate the meetings by a Poisson distribution. When all sellers quote price q the queue length (number of buyers that a seller expects) is θ.

We find that the equilibrium price in stage two, when all agents have chosen the same capacity in the first period, is given by proposition 1.

Proposition 1 *When all sellers have capacity k the equilibrium price q is given by*

$$q = \frac{k\,(1 - F_\theta\,(k))}{k\,(1 - F_\theta(k)) + \theta F_\theta\,(k-1)}. \tag{3}$$

The idea of the proof is the following. Let q be the equilibrium price. To find equilibrium we need to be able to derive the queue length that a deviator faces, i.e. the queue length off the equilibrium path. To do this we assume that a small proportion s of sellers deviate from the proposed equilibrium and quote price $\widetilde{q}$. The buyers must be indifferent between going to the deviating sellers and going to the non-deviating sellers. Letting s tend to zero while the buyers' indifference condition must be satisfied we find the rate that a single deviator faces.[2]

We then solve for the deviators' first order condition and impose the equilibrium requirement that $q = \widetilde{q}$. After having obtained the local necessary conditions this way we show that there are no other profitable deviations from the equilibrium candidate.

Proof. The buyers must be indifferent between going to a deviating seller or going to a non-deviating seller. Assume that a buyer goes to the deviator with probability ω and to the non-deviators with

[2]There are at least three other approaches in the directed search litterature that can be used for finding the off equilibrium expected queue lenghts. Galenianos and Kircher (2009) and Kirtcher (2009) introduces a fraction of ϵ of noice sellers that post every price in [0, 1] and lets ϵ tend to zero and can therefore define the meeting rates and thus expected profits for any price in the support. Burdet, Shi and Wright (2001) solve for subgame perfect equilibria in a finite model and let the number of buyers and sellers tend to infinity keeping their ratio constant. In the market utility approach that is used e.g. by Moen (1997) and Shimer (2005) buyers respond to sellers deviations so that they recieve the same utility by going to the deviators as going to the non-deviators, i.e. the market utility. Our approach yields similar results as both the market utility approach and to the approach of Kircher.

probability $1-w$. The deviator's queue length is $\beta = \frac{w}{s}\theta$, and the non-deviators' queue length is $\alpha = \frac{1-w}{1-s}\theta$. The indifference condition of the buyers is thus

$$(1-q)\left[F_\alpha(k-1) + \frac{k}{\alpha}\left(1-F_\alpha(k)\right)\right] = (1-\widetilde{q})\left[F_\beta(k-1) + \frac{k}{\beta}\left(1-F_\beta(k)\right)\right]. \tag{4}$$

The firms deviating to price $\widetilde{q}$ maximize

$$\max_{\widetilde{q}} \widetilde{q}\left[\sum_{1}^{k} e^{-\beta}\frac{\beta^i}{i!}i + \sum_{k+1}^{\infty} e^{-\beta}\frac{\beta^i}{i!}k\right]. \tag{5}$$

The first order condition is

$$\sum_{1}^{k} e^{-\beta}\frac{\beta^i}{i!}i + \sum_{k+1}^{\infty} e^{-\beta}\frac{\beta^i}{i!}k +$$
$$\widetilde{q}\left[-e^{-\beta}\sum_{1}^{k}\frac{\beta^i}{i!}i - e^{-\beta}\sum_{k+1}^{\infty}\frac{\beta^i}{i!}k + e^{-\beta}\sum_{1}^{k} i\frac{\beta^{i-1}}{i!}i + e^{-\beta}\sum_{k+1}^{\infty} i\frac{\beta^{i-1}}{i!}k\right]\frac{\theta}{s}\frac{dw}{d\widetilde{q}} = 0 \tag{6}$$

and simplifies to

$$\beta F_\beta(k-1) + (1-F_\beta(k))k + \widetilde{q}F_\beta(k-1)\frac{\theta}{s}\frac{dw}{d\widetilde{q}} = 0. \tag{7}$$

To solve the first order condition (7) we first need to solve for $\frac{1}{s}\frac{dw}{d\widetilde{q}}$. We do so by totally differentiating the buyers' indifference condition (4) noting that $\frac{d\alpha}{dw} = -\frac{\theta}{1-s}$ and $\frac{d\beta}{dw} = \frac{\theta}{s}$. We get

$$d\widetilde{q}\left[F_\beta(k-1) + \frac{k}{\beta}(1-F_\beta(k))\right] +$$
$$dw\left[\begin{array}{c} (1-q)\left[\frac{d}{d\alpha}F_\alpha(k-1) + \frac{k}{\alpha^2}\left(1-F_\alpha(k)\right) - \frac{k}{\alpha}\frac{d}{d\alpha}F_\alpha(k)\right]\left(-\frac{\theta}{1-s}\right) \\ -(1-\widetilde{q})\left[\frac{d}{d\beta}F_\beta(k-1) + \frac{k}{\beta^2}(1-F_\beta(k)) - \frac{k}{\beta}\frac{d}{d\beta}F_\beta(k)\right]\frac{\theta}{s} \end{array}\right] = 0. \tag{8}$$

Now we impose $\widetilde{q} = q$ which leads to $\beta = \alpha = \theta$ Thus

$$\frac{1}{s}\frac{dw}{d\widetilde{q}} = -\frac{\theta F_\theta(k-1) + k(1-F_\theta(k))}{(1-q)\left[k\left(1-F_\theta(k)\right)\frac{s}{1-s} + k\left(1-F_\theta\left(k\right)\right)\right]} \tag{9}$$

Inserting (9) in the first order condition and simplifying and letting s tend towards zero the first order condition becomes

$$\theta F_{\theta}(k-1) + (1 - F_{\theta}(k))\, k - \theta F_{\theta}(k-1) \frac{[\theta F_{\theta}(k-1) + k(1 - F_{\theta}(k))]}{(1-q)\, k\,(1 - F_{\theta}(k))} q = 0,$$

which simplifies to

$$(1-q)\, k\,(1 - F_{\theta}(k)) - \theta F_{\theta}(k-1) = 0. \tag{10}$$

We solve for

$$q = \frac{k\,(1 - F_{\theta}(k))}{k\,(1 - F_{\theta}(k)) + \theta F_{\theta}(k-1)}. \tag{11}$$

In the appendix we show that there are no profitable deviations from q implying that q is indeed the equilibrium price. ■

3.1 Prices with different capacities

When all sellers choose the same capacity in stage one there is an equilibrium where they choose the same price in stage two. Next we look at prices in equilibrium in a situation where sellers have chosen two different capacities in stage one. We need these to understand pricing with different capacities.

We solve the equilibrium prices in a similar way as in the last section. First we postulate that price q is the price for firms with capacity k and r is the price for firms with capacity l. In this prospective equilibrium the buyers distribute themselves so that they are indifferent between which seller to visit. We then look at deviations from this prospective equilibrium. As the capacities have been decided in stage one the deviations in period two are in price only. To find the expected utility of a deviator from capacity price pair (k, q) we let a proportion ϵ of sellers deviate and quote price $\widetilde{q}$. We solve for the deviators' first order condition. Then we impose as equilibrium condition that $q = \widetilde{q}$ and let the proportion of deviators ϵ approach zero to get the equilibrium price. One difference to the proof in the last section is that there are two indifference conditions that have to be satisfied as the buyers have to be indifferent between the three kinds of sellers. We thus have three groups of buyers; those who go to sellers with price-quantity pair (r, l), those with price-quantity pair (q, k) and those who go to sellers with price-quantity pair $(\widetilde{q}, k)$.

Proposition 2 *When some sellers have capacity k and other sellers have capacity l the equilibrium prices q and r are given by*

$$q=\frac{k\left(1-F_{\alpha}\left(k\right)\right)}{k\left(1-F_{\alpha}\left(k\right)\right)+\alpha F_{\alpha}(k-1)} \tag{12}$$

and

$$r=\frac{l\left(1-F_{\beta}\left(l\right)\right)}{l\left(1-F_{\beta}\left(l\right)\right)+\beta F_{\beta}(l-1)} \tag{13}$$

where the subscript α and β refer to the queue lengths (Poisson rates that govern the meeting process).. Sellers with capacity price pair (l,r) have expected queue length $\beta=\frac{w}{s}\theta$, and sellers with capacity price pair (k,q) have expected queue length $\alpha=\frac{1-w}{1-s}\theta$, where w is the proportion of buyers going to capacity l sellers and s is the proportion of sellers with capacity l.

Proof. The proof of the proposition is similar to the proof of Proposition 1 and can be found in the appendix. ■

It immediately follows from Proposition 2 that there is price dispersion when there are sellers with different capacities. Translated to a model with job applicants and firms Proposition 2 means that in equilibrium similar workers are paid different wages when there are firms of different size.

3.1.1 Sellers' profit and buyers' expected utility

If there are sellers of two different capacities, k and l, then their expected profits are determined by the following expressions in equilibrium. The expected profit of a seller with capacity k is

$$\pi(k)=q\left[\alpha F_{\alpha}(k-1)+k(1-F_{\alpha}(k))\right]-c(k)=k\left(1-F_{\alpha}(k)\right)-c(k), \tag{14}$$

where $[\alpha F_{\alpha}(k-1)+k(1-F_{\alpha}(k))]$ is the expected number of trades and c is the cost function. The profit of sellers with capacity l is

$$\pi(l)=r\left[\beta F_{\beta}(l-1)+l(1-F_{\beta}(l))\right]-c(l)=l\left(1-F_{\beta}(l)\right)-c(l). \tag{15}$$

To derive the buyers' indifference condition, that determines the queue lengths α and β, first note that the buyers' utility if they manage to buy a good for price q is

$$(1-q) = \frac{k(1-F_\alpha(k)) + \alpha F_\alpha(k-1) - k(1-F_\alpha(k))}{k(1-F_\alpha(k)) + \alpha F_\alpha(k-1)}$$
$$= \frac{\alpha F_\alpha(k-1)}{k(1-F_\alpha(k)) + \alpha F_\alpha(k-1)}. \tag{16}$$

Thus the expected utility for the buyers visiting a seller with capacity-price pair (k, q) is

$$\frac{\alpha F_\alpha(k-1)}{k(1-F_\alpha(k)) + \alpha F_\alpha(k-1)} \cdot \left[F_\alpha(k-1) + \frac{k}{\alpha}(1-F_\alpha(k))\right] = F_\alpha(k-1). \tag{17}$$

Similarly the expected utility from visiting a firm with capacity-price pair (l, r) is

$$\frac{\beta F_\beta(l-1)}{l(1-F_\beta(l)) + \beta F_\beta(l-1)} \cdot \left[F_\beta(l-1) + \frac{l}{\beta}(1-F_\beta(l))\right] = F_\beta(l-1) \tag{18}$$

The buyers' indifference condition is thus

$$F_\alpha\,(k-1) = F_\beta\,(l-1)\,. \tag{19}$$

4 Capacity

We find the equilibrium capacity under free entry. First we, however, show that there cannot exist an equilibrium (including a free entry equilibrium) with linear costs.

Assume that sellers have capacity k in a prospective equilibrium. Their profit is then given by $\pi(k) = k\,(1 - F_\theta(k)) - c(k)$. Then let a proportion s of sellers deviate from this to capacity l. Proportion w of the buyers go to the deviators and proportion $(1-w)$ go to the non-deviators.

The deviators' profit is given by

$$\pi(l) = l\left(1 - F_\beta(l)\right) - c(l),$$

Notice that the buyers indifference condition between visiting sellers of the different capacities

$$F_\alpha\,(k-1) = F_\beta\,(l-1)$$

gives us the relationship between β, the expected queue length the deviators face, and α, the expected queue length the non-deviators face.

Assume that $l > k$. As

$$\partial \frac{F_{\gamma}\ (k-1)}{\partial \gamma} = -\frac{\gamma^{\ k} e^{-\gamma}}{k!}\ < 0$$

β has to be bigger than θ for the deviation to be profitable and to satisfy the buyers indifference condition. As $\beta = \frac{w}{s}\theta$ this implies that $w > s$. i.e. that $\frac{w}{s}\theta > \theta > \frac{1-w}{1-s}\theta$. Letting the proportion of deviating sellers s tend towards zero, w still has to be such that the indifference condition is satisfied, $\frac{1-w}{1-z}$ however tends to one which means that $\alpha \to \theta$. This means that at the limit the indifference condition will be $F_{\theta}\ (k-1) = F_{\beta}\ (l-1)$. As θ is known we can solve the indifference condition for β.

A non-existence result with a linear cost function One easily sees that in order for there to be an equilibrium the marginal cost of capacity must be increasing. With constant marginal cost there exists a profitable deviation to a higher quantity l from any candidate equilibrium (with capacity k). The deviation is profitable whenever

$$k\,(1 - F_{\theta}(k)) - ck\ < l\left(1 - F_{\beta}(l)\right) - cl. \tag{20}$$

Dividing both sides of the inequality with k we get $(1 - F_{\theta}(k)) - c\ < \frac{l\left(1-F_{\beta}(l)\right)-cl}{k}$. But as $0\ < k\ < l$ and $k\,(1 - F_{\theta}(k)) - ck \geq 0$ we know that if the inequality holds for $(1 - F_{\alpha}(k)) - c\ < \left(1 - F_{\beta}(l)\right) - c$ it certainly holds for (21). Thus we can look at per unit cost (marginal cost), which is a stricter condition. We can show that the following inequality, implying Proposition 3, always holds:

$$F_{\theta}\ (k) > F_{\beta}\ (l)\,. \tag{21}$$

Proposition 3 *There is no equilibrium with constant per unit cost.*[3] [4]

Proof. The buyers' indifference condition

$$F_{\theta}\ (k-1) = F_{\beta}\ (l-1) \tag{22}$$

can be written with the help of upper incomplete gamma functions as

$$Q(k,\theta) = Q(l,\beta),$$

[3]For the trivial case of a per unit cost of $c \geq 1$ it is obvious that producing $k = 0$ is an equilibrium.

[4]Note that the non-excistence result holds for any potential equilibrium with linear costs including the free entry equilibrium.

where

$$Q(k,\theta)=\frac{\Gamma\left(k,\theta\right)}{\Gamma\left(k\right)}=\frac{1}{(k-1)!}\int_{\theta}^{\infty}y^{k-1}e^{-y}dy=\sum_{i=0}^{k-1}e^{-\theta}\frac{\theta^{i}}{i!}=F_{\theta}\left(k-1\right), \tag{23}$$

and

$$Q(l,\beta)=\frac{\Gamma\left(l,\beta\right)}{\Gamma\left(l\right)}=\frac{1}{(l-1)!}\int_{\beta}^{\infty}y^{l-1}e^{-y}dy=\sum_{i=0}^{l-1}e^{-\beta}\frac{\beta^{i}}{i!}=F_{\beta}\left(l-1\right).$$

As both k and θ are known we can treat the probability $Q(k,\theta)$ as a constant $t\in(0,1)$.Thus

$$Q(k,\theta)=Q(l,\beta)=t. \tag{24}$$

Now the inverted-regularized incomplete gamma function $Q^{-1}(l,t)$ is the solution in β to $Q(l,\beta)=t$. Next let's define

$$C_{c}(l,t)=Q\left(l+d,Q^{-1}(l,t)\right), \tag{25}$$

where $d>0$. In the current setting $d=1$. Furman and Zitikis (2008) show that the function $C_c(l,t)$ is decreasing in l. To see that this implies that (22) holds when β is the solution to (23) we first note that for $l=k$ we get $Q^{-1}(l,t)=\theta$. Thus

$$C_{c}(k,t)=Q\left(k+1,\theta\right)=F_{\theta}\left(k\right). \tag{26}$$

Similarly $l=k+1$ gives us $Q^{-1}(k+1,t)=\beta$ and we have

$$C_{c}(k+1,t)=Q\left(k+2,\beta\right)=F_{\beta}\left(k+1\right). \tag{27}$$

$$C_{c}(k+1,t)=Q\left(k+2,\beta\right)=F_{\beta}\left(k+1\right). \tag{28}$$

Then by $C_c(l,t)$ being decreasing in l it follows that $F_{\theta}(k)>F_{\beta}(k+1)$ implying that the inequality (22) always holds. This means that given any potential equilibrium[5] in capacities there always exists a profitable deviation. Thus there cannot exist an equilibrium given linear costs. ■

To interpret the nonexistence result first note that an economy with the same number of buyers and goods behaves very differently depending on how many sellers are active in the economy. If there were only one large seller the market would be frictionless. A deviation to a larger capacity by a firm

[5]It is easy to see that the result also holds for deviations from mixed equilibria in capacities as long as the supports are finite.

increases the amount of goods in the economy without affecting the number of active sellers. Thus there are less frictions in the market than before. Buyers are willing to pay a higher price at locations with higher capacity in addition to increasing their visiting probability. This more than offsets the linear cost increase.

A few words of caution regarding the interpretation of the result are appropriate here. Note that the nonexistence result regarding the symmetric equilibrium is due to there being an infinite number of buyers in the market. It is immediate that in finite markets there exists a equilibrium capacity with linear costs as there is no incentive for a seller to deviate and offer a capacity larger than the number of buyers in the market as this doesn't increase the probability of trade for buyers visiting him. When the number of buyers in the market is infinite this boundary is never reached. Thus, in order for us to have a symmetric equilibrium we must assume that costs are convex. This assumption can be seen as making the sellers set of feasible capacities finite.

When marginal cost is strictly increasing and without upper bound the set of profitable capacities is finite. The result is immediate[6] as the buyers valuation of a good is unity.

4.1 Equilibrium with convex costs

In this section we find equilibrium when the cost function is increasing and convex. First we define equilibrium in a standard way (see e.g. Lester(2010)). Then we show existence and describe an algorithm that finds equilibrium. After this we discuss efficiency properties of equilibrium.

To define a free entry equilibrium we first define

a) Buyers' equilibrium choice.

Given the distribution of different capacity-price pairs $F(\kappa^j, p^j)$ the buyers adjust their behavior in equilibrium so that they are indifferent between the different capacity-price pairs and receive the market utility M from from each type of seller when the ratio of buyers to sellers in the whole economy

[6] For an illustration assume e.g. linear marginal cost ax, with $0 < a \leq 1$. Unit costs would then be above one from the $k^{th}_{\max}$ unit onwards with $k_{\max} = \left\lfloor \frac{1}{a} \right\rfloor$, where $\lfloor x \rfloor$ is the floor function and denotes the largest integer below x. Furthermore average cost per unit would be above one from $k_{2\max} = \left\lfloor \frac{2}{a} \right\rfloor$ onwards. As no buyer is willing to pay a price higher than one it is clear that a seller would make a negative profit by producing more than $k_{\max}$. We can thus restrict the choice set of the sellers in the capacity subgame to integers from zero to $k_{\max}$. The set of possible equilibrium capacities is thus finite.

is θ.This adjustment of behavior leads to different expected queue lengths $\beta_{\kappa,p}(\theta, M)$ to the different types of sellers such that the expected utility for all buyers is M and $\int \beta_{\kappa,p}(\theta, M)dF(\kappa^j, p^j) = \theta$.

b) A seller's equilibrium price $p^*(\kappa)$ given his capacity is following section 2.2 equal to

$$\arg\max_p \left[\beta_{\kappa,p}(\theta, M)F_{\beta_{\kappa,p}(\theta,M)}(k-1) + (1 - F_{\beta_{\kappa,p}(\theta)}(\kappa))\kappa\right].$$

c) A sellers equilibrium choice of capacity κ^* is, following section 2.4, equal to

$$\begin{aligned}\arg\max_\kappa \pi_\kappa(M) &= \arg\max_\kappa \left\{p^*(\kappa)\left[\beta_{\kappa,p}(\theta, M)\cdot F_{\beta_{\kappa,p(\theta,M)}}(\kappa-1) + \kappa(1 - F_{\beta_{\kappa,p(\theta,M)}}(\kappa))\right] - c(\kappa)\right\} \\ &= \arg\max_\kappa \left\{\kappa\left(1 - F_{\beta_{\kappa,p(\theta,M)}}(\kappa)\right) - c(\kappa)\right\}\end{aligned}$$

when the solution exists.

d) Free entry of sellers

The free entry condition implies that the measure of active sellers $\theta^{-1}$is such that the expected profit of sellers $\pi_\kappa(\theta, M)$ is zero.

Definition 4 *Let $\pi^*(M) = max\{\pi_\kappa(M)\}$. An free entry equilibrium is a distribution $F^*(\kappa, p)$ of capacities and prices across firms, a market utility M, queue lengths $\beta_{\kappa,p}(\theta, M)$ such that (i) $\pi^j_\kappa(p^j, M) = \pi^*_\kappa(M)$ for all (κ^j, p^j) such that $dF^*(\kappa^j, p^j) > 0$; (ii) $\pi^j_\kappa(p^j, M) \leq \pi^*_\kappa(M)$ for all (κ^j, p^j) such that $dF^*(\kappa^j, p^j) = 0$; (iii) M and $\beta_{\kappa,p}(\theta, M)$ constitute a symmetric equilibrium of the third stage subgame; and (iv) the expected profit of all sellers is zero $\pi_\kappa(\theta, M) = 0$.*

4.2 Planner's solution

In order to show the existence of equilibrium we first introduce a matching function and a measure of social welfare. When all sellers have the same capacity the number of matches in the economy $m(\theta, k)$, where the ratio of sellers to buyers is θ and each seller has k goods, is given by the probability that a buyer makes a trade times the number of buyers in the economy. This is

$$m(\theta, k) = F_\theta(k-1) + \frac{k}{\theta}(1 - F_\theta(k)) = [\theta F_\theta(k-1) + k(1 - F_\theta(k))]\theta^{-1} \tag{29}$$

We define pure restricted social welfare $S(\theta, k)$ as the total value of the matches minus the total cost of the capacity.

$$S(\theta,k) = m(\theta,k) - \theta^{-1}c(k) = \left[\theta F_{\theta}(k-1) + k\left(1 - F_{\theta}\left(k\right)\right)\right]\theta^{-1} - c(k)\theta^{-1} \tag{30}$$

The social planner maximizes $S(\theta,k)$. The planner chooses θ^{-1} as well as the distribution of capacities over the sellers. We show in the appendix that the planner always has a solution where all sellers have identical capacities. Therefore the planner maximizes

$$max_{\theta,k}S(\theta,k).$$

The FOC with respect to θ is

$$\left[c(k) - k\left(1 - F_{\theta}\left(k\right)\right)\right]\theta^{-2} = 0.$$

It is zero when

$$c(k) - k\left(1 - F_{\theta}\left(k\right)\right) = 0. \tag{31}$$

This is precisely when the zero profit condition of free entry holds.

Note that for different values of k the θs making the condition true are different. Thus, to avoid confusion, we henceforth let θ_k denote the value of θ making the zero profit condition true when all active firms produce k goods.

The social planner's maximization problem can be expressed as

$$max_k S(\theta_k,k) = max_k\left[\theta_k F_{\theta_k}(k-1) + k\left(1 - F_{\theta_k}\left(k\right)\right)\right]\theta_k^{-1} - c(k)\theta_k^{-1} \tag{32}$$

Given the first order conditions () this is equivalent to

$$max_k\left[F_{\theta_k}(k-1)\right]. \tag{33}$$

where θ_k is given by the first order condition.

Proposition 5 *The pair (k^*,θ_{k^*}) is an equilibrium iff it is the planners solution.*

Proof. The proof can be found in the appendix ■

The algorithm for finding the free entry equilibrium is the following. Assume all sellers produce $k=0$ goods. If there is a profitable deviation to capacity $k=1$ look at a situation where all choose capacity $k=1$. The free entry condition implies that the amount of active sellers θ^{-1},will adjust until

$$k\left(1 - F_{\theta}(k)\right) - c(k) = 0. \tag{34}$$

This is however not necessarily an equilibrium as there might exist a profitable deviation to capacity $k+1$ for any seller

$$(k+1)\left(1-F_{\beta}(k+1)\right)-c(k+1) > (k)\left(1-F_{\theta}(k)\right)-c(k), \tag{35}$$

where, as before, β is given by the buyers indifference condition.

If the deviation is profitable consider next $k+1$ as a potential equilibrium and again let supply be determined by the free entry condition

$$(k+1)\left(1-F_{\theta'}(k+1)\right)-c(k+1)=0.$$

This process is repeated until one finds a k^* such that

$$\left(k^*+1\right)\left(1-F_{\beta}(k^*+1)\right)-c(k^*+1)\leq k^*\left(1-F_{\theta}(k^*)\right)-c(k^*) \tag{36}$$

Analogously, if

$$(k-1)\left(1-F_{\alpha}(k-1)\right)-c(k-1)\geq(k)\left(1-F_{\theta}(k)\right)-c(k)$$

try $k-1$ as a potential equilibrium again letting supply be determined by the free entry condition and repeat the procedure until one finds findsk_* such that

$$(k_*-1)\left(1-F_{\alpha}(k_*-1)\right)-c(k_*-1) < (k_*)\left(1-F_{\theta}(k_*)\right)-c(k_*) \tag{37}$$

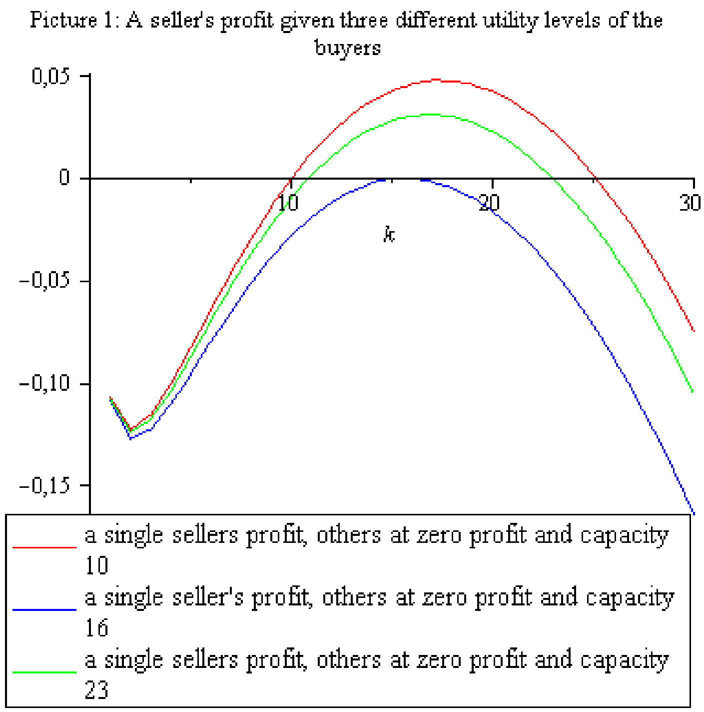
Under free entry () and () must both be satisfied in equilibrium. This is, however, not yet enough as there might be larger deviations from the proposed equilibrium.

Proposition 5 implies that if a capacity k satisfying free entry is not the planners solution then there exists a profitable deviation from it for some seller. Assume that there is a profitable deviation from k to $k+d$. Then, following the steps above, there exists pair $(k+d,\theta_{k+d})$, satisfying the free entry condition, for which $F_{\theta_k}(k-1) < F_{\theta_{k+d}}(k+d-1)$. Thus capacity $(k+d,\theta_{k+d})$ is the new candidate free entry equilibrium. The algorithm from this point onwards is a repetition of the previous steps. To see that the algorithm eventually comes to a halt note again that unit costs $\frac{c(x)}{x}$ are increasing without bound driving the average costs to above unity when capacity is above some $\widehat{k}$. Thus there are a finite number of possible capacities satisfying the free entry condition. The algorithm is thus finite and comes to an end at k^*.

We have show that the that the free entry equilibrium always exists as the set of feasible capacities is finite. In addition the free entry equilibrium is efficient as it is the solution in k to $max_k \left[F_{\theta_k}(k-1)\right]$. where $F_{\theta_k}(k-1)$ is the buyers expected utility from visiting a seller when all sellers have capacity k and θ_k is derived from the free entry condition. The maximum can always be found as there are only a finite number of capacities leading to non-negative profits even if a seller could sell all his goods.´

Picture 1 and picture 2 illustrate the profits from deviations to a seller when all other sellers have capacity 10, 16 and 23 when the cost function is $0.2k^{1.15}$. (Recall that the measure of sellers has adjusted to satisfy the zero profit condition for these capacities .) Also note that all points on each of the profit curves satisfy the buyers' indifference condition. The curves can thus be seen as profit curves for a seller keeping the buyers expected utility fixed (see picture 2). One quickly sees that there are profitable deviations towards capacity $k = 16$ from both 10 and 23. In fact the only capacity satisfying the sellers' zero profit condition from which there is no profitable deviation is the free entry equilibrium capacity $k = 16$ as the algorithm and Proposition (5) suggest.

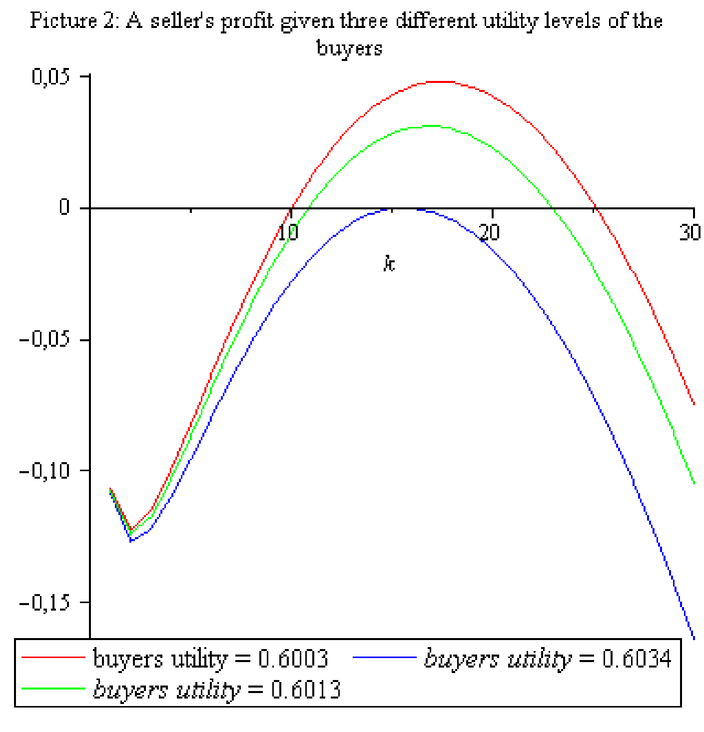
1



Picture 1: A seller's profit given three different utility levels of the buyers
0,05
0
−0,05
−0,10
−0,15
10
20
30
k
a single sellers profit, others at zero profit and capacity 10
a single seller's profit, others at zero profit and capacity 16
a single sellers profit, others at zero profit and capacity 23

1.pdf

2



Picture 2: A seller's profit given three different utility levels of the buyers
0,05
0
−0,05
−0,10
−0,15
10
20
30
k
buyers utility = 0.6003
buyers utility = 0.6034
buyers utility = 0.6013

2.pdf

5 Cost, endogenous capacity and welfare

Directed search is often motivated as a way to model an economy with frictions that arise because buyers cannot coordinate their actions. An interesting question is how much the inefficiencies are alleviated when we let the sellers choose their capacity compared to the standard directed search model where all sellers are assumed to have only one good. We use examples to shed some light on the issue.

Example 6 *Assume that the cost function is* $0.2k^{1.25}$. *Then by Proposition 5 the free entry capacity is given by*

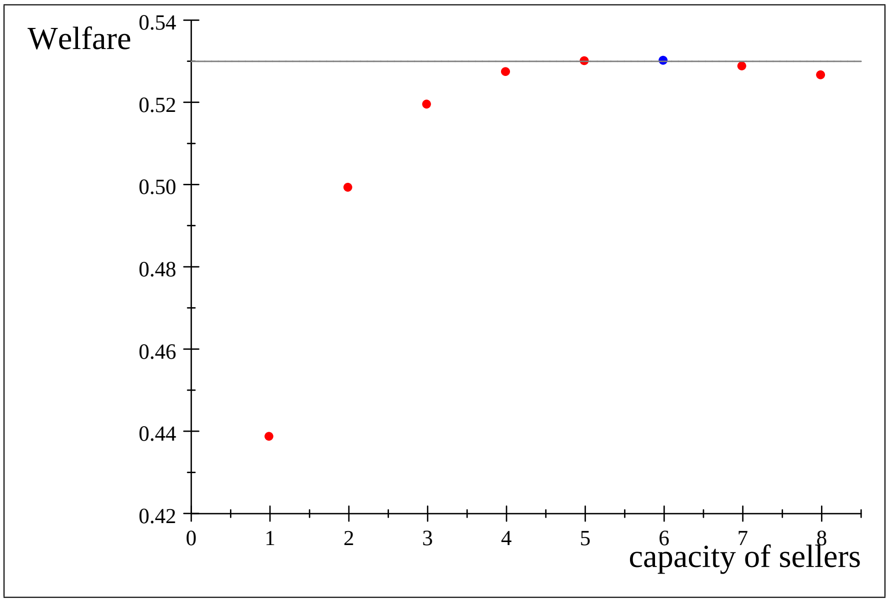
$$\arg\max_k \left[F_{\theta_k}(k-1)\right]$$

where θ_k *satisfies the free entry condition*

$$k\left(1 - F_{\theta_k}(k)\right) - c(k) = 0$$

We find (with the help of Maple) that the free entry equilibrium capacity is $k = 6$ *and the measure of active sellers is* $\theta_6^{-1} = 0.182$ *The expected utility of the buyers i.e. the welfare is* 0.53 *and the price of the good is* 0.39. *A free entry model where the sellers' capacity is restricted to one i.e.* $k = 1$ *gives* $\theta_1^{-1} = 1.213$. *The expected utility of the buyers, i.e.,. the welfare is* 0.44 *and the price is* 0.36.*The welfare is* 20.8% *higher and the price is* 10.1% *higher with endogenous capacity choice than the welfare when capacity is restricted to one.*

The illustration below shows the buyers expected utility for different capacities of the sellers satisfying the free entry condition. The free entry equilibrium capacity$k = 6$ is shown as a blue dot.



Planners choice

Example 7 *With cost function* $0.2k^{1.15}$ *the free entry equilibrium capacity is* $k = 16$, *the measure of sellers* $\theta_{16}^{-1} = 6.8237 \times 10^{-2}$. *The price is* 0.35 *and welfare is* 0.60. *When capacity is restricted to one the measure of sellers and price is as in the earlier example.*

The welfare with endogenous capacity choice is now 37.6% *higher than the welfare when capacity is restricted to one. The price under endogenous capacity is* 0.66% *of the price with capacity restricted to one. (This decrease in price is due to there being fewer sellers than buyers under free entry with capacity one.).*

Example 8 *With cost function* $0.2k^{1.05}$ *the free entry equilibrium capacity is* $k = 162$ *and the measure of active sellers is 0.00647. Welfare is* 0.716 *which is almost* 73% *higher then when capacity is restricted to one.*

It might be interesting to compare the welfare in the examples to the welfare without frictions. With the above cost function there would be an equal number of buyers and sellers and the welfare would be 0.8.[7]

The examples demonstrate that allowing firms to adjust their capacity can significantly alleviate the inefficiencies due to frictions compared to when sellers are of size one. The standard directed search

[7] In a Walrasian equilibrium the price would be 0.2.

model thus exaggerates the costs that are due to frictions. The less convex the cost function is, the higher is equilibrium capacity and welfare.

6 Extension: Equivalence of auctions and posted prices

It is known that when sellers have capacity one the equilibrium expected prices and expected utilities are equivalent when sellers post prices and when prices are determined in auctions where each seller auctions his good to the buyers who visit him (see e.g. Kultti (1999)). We next generalize this result and show that it holds also when sellers are free to choose any capacity.

Assume that we have auctions with quantities k and l and the proportions of buyers to sellers are as before. The buyers who chose to go to an auction with capacity k are randomly distributed on the auctions with capacity k. The buyers' expected utility from going to a capacity k auction (seller) is therefore

$$e^{-\alpha}\sum_{i=0}^{k-1}\frac{\alpha^i}{i!}\cdot 1+0=F_\alpha(k-1). \tag{38}$$

The expression captures the idea that if there are less than k buyers visiting a seller then the buyers will bid zero and thus receive the good at cost zero whereas if there is k or more other buyers then the buyers will compete the with each other and bid up the price to one. Likewise if the buyer goes to a capacity l seller his expected utility is

$$e^{-\beta}\sum_{i=0}^{l-1}\frac{\beta^i}{i!}\cdot 1+0=F_\beta(l-1). \tag{39}$$

Note that $\alpha=\frac{1-w}{1-s}\theta$ and $\beta=\frac{w}{s}\theta$ are determined by the buyers' indifference condition $F_\alpha\left(k-1\right)=F_\beta\left(l-1\right)$. The buyers' expected utility is thus the same when prices are determined in auctions as when prices are posted. This can be expressed as

$$(1-q)\left[F_\alpha(k-1)+\frac{k}{\alpha}(1-F_\alpha(k))\right]=F_\alpha(k-1), \tag{40}$$

from which we can solve

$$q=\frac{F_{\alpha}(k-1)+\frac{k}{\alpha}(1-F_{\alpha}(k))-F_{\alpha}(k-1)}{F_{\alpha}(k-1)+\frac{k}{\alpha}(1-F_{\alpha}(k))}.$$

We thus get

$$q=\frac{k\left(1-F_{\alpha}\left(k\right)\right)}{k\left(1-F_{\alpha}\left(k\right)\right)+\alpha F_{\alpha}(k-1)} \tag{41}$$

and similarly

$$r=\frac{l\left(1-F_{\beta}\left(l\right)\right)}{l\left(1-F_{\beta}\left(l\right)\right)+\beta F_{\beta}(l-1)} \tag{42}$$

Proposition 9 *Posted prices in equilibrium are equivalent to expected prices when the prices are determined by auctions at the different locations.*

Proof. The proof above is by construction. ■

7 Conclusion

We endogenize capacity choice of sellers in a directed search type model by allowing free entry by the sellers. The free entry equilibrium is constrained efficient, a social planner can not improve upon it. We find that the welfareloss due to frictions becomes surprisingly much less severe when firms are free to chose their own capacity compared to the standard models. We also show the equivalence of expected profits as well as of expected utilities between a market where firms post prices and a market where prices are determined in auctions at the sellers locations. In the still incomplete extension we show that equilibrium exists also with a fixed measure of sellers in the economy. The equilibrium capacities and prices are constrained efficient and price dispersion occurs in equilibrium.

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8 Appendix

A1 Proof that there are noprofitable deviations from the equilibrium price q in proposition 1

Proof. Price q gives expected revenue

$$q\left[\theta F_\theta(k-1)+k(1-F_\theta(k))\right]=k\left(1-F_\theta(k)\right) \tag{43}$$

deviating to some $\overline{q}$ gives

$$\overline{q}\left[\gamma\, F_\gamma\,(k-1)+k(1-F_\gamma\,(k))\right]$$

γ is derived from the buyers indifference condition

$$(1-q)\left[F_\theta(k-1)+\frac{k}{\theta}\left(1-F_\theta(k)\right)\right]=(1-\overline{q})\left[F_\gamma(k-1)+\frac{k}{\gamma}\left(1-F_\gamma(k)\right)\right]$$

Rewriting we get

$$\frac{\theta F_\theta(k-1)}{k\left(1-F_\theta(k)\right)+\theta F_\theta(k-1)}\left[F_\theta(k-1)+\frac{k}{\theta}\left(1-F_\theta(k)\right)\right]=(1-\overline{q})\left[F_\gamma(k-1)+\frac{k}{\gamma}\left(1-F_\gamma(k)\right)\right]$$

$$F_\theta(k-1)=(1-\overline{q})\left[F_\gamma(k-1)+\frac{k}{\gamma}\left(1-F_\gamma(k)\right)\right]$$

Thus

$$\overline{q}=\frac{F_\gamma(k-1)+\frac{k}{\gamma}\left(1-F_\gamma(k)\right)-F_\theta(k-1)}{F_\gamma(k-1)+\frac{k}{\gamma}\left(1-F_\gamma(k)\right)}$$

A deviator expects to get

$$\begin{aligned}&\frac{F_\gamma(k-1)+\frac{k}{\gamma}\left(1-F_\gamma(k)\right)-F_\theta(k-1)}{F_\gamma(k-1)+\frac{k}{\gamma}\left(1-F_\gamma(k)\right)}\left[\gamma F_\gamma(k-1)+k(1-F_\gamma(k))\right]\\&=\gamma F_\gamma(k-1)+k\left(1-F_\gamma(k)\right)-\gamma F_\theta(k-1)\end{aligned}\tag{44}$$

for q to be an equilibrium we need

$$k\left(1-F_\theta(k)\right)\geq\gamma F_\gamma(k-1)+k\left(1-F_\gamma(k)\right)-\gamma F_\theta(k-1)\tag{45}$$

Differentiating the RHS with respect to γ

$$\begin{aligned}&\frac{d}{d\gamma}\left(\gamma F_\gamma(k-1)+k\left(1-F_\gamma(k)\right)-\gamma F_\theta(k-1)\right)\\&=\frac{d}{d\gamma}\left(\sum_1^k e^{-\gamma}\frac{\gamma^i}{i!}i+\sum_{k+1}^\infty e^{-\gamma}\frac{\gamma^i}{i!}k-\sum_1^k e^{-\theta}\frac{\theta^{i-1}}{i!}i\gamma\right)\\&=-e^{-\gamma}\sum_1^k\frac{\gamma^i}{i!}i-e^{-\gamma}\sum_{k+1}^\infty\frac{\gamma^i}{i!}k+e^{-\gamma}\sum_1^k i\frac{\gamma^{i-1}}{i!}i+e^{-\gamma}\sum_{k+1}^\infty i\frac{\gamma^{i-1}}{i!}k-e^{-\theta}\sum_1^k\frac{\theta^{i-1}}{i!}i\\&=F_\gamma(k-1)-F_\theta(k-1)\end{aligned}\tag{46}$$

This is zero when $\gamma=\theta$

The second order condition

$$\frac{d}{d\gamma}\left(F_{\gamma}\left(k-1\right)-F_{\theta}\left(k-1\right)\right)=-e^{-\gamma}\sum_{0}^{k-1}\frac{\gamma^{\,i}}{i!}+e^{-\gamma}\sum_{0}^{k-1}\frac{\gamma^{\,i-1}}{(i-1)!}$$
$$=-e^{-\gamma}\frac{\gamma^{\,k-1}}{(k-1)!}<0 \tag{47}$$

is negative meaning that $\gamma=\theta$ is a global maximum. Thus there is no profitable deviation from

$$q=\frac{k\left(1-F_{\theta}(k)\right)}{k\left(1-F_{\theta}(k)\right)+\theta F_{\theta}\left(k-1\right)}$$

and q is the equilibrium price. ■

A2 We show the necessary conditions for Proposition 2.

Proof. Again in a prospective equilibrium the buyers should be indifferent as to which type of seller they contact. Sellers with price-quantity pair (r,l) face expected queue length $\beta=\frac{w}{s}\theta$, and sellers with price-quantity pair (q,k) face expected queue length $\alpha=\frac{1-w}{1-s}\theta$. The indifference condition of the buyers is thus as follows:

$$(1-q)\left[F_{\alpha}(k-1)+\frac{k}{\alpha}\left(1-F_{\alpha}(k)\right)\right]$$
$$=(1-r)\left[F_{\beta}(l-1)+\frac{l}{\beta}\left(1-F_{\beta}(l)\right)\right]. \tag{48}$$

From this expression we can solve for s, the proportion of buyers going to sellers with price quantity pair (r,l).Next we show that there are no profitable deviations (by a small number of sellers) in price from this prospective equilibrium. We do this by first showing that there are no profitable deviations for sellers with capacity k and then showing the same for sellers with capacity l.

To look at the former, assume, that proportion ϵ of sellers with capacity k deviate and produces $\widetilde{q}$. Now buyers have to be indifferent between the three kinds of sellers. We thus have three groups of buyers; those who go to sellers with price-quantity pair (l,r), those with price-quantity pair (k,q) and those who go to sellers with price-quantity pair $(k,\widetilde{q})$. We call these groups of buyers w_1, w_2, and w_3,with $w_1+w_2+w_3=1$.

We get the following expected queue lengths determining the meetings of buyers and sellers for the meeting of buyers and sellers in the different submarkets:.$\alpha_1=\frac{w_1}{s}\theta$, $\beta_1=\frac{w_2}{(1-s)(1-\epsilon)}\theta$, and $\gamma_1=\frac{w_3}{(1-s)\epsilon}\theta$.

Buyer's utility when he goes to (q, k) is

$$e^{-\beta_1}\sum_{i=0}^{k-1}\frac{\beta_1^i}{i!}(1-q)+e^{-\beta_1}\sum_{i=k}^{\infty}\frac{\beta_1^i}{i!}\frac{k}{i+1}(1-q)=(1-q)\left[F_{\beta_1}(k-1)+k\frac{1}{\beta_1}\left(1-F_{\beta_1}(k)\right)\right] \quad (49)$$

In equilibrium

$$N(w_2, w_3,\widetilde{q})=(1-q)\left[F_{\beta_1}(k-1)+\frac{k}{\beta_1}\left(1-F_{\beta_1}(k)\right)\right]-(1-\widetilde{q})\left[F_{\gamma_1}(k-1)+\frac{k}{\gamma_1}\left(1-F_{\gamma_1}(k)\right)\right]=0 \quad (50)$$

and

$$M(w_2, w_3,\widetilde{q})=(1-r)\left[F_{\alpha_1}(l-1)+\frac{l}{\alpha_1}\left(1-F_{\alpha_1}(l)\right)\right]-(1-\widetilde{q})\left[F_{\gamma_1}(k-1)+\frac{k}{\gamma_1}\left(1-F_{\gamma_1}(k)\right)\right]=0, \quad (51)$$

The firms deviating from (k, q) to $(k, \widetilde{q})$ maximize

$$\max_{\widetilde{q}}\widetilde{q}\left[\sum_{1}^{k}e^{-\gamma_1}\frac{\gamma_1^i}{i!}i+\sum_{k+1}^{\infty}e^{-\gamma_1}\frac{\gamma_1^i}{i!}k\right], \quad (52)$$

we get first order condition

$$\begin{aligned}&\left[\sum_{1}^{k}e^{-\gamma_1}\frac{\gamma_1^i}{i!}i+\sum_{k+1}^{\infty}e^{-\gamma_1}\frac{\gamma_1^i}{i!}k\right]\\&+\widetilde{q}\left[-e^{-\gamma_1}\sum_{1}^{k}\frac{\gamma_1^i}{i!}i-e^{-\gamma_1}\sum_{k+1}^{\infty}\frac{\gamma_1^i}{i!}k+e^{-\gamma_1}\sum_{1}^{k}i\frac{\gamma_1^{i-1}}{i!}i+e^{-\gamma_1}\sum_{k+1}^{\infty}i\frac{\gamma_1^{i-1}}{i!}k\right]\frac{\theta}{(1-s)\epsilon}\frac{dw_3}{d\widetilde{q}}\\&=0\end{aligned} \quad (53)$$

This simplifies to

$$\gamma_1F_{\gamma_1}(k-1)+(1-F_{\gamma_1}(k))k+\widetilde{q}F_{\gamma_1}(k-1)\frac{\theta}{(1-s)\epsilon}\frac{dw_3}{d\widetilde{q}}=0 \quad (54)$$

In order to solve this we first need to derive the expression for $\frac{1}{\epsilon}\frac{dw_3}{d\widetilde{q}}$. We get it by totally differentiating the indifference conditions $M(w_2, w_3, \widetilde{q})$ and $N(w_2, w_3, \widetilde{q})$ with respect to $\widetilde{q}$, w_2 and w_3. We then show that there are no deviations from price-quantity pair (k, q) by a small number of sellers and derive $q(k)$. Note that $\frac{d\alpha_1}{dw_3}=-\frac{\theta}{s}$and $\frac{d\beta_1}{dw_3}=-\frac{\theta}{(1-s)(1-\epsilon)}$.and $\frac{d\gamma_1}{dw_3}=\frac{\theta}{(1-s)\epsilon}$ and $w_1=1-w_2-w_3$.

Also note that

$$\frac{dF_{\beta_1}}{dw_2}(k) = \frac{d}{d\beta_1}\sum_{i=0}^{k}\frac{\beta_1^i}{i!} = -F_{\beta_1}(k) + e^{-\beta_1}\sum_{i=1}^{k}\frac{\beta_1^{i-1}}{(i-1)!} = -F_{\beta_1}(k) + e^{-\beta_1}\sum_{i=0}^{k-1}\frac{\beta_1^i}{i!} = -e^{-\beta_1}\frac{\beta_1^k}{k!}$$

Totally differentiating $M(w_2, w_3, \widetilde{q})$ and $N(w_2, w_3, \widetilde{q})$ we get

$$M_{w_2}dw_2 + M_{w_3}dw_3 + M_{\widetilde{q}}d\widetilde{q} = 0 \tag{55}$$

$$N_{w_2}dw_2 + N_{w_3}dw_3 + N_{\widetilde{q}}d\widetilde{q} = 0 \tag{56}$$

By using Cramer's rule we get

$$\frac{dw_3}{d\widetilde{q}} = -\frac{\begin{vmatrix} M_{w_2} & M_{\widetilde{q}} \\ N_{w_2} & N_{\widetilde{q}} \end{vmatrix}}{\begin{vmatrix} M_{w_2} & M_{w_3} \\ N_{w_2} & N_{w_3} \end{vmatrix}} \tag{57}$$

where

$$\begin{aligned} M_{w_2} &= (1-r)\left[-e^{-\alpha_1}\frac{\alpha_1^{l-1}}{(l-1)!} - \frac{l}{\alpha_1^2}\left(1-F_{\alpha_1}(l)\right) + \frac{l}{\alpha_1}e^{-\alpha_1}\frac{\alpha_1^l}{l!}\right]\left(-\frac{\theta}{s}\right) = \\ &= (1-r)\left\{-\frac{l}{\alpha_1^2}\left(1-F_{\alpha_1}(l)\right)\right\}\left(-\frac{\theta}{s}\right) \end{aligned} \tag{58}$$

$$M_{w_3} = M_{w_2} - (1-\widetilde{q})\left\{-\frac{k}{\gamma_1^2}\left(1-F_{\gamma_1}(k)\right)\right\}\left(\frac{\theta}{(1-s)\,\epsilon}\right) \tag{59}$$

$$M_{\widetilde{q}} = \left[F_{\gamma_1}(k-1) + \frac{k}{\gamma_1}\left(1-F_{\gamma_1}(k)\right)\right] \tag{60}$$

$$N_{w_2} = (1-q)\left\{-\frac{k}{\beta_1^2}\left(1-F_{\beta_1}(k)\right)\right\}\left(\frac{\theta}{(1-s)\,(1-\epsilon)}\right) \tag{61}$$

$$N_{w_3} = (1-\widetilde{q})\left\{-\frac{k}{\gamma_1^{\,2}}\left(1-F_{\gamma_1}(k)\right)\right\}\left(\frac{\theta}{(1-s)\,\epsilon}\right) \tag{62}$$

$$N_{\widetilde{q}} = M_{\widetilde{q}} \tag{63}$$

$$\begin{vmatrix} M_{w_2} & M_{w_3} \\ N_{w_2} & N_{w_3} \end{vmatrix} = M_{w_2}N_{w_3} - M_{w_3}N_{w_2}$$

and

$$\begin{vmatrix} M_{w_2} & M_{\widetilde{q}} \\ N_{w_2} & N_{\widetilde{q}} \end{vmatrix} = M_{\widetilde{q}}\left(M_{w_2} - N_{w_2}\right)$$

Therefore

$$\frac{1}{\epsilon}\frac{dw_3}{d\widetilde{q}} = -\frac{1}{\epsilon}\left\{\frac{\begin{array}{c}(1-r)\,\frac{l}{\alpha_1^2}\left(1-F_{\alpha_1}(l)\right)\left(\frac{\theta}{s}\right)\left[F_{\gamma_1}(k-1)+\frac{k}{\gamma_1}\left(1-F_{\gamma_1}(k)\right)\right] \\ +(1-q)\,\frac{k}{\beta_1^2}\left(1-F_{\beta_1}(k)\right)\left(\frac{\theta}{(1-s)(1-\epsilon)}\right)\left[F_{\gamma_1}(k-1)+\frac{k}{\gamma_1}\left(1-F_{\gamma_1}(k)\right)\right]\end{array}}{\begin{array}{c}(1-\widetilde{q})\left\{\frac{k}{\gamma_1^2}\left(1-F_{\gamma_1}(k)\right)\right\}\left(\frac{\theta}{(1-s)\epsilon}\right)\cdot(1-r)\,\frac{l}{\alpha_1^2}\left(1-F_{\alpha_1}(l)\right)\left(\frac{\theta}{s}\right) \\ +(1-q)\,\frac{k}{\beta_1^2}\left(1-F_{\beta_1}(k)\right)\left(\frac{\theta}{(1-s)(1-\epsilon)}\right)\cdot(1-r)\,\frac{l}{\alpha_1^2}\left(1-F_{\alpha_1}(l)\right)\left(\frac{\theta}{s}\right) \\ +(1-\widetilde{q})\left\{\frac{k}{\gamma_1^2}\left(1-F_{\gamma_1}(k)\right)\right\}\left(\frac{\theta}{(1-s)\epsilon}\right)\end{array}}\right\}$$

From this we get

$$\frac{1}{\epsilon}\frac{dw_3}{d\widetilde{q}} = \frac{\begin{array}{c}(1-r)\,\frac{l}{\alpha_1^2}\left(1-F_{\alpha_1}(l)\right)\left(\frac{\theta}{s}\right)\left[F_{\gamma_1}(k-1)+\frac{k}{\gamma_1}\left(1-F_{\gamma_1}(k)\right)\right] \\ +(1-q)\,\frac{k}{\beta_1^2}\left(1-F_{\beta_1}(k)\right)\left(\frac{\theta}{(1-s)(1-\epsilon)}\right)\left[F_{\gamma_1}(k-1)+\frac{k}{\gamma_1}\left(1-F_{\gamma_1}(k)\right)\right]\end{array}}{\begin{array}{c}(1-\widetilde{q})\left\{\frac{k}{\gamma_1^2}\left(1-F_{\gamma_1}(k)\right)\right\}\left(\frac{\theta}{(1-s)}\right)\cdot(1-r)\,\frac{l}{\alpha_1^2}\left(1-F_{\alpha_1}(l)\right)\left(\frac{\theta}{s}\right) \\ +(1-q)\,\frac{k}{\beta_1^2}\left(1-F_{\beta_1}(k)\right)\left(\frac{\theta}{(1-s)(1-\epsilon)}\right)\cdot(1-r)\,\frac{l}{\alpha_1^2}\left(1-F_{\alpha_1}(l)\right)\left(\frac{\theta*\epsilon}{s}\right)+(1-\widetilde{q})\left\{\frac{k}{\gamma_1^2}\left(1-F_{\gamma_1}(k)\right)\right\}\left(\frac{\theta}{(1-s)}\right)\end{array}} \tag{64}$$

Inserting this expression in the first order condition and simplifying the first order conditions in equilibrium and letting ϵ tend towards zero the first order conditions become

$$\alpha F_\alpha(k-1) + k(1-F_\alpha(k))$$

$$+ qF_\alpha(k-1)\frac{\theta}{(1-s)} \tag{65}$$

$$\cdot \frac{\left[(1-r)\frac{l}{\beta^2}\left(1-F_\beta(l)\right)\left(\frac{\theta}{s}\right)+(1-q)\frac{k}{\alpha^2}\left(1-F_\alpha(k)\right)\left(\frac{\theta}{(1-s)}\right)\right]\left[F_\alpha(k-1)+\frac{k}{\alpha}\left(1-F_\alpha(k)\right)\right]}{(1-q)\left\{\frac{k}{\alpha^2}\left(1-F_\alpha(k)\right)\right\}\left(\frac{\theta}{(1-s)}\right)\cdot\left[(1-r)\frac{l}{\beta^2}\left(1-F_\beta(l)\right)\left(\frac{\theta}{s}\right)+(1-q)\frac{k}{\alpha^2}\left(1-F_\alpha(k)\right)\left(\frac{\theta}{(1-s)}\right)\right]} \tag{66}$$

$$= 0$$

This can be simplified to

$$\begin{aligned}&[\alpha F_\alpha(k-1) + k(1-F_\alpha(k))](1-q)\frac{k}{\alpha^2}\left(1-F_\alpha(k)\right)\frac{\theta}{(1-s)}\\ &- qF_\alpha(k-1)\frac{\theta}{(1-s)}\left[F_\alpha(k-1)+\frac{k}{\alpha}\left(1-F_\alpha(k)\right)\right]\\ &= 0\end{aligned} \tag{67}$$

and further to

$$\alpha(1-q)\frac{k}{\alpha^2}\left(1-F_\alpha(k)\right)\frac{\theta}{(1-s)} - qF_\alpha(k-1)\frac{\theta}{(1-s)} = 0 \tag{68}$$

Using the fact that $\alpha\frac{1-s}{1-w} = \theta$ we get

$$\frac{k}{\alpha}\left(1-F_\alpha(k)\right)\frac{\alpha}{(1-w)} - q\frac{k}{\alpha}\left(1-F_\alpha(k)\right)\frac{\alpha}{(1-w)} - qF_\alpha(k-1)\frac{\alpha}{(1-w)} = 0 \tag{69}$$

which can be rewritten as

$$k\left(1-F_\alpha(k)\right) = q\left\{k\left(1-F_\alpha(k)\right)+\alpha F_\alpha(k-1)\right\} \tag{70}$$

We get the equilibrium price

$$q^E = \frac{k\left(1-F_\alpha(k)\right)}{\left\{k\left(1-F_\alpha(k)\right)+\alpha F_\alpha(k-1)\right\}} \tag{71}$$

Similarly we get

$$r^E = \frac{l\left(1 - F_\beta\left(l\right)\right)}{\left\{l\left(1 - F_\beta\left(l\right)\right) + \beta F_\beta(l-1)\right\}}$$

The second order conditions and equilibrium conditions are derived similarly as in Proposition 1. ■

A3 Proof of Proposition 5

Proof. Assume that the planner chooses capacity k^* and queue length, θ_{k^*} i.e. they are the solutions to

$$\arg\max_k \left[F_{\theta_k}(k-1)\right]$$

where θ_k satisfies the free entry condition

$$k\left(1 - F_{\theta_k}\left(k\right)\right) - c(k) = 0$$

We show by contradiction that there cannot exist a profitable deviation from k^* for a seller.

Let there exist a profitable deviation from k^* to quantity $k^* + d$. Then

$$\overbrace{\left(k^*\right)\left(1 - F_{\theta_{k^*}}(k^*)\right) - c(k^*)}^{=0} < \left(k^* + d\right)\left(1 - F_{\beta_{k^*,d}}(k^* + d)\right) - c(k^* + d). \tag{72}$$

The RHS of () is then also higher than the profit under the free entry condition for capacity $k^* + d$ i.e.

$$\overbrace{\left(k^* + d\right)\left(1 - F_{\theta_{k^*+d}}(k^* + d)\right) - c(k^* + d)}^{=0} < \left(k^* + d\right)\left(1 - F_{\beta_{k^*,d}}(k^* + d)\right) - c(k^* + d).$$

But as

$$\partial\frac{F_\gamma\ (k)}{\partial\gamma} = -\frac{\gamma^k e^{-\gamma}}{k!} < 0,$$

this means that

$$\beta_{k,d} > \theta_{k+d}.$$

Next we define $t_k = F_{\theta_{k^*}}(k^*-1)$ and $t_{k^*+d} = F_{\theta_{k^*+d}}(k^*+d-1)$ and note that $\beta_{k^*,d} = Q^{-1}(k^*+d, t_{k^*})$ and $\theta_{k^*+d} = Q^{-1}(k^* + d, t_{k^*+d})$. As

$$\frac{\partial Q^{-1}(k+d,t)}{\partial t} < 0$$

$\beta_{k,d} > \theta_{k+d}$ implies that

$$F_{\theta_k^*}(k^* - 1) < F_{\theta_{k^*+d}}(k^* + d - 1)$$

contradicting the claim that k^* is the solution to

$$\arg\max_k \left[F_{\theta_k}(k-1)\right].$$

■

A4 Sellers have identical capacities in planner's choice when the planner can choose θ and k

(proof still under construction)

The planner chooses θ^{-1} as well as the distribution of capacities over the sellers. We show that there always exists a planners solution such that all sellers have the same capacity.

Assume that a planners solution is such that there are two capacities, k and l. The planner maximizes the value of the matches minus the cost of capacity.

$$Max_s \left[\begin{array}{c} \left[F_\alpha(k-1) + \frac{k}{\alpha}\left(1 - F_\alpha\left(k\right)\right)\right](1-\omega) - \frac{(1-s)}{\theta}c(k) \\ + \left[F_\beta(l-1) + \frac{l}{\beta}\left(1 - F_\beta\left(l\right)\right)\right]\omega - \frac{s}{\theta}c(l) \end{array} \right] \tag{73}$$

s.t.

$$F_\alpha\left(k-1\right) = F_\beta\left(l-1\right)$$

where $\alpha = \frac{1-\omega}{1-s}\theta$ and $\beta = \frac{\omega}{s}\theta$.

We get the following first order condition with respect to s

$$\begin{aligned} &\frac{k}{\alpha^2}\left(1 - F_\alpha\left(k\right)\right)\left[\frac{-\theta}{1-s}\frac{d\omega}{ds} + \frac{(1-\omega)\theta}{(1-s)^2}\right](1-\omega) - \left[F_\alpha(k-1) + \frac{k}{\alpha}\left(1 - F_\alpha\left(k\right)\right)\right]\frac{d\omega}{ds} \\ &+ \theta^{-1}c(k) - \frac{l}{\beta^2}\left(1 - F_\beta\left(l\right)\right)\left[\frac{\theta}{s}\frac{d\omega}{ds} - \frac{\omega\theta}{s^2}\right]\omega - \theta^{-1}c(l) \\ &+ \left[F_\beta(l-1) + \frac{l}{\beta}\left(1 - F_\beta\left(l\right)\right)\right]\frac{d\omega}{ds} \end{aligned}$$

$$\begin{aligned} &= \frac{k}{\alpha}\left(1 - F_\alpha\left(k\right)\right)\frac{d\omega}{ds} - \frac{k}{\theta}\left(1 - F_\alpha\left(k\right)\right) - F_\alpha(k-1)\frac{d\omega}{ds} - \frac{k}{\alpha}\left(1 - F_\alpha\left(k\right)\right)\frac{d\omega}{ds} + \theta^{-1}c(k) \\ &- \frac{l}{\beta}\left(1 - F_\beta\left(l\right)\right)\frac{d\omega}{ds} + \frac{l}{\theta}\left(1 - F_\beta\left(l\right)\right) + F_\beta(l-1)\frac{d\omega}{ds} \\ &+ \frac{l}{\beta}\left(1 - F_\beta\left(l\right)\right)\frac{d\omega}{ds} - \theta^{-1}c(l) \end{aligned}$$

Utilizing $F_\alpha(k-1) = F_\beta(l-1)$ the $\frac{d\omega}{ds}$ terms cancel each other out the FOC simplifies to

$$= -\frac{k}{\theta}(1 - F_\alpha(k)) + \theta^{-1}c(k) + \frac{l}{\theta}\left(1 - F_\beta(l)\right) - \theta^{-1}c(l) = 0. \tag{74}$$

Thus

$$-k(1 - F_\alpha(k)) + c(k) + l\left(1 - F_\beta(l)\right) - c(l) = 0. \tag{75}$$

Now let the planner maximize

$$Max_\theta \left[\begin{array}{c} \left[F_\alpha(k-1) + \frac{k}{\alpha}(1 - F_\alpha(k))\right](1-\omega) - \frac{(1-s)}{\theta}c(k) \\ + \left[F_\beta(l-1) + \frac{l}{\beta}\left(1 - F_\beta(l)\right)\right]\omega - \frac{s}{\theta}c(l) \end{array} \right] \tag{76}$$

s.t.

$$F_\alpha(k-1) = F_\beta(l-1)$$

where $\alpha = \frac{1-\omega}{1-s}\theta$ and $\beta = \frac{\omega}{s}\theta$.

We get the following first order condition

$$\frac{(1-s)}{\theta^2}\left[-k(1 - F_\alpha(k)) + c(k)\right] + \frac{s}{\theta^2}\left[-l\left(1 - F_\beta(l)\right) + c(l)\right] = 0$$

Now inserting () we get

$$\frac{1}{\theta^2}\left[-k(1 - F_\alpha(k)) + c(k)\right] = 0 \tag{77}$$

or

$$\frac{1}{\theta^2}\left[-k\left(1 - F_\beta(l)\right) + c(l)\right] = 0.$$

One can see that this implies that $\alpha = \theta_k$ and $\beta = \theta_l$. But this means that a social planner can replicate any expected utility from having two capacities by a choice of θ and k. This result generalizes to include any planners solution over n capacities.

9 Supplementary material: Equilibrium with fixed θ

(Under construction)

When the measure of sellers θ^{-1} is fixed in the economy we find equilibrium in the following way. First, note that when the measure of sellers in the economy is too large compared to the free entry outcome, $\theta^{-1} > \theta_k^{-1}$, the sellers make negative profits. In this case there exists a profitable deviation to capacity zero for some seller. In equilibrium the sellers distribute themselves between capacity zero and the free entry capacity k so that exactly θ_k^{-1} (the free entry measure) of the sellers have capacity k and the rest have capacity zero. The sellers are indifferent between the capacities as their expected profit in both cases is zero. Note that there exists no profitable deviations for any seller to any capacity as the pair (θ_k^{-1}, k) is the free entry equilibrium.

When $\theta^{-1} < \theta_k^{-1}$ finding equilibrium is harder. In the next section we derive equilibrium and show that it is efficient.

Planners solution

In order to solve the game for an arbitrary θ^{-1} we first show that the planners solution always exists. Then we show that when the planners solution yields non-negative expected profits to the sellers it is also the equilibrium of the competitive game. We then conclude with a few examples.

Proposition 10 *The planners solution always exists*

With convex costs there exists a capacity $k^{\max}$ after which the costs per unit of capacity is higher than unity and even a sure trade leads to a net loss. The planner clearly has a profitable deviation from such a capacity for a seller to zero. Thus the planner maximizes by distributing sellers over a finite set of capacities. It is clear that a maximum exists.

Proposition 11 *When the planners solution gives sellers positive expected profits no seller has a profitable deviation from it and the planners solution is the competitive solution.*

Assume that the planners solution involves positive mass on m out of n possible capacities. This implies that a planner maximizing his utility over n capacities chooses a corner solution where he distributes zero mass on all other than the m capacities chosen. As the first order conditions of the planner and a single sellers indifference conditions coincide it is immediate that there can exist no deviation from a planners solution as long as it guarantees the sellers a non negative expected profit.

Next we show that if the planners solution is such that only capacities k and l are offered and sellers make a positive expected profit then no single seller has a profitable deviation from it.

If the planners solution is such that only two capacities k and l are offered the planners choice is the solution to the following maximization problem. The buyers move after the planner. Thus if the planner offers capacities k and l then the buyers indifference condition $F_{\alpha}(k-1) = F_{\beta}(l-1)$ determines α and β. The planner maximizes the value of the matches minus the cost of capacity.

$$Max_s \left[\begin{array}{c} \left[F_{\alpha}(k-1) + \frac{k}{\alpha}(1 - F_{\alpha}(k))\right](1-\omega) - \frac{(1-s)}{\theta}c(k) \\ + \left[F_{\beta}(l-1) + \frac{l}{\beta}\left(1 - F_{\beta}(l)\right)\right]\omega - \frac{s}{\theta}c(l) \end{array} \right] \tag{78}$$

s.t.

$$F_{\alpha}(k-1) = F_{\beta}(l-1)$$

where $\alpha = \frac{1-\omega}{1-s}\theta$ and $\beta = \frac{\omega}{s}\theta$.

We get the following first order condition with respect to s

$$\begin{aligned} &\frac{k}{\alpha^2}(1 - F_{\alpha}(k))\left[\frac{-\theta}{1-s}\frac{d\omega}{ds} + \frac{(1-\omega)\theta}{(1-s)^2}\right](1-\omega) - \left[F_{\alpha}(k-1) + \frac{k}{\alpha}(1 - F_{\alpha}(k))\right]\frac{d\omega}{ds} \\ &+ \theta^{-1}c(k) - \frac{l}{\beta^2}\left(1 - F_{\beta}(l)\right)\left[\frac{\theta}{s}\frac{d\omega}{ds} - \frac{\omega\theta}{s^2}\right]\omega - \theta^{-1}c(l) \\ &+ \left[F_{\beta}(l-1) + \frac{l}{\beta}\left(1 - F_{\beta}(l)\right)\right]\frac{d\omega}{ds} \end{aligned}$$

$$\begin{aligned} &= \frac{k}{\alpha}(1 - F_{\alpha}(k))\frac{d\omega}{ds} - \frac{k}{\theta}(1 - F_{\alpha}(k)) - F_{\alpha}(k-1)\frac{d\omega}{ds} - \frac{k}{\alpha}(1 - F_{\alpha}(k))\frac{d\omega}{ds} + \theta^{-1}c(k) \\ &- \frac{l}{\beta}\left(1 - F_{\beta}(l)\right)\frac{d\omega}{ds} + \frac{l}{\theta}\left(1 - F_{\beta}(l)\right) + F_{\beta}(l-1)\frac{d\omega}{ds} \\ &+ \frac{l}{\beta}\left(1 - F_{\beta}(l)\right)\frac{d\omega}{ds} - \theta^{-1}c(l) \end{aligned}$$

Utilizing $F_{\alpha}(k-1) = F_{\beta}(l-1)$ the $\frac{d\omega}{ds}$ terms cancel each other out the FOC simplifies to

$$= -\frac{k}{\theta}(1 - F_{\alpha}(k)) + \theta^{-1}c(k) + \frac{l}{\theta}\left(1 - F_{\beta}(l)\right) - \theta^{-1}c(l) = 0. \tag{79}$$

Thus

$$-k(1 - F_{\alpha}(k)) + c(k) + l\left(1 - F_{\beta}(l)\right) - c(l) = 0. \tag{80}$$

Note that $\pi(k) = k(1 - F_\alpha(k)) - c(k)$ is the expected profit of a seller with capacity k and $\pi(l) = l\left(1 - F_\beta(l)\right) - c(l)$ is the expected profit of a seller with capacity l. In an interior solution sellers are thus indifferent between offering capacity l and k.

We assume that the planners solution is such that in equilibrium there are only two capacities k and l. This implies that a planner maximizing his utility by choosing in which proportion sellers offer capacities k, l and (any) m chooses a corner solution where only capacities k and l are offered. We next show that at this cornersolution no single seller would like to deviate to capacity m. But this implies that there is a competitive equilibrium where only capacities k and l are offered.

A planner choosing the proportion of sellers with capacities k, l and m maximizes the number of matches in the economy minus the costs of producing these matches

$$Max_{s,r}\left[\begin{array}{c} \left[F_\alpha(k-1) + \frac{k}{\alpha}(1 - F_\alpha(k))\right]\psi - \frac{r}{\theta}c(k) \\ + \left[F_\beta(l-1) + \frac{l}{\beta}\left(1 - F_\beta(l)\right)\right]\omega - \frac{s}{\theta}c(l) \\ + \left[F_\gamma(m-1) + \frac{m}{\gamma}\left(1 - F_\gamma(m)\right)\right](1 - \psi - \omega) - \frac{1-r-s}{\theta}c(m) \end{array}\right] \tag{81}$$

s.t. $F_\alpha(k-1) = F_\gamma(m-1)$ and $F_\beta(l-1) = F_\gamma(m-1)$

where $\alpha = \frac{\psi}{r}\theta$ and $\beta = \frac{\omega}{s}\theta$ and $\gamma = \frac{1-\psi-\omega}{1-s-r}$

The foc with respect to r is

$$\begin{aligned} &-\frac{k}{\alpha^2}(1 - F_\alpha(k))\left[\frac{\theta}{r}\frac{d\psi}{dr} - \frac{\psi\theta}{r^2}\right]\psi - \left[F_\alpha(k-1) + \frac{k}{\alpha}(1 - F_\alpha(k))\right]\frac{d\psi}{dr} \\ &- \theta^{-1}c(k) - \frac{m}{\psi^2}\left(1 - F_\psi(m)\right)\left[-\frac{\theta}{(1-s-r)^2}\frac{d\psi}{dr} + \frac{1-\psi-\omega}{(1-s-r)^2}\right](1 - \psi - \omega) \\ &+ \left[F_\gamma(m-1) + \frac{m}{\gamma}\left(1 - F_\gamma(m)\right)\right]\frac{d\psi}{dr} - \theta^{-1}c(m) = 0 \end{aligned}$$

The first order conditions simplify to

$$\frac{k}{\theta}(1 - F_\alpha(k)) - \theta^{-1}c(k) - \frac{m}{\theta}\left(1 - F_\gamma(m)\right) + \theta^{-1}c(m) = 0 \tag{82}$$

and

$$\frac{l}{\theta}\left(1-F_{\beta}\left(l\right)\right)-\theta^{-1}c(l)-\frac{m}{\theta}\left(1-F_{\gamma}\ \left(m\right)\right)+\theta^{-1}c(m)=0.$$

But as we assumed only capacities k and l are offered in equilibrium it must be that when $1-\psi-\omega$ tends to zero

$$\frac{k}{\theta}\left(1-F_{\alpha}\left(k\right)\right)-\theta^{-1}c(k)-\frac{m}{\theta}\left(1-F_{\gamma}\ \left(m\right)\right)+\theta^{-1}c(m)\geq 0 \tag{83}$$

and

$$\frac{l}{\theta}\left(1-F_{\beta}\left(l\right)\right)-\theta^{-1}c(l)-\frac{m}{\theta}\left(1-F_{\gamma}\ \left(m\right)\right)+\theta^{-1}c(m)\geq 0. \tag{84}$$

But then no seller has a profitable deviation to a capacity m as () and () imply that both $\pi\left(k\right)\geq\pi\left(m\right)$ and $\pi\left(l\right)\geq\pi\left(m\right)$ hold. The claim follows.