STANDING FACILITIES VERSUS OPEN MARKET OPERATIONS: EQUIVALENCE RESULTS

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ABSTRACT. In this paper we compare two mechanisms for implementing monetary policy: standing facilities and open market operations. We show that any equilibrium allocation that can be attained with open market operations can be replicated with standing facilities, but that the converse is not true. Furthermore, the set of equilibrium allocations that can be attained with standing facilities is weakly dominated in terms of welfare by the set of equilibrium allocations that can be attained with open market operations. However, if the central bank chooses the optimal policy for each mechanism, the allocations are equivalent.

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1. Introduction

An increasing number of central banks are using channel systems or at least some features of the channel system for implementing monetary policy.[1] Despite its popularity, we know little about optimal policy in a channel system. For example, why do all central banks that operate standing facilities (SF) choose a strictly positive interest-rate spread? Why did they change their operating procedures in the first place.[2] What are the allocative implications if a central bank implements monetary policy via SF rather than via open market operations (OMO)?

The purpose of this paper is to shed some light on these questions. In particular, we want to investigate the allocative implications of implementing monetary policy via SF rather than via OMO in general equilibrium. For this purpose, we construct a dynamic general equilibrium model with microfoundations for the demand for liquid assets to compare the allocations under these two mechanisms.

The following key results emerge from our model. First, any equilibrium allocation that can be attained with OMO can be replicated with SF, but that the converse is not true. Second, the set of equilibrium allocations that can be attained with SF is

[1] In a channel system, a central bank offers two facilities: a lending facility whereby it is ready to supply money overnight at a given lending rate against collateral and a deposit facility whereby banks can make overnight deposits to earn a deposit rate. The interest-rate corridor is chosen to keep the overnight interest rate in the money market close to the target rate. In a pure channel system, a change in policy is implemented by simply changing the corridor without any open market operations.
Versions of a channel system are operated by the Bank of Canada, the Bank of England, the European Central Bank, the Reserve Bank of Australia, and the Reserve Bank of New Zealand. The US Federal Reserve System recently modified the operating procedures of its discount window facility in such a way that it now shares elements of a standing facility. Prior to 2003, the discount window rate was set below the target federal funds rate, but banks faced penalties when accessing the discount window. In 2003, the Federal Reserve decided to set the discount window rate 100 basis points above the target federal funds rate and eased access conditions to the discount window. The resulting framework is similar to a channel system, where the deposit rate is zero and the lending rate 100 basis points above the target rate.

[2] One reason put forward by central banks is that SF are simply easier to operate than open market operations.

weakly dominated in terms of welfare by the set of equilibrium allocations that can be attained with OMO. Third, if the central bank chooses the optimal policy for each mechanism, the allocations are equivalent.

Our framework builds on the divisible money model of Lagos and Wright (2005). Their framework is useful because it allows us to introduce heterogeneous preferences while still keeping the distribution of money balances analytically tractable. Our changes to their framework are motivated by the functioning of existing channel systems.[3] In each day, three markets open sequentially. The first market is a settlement market, where all claims from the previous day are settled. The second market is a secondary bond market. Finally, at the beginning of the third market, after the close of the bond market, the SF opens.[4]

2. Related Literature

Berentsen and Monnet (2008) were the first to study optimal policy in channel system in a general equilibrium framework. Their key result is that it is socially optimal for a central bank to implement a strictly positive interest-rate spread. In contrast, in our model it is optimal to have a zero spread. These differences are due to the different properties of the assets that serve as collateral in the two models. In our model, the central bank's borrowing facility accepts government bonds as collateral, which is essentially a piece of paper that is costlessly to produce. In contrast, in Berentsen and Monnet (2008) the collateral is a real asset. Furthermore, due to its

[3]See Berentsen and Monnet (2008) who developed this modified framework.

[4]For example, as shown in Berentsen and Monnet (2008) the key features of the ECB's implementation framework and of the Euro money market are the following. First, any outstanding overnight loans at the ECB are settled at the beginning of the day. Second, most lending in the Euro money market and all credit obtained at the ECB's standing facility is collateralized. Third, the Euro money market operates between 7 am and 5 pm. Fourth, after the money market has closed, market participants can access the ECB's facilities for an additional 30 minutes. This means that after the close of the money market, the ECB's lending facility is the only possibility for obtaining overnight liquidity. Also, any late payments received can still be deposited at the deposit facility of the ECB.

liquidity premium the social return of this real asset is lower than the marginal return to an agent. From a social point of view, this results in an overaccumulation of the collateral and by implementing a positive spread the central bank can discourage the use of its lending facility and thereby reduce the stock of collateral in the economy.

With the exception of Berentsen and Monnet (2008) and Martin and Monnet (2009) previous studies on channel systems or aspects of channel systems were conducted in partial equilibrium models. Berentsen and Monnet (2008) contains a discussion of these models. Furthermore, with the exception of Martin and Monnet (2009) we are not aware of any other paper that compares the allocative implications of implementing monetary policy via a channel system rather than with plain-vanilla OMO. Martin and Monnet (2009) also study SF and OMO in the Lagos-Wright framework. Our model, however, differs from Martin-Monnet (2009) in several aspects. We assume a common environment when we compare the two mechanism, while in their paper this is not the case. Furthermore, we have a richer type structure that they do.

The structure of the paper is the following. Section 3 describes the environment. The economy with SF is analyzed in Section 4. Section 5 studies the economy with OMO. The results are presented in Section 6. Section 7 concludes. All proofs are in the Appendix.

3. Environment

The common environment is based on the divisible money model developed in Lagos and Wright (2005). This model is useful because it allows us to introduce heterogeneous preferences while still keeping the distribution of money balances analytically tractable.[5] Time is discrete, and in each period there are three perfectly competitive markets that open sequentially. The first market is a settlement market, where all claims from the previous period are settled. The second market is a bond market, where households after receiving idiosyncratic preference shocks trade government bonds. The third market, is a goods market where agents trade goods and make their financial decisions.

The economy is populated by two types of infinitely-lived households: buyers and sellers. Each type of household has measure 1. There are two nonstorable and perfectly divisible consumption goods at each date: market 1 goods and market 3 goods. Nonstorable means that they cannot be carried from one market to the next. Buyers consume in market 3 and consume and produce in market 1. Sellers produce in market 3 and consume in market 1. Furthermore, the buyers receive an idiosyncratic preference shock which determines their liquidity need in the goods market. A fraction n receive the preference shock at the beginning of the bonds market; we call them the $e-$buyers. The remaining buyers receive the preference shock at the beginning of the goods market after the bonds market has closed; we call them the $\varepsilon-$buyers.[6]

We now discuss these markets in detail, starting with the settlement market, which opens at the beginning of the period. The discount factor across periods is $\beta = (1+r)^{-1} < 1$ where r is the time rate of discount. In the settlement market, buyers produce general goods, repay loans, redeem deposits, and adjust their money balances. General goods are produced solely from inputs of labor according to a constant return to scale production technology where one unit of the consumption good is produced

[5]An alternative framework would be Shi (1997) which we could amend with preference and technology shocks to generate the same results.

[6]The interpretation for these shocks in the ECB-framework described above is that $e-$buyers are banks that receive liquidity shocks during the day and so can use the bond market to readjust their portfolio. In contrast, the $\varepsilon-$buyers are banks that receive liquidity shocks so late in the day that they have to rely on the ECB's SF to readjust their portfolio.

with one unit of labor generating one unit of disutility. Thus, producing h units of the general good implies disutility $-h$, while consuming h units gives utility h. Sellers do not produce in the settlement market but they can consume. Their utility of consuming x units of general goods is $U(x) = x$.[7]

At the beginning of the bond market, with probability n a buyer receives the idiosyncratic preference shocks $e \in [0, \infty)$ which determines the marginal utility of consumption in the goods market: we refer to these households as $e-$buyers. With probability $1 - n$ he does not learn yet his preference for the consumption good in the current period. After buyers learn their type, they trade bonds. The preference shock e has a continuous distribution $F(e)$ with support $[0, \infty)$, is iid across $e-$buyers and serially uncorrelated. At the beginning of the goods market, $\varepsilon-$buyers receive their preference shock $\varepsilon \in [0, \infty)$. The preference shock ε has a continuous distribution $F(\varepsilon)$ with support $[0, \infty)$, is iid across $\varepsilon-$buyers and serially uncorrelated.

In the goods market, an $e-$buyer ($\varepsilon-$buyer) gets utility $eu(q_e)$ ($\varepsilon u(q_\varepsilon)$) from q_e (q_ε) consumption in the third market, where $u(q) = \log(q)$. Sellers incur a utility cost $c(q_s) = q_s$ from producing q_s units of output.

3.1. **First-best allocation.** We assume without loss in generality that the planner treats all sellers symmetrically. He also treats all buyers experiencing the same preference shock symmetrically. Given this assumption, the weighted average of expected

[7] The idiosyncratic preference shocks play a similar role as random matching and bargaining in Lagos and Wright (2005). Depending on the realization of these shocks, households will spend different amounts of money in the goods market. Then, without quasilinear preferences and unbounded hours in market 3, the preference shocks would generate a nondegenerate distribution of money holdings since the money holdings of individual households would depend on their history of shocks. For tractability, we therefore impose assumptions, as in Lagos and Wright (2005), that generate a degenerate distribution of money holdings at the beginning of a period.

steady state lifetime utility of households and firms can be written as follows

$$(3.1) \qquad \begin{aligned}(1-\beta)\mathcal{W} &= n\int_0^\infty \left[eu(q_e)-h_e\right]dF(e) \\ &\quad +(1-n)\int_0^\infty \left[\varepsilon u(q_\varepsilon)-h_\varepsilon\right]dF(\varepsilon)+x-q_s.\end{aligned}$$

where h_e (h_ε) is hours worked by an $e-$household ($\varepsilon-$household) in the settlement market and q_e (q_ε) is consumption of an $e-$household ($\varepsilon-$household) in the goods market. The planner maximizes (3.1) subject to the feasibility constraints

$$(3.2) \qquad n\int_0^\infty q_e dF(e)+(1-n)\int_0^\infty q_\varepsilon dF(\varepsilon)-q_s \le 0.$$

$$(3.3) \qquad x-n\int_0^\infty h_e dF(e)-(1-n)\int_0^\infty h_\varepsilon dF(\varepsilon) \le 0$$

where x is consumption by a seller in the settlement market. The first-best allocation satisfies

$$(3.4) \qquad \begin{aligned} eu'(q_e^*) &= 1 \text{ for all } e. \\ \varepsilon u'(q_\varepsilon^*) &= 1 \text{ for all } \varepsilon.\end{aligned}$$

These are the quantities chosen by a social planner who could dictate production and consumption in the goods market.

3.2. **Information frictions, money and bonds.** There are two perfectly divisible financial assets: money and bonds. Both are intrinsically useless since they are neither arguments of any utility function nor are they arguments of any production function. Money is issued by a "central bank" as described below. We assume that the bonds are one-period, nominal discount bonds issued by a "government". They are payable to the bearer and default free. One bond pays off one unit of currency at maturity. The government is assumed to have a record-keeping technology over bond trades and bonds are book-keeping entries – no physical object exists. This implies that households are not anonymous to the government. Nevertheless, despite having a record-keeping technology over bond trades, the government has no record-keeping technology over goods trades.

At time t, the government sells one-period, nominal discount bonds in market 1 and redeems bonds that were sold in $t-1$. Then households trade bonds and money in market 2. The government acts as the intermediary for these trades, recording purchases/sales of bonds and redistributes money. Private households are anonymous to each other and cannot commit to honor inter-temporal promises. Since bonds are intangible objects, they are incapable of being used as media of exchange in market 3, hence they are illiquid.[8] Since households are anonymous and cannot commit, a household's promise in market 2 to deliver outside bonds to a seller in market 3 is not credible.

To motivate a role for fiat money, search models of money typically impose three assumptions on the exchange process (Shi 2008): a double coincidence problem, anonymity, and costly communication. First, our preference structure creates a single-coincidence problem in market 3 since households do not have a good desired by sellers. Second, agents in market 3 are anonymous which rules out trade credit between individual buyers and sellers. Third, there is no public communication of individual trading outcomes (public memory), which in turn eliminates the use of social punishments in support of gift-giving equilibria. The combination of these frictions implies that sellers require immediate compensation from buyers. In short, there must be immediate settlement with some durable asset and money is the only durable asset. These are the micro-founded frictions that make money essential for trade in market

[8]As Kocherlakota (2003), we simply assume that bonds are illiquid and we show among other things that it is welfare improving if they are illiquid.

3. Kocherlakota (1998), Wallace (2001), and Aliprantis et al. (2007) provide a more detailed discussion of the features that generate an essential role for money. In contrast, in market 1 all agents can produce for their own consumption or use money balances acquired earlier. In this market, money is not essential for trade.[9]

4. Standing Facilities

In this section, we analyze the economy when monetary policy is implemented with a deposit facility and a borrowing facility. For notational ease, variables corresponding to the next period are indexed by $+$, and variables corresponding to the previous period are indexed by $-$. The money price of goods in market 1 is P, implying that the goods price of money in market 1 is $\phi = 1/P$. Let p be the money price of goods in market 3; ρ_S the money price of newly issued bonds in market 1; and ρ_T the money price of bonds in market 2.

At the beginning of the goods market, after all preference shocks are observed, the central bank offers a borrowing and a deposit facility. The central bank operates at zero cost and offers nominal loans ℓ at an interest rate i_ℓ and promises to pay interest rate i_d on nominal deposits d with $i_\ell \geq i_d$. Since we focus on facilities provided by the central bank, we restrict financial contracts to overnight contracts. An agent who borrows ℓ units of money from the central bank in market 3 repays $(1 + i_\ell)\,\ell$ units of money in market 1 of the following period. Also, an agent who deposits d units of money at the central bank in market 3 receives $(1 + i_d)\,d$ units of money in market 1 of the following period.

In a channel system, the money stock evolves as follows

$$M^+ = M - i_\ell L + i_d D + \tau M \tag{4.1}$$

[9]One can think of agents being able to barter perfectly in this market. Obviously in such an environment, money is not needed.

where M denotes the stock of money at the beginning of period t. In the first market, total loans L are repaid and total deposits D are redeemed. Since interest-rate payments by the agents are $i_\ell L$, the stock of money shrinks by this amount. Interest payments by the central bank on total deposits are $i_d D$. The central bank simply prints additional money to make these interest payments, causing the stock of money to increase by this amount. Finally, the central bank can also change the stock of money via lump-sum transfers $T = \tau M$ in market 1. Since central banks have no fiscal authority, we restrict these lump-sum transfers to be positive, that is, $\tau \geq 0$. Note that the central bank has three policy instruments: the deposit rate i_d, the lending rate i_ℓ, and lump-sum transfers $T = \tau M$.

4.1. **Settlement market.** $V_S(m, b, \ell, d)$ denotes the expected value of entering the settlement market with m units of money, b bonds, ℓ loans, and d deposits. $V_T(m, b)$ denotes the expected value from entering the bond market with m units of money and b collateral. For notational simplicity, we suppress the dependence of the value function on the time index t. In what follows, we solve the model backward from the settlement to the bond market.

In the settlement market, the problem of the representative buyer is:

$$V_S(m, b, \ell, d) = \max_{h, m^+, b^+} -h + V_T(m', b')$$

$$s.t. \quad \phi m' + \phi \rho_S b' = h + \phi m + \phi b + \phi (1 + i_d) d - \phi (1 + i_\ell) \ell + \phi T.$$

where h is hours worked in market 1, m' is the amount of money brought into the bond market, and b' is the amount of bonds brought into the bond market. Using the budget constraint to eliminate h in the objective function, one obtains the first-order conditions

$$V_T^{m'} \leq \phi \;(= \text{ if } m' > 0\;) \tag{4.2}$$

$$V_T^{b'} \leq \phi\rho_S \;(= \text{ if } b' > 0\;) \tag{4.3}$$

$V_T^{m'} \equiv \frac{\partial V_T(m',b')}{\partial m'}$ is the marginal value of taking an additional unit of money into the bond market. Since the marginal disutility of working is one, $-\phi$ is the utility cost of acquiring one unit of money in the settlement market. $V_T^{b'} \equiv \frac{\partial V_T(m',b')}{\partial b'}$ is the marginal value of taking additional bonds into the bond market. Since the marginal disutility of working is 1, $-\phi\rho_S$ is the utility cost of acquiring one unit of bonds in the settlement market. The implication of (4.2) and (4.3) is that all agents enter the bond market with the same amount of money and the same quantity of collateral (which can be zero).

The envelope conditions are

$$V_S^m = V_S^b = \phi; V_S^d = \phi\,(1 + i_d)\,; V_S^\ell = -\phi\,(1 + i_\ell) \tag{4.4}$$

where V_S^j is the partial derivative of $V_S(m, b, \ell, d)$ with respect to $j = m, b, \ell, d$.

4.2. **Goods Market.** We first consider the problem solved by sellers, then the one solved by the $\varepsilon-$buyers, and finally the one solved by the $e-$buyers. During the goods market the central bank operates SF which allows all buyers to borrow at rate i_ℓ and deposit excess money at rate i_d.

4.2.1. *Decisions by sellers.* Sellers produce goods in the goods market with linear cost $c\,(q) = q$ and consume in the settlement market obtaining linear utility $U(x) = x$. It is straightforward to show that that sellers are indifferent as to how much they sell in the goods market if

$$p\beta\phi^{+}(1+i_d)=1. \tag{4.5}$$

Since we focus on a symmetric equilibrium, we assume that all sellers produce the same amount. With regard to bond holdings, it is straightforward to show that, in equilibrium, firms are indifferent to holding any bonds if the Fisher equation holds and will hold no bonds if the yield on the bonds does not compensate them for inflation or time discounting. Thus, for brevity of analysis, we assume sellers carry no bonds across periods.

It is also clear that sellers always deposit their proceeds from sales at the deposit facility since they can earn the interest rate i_d.

4.2.2. *Decisions by $\varepsilon-$buyer.* The indirect utility function of a $\varepsilon-$buyer in the goods market is

$$V_G(m,b|\,\varepsilon)=\max_{q_\varepsilon,d_\varepsilon,\ell_\varepsilon}\varepsilon u\left(q_\varepsilon\right)+\beta V_S\left(m+\ell_\varepsilon-pq_\varepsilon-d_\varepsilon,b,\ell_\varepsilon,d_\varepsilon|\,\varepsilon\right)$$

$$\text{s.t.}\quad m+\ell_\varepsilon-pq_\varepsilon-d_\varepsilon\geq 0$$

$$\frac{b}{1+i_\ell}-\ell_\varepsilon\geq 0$$

where d_ε is the amount of money a ε-type household deposits at the central bank and ℓ_ε is the loan received from the central bank. The first inequality is the buyer's budget constraint. The second inequality is the collateral constraint. Let $\beta\phi^{+}\lambda_\varepsilon$ denote the Lagrange multiplier for the first inequality and denote $\beta\phi^{+}\lambda_\ell$ the Lagrange multiplier of the second inequality. Then, using (4.4) to replace V_S^m, V_S^ℓ and V_S^d, the first-order conditions for q_ε, d_ε, and ℓ_ε can be written as follows:

$$\begin{aligned} \varepsilon u'(q_\varepsilon) - \beta p \phi^{+}(1+\lambda_\varepsilon) &= 0 \\ i_d - \lambda_\varepsilon &\leq 0 \qquad (= 0 \text{ if } d_\varepsilon > 0) \\ -i_\ell + \lambda_\varepsilon - \lambda_\ell &\leq 0 \qquad (= 0 \text{ if } \ell_\varepsilon > 0) \end{aligned} \tag{4.6}$$

Lemma 1 below characterizes the optimal borrowing and lending decisions by a $\varepsilon-$buyer and the quantity of goods obtained by the $\varepsilon-$buyer:

LEMMA 1. *There exist critical values ε_d, ε_ℓ, $\varepsilon_{\bar{\ell}}$, with $0 \leq \varepsilon_d \leq \varepsilon_\ell \leq \varepsilon_{\bar{\ell}}$, such that the following is true: if $0 \leq \varepsilon < \varepsilon_d$, a buyer deposits money at the central bank; if $\varepsilon_\ell < \varepsilon \leq \varepsilon_{\bar{\ell}}$, he borrows money and the collateral constraint is nonbinding; if $\varepsilon_{\bar{\ell}} \leq \varepsilon$, he borrows money and the collateral constraint is binding; and if $\varepsilon_d \leq \varepsilon \leq \varepsilon_\ell$, he neither borrows nor deposits money. The critical values solve:*

$$\varepsilon_d = (1+i_d)\beta\phi^{+}m, \; \varepsilon_\ell = (1+i_\ell)\beta\phi^{+}m, \; \varepsilon_{\bar{\ell}} = (1+i_\ell)\beta\phi^{+}m + \beta\phi^{+}b. \tag{4.7}$$

In any equilibrium, the amount of borrowing and depositing by a buyer with a taste shock ε and the amount of goods purchased by the buyer satisfy:

$$\begin{array}{llll} q_\varepsilon = \varepsilon, & d_\varepsilon = p(\varepsilon_d - \varepsilon), & \ell_\varepsilon = 0, & \text{if } 0 \leq \varepsilon \leq \varepsilon_d \\ q_\varepsilon = \varepsilon_d, & d_\varepsilon = 0, & \ell_\varepsilon = 0, & \text{if } \varepsilon_d \leq \varepsilon \leq \varepsilon_\ell \\ q_\varepsilon = \Delta\varepsilon, & d_\varepsilon = 0, & \ell_\varepsilon = p(\Delta\varepsilon - \varepsilon_d), & \text{if } \varepsilon_\ell \leq \varepsilon \leq \varepsilon_{\bar{\ell}}, \\ q_\varepsilon = \Delta\varepsilon_{\bar{\ell}}, & d_\varepsilon = 0, & \ell_\varepsilon = \frac{b}{1+i_\ell}, & \text{if } \varepsilon_{\bar{\ell}} \leq \varepsilon, \end{array} \tag{4.8}$$

where $\Delta \equiv \frac{1+i_d}{1+i_\ell}$.

The optimal borrowing and lending decisions follow the cut-off rules according to the realization of the taste shock. The cut-off levels, ε_d, ε_ℓ, and $\varepsilon_{\bar{\ell}}$ partition the set of taste shocks into three regions. For shocks lower than ε_d, a buyer deposits money at the SF; for shocks higher than ε_ℓ, the buyer borrows at the SF. For values between

ε_d and ε_ℓ,, the buyer does not use the central bank's SF. Finally, the cutoff value $\varepsilon_{\bar{\ell}}$ determines whether a buyer's collateral constraint is binding or not.

Then, using (4.4) to replace V_S^m, V_S^ℓ and V_S^d, the envelope conditions for buyers can be written as

$$
\begin{aligned}
V_G^m(m, b|\,\varepsilon) &= \beta\phi^+\left(1+\lambda_\varepsilon\right) \\
V_G^b(m, b|\,\varepsilon) &= \beta\phi^+\left(1+\tfrac{\lambda_\ell}{1+i_\ell}\right)
\end{aligned}
\tag{4.9}
$$

4.2.3. *Decisions by $e-$buyer.* The indirect utility function of a $e-$buyer in the goods market is

$$
\begin{gathered}
V_G(m, b|\,e) = \max_{q_e, d_e, \ell_e} u\left(q_e\right) + \beta V_S\left(m - pq_e + l_e - d_e, b, l_e, d_e|\,p\right) \\
\text{s.t.} \quad m - pq_e + l_e - d_e \geq 0 \text{ and } \frac{b}{1+i_\ell} - \ell_e \geq 0
\end{gathered}
$$

where q_e is the quantity of goods he consumes, and d_e the amount of money he deposits at the central bank. The first constraint means that they cannot deposit more money than what they carry from the previous period minus what he spend in goods purchases plus what he borrows from the central bank. The second short selling constraint means that he cannot borrow more money than what he can afford by pledging his bonds holdings. Using (4.4) to replace V_S^m and V_S^d, the first-order conditions can be rewritten as follows:

$$
\begin{aligned}
eu'\left(q_e\right) - \beta p\phi^+\left(1+\lambda_e\right) &= 0 \\
i_d - \lambda_e &\leq 0 \qquad (= 0 \text{ if } d_e > 0) \\
-i_\ell + \lambda_e - \lambda_\ell &\leq 0 \qquad (= 0 \text{ if } \ell_e > 0)
\end{aligned}
\tag{4.10}
$$

Note that early buyers never deposit money at the central bank nor do they borrow from the central bank, that is $d_e = \ell_e = 0$ implying that

$$i_d \leq \lambda_e \leq i_\ell$$

Lemma 2 below characterizes the optimal borrowing and lending decisions by a $e-$buyer and the quantity of goods obtained by the $e-$buyer:

LEMMA 2. *There exists a critical value $\tilde{e}$ with $0 < \tilde{e} < \infty$, such that the following is true: for any e the buyer neither deposits nor borrows money at the central bank. The critical value solve:*

$$\tilde{e} = (1 + i_T)\, \beta\phi^+ m + \beta\phi^+ b. \tag{4.11}$$

In any equilibrium, the amount of borrowing and depositing by a buyer with a taste shock e and the amount of goods purchased by the buyer satisfy:

$$\begin{aligned} q_e &= e/\Delta_T, \quad d_e = 0, \quad \ell_e = 0, \quad \text{if } 0 \leq e \leq \tilde{e} \\ q_e &= \tilde{e}/\Delta_T, \quad d_e = 0, \quad \ell_e = 0, \quad \text{if } \tilde{e} \leq e \end{aligned} \tag{4.12}$$

where $\Delta_T \equiv \frac{1+i_T}{1+i_d}$.

Replacing V_S^m, V_S^b and $\beta\phi^+\lambda_d$ using (4.4) and (4.10), we obtain the following envelope condition for a $e-$buyer:

$$V_G^m(m, b|\, e) = \frac{\varepsilon u'(q_e)}{p}, \text{ and } V_G^b(m, b|\, e) = \beta\phi^+ \tag{4.13}$$

4.3. **Treasury market.** In the bond market households are subject to an idiosyncratic preference and technology shock. With probability n they learn their type at the beginning of the bond market, and with probability $1 - n$ they learn their type at the beginning of the goods market. The indirect expected utility of a buyer entering the bond market with a portfolio (m, b) is

$$V_T(m, b) = n \int_0^\infty V_T(m, b|\, e)\, dF(e) + (1 - n)\, V_T(m, b|\, .)$$

where

$$\begin{gathered} V_T(m, b|\, .) = \max_y \int_0^\infty V_G(m - \rho_T y, b + y|\, \varepsilon)\, dF(\varepsilon) \\ \text{s.t. } m - \rho_T y \geq 0 \text{ and } b + y \geq 0 \end{gathered} \tag{4.14}$$

and

$$\begin{gathered} V_T(m, b|\, e) = \max_{y_e} \int_0^\infty V_G(m - \rho_T y_e, b + y_e|\, e)\, dF(e) \\ m - \rho_T y_e \geq 0 \text{ and } b + y_e \geq 0 \end{gathered} \tag{4.15}$$

A household who learns its preference shock solves problem (4.15) while a household who receives the preference shock later solves (4.14). In both programs, the first inequality reflects the fact that a buyer cannot sell more money for bonds that what he brought into the bond market, and the second inequality reflects the fact that he cannot sell more bonds for money that what he brought into the bond market.

4.4. **Equilibrium.** We focus on symmetric stationary equilibria where households participate in the financial market and money is used as a medium of exchange. Such equilibria meet the following requirements: (i) Buyers' decisions are optimal given prices; (ii) The decisions are symmetric across all buyers with the same preference shocks; (iii) The relative price between market 2 goods and market 3 goods satisfies (4.5); (iv) The goods and bond markets clear; (v) All real quantities are constant across time; (vi) The law of motion for the stock of money (4.1) holds in each period.

Point (v) requires that the real stock of money is constant; i.e.,

$$\phi M = \phi^{+} M^{+}. \tag{4.16}$$

This implies that $\phi/\phi^{+} = M^{+}/M \equiv \gamma$ where γ is the gross steady-state money growth rate. Symmetry requires $m = M$ and $b = B$. The restriction that there is a positive demand for money and bonds requires that Lemma 3 holds.

LEMMA 3. *In any equilibrium*

$$\rho_S = \rho_T$$

Lemma 3 relies on an arbitrage argument. The prices for bonds must be the same in the two markets. Otherwise agents could buy bonds in one of the markets and sell them at a profit in the other market. For what follows we ignore the index and write ρ for the price of bonds in markets 1 and 2. Moreover, in a stationary equilibrium the bonds price ρ has to be constant. This can be seen for example from (4.12), where a changing ρ involves a non-stationary path for consumption. A constant bond price then implies that the bond-money ratio has to be constant, and this can be only achieved when the growth rates of money and bonds are equal. We assume there are positive initial stocks of money M_0 and outside bonds B_0.[10]

Market clearing in market 2 and market 3 requires

$$n \int_0^\infty y_e dF(e) + (1-n) \int_0^\infty y dF(\varepsilon) = 0 \tag{4.17}$$

$$q_s - n \int_0^\infty q_e dF(e) - (1-n) \int_0^\infty q_\varepsilon dF(\varepsilon) = 0 \tag{4.18}$$

where q_s is aggregate production by sellers in market 3.

[10]Since the assets are nominal objects, the government can start the economy off by one-time injections of cash M_0 and bonds B_0.

PROPOSITION 1. *For the SF economy, an equilibrium is a policy* (i_d, i_ℓ, τ) *and endogenous variables* $(\gamma, \rho, \varepsilon_d)$ *satisfying*

$$\frac{\rho\gamma}{\beta} = n\left[\int_0^{\tilde{e}} dF(e) + \int_{\tilde{e}}^{\infty} \frac{e}{\tilde{e}} dF(e)\right] \tag{4.19}$$

$$+\rho(1+i_d)(1-n)\left[\int_0^{\varepsilon_d} dF(\varepsilon) + \int_{\varepsilon_d}^{\varepsilon_\ell} \frac{\varepsilon}{\varepsilon_d} dF(\varepsilon) + \int_{\varepsilon_\ell}^{\varepsilon_{\bar{\ell}}} \frac{1}{\Delta} dF(\varepsilon) + \int_{\varepsilon_{\bar{\ell}}}^{\infty} \frac{\varepsilon}{\varepsilon_{\bar{\ell}}\Delta} dF(\varepsilon)\right]$$

$$\int_0^{\varepsilon_{\bar{\ell}}} \frac{1}{\Delta} dF(\varepsilon) + \int_{\varepsilon_{\bar{\ell}}}^{\infty} \frac{\varepsilon}{\varepsilon_{\bar{\ell}}\Delta} dF(\varepsilon) \tag{4.20}$$

$$= \rho(1+i_\ell)\left[\int_0^{\varepsilon_d} dF(\varepsilon) + \int_{\varepsilon_d}^{\varepsilon_\ell} \frac{\varepsilon}{\varepsilon_d} dF(\varepsilon) + \int_{\varepsilon_\ell}^{\varepsilon_{\bar{\ell}}} \frac{1}{\Delta} dF(\varepsilon) + \int_{\varepsilon_{\bar{\ell}}}^{\infty} \frac{\varepsilon}{\varepsilon_{\bar{\ell}}\Delta} dF(\varepsilon)\right]$$

and

$$\gamma = 1 + i_d + \frac{(1-n)(i_d - i_\ell)}{\varepsilon_d}\left[\int_{\varepsilon_\ell}^{\varepsilon_{\bar{\ell}}} (\Delta\varepsilon - \varepsilon_d) dF(\varepsilon) + \int_{\varepsilon_{\bar{\ell}}}^{\infty} (\Delta\varepsilon_{\bar{\ell}} - \varepsilon_d) dF(\varepsilon)\right] + \tau \tag{4.21}$$

A detailed derivation of (4.19)-(4.21) is relegated in the appendix. Equation (4.19) comes from the marginal values of money. Equation (4.20) is obtained from the $\varepsilon-$buyer's optimization problem using the fact that he never sells bonds for money, or money for bonds, in the treasury market. Equation (4.21) is obtained from the law of motion of money (4.1).

5. OPEN MARKET OPERATIONS

When using OMO, the central bank stands ready to trade money for bonds in the goods market. We assume that households cannot trade bonds with each other directly. They can only trade money for bonds with the central bank. Buyers with a high $e-$shock sell bonds to the central bank, while buyers with a low $e-$shock buy bonds from the central bank. In what follow, we focus on a representative period t.

Let ρ_S, ρ_T, and ρ_G be the unit price of bonds in the settlement market, in the bond market and in the goods market, respectively. There is a fixed quantity $\bar{B}$ of discount securities that yield a per unit nominal return $1-\rho_S$ in each period. The issuance of securities is financed by government lump-sum taxes.

Let B_T and b_T denote the stock of securities held by the central bank and by the households at the beginning of the bond market, so $\bar{B} \equiv B_T + b_T$. Note that B_T and b_T are not affected by bonds trade at the bond market since the central bank cannot conduct OMO in this market; i.e., $B_T = B_G$ and $b_T = b_G$. In the goods market, let Y_G be the central bank's stock increase of bonds. Then, it holds that $B_S^+ = B_G + Y_G$ and $b_S^+ = b_G - Y_G$, where B_S^+ and b_S^+ is the stock of bonds held by the central bank, respectively by the households, at the beginning of the next-period settlement market. In the settlement stage, households receive interests on their bonds, the central bank makes lump sum transfers, buy and sell bonds.

Let M_G denote the stock of money at the beginning of the goods market. The stock of money increases by $\rho_G Y_G$ if the central bank increases its stock of bonds by Y_G. Hence, the stock of money at the beginning of the next-period settlement market is

$$M_S^+ = M_G + \rho_G Y_G$$

Let M_T^+ be the stock of money at the beginning of the next-period bond market. Then the following holds

$$M_T^+ = M_S^+ - Y_G - (1-\rho_S)\, B_G + \pi^+$$

where $\pi^{+} = (1-\rho_S)\,B_G + \tau M_T$ is the lump sum money transfer from the central bank to the households and the settlement market. We are assuming, for simplicity, that the central bank redistributes the interests she receives from the stock of bonds outstanding at the beginning of the goods market.

The central bank cannot conduct OMO in the settlement market. Nevertheless, the stock of money changes in this market since the central bank receives (pays) the interests on the bonds she acquired (sold) in the previous period goods market. The central bank cannot conduct OMO in the bond market, so the stock of money is invariant in this market, i.e. $M_G^{+} = M_T^{+}$ must hold. Combining the previous two equations, and replacing M_G with M_T, the stock of money evolves according to the following rule

$$M_T^{+} = M_T - (1-\rho_G)\,Y_G + \tau M_T \tag{5.1}$$

In period t, let $\phi \equiv 1/P$ be the real price of money in the settlement market, where P is the price of goods in the settlement market. Let M be the stock of money at the beginning of the settlement market. We focus on symmetric and stationary equilibria, where all agents follow identical strategies and where the real allocation is constant over time. In a stationary equilibrium, beginning-of-period real money balances are time-invariant

$$\phi M = \phi^{+} M^{+} \tag{5.2}$$

which implies that $\phi/\phi^{+} = M^{+}/M = \gamma$, where γ is the gross rate of growth of money.

5.1. **Settlement Market.** The functioning of this market is similar to the one in the previous section. Let $V_S(m,b)$ be the expected value of entering the first market with m units of money, and b bonds. $V_T(m,b)$ denotes the expected value from entering the bond market with m units of money and b bonds. In the settlement market, the problem of the representative buyer is:

$$V_S(m,b) = \max_{h,m',b'} -h + V_T\left(m',b'\right)$$
$$s.t. \quad \phi m' + \phi \rho_S b' = h + \phi m + \phi b + \phi T.$$

where h is hours of work in the settlement market, m' is the amount of money brought into the next bond market, and b' is the amount of bonds brought into the bond market. Using the budget constraint to eliminate h in the objective function, the first-order conditions are

$$V_T^{m'} \leq \phi \; (= \text{ if } m' > 0 \;) \tag{5.3}$$

$$V_T^{b'} \leq \phi \rho_S \; (= \text{ if } b' > 0 \;) \tag{5.4}$$

$V_T^{m'} \equiv \frac{\partial V_T(m',b')}{\partial m'}$ is the marginal value of taking an additional unit of money into the bond market. Since the marginal disutility of working is one, $-\phi$ is the utility cost of acquiring one unit of money in the third market. $V_T^{b'} \equiv \frac{\partial V_T(m',b')}{\partial b'}$ is the marginal value of taking additional bonds into the next period. Since the marginal disutility of working is 1, $-\phi\rho_S$ is the utility cost of acquiring one unit of bonds in the third market.

The envelope conditions are

$$V_S^m = V_S^b = \phi \tag{5.5}$$

where V_S^j is the partial derivative of $V_S(m,b)$ with respect to $j = m, b$.

5.2. **Goods Market.** We know describe the agents decision in the goods market. We consider the problem solved by sellers, $\varepsilon-$buyers, and $e-$buyers, sequentially. In the goods market the central bank stands ready to buy and sell bonds at any time. The central bank can increase or decrease its beginning-of-period stock of bonds in this market —that is it can conduct OMO.

5.2.1. *Decision by sellers.* Seller produce in the goods market so they don't bring any cash into this market. It is easy to show that, if the Fisher equation holds, then sellers are indifferent to bringing any amount of bonds into the goods market. For simplicity, we assume that they carry no bonds. In the goods market, they are indifferent as to how much to produce if the following condition is satisfied

$$\beta p \phi^{+} = \rho_G \tag{5.6}$$

It is also evident that they invest all their proceeds from sales in bonds if the return on bonds is strictly positive (i.e. $1 - \rho_G > 0$), since they can earn interests by doing so.

5.2.2. *Decision by $\varepsilon-$buyers.* The indirect utility function of a $\varepsilon-$buyer in the goods market is

$$V_G(m, b|\,\varepsilon) = \max_{q_\varepsilon, a_\varepsilon} \varepsilon u\,(q_\varepsilon) + \beta V_S\,(m - pq_\varepsilon - \rho a_\varepsilon, b + a_\varepsilon|\,\varepsilon)$$

$$\text{s.t.} \quad m - pq_\varepsilon - \rho_G a_\varepsilon \geq 0, \text{ and}$$

$$b + a_\varepsilon \geq 0$$

where a_ε is the amount of bonds a ε-type buyers buys at the central bank. The first inequality is the households budget constraint. The second inequality is the short selling constraint for bonds. It means that a buyer cannot sell more bonds than what he has. By Inada conditions buyers always consume a strictly positive quantity of goods.

Let $\beta\phi^+\lambda_\varepsilon$ denote the Lagrange multiplier for the first inequality and denote $\beta\phi^+\lambda_a$ the Lagrange multiplier of the second inequality. Then, using (5.5) to replace V_S^m and V_S^b we obtain the following envelope conditions for a buyer in the goods market

$$\begin{aligned} \varepsilon u'(q_\varepsilon) - \beta p\phi^+(1+\lambda_\varepsilon) &= 0 \\ 1+\lambda_a - \rho_G(1+\lambda_\varepsilon) &= 0 \end{aligned} \tag{5.7}$$

Lemma 4. *There exists two critical values for the taste index, denoted ε_b and $\bar{\varepsilon}$ with $0<\varepsilon_b<\bar{\varepsilon}$, such that the following is true: if $\varepsilon \le \varepsilon_b$, a $\varepsilon-$buyer acquires bonds; if $\varepsilon_b \le \varepsilon \le \bar{\varepsilon}$, he sells bonds and his bond constraint is not binding ($\lambda_a = 0$); if $\varepsilon \ge \bar{\varepsilon}$, he sells bonds and his bond constraint is binding ($\lambda_a > 0$). The critical values solve*

$$\varepsilon_b = \frac{\beta\phi^+ m}{\rho}, \text{ and } \bar{\varepsilon} = \frac{\beta\phi^+ m}{\rho} + \beta\phi^+ b,$$

In equilibrium, the quantity of goods consumed, and the quantity of bonds acquired, by a $\varepsilon-$buyer satisfy:

$$\begin{array}{lll} q_\varepsilon = \varepsilon, & a_\varepsilon > 0, & \text{if } \ \varepsilon \le \varepsilon_b \\ q_\varepsilon = \varepsilon, & a_\varepsilon \in [-b, 0], & \text{if } \ \varepsilon_b \le \varepsilon \le \bar{\varepsilon} \\ q_\varepsilon = \bar{\varepsilon}, & a_\varepsilon = -b, & \text{if } \ \bar{\varepsilon} \le \varepsilon \end{array} \tag{5.8}$$

Replacing V_S^m, V_S^b and $\beta\phi^+\lambda_a$ from (5.7) the envelope conditions for a buyer in the goods market are

$$\begin{aligned} V_G^m(m,b|\,\varepsilon) &= \beta\phi^+(1+\lambda_\varepsilon), \text{ and} \\ V_G^b(m,b|\,\varepsilon) &= \rho_G\beta\phi^+(1+\lambda_\varepsilon). \end{aligned} \tag{5.9}$$

5.2.3. *Decision by $e-$buyers.* The indirect utility function of a $e-$buyer in the goods market is

$$V_G(m,b|\,e) = \max_{q_e,a_e} eu\left(q_e\right) + \beta V_S\left(m - pq_e - \rho a_e, b + a_e|\,\varepsilon\right)$$
$$\text{s.t.} \quad m - pq_e - \rho_G a_e \geq 0, \text{ and}$$
$$b + a_e \geq 0$$

where a_e is the amount of bonds a e-type buyers buys at the central bank. The first inequality is the households budget constraint. The second inequality is the short selling constraint for bonds. It means that a buyer cannot sell more bonds than what he has. By Inada conditions buyers always consume a strictly positive quantity of goods.

Let $\beta\phi^+\lambda_{a,e}$ denote the Lagrange multiplier for the first inequality and denote $\beta\phi^+\lambda_{a,e}$ the Lagrange multiplier of the second inequality. Then, using (5.5) to replace V_S^m and V_S^b we obtain the following envelope conditions for a buyer in the goods market

$$\begin{aligned} eu'\left(q_e\right) - \beta p\phi^+\left(1+\lambda_e\right) &= 0 \\ 1 + \lambda_{a,e} - \rho_G\left(1+\lambda_e\right) &= 0 \end{aligned} \tag{5.10}$$

LEMMA 5. *There exists two critical values for the taste index, denoted e_b and $\bar{e}$ with $0 < e_b < \bar{e}$, such that the following is true: if $e \leq e_b$, a $e-$buyer acquires bonds; if $e_b \leq e \leq \bar{e}$, he sells bonds and his bond constraint is not binding ($\lambda_{a,e} = 0$); if $e \geq \bar{e}$, he sells bonds and his bond constraint is binding ($\lambda_{a,e} > 0$). The critical values solve*

$$e_b = \frac{\beta\phi^+ m}{\rho}, \text{ and } \bar{e} = \frac{\beta\phi^+ m}{\rho} + \beta\phi^+ b,$$

In equilibrium, the quantity of goods consumed, and the quantity of bonds acquired, by a $e-$buyer satisfy:

$$
\begin{array}{lll}
q_e = e, & a_e > 0, & \text{if } \; e \leq e_b \\
q_e = e, & a_e \in [-b, 0]\,, & \text{if } \; e_b \leq e \leq \bar{e} \\
q_e = \bar{e}, & a_e = -b, & \text{if } \; \bar{e} \leq e
\end{array}
\tag{5.11}
$$

Replacing V_S^m, V_S^b and $\beta\phi^+\lambda_{a,e}$ from (5.10) the envelope conditions for a buyer in the goods market are

$$
\begin{aligned}
V_G^m(m, b|\, e) &= \beta\phi^+ \left(1 + \lambda_e\right), \text{ and} \\
V_G^b(m, b|\, e) &= \rho_G \beta\phi^+ \left(1 + \lambda_e\right).
\end{aligned}
\tag{5.12}
$$

If the price of bonds in the bond market is equal to the price of bonds in the goods market —a conjecture that need to be verified— then $e-$buyers and $\varepsilon-$buyers make the same decisions in the goods market. Hence, the problem of $e-$buyers in the goods market is the same as the one above with ε replaced by e. That is, for $e-$buyers we have the same relations (5.7)-(5.9), with ε replaced by e.

5.3. **Treasury market.** In the bond market buyers are subject to an idiosyncratic shock. With probability n they learn their type e at the beginning of the bond market, we refer to these buyers as $e-$buyers, and with probability $1-n$ they learn their type ε at the beginning of the goods market, we refer to these buyers as $\varepsilon-$buyers.

The expected utility function of a buyer entering the bond market with a portfolio (m, b) is

$$
V_T\left(m, b\right) = n \int_0^\infty V_T\left(m, b|\, e\right) dF\left(e\right) + \left(1-n\right) V_T\left(m, b|\, .\right)
\tag{5.13}
$$

where

$$V_T\left(m,b\right|.) = \max_y \int_0^\infty V_G\left(m-\rho_T y, b+y\right|\varepsilon)\, dF\left(\varepsilon\right)$$

$$\text{s.t. } m-\rho_T y \geq 0, \text{ and } b+y \geq 0$$

is the $\varepsilon-$buyer's problem, and

$$V_T\left(m,b\right|e) = \max_{y_e} \int_0^\infty V_G\left(m-\rho_T y_e, b+y_e\right|e)\, dF\left(\varepsilon\right)$$

$$\text{s.t. } m-\rho_T y_e \geq 0, \text{ and } b+y_e \geq 0$$

is the $e-$buyer's problem. In both problems, the first inequality and the second inequality are the short selling constraints for money holdings and bonds holdings, respectively. This means that a buyer cannot sell more money, respectively more bonds, than what he carries into the bond market. Sellers are idle in the bond market since they don't carry neither money nor bonds.

5.4. **Equilibrium.** We focus on symmetric stationary equilibria where households participate in the financial market and money is used as a medium of exchange. Such equilibria meet the following requirements: (i) Buyers' decisions are optimal given prices; (ii) The decisions are symmetric across all buyers with the same preference shocks; (iii) The relative price between market 1 goods and market 3 goods satisfies (5.6); (iv) The goods and bond markets clear; (v) All real quantities are constant across time; (vi) The law of motion for the stock of money (5.1) holds in each period.

Point (v) requires that the real stock of money is constant; i.e.,

$$\phi M = \phi^+ M^+. \tag{5.14}$$

This implies that $\phi/\phi^{+} = M^{+}/M \equiv \gamma$ where γ is the gross steady-state money growth rate. Symmetry requires $m = M$ and $b = B$. In equilibrium, the following holds:

LEMMA 6. *In equilibrium the following relation must hold*

$$\rho_T = \rho_G. \tag{5.15}$$

Lemma 6 is motivated by arbitrage arguments (the proof is in the appendix). It means that, in equilibrium, the price of bonds must be the same in the goods market and in the bond market. Otherwise all buyers wish to sell (buy) bonds for money in one market and buy (sell) bonds for money in the other market, which cannot constitute an equilibrium. Since the price of bonds in the two markets is the same, $\varepsilon-$buyers and $e-$buyers are indifferent between trading bonds for money in the bond market or in the goods market.

In what follows we drop the subscript on the bond price and use the notation $\rho \equiv \rho_S = \rho_T = \rho_G$.

PROPOSITION 2. *For the OMO economy, a steady state equilibrium is a policy* (ρ, τ) *and endogenous variables* $(\gamma, \bar{e})$ that solve

$$\frac{\rho\gamma}{\beta} = \int_0^{\bar{e}} dF(e) + \int_{\bar{e}}^{\infty} \frac{e}{\bar{e}} dF(e) \text{ and} \tag{5.16}$$

$$\gamma = 1/\rho + \tau. \tag{5.17}$$

The first equilibrium condition comes from the marginal value of money. The second equilibrium equation comes from the law of motion of money holdings (5.1).

It is straightforward to show that in the OMO economy it is optimal to choose policy $(\rho, \tau) = (1, 0)$.

6. OMO vs SF

We now state our first proposition that relates the set of feasible allocations for the two mechanisms.

PROPOSITION 3. *Any equilibrium allocation in the OMO economy can be replicated in the SF economy by choosing $\Delta = 1$. The converse is not true.*

Proposition 3 states that the set of equilibrium allocations in the SF economy is a subset of the set of equilibrium allocations in the OMO economy. The reason for this result is as follows. In the OMO economy, the money growth rate is the only policy instrument. The multiplicity of policy instruments in the SF economy is what drives the converse part of the proposition – in the OMO economy the government only has γ as a policy instrument to affect the allocation. So in general, it is not possible to replicate the allocation occurring in the outside bond economy via a choice of γ alone.

Proposition 4 below contains a welfare ranking of these sets.

PROPOSITION 4. *The set of allocations that can be implemented in the SF economy is weakly dominated in terms of welfare by the equilibrium allocation in the OMO economy.*

The proof in the Appendix first shows that in the SF economy for a given value of γ welfare is strictly decreasing in Δ. Then, since we can replicate the allocation of the OMO economy by setting $\Delta = 1$, it has to be the case that an equilibrium in the SF economy with $\Delta > 1$ is strictly dominated in terms of welfare by the equilibrium allocation under the same γ in the OMO economy.

This result is in contrast with the one in Martin-Monnet (2009). They show that SF can do better than OMO if the inflation rate is sufficiently high (Proposition 9). This is a direct consequence of the assumption that, in their paper, producers are allowed to deposit cash from good sales, and earn interests on it, while they are not allowed to use it to buy bonds. This bias in the timing of the events exposes sellers in the OMO-economy to the inflation tax, which makes SF more welfare improving if inflation is sufficiently high. By assuming the same timing of events, instead, we allow producers to use their sales revenues to buy bonds, which eliminates the inefficiency associated to the inflation tax —nobody holds cash after the goods market activity. In this environment SF can never do better than OMO.

The optimal policy in the SF economy is characterized by the following:

Proposition 5. *The allocations are the same in both economies under optimal policies.*

This result is closely related to Propositions 3 and 4, and it states that the optimal allocation in the SF economy is implemented when the interest rate band is set to zero.

7. Conclusions

To be added

Appendix

Proof of Lemma 1. We first derive the cutoff values ε_d and ε_ℓ. For this proof, to the notation of the consumption level of a buyer, we add a subscript d if the buyer deposits money at the central bank, a subscript ℓ if the buyer takes out a loan and the collateral constraint is nonbinding, a subscript $\bar{\ell}$ if the buyer takes out a loan and the collateral constraint is binding,and a subscript 0 if the buyer does neither.

From (4.6), the consumption level of a buyer who enters the loan market satisfies:

$$q_d(\varepsilon) = \frac{\varepsilon}{p\beta\phi^+(1+i_d)}, \quad q_\ell(\varepsilon) = \frac{\varepsilon}{p\beta\phi^+(1+i_\ell)}. \tag{7.1}$$

A buyer who does not use the deposit facilities will spend all his money on goods – If he anticipated that he will have idle cash after the goods trade, it would be optimal to deposit the idle cash in the intermediary, provided $i_d > 0$. Thus, consumption of such a buyer is:

$$q_0(\varepsilon) = \frac{m}{p}. \tag{7.2}$$

At $\varepsilon = \varepsilon_d$, the household is indifferent between depositing and not depositing. We can write this indifference condition as:

$$\varepsilon_d u(q_d) - \beta\phi^+ (pq_d - i_d d) = \varepsilon_d u(q_0) - \beta\phi^+ pq_0.$$

By using (7.1), (7.2), and $d = m - pq_d$, we can write the equation further as

$$\varepsilon_d \ln\left[\frac{\varepsilon_d}{\beta\phi^+ (1 + i_d) m}\right] = \varepsilon_d - (1 + i_d) \beta\phi^+ m.$$

The unique solution to this equation is $\varepsilon_d = (1 + i_d) \beta\phi^+ m$, which implies that $\beta\phi^+ m < \varepsilon_d$.

At $\varepsilon = \varepsilon_\ell$, the household is indifferent between borrowing and not borrowing. We can write this indifference condition as

$$\varepsilon_\ell u(q_\ell) - \beta\phi^+ (pq_\ell + i_\ell \ell) = \varepsilon_\ell u(q_0) - \beta\phi^+ pq_0.$$

Using (7.1), (7.2) and $\ell = pq_\ell - m$, we can write this equation further as

$$\varepsilon_\ell \ln\left[\frac{\varepsilon_\ell}{(1 + i_\ell) \beta\phi^+ m}\right] = \varepsilon_\ell - (1 + i_\ell) \beta\phi^+ m$$

The unique solution to this equation is $\varepsilon_\ell = (1 + i_\ell)\,\beta\phi^+ m$. Using the expression for ε_d we get

$$\varepsilon_\ell = \varepsilon_d/\Delta$$

We now calculate $\varepsilon_{\bar{\ell}}$. There is a critical buyer who enters the goods market and wants to take out a loan who's collateral constraint is just binding. From (4.6), for this buyer we have the following equilibrium conditions

$$\begin{aligned} q_{\bar{\ell}} &= \frac{\varepsilon_{\bar{\ell}}}{(1 + i_\ell)\,\beta\phi^+ p} \\ pq_{\bar{\ell}} &= m + b/(1 + i_\ell) \end{aligned}$$

Eliminating $q_{\bar{\ell}}$ we get

$$\varepsilon_{\bar{\ell}} = (1 + i_\ell)\,\beta\phi^+ m + \beta\phi^+ b$$

Using the expression for ε_ℓ we get

$$\varepsilon_{\bar{\ell}} = (\varepsilon_d/\Delta)\left(1 + \frac{b}{m\,(1 + i_\ell)}\right)$$

It is then evident that

$$\varepsilon_d \leq \varepsilon_\ell \leq \varepsilon_{\bar{\ell}}$$

In order to derive an expression for $\varepsilon_{\bar{\ell}}$ that only depends on the endogenous variables ρ_T and ε_d we need to derive the ratio b/m. Recall that M is the quantity of money a household holds at the beginning of the bond market and B the amount of bonds at the beginning of the bond market. Furthermore, $\varepsilon-$buyers do not trade in the bond market. The reason is that in any equilibrium we have $\rho_S = \rho_T$ so any desirable portfolio trade they could have done in the previous settlement market already. Hence,

$$b/m = B/M$$

which allows us to rewrite $\varepsilon_{\bar{\ell}}$ as follows

$$\varepsilon_{\bar{\ell}} = (\varepsilon_d/\Delta)\left(1 + \frac{B}{M(1+i_\ell)}\right).$$

□

Proof of Lemma 2. For easier reference we replicate the $e-$buyer's optimization problem in the bond market here:

$$\max_{y_e} \int_0^\infty V_G(m - \rho_T y_e, b + y_e | e)\, dF(e)$$
$$m - \rho_T y_e \geq 0 \text{ and } b + y_e \geq 0$$

Because of the Inada condition on consumption utility, the first constraint is never binding if $i_\ell > i_T$ and, therefore, we ignore it. Denote $\beta\phi^+\lambda$ the multiplier for the second inequality. Then the FOC is

$$-\rho_T V_G^m(m - \rho_T y_e, b + y_e | e) + V_G^b(m - \rho_T y_e, b + y_e | e) + \beta\phi^+\lambda = 0 \tag{7.3}$$

Use (4.13) to replace $V_G^m(m - \rho_T y_e, b + y_e | e)$ and $V_G^b(m - \rho_T y_e, b + y_e | e)$ to get

$$-\rho_T\beta\phi^+(1+\lambda_e) + \beta\phi^+ + \beta\phi^+\lambda = 0$$

which implies that

$$1 + \lambda_e = (1+\lambda)(1+i_T)$$

where $i_T \equiv (1-\rho_T)/\rho_T$. We can use this relation to rewrite the first-order condition in the goods market as follows

$$eu'(q_e) - \beta p\phi^+(1+\lambda)(1+i_T) = 0$$

Finally, use the sellers' first-order condition to get

$$eu'(q_e) - (1+\lambda)\Delta_T = 0$$

where

$$\Delta_T = \frac{1+i_T}{1+i_d}$$

The optimal buying and selling decisions in the bond market follow the cut-off rule according to the realization of the taste shock. The cut-off level $\tilde{e}$ partitions the set of taste shocks into two regions. If $e < \tilde{e}$, the short-selling constraint on bonds is non-binding, i.e., $\lambda = 0$, implying that

$$eu'(q_e) - \Delta_T = 0$$

If $e \geq \tilde{e}$, the short-selling constraint on bonds is binding. The e-buyer sells all bonds for money and consumes

$$pq_e = m + \rho_T b$$

Note that for a very high e the buyer would like to borrow at the SF but he can't because he has no collateral left. □

Proof of Lemma 3. For easier reference, we replicate the $\varepsilon-$buyer's optimization problem in the bond market here:

$$\max_y \int_0^\infty V_G(m - \rho_T y, b + y|\,\varepsilon)\,dF(\varepsilon)$$
$$\text{s.t. } m - \rho_T y \geq 0 \text{ and } b + y \geq 0$$

Note that an ε-buyer will never sell all his money for bonds or bonds for money which implies that in any equilibrium the following first-order condition holds

$$\rho_T = \frac{\int_0^\infty V_G^b\left(m - \rho_T y, b + y|\,\varepsilon\right) dF\left(\varepsilon\right)}{\int_0^\infty V_G^m\left(m - \rho_T y, b + y|\,\varepsilon\right) dF\left(\varepsilon\right)} \tag{7.4}$$

Differentiate $V_T\left(m, b\right)$ with respect to m and b to get

$$\begin{aligned} V_T^m\left(m, b\right) &= n \int_0^\infty V_G^m\left(m - \rho_T y_e, b + y_e|\,e\right) dF\left(e\right) \\ &\quad + \left(1 - n\right) \int_0^\infty V_G^m\left(m - \rho_T y, b + y|\,\varepsilon\right) dF\left(\varepsilon\right) \end{aligned}$$

and

$$\begin{aligned} V_T^b\left(m, b\right) &= n \int_0^\infty \left[V_G^b\left(m - \rho_T y_e, b + y_e|\,e\right) + \beta\phi^+\lambda\right] dF\left(e\right) \\ &\quad + \left(1 - n\right) \int_0^\infty V_G^b\left(m - \rho_T y, b + y|\,\varepsilon\right) dF\left(\varepsilon\right) \end{aligned}$$

respectively. Use (7.3) to eliminate $\beta\phi^+\lambda$ in the second equality and (7.4) eliminate $V_G^b\left(m - \rho_T y, b + y|\,\varepsilon\right)$ to get

$$\rho_T V_T^m\left(m, b\right) = V_T^b\left(m, b\right)$$

Use (4.2) and (4.3) to replace $V_T^m\left(m, b\right)$ and $V_T^b\left(m, b\right)$ to get

$$\rho \equiv \rho_T = \rho_S.$$

□

Proof of Proposition 1. In this proof we show that how to derive two equations that solve for the endogenous variables ρ and ε_d. We first derive the envelope condition. Use (7.3) to eliminate $\beta\phi^+\lambda$ in the second equality and (7.4) eliminate $V_G^b(m-\rho_T y, b+y|\,\varepsilon)$ to get

$$\begin{aligned} V_T^m(m,b) &= n\int_0^\infty V_G^m(m-\rho_T y_e, b+y_e|\,e)\,dF(e) \\ &\quad +(1-n)\int_0^\infty V_G^m(m-\rho_T y, b+y|\,\varepsilon)\,dF(\varepsilon) \end{aligned}$$

and

$$\begin{aligned} \frac{V_T^b(m,b)}{\rho} &= n\int_0^\infty V_G^m(m-\rho_T y_e, b+y_e|\,e)\,dF(e) \\ &\quad +(1-n)\int_0^\infty V_G^m(m-\rho_T y, b+y|\,\varepsilon)\,dF(\varepsilon) \end{aligned}$$

respectively. Use equations (4.13) and (4.9) to replace $V_G^m(m-\rho_T y_e, b+y_e|\,e)$ and $V_G^m(m-\rho_T y, b+y|\,\varepsilon)$ to get

$$\begin{aligned} V_T^m(m,b) &= n\int_0^\infty \frac{eu'(q_e)}{p}dF(e)+(1-n)\int_0^\infty \frac{\varepsilon u'(q_\varepsilon)}{p}dF(\varepsilon) \\ \frac{V_T^b(m,b)}{\rho} &= n\int_0^\infty \frac{eu'(q_e)}{p}dF(e)+(1-n)\int_0^\infty \frac{\varepsilon u'(q_\varepsilon)}{p}dF(\varepsilon) \end{aligned}$$

Finally, use the first-order condition for q_s to get:

$$V_T^m(m,b)=\beta\phi^+(1+i_d)\left[n\int_0^\infty eu'(q_e)\,dF(e)+(1-n)\int_0^\infty \varepsilon u'(q_\varepsilon)\,dF(\varepsilon)\right]$$

We now prove that this equation is an equation in the two endogenous variables ρ and ε_d only. Using (4.2) to replace $V_T^m(m,b)$ and (5.14) to replace the ϕ^+/ϕ yields

$$\frac{\rho\gamma}{\beta} = \rho(1+i_d)\left[n\int_0^\infty eu'(q_e)\,dF(e) + (1-n)\int_0^\infty \varepsilon u'(q_\varepsilon)\,dF(\varepsilon)\right]$$

Finally, note that $u'(q) = 1/q$ and replace q_ε using Lemma 2 and q_e using Lemma 1 to get

$$\begin{aligned}
&\Theta(\rho,\varepsilon_d) \equiv -\rho\gamma/\beta + n\left[\int_0^{\tilde{e}} dF(e) + \int_{\tilde{e}}^\infty \tfrac{e}{\tilde{e}}dF(e)\right] \\
&+\rho(1+i_d)(1-n)\left[\int_0^{\varepsilon_d} dF(\varepsilon) + \int_{\varepsilon_d}^{\varepsilon_\ell} \tfrac{\varepsilon}{\varepsilon_d}dF(\varepsilon) + \int_{\varepsilon_\ell}^{\varepsilon_{\bar{\ell}}} \tfrac{1}{\Delta}dF(\varepsilon) + \int_{\varepsilon_{\bar{\ell}}}^\infty \tfrac{\varepsilon}{\Delta\varepsilon_{\bar{\ell}}}dF(\varepsilon)\right] \\
&= 0
\end{aligned} \tag{7.5}$$

Note that since $\varepsilon_\ell = \varepsilon_d/\Delta$, $\varepsilon_{\bar{\ell}} = (\varepsilon_d/\Delta)\left(1+\frac{B}{M(1+i_\ell)}\right)$, and $\tilde{e} = (\varepsilon_d/\Delta_T)\left(1+\frac{B}{M(1+i_T)}\right)$ the cutoff values are functions of ρ and ε_d only. Hence, the first-order condition is a function of ρ and ε_d only.

We obtain a second equation in ρ and ε_d from (7.4). Use equation (4.9) to replace $V_G^b(m-\rho y, b+y|\,\varepsilon)$ and $V_G^m(m-\rho y, b+y|\,\varepsilon)$ to get

$$\rho = \frac{\int_0^\infty \left(1+\frac{\lambda_\ell}{1+i_\ell}\right)dF(\varepsilon)}{\int_0^\infty (1+\lambda_\varepsilon)\,dF(\varepsilon)}$$

Use (4.6) to replace λ_ℓ and rearrange to get

$$\rho(1+i_\ell)\int_0^\infty (1+\lambda_\varepsilon)\,dF(\varepsilon) = \int_0^{\varepsilon_{\bar{\ell}}} (1+i_\ell)\,dF(\varepsilon) + \int_{\varepsilon_{\bar{\ell}}}^\infty (1+\lambda_\varepsilon)\,dF(\varepsilon)$$

Use (4.6) to replace $(1+\lambda_\varepsilon)$ and rearrange to get

$$\rho\left(1+i_\ell\right)\int_0^\infty \frac{\varepsilon u'\left(q_\varepsilon\right)}{\beta\phi^+ p}dF\left(\varepsilon\right)=\int_0^{\varepsilon_{\bar\ell}}\left(1+i_\ell\right)dF\left(\varepsilon\right)+\int_{\varepsilon_{\bar\ell}}^\infty \frac{\varepsilon u'\left(q_\varepsilon\right)}{\beta\phi^+ p}dF\left(\varepsilon\right)$$

Multiply by $\beta\phi^+ p$ and rearrange to get

$$\rho\left(1+i_\ell\right)\int_0^\infty \varepsilon u'\left(q_\varepsilon\right)dF\left(\varepsilon\right)=\int_0^{\varepsilon_{\bar\ell}}\left(1/\Delta\right)dF\left(\varepsilon\right)+\int_{\varepsilon_{\bar\ell}}^\infty \varepsilon u'\left(q_\varepsilon\right)dF\left(\varepsilon\right)$$

Note that $u'\left(q_\varepsilon\right)=1/q_\varepsilon$ and replace q_ε using Lemma 1 to get

$$\rho\left(1+i_\ell\right)\left[\int_0^{\varepsilon_d}dF\left(\varepsilon\right)+\int_{\varepsilon_d}^{\varepsilon_\ell}\frac{\varepsilon}{\varepsilon_d}dF\left(\varepsilon\right)+\int_{\varepsilon_\ell}^{\varepsilon_{\bar\ell}}\frac{1}{\Delta}dF\left(\varepsilon\right)+\int_{\varepsilon_{\bar\ell}}^\infty\frac{\varepsilon}{\varepsilon_{\bar\ell}\Delta}dF\left(\varepsilon\right)\right]$$
$$=\int_0^{\varepsilon_{\bar\ell}}\frac{1}{\Delta}dF\left(\varepsilon\right)+\int_{\varepsilon_{\bar\ell}}^\infty\frac{\varepsilon}{\varepsilon_{\bar\ell}\Delta}dF\left(\varepsilon\right) \tag{7.6}$$

Note that since $\varepsilon_\ell=\varepsilon_d/\Delta$, $\varepsilon_{\bar\ell}=\left(\varepsilon_d/\Delta\right)\left(1+\frac{B}{M(1+i_\ell)}\right)$ the cutoff values are functions of ρ and ε_d only. Hence, Ψ is a function of ρ and ε_d only.

We now derive the third equilibrium equation which comes from the law of motion of money. The first step of the proof is to derive an expression for the price of goods p. From the $\varepsilon-$buyer with $\varepsilon_d\leq\varepsilon\leq\varepsilon_\ell$, we know that $d_\varepsilon=\ell_\varepsilon=0$. Hence, from their budget cash constraint in the goods market we know that they spends all their money buying goods, that is

$$pq_\varepsilon=m \tag{7.7}$$

Replace $q_\varepsilon = \varepsilon_d$ to get

$$p = \frac{M}{\varepsilon_d}. \tag{7.8}$$

The second step of the proof is to derive aggregate loans, L.

From Lemma 1, we know that the agents who borrow are buyers with a shock $\varepsilon \geq \varepsilon_\ell$. Thus aggregate lending is

$$L = (1-n)\int_{\varepsilon_\ell}^{\infty} \ell_\varepsilon dF(\varepsilon)$$

From Lemma 1, we also know that $\ell_\varepsilon = p(\Delta\varepsilon - \varepsilon_d)$ if $\varepsilon_\ell \leq \varepsilon \leq \varepsilon_{\bar{\ell}}$, and $\ell_\varepsilon = b/(1+i_\ell) = p(\Delta\varepsilon_{\bar{\ell}} - \varepsilon_d)$ if $\varepsilon \geq \varepsilon_{\bar{\ell}}$. Thus, splitting the integral and replacing ℓ_ε, the last expression can be rewritten as

$$L = (1-n)\frac{M}{\varepsilon_d}\left[\int_{\varepsilon_\ell}^{\varepsilon_{\bar{\ell}}} (\Delta\varepsilon - \varepsilon_d)\, dF(\varepsilon) + \int_{\varepsilon_{\bar{\ell}}}^{\infty} (\Delta\varepsilon_{\bar{\ell}} - \varepsilon_d)\, dF(\varepsilon)\right] \tag{7.9}$$

where we have also used (7.8) to pin down p. The above equation is a function of ε_d.

The third step of the proof is to derive aggregate deposits, D.

From Lemma 1, we know that depositors are: (i) sellers and (ii) buyers with a shock $\varepsilon < \varepsilon_d$. We also know that sellers deposit all their proceeds from sales in the goods markets. Hence aggregate deposits are

$$D = pq_s + (1-n)\int_0^{\varepsilon_d} d_\varepsilon dF(\varepsilon)$$

From Lemma 1, we also know that $d_\varepsilon = p(\varepsilon_d - \varepsilon)$. Thus, replacing d_ε, the last expression can be rewritten as

$$D = pq_s + (1-n)\frac{M}{\varepsilon_d}\int_0^{\varepsilon_d} (\varepsilon_d - \varepsilon)\, dF(\varepsilon)$$

where we have also eliminated p using (7.8). Clearing condition in the goods market implies

$$q_s = n\int_0^\infty q_e dF(e) + (1-n)\int_0^\infty q_\varepsilon dF(\varepsilon)$$

Splitting the integrals and replacing the terms q_e and q_ε using Lemma 1 and Lemma 2, the above expression can be rewritten as

$$\begin{aligned} q_s &= n\int_0^\infty q_e dF(e) \\ &\quad + (1-n)\left[\int_0^{\varepsilon_d} \varepsilon dF(\varepsilon) + \int_{\varepsilon_d}^{\varepsilon_\ell} \varepsilon_d dF(\varepsilon) + \int_{\varepsilon_\ell}^{\varepsilon_{\bar{\ell}}} \Delta\varepsilon dF(\varepsilon) + \int_{\varepsilon_{\bar{\ell}}}^\infty \Delta\varepsilon_{\bar{\ell}} dF(\varepsilon)\right] \end{aligned}$$

Replacing q_s using the last equation, aggregate deposits are

$$\begin{aligned} D &= (1-n)\frac{M}{\varepsilon_d}\int_0^{\varepsilon_d} (\varepsilon_d - \varepsilon) dF(\varepsilon) + n\int_0^\infty p q_e dF(e) \\ &\quad + (1-n)\frac{M}{\varepsilon_d}\left[\int_0^{\varepsilon_d} \varepsilon dF(\varepsilon) + \int_{\varepsilon_d}^{\varepsilon_\ell} \varepsilon_d dF(\varepsilon) + \int_{\varepsilon_\ell}^{\varepsilon_{\bar{\ell}}} \Delta\varepsilon dF(\varepsilon) + \int_{\varepsilon_{\bar{\ell}}}^\infty \Delta\varepsilon_{\bar{\ell}} dF(\varepsilon)\right] \end{aligned}$$

where we have also used (7.8) to pin down p. Simplifying terms, the first expression can be rewritten as

$$\begin{aligned} D &= (1-n)\frac{M}{\varepsilon_d}\int_0^{\varepsilon_d} \varepsilon_d F(\varepsilon) + n\int_0^\infty p q_e dF(e) \\ &\quad + (1-n)\frac{M}{\varepsilon_d}\left[\int_{\varepsilon_d}^{\varepsilon_\ell} \varepsilon_d dF(\varepsilon) + \int_{\varepsilon_\ell}^{\varepsilon_{\bar{\ell}}} \Delta\varepsilon dF(\varepsilon) + \int_{\varepsilon_{\bar{\ell}}}^\infty \Delta\varepsilon_{\bar{\ell}} dF(\varepsilon)\right] \end{aligned}$$

Replacing $\int_{0}^{\varepsilon_d} \varepsilon_d F(\varepsilon)$ with $\varepsilon_d - \int_{\varepsilon_d}^{\infty} \varepsilon_d F(\varepsilon)$ and rearranging terms, the last expression can be rewritten as

$$\begin{aligned} D &= (1-n) M + n \int_{0}^{\infty} pq_e dF(e) \\ &+ (1-n) \frac{M}{\varepsilon_d} \left[\int_{\varepsilon_\ell}^{\varepsilon_{\bar{\ell}}} (\Delta\varepsilon - \varepsilon_d) dF(\varepsilon) + \int_{\varepsilon_{\bar{\ell}}}^{\infty} (\Delta\varepsilon_{\bar{\ell}} - \varepsilon_d) dF(\varepsilon) \right] \end{aligned}$$

Note that the amount of money spent by a $e-$buyer in the goods market is given by the amount of money he enters the treasury market minus the amount of money spent in bonds purchases, that is $pq_e = M - \rho y_e$. Replacing pq_e into the above equation yields

$$\begin{aligned} D &= (1-n) M + n \int_{0}^{\infty} M - \rho y_e dF(e) \\ &+ (1-n) \frac{M}{\varepsilon_d} \left[\int_{\varepsilon_\ell}^{\varepsilon_{\bar{\ell}}} (\Delta\varepsilon - \varepsilon_d) dF(\varepsilon) + \int_{\varepsilon_{\bar{\ell}}}^{\infty} (\Delta\varepsilon_{\bar{\ell}} - \varepsilon_d) dF(\varepsilon) \right] \end{aligned}$$

Since $\varepsilon-$buyers do not trade bonds in the treasury market, clearing condition implies

$$\int_{0}^{\infty} y_e dF(e) = 0$$

Replacing $\int_{0}^{\infty} y_e dF(e) = 0$ into the aggregate deposits we obtain

$$D = M + (1-n) \frac{M}{\varepsilon_d} \left[\int_{\varepsilon_\ell}^{\varepsilon_{\bar{\ell}}} (\Delta\varepsilon - \varepsilon_d) dF(\varepsilon) + \int_{\varepsilon_{\bar{\ell}}}^{\infty} (\Delta\varepsilon_{\bar{\ell}} - \varepsilon_d) dF(\varepsilon) \right] \tag{7.10}$$

which is an expression in ε_d. The last equation means that the total amount of deposits is equal to the sum of two components. The first component is the total amount of money, M, held by buyers at the beginning of the goods market, and the second component is the total amount of lending, L.

Divide both sides of (4.1) by M and get

$$\gamma = 1 + \frac{i_d D - i_\ell L}{M} + \tau$$

Eliminating D and L using (7.10), respectively (7.9), the last expression can be re-written as

$$\begin{aligned}\gamma &= 1 + i_d \left[1 + \frac{(1-n)}{\varepsilon_d} \left[\int_{\varepsilon_\ell}^{\varepsilon_{\bar{\ell}}} (\Delta\varepsilon - \varepsilon_d)\, dF(\varepsilon) + \int_{\varepsilon_{\bar{\ell}}}^{\infty} (\Delta\varepsilon_{\bar{\ell}} - \varepsilon_d)\, dF(\varepsilon) \right] \right] \\ &\quad - \frac{(1-n)\, i_\ell}{\varepsilon_d} \left[\int_{\varepsilon_\ell}^{\varepsilon_{\bar{\ell}}} (\Delta\varepsilon - \varepsilon_d)\, dF(\varepsilon) + \int_{\varepsilon_{\bar{\ell}}}^{\infty} (\Delta\varepsilon_{\bar{\ell}} - \varepsilon_d)\, dF(\varepsilon) \right] + \tau\end{aligned}$$

Simplifying terms, the last equation can be rewritten as

$$\gamma = 1 + i_d + \frac{(1-n)(i_d - i_\ell)}{\varepsilon_d} \left[\int_{\varepsilon_\ell}^{\varepsilon_{\bar{\ell}}} (\Delta\varepsilon - \varepsilon_d)\, dF(\varepsilon) + \int_{\varepsilon_{\bar{\ell}}}^{\infty} (\Delta\varepsilon_{\bar{\ell}} - \varepsilon_d)\, dF(\varepsilon) \right] + \tau \tag{7.11}$$

which is a function in γ and ε_d.

Accordingly, we can solve (7.5), (7.6) and (7.11) for ρ, ε_d and γ given policy $(B/M, i_d, i_{i_\ell}, \tau)$ and exogenous variables. □

Proof of Lemma 4. We first derive the cut-off value ε_b. In what follows we use the following notation: a subscript $\mathcal{B}$ if the buyer buys bonds at the central bank, a subscript $\mathcal{S}$ if the buyer sell bonds and the bonds constraint is nonbinding, an overbar

"−" if the bonds constraint is binding, and a subscript 0 if the buyer does neither. Let the subscript b refers to buyers whose bonds constraint is nonbinding.

From (5.7), the consumption level of a buyer whose bond constraint is not binding:

$$q_{\mathcal{B}}(\varepsilon) = q_{\mathcal{S}}(\varepsilon) = \frac{\varepsilon \rho_G}{p\beta\phi^+}. \tag{7.12}$$

A buyer who neither buys nor sells bonds spends all his money in goods purchases:

$$q_0 = \frac{m}{p}. \tag{7.13}$$

At $\varepsilon = \varepsilon_{\mathcal{B}}$, the household is indifferent between buying bonds and not buying, and at $\varepsilon = \varepsilon_{\mathcal{S}}$ she is indifferent between selling bonds and not selling. We can write these indifferent conditions as

$$\varepsilon_{\mathcal{B}} u(q_{\mathcal{B}}) - \beta\phi^+ \left(pq_{\mathcal{B}} - \frac{1-\rho_G}{\rho_G} a_{\mathcal{B}} \right) = \varepsilon_{\mathcal{B}} u(q_0) - \beta\phi^+ pq_0$$

and

$$\varepsilon_{\mathcal{S}} u(q_{\mathcal{S}}) - \beta\phi^+ \left(pq_{\mathcal{S}} - \frac{1-\rho_G}{\rho_G} a_{\mathcal{S}} \right) = \varepsilon_{\mathcal{S}} u(q_0) - \beta\phi^+ pq_0$$

respectively. Using (7.12), (7.13), and $a_{\mathcal{B}} = m - pq_{\mathcal{B}}$ and $a_{\mathcal{S}} = m - pq_{\mathcal{S}}$, the last two equations can be rewritten as

$$\varepsilon_{\mathcal{B}} \ln\left(\frac{\varepsilon_{\mathcal{B}}\rho_G}{\beta\phi^+ m}\right) = \varepsilon_{\mathcal{B}} - \frac{\beta\phi^+ m}{\rho_G}$$

and

$$\varepsilon_{\mathcal{S}} \ln\left(\frac{\varepsilon_{\mathcal{S}}\rho_G}{\beta\phi^+ m}\right) = \varepsilon_{\mathcal{S}} - \frac{\beta\phi^+ m}{\rho_G}$$

The unique solution to the last two equations is

$$\varepsilon_{\mathcal{B}} = \frac{\beta\phi^+ m}{\rho_G}$$

and

$$\varepsilon_{\mathcal{S}} = \frac{\beta\phi^{+}m}{\rho_G}$$

respectively. Which implies $\varepsilon_b \equiv \varepsilon_{\mathcal{B}} = \varepsilon_{\mathcal{S}}$. Hence at $\varepsilon = \varepsilon_b$ buyers are indifferent between buying bonds, selling bonds and doing neither.

We now calculate $\bar{\varepsilon}$. There is a critical buyer who enters the goods market and wants to take out a loan who's collateral constraint is just binding. From (4.6), for this buyer we have the following equilibrium conditions

$$\begin{aligned} \bar{q} &= \frac{\rho_G \bar{\varepsilon}}{\beta\phi^{+}p} \\ p\bar{q} &= m + b\rho_G \end{aligned}$$

Eliminating $\bar{q}$ we get

$$\bar{\varepsilon} = \frac{\beta\phi^{+}m}{\rho_G} + \beta\phi^{+}b$$

Using the expression for ε_b we get

$$\bar{\varepsilon} = \varepsilon_b \left(1 + \frac{b\rho_G}{m}\right)$$

It is then evident that

$$\varepsilon_b \leq \bar{\varepsilon}$$

In order to derive an expression for $\bar{\varepsilon}$ that only depends on the endogenous variables ρ_G and ε_b we need to derive the ratio b/m. Recall that M is the quantity of money a household holds at the beginning of the bond market and B the amount of bonds at the beginning of the bond market. Furthermore, $\varepsilon-$buyers and $e-$buyers do not trade in the bond market. The former do not trade because they could have trade bonds in the previous settlement market already, since in any equilibrium $\rho_S = \rho_T$. The latter do not trade because they can always trade bonds in the goods market , since in any equilibrium $\rho_T = \rho_G$. Hence,

$$b/m = B/M$$

which allows us to rewrite $\bar{\varepsilon}$ as follows

$$\bar{\varepsilon} = \varepsilon_b \left(1 + \frac{B\rho_G}{M}\right)$$

□

Proof of Lemma 5. The Proof of Lemma 5 is similar as the Proof of Lemma 4 and is omitted. □

Proof of Lemma 6. Assume $\rho_T > \rho_G$. Then a buyer can get 1 unit of money by selling $1/\rho_T$ units of bonds in the bond market, and he can buy $1/\rho_T$ unit of bonds with $\rho_G/\rho_T < 1$ units of money, thus making a profit. This implies that all buyers want to sell bonds for money in the bond market, which cannot constitute an equilibrium.

Assume $\rho_T < \rho_G$. Then a buyer can get 1 unit of bonds by spending ρ_T units of money in the bond market, while he can sell 1 unit of bond and receive $\rho_G > \rho_T$ units of money in the goods market, and making a profit by doing so. This implies that all buyers wish to buy bonds in the bond market, which cannot constitute and equilibrium. □

Proof of Proposition 2. For easier reference we rewrite here the buyers' problem in the bond market. We focus on $\varepsilon-$buyers first, then we describe the $e-$buyers' decisions.

The $\varepsilon-$buyer's problem in the bond market is

$$V_T(m,b|\,.) = \max_y \int_0^\infty V_G(m-\rho_T y, b+y|\,\varepsilon)\,dF(\varepsilon)$$

$$\text{s.t. } m-\rho_T y \geq 0, \text{ and}$$

$$b+y \geq 0$$

Let $\phi\lambda_m$ and $\phi\lambda_b$ denote the multipliers for the first inequality and the second inequality, respectively. Then, the FOC is

$$-\rho_T \int_0^\infty V_G^m + \int_0^\infty V_G^b - \rho_T \phi\lambda_m + \phi\lambda_b = 0$$

Replace V_G^b and V_G^m using (5.9), and use Lemma 6, to get $\rho_T \lambda_m = \lambda_b$, which holds if and only if $\lambda_m = \lambda_b = 0$ since both constraints cannot bind at the same time. That is, $\varepsilon-$buyers are indifferent between buying and selling bonds in the bond market, since the price of bonds is the same in the bond market and in the goods market.

In order to derive the envelope condition, we take the derivative of $V_T(m,b|\,.)$ with respect to m and b and obtain

$$\begin{aligned} V_T^m(m,b|\,.) &= \int_0^\infty V_G^m(m-\rho_T y, b+y|\,\varepsilon)\,dF(\varepsilon)\,, \text{ and} \\ V_T^b(m,b|\,.) &= \int_0^\infty V_G^b(m-\rho_T y, b+y|\,\varepsilon)\,dF(\varepsilon) \end{aligned}$$

respectively. Replacing V_G^m and V_G^b using (5.9), the last two equations can be rewritten as follows

$$
\begin{aligned}
(7.14)\qquad V_T^m(m,b|\,.) &= \int_0^\infty \beta\phi^+ (1+\lambda_\varepsilon)\, dF(\varepsilon)\,, \text{ and} \\
V_T^b(m,b|\,.) &= \int_0^\infty \rho_G\beta\phi^+ (1+\lambda_\varepsilon)\, dF(\varepsilon)\,.
\end{aligned}
$$

The problem for a $e-$buyer in the bond market is

$$
V_T(m,b|\,e) = \max_{y_e} \int_0^\infty V_G(m-\rho_T y_e, b+y_e|\,e)\, dF(e)
$$

$$
\text{s.t. } m-\rho_T y_e \geq 0, \text{ and}
$$

$$
b+y_e \geq 0
$$

Let $\phi\lambda_{m,e}$ and $\phi\lambda_{b,e}$ denote the multipliers for the first inequality and the second inequality, respectively. Then, the FOC is

$$
-\rho_T \int_0^\infty V_G^m dF(e) + \int_0^\infty V_G^b dF(e) - \rho_T\phi\lambda_{m,e} + \phi\lambda_{b,e} = 0
$$

Replace V_G^b and V_G^m using (5.12), and use Lemma 6, to get $\rho_T\lambda_{m,e} = \lambda_{b,e}$, which holds if and only if $\lambda_{m,e} = \lambda_{b,e} = 0$ since both constraints cannot bind at the same time. That is, $e-$buyers are indifferent between buying and selling bonds in the bond market, since the price of bonds is the same in the bond market and in the goods market.

In order to derive the envelope conditions, we take the derivative of $V_T(m, b|\, e)$ with respect to m and b and obtain

$$\begin{aligned} V_T^m(m, b|\, e) &= \int_0^\infty V_G^m(m - \rho_T y, b + y|\, e)\, dF(e), \text{ and} \\ V_T^b(m, b|\, e) &= \int_0^\infty V_G^b(m - \rho_T y, b + y|\, e)\, dF(e) \end{aligned}$$

respectively. Replacing V_G^m and V_G^b using (5.12), the last two equations can be rewritten as follows

$$\begin{aligned} V_T^m(m, b|\, e) &= \int_0^\infty \beta\phi^+ (1 + \lambda_e)\, dF(e), \text{ and} \\ V_T^b(m, b|\, e) &= \int_0^\infty \rho_G \beta\phi^+ (1 + \lambda_e)\, dF(e). \end{aligned} \tag{7.15}$$

In order to derive the envelope conditions for a buyer in the bond market, take the derivative of (5.13) with respect to m and b and get

$$\begin{aligned} V_T^m(m, b) &= n \int_0^\infty V_T^m(m, b|\, e)\, dF(e) + (1 - n)\, V_T^m(m, b|\, .), \text{ and} \\ V_T^b(m, b) &= n \int_0^\infty V_T^b(m, b|\, e)\, dF(e) + (1 - n)\, V_T^b(m, b|\, .) \end{aligned}$$

Replacing V_T^m and V_T^b using (7.14) and (7.15), we can rewrite the last two equations as

$$\begin{aligned} V_T^m(m,b) &= n\int_0^\infty \beta\phi^+(1+\lambda_e)\,dF(e) + (1-n)\int_0^\infty \beta\phi^+(1+\lambda_\varepsilon)\,dF(\varepsilon)\,, \text{ and} \\ \frac{V_T^b(m,b)}{\rho_G} &= n\int_0^\infty \beta\phi^+(1+\lambda_e)\,dF(e) + (1-n)\int_0^\infty \beta\phi^+(1+\lambda_\varepsilon)\,dF(\varepsilon) \end{aligned}$$

Dividing by $\beta\phi^+$, and replacing $V_T^m(m,b)$ and $V_T^b(m,b)$ using (4.2) and (4.3), the last two equations implies $\rho_S = \rho_G = \rho$ and

$$\frac{\gamma}{\beta} = n\int_0^\infty (1+\lambda_e)\,dF(e) + (1-n)\int_0^\infty (1+\lambda_\varepsilon)\,dF(\varepsilon)\,,$$

Replacing $(1+\lambda_e)$ and $(1+\lambda_\varepsilon)$ using (5.6), (5.7) and (5.10), the last equation can be rewritten as

$$\frac{\rho\gamma}{\beta} = n\int_0^\infty eu'(q_e)\,dF(e) + (1-n)\int_0^\infty \varepsilon u'(q_\varepsilon)\,dF(\varepsilon)\,.$$

Finally, using the fact that $u'(q) = 1/q$, and replacing q_ε and q_e using Lemma 4 and Lemma 5, the last equation can be rewritten as

$$\frac{\rho\gamma}{\beta} = n\left[\int_0^{\bar{e}} dF(e) + \int_{\bar{e}}^\infty \frac{e}{\bar{e}}dF(e)\right] + (1-n)\left[\int_0^{\bar{\varepsilon}} dF(\varepsilon) + \int_{\bar{\varepsilon}}^\infty \frac{\varepsilon}{\bar{\varepsilon}}dF(\varepsilon)\right]$$

From Lemma 4 and Lemma 5, $\bar{\varepsilon} = \bar{e}$. Then, the last expression can be rewritten as

$$\frac{\rho\gamma}{\beta} = \int_0^{\bar{e}} dF(e) + \int_{\bar{e}}^\infty \frac{e}{\bar{e}}dF(e)$$

This is one equation in three endogenous variables $(\rho, \gamma, \bar{e})$.

We know derive (5.17). From the market clearing condition in the goods market

$$a_{G,p} + n\int_0^{\infty} a_{G,e}dF(e) + (1-n)\int_0^{\infty} a_{G,\varepsilon}dF(\varepsilon) + Y_G = 0 \tag{7.16}$$

where $a_{G,p}$, $a_{G,\varepsilon}$ and $a_{G,e}$ are the quantity of bonds acquired in the goods market by sellers, $\varepsilon-$buyers and $e-$buyers, respectively. Since agents exit the goods market with no money, from their cash constraint we know that

$$\begin{aligned} a_{G,p} &= \frac{pq}{\rho}, \text{ and} \\ a_{G,\varepsilon} &= \frac{m - pq_\varepsilon}{\rho}, \text{ and} \\ a_{G,e} &= \frac{m - pq_e}{\rho} \end{aligned}$$

where m is the amount of money held by buyers at the beginning of the goods market. Replacing $a_{G,p}$, $a_{G,\varepsilon}$ and $a_{G,e}$ using the last two equations, we can rewrite (7.16) as

$$\frac{pq}{\rho} + n\int_0^{\infty} \frac{m - pq_e}{\rho}dF(e) + (1-n)\int_0^{\infty} \frac{m - pq_\varepsilon}{\rho}dF(\varepsilon) + Y_G = 0$$

The other market clearing condition in the goods market is

$$q = n\int_0^{\infty} q_e dF(e) + (1-n)\int_0^{\infty} q_\varepsilon dF(\varepsilon)$$

Combining the last two equations we get

$$
\begin{aligned}
n\int_0^\infty \frac{m}{\rho}dF(\varepsilon)+(1-n)\int_0^\infty \frac{m}{\rho}dF(\varepsilon)+Y_G &= 0\\
\frac{1}{\rho}\left[n\int_0^\infty m dF(\varepsilon)+(1-n)\int_0^\infty m dF(\varepsilon)\right]+Y_G &= 0\\
\frac{m}{\rho}+Y_G &= 0
\end{aligned}
$$

which implies

$$-\rho Y = M \tag{7.17}$$

where we have replaced M_G and Y_G with M and Y, respectively. The last equation means that the central bank buys all the money. This is the only policy consistent with the fact that all agents exit the goods market with no cash. Since in any equilibrium ρ is constant over time, also the ratio M/Y must be constant, thus $M/Y = M_0/Y_0$. In order to derive the equilibrium equation (5.16), divide (5.1) by M_T and rearrange terms to get

$$\gamma = 1-(1-\rho)\frac{Y_0}{M_0}+\tau$$

or, replacing Y_0/M_0 with $-1/\rho$

$$\gamma = 1/\rho+\tau$$

where we have used the fact that the stock of money supply does not change in the bond market, that is $M_T = M_G \equiv M$. The proof is finished. □

Proof of Proposition 3. Let $\mathcal{Q}^{OMO}$ be the allocation in the OMO economy,

$$\mathcal{Q}^{OMO}\equiv n\left[\int_0^{\bar{e}} e dF(e)+\int_{\bar{e}}^\infty \bar{e} dF(e)\right]+(1-n)\left[\int_0^{\bar{\varepsilon}} \varepsilon dF(\varepsilon)+\int_{\bar{\varepsilon}}^\infty \bar{\varepsilon} dF(\varepsilon)\right] \tag{7.18}$$

The first part of the $\mathcal{Q}^{OMO}$ is total consumption by $e-$buyers in the OMO economy and the second part is total consumption by $\varepsilon-$buyers. Let $\mathcal{Q}^{SF}$ be the allocation in the SF economy

$$\begin{aligned}\mathcal{Q}^{SF} &\equiv n\left[\int_0^{\tilde{e}} \frac{e}{\Delta_T} dF(e) + \int_{\tilde{e}}^{\infty} \frac{\tilde{e}}{\Delta_T} dF(e)\right] \\ &+ (1-n)\left[\int_0^{\varepsilon_d} \varepsilon dF(\varepsilon) + \int_{\varepsilon_d}^{\varepsilon_\ell} \varepsilon_d dF(\varepsilon) + \int_{\varepsilon_\ell}^{\varepsilon_{\bar{\ell}}} \Delta\varepsilon dF(\varepsilon) + \int_{\varepsilon_{\bar{\ell}}}^{\infty} \Delta\varepsilon_{\bar{\ell}} dF(\varepsilon)\right] \qquad (7.19)\end{aligned}$$

where the first line in the right hand side is total consumption by $e-$buyers and the second line is total consumption by $\varepsilon-$buyers. Set $\Delta = 1$, which implies $\Delta_T = 1$. Then $\mathcal{Q}^{SF}$ reduces to

$$\mathcal{Q}^{SF} \equiv n\left[\int_0^{\tilde{e}} e dF(e) + \int_{\tilde{e}}^{\infty} \tilde{e} dF(e)\right] + (1-n)\left[\int_0^{\varepsilon_{\bar{\ell}}} \varepsilon dF(\varepsilon) + \int_{\varepsilon_{\bar{\ell}}}^{\infty} \Delta\varepsilon_{\bar{\ell}} dF(\varepsilon)\right]$$

where we have also used $\varepsilon_d = \varepsilon_\ell$. Since $\tilde{e} = \bar{e} = \bar{\varepsilon} = \Delta\varepsilon_{\bar{\ell}}$. The last two equations imply $\mathcal{Q}^{OMO} = \mathcal{Q}^{SF}$. We have shown that an allocation with OMO can be replicated in the SF economy by choosing $\Delta = 1$. We now prove that the converse is not true.

We know that $\Delta_T = 1/[\rho(1+i_d)] \geq 1$, and $\Delta \equiv (1+i_d)/(1+i_\ell) \leq 1$. From Lemmas 1-2 and Lemmas 4-5, it holds that $\varepsilon_d \leq \bar{\varepsilon} = \bar{e} = \tilde{e} \leq \varepsilon_{\bar{\ell}}$. In what follows, let us use $\bar{e}$ instead of $\bar{\varepsilon}$ and $\tilde{e}$. We can have three possible cases: $\bar{e} > \varepsilon_\ell$, or $\bar{e} < \varepsilon_\ell$, or $\bar{e} = \varepsilon_\ell$.

Case 1: $\bar{e} > \varepsilon_\ell$. This implies the following ordering for the cut-off values, $0 < \varepsilon_d < \varepsilon_\ell < \bar{e} < \varepsilon_{\bar{\ell}} < \infty$. Splitting the integrals according to the above ordering, (7.19) and

(7.18) can be rewritten as

$$
\begin{aligned}
Q^{SF} \equiv \; & n\left[\int_{0}^{\varepsilon_d} \frac{e}{\Delta_T} dF(e) + \int_{\varepsilon_d}^{\varepsilon_\ell} \frac{e}{\Delta_T} dF(e) + \int_{\varepsilon_\ell}^{\bar{e}} \frac{e}{\Delta_T} dF(e) + \int_{\tilde{e}}^{\varepsilon_{\bar{\ell}}} \frac{\bar{e}}{\Delta_T} dF(e) + \int_{\varepsilon_{\bar{\ell}}}^{\infty} \frac{\bar{e}}{\Delta_T} dF(e)\right] \\
& + (1-n)\left[\int_{0}^{\varepsilon_d} \varepsilon dF(\varepsilon) + \int_{\varepsilon_d}^{\varepsilon_\ell} \varepsilon_d dF(\varepsilon) + \int_{\varepsilon_\ell}^{\bar{e}} \Delta\varepsilon dF(\varepsilon) + \int_{\bar{e}}^{\varepsilon_{\bar{\ell}}} \Delta\varepsilon dF(\varepsilon) + \int_{\varepsilon_{\bar{\ell}}}^{\infty} \Delta\varepsilon_{\bar{\ell}} dF(\varepsilon)\right]
\end{aligned}
$$

and

$$
\begin{aligned}
Q^{OMO} \equiv \; & n\left[\int_{0}^{\varepsilon_d} e dF(e) + \int_{\varepsilon_d}^{\varepsilon_\ell} e dF(e) + \int_{\varepsilon_\ell}^{\bar{e}} e dF(e) + \int_{\bar{e}}^{\varepsilon_{\bar{\ell}}} \bar{e} dF(e) + \int_{\varepsilon_{\bar{\ell}}}^{\infty} \bar{e} dF(e)\right] \\
& + (1-n)\left[\int_{0}^{\varepsilon_d} \varepsilon dF(\varepsilon) + \int_{\varepsilon_d}^{\varepsilon_\ell} \varepsilon dF(\varepsilon) + \int_{\varepsilon_\ell}^{\bar{e}} \varepsilon dF(\varepsilon) + \int_{\bar{e}}^{\varepsilon_{\bar{\ell}}} \bar{e} dF(\varepsilon) + \int_{\varepsilon_{\bar{\ell}}}^{\infty} \bar{e} dF(\varepsilon)\right]
\end{aligned}
$$

From Lemmas 1-2 and Lemmas 4-5, it holds that $\Delta\varepsilon_{\bar{\ell}} = (1+i_d)\,\beta\phi^+ m + \Delta\beta\phi^+ b \leq \left(\beta\phi^+ m/\rho\right) + \beta\phi^+ b = \bar{e}$, since $\Delta \leq 1$. Now subtract $\mathcal{Q}^{OMO} - \mathcal{Q}^{SF}$ and obtain

$$\begin{aligned}
&Q^{OMO}-Q^{SF} \\
= \;& n\left[\int_0^{\varepsilon_d} edF(e) + \int_{\varepsilon_d}^{\varepsilon_\ell} edF(e) + \int_{\varepsilon_\ell}^{\bar{e}} edF(e) + \int_{\bar{e}}^{\varepsilon_{\bar{\ell}}} \bar{e}dF(e) + \int_{\varepsilon_{\bar{\ell}}}^{\infty} \bar{e}dF(e)\right] \\
& + (1-n)\left[\int_0^{\varepsilon_d} \varepsilon dF(\varepsilon) + \int_{\varepsilon_d}^{\varepsilon_\ell} \varepsilon dF(\varepsilon) + \int_{\varepsilon_\ell}^{\bar{e}} \varepsilon dF(\varepsilon) + \int_{\bar{e}}^{\varepsilon_{\bar{\ell}}} \bar{e}dF(\varepsilon) + \int_{\varepsilon_{\bar{\ell}}}^{\infty} \bar{e}dF(\varepsilon)\right] \\
& - n\left[\int_0^{\varepsilon_d} \frac{e}{\Delta_T}dF(e) + \int_{\varepsilon_d}^{\varepsilon_\ell} \frac{e}{\Delta_T}dF(e) + \int_{\varepsilon_\ell}^{\bar{e}} \frac{e}{\Delta_T}dF(e) + \int_{\bar{e}}^{\varepsilon_{\bar{\ell}}} \frac{\bar{e}}{\Delta_T}dF(e) + \int_{\varepsilon_{\bar{\ell}}}^{\infty} \frac{\bar{e}}{\Delta_T}dF(e)\right] \\
& - (1-n)\left[\int_0^{\varepsilon_d} \varepsilon dF(\varepsilon) + \int_{\varepsilon_d}^{\varepsilon_\ell} \varepsilon_d dF(\varepsilon) + \int_{\varepsilon_\ell}^{\bar{e}} \Delta\varepsilon dF(\varepsilon) + \int_{\bar{e}}^{\varepsilon_{\bar{\ell}}} \Delta\varepsilon dF(\varepsilon) + \int_{\varepsilon_{\bar{\ell}}}^{\infty} \Delta\varepsilon_{\bar{\ell}} dF(\varepsilon)\right]
\end{aligned}$$

Since $\Delta_T \geq 1$, it is evident that the sum of the integrals in line 2 is bigger than or equal to the sum of the integrals in line 4, with strict inequality when $\Delta_T > 1$. Since $\bar{e} \geq \Delta\varepsilon_{\bar{\ell}}$, it is also evident that the sum of the integrals in line 3 is greater than or equal to the sum of the integrals in line 5, with strict inequality when $\Delta < 1$.

Case 2: $\bar{e} < \varepsilon_\ell$. This implies the following ordering for the cut-off values, $0 < \varepsilon_d < \bar{e} < \varepsilon_\ell < \varepsilon_{\bar{\ell}} < \infty$. Hence (7.19) and (7.18) can be rewritten as

$$\begin{aligned}
Q^{SF} \;\equiv\; & n\left[\int_0^{\varepsilon_d} \frac{e}{\Delta_T}dF(e) + \int_{\varepsilon_d}^{\bar{e}} \frac{e}{\Delta_T}dF(e) + \int_{\bar{e}}^{\varepsilon_\ell} \frac{\bar{e}}{\Delta_T}dF(e) + \int_{\varepsilon_\ell}^{\varepsilon_{\bar{\ell}}} \frac{\bar{e}}{\Delta_T}dF(e) + \int_{\varepsilon_{\bar{\ell}}}^{\infty} \frac{\bar{e}}{\Delta_T}dF(e)\right] \\
& + (1-n)\left[\int_0^{\varepsilon_d} \varepsilon dF(\varepsilon) + \int_{\varepsilon_d}^{\bar{e}} \varepsilon_d dF(\varepsilon) + \int_{\bar{e}}^{\varepsilon_\ell} \varepsilon_d dF(\varepsilon) + \int_{\varepsilon_\ell}^{\varepsilon_{\bar{\ell}}} \Delta\varepsilon dF(\varepsilon) + \int_{\varepsilon_{\bar{\ell}}}^{\infty} \Delta\varepsilon_{\bar{\ell}} dF(\varepsilon)\right]
\end{aligned}$$

and

$$Q^{OMO} \equiv n\left[\int_{0}^{\varepsilon_d} edF\left(e\right) + \int_{\varepsilon_d}^{\bar{e}} edF\left(e\right) + \int_{\bar{e}}^{\varepsilon_\ell} \bar{e}dF\left(e\right) + \int_{\varepsilon_\ell}^{\varepsilon_{\bar{\ell}}} \bar{e}dF\left(e\right) + \int_{\varepsilon_{\bar{\ell}}}^{\infty} \bar{e}dF\left(e\right)\right]$$

$$+\left(1-n\right)\left[\int_{0}^{\varepsilon_d} \varepsilon dF\left(\varepsilon\right) + \int_{\varepsilon_d}^{\bar{e}} \varepsilon dF\left(\varepsilon\right) + \int_{\bar{e}}^{\varepsilon_\ell} \bar{e}dF\left(\varepsilon\right) + \int_{\varepsilon_\ell}^{\varepsilon_{\bar{\ell}}} \bar{e}dF\left(\varepsilon\right) + \int_{\varepsilon_{\bar{\ell}}}^{\infty} \bar{e}dF\left(\varepsilon\right)\right]$$

Subtracting the last two equations we obtain

$$\begin{aligned}
& Q^{OMO} - Q^{SF} \\
= \; & n\left[\int_{0}^{\varepsilon_d} edF\left(e\right) + \int_{\varepsilon_d}^{\bar{e}} edF\left(e\right) + \int_{\bar{e}}^{\varepsilon_\ell} \bar{e}dF\left(e\right) + \int_{\varepsilon_\ell}^{\varepsilon_{\bar{\ell}}} \bar{e}dF\left(e\right) + \int_{\varepsilon_{\bar{\ell}}}^{\infty} \bar{e}dF\left(e\right)\right] \\
& +\left(1-n\right)\left[\int_{0}^{\varepsilon_d} \varepsilon dF\left(\varepsilon\right) + \int_{\varepsilon_d}^{\bar{e}} \varepsilon dF\left(\varepsilon\right) + \int_{\bar{e}}^{\varepsilon_\ell} \bar{e}dF\left(\varepsilon\right) + \int_{\varepsilon_\ell}^{\varepsilon_{\bar{\ell}}} \bar{e}dF\left(\varepsilon\right) + \int_{\varepsilon_{\bar{\ell}}}^{\infty} \bar{e}dF\left(\varepsilon\right)\right] \\
& -n\left[\int_{0}^{\varepsilon_d} \frac{e}{\Delta_T}dF\left(e\right) + \int_{\varepsilon_d}^{\bar{e}} \frac{e}{\Delta_T}dF\left(e\right) + \int_{\tilde{e}}^{\varepsilon_\ell} \frac{\bar{e}}{\Delta_T}dF\left(e\right) + \int_{\varepsilon_\ell}^{\varepsilon_{\bar{\ell}}} \frac{\bar{e}}{\Delta_T}dF\left(e\right) + \int_{\varepsilon_{\bar{\ell}}}^{\infty} \frac{\bar{e}}{\Delta_T}dF\left(e\right)\right] \\
& -\left(1-n\right)\left[\int_{0}^{\varepsilon_d} \varepsilon dF\left(\varepsilon\right) + \int_{\varepsilon_d}^{\bar{e}} \varepsilon_d dF\left(\varepsilon\right) + \int_{\bar{e}}^{\varepsilon_\ell} \varepsilon_d dF\left(\varepsilon\right) + \int_{\varepsilon_\ell}^{\varepsilon_{\bar{\ell}}} \Delta\varepsilon dF\left(\varepsilon\right) + \int_{\varepsilon_{\bar{\ell}}}^{\infty} \Delta\varepsilon_{\bar{\ell}} dF\left(\varepsilon\right)\right]
\end{aligned}$$

On the same lines of Case 1, it is easy to see that $\mathcal{Q}^{OMO} - \mathcal{Q}^{SF} \geq 0$, with strict inequality if $\Delta < 1$.

Case 3: $\bar{e} = \varepsilon_\ell$. This implies the following ordering for the cut-off values, $0 < \varepsilon_d < \bar{e} = \varepsilon_\ell < \varepsilon_{\bar{\ell}} < \infty$. Hence (7.19) and (7.18) can be rewritten as

$$\begin{aligned} Q^{SF} \equiv\ & n\left[\int_0^{\varepsilon_d} \frac{e}{\Delta_T} dF(e) + \int_{\varepsilon_d}^{\bar{e}} \frac{e}{\Delta_T} dF(e) + \int_{\bar{e}}^{\varepsilon_{\bar{\ell}}} \frac{\bar{e}}{\Delta_T} dF(e) + \int_{\varepsilon_{\bar{\ell}}}^{\infty} \frac{\bar{e}}{\Delta_T} dF(e)\right] \\ & + (1-n)\left[\int_0^{\varepsilon_d} \varepsilon dF(\varepsilon) + \int_{\varepsilon_d}^{\bar{e}} \varepsilon_d dF(\varepsilon) + \int_{\bar{e}}^{\varepsilon_{\bar{\ell}}} \Delta\varepsilon dF(\varepsilon) + \int_{\varepsilon_{\bar{\ell}}}^{\infty} \Delta\varepsilon_{\bar{\ell}} dF(\varepsilon)\right] \end{aligned}$$

and

$$\begin{aligned} Q^{OMO} \equiv\ & n\left[\int_0^{\varepsilon_d} e dF(e) + \int_{\varepsilon_d}^{\bar{e}} e dF(e) + \int_{\bar{e}}^{\varepsilon_{\bar{\ell}}} \bar{e} dF(e) + \int_{\varepsilon_{\bar{\ell}}}^{\infty} \bar{e} dF(e)\right] \\ & + (1-n)\left[\int_0^{\varepsilon_d} \varepsilon dF(\varepsilon) + \int_{\varepsilon_d}^{\bar{e}} \varepsilon dF(\varepsilon) + \int_{\bar{e}}^{\varepsilon_{\bar{\ell}}} \bar{e} dF(\varepsilon) + \int_{\varepsilon_{\bar{\ell}}}^{\infty} \bar{e} dF(\varepsilon)\right] \end{aligned}$$

where we have used $\bar{e}$ instead of ε_ℓ. Subtracting the last two equations we obtain

$$
\begin{aligned}
& Q^{OMO} - Q^{SF} \\
= \; & n\left[\int_0^{\varepsilon_d} e dF(e) + \int_{\varepsilon_d}^{\bar{e}} e dF(e) + \int_{\bar{e}}^{\varepsilon_{\bar{\ell}}} \bar{e} dF(e) + \int_{\varepsilon_{\bar{\ell}}}^{\infty} \bar{e} dF(e)\right] \\
& + (1-n)\left[\int_0^{\varepsilon_d} \varepsilon dF(\varepsilon) + \int_{\varepsilon_d}^{\bar{e}} \varepsilon dF(\varepsilon) + \int_{\bar{e}}^{\varepsilon_{\bar{\ell}}} \bar{e} dF(\varepsilon) + \int_{\varepsilon_{\bar{\ell}}}^{\infty} \bar{e} dF(\varepsilon)\right] \\
& - n\left[\int_0^{\varepsilon_d} \frac{e}{\Delta_T} dF(e) + \int_{\varepsilon_d}^{\bar{e}} \frac{e}{\Delta_T} dF(e) + \int_{\bar{e}}^{\varepsilon_{\bar{\ell}}} \frac{\bar{e}}{\Delta_T} dF(e) + \int_{\varepsilon_{\bar{\ell}}}^{\infty} \frac{\bar{e}}{\Delta_T} dF(e)\right] \\
& - (1-n)\left[\int_0^{\varepsilon_d} \varepsilon dF(\varepsilon) + \int_{\varepsilon_d}^{\bar{e}} \varepsilon_d dF(\varepsilon) + \int_{\bar{e}}^{\varepsilon_{\bar{\ell}}} \Delta\varepsilon dF(\varepsilon) + \int_{\varepsilon_{\bar{\ell}}}^{\infty} \Delta\varepsilon_{\bar{\ell}} dF(\varepsilon)\right]
\end{aligned}
$$

On the same lines of Case 1, it is easy to see that $\mathcal{Q}^{OMO} - \mathcal{Q}^{SF} \geq 0$, with strict inequality if $\Delta < 1$. To show that the converse is not true, set $\Delta < 1$. Then the allocation with OMO is greather than the allocation with SF in all cases 1-3. The proof is finished. □

Proof of Proposition 4. To prove Proposition 4 it is sufficient to show that $\mathcal{Q}^{OMO} \geq \mathcal{Q}^{SF}$ if $\Delta \leq 1$, with strict inequality if $\Delta < 1$ (see Proof of Proposition 3). □

Proof of Proposition 5. From (7.19), $\mathcal{Q}^{SF}$ is an increasing and continuous function of Δ, with $\Delta \in (0, 1]$. Then, the optimal allocation in the SF economy can be implemented by setting $\Delta = 1$. □

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