Discretionary Policy and Multiple Equilibria in LQ RE Models*

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Abstract

We study discretionary equilibria in dynamic linear-quadratic rational expectations models. In contrast to the assumptions that pervade this literature we show that these models *do* have multiple equilibria in some situations. We demonstrate the existence of multiple discretionary equilibria by example. We investigate general properties of discretionary equilibria.

Key Words: Discretion, Multiple Equilibria

JEL References: E31, E52, E58, E61, C61

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1 Introduction

In this paper we study discretionary policy in the class of infinite-horizon discrete-time linear dynamic models that is typically used to study aggregate fluctuations in macroeconomics. The policy maker has quadratic intra-temporal objectives and the optimal behaviour of the private sector is characterized by a forward-looking implementability constraint. We discuss and illustrate the potential for existence of multiple rational expectations equilibria in this class of problems, that is usually termed as Linear Quadratic Rational Expectations (LQ RE) models. The class of LQ RE models is of considerable applied interest because it can – under appropriate conditions – be interpreted as a quadratic approximation to the underlying fully non-linear optimal policy problem. Arising multiplicity of equilibria can generate rich dynamics with switches between periods of high and low volatility of aggregate macroeconomic variables.

Albanesi et al. (2003) and King and Wolman (2004) demonstrate how discretionary policy generates multiplicity of equilibria in non-linear models, where the non-linearity gives rise to dynamic complementarities of pricing decisions of different firms in the private sector. In standard LQ RE models the private sector is aggregate and these types of equilibria do not exist. Because of their linearity, LQ RE models are often treated as immune to multiplicity.[1] These models are routinely used in policy analysis; the standard New Keynesian model is frequently used to demonstrate results that are presumed valid for the whole class of LQ RE models under discretion.[2]

In contrast to the assumption that pervades the literature, we show that LQ RE models *do* have multiple discretionary equilibria in some situations. We describe all types of equilibria that can exist in non-degenerate LQ RE models and show how this can help to locate them.

Our definition of discretionary policy is conventional and is widely used in the monetary policy literature, see Backus and Driffill (1986), Oudiz and Sachs (1985), Clarida et al. (1999), and Woodford (2003a) to mention only few. At the beginning of each time period the policy maker observes the state of the economy and optimizes; it is also known by all agents that the policy maker will apply the same procedure in every subsequent period. When the private sector expects the policy maker to pursue the discretionary policy in each future period, the best the policy maker can do is to pursue the discretionary policy: there is no temptation for the policy maker to deviate from it and the private sector's expectations are rational. Discretionary policy is credible by construction.

This very property of credibility creates a potential for multiplicity of discretionary equilibria and makes it impossible for the policymaker to choose the best one if there are multiple equilibria.

When the state is observed, and the policy maker acts first, the private sector observes the policy action and takes an optimal reaction given the state of the economy and the action of the policy maker. The optimizing policy maker expects the private sector to react rationally and the policy is chosen accordingly. Crucially, the rational reaction of the private sector is determined by the expectations of the private sector about the future state of the economy, that, in its turn, is

[1] In the seminal paper on discretionary policy Oudiz and Sachs (1985, p. 288) suggest an algorithm to find an equilibrium and remark: "Although we cannot prove that the resulting function is the unique memoryless, time-consistent equilibrium, we suspect that it is in fact unique, in view of the linear-quadratic structure of the underlying problem."

[2] See Woodford (2003b), Vestin (2006), Walsh (2003) *[add something on learning]* among many others.

affected by the future policy. So the policy acts taking into account the effect of the future policy decisions as *expected* by the private sector. First, if there are complementarities between decisions of the private sector and the policy maker, so the optimal decision of the policy maker reinforces the decision of the private sector, then multiple equilibria can arise. The presence of a state that can be affected by agents' decisions ensures the importance of future policy decisions for current actions and so is crucial for multiplicity. Second, when choosing the optimal policy the current policy maker reacts indirectly to past actions of the private sector through the observations of the state of the economy. If there is multiplicity of equilibria and, for some reasons, the past-period private sector has changed expectations about the current and future policy after the past-period policy acted, then the current policy maker finds it optimal to validate these expectations and has no power to choose the equilibrium.

We present a line of three models. As the first example we use the Canonical New Keynesian model that is often used as a benchmark model for results on discretionary policy. We show *the uniqueness* of discretionary equilibrium as a consequence of the disconnect between consequent periods. This disconnect and the absence of important interactions between current and future policy and the private sector decisions make this model a very special case of LQ RE models of discretionary policy. The uniqueness of equilibria under discretionary policy, therefore, is a very special case and many policy implication derived from the Canonical New Keynesian model under discretion may be difficult to generalize.

In the second example we demonstrate *the existence* of multiple discretionary equilibria. We introduce an *endogenous state*, the government debt, into the New Keynesian model. This model is known to generate multiplicity of determinate regimes under rules-based policy (Leeper (1991)) that makes it appealing to investigate the existence of different equilibria under discretion. We demonstrate how dynamic complementarities between decisions of the private sector and the policy maker lead to multiplicity of equilibria. We also argue that the discovered equilibria are consistent with empirical evidence documented in a number of studies.

In the third example of the New Keynesian model with capital accumulation we demonstrate that discretionary equilibria may also arise because of dynamic complementarities between different decisions of the aggregate private sector, and the nature of these equilibria is different in important ways. There can be several time-invariant responses of the aggregate private sector to the same policy action; King and Wolman (2004) call similar equilibria 'point-in-time' equilibria. Because of linearity of our model complementarities can only arise between different decisions of the aggregate private sector, such as between consumption and investment decisions in their effect on marginal cost of firms.

We generalize some of our results for the general class of LQ RE models. First, we prove that in an economy without endogenous predetermined endogenous state variables the discretionary equilibrium is unique. Second, we say how many policy-induced 'point-in-time' private sector equilibria exist and how to find all of them. This result eliminates the need to search for dynamic complementarities between the private sector's actions in order to assess the possibility of multiplicity of policy-induced private sector equilibria. Third, we prove that for a given private sector equilibrium, the policy response is unique. Fourth, we demonstrate that, for most economic applications, there is a finite number of locally isolated, or determinate, equilibria. We demonstrate how these results can be used in a numerical procedure to locate those equilibria that cannot be found with conventional routines.

The paper is organized as follows. In the next section we present the three examples. In Section 3 we formally define the discretionary optimization problem and derive the first order conditions to the optimization problem. In Section 4 we discuss properties of discretionary equilibria. In Section 5 we discuss numerical algorithms to find discretionary equilibria. Section 6 concludes.

2 Examples

2.1 Standard New Keynesian Model

We start with the Standard New Keynesian model that we adopt from Clarida et al. (1999), Sec. 3. This model has become a workhorse for policy analysis, including analysis of discretionary policy.[3] This model does *not* have multiple discretionary equilibria. We use this model to introduce all notions and definitions, and to uncover the features that preclude multiple equilibria in this model. It will also become apparent that these features are unlikely to exist in the general class of LQ RE models under discretionary policy.

We consider deterministic model. The law of motion of the aggregate economy can be written as:

$$\pi_t = \beta\pi_{t+1} + \lambda c_t + \nu b_t, \tag{1}$$
$$b_{t+1} = \rho b_t, \tag{2}$$

and the initial state $\bar{b}$ is known to all agents. The only predetermined state variable in this economy is the *exogenous* autoregressive process b_t. Equation (1) is a New Keynesian Phillips curve that relates inflation π_t positively to the output c_t. Parameter β is the private sector discount factor and λ is the slope of Phillips curve.[4] Parameter ν scales the effect of exogenous state on inflation. The aggregate agent's state control variable is inflation, π_t.

Following Clarida et al. (1999), we assume that the policy maker chooses output c_t and then, conditional on optimal behavior of the economy as described by c_t and π_t, the policy maker decides on the value of interest rate that supports c_t and π_t.[5]

The inter-temporal policy maker's criterion is defined by the quadratic loss function:

$$L_t = \frac{1}{2}\sum_{s=t}^{\infty}\beta^{s-t}\left(\pi_s^2 + \alpha c_s^2\right). \tag{3}$$

The policy maker knows the law of motion (1)-(2) of the aggregate economy and takes it into account when formulating its policy. The policy maker finds the best action every period, knows

[3] See e.g. Woodford (2003b), Vestin (2006), Walsh (2003).

[4] The Phillips curve evolves from staggered nominal price setting as in Calvo (1983). The individual firm price-setting decision, which provides the basis for the aggregate relation, is derived from an explicit optimization problem. Firms are monopolistically-competitive: When it has the opportunity, each firm chooses its nominal price to maximize profits subject to constraints on the frequency of future price adjustments. Parameter λ is a function of frequency of price adjustments.

[5] Using interest rate as an instrument implies that consumption and price-setting decisions are made simultaneously, while in this model they are consequent. Here and in the next example this makes no difference for results on multiplicity.

that future policy makers have freedom to change policy, and knows that future policy makers will apply the same decision process.

We assume the following timing of events. At the beginning of each period the state is observed by all agents. Then the policy maker truthfully announces the policy or make a decision and then the private sector reacts.

We assume that the policy maker acts in a *discretionary* way in the following sense. At every point t in time the private sector *observes* policy that reacts only to the current state[6],

$$c_t = c_b b_t. \tag{4}$$

The private sector *expects* that future policy makers will apply the same decision process and will react to the contemporary state only, and implement decision (4).

At any time t, the policy maker reacts to the current state (4), knows that the private sector observes its action, and knows that the private sector expects all future policy makers will apply the same decision process and implement decision (4).

We assume that the aggregate decision of the private sector can be written as a linear feedback function:

$$\pi_t = \pi_b b_t. \tag{5}$$

Herewith we shall refer to parameters c_b and π_b, that define policy and private sector decisions, as to 'decisions'.

Note that from

$$\pi_{t+1} \overset{eq.(5)}{=} \tilde{\pi}_b b_{t+1} \overset{eq.(2)}{=} \tilde{\pi}_b \rho b_t \overset{eq.(1)}{=} \frac{1}{\beta}\pi_t - \frac{\lambda}{\beta}c_t - \frac{\nu}{\beta}b_t$$

it follows that the private sector's decision can also be written as

$$\pi_t = (\beta\rho\tilde{\pi}_b + \nu)\, b_t + \lambda c_t. \tag{6}$$

Here we denoted the response of the next-period private sector to the next-period state b_{t+1} with 'tilde'.[7] If there is a disturbance $b_t > 0$ and firms raise inflation to $\pi_b b_t$ then this rise is the result of reaction to state, $(\beta\rho\tilde{\pi}_b + \nu)\, b_t$, and to the policy, λc_t. Because the policy maker moves first within each period and because the private sector observes the policy, the private sector takes into account the 'instantaneous' influence of the policy choice, that is measured by λ. The first term in (6) shows that the response to the state is also determined by the *next-period* response of the private sector to the state, $\tilde{\pi}_b$.

Policy determined by (4) is *discretionary* if the policy maker *finds it optimal* to follow (4) each time $s > t$, given the private sector (i) observes policy (4), (ii) knows that future policy makers re-optimize and use the same decision process, (iii) expects policy (4) will be implemented in all future periods.[8]

[6] We restrict ourselves to 'memoryless' or Markov equilibria, so agents' decisions are functions of current state only. We also assume linear form of contemporary relationship.

[7] We shall use this notation in all examples to denote next-period private sector decisions.

[8] In the language of game theory we restrict attention to time-consistent feedback equilibria with intra-period leadership, see e.g. de Zeeuw and van der Ploeg (1991), Oudiz and Sachs (1985), Cohen and Michel (1988). Here and below we term these equilibria as 'discretionary'.

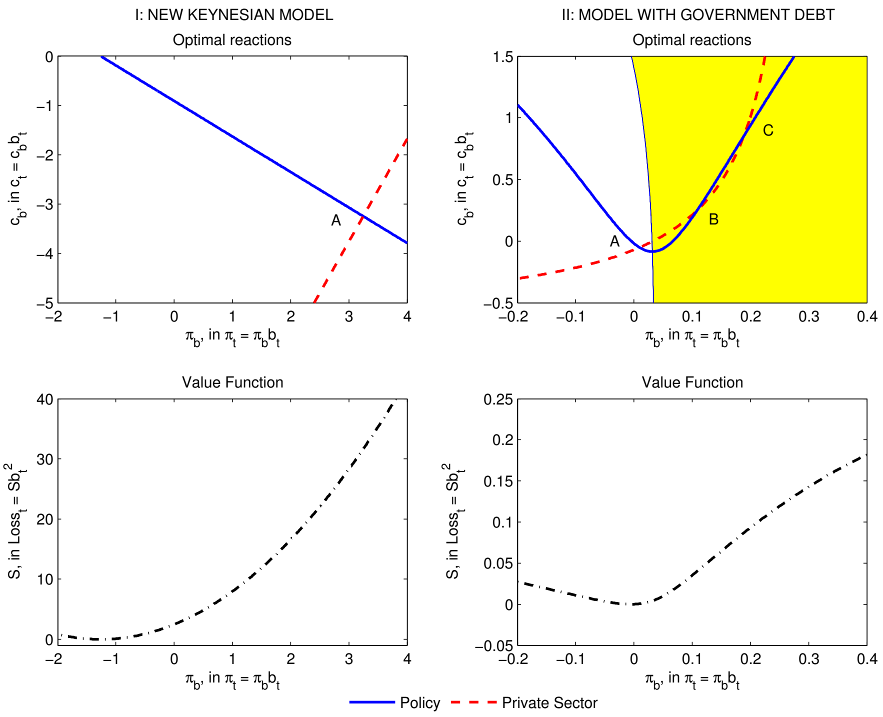

Figure 1: Discretionary Equilibria

We can write the criterion for optimality as

$$L_t(b_t) = \min_{c_t} \left(\frac{1}{2} \left(((\beta\rho\tilde{\pi}_b + \nu) b_t + \lambda c_t)^2 + \alpha c_t^2 \right) + \beta L_{t+1}(\rho b_t) \right),$$

where we have taken the intra-period leadership of the policy into account and substituted constraint (6). Because of the quadratic form of a flow-objective in (3) and because decisions of the policy maker and of the private sector are linear in state, the discounted loss will necessarily be quadratic in state:

$$L_t(b_t) = \frac{1}{2} S b_t^2.$$

The Bellman equation characterizing discretionary policy, therefore, becomes

$$S b_t^2 = \min_{c_t} \left(\left(((\beta\rho\tilde{\pi}_b + \nu) b_t + \lambda c_t)^2 + \alpha c_t^2 \right) + \beta \tilde{S} (\rho b_t)^2 \right). \tag{7}$$

Differentiation of (7) with respect to c_t yields the optimal policy response:

$$c_t = -\frac{(\beta\rho\tilde{\pi}_b + \nu)\lambda}{(\lambda^2 + \alpha)} b_t = c_b b_t. \tag{8}$$

The feedback coefficient in (8) determines the optimal policy feedback on predetermined state, b_t, and the feedback coefficient is independent of $\tilde{S}$. When the discretionary policy maker chooses the best policy in period t, it knows that future policy makers will re-optimize. In this model this constraint is not binding. This implies that the optimal policy feedback coefficient c_b is a *linear* function of the future private sector reaction $\tilde{\pi}_b$. It is immediately apparent that the disconnect between *policy decisions* in times t and s for any $t \neq s$ only happens if the predetermined state variable is exogenous, so policy neither affects the state directly nor via its effect on private sector's decisions.

We plot it with the solid line in the top chart in Panel I in Figure 1.[9]

It remains to present optimal decisions of the private sector in the feedback form of (5). Substitute (4) into (6) to obtain:

$$\pi_t = \left(\beta\rho\tilde{\pi}_b + \nu + \lambda c_b\right) b_t = \pi_b b_t. \tag{9}$$

The next-period response to disturbances positively affects the current-period response to disturbances, but the effect is linear in decisions of both agents. The effect is linear because the future state is determined by parameter ρ alone and depends on neither policy nor private sector decisions.

If the policy is expected to be the same in the next period, the time-invariant aggregate private sector response is the same, $\tilde{\pi}_b = \pi_b$, and there is a unique solution $\pi_b = \pi_b\left(c_b\right)$ which is a linear function:

$$\pi_b = \frac{\nu + \lambda c_b}{1 - \beta\rho} \tag{10}$$

We plot (10) with the dashed line in the top chart in Panel I in Figure 1.

To summarize, we have two linear dependencies, $c_b = c_b\left(\pi_b\right)$ and $\pi_b = \pi_b\left(c_b\right)$ given by (8) and (10). There is a unique solution, given by

$$c_b = -\frac{\lambda\nu}{\lambda^2 + \alpha\left(1 - \beta\rho\right)}, \qquad \pi_b = \frac{\alpha\nu}{\lambda^2 + \alpha\left(1 - \beta\rho\right)},$$

that was also obtained by Clarida et al. (1999), formulae (3.4) and (3.5).[10]

In order to find the value function S we substitute the optimal solution (8) into the Bellman equation (7) and, using that the next-period policy is expected to be the same and so $S = \tilde{S}$, obtain:

$$S = \frac{\alpha\left(\beta\rho\pi_b + \nu\right)^2}{\left(\alpha + \lambda^2\right)\left(1 - \beta\rho^2\right)},$$

that is a quadratic function of the private sector's response, π_b. The stronger the response of firms to disturbances, π_b, the more costly it is to stabilize the economy as the optimal policy has

[9] We use the following calibration: $\beta = 0.99$, $\lambda = 0.1$, $\alpha = 0.1$, $\rho = 0.8$.

[10] Clarida et al. (1999), Sec. 3 do not discuss details of interactions between the private sector and the policymaker, primerely because they are absent in this simple model. Because of this simplicity the authors demonstrate a simple way of solving the model that is, however, not valid in the general case, see their footnote 27.

to offset the effect on prices with lower output c_t and so bigger value of c_b. The effect on π_b is linear in c_b and the overall loss, as measured by formula (3), rises as π_b^2. We plot it in the bottom chart in Panel I in Figure 1.

Coefficients π_b, c_b and S describe the solution to the problem of discretionary optimization outlined above. Coefficients π_b, c_b and S uniquely define trajectories $\{b_t, \pi_t, c_t\}_{t=0}^{\infty}$ for a given $b_0 = \bar{b}$. If the sequence $\{b_t, \pi_t, c_t\}_{t=0}^{\infty}$ solves the discretionary policy outlined above then there is a unique triplet $\{\pi_b, c_b, S\}$ that satisfies equations (4), (5) and (7). In what follows we call the triplet of coefficients $\{\pi_b, c_b, S\}$ as *discretionary equilibrium.*

Because of linearity of agents' decisions the discretionary equilibrium always exists and unique.

It is instructive to compute welfare, equal to minus loss (3), on *arbitrary* decisions c_b and π_b, but exploiting the intra-period order of moves:

$$\begin{aligned} W_t &= -\frac{1}{2}\sum_{s=t}^{\infty}\beta^{s-t}\left(\pi_s^2+\alpha c_s^2\right) = -\frac{1}{2}\left(\left((\beta\rho\pi_b+\nu)+\lambda c_b\right)^2+\alpha c_b^2\right)\sum_{s=t}^{\infty}\beta^{s-t}b_s^2 \\ &= -\frac{1}{2}\frac{\left((\beta\rho\pi_b+\lambda c_b+\nu)^2+\alpha c_b^2\right)}{1-\beta\rho^2}b_t^2. \end{aligned}$$

The second derivative

$$\frac{\partial^2 W_t}{\partial\pi_b\partial c_b} = -\frac{\beta\rho\lambda}{1-\beta\rho^2}b_t^2$$

is negative for any π_b, c_b. If it is expected that firms increase inflation in response to a disturbance b_t then this reduces the marginal return to the policy that increases output in response to the same disturbance. Cooper and John (1988) define that decisions of the private sector and the policy maker are *dynamic substitutables* if the optimal decision of the policy maker is decreasing in decision of the private sector, i.e. it is optimal to reduce output if inflation is expected to rise. Condition $\frac{\partial^2 W_t}{\partial\pi_b\partial c_b} < 0$ implies dynamic substitutability. In this model the uniqueness is ensured by the linearity of responses, but the global dynamic substitutability and existence of the equilibrium would also ensure its uniqueness.

The speed of stabilization of the economy is determined by the persistence parameter ρ and exogenous.

2.2 New Keynesian Model with Government Debt

2.2.1 Quick Overview of the Example

In the previous example the discretionary equilibrium was unique. In the Canonical New Keynesian model without endogenous state variable there was a complete disconnect between the current and future decisions of agents. This implied linearity of agents' decisions and uniqueness of the solution.

In this section we present an example *that has* multiple discretionary equilibria. Again, we consider a deterministic New Keynesian model, but instead of an exogenous state variable we assume that agents observe and *can affect* the accumulation of real government debt. Again, we study interactions of a single monetary policy maker and the aggregated private sector.

The accumulation of the government debt also depends on fiscal stance. There is a *non-optimizing* fiscal authority that faces a stream of exogenous public consumption. These expenditures are financed by levying income taxes and by issuing one-period risk-free nominal bonds. We assume that the fiscal authority imposes a simple proportional rule for the tax rate: if the real debt is higher/lower than in steady state then tax rate rises/falls. We shall refer to the tax rate as to 'taxes' and to the parameter of the proportional rule as to the 'fiscal feedback'. The size of the fiscal feedback measures the 'strength of fiscal stabilization' of debt and plays an important role in the model. The presence of non-optimizing fiscal authority in the economy can be described by a single parameter, the fiscal feedback.

If this parameter is relatively large then an increase in public debt is practically eliminated by fiscal policy within few periods. The equilibrium behavior of the discretionary monetary policy maker and of the private sector is, therefore, similar to the one in the Standard New Keynesian model. In order to reduce the implied effect of temporarily high taxes on inflation the policy maker generates low demand by reducing consumption. This is a low-inflation-volatility equilibrium as firms set relatively low inflation as they face low consumption in the future.

If the fiscal feedback on debt is zero, then an initial increase in public debt results in a debt spiral unless the agents intervene. The policy maker intervenes: the debt is stabilized when higher consumption-fuelled demand raises tax revenues. This is a high-inflation-volatility equilibrium as firms set inflation relatively high facing high demand in the future.

We shall demonstrate that if the fiscal feedback on debt is moderate – and so debt is only slow stabilized by fiscal policy alone – then both equilibria are possible. If the private sector *believes* that future demand will be low and sets low inflation, the next-period policy maker *finds it optimal to validate these beliefs* and will reduce demand; if the private sector *believes* that future demand will be high and inflates, the next-period policy maker *finds it optimal to validate these beliefs* and will increase demand. Following an initial debt displacement the economy can follow one of several paths, each of which satisfies conditions imposed on policy, i.e. time-invariance and optimality, given that future policy makers re-optimize.

As we discuss later in Section 2.2.4 the described economic behavior is familiar from the literature on the 'fiscal theory of the price level', both theoretical and empirical.[11] Crucially, we study discretionary policy, not simple rules; but because of similarity with rules-based models, we expect to find a switch from one equilibrium to another when we vary the strength of fiscal feedback. This similarity and the empirical evidence makes this example appealing to demonstrate the existence of multiple equilibria.

2.2.2 Discretionary Equilibria

Aggregate Behavior of the Private Agents and the Policy Maker's Actions We adopt the model from Woodford (2001), Benigno and Woodford (2004).[12] We delegate all technical details to the Online Appendix and only present log-linearized about the steady state equations

[11] See e.g. Leeper (1991), Woodford (2001), Davig and Leeper (2006), Favero Monacelli 2005.

[12] It is also a log-linearision of the model in Schmitt-Grohe and Uribe (2007); we take its version with cashless economy with constant stock of capital and labour income taxes.

that describe aggregate decisions of the private sector and evolution of the state.[13] As before, we assume that the policy maker chooses consumption c_t.

We assume that *nominal* debt is observed at the beginning of period t, and define real debt as nominal beginning-of-period debt deflated by the end-of-the-previous-period price. The begining-of-period real debt b_t is the aggregate predetermined state variable in period t. The law of motion of the economy can be described as:

$$\pi_t = \beta\pi_{t+1} + \lambda c_t + \nu b_t, \tag{11}$$

$$b_{t+1} = \rho b_t - \eta c_t, \tag{12}$$

and the initial state $\bar{b}$ is known to all agents. The aggregate agents' state control variable is inflation, π_t.

Equation (11) is a New Keynesian Phillips curve.[14] Equation (12) describes evolution of the real debt.[15] In contrast to the Standard New Keynesian model the state is affected by the policy.

The inter-temporal criterion of the policy maker is defined by the quadratic loss function[16]

$$L_t = \frac{1}{2}\sum_{s=t}^{\infty}\beta^{s-t}\left(\pi_s^2 + \alpha c_s^2\right). \tag{13}$$

The policy maker knows the law of motion (11)-(12) of the aggregate economy and takes it into account when formulating its policy. The policy maker finds the best action every period, knows that future policy makers have freedom to change policy, and knows that future policy makers will apply the same decision process.

We assume that the policy maker acts under discretion. At every point t in time the decision of each agent can be written as a linear function of the current state:

$$c_t = c_b b_t, \tag{14}$$

$$\pi_t = \pi_b b_t. \tag{15}$$

[13] The Online Appendix, along with all relevant MATLAB programs, is available from www.people.ex.ac.uk/tkirsano or upon the request from the authors.

[14] In contrast to the previous Example, the marginal cost is a function of consumption and taxes, but taxes are determined by debt, and so we have the additional term νb_t in (11). If the underlying fiscal feedback rule is $\tau_t = \mu b_t$ where τ_t is log-linearized tax rate with steady state level τ_o, then $\nu = \mu\kappa\tau_o/\left(1-\tau_o\right)$ and $\lambda = \kappa\left(1/\sigma + \theta/\psi\right)$ where σ and ψ are parameters of private sector utility function, β is the private sector discount factor, θ is steady-state consumption to output ratio and λ is the slope of Phillips curve. Public spending are a part of the marginal cost too, but they are assumed to be constant and so do not enter the log-linearized version of the model.

[15] It states that real debt at the beginning of the next period $t+1$ is equal to the real debt at the beginning of period t minus taxes collected during period t, and this accumulates at rate $1/\beta$, that is the steady state real interest rate. Collected taxes change because of either change in the tax rate, or because of change in the tax base. Parameter $\rho = \left(1-\mu\tau_o\right)/\beta$ is a function of the tax rate and with stronger feedback μ the debt is stabilized faster. Parameter $\eta = \theta\tau_o/\beta$ describes sensitivity of debt to the tax base, the base is proportional to consumption. We assume that the steady state level of debt is zero that will eliminate first-order effects of interest rate and inflation on debt in equation (12).

[16] The criterion is derived under the assumption of steady state labour subsidy. Here parameter α is a function of model parameters, $\alpha = \theta\lambda/\epsilon$, and ϵ is elasticity of substitution between any pair of monopolistically produced goods.

Because of

$$\pi_{t+1} \overset{eq.(15)}{=} \tilde{\pi}_b b_{t+1} \overset{eq.(12)}{=} \tilde{\pi}_b \left(\rho b_t - \eta c_t\right) \overset{eq.(11)}{=} \frac{1}{\beta}\pi_t - \frac{\lambda}{\beta}c_t - \frac{\nu}{\beta}b_t,$$

the private sector reaction function can also be written as

$$\pi_t = \left(\beta\rho\tilde{\pi}_b + \nu\right) b_t + \left(\lambda - \beta\eta\tilde{\pi}_b\right) c_t. \tag{16}$$

So, because the policy maker moves first within each period and because the private sector observes the policy, the private sector takes into account the 'instantaneous' influence of the policy choice, that is measured by $(\lambda - \beta\eta\tilde{\pi}_b)$. The current-period responses of the private sector to the state and to the policy action are determined by the *next-period* response of the private sector to the state, $\tilde{\pi}_b$.

The Bellman equation characterizing discretionary policy becomes

$$Sb_t^2 = \min_{c_t}\left(\left(\left(\left(\beta\rho\tilde{\pi}_b + \nu\right) b_t + \left(\lambda - \beta\eta\tilde{\pi}_b\right) c_t\right)^2 + \alpha c_t^2\right) + \beta\tilde{S}\left(\rho b_t - \eta c_t\right)^2\right). \tag{17}$$

Discretionary Policy. Differentiation of (17) with respect to c_t yields the optimal policy response:

$$c_t = -\frac{\left(\left(\beta\rho\tilde{\pi}_b + \nu\right)\left(\lambda - \tilde{\pi}_b\eta\beta\right) - \beta\rho\eta\tilde{S}\right)}{\left(\left(\lambda - \beta\eta\tilde{\pi}_b\right)^2 + \alpha + \beta\eta^2\tilde{S}\right)} b_t = c_b b_t. \tag{18}$$

The feedback coefficient c_b in (18) determines the optimal policy feed back on predetermined state, b_t, but the feedback coefficient is a function of $\tilde{S}$. $\tilde{S}$ determines the next-period loss that depends on the whole future path of the state, that is a function of future policy decisions. When the current-period policy maker chooses the best policy, it knows that the next-period policy maker will re-optimize. In this model this constraint is binding because $\eta \neq 0$. It ensures the non-linearity of $c_b = c_b\left(\tilde{\pi}_b\right)$.

In order to find the value function S we substitute the optimal solution (18) into the Bellman equation (17) and, under the equilibrium condition $S = \tilde{S}$, obtain equation for the value function S:

$$S = \left(\beta\rho\tilde{\pi}_b + \nu\right)^2 + \beta\rho^2 S - \frac{\left(\left(\beta\rho\tilde{\pi}_b + \nu\right)\left(\lambda - \beta\eta\tilde{\pi}_b\right) - \beta\eta\rho S\right)^2}{\left(\left(\lambda - \beta\eta\tilde{\pi}_b\right)^2 + \alpha + \beta\eta^2 S\right)}.$$

After straightforward manipulations we obtain a quadratic equation for S with positive leading coefficient and negative constant term. This equation has a unique solution $S > 0$:

$$\begin{aligned} S \quad &= \quad \frac{1}{2\beta\eta^2}\left(-\left(\left(\lambda - \beta\eta\tilde{\pi}_b\right)^2 - \beta\left(\rho\lambda + \nu\eta\right)^2 - \alpha\left(\beta\rho^2 - 1\right)\right)\right. \\ &\quad \left. + \sqrt{\left(\left(\lambda - \beta\eta\tilde{\pi}_b\right)^2 - \beta\left(\rho\lambda + \nu\eta\right)^2 - \alpha\left(\beta\rho^2 - 1\right)\right)^2 + 4\alpha\beta\eta^2\left(\beta\rho\tilde{\pi}_b + \nu\right)^2}\right) \geq 0. \end{aligned}$$

We plot S as a function of the equilibrium private sector decision $\tilde{\pi}_b = \pi_b$ in the lower chart of Panel III in Figure 1.[17]

In order to obtain the optimal policy we substitute the equilibrium value of S out of the feedback coefficient in (18):

$$\begin{aligned} c_b \;=\; & -\frac{(\rho\lambda+\nu\eta)\left(\beta\rho\tilde{\pi}_b+\nu\right)}{\left(\rho\alpha+\left(\lambda-\beta\eta\tilde{\pi}_b\right)\left(\rho\lambda+\nu\eta\right)\right)}+\Big(-\beta\left(\left(\lambda-\beta\eta\tilde{\pi}_b\right)^2-\beta\left(\rho\lambda+\nu\eta\right)^2-\alpha\left(\beta\rho^2-1\right)\right) \\ & +\sqrt{\left(\left(\lambda-\tilde{\pi}_b\eta\beta\right)^2-\beta\left(\rho\lambda+\nu\eta\right)^2-\alpha\left(\beta\rho^2-1\right)\right)^2+4\beta\alpha\eta^2(\beta\rho\tilde{\pi}_b+\nu)^2}\Big) \\ & /\left(2\eta\beta\left(\rho\alpha+\left(\lambda-\beta\eta\tilde{\pi}_b\right)\left(\rho\lambda+\nu\eta\right)\right)\right). \end{aligned} \tag{19}$$

The optimal policy feedback coefficient as function of the equilibrium reaction of the private sector, $\tilde{\pi}_b = \pi_b$, is plotted in the top chart in Panel III in Figure 1. It is U-shaped in $\tilde{\pi}_b$ as we now explain.

Suppose that debt b_t is positive and the policy maker knows that firms believe that the next-period firms reacts with higher inflation, $\tilde{\pi}_b \gg 0$. The policy maker also knows that the current-period firms take into account the 'instantaneous' influence of the policy choice, that is measured by $\left(\lambda - \beta\eta\tilde{\pi}_b\right)$. The profit maximization problem for firms implies that if future response of inflation to the state is strong then higher demand (consumption) requires to reduce inflation, $\left(\lambda - \beta\eta\tilde{\pi}_b\right) < 0$ if $\tilde{\pi}_b \gg 0$. Hence, in order to reduce the cost of inflation the optimal policy has to increase consumption in response to higher debt, $c_b > 0$. This also contributes to the debt stabilization. But too large movements in consumption are also costly, and this determines a finite optimal value of c_b. In equilibrium, if π_b is large then c_b also rises. The social loss S is high, as in a response to the positive debt both consumption and inflation change by much.

Suppose the policy maker knows that firms believe that the next-period firms react to positive debt with lower inflation, $\tilde{\pi}_b \ll 0$. The profit maximization problem for firms implies that if future inflation is low then an increase of demand (consumption) raises inflation, $\left(\lambda - \beta\eta\tilde{\pi}_b\right) > 0$ if $\tilde{\pi}_b \ll 0$. Hence, the optimal policy is to increase consumption, $c_b > 0$. This will stabilize debt and moderate the fall in inflation. This regime is also characterized by high social loss S because of large response of consumption to debt.

Suppose there is an 'intermediate' scenario and the next-period firms do not react to debt, or react only weakly, $\tilde{\pi}_b \simeq 0$. The current-period firms will keep inflation low if the effect of higher taxation on the marginal cost (via ν) is nearly completely offset by the negative effect of demand. This can only happen if consumption falls, i.e. $c_b < 0$. A fall in demand slows down the debt stabilization, but still ensures debt stability. As the effect of taxation is going to reduce in the future, the next-period profit-optimizing firms will keep inflation low. Hence a small $c_b < 0$ for small $\pi_b \simeq 0$ can characterize an optimal policy. The loss is small as in a response to the higher b_t the implied responses of π_t and c_t are small.

Private sector response. In order to obtain the time-invariant optimal decision of the private sector we substitute equation (14) into equation (16) and obtain:

$$\pi_t = \left(\beta\rho\tilde{\pi}_b + \nu + \left(\lambda - \beta\eta\tilde{\pi}_b\right)c_b\right)b_t = \pi_b b_t. \tag{20}$$

[17] The implied values of parameters are: $\beta = 0.99$, $\lambda = 0.0582$, $\nu = 0.0025$, $\xi = 0.925$, $\eta = 0.1875$, $\alpha = 0.0087$. We explain calibration of deep structural parameters in the *Appendix*.

Future decisions of the private sector $\tilde{\pi}_b$ affect the current decisions of the private sector π_b. In a discretionary equilibrium $\tilde{\pi}_b = \pi_b$ that yields:

$$\pi_b = \frac{\nu + \lambda c_b}{\left(1 - \beta\left(\rho - \eta c_b\right)\right)}. \tag{21}$$

The response of inflation to the state b_t is determined by policy c_b. Higher demand increases inflation, but current price-setting decisions of firms depend on their future decisions, that depend on future policy and on the future state. With stronger reaction of consumption to debt, c_b, parameter $0 < \beta\left(\rho - \eta c_b\right) < 1$ is smaller and debt is stabilized faster, see equation (12), and so the discounted effect of future debt on current inflation is smaller. Hence, the total effect of debt on inflation $\pi_b = \pi_b\left(c_b\right)$ is *non-linear* and concave. We plot the the optimal response of the private sector using a solid line in the top chart of Panel III in Figure 1.

Multiple Discretionary Equilibria. As in the previous example, it is instructive to search for dynamic complementarities. Following Cooper and John (1988), we say that we have a *dynamic complementarity* if the optimal decision of one agent is increasing in decisions of the other. In order to demonstrate complementarities it is convenient to look at properties of the welfare function.

The welfare, given each agent's response along the debt-stabilizing solution that starts at given b_t, can be computed as:

$$\begin{aligned} W_t &= -\frac{1}{2}\sum_{s=s}^{\infty}\beta^{s-t}\left(\pi_s^2 + \alpha c_s^2\right) \overset{eqs.(14,16)}{=} -\frac{1}{2}\left(\left(\beta\rho\pi_b + \nu + \lambda c_b - \beta\eta\pi_b c_b\right)^2 + \alpha c_b^2\right)\sum_{s=s}^{\infty}\beta^{s-t}b_s^2 \\ &\overset{eqs.(12,14,15)}{=} -\frac{1}{2}\left(\left(\beta\rho\pi_b + \nu + \lambda c_b - \beta\eta\pi_b c_b\right)^2 + \alpha c_b^2\right) b_t^2 \sum_{s=t}^{\infty}\left(\beta\left(\rho - \eta c_b\right)^2\right)^{s-t} \\ &= -\frac{1}{2}\frac{\left(\left(\beta\rho\pi_b + \lambda c_b + \nu - \beta\eta\pi_b c_b\right)^2 + \alpha c_b^2\right)}{\left(1 - \beta\left(\rho - \eta c_b\right)^2\right)} b_t^2. \end{aligned}$$

We also assumed that welfare is finite: $\left(\rho - \eta c_b\right)^2 < 1/\beta$. We can compute the second derivative:

$$\frac{\partial^2 W_t}{\partial\pi_b\partial c_b} = \left(-\frac{\beta\rho\lambda + \beta\nu\eta}{1 - \beta\left(\rho - \eta c_b\right)^2} + 2\frac{\beta^2\eta\left(\left(\nu + \beta\rho\pi_b\right) + \left(\lambda - \beta\eta\pi_b\right)c_b\right)}{\left(1 - \beta\left(\rho - \eta c_b\right)^2\right)^2}\right) b_t^2. \tag{22}$$

If $\frac{\partial^2 W_t}{\partial\pi_b\partial c_b} > 0$ then we have dynamic complementarities between the private sector and the policy actions: a higher inflation set by firms in response to higher debt level increases the marginal return to the policy that increases consumption in response to higher debt level. It is apparent that global dynamic complementarities are impossible in this model. The shaded area in the top chart of Panel III in Figure 1 shows where decisions of the private sector and the policy maker are complementary while in the un-shaded area decisions are substitutable. We can only guarantee

Table 1: New Keynesian Model with Debt Accumulation: Characteristics of Discretionary Equilibria

Discretionary Equilibria		'Active' A	Weakly 'passive' B	'Passive' C
Private	π_b	0.0095	0.1088	0.1885
sector	$\frac{\partial \pi_t}{\partial b_t} = (\beta\rho\pi_b + \nu)$	0.0124	0.0993	0.1690
response	$\frac{\partial \pi_t}{\partial c_t} = (\lambda - \beta\eta\pi_b)$	0.0565	0.0378	0.0229
Policy	c_b	-0.0513	0.2514	0.8505
Speed of adjustment	b_b	0.8935	0.8362	0.7228
Loss	S	0.0005	0.0403	0.0867
Complementarities	$\text{sign}\left(\partial^2 W_t / \partial \pi_b \partial c_b\right)$	$-$	$+$	$+$
Numerical methods		OS, BD, PP	–	PP, BD*

Notes:
* Obtained with initialization $S = 1$, $\pi_b = 0$, Söderlind (1999).

a unique equilibrium if actions are dynamically substitutable globally *(CJ Theorem X)* which is not the case here.

As the first chart in Panel III in Figure 1 demonstrates we have three discretionary equilibria. The two of them, labelled B and C, are in the area of dynamic complementarity of decisions of the aggregate private sector and the policy maker, and the remaining equilibrium A is in the area where decisions are substitutable. Table 1 reports numerical characteristics of these equilibria.

If the policy maker knows that the next-period firms will raise inflation *sufficiently high* in a response to a positive debt, then it is optimal for the policy maker to increase consumption. The higher future inflation is going to be, the higher consumption needs to be set; the higher consumption increases demand and thus further increases inflation, but higher consumption also slows down the rise of inflation as the firms' response to lower *future* taxes will reduce the response to higher current demand. The dynamic complementarities are at work and two discretionary equilibria, B and C arise.

If the policy maker knows that the next-period firms will raise inflation only weakly or even deflate in a response to a positive debt, then with stronger *deflationary* decisions by firms, the policy maker will optimally raise consumption and demand and stabilize inflation. Consumption and inflation are dynamic substitutes and only one equilibrium A arises.

Equilibrium Trajectories. An initial deviation of debt from its steady state value, $\bar{b} \neq 0$ is the only reason for future dynamic adjustment of the economy in our model. Panel I in Figure 2 plots responses of the economy to an initial unit-deviation of debt from its steady state value. The three solutions discussed above correspond to the three different paths towards the steady state.

In equilibrium A the policy maker expects that firms expect that the next-period firms set low inflation and optimally respond with low consumption. The slow convergence of debt is consistent with depressed demand and low inflation along the path.

In equilibria B and C the policy maker expects that firms expect that the next-period firms set high inflation and so consumption optimally increases. Higher consumption contributes to faster debt stabilization. All responses are stronger in equilibrium C than in equilibrium B, as we discussed above.

Numerical Solution. Because of the model simplicity we were able to substitute (21) into (19) and obtain the resulting expression in the form of a univariate polynomial with given coefficients. We then employed standard numerical technique to find roots of this polynomial.[18] Instead, we could specify parameters of the dynamic system (11)-(12) and of the policy objective (13) and employ numerical routines to obtain *discretionary* equilibria directly. The two commonly used routines by Oudiz and Sachs (1985) and Backus and Driffill (1986) employ essentially the same *iterative algorithm* that requires to initialize values for π_b and S and then updates them *simultaneously*.[19] Oudiz and Sachs (1985) suggest to use some particular initialization to start iterations, while Backus and Driffill (1986) suggest that any initialization would work. If the equilibrium is unique, as Oudiz and Sachs (1985) conjecture, and the algorithm converges, then the initialization should not matter; and Söderlind (1999), who presents a popular implementation of this algorithm, suggests to initialize with any positive S and zero π_b.

The last line in Table 1 reports whether the equilibrium can be obtained by numerical routines. In addition to the Oudiz and Sachs (1985) and Backus and Driffill (1986) algorithm with different initialization, herewith labelled 'OS' and 'BD' correspondingly, we checked which equilibria we obtain if for an arbitrary policy c_b we find π_b using (21) and then update c_b using (19). We label the latter algorithm 'PP' for policy–private sector iterations and it is based on *consequent* update of agents' decisions. Again, different initialization of c_b can generate different equilibria, not necessarily those that are obtained by the OS/BD algorithm.

Only two equilibria A and C can be obtained by these iterative routines. A dash in the cell for the 'Weakly Passive' equilibrium B means that even the initialization that is only different by 10^{-6} from the exact solution does not lead to convergence of any algorithm we used, as the difference with the exact solution rises with the number of iterations.

2.2.3 Moves between Equilibria

This example explicitly shows that the optimizing policy maker at time t chooses policy c_b knowing that the private sector's decision π_b in period t explicitly depends on the private sector's decision $\tilde{\pi}_b$ in period $t+1$, and also knowing that at any time $s > t$ the optimizing policy maker will choose the best policy, based on period-s decision rule of the private sector, that itself depends on the period-$s+1$ decision rule of the private sector. The next-period decision of the private sector $\tilde{\pi}_b$ will be a function of the next-period policy $\tilde{c}_b$ and of the period-$s+2$ private sector decisions. The policy maker, thus, chooses policy c_b that is conditioned on the aggregated private sector's beliefs about the future policy $\tilde{c}_b$. Dynamic complementarities between decisions of the private sector and the policy maker create the possibility of different beliefs about the future course of policy and multiple discretionary equilibria arise.

In contrast to our first example, the policy maker's actions affect evolution of the state. The policy maker can affect the state and, therefore, future decisions of both agents in different ways

[18] We used standard routines in MATLAB.

[19] Currie and Levine (1993) suggest a similar algorithm to solve continuous-time problems.

that are consistent with different beliefs. When multiplicity arises, the different sets of beliefs correspond to different paths of adjustment, that differ by the speed of adjustment. As in linear models all variables are adjusting at the same speed, that was uniquely determined by ρ in the Canonical New Keynesian model, different paths are only possible if the state can be controlled in different ways, so the presence of an *endogenous* state variable is necessary for multiplicity.

In this model, the dynamic structure of interactions between the private sector and the policy maker implies an easy move of the economy from one equilibrium to another. Suppose that in the response to positive b_t the current policy maker reduces demand because (i) it knows the current-period firms expect tight monetary policy in the future and (ii) it expects the current-period firms will rationally lower inflation given firms expectations of tight future policy. The next-period debt remains high. Suppose that after the current-period policy maker has acted, and *if multiple equilibria exist*, the current-period firms change their beliefs about the future policy and increase inflation. They react to the observed policy and to the state, but the strength of their reaction depends on expectations of future policy. The next-period policy maker will observe high debt and high inflation and will know that firms expect lax policy in all consequent periods, and will expect that the next-period firms increase inflation after the policy maker moves. We have shown that it will be optimal for the next-period policy maker to validate the beliefs that will prevail at the time of decision.

Firms choose inflation based on beliefs, but they are free to change beliefs when making decisions. A change in the private sector beliefs can occur because of some exogenous event, a 'sunspot', that arrives at the time of decisions of firms. Albanesi et al. (2003) and King and Wolman (2004) described this situation as 'expectations traps'. Although the policy maker moves first within each period, it cannot manipulate the beliefs of the private sector about *future* policy.

This model does not suggest which equilibria are more realistic and should be preferred. Nothing in the model suggests how the private sector sets its beliefs about the future policy if a sunspot realizes. The private sector knows that if they change the belief that it will be optimal for the policymaker change the policy. If some 'bad' sunspot realizes the welfare-dominant equilibrium becomes unattainable.

2.2.4 Robustness of Results and their Empirical Relevance

The strength of fiscal control of debt and 'active'/'passive' policies. Parameters ν and ρ of the model in (11)-(12) are functions of parameter μ, that determines the strength of fiscal feedback in the underlying tax rule $\tau_t = \mu b_t$. We vary μ between zero and some relatively large number. We plot π_b, c_b and welfare against the strength of fiscal feedback in Panel II in Figure 2.[20] Different discretionary equilibria as functions of μ correspond to different branches. Panel II in Figure 2 demonstrates that only equilibrium C survives for small μ, in particular for $\mu = 0$; if μ is sufficiently large, then only equilibrium A survives. If μ is sufficiently large then monetary policy reacts to debt only weakly and stabilizes inflation in a conventional way, so can be classified as 'active'. Fiscal policy can be classified as 'passive' as it is totally devoted to control of domestic debt. If μ is zero or relatively small then monetary policy controls debt tightly in order to ensure

[20] We plot welfare as minus loss, hence negative numbers. The loss is measured in percentage of steady state consumption; the loss is small as we work with the deterministic model, assuming that the only sourse of displacement is the higher debt level (1%) in the initial moment.

that it will converge back to equilibrium, so it can be classified as 'passive'. Fiscal policy can be classified as 'active' as it pursues some other targets but not the control of debt. In this regime inflation is accommodated as this helps to reduce real debt. Our classification resembles the one in Leeper (1991), but it differs in policy design: we consider discretionary monetary policy, not a policy formulated in terms of simple rules.

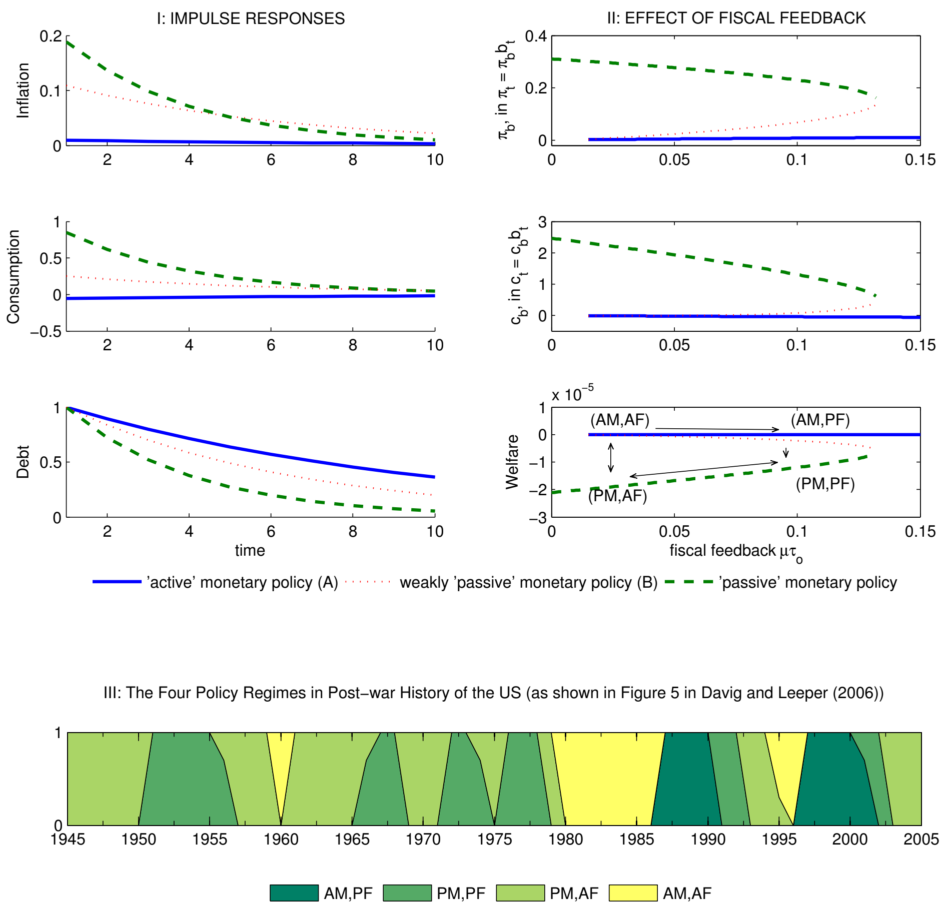

Figure 2: Discretionary Equilibria and Policy Regimes

We discover three equilibria for a weak fiscal feedback $0 < \mu \ll \infty$. Equilibria B and C are qualitatively similar, and only differ by the strength of reactions. It may be difficult to distinguish them empirically. One of them may be implausible and further research in equilibrium selection mechanisms may help to eliminate one of them. Because the issues of equilibrium selection are beyond the scope of this paper, in the rest of this section we simply distinguish between two types of interactions, and use equilibrium C to illustrate 'passive' monetary policy, as equilibrium B does not survive for very small values of μ.

Empirical relevance. Davig and Leeper (2006) (herewith DL) document fluctuating policies is the post-war United States. We argue that a Markov-switching model in DL implies the same regimes that we identified as equilibria A and C.

DL identify four different regimes of interactions of monetary and fiscal policy. They assume that economic policy is formulated in terms of simple rules and each policy maker can implement either an 'active' or a 'passive' rule. As a result, there are four possible combinations of active/passive monetary policy (we label them as AM and PM correspondingly) and active/passive fiscal policy (labelled AF and PF). The authors estimate these rules and document the sequence of movements from one such regime to another in the post-war US history. We plot the sequence of regimes in Panel III in Figure 2. This picture is adapted from Figure 5 in DL. For every year the relative width of every color corresponds to probability that each regime prevails. For example, in 1956 we have approximately a 70% probability of (PM, PF) regime and a 30% probability of (PM, AF) while in 1985 it is a 100% probability of (AM, AF). In 1945 the US economy starts at (PM, AF) and it arrives to the same state in 2005.

Using our model and the resulting discretionary equilibria we can interpret these movements as switches between the two discretionary equilibria. Our fiscal feedback parameter μ is similar (in effects) to the fiscal feedback on output in the estimated fiscal rules in DL. We use the bottom chart in Panel II in Figure 2 and label where the four policy regimes identified by DL might lie. Arrows show all realized movements between the states that happened in the post-war period in the US. Some of the changes in regimes can be seen as switches between the two equilibria. We do not discuss what exact signal can be interpreted as a sunspot in each particular case, but note instead the very particular sequence of 'jumps' and 'falls' between the two equilibria. The best equilibrium has been attained in 'jumps' from the worst state (PM, AF) only. There are three such 'jumps' from (PM, AF) to (AM, AF), in 1959, 1980 and in 1994, one of which resulted in an immediate 'fall' back. It is also apparent that the route from the best to the worst equilibrium is via (AM, PF) to (PM, PF), and this route is with the minimal loss in welfare as the two branches in the welfare chart in Panel II of Figure 2 are closest to each other. These moves happened in 1991 and 2001, both post-recession years in which the FED lowered interest rates. The observed regularity invites new research to explain the pattern, but further explorations are beyond the scope of this paper.

2.3 New Keynesian Model with Capital Accumulation

The previous model demonstrates the *existence* of multiple equilibria in LQ RE models under discretion and the role of predetermined endogenous states. It was chosen to be simple to do this in an effective way. More realistic models, however, are likely to have several decision variables of the agents and several states. In most cases we invariably resort to numerical methods to find discretionary equilibria.

This example shows that LQ RE models can have *two different types* of multiple equilibria. We have shown that multiple equilibria can arise because of dynamic complementarity of decisions of the private sector and the policy maker, we show in this section that dynamic complementarity between different decisions of the private sector can also lead to multiple equilibria. King and Wolman (2004) obtain equilibria of the second type in a non-linear model; they call them multiple 'point-in-time' equilibria.

The point-in-time equilibria are likely to happen in any moderately complex model, but as we emphasize here, conventional numerical algorithms do not find them. We need to develop different algorithms but we need to know properties of equilibria they search for.

The main aim of this Example is to demonstrate the existence of multiple point-in-time equilibria and show how their existence implies the existence of multiple discretionary equilibria. We shall generalize our results in Section 4. These results allow us to develop a numerical algorithm that finds *all* point-in-time equilibria and reduces the number of discretionary equilibria that remain undiscovered by conventional algorithms.

In order to demonstrate this type of multiplicity we have to work model that has several decision variables of the private sector. We use the New Keynesian model with capital accumulation, adopted from Woodford (2003a), Woodford (2005) and Sveen and Weinke (2005) that has several decision variables of the private sector. As before, we delegate all technical details to the Online Appendix.

2.3.1 Discretionary Policy

The log-linearized about the steady state equations that describe aggregate decisions of the private sector and evolution of the state can be written as:

$$\begin{aligned} c_t &= c_{t+1} - \sigma(i_t - \pi_{t+1}), && (23) \\ \Delta k_{t+1} &= \beta \Delta k_{t+2} + \frac{1}{\varepsilon_\psi}\left(\left(1-\beta\left(1-\delta\right)\right) ms_{t+1} - \left(i_t - \pi_{t+1}\right)\right), && (24) \\ \pi_t &= \beta \pi_{t+1} + \lambda_c c_t + \lambda_o k_{t+1} - \lambda_k k_t. && (25) \end{aligned}$$

The aggregate state variable in this economy is the real capital stock, k_t and the initial state $\bar{k}$ is known to all agents. Equation (23) is standard Euler equation for aggregate consumption c_t, i_t is nominal interest rate and π_t is inflation. Equation (24) describes capital accumulation with depreciation rate δ, adjustment cost parameter ε_ψ, and real marginal savings $ms_t = \frac{\phi+1}{(1-\alpha)}\left(\zeta c_t + \frac{1-\zeta}{\delta}\left(k_{t+1} - \left(1-\delta\right)k_t\right)\right) + \frac{1}{\sigma}c_t - \frac{\alpha\phi+1}{(1-\alpha)}k_t$.[21] Equation (25) is a New Keynesian Phillips curve. The private sector chooses consumption c_t, inflation π_t, and the the next-period capital k_{t+1}. System (23)-(25) defines the private sector equilibrium.

The policy maker's control variable is nominal interest rate i_t. The inter-temporal policy maker's criterion is defined by the quadratic loss function

$$L_t = \frac{1}{2}\sum_{s=t}^{\infty}\beta^{s-t}\left(\pi_s^2 + \omega y_s^2\right) = \frac{1}{2}\sum_{s=t}^{\infty}\beta^{s-t}\left(\pi_s^2 + \omega\left(\zeta c_s + \frac{1-\zeta}{\delta}\left(k_{s+1} - \left(1-\delta\right)k_s\right)\right)^2\right),$$

[21] Parameters ζ, ϕ, σ and α are the steady state consumption to output ratio, elasticity of labour supply, inverse elasticity of inter-temporal substitution of consumption, and capital share in production function, correspondingly. Parameter β is the private sector discount factor. The system is log-linearised about the zero-inflation steady state. We assume capital can be rented.

where we substituted out output y_t using the national income identity. Parameter ω is given.[22]

The policy maker knows the law of motion (23)-(25) of the aggregate economy and takes it into account when formulating its policy. The policy maker finds the best action every period, knows that future policy makers have freedom to change policy, and knows that the thought process by which all policy makers decide is the same.

The policy maker acts under discretion. At every point t in time instruments of each agent are linear functions of the current state:

$$\begin{aligned} i_t &= \iota_k k_t, && (26)\\ k_{t+1} &= k_k k_t && (27)\\ c_t &= c_k k_t && (28)\\ \pi_t &= \pi_k k_t. && (29) \end{aligned}$$

We take (27)-(29) one step forward and use (23)-(25) to obtain decomposition of the private sector decisions into the response to the state and response to policy:

$$\begin{aligned} k_{t+1} &= \tilde{k}_S\left(\tilde{k}_k, \tilde{c}_k, \tilde{\pi}_k\right) k_t + \tilde{k}_P\left(\tilde{k}_k, \tilde{c}_k, \tilde{\pi}_k\right) i_t, && (30)\\ \pi_t &= \tilde{\pi}_S\left(\tilde{k}_k, \tilde{c}_k, \tilde{\pi}_k\right) k_t + \tilde{\pi}_P\left(\tilde{k}_k, \tilde{c}_k, \tilde{\pi}_k\right) i_t, && (31)\\ c_t &= \tilde{c}_S\left(\tilde{k}_k, \tilde{c}_k, \tilde{\pi}_k\right) k_t + \tilde{c}_P\left(\tilde{k}_k, \tilde{c}_k, \tilde{\pi}_k\right) i_t, && (32) \end{aligned}$$

where we use subscripts 'S' for state and 'P' for policy. We also put tildes over the coefficients to demonstrate that they depend on *next-period* decisions of the private sector, $\tilde{k}_k, \tilde{c}_k, \tilde{\pi}_k$. This decomposition allows us to account for the 'instantaneous' influence of policy on private sector decisions. As in previous examples, coefficients are rational functions of next-period decisions $\tilde{k}_k, \tilde{c}_k$, and $\tilde{\pi}_k$, we report them in Appendix A.

The Bellman equation characterizing discretionary policy becomes

$$\begin{aligned} Sk_t^2 = \min_{i_t} \Bigg(& \left(\tilde{\pi}_S k_t + \tilde{\pi}_P i_t\right)^2 + \omega \Bigg(\left(\zeta \tilde{c}_S + \frac{1-\zeta}{\delta}\left(\tilde{k}_S - (1-\delta)\right)\right) k_t \\ & + \left(\zeta \tilde{c}_P + \frac{1-\zeta}{\delta}\tilde{k}_P\right) i_t\Bigg)^2 + \beta \tilde{S}\left(\tilde{k}_S k_t + \tilde{k}_P i_t\right)^2\Bigg). \end{aligned} \qquad (33)$$

Differentiation of (33) with respect to i_t yields the optimal policy response:

$$i_t = -\frac{\left(\tilde{\pi}_P \tilde{\pi}_S + \left(\zeta \tilde{c}_P + \frac{1-\zeta}{\delta}\tilde{k}_P\right)\omega\left(\zeta \tilde{c}_S + \frac{1-\zeta}{\delta}\left(\tilde{k}_S - (1-\delta)\right)\right) + \beta \tilde{S}\tilde{k}_P \tilde{k}_S\right)}{\left(\tilde{\pi}_P^2 + \omega\left(\zeta \tilde{c}_P + \frac{1-\zeta}{\delta}\tilde{k}_P\right)^2 + \beta \tilde{S}\tilde{k}_P^2\right)} k_t = \iota_k k_t. \qquad (34)$$

The feedback coefficient in (34) determines the optimal policy feedback on predetermined state, k_t, but the feedback coefficient is a function of $\tilde{S}$. $\tilde{S}$ determines the next-period loss that depends on the whole future path of the state, that is a function of future policy decisions. When the

[22] We use 'traditional' welfare metric. The choice is unimportant for our results and this is a reasonable metric an actual policymaker is likely to choose for a complex model.

current-period policy maker chooses the best policy, it knows that the next-period policy maker will re-optimize. In this model this constraint is binding. It ensures the non-linearity of $\iota_k = \iota_k\left(\tilde{k}_k, \tilde{c}_k, \tilde{\pi}_k\right)$.

In order to find the value function S we substitute the optimal solution (34) into the Bellman equation (33) and, under the equilibrium condition $S = \tilde{S}$, obtain the following quadratic equation for the value function S with negative constant term:

$$\beta S^2 + \mu S - \omega\left(\frac{(1-\zeta)}{\delta}\left(\tilde{\pi}_S - \left(\tilde{k}_S - (1-\delta)\right)\frac{\tilde{\pi}_P}{\tilde{k}_P}\right) + \zeta\frac{(\tilde{\pi}_S\tilde{c}_P - \tilde{\pi}_P\tilde{c}_S)}{\tilde{k}_P}\right)^2 = 0,$$

where coefficient $\mu = \mu\left(\tilde{k}_k, \tilde{c}_k, \tilde{\pi}_k\right)$ is given in Appendix A. This equation has a unique solution $S \geq 0$.

$$\begin{aligned} S &= -\frac{1}{2\beta}\left(\mu + \sqrt{\mu^2 + 4\beta\omega\left(\frac{(1-\zeta)}{\delta}\left(\tilde{\pi}_S - \left(\tilde{k}_S - (1-\delta)\right)\frac{\tilde{\pi}_P}{\tilde{k}_P}\right) + \zeta\frac{(\tilde{\pi}_S\tilde{c}_P - \tilde{\pi}_P\tilde{c}_S)}{\tilde{k}_P}\right)^2}\right) \\ &= S\left(\tilde{k}_k, \tilde{c}_k, \tilde{\pi}_k\right) = S\left(k_k, c_k, \pi_k\right). \end{aligned}$$

The last equality holds in the time-invariant private sector equilibrium.

In order to obtain the optimal policy we substitute S into (34) and obtain

$$\iota_k = \iota_k\left(\tilde{k}_k, \tilde{c}_k, \tilde{\pi}_k\right) = \iota_k\left(k_k, c_k, \pi_k\right). \tag{35}$$

By construction, for every triplet $\{k_k, c_k, \pi_k\}$ that describes the time-invariant private sector response we obtain a *unique* ι_k that describes the policy decision.

2.3.2 Private Sector Response: Point-in-Time equilibria

We substitute equation (26) into (30)-(32) and, after some manipulations, obtain the following system that describes time-invaiant optimal response of the private sector:

$$k_k = \frac{1}{\beta\nu_k + \lambda_k\nu_r}\left(\left(\beta\left(1 - \nu_o\tilde{k}_k\right) + \lambda_o\nu_r\right)k_k - \nu_r\pi_k + (\lambda_c\nu_r - \beta\nu_c)\,c_k + \beta\nu_r\iota_k\right), \tag{36}$$

$$\pi_k = (\beta\tilde{\pi}_k + \lambda_o)\,k_k + \lambda_c c_k - \lambda_k, \tag{37}$$

$$c_k = \frac{1}{\beta + \sigma\lambda_c}\left((\beta\tilde{c}_k - \sigma\lambda_o)\,k_k + \sigma\pi_k + \sigma\left(\lambda_k - \beta\iota_k\right)\right). \tag{38}$$

where all coefficients ν are given in *Appendix*. From the first two equations it is immediately apparent that for a given policy ι_k, in a response to a positive state k_t higher consumption raises inflation but it also makes profit optimizing firms to increase next-period capital stock in order to meet the increased demand; and higher next-period capital raises inflation too. Decisions to raise consumption and to increase the next-period capital stock are dynamic complements.

In the previous example the decision of the private sector was unique given policy. Here we have complementarity between the effects of different instruments of the private sector and the multiplicity of *point-in-time equilibria* is likely to arise: there can be several private sector responses to *the same* policy decision.

We simplify system (36)-(38) assuming equilibrium conditions $k_k = \tilde{k}_k$, $\pi_k = \tilde{\pi}_k$ and $c_k = \tilde{c}_k$. We solve the last equation with respect to π_k :

$$\pi_k = \frac{1}{\sigma}\left(\sigma\lambda_o k_k + (\beta + \sigma\lambda_c - \beta k_k)\, c_k + \sigma\left(\beta\iota_k - \lambda_k\right)\right), \tag{39}$$

and substitute into the first two equations. We obtain the following system:

$$c_k = \sigma\frac{\left(\nu_k - k_k + \nu_o k_k^2\right)}{\left(\nu_r k_k - (\nu_r + \sigma\nu_c)\right)} \tag{40}$$

$$c_k = \sigma\frac{\left(\lambda_o k_k^2 - (\lambda_k - \beta\iota_k)\, k_k - \iota_k\right)}{\left(\beta k_k^2 - (1 + \beta + \sigma\lambda_c)\, k_k + 1\right)} \tag{41}$$

Solution to this system for some value of ι_k is plotted in Panel I in Figure 3. We plot dependence $c_k(k_k)$, given by equation (40), with a solid line. We plot dependence $k_k(c_k)$ given by equation (41) with a dash-dotted line.[23] There are four pairs $\{c_k, k_k\}$ that solve system (40)-(41) and they are labelled as private sector equilibria A, B, C and D. For every pair $\{c_k, k_k\}$ we restore the unique $\pi_k(c_k, k_k)$ using (39). For all values of ι_k in equilibria A and B the economy is stable as $|k_k| < 1$, while it is unstable in equilibria C and D with $k_k > 1$. We do not consider equilibria C and D further.

We plot how $\{c_k, k_k, \pi_k\}$ change with ι_k in equilibria A and B in Panel II in Figure 3. We use dotted and solid lines for equilibria A and B correspondingly.

To summarize, for a given policy $\iota_k$we have found two stable time-invariant point-in-time private sector equilibria. Suppose the capital stock is higher than in the steady state. In response to policy ι_k the aggregated private sector can coordinate on one of the two equilibria. In equilibrium A consumption and investment fall so that excessive capital stock is quickly reduced. Inflation falls by much because and demand is low and the high level of productive capital has deflationary effect. In equilibrium B consumption rises by little and the next period capital is chosen to ensure slow reduction of the excessive stock. A combination of high level of productive capital with high demand ensures only slight fall in inflation. All variables converge back to the steady state only slow.

For a given ι_k we compute two triplets $\{c_k(\iota_k), k_k(\iota_k), \pi_k(\iota_k)\}$ that describe private sector equilibria A and B. However, knowing the beliefs of the aggregated private sector the *optimal* discretionary policy will choose the *unique* $\iota_k^* = \iota_k(c_k, k_k, \pi_k)$, see equation (35). We plot $\iota_k^* = \iota_k^*(c_k(\iota_k), k_k(\iota_k), \pi_k(\iota_k)) \equiv \iota_k^*(\iota_k)$ with dotted and solid lines for private sector equilibria A and B correspondingly in the right chart in Figure 3. Points of intersection of lines $\iota_k^* = \iota_k^*(\iota_k)$ with the 45-degree line are points of *discretionary policy equilibria.*

The dotted line (private sector equilibrium A) intersects the 45-degree line once. If the policy maker expects that the private sector equilibrium A will prevail, it will choose positive ι_k. Consistent with current and future positive response of interest rate to higher-than-steady-state-level of capital the private sector coordinates on the equilibrium with large contraction in demand and investment and with low inflation. Capital is stabilized quickly in this equilibrium.[24]

[23] We calibrate the model as

[24] The dotted line also intersects the solid line (private sector equilibrium B): for some ι_k the optimal response of policy to *different* private sector equilibria is the same ι_k^*. Howevere this is not a *discretionary equilibrium* because the private sector will react in two different ways and the optimal response of policy will differ from ι_k^*.

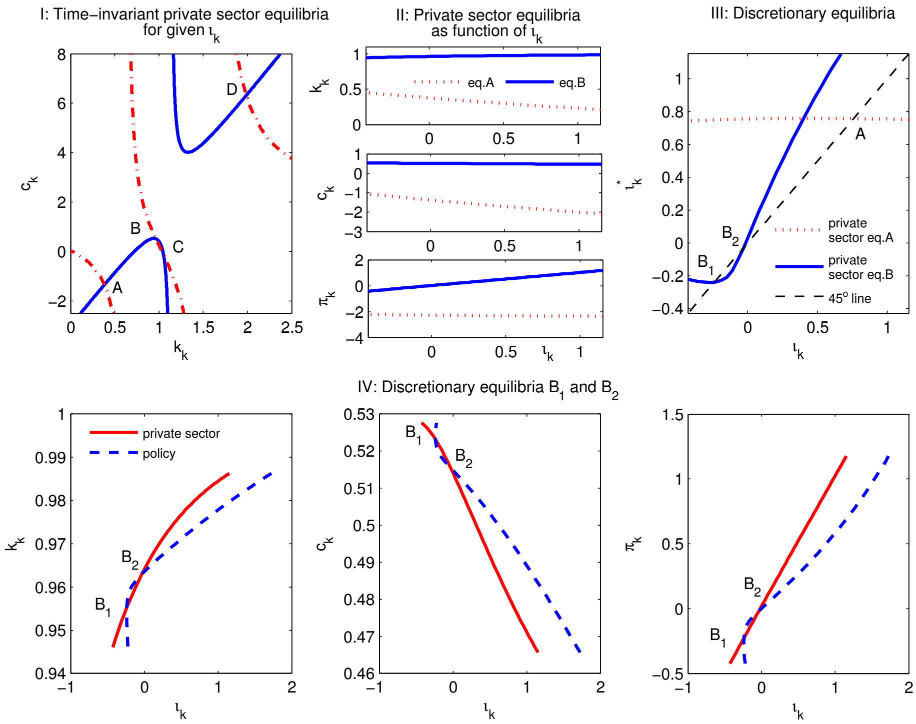

Figure 3: Discretionary Equilibria in the New Keynesian Model with Capital Accumulation

The solid line (private sector equilibrium B) intersects the 45-degree line twice, in points labelled B_1 and B_2 in the right chart in Figure 3. If the policy maker expects that the private sector equilibrium B will prevail, it chooses negative ι_k. Consistent with current and future negative response of interest rate to higher-than-steady-state-level of capital the private sector coordinates on equilibrium with small expansion in demand, small contraction in future capital stock and small negative inflation. However, there are dynamic complementarities between the decisions of the private sector and the policy maker, so two discretionary equilibria arise, similar to the previous example with government debt. The mechanism is also similar: lower marginal cost results in lower inflation that implies lower interest rate that results in lower marginal cost.

Specifically, suppose the real capital stock is above the steady state, $k_t > 0$. Suppose that the policy maker knows that the private sector believes that the future policy maker lowers interest rate in response to a positive k_t. Suppose the policy maker expects that in a response to the current-period fall in interest rate the aggregated private sector coordinates on equilibrium B with higher consumption, slightly lower next-period capital and a small negative inflation. It will be optimal for the policy maker to lower interest rate and satisfy the beliefs of the private sector. However, assume a further fall in interest rate. It results in even higher consumption, that crowds out investment even more and so the next-period capital is even lower. The deflationary effect of even lower next-period capital on marginal cost outweighs the effect of consumption,

Table 2: New Keynesian Model with Capital Accumulation: Characteristics of Discretionary Equilibria

Discretionary Equilibria		'Fast' A	'Moderately Slow' B_1	'Slow' B_2
Private	k_k	0.2625	0.9552	0.9630
sector	c_k	-2.3367	-0.2257	-0.0145
response	π_k	-1.8590	0.5232	0.5155
Policy	ι_k	0.7577	-0.2391	-0.0331
Speed of adjustment	k_k	0.2625	0.9552	0.9630
Loss	S	6.4064	0.5302	0.0102
Complementarities	$\text{sign}\left(\frac{\partial^2 W_t}{\partial k_k \partial \iota_k}, \frac{\partial^2 W_t}{\partial \pi_k \partial \iota_k}, \frac{\partial^2 W_t}{\partial c_k \partial \iota_k}\right)$	$(+,-,+)$	$(+,+,-)$	$(+,+,+)$
Numerical methods		BD*, PP	PP	OS, BD

Notes:
* Obtained with initialization $S = 1$, $\pi_k = c_k = k_k = 0$, Söderlind (1999).

so inflation falls by more, that makes the assumed further fall in interest rate be rational and optimal. Multiple discretionary policy equilibria B_1 and B_2 arise, see also Panel IV in Figure 3.[25] All characteristics of equilibria A, B_1 and B_2 are reported in Table 2.

As in previous examples we can formally check whether the private sector and the policy maker decisions are dynamic complements. Straightforward substitutions yield the following welfare function as function of ι_k, k_k, c_k, and π_k:

$$\begin{aligned} W_t &= -\frac{1}{2}\sum_{s=t}^{\infty}\beta^{s-t}\left(\pi_s^2 + \omega\left(\zeta c_s + \frac{1-\zeta}{\delta}\left(k_{s+1} - (1-\delta)\,k_s\right)\right)^2\right) \\ &= -\frac{1}{2}\left(\left(\pi_S + \pi_P \iota_k\right)^2 + \omega\left(\zeta\left(c_S + c_P\iota_k\right) + \frac{1-\zeta}{\delta}\left(k_S + k_P\iota_k - (1-\delta)\right)\right)^2\right)\frac{k_t^2}{1-\beta k_k^2}. \end{aligned}$$

We compute the sign of each of the second order derivatives $\frac{\partial^2 W_t}{\partial k_k \partial \iota_k}$, $\frac{\partial^2 W_t}{\partial \pi_k \partial \iota_k}$ and $\frac{\partial^2 W_t}{\partial c_k \partial \iota_k}$ in discretionary equilibria A, B_1 and B_2 and report them in Table 2. It is apparent that in each equilibrium at least one derivative is positive. Hence, private sector and the policy maker decisions are dynamic complements and each pont-in-time equilibrium can generate more than one discretionary equilibrium, as we demonstrate.

2.3.3 Numerical Solution

As in the previous example, we checked which equilibria can be obtained with numerical iterative algorithms. We present the result in the last line of Table 2 The OS and BD initialization of the *conventional iterative algorithm* can deliver equilibria A and B_2.

[25] Because of multi-variate private sector's response, Panel IV plots optimal policy given the private sector's *response to a policy.* In the previous example in *Figure X* we plotted the optimal policy for an *arbitrary* decision of the private sector.

We can also compute *all* point-in-time equilibria for some policy decision ι_k.[26] Then, for *each* point-in-time equilibrium that stabilizes the economy we can use (34) and compute optimal policy ι_k, and for this policy compute *the same* point-in-time equilibrium.[27] If this iterative procedure converges, then we label the resulting equilibrium with 'PP' in Table 2. We are able to get equilibria A and B_1.

Note that the PP iterative algorithm is different from the OS and BD algorithm in convergence properties: Identical initialisations of the PP and of the OS/BD algorithm can lead to different equilibria. It is also important that we know *how many* point-in-time equilibria exist before searching for them, their multiplicity means that *iterative* algorithms may not be able to find all of them, regardless of their intialiazation. The knowledge of the *nature* of the solution we seek may help to design different algorithms, with either different convergence properties or not even recoursive.

3 The General Framework

3.1 Discretionary Policy

We assume a non-singular linear deterministic rational expectations model, augmented by a vector of control instruments. Specifically, the evolution of the economy is explained by the following system:

$$\begin{bmatrix} y_{t+1} \\ x_{t+1} \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} y_t \\ x_t \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} [u_t], \tag{42}$$

where y_t is an n_1-vector of predetermined variables with initial conditions y_0 given, x_t is n_2-vector of non-predetermined (or jump) variables with $\lim_{t\to\infty} x_t = 0$, and u_t is a $k-$vector of policy instruments of the policy maker. For notational convenience we define the n-vector $z_t = (y_t', x_t')'$ where $n = n_1 + n_2$. We assume invertibility of A_{22}.

Typically, the second block of equations in this system represents aggregation of the first order conditions to optimization problem for the private sector. The private sector has decision variables, represented by x_t. Additionally, there is an equation explaining the evolution of the predetermined state variables y_t. These two equations together describe the 'evolution of the economy' as observed by the policy maker, or the private sector equilibrium.

The inter-temporal policy maker's criterion is defined by the quadratic loss function:

$$L_t = \frac{1}{2}\sum_{s=t}^{\infty} \beta^{s-t} g_s' \mathcal{Q} g_s = \frac{1}{2}\sum_{s=t}^{\infty} \beta^{s-t} \left(z_s' Q z_s + 2 z_s' P u_s + u_s' R u_s \right). \tag{43}$$

The elements of vector g_s are the goal variables of the policy maker, $g_s = \mathcal{C}(z_s', u_s')'$. The matrix $\mathcal{Q}$ is assumed to be symmetric and positive semi-definite.[28]

[26] The simplicity of this particular model and the prior knowledge of the number and location of the equilibria implies that we can use polynomial root-finding algorithms to find them. The method that finds all points-in-time equilibria is suggested by Proposition 4 that we discuss in the next section.

[27] In this model we order all point-in-time equilibria by the implied speed of convergence.

[28] It is standard to assume that R is symmetric positive definite, respectively (see Anderson et al. (1996), for

Assumption 1 *Suppose at any time t the private sector and the policy maker only respond to the current state:*

$$u_t = \mathcal{F}(y_t) = -Fy_t \quad (44)$$
$$x_t = \mathcal{N}(y_t) = -Ny_t \quad (45)$$

This assumption rules out non-stationarity of policy and private sector decisions, i.e. the time-dependence in a more general formulation $u_t = \mathcal{F}(t; y_t, y_{t-1}, .., y_{t-k}, ..)$, $x_t = \mathcal{N}(t; y_t, y_{t-1}, .., y_{t-k}, ..)$, and restricts policy decisions to memoryless feedback rules. We also assume that rules are *linear* in state.[29]

We define discretionary policy as satisfying several constraints. We want to assume that the policy maker can implement (or truthfully announce) at each point of time its policy decision *before* the private sector selects its own action x_t.

Assumption 2 *At each time t the private sector observes the current decision u_t and expects that future policy makers at any time $s > t$ will re-optimize, will apply the same decision process and implement decision (44).*

Proposition 1 *Given Assumption (2) the current aggregate decision of the private sector can be written as a linear feedback function:*

$$x_t = -Jy_t - Ku_t \quad (46)$$

where

$$J(N) = (A_{22} + NA_{12})^{-1}(A_{21} + NA_{11}), \quad (47)$$
$$K(N) = (A_{22} + NA_{12})^{-1}(B_2 + NB_1). \quad (48)$$

Proof. Relationship (45) can be taken with one lead forward and y_{t+1} is substituted from the first equation (42). We obtain:

$$\begin{aligned} x_{t+1} &= -\tilde{N}y_{t+1} = -\tilde{N}(A_{11}y_t + A_{12}x_t + B_1u_t) \\ &= A_{21}y_t + A_{22}x_t + B_2u_t, \end{aligned} \quad (49)$$

from where it follows:

$$x_t = -(A_{22} + \tilde{N}A_{12})^{-1}[(A_{21} + \tilde{N}A_{11})y_t + (B_2 + \tilde{N}B_1)u_t] = -J\left(\tilde{N}\right)y_t - K\left(\tilde{N}\right)u_t. \quad (50)$$

where $J\left(\tilde{N}\right)$ and $K\left(\tilde{N}\right)$ are defined as in (47)-(48). Invertibility of A_{22} ensures invertibility of $A_{22} + \tilde{N}A_{12}$ almost surely. ∎

Proposition 1 implies that the policymaker which moves before the private sector takes into account its 'instantaneous' influence on the choice of x_t which is measured by $-K\left(\tilde{N}\right)$ and which also depends on the future response of the private sector to the state, $\tilde{N}$.

example). However, since many economic applications involve a loss function that places no penalty on the control variables, we note that the requirement of $\mathcal{Q}$ being positive definite can be weakened to $\mathcal{Q}$ being positive semi-definite if additional assumptions about other system matrices are met (Clements and Wimmer (2003)). The analysis in this paper is valid for $R \equiv 0$.

[29] Here and below the word 'rule' is used with meaning 'function'. No 'precommitment to rules' is assumed.

Assumption 3 *At each point of time t the policy maker knows Assumptions 1 and 2.*

Definition 1 *Policy determined by (44) is discretionary if the policy maker finds it optimal to follow (44) each time $s > t$ given the Assumptions (1) – (3).*

Hence, we look for a policy u_t that satisfies the following Bellman equation[30]

$$L_t(y_t) = \min_{u_t} \left(\frac{1}{2} \left(y_s' Q^* y_t + 2y_t' P^* u_t + u_t' R^* u_s \right) + \beta L_{t+1} \left(A^* y_t + B^* u_t \right) \right),$$

with

$$\begin{aligned} Q^* &= Q_{11} - Q_{12}J - J'Q_{21} + J'Q_{22}J, \qquad P^* = J'Q_{22}K - Q_{12}K + P_1 - J'P_2, && (51) \\ R^* &= K'Q_{22}K + R - K'P_2 - P_2'K, \qquad A^* = A_{11} - A_{12}J, \qquad B^* = B_1 - A_{12}K. && (52) \end{aligned}$$

Because of the quadratic form of a flow-objective in (43) and because policies and private sector decisions are linear in state, the discounted loss will necessarily have quadratic form in state:

$$L_t(y_t) = \frac{1}{2} y_t' S y_t. \tag{53}$$

The Bellman equation characterizing discretionary policy, therefore, becomes:

$$y_t' S y_t = \min_{u_t} \left(y_s' \left(Q^* + \beta A^{*\prime} S A^* \right) y_t + 2y_t' \left(P^* + \beta A^{*\prime} S B^* \right) u_t + u_t' \left(R^* + \beta B^{*\prime} S B^* \right) u_t \right) \tag{54}$$

We have outlined a deterministic setup. None of the results depend on this, as we can always add an appropriate vector of shocks and appeal to the certainty equivalence property of LQ models. This, however, would complicate the analysis unnecessarily in order to demonstrate the main point.[31]

For a policy F and the private sector response N, the evolution of state variable satisfies the following equation:

$$y_{t+1} = M y_t, \tag{55}$$

where $M = A_{11} - A_{12}N - B_1 F$.

3.2 Discretionary equilibrium as a 'triplet' of matrices

Given y_0 and system matrices A and B, matrices N and F define trajectories $\{y_s, x_s, u_s\}_{s=t}^{\infty}$ in a unique way and vise versa: if we know that $\{y_s, x_s, u_s\}_{s=t}^{\infty}$ solve discretionary optimization problem stated above then, by construction, there are unique time-invariant linear relationships between them which we label by N and F. Matrix S defines the cost-to-go along a trajectory. Given the one-to-one mapping between equilibrium trajectories and $\{y_s, x_s, u_s\}_{s=t}^{\infty}$ and the triplet of matrices $\mathcal{T} = \{N, S, F\}$, it is convenient to continue with definition of policy equilibrium in terms of $\mathcal{T}$, not trajectories.

The following Proposition derives first order conditions for a discretionary optimization problem.

[30] A more general notion of time-consistency requires policy to be optimal but without specific assumption about intra-period order of moves.

[31] Shocks can be included into vector y_t, see e.g. Anderson et al. (1996).

Proposition 2 (First order conditions) *The first-order conditions to the discretionary optimization problem (42) – (43) can be written in the following form:*

$$\begin{aligned} N &= (A_{22} + NA_{12})^{-1}((A_{21} - B_2F) + N(A_{11} - B_1F)) && (56) \\ F &= (R^* + \beta B^{*\prime}SB^*)^{-1}\left(P^{*\prime} + \beta B^{*\prime}SA^*\right) && (57) \\ S &= Q^* + F'R^*F - F'P^{*\prime} - P^*F + \beta\left(A^* - B^*F\right)' S\left(A^* - B^*F\right) && (58) \end{aligned}$$

where matrices Q^*, P^*, R^*, A^*, and B^* are defined in (47)-(48) and (51)-(52)

Proof. From relationships (45) and (46) it immediately follows that

$$N = J - KF. \tag{59}$$

A straightforward substitution of (47)-(46) into (59) leads to (56)

The discretionary policy can be determined from (54) by differentiating with respect to u_t:

$$u_t = -(R^* + \beta B^{*\prime}SB^*)^{-1}(P^{*\prime} + \beta B^{*\prime}SA^*)y_t = -Fy_t,$$

from where the policy maker's reaction function is defined by (57). Now, we substitute policy $u_t = -Fy_t$ into (54) and obtain equations (58) for S. ■

Definition 2 *The triplet $\mathcal{T} = \{N, S, F\}$ is a discretionary equilibrium if it satisfies the system of FOCs (56)-(58).*

Definition 2 implicitly assumes that the first order conditions are necessary and sufficient conditions of optimality. We proceed with this assumption and demonstrate later in Proposition 5 that, under the assumption of symmetric positive semi-definite $\mathcal{Q}$, the second order conditions for the minimum are always satisfied.

4 Properties of discretionary equilibria

This section describes some properties of discretionary equilibria that can help to locate them.

We start the section on *multiple* equilibria with demonstration of a particular case where the discretionary equilibrium is *unique.* [32]

Proposition 3 (A Special Case) *Suppose A_{22} is non-singular, $A_{12} = 0$ and $B_1 = 0$. Then if the discretionary equilibrium exists it is unique.*

Proof. Formulae (46) suggests $K = A_{22}^{-1}B_2$ so it does not depend on N. It follows also that R^* does not depend on N and, generally speaking, is non-singular.

Equation (59) can be written as

$$A_{22}N - NA_{11} = A_{21} - B_2F,$$

[32] Proposition 3 proves what appears to be a well known fact, but we were unable to find a published proof. Typically, when dealing with a particular problem in this class of models, researchers easily find the particular solution, and it is clear that it is unique by construction, see e.g. Clarida et al. (1999).

so there is no quadratic term in N. Similarly, equation (57) collapses to

$$F = (R^*)^{-1}\left(\left(B_2' A_{22}^{-1\prime} \mathcal{Q}_{22}' - \mathcal{P}_2'\right)\left(N + A_{22}^{-1} B_2 F\right) - \left(\mathcal{Q}_{12} A_{22}^{-1} B_2 - \mathcal{P}_1\right)'\right),$$

and hence S does not affect F. Both equations together constitute a linear in coefficients of F and N system of $(k+n_2)n_1$ equations (after applying the vec-operator). If the system is non-singular, the solution is unique. Having determined F we can find corresponding S from equation (58), which always has a unique symmetric solution, as we demonstrate in later in this section. ■

In this example, $B_1 = 0$ suggests that predetermined state variables cannot be affected by policy and $A_{12} = 0$ suggests that they cannot be affected by private sector's decisions. A typical example of models in this class is a system where the only predetermined variables are potentially (auto-)correlated shocks, which are *exogenous* predetermined state variables. This is by no means uncommon in models that omit potentially important *endogenous* predetermined state variables such as capital or debt for the sake of simplicity.

Formally, our assumptions result in two of three non-linear first order conditions, (56)-(58), becoming linear and disconnected from the third equation in this special case. It is clear that this is unlikely to happen under more general conditions. In what follows, we shall study the first-order conditions in their most general form. Proposition 3 suggests that the model has to have predetermined endogenous state variables in order to be able to generate multiple equilibria under discretion.

The absence of endogenous predetermined state variables ensures complete disconnect between time periods. Discretionary policy maker knows that all future policy makers will re-optimize but without endogenous states this has no implications for the current policy choice.

Example in Section 2.3 demonstrates existence of multiple point-in-time equilibria. We explain their existence by dynamic complementarities of decisions of the private sector, given policy. If there are complementarities then multiple equilibria may arise. But if multiple equilibria do arise then there must be complementarities between (appropriately normalized) private sector's decisions.[33] It is more difficult to find complementarities among private sector's decisions than to find *all* point-in-time equilibria directly.

For a given policy response written in the form of linear rule $u_t = -Fy_t$ the coefficients of (56) depend only on the structural system matrices A and B. The next Proposition *describes* all solutions $N = N(F)$.

Proposition 4 *Under the following conditions:*

i) matrix $C = \begin{bmatrix} A_{11} - B_1 F & A_{12} \\ A_{21} - B_2 F & A_{22} \end{bmatrix}$ *has all distinct eigenvalues, and* V_{22} *in diagonalization* $C = V^{-1}\Lambda V$ *is invertible for any permutation of eigenvalues in* Λ.

ii) for some $\rho \in \mathbb{R}$, $\rho > 0$, m *is the number of eigenvalues of* C *that are strictly less than* ρ *in modulus,*

one of the three situations is almost always possible:

[33] See Vives (2005), p. XXX.

1. *If $m = n_2$ then there is a unique solution N to (56) such that all eigenvalues of transition matrix $M = A_{11} - A_{12}N - B_1F$ in:*

$$y_{t+1} = (A_{11} - B_1F)\, y_t + A_{12}x_t = My_t,$$

are strictly less than ρ in modulus.

2. *If $m > n_2$ then there are no solutions to (56) such that all eigenvalues of matrix $M(N)$ are strictly less than ρ in modulus.*

3. *If $m < n_2$ then there are at most $\binom{n-m}{n_2-m}$ of different solutions N to (56) such that all eigenvalues of matrix $M(N)$ are strictly less than ρ in modulus. All solutions can be found using standard methods of eigenvalue decomposition.*

This proposition generalizes the well known Blanchard and Kahn (1980) condition for rational expectations equilibrium to the class of time-invariant solutions.

Condition i) rules out continuum of solutions to (56), as shown by Freiling (2002). Condition ii) leaves the choice of parameter ρ to a researcher. Parameter ρ defines the asymptotic growth rate of y_t. Because in infinite-horizon optimization problems transversality conditions are *necessary* conditions, i.e. they follow from optimization, setting $\rho = 1/\sqrt{\beta}$ is the most nonrestrictive.[34]

In all examples in Section 2 the policy maker had only one instrument, either demand or interest rate. For these examples we demonstrated the uniqueness of optimal policy response, given time-invariant private sector equilibrium. The next Proposition proves that the result holds even for multi-variate policy instrument.

Proposition 5 *Suppose N is given. The following two results hold:*

1. *There is a unique symmetric positive semi-definite solution S to (58) if the matrix pair (A^*, B^*) is controllable, i.e. if the controllability matrix $[B^*, A^*B^*, A^{*2}B^*, ..., A^{*n_1-1}B^*]$ has full row rank.*[35]

2. *The policy function F, which is uniquely determined from (57) for given S, is such that all eigenvalues of transition matrix M (that defines the evolution of the system under control):*

$$\begin{aligned} y_{t+1} &= A_{11}y_t + A_{12}x_t + B_1u_t = (A_{11} - A_{12}J)\, y_t + (B_1 - A_{12}K)\, u_t \\ &= (A^* - B^*F)\, y_t = M\,(F(S))\, y_t, \end{aligned} \tag{60}$$

are strictly less than $1/\sqrt{\beta}$ in modulus.

It follows that $\lim_{t\to\infty} \beta^{-t/2} y_t = 0$. Thus, the policy reaction function ensures finite loss. It also follows that necessary conditions for optimality (58)-(57) are sufficient, because with symmetric and positive semi-definite matrix S the second-order conditions for minimum are always satisfied.

[34] It is sometimes suggested to set $\rho = 1$ (which is the case in Blanchard and Kahn (1980)). The over-restrictiveness of $\rho = 1$ is discussed, for example, in Sims (2001) for a similar class of LQ RE problems.

[35] The requirement of controllability of (A^*, B^*) is standard for the linear-quadratic optimal control. We use this condition as a sufficient condition. We do not discuss whether this is necessary condition.

Propositions 4 and 5 demonstrate that the private sector and the policy maker are 'non-symmetric' agents: while multiple private sector equilibria are possible if there are several decision instruments, the optimal response of the policy maker is always unique regardless the number of policy instruments.

Corollary 1 *Propositions 4 and 5 suggest that distinct discretionary equilibria have different matrices N and vice versa. Therefore, we can label a discretionary equilibrium not by a triplet of matrices, $T = \{N, S, F\}$ but by a single matrix 'identifier' N.*

Finally, we claim that all discretionary equilibria are determinate in the following sense.

Definition 3 *Discretionary equilibrium is determinate if, for given initial conditions, $(z_t, \ t = 0, 1, 2, ..)$ is a path under optimal discretionary policy then there exist no other path $(\tilde{z}_t, \ t = 0, 1, 2, ..)$ such that $\|z_t - \tilde{z}_t\| < \varepsilon$ in each period, where $\varepsilon > 0$ is any arbitrary small real number.*

Thus determinacy is viewed as a property of trajectories and not of their limit points.

Proposition 6 *Suppose we can find a discretionary equilibrium and compute the Jacobian of the system of first order conditions (56)-(58), $\mathcal{J}$. If $\det(\mathcal{J}) \neq 0$ then:*

1. *There can be at most a finite number of other discretionary equilibria.*

2. *All discretionary equilibria are determinate.*

Proof. There is a one-to-one correspondence between a trajectory and a triplet $T = \{N, S, F\}$. $q = (n + k) \times n_1$ coefficients of T solve polynomial system (56)-(58) of q equations. If the determinant of the Jacobian of the *polynomial* system of first order conditions is not equal to zero *identically*, then the system can *only* have a finite number of locally isolated solutions T. The local isolation is equivalent to determinacy. ■

We can say nothing if $\det(\mathcal{J}) = 0$ as it can either be equal to zero *identically* and so a continuum of solutions is possible, or it might be that $\det(\mathcal{J})$ has an isolated zero in this point and, again, we have the finite number of locally isolated discretionary equilibria. Condition $\det(\mathcal{J}) \neq 0$ is likely to be satisfied in most economic applications.[36]

5 Finding Equilibria

Examples 2 and 3 in Section 2 demonstrate that the number of discretionary equilibria can be greater than the number of point-in-time equilibria if there are dynamic complementarities between policy decisions and decisions of the private sector. Although a search for complementarities can help to understand *why* multiplicity arises, it is difficult to use this information in order to *find* the equilibria, or to prove that there is no more than $\binom{n-m}{n_2-m}$ equilibria, in particular in complex models.

Our examples show that even simple models lead to very complex system of polynomial equations to solve. Practically, we have to rely on numerical solutions. The current literature on

[36] The Online Appendix gives a formula for the Jacobian.

discretionary policy that uses numerical algorithms to find discretionary policy, invariably uses (variations of) the OS and BD iterative algorithm, that is essentially the same algorithm, based on simultaneous update of matrices S and N. We have demonstrated in Section 2 that *different initialization* of the same algorithm can converge to different solutions.

However, *different algorithms*, that are based on different updating schemes, may converge to different solutions even if we start with the same initialisation. Propositions 4 and 5 have practical implications for the design of different iterative routines. Proposition 4 states how to find all point-in-time equilibria for an arbitrary policy F. The proposition implies that *iterative* algorithms to find N may be unhelpful, but appropriate eigenvalue decomposition methods will find all point-in-time equilibria. Proposition 5 implies that for any private sector equilibrium N we can use *any* algorithm to search for for the optimal policy F as the solution is unique, no local extrema exists and the initialization does not matter. One can iterate between F and the *corresponding* N as we did in examples (the PP algorithm), and find discretionary equilibria that are impossible to locate with the OS/BD algorithm.[37] Propositions 4 and 5 can help in the future research to design better numerical algorithms to find all equilibria for particular models.

Finally, when solving a stochastic model we can substantionally reduce the size of the problem. In stochastic LQ RE models shocks are part of the vector of predetermined state variables and the number of predetermined states may be substantial.[38] However, because exogenous state variables do not create multiplicity in LQ RE models, we prove in the Online Appendix the following Proposition.

Proposition 7 *A stochastic LQ RE model can be solved in two steps. First, we solve the deterministic model where all variables are endogenous. Second, we restore feedback coefficients on stochastic components for matrices N and F and compute the necessary components of matrix S in the unique way.*

Hence, the search for multiple solutions in models of discretionary policy can only involve their endogenous deterministic components; this keeps the size of the problem relatively low.

6 Conclusion

We have described discretionary equilibria in the general class of LQ RE models. We illustrated the potential for existence of multiple rational expectations equilibria. Because decisions of the policymaker depend on expectations of the private sector that are based on the future policy, dynamic complementarities between decisions of agents can create multiplicity, and different beliefs about the future policy correspond to different discretionary equilibria. The policy maker cannot control expectations of the private sector about the future policy and current policy decisions have to accommodate expectations set by the past-period private sector. A sunspot, that changes private sector beliefs about the future policy, can decide which equilibrium will realize.

[37] There can be different ways to rank point-in-time equilibria and identify the correspondence. This might depend on the adopted iterative algorithm. In our examples we used simple iterations, but some models might require more elaborate algorithms to find 'intersecion' of particular point-in-time equilibrium with optimal policy response. Proposion 4 ensures all solutions can be found 'along' the corresponding point-in-time equilibrium.

[38] See e.g. Anderson et al. (1996).

These interactions and sunspot-driven changes between equilibria can generate rich dynamics that is often observed in the data on aggregate fluctuations.

We generalised several results for the general class of LQ RE models. We described all types of equilibria that can arise in these models. Our analysis can be used to develop numerical methods that find most of discretionary equilibria in complex models.

A Parameters of Models in Examples

$$\begin{aligned}
\lambda_c &= \kappa\left(\frac{(\phi+\alpha)\zeta}{1-\alpha}+\frac{1}{\sigma}\right), \lambda_o=\kappa\frac{(\phi+\alpha)(1-\zeta)}{(1-\alpha)\delta}, \lambda_k=\kappa\left(\frac{(\phi+\alpha)(1-\zeta)(1-\delta)}{\delta(1-\alpha)}+\frac{\alpha(1+\phi)}{(1-\alpha)}\right)\\
\bar{\nu} &= \left(\varepsilon_\psi(1+\beta)+\frac{(1-\beta(1-\delta))}{1-\alpha}\left(\frac{(\phi+1)(1-\zeta)(1-\delta)}{\delta}+\alpha\phi+1\right)\right)^{-1}, \nu_k=\varepsilon_\psi\bar{\nu},\\
\nu_o &= \left(\varepsilon_\psi\beta+\frac{(1-\beta(1-\delta))(\phi+1)(1-\zeta)}{(1-\alpha)\delta}\right)\bar{\nu}, \nu_c=(1-\beta(1-\delta))\left(\frac{(\phi+1)\zeta}{1-\alpha}+\frac{1}{\sigma}\right)\bar{\nu},\\
\nu_r &= \left(1-(1-\beta(1-\delta))\left(\frac{(\phi+1)\zeta\sigma}{(1-\alpha)}+1\right)\right)\bar{\nu}, \xi=1-(\nu_r+\sigma\nu_c)\pi_k-\nu_c c_k-\nu_o k_k,\\
k_S &= \frac{\nu_k}{\xi}, k_P=-\frac{(\nu_r+\sigma\nu_c)}{\xi}, c_S=\frac{\nu_k c_k+\nu_k\sigma\pi_k}{\xi}, c_P=\frac{-\sigma-\nu_r c_k+\sigma\nu_o k_k}{\xi},\\
\pi_S &= \frac{1}{\xi}\left(-\lambda_k+\nu_k\lambda_o+((\beta+\sigma\lambda_c)\nu_k+(\nu_r+\sigma\nu_c)\lambda_k)\pi_k+(\lambda_c\nu_k+\nu_c\lambda_k)c_k+\lambda_k\nu_o k_k\right),\\
\pi_P &= -\frac{1}{\xi}\left(\sigma\lambda_c+\lambda_o\nu_r+\sigma\nu_c\lambda_o+\beta(\nu_r+\sigma\nu_c)\pi_k+\lambda_c\nu_r c_k-\sigma\lambda_c\nu_o k_k\right)\\
\mu &= \left(\frac{\pi_P^2}{k_P^2}+\omega\left(\eta\frac{c_P}{k_P}+\gamma\right)^2-\beta\left(\left(\pi_P\frac{k_S}{k_P}-\pi_S\right)^2+\omega\left(\eta\left(c_S-c_P\frac{k_S}{k_P}\right)-\gamma(1-\delta)\right)^2\right)\right)
\end{aligned}$$

B Proof of Proposition (4)

First, all solutions of equation (56) also solve the Non-Symmetric Continuous Algebraic Riccati Equation:

$$NC_{12}N+C_{22}N-NC_{11}-C_{21}=0. \tag{61}$$

Indeed, we multiply both sides of (56) by $(A_{22}+NA_{12})$ and, at most, we also acquire all solutions of $A_{22}+NA_{12}=0$. Matrix C was defined in Proposition (4).

By assumption, matrix C can be diagonalized as $C=V^{-1}\Lambda V$.[39] Matrix V is the matrix of left eigenvectors which correspond to eigenvalues Λ. Arrange the eigenvalues so that Λ_u is a diagonal matrix of size n_2 and Λ_s a diagonal matrix of size $n_1=n-n_2$. Rearrange similarly V

[39] In what follows we always assume that matrix C is *simple*, i.e. all its eigenvalues are of geometric multiplicity one, and the column rank of M is equal to n. This case is of practical interest; but Freiling (2002) discusses implications of higher geometric multiplicity.

and partition it to give

$$\Lambda = \begin{bmatrix} \Lambda_s & 0 \\ 0 & \Lambda_u \end{bmatrix} \text{ and } V = \begin{bmatrix} V_{11} & V_{12} \\ V_{21} & V_{22} \end{bmatrix}.$$

Now, construct $N = V_{22}^{-1}V_{21}$. Matrix V_{22} is invertible by assumption, but this assumption is unlikely to be restrictive.

It is known from the control literature (see e.g. Medanic (1982) Th. 1, Freiling (2002) Th. 3.3) that *any* solution of (61) can be represented in the form of $V_{22}^{-1}V_{21}$ for some adequate Jordan basis of C. If all eigenvalues of C are simple then there are at most $\binom{n}{n_2}$ of different solutions N. Note that we did not make any assumptions about matrices Λ_s and Λ_u apart from assuming that they are of particular size.

We fix some value $\rho > 0$ and rearrange rows of V such that Λ_u collects all eigenvalues that are greater than ρ in modulus. Suppose there are $m \leq n_2$ of them, so Λ_u might also have $n_2 - m$ eigenvalues that are not greater than ρ in modulus. For any solution N in the form of $V_{22}^{-1}V_{21}$ matrix $M = C_{11} - C_{12}V_{22}^{-1}V_{21}$. Freiling (2002), Blake (2004) prove that the eigenvalues of $C_{11} - C_{12}V_{22}^{-1}V_{21}$ are Λ_s and the eigenvalues of $C_{11} + C_{22}V_{22}^{-1}V_{21}$ are Λ_u.

It follows that if $m = n_2$ then Λ_u is uniquely determined and $V_{22}^{-1}V_{21}$ is a unique solution (as in Blanchard and Kahn (1980)). If $m < n_2$ we can construct Λ_u in at most $\binom{n-m}{n_2-m}$ ways, collecting different combinations of smaller than ρ in modulus eigenvalues into Λ_u, and correspondingly rearranging rows of matrix V.

C Proof of Proposition (5)

First, equation (58) is equivalent to the following Discrete Algebraic Riccati Equation:

$$S = Q^* + \beta A^{*\prime}SA^* - \left(P^{*\prime} + \beta B^{*\prime}SA^*\right)' \left(R^* + \beta B^{*\prime}SB^*\right)^{-1} \left(P^{*\prime} + \beta B^{*\prime}SA^*\right), \tag{62}$$

provided we can use (57). Indeed, start with (62) and add and subtract additional terms:

$$\begin{aligned} S &= Q^* + \beta A^{*\prime}SA^* - (\beta B^{*\prime}SA^* + P^{*\prime})'F = Q^* + F'R^*F - F'P^{*\prime} - P^*F \\ &\quad + \beta\left(A^* - B^*F\right)' S \left(A^* - B^*F\right) + F'P^{*\prime} - F'R^*F + \beta F'B^{*\prime}S\left(A^* - B^*F\right) \\ &= Q^* + F'R^*F - F'P^{*\prime} - P^*F + \beta\left(A^* - B^*F\right)' S\left(A^* - B^*F\right) \\ &\quad + F'\left[\beta B^{*\prime}S\left(A^* - B^*F\right) + P^{*\prime} - R^*F\right]. \end{aligned}$$

The term in square brackets is zero because of (57) and we obtain (58).

Second, properties of solutions to equation (62) are known from the control literature. If the pair matrices $(\beta^{\frac{1}{2}}A^*, \beta^{\frac{1}{2}}B^*)$ in (62) is controllable (i.e. if the $k \times n_1$ controllability matrix $[\beta^{\frac{1}{2}}B^*, \beta^{\frac{2}{2}}A^*B^*, \beta^{\frac{3}{2}}A^{*2}B^*, ..., \beta^{\frac{n_1}{2}}A^{*n_1-1}B^*]$ has rank n_1 or full row rank) then the solution pair $\{S, F\}$ to (62) and (57) exists and unique, and all eigenvalues of matrix $\beta^{\frac{1}{2}}M = \beta^{\frac{1}{2}}\left(A^* - B^*F\right)$ are strictly inside the unit circle, see e.g. (Kwakernaak and Sivan (1972, Ch. 6)). The controllability of $(\beta^{\frac{1}{2}}A^*, \beta^{\frac{1}{2}}B^*)$ is equivalent to the controllability of (A^*, B^*).

Third, it is also a textbook result that matrix S is symmetric and positive semi-definite if $\hat{Q}^* = \begin{bmatrix} Q^* & P^* \\ P^{*\prime} & R^* \end{bmatrix}$ is symmetric and positive semi-definite. One can easily demonstrate that

$\hat{Q}^* = (\mathcal{C}\Psi)' \mathcal{Q} (\mathcal{C}\Psi)$, where $\Psi = \begin{bmatrix} I & 0 \\ -J & -K \\ 0 & I \end{bmatrix}$ and $\mathcal{C}$ is defined in Section 3.1. Because $\mathcal{Q}$ is symmetric and positive semi-definite by assumption then $\hat{Q}^*$ has the same properties. Hence S is symmetric and positive semi-definite.

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