Sources of Entropy in Dynamic Representative Agent Models

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Understanding dynamic models

- Are dynamic features important for the disaster story?
- More generally, how does one discern critical features of modern dynamic models?
  - The size of equity premium is no longer an overidentifying restriction

Market-adjusted excess returns

Asset Class	Value	Momentum
US stocks	4.3%	6.1%
UK stocks	2.7%	10.8%
Euro stocks	4.2%	10.9%
Jpn stocks	11.3%	4.2%
FX	4.9%	2.7%
Bonds	0.3%	0.3%
Commodities	6.4%	8.8%

- Annualized alphas relative to the MSCI world equity index in excess of the US Treasury Bill rate
- Source: Asness, Moskowitz, and Pedersen (2009)

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Understanding dynamic models

- Are dynamic features important for the disaster story?
- More generally, how does one discern critical features of modern dynamic models?
  - The size of equity premium is no longer an overidentifying restriction
  - The models are built up from different state variables
  - Which pieces are most important quantitatively?
- We start by thinking about how risk is priced in these models
  - What is the source of the evident high entropy in the data?
  - We use ACE to characterize this

AJ bound, non-i.i.d. case

- AJ bound

$$L(m) \quad \geq \quad E\left(\log r^j - \log r^1\right) + \underbrace{L(q^1)}_{\text{non-i.i.d. piece}}$$

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$$L(m) \quad \geq \quad E\left(\log r^j - \log r^1\right) + \underbrace{L(q^1)}_{\text{non-i.i.d. piece}}$$

- Conditional entropy:

$$L_t(m_{t+1}) \quad = \quad \log E_t m_{t+1} - E_t \log m_{t+1}$$

- Average conditional entropy (ACE)

$$\begin{aligned} L(m) &= EL_t(m_{t+1}) + L(E_t(m_{t+1})) = EL_t(m_{t+1}) + L(q^1) \\ EL_t(m_{t+1}) &\geq E\left(\log r^j - \log r^1\right) \end{aligned}$$

Advantages of average conditional entropy (ACE)

- Transparent lower bound: expected excess return (in logs)
- Alternatively, ACE measures the highest risk premium in the economy
- Conditional entropy is easy to compute; to compute ACE evaluate conditional entropy at steady-state values
- ACE is comparable across different models with different state variables, preferences, etc.

Key models

- **External habit**
- **Recursive preferences**
- Heterogeneous preferences
- ....

A change in notation

- α is replaced by $1 - \alpha$
- Example: CRRA preferences; RA$= 5$
- Old $\alpha = 5$
- New $\alpha = -4$

External habit

- Equations (Abel/Campbell-Cochrane/Chan-Kogan/Heaton)

$$
\begin{aligned}
U_t &= \sum_{j=0}^{\infty} \beta^j u(c_{t+j}, x_{t+j}), \\
u(c_t, x_t) &= \left(f(c_t, x_t)^{\alpha} - 1\right)/\alpha.
\end{aligned}
$$

- Habit is a function of past consumption: $x_t = h(c^{t-1})$, e.g., Abel: $x_t = c_{t-1}$.
- Dependence on habit
  - Abel: $f(c_t, x_t) = c_t / x_t$
  - Campbell-Cochrane: $f(c_t, x_t) = c_t - x_t$
- Pricing kernel:

$$
m_{t+1} = \beta \frac{u_c(c_{t+1}, x_{t+1})}{u_c(c_t, x_t)} = \beta \left(\frac{f(c_{t+1}, x_{t+1})}{f(c_t, x_t)} \right)^{\alpha - 1} \left(\frac{f_c(c_{t+1}, x_{t+1})}{f_c(c_t, x_t)} \right)
$$

Example 1:
Abel (1990) + Chan and Kogan (2002)

- Preferences: $f(c_t, x_t) = c_t / x_t$
- Chan and Kogan have extended the habit formulation:

$$\log x_{t+1} \quad = \quad (1-\phi)\sum_{i=0}^{\infty}\phi^i \log c_{t-i} = \phi \log x_t + (1-\phi)\log c_t$$

- Relative (log) consumption

$$\log s_t \equiv \log(c_t / x_t) = \phi \log s_{t-1} + \log g_t$$

- Pricing kernel:

$$\begin{aligned}\log m_{t+1} &= \log\beta + (\alpha-1)\log g_{t+1} - \alpha \log(x_{t+1}/x_t) \\ &= \log\beta + (\alpha-1)\log g_{t+1} - \alpha(1-\phi)\log s_t\end{aligned}$$

ACE: Abel-Chan-Kogan

- Pricing kernel:

$$\log m_{t+1} = \log \beta + (\alpha - 1)\log g_{t+1} - \alpha(1-\phi)\log s_t$$

- Conditional entropy: $L_t(m_{t+1}) = \log E_t e^{\log m_{t+1}} - E_t \log m_{t+1}$

$$\begin{aligned}
\log E_t e^{\log m_{t+1}} &= \log \beta + k(\alpha - 1; \log g) - \alpha(1-\phi)\log s_t (= -\log r^1) \\
E_t \log m_{t+1} &= \log \beta + (\alpha - 1)\kappa_1(\log g) - \alpha(1-\phi)\log s_t \\
L_t(m_{t+1}) &= k(\alpha - 1; \log g) - (\alpha - 1)\kappa_1(\log g)
\end{aligned}$$

- ACE: $EL_t(m_{t+1}) = k(\alpha - 1; \log g) - (\alpha - 1)\kappa_1(\log g)$

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\end{aligned}$$

- ACE: $EL_t(m_{t+1}) = k(\alpha - 1; \log g) - (\alpha - 1)\kappa_1(\log g)$
- It is exactly the same as in the CRRA case

Example 2: Campbell and Cochrane (1999)

- Preferences: $f(c_t, x_t) = c_t - x_t$
- Campbell and Cochrane specify (log) surplus consumption ratio directly:

$$\begin{aligned} \log s_t &= \log[(c_t - x_t)/c_t] \\ \log s_t &= \phi(\log s_{t-1} - \log \bar{s}) + \lambda(\log s_{t-1})(\log g_t - \kappa_1(\log g)). \end{aligned}$$

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- Compare to relative (log) consumption in Chan and Kogan

$$\log s_t \equiv \log(c_t/x_t) = \phi \log s_{t-1} + \log g_t$$

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- Pricing kernel:

$$\begin{aligned} \log m_{t+1} &= \log \beta + (\alpha - 1)\log g_{t+1} + (\alpha - 1)\log(s_{t+1}/s_t) \\ &= \log \beta - (\alpha - 1)\lambda(\log s_t)\kappa_1(\log g) \\ &+ (\alpha - 1)(1 + \lambda(\log s_t))\log g_{t+1} \\ &+ (\alpha - 1)(\phi - 1)(\log s_t - \log \bar{s}) \end{aligned}$$

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&+ (\alpha-1)(1+\lambda(\log s_t))\log g_{t+1} \\
&+ (\alpha-1)(\phi-1)(\log s_t - \log\bar{s})
\end{aligned}$$

- Conditional entropy:

$$\begin{aligned}
L_t(m_{t+1}) &= k((\alpha-1)(1+\lambda(\log s_t));\log g) \\
&- (\alpha-1)(1+\lambda(\log s_t))\kappa_1(\log g)
\end{aligned}$$

Additional assumptions

- To compute ACE, we have to specify λ and $\log g$
- Conditional volatility of the consumption surplus ratio

$$\lambda(\log s_t) = \frac{1}{\sigma}\sqrt{\frac{1-\phi-b/(1-\alpha)}{1-\alpha}}\sqrt{1-2(\log s_t - \log \bar{s})} - 1$$

  - In discrete time, there is an upper bound on $\log s_t$ to ensure positivity of λ
  - In continuous time, this bound never binds so we will ignore it
  - In Campbell and Cochrane, $b = 0$ to ensure a constant $\log r$[1]
- Consumption growth is i.i.d.
  - Case 1. $\log g_{t+1} = w_{t+1}$, $w_{t+1} \sim \mathcal{N}(\mu, \sigma^2)$
  - Case 2. $\log g_{t+1} = w_{t+1} - z_{t+1}$, $z_{t+1}|j \sim Gamma(j, \theta^{-1})$, $\bar{j} = \omega$

ACE: Campbell and Cochrane, Case 1

- Conditional entropy:

$$L_t(m_{t+1}) = ((\alpha - 1)(\phi - 1) - b)/2 + b(\log s_t - \log \bar{s})$$

- ACE: $EL_t(m_{t+1}) = ((\alpha - 1)(\phi - 1) - b)/2$

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- All authors use $\alpha = -1$
- ACE for different calibrations (quarterly)

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    $EL_t(m_{t+1}) = 0.0300$ (0.120 annual)

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  - Wachter (2006): $\phi = 0.97$, $b = 0.011$; $EL_t(m_{t+1}) = 0.0245$ (0.098 annual)

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  - Verdelhan (2009): $\phi = 0.99$, $b = -0.011$; $EL_t(m_{t+1}) = 0.0155$ (0.062 annual)

ACE: Campbell and Cochrane, Case 2

- Conditional entropy:

$$
\begin{aligned}
L_t(m_{t+1}) &= (\alpha-1)(1+\lambda(\log s_t))\omega\theta \\
&+ ((1+(\alpha-1)(1+\lambda(\log s_t))\theta)^{-1}-1)\omega \\
&+ ((\alpha-1)(\phi-1)-b)/2+b(\log s_t-\log\bar{s})
\end{aligned}
$$

- ACE: use log-linearization around $\log\bar{s}$

$$
\begin{aligned}
EL_t(m_{t+1}) &= \omega d^2/(1+d)+((\alpha-1)(\phi-1)-b)/2 \\
d &= \frac{\theta}{\sigma}\sqrt{(\alpha-1)(\phi-1)-b}
\end{aligned}
$$

ACE: Campbell and Cochrane, Case 2

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- Calibration as above + vol of $\log g$ + jump parameters:
  - $\sigma^2=(0.035)^2/4-\omega\theta^2$
  - BNSU: $\omega=0.01/4$, $\theta=0.15$
  - BCM: $\omega=1.3864/4$, $\theta=0.0229$

ACE: Campbell and Cochrane, Case 2

Calibration	ACE	ACE (case 1)	ACE jumps
CC + BNSU	0.0341	0.0300	0.0041
W + BNSU	0.0281	0.0245	0.0036
V + BNSU	0.0181	0.0155	0.0026
CC + BCM	0.0883	0.0300	0.0583
W + BCM	0.0737	0.0245	0.0492
V + BCM	0.0487	0.0155	0.0332

Time dependence via external habit

- No time-dependence in consumption growth

- Nevertheless: habit with varying volatility may have a substantial impact on the entropy of the pricing kernel

- Could be relevant for option prices (Du, 2008)

Recursive preferences: traditional version

- Equations (Kreps-Porteus/Epstein-Zin/Weil)

$$
\begin{aligned}
U_t &= \left[(1-\beta)c_t^{\rho} + \beta \mu_t(U_{t+1})^{\rho}\right]^{1/\rho} \\
\mu_t(U_{t+1}) &= \left(E_t U_{t+1}^{\alpha}\right)^{1/\alpha} \\
IES &= 1/(1-\rho) \\
CRRA &= 1-\alpha \\
\alpha &= \rho \quad \Rightarrow \quad \text{additive preferences}
\end{aligned}
$$

Recursive preferences: pricing kernel

- Scale problem by c_t ($u_t = U_t/c_t$, $g_{t+1} = c_{t+1}/c_t$)

$$u_t \quad = \quad [(1-\beta) + \beta\mu_t(g_{t+1}u_{t+1})^{\rho}]^{1/\rho}$$

- Pricing kernel (mrs)

$$\begin{aligned} m_{t+1} &= \beta \left(\frac{c_{t+1}}{c_t}\right)^{\rho-1} \left(\frac{U_{t+1}}{\mu_t(U_{t+1})}\right)^{\alpha-\rho} \\ &= \beta\, g_{t+1}^{\rho-1} \left(\frac{g_{t+1}u_{t+1}}{\mu_t(g_{t+1}u_{t+1})}\right)^{\alpha-\rho} \end{aligned}$$

Loglinear approximation

- Loglinear approximation

$$\begin{aligned}\log u_t &= \rho^{-1}\log\left[(1-\beta)+\beta\mu_t(g_{t+1}u_{t+1})^{\rho}\right]\\ &= \rho^{-1}\log\left[(1-\beta)+\beta e^{\rho\log\mu_t(g_{t+1}u_{t+1})}\right]\\ &\approx b_0+b_1\log\mu_t(g_{t+1}u_{t+1}).\end{aligned}$$

- Exact if $\rho = 0$: $b_0 = 0$, $b_1 = \beta$

- Solve by guess and verify

Example 1: Bansal-Yaron

- Consumption growth

$$\begin{aligned} \log g_t &= g + \gamma(L) v_{t-1}^{1/2} w_{1t} \\ v_t &= v + \nu(L) w_{2t} \\ (w_{1t}, w_{2t}) &\sim \text{NID}(0, I) \end{aligned}$$

- Guess value function

$$\log u_t = u + \omega_g(L) v_{t-1}^{1/2} w_{1t} + \omega_v(L) w_{2t}$$

- Solution includes

$$\begin{aligned} \omega_{g0} + \gamma_0 &= \gamma(b_1) \equiv \sum_{i=0}^{\infty} b_1^i \gamma_i \\ \omega_{v0} &= b_1(\alpha/2)\gamma(b_1)^2 \nu(b_1) \end{aligned}$$

ACE: Bansal-Yaron

- Pricing kernel

$$
\begin{aligned}
\log m_{t+1} &= \log\beta + (\rho-1)g - (\alpha-\rho)(\alpha/2)\omega_{v0}^2 \\
&\quad + (\rho-1)[\gamma(L)/L]_+ v_{t-1}^{1/2} w_{1t} - (\alpha-\rho)(\alpha/2)\gamma(b_1)^2 v_t \\
&\quad + [(\rho-1)\gamma_0 + (\alpha-\rho)\gamma(b_1)] v_t^{1/2} w_{1t+1} \\
&\quad + (\alpha-\rho)\omega_{v0}^2 w_{2t+1}
\end{aligned}
$$

- Conditional entropy

$$
L_t(m_{t+1}) = [(\rho-1)\gamma_0 + (\alpha-\rho)\gamma(b_1)]^2 v_t/2 + (\alpha-\rho)^2\omega_{v0}^2/2
$$

- ACE (Bansal, Kiku, Yaron, 2007; monthly)

$$
\begin{aligned}
0.0218 &= 0.0065 + 0.0153 \\
0.0026 &= 0.0026 + 0.0000 \text{ if } \rho = \alpha
\end{aligned}
$$

Example 2: Wachter

- Consumption growth

$$
\begin{aligned}
\log g_t &= g + \sigma w_{1t} + z_t \\
\lambda_t &= (1-\varphi)\lambda + \varphi\lambda_{t-1} + \sigma_\lambda w_{2t} \\
(w_{1t}, w_{2t}) &\sim \text{NID}(0, I) \\
z_t | j &\sim \mathcal{N}(j\theta, j\delta^2) \\
j &\geq 0 \text{ has jump intensity } \lambda_{t-1}
\end{aligned}
$$

- Guess value function

$$
\log u_t = u + \omega_\lambda \lambda_t
$$

- Solution includes

$$
\omega_\lambda = (1 - b_1\varphi)^{-1} b_1 \left[e^{\alpha\theta + (\alpha\delta)^2/2} - 1 \right] / \alpha
$$

ACE: Wachter

- Pricing kernel

$$
\begin{aligned}
\log m_{t+1} &= \log\beta + (\rho-1)x - (\alpha-\rho)(\alpha/2)[\sigma^2 + (\omega_\lambda\sigma_\lambda)^2] \\
&\quad - \lambda_t(e^{\alpha\theta+(\alpha\delta)^2/2} - 1)/\alpha \\
&\quad + (\alpha-1)(\sigma w_{1t+1} + z_{t+1}) + (\alpha-\rho)(\omega_\lambda\sigma_\lambda)w_{2t+1}
\end{aligned}
$$

- Conditional entropy (monthly)

$$
\begin{aligned}
L_t(m_{t+1}) &= (\alpha-1)^2\sigma^2/2 + (\alpha-\rho)^2(\omega_\lambda\sigma_\lambda)^2/2 \\
&\quad + \lambda_t\left\{[e^{(\alpha-1)\theta+(\alpha-1)^2\delta^2/2} - 1] - (\alpha-1)\theta\right\}
\end{aligned}
$$

- ACE (monthly)

$$
\begin{aligned}
0.0100 &= 0.0001 + 0.0087 + 0.0012 \\
0.0013 &= 0.0001 + 0.0000 + 0.0012 \text{ if } \rho = \alpha
\end{aligned}
$$

Time dependence via recursive preferences

- Little time-dependence in pricing kernel

- Nevertheless: interaction of (modest) dynamics in consumption growth and recursive preferences can have a substantial impact on the entropy of the pricing kernel

- Not clear it's relevant to option prices, but it's a route to magnify the impact of disasters on excess returns