Competitive on-the-job search

Pietro Garibaldi
University of Torino and Collegio Carlo Alberto

Espen R. Moen
Norwegian School of Management

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Abstract

The paper proposes a model of on- and off-the-job search that combines convex hiring costs and directed search. Firms permanently differ in productivity levels, their production function features constant or decreasing returns to scale, and search costs are convex in search intensity. Wages are determined in a competitive manner, as firms advertise wage contracts (expected discounted incomes) so as to balance wage costs and search costs (queue length). An important assumption is that a firm is able to sort out its coordination problems with their employees in such a way that the on-the-job search behavior of workers maximizes the match surplus. Our model has several interesting features. First, it is close in spirit to the competitive model, with a tractable and unique equilibrium, and is therefore useful for empirical testing. Second, the resulting equilibrium gives rise to an efficient allocation of resources. Third, the equilibrium is characterized by a job ladder: unemployed workers search for low-productivity, low-wage firms. Workers in low-wage firms search for firms slightly higher on the productivity/ ladder, and so forth up to the workers in the second most productive firms who only apply to the most productive firms. Finally, the model rationalizes empirical regularities of on-the-job search and labor turnover. First, job-to job mobility falls with average firm tenure and firm size. Second, wages increase with firm size, and wage growth is larger in fast-growing firms.

1 Introduction

In the real economy, firm-and industry dynamics play an important role. Firms are born, expand and contract. Resources are allocated from less productive to more productive firms, and thereby improve the allocation of resources. There is substantial evidence that reallocation of resources on firms is important for economic growth, and Baily, Hulten and Campbell (1992) argues the about half of overall productivity growth in the U.S. manufacturing in the 80ies can be attributed to this. Existing empirical evidence also shows that industry dynamics is associated with large worker flows, not only in and out of unemployment, but even more importantly as direct job to job movements (Haltiwanger, 1999; Foster et al., 2007; Bartelsman et al; 2005). Lentz and Mortensen (2005, 2006) decompose the effect of firm selection on the growth rate, and then estimate that it accounts for 58 percent of the growth rate.

In this paper, our objective is to provide a simple competitively flavour model consistent with a variety of facts described in longitudinal data sets on firms and workers flows. In particular, the model should be able to explain that 1) firms with different productivities coexist in the labor market, 2) On-the-job search is prevalent and worker flows between firms are large, and 3) more productive firms are larger and pay higher wages than less productive firms.

Our model contains three key elements. First, it applies the *competitive search equilibrium* concept, initially proposed by Moen (1997). Thus, firms publicly advertise wages, and workers observe the advertised wages prior to making their decision where to direct their search. We show that the labor market is endogenously separated into submarkets so that in each submarket, all agents at the same side of the market are identical. Second, we assume that firms have access to a search technology with *convex costs of maintaining vacancies* (Bertola and Caballero, 1992; Bertola and Garibaldi, 2001). In the traditional search model (Mortensen and Pissarides, 1994) the costs of maintaining vacancies are linear. Together with constant returns to scale in production, this implies that the size of the firms typically is undefined. Third, we follow Moen and Rosen (2004) and allow for *efficient contracting*. The contracts are thus designed so as to resolve any agency problems between employers and employees so that their joint income is maximized. In particular, this implies that the workers' on-the-job search behavior maximizes the joint surplus of the worker and the firm. This assumption simplifies the model enormously. Without this assumption, a worker's current wage will influence his search behavior. As shown by Shimer (2006), this opens up for multiple equilibria and generally makes on-the-job search models intractable.

Our analysis thus delivers a tractable model of on-the-job search, closely related to the competitive model, in which on-the-job search and wage differentials for identical workers is an optimal response to search frictions and heterogenous firms. As a tool for empirical analysis our model is interesting because it includes, in a simple way, the effects of search frictions for industry dynamics. Finally, as our model gives rise to a (constrained) efficient allocation of resources, and hence is well suited as a benchmark for welfare analysis.

The equilibrium of the model is characterized by a sluggish employment growth toward a steady-state employment level. Low productivity firms pay low wages, face high turnover rates, and grow slowly towards a steady state with low employment. More efficient firms pay higher wages, post more vacancies, and grow more quickly to a steady state with a higher employment level. The equilibrium features a job ladder: unemployed workers disproportionately search for firms with the lowest productivity. Workers employed in these firms, in turn, search only for firms with higher productivity. Hence our model easily explain a set of stylized facts about industry dynamics and worker flows: 1) productivity differences between firms are large and persistent, 2) workers move from low-wage to high-wage occupations, 3) more productive firms are larger and pay higher wages than less productive firms, 4) job-to-job mobility falls with average firm size and worker tenure, 5) wages increase with firm size, and 6) wages are higher in fast-growing firms .

We show that convex costs of maintaining vacancies is a neccessary assumption for our model to deliver the stylized facts. We do this by letting the maintenance costs converge to the linear case. As the costs becomes more linear, the productive firms grow on the expense of the non-productive firms. In the limit, only the most productive firms post vacancies and hire workers. At some point (before we reach the limit) only the most productive firms operate in the market, and there is no on-the-job search.

We also augment the model by including convex hiring costs, as in Sargent (1987). Hiring costs are a function of the flow of new hires, and are thus internal, in the sense that they are independent of labor market conditions such as the tightness of the labor market. These costs come in addition to linear vacancy costs. We show that in equilibrium, firms of different productivities may coexist in the market. However, the optimal wage is independent of the productivity of the firm in question, and there is no on-the-job search.

Related literature

Several recent papers analyze models of industry dynamics (Hopenayn, 1992; Hopenayn and Rogerson, 1993, Melitz 2003, Klette and Kortum 2004). However, these papers typically do not take into account that the factor markets, and in particular the labor market, may contain frictions. An exception here is Lents and Mortensen (2007), who do include a frictional labour market in a Klette-Kortum model of innovation-driven industry dynamics.

Pissarides (1994) was the first paper that studied on-the-job search in a Diamond-Mortensen-Pissarides type of matching model. As show by Shimer (2006), a problem with this model is that the bargaining is not convex, and this may give rise to a continuum of wages. Current papers by Bagger and Lentz, Lise et. al. (2008) still uses this model, and get around the problem of a non-convex bargaining set by introducing competitive bidding for the worker after successful on-the-job search.

Maybe the most used model of on-the-job search in empirical research is Burdett and Mortensen (1998) with its many follow-ups, for instance Postel-Vinay and Robin (2002). Moen and Rosen (2004) are the first to analyze competitive on-the-job search and the first to assume efficient on-the-job search. Shi (2008) studies competitive on-the-job search in a model with wage tenure contracts. Menzio and Shi (2008), in a paper written simultaneously and independently of the current paper, study the effects of business cycle fluctuations in a model with competitive on-the-job search. Kiyotaki and Lagos (2006) study optimal assignment of workers to jobs in a model where matches differ in quality, but without entry of firms. Delacroix and Shi (2006) analyzes on-the-job search in an urn-ball type of model of the labor market, and also obtain a job ladder in a similar way as we do. However, in their model all agents on both side of the market are homogenous, and firms at most hire one worker, hence their model is ill suited to analyze industry dynamics. Currently, several authors simultaneously study models of on-the-job search, Menzio and Shi (2008), Lentz and Mortensen (2007) and Moscarini and Postel-Vinay (2009).

The paper proceeds as flows. Section 2 briefly describes the empirical regularities we are interested and reviews the relevant literature. Section 3 introduces the structure of the model while sections 4 and 5 derive the main formulation of the model for different type of firms. Section 6 introduces the general equilibrium and spells out some key results. Section 7 presents the baseline simulation.

2 A Brief Look at some empirical regularities

We briefly review some of the key empirical regularities linked to industry dynamics and worker flows. We are certainly not meant to be exhaustive in this review, and the selection of facts outlined in closely associated to the theoretical approach we will propose in the rest of the paper.

1. *In any industry there is a large scale reallocation of input and output across producers.* The work of Davis and Haltiwanger (1999) summarize much of the literature on gross job flows that was carried during the last decade; they note that in the United States more than 1 in 10 jobs is created in a given year and more than 1 in 10 jobs is destroyed. Much of this reallocation reflects reallocation within narrowly defined sectors. Note that not only labor is being reallocated but also capital and output (Haltiwanger, 2000).A large fraction of the input and output gross creation is associated with entry of firms and a large fraction output and input gross destruction is associated to firm.

2. *Productivity differences between firms are large and persistent.* Bartelsman and Dome (2000) summarize most of the evidence based on longitudinal micro data on firm level productivity differential. They clearly argue that the most significant finding of this vast literature is the heterogeneity across establishments and firms in productivity in nearly all industries examined. Bernard et al. (2003) report the distribution across plants of value added per worker relative to the overall mean, and show that a substantial number of plants have productivity either less than a fourth or more than four times the average. These differences are also very persistent over time. Danish data analyzed by Mortensen (2007) provide similar patterns.

3. *More productive firms are larger and pay higher wages.* Wage differentials across observationally equivalent workers are both sizable and persistent . The employer size-wage effect is perhaps the strongest such stylized fact: larger firms or plants pay higher wages. The literature includes especially influential work by Krueger and Summers (1988) and Brown and Medoff (1989), and has been surveyed by Oi and Idson (1999).

4. *Job to job mobility falls with worker tenure.* There is a well established relationship between job duration and tenure. Farber (1999) carefully reviews this literature and reports that a monotonically declining survival rates is one of the mot robust stylized facts in the labour market. Such monotonicity holds regardless of the reasons beyond job termination and is particularly significant for voluntary job to job movements.

5. *Job to job movements are associated to wage gains* . Bartel and Borjas (1978) early work showed that young men who quit experience significant wage gains compared both to job stayers and to their own wage growth prior to the job change. More recently Light (2005) summarize empirical evidence based on the National Longitudinal Survey of Youth. He shows that the typical worker holds about five jobs in the first 8 years of the career, but that workers vary considerably in their mobility rates. He also reports evidence that workers who change jobs voluntarily through a job to job transition receive significant contemporaneous wage boosts that, on average, are at least as large as the wage gains received by job stayers.

6. *and wages are higher in fast growing firms.* Belzil (2000) uses Danish data and shows that after controlling for individual and business cycle effects, job creation at the firm level is found to increase male wages.

7. *Wage differentials are associated to productivity differentials* . The evidence on this link is more scarce, since data-set able to observe output and wages for a variety of workers are not readily available. Iranzo et al. (2007) using Italian data show that the level of labour productivity is clearly associated to larger wages for both production and non production workers.

3 Model and equilibrium

The structure of our model is as follows

- Labor is the only factor of production. The labor market is populated by a measure 1 of identical workers. Individuals are neutral, infinitely lived, and discount the future at rate r.

- The technology requires an entry cost equal to K. Conditional upon entry, the firm learns its productivity, which can take any value between y_1, and y_n with y_2, $y_1 < y_2 < ... < y_n$. The probability that each productivity is selected is α_i with $\sum_i \alpha_i = 1$. The productivity of a firm is a fixed throughout its life. Unemployed workers have access to an income flow y_0, which may denote unemployment benefits, the value of leisure, or the income when self-employed. We assume that $y_0 < y_1$.

- Firms post vacancies and wages to maximize expected profits. Vacancy costs $c(v)$ are convex in the number of vacancies posted. Unless otherwise stated we assume that

$c(0) = c'(0) = 0$. In the numerical analysis we assume that the vacancy costs are zero up to a point ν, at which they become infinite.[1]

- Firms die at rate δ. In addition, workers separates from firms at an exogenous rate s.
- Wage contracts are complete, and resolve any agency problems between employers and employees. In particular, the wage contract ensures efficient on-the-job search.

The last point deserves a comment. In this context, efficient on-the-job search implies that the worker, when searching on the job, searches for the jobs that maximizes the joint surplus of the worker and the firm. The are various wage contracts that implement this behavior. For example, the worker pays the firm its entire pdv value up front and then gets a wage equal to y_i. In other words, the worker buys the job from the firm and acts thereafter a residual claimant. As an alternative contract, the worker gets a constant wage and pays a quit fee equal to the continuation value of the firm if a new job is accepted (see Moen and Rosen (2004) for more examples). In principle the worker and the firm may also write directly into the wage contract the search behavior of the worker. In any event, the wages paid to the worker in the current job do not influence her on-the-job search behavior.[2]

A submarket is characterized by an aggregate matching functions, brining together the searching workers and the vacant firms in that submarket. As will be clear below, the labor market endogenously separates into submarkets with identical agents on each side of the market. Suppose N workers search for V vacancies. We assume a constant-returns-to-scale matching function $x(N, V)$, mapping the stocks of searching workers and firms into a flow of new matches. Define $p(\theta) = x(u, V)/u = x(1, \theta)$ and $q(\theta) = x(u, V)/V = x(1/\theta, 1)$. Finally. let $\eta = |q'(\theta)\theta/q|$ denote the absolute value of the elasticity of η with respect to θ. It is convenient to assume that $\eta(\theta)$ is non-decreasing in θ.

3.1 Worker search

In competitive search equilibrium, firms advertise contracts and workers apply to one of the contracts. For any given contract σ, let $W(\sigma)$ denote the associated net present income of the worker that obtains the job. Consider an economy where a countable set of NPV wages $W_1, ..., W_k...$ are advertised, each by a strictly positive measure of firms. The firms that advertise a given wage and the workers that apply to those firms form a submarket. For any given submarket, let θ_k denote the labor market tightness (fraction of vacancies to workers) in that submarket. The set of pairs (θ_k, W_k) is denoted by Ω. We require that $W \leq y_n$.

[1] Convex costs of maintaining vacancies may be rationalized be decreasing returns to scale in the firm's recruitment department. Convex hiring costs can be seen as a generalization of Pissarides (2000) and Burdet Mortensen (1999), where the number of vacancies is exogenously fixed. Ananlogusly, the search costs of workers are usually assumed to be convex (Pissarides 2000). Note also that the structural estimates provided by Yashiv (2000a,b) are fully consistent with convex costs of maintaining vacancies.

[2] It follows from this that a worker employed in a firm of type i will never search for a job in another firm of type $j \leq i$. Such jobs cannot profitably offer a wage that exceeds the productivity in the current firm.

Let M_i $i = 0, 1, ..., n$ denote the joint expected discounted income flow of a worker and a job in a firm of type i, when the worker searches on-the-job in a submarket that offers an NPV wage of W and an arrival rate of offers p. M_0 denotes the expected net present income for an unemployed worker. Then

$$rM_i = y_i + (s + \delta)(M_0 - M_i) + p(\theta)[W - M_i] \tag{1}$$

The first term is the flow production value created on the job. Since the wage paid by firm is a pure transfer to the worker, it does not appear in the previous expression. In addition, the current job can be destroyed for exogenous reasons at rate $s+\delta$. In that case the worker turns into unemployment and receives M_0 while the firm gets zero, as reported in the equation. Finally, the worker will with probability rate $p(\theta)$ find a new job. In this event, the joint income is lost and the worker earn the present discounted value. Note that for $i = 0$ the second term is zero.

As discussed above, we require that on-the-job search is efficient, and hence that for all submarkets (p, W),

$$rM_i = y_i + (s + \delta)(M_0 - M_i) + \max_k p(\theta_k)[W_k - M_i] \tag{2}$$

with equality for all submarkets (θ, W) that attract workers hired in type i firms, hereafter referred to as a type $i-$searching worker or just type i worker (note that all worker "types" are equally productive).

The indifference curve of a type-i searching worker shows combinations of θ and W that gives a joint income equal to M_i. We can represent this as $\theta_i = f_i(W; M)$. From (1) we have that for $M_i > W_i$

$$f_i(W; M) = p^{-1}\left(\frac{(r + s + \delta)M_i - y_i - (s + \delta)M_0}{W - M_i}\right) \tag{3}$$

(strictly speaking, f_i only depends on M_i and M_0, but we write it as a function of the vector M for convenience). The indifference curve is defined for all W, not only the values advertised in equilibrium. Define

$$f(W; M) = \min_{i \in \{0,1,...,n\}} f_i(W; M) \tag{4}$$

Let W^s denote the supremum of wages in Ω. If $M_j \geq W^s$, workers employed in firms of type j do not search. In this case the nominator in (3) is zero, and $f_i(W, M) = 0$ for $W > W^s$. If $M_i < W^s$, type i-searching workers will search, hence $f_i(W, M) = f(W, M)$ for some W (at least the wages they actually search for). The nominator in (3) is then strictly positive, hence $f_i(W; M) > 0$ for all W.

Lemma 1 *For any asset value vector M that satisfies (2) the following holds*

a) $y_0/r \leq M_0 < M_1, ... < M_n$ and $(r + s + \delta)M_i - y_i$ is decreasing in i

b) Single crossing: For any $i, j,\ i < j,\ M_j < W^s$, the equation $f_i(W; M) = f_j(W; M)$ has exactly one solution, and at this point

$$|\frac{df_i(W; M)}{dW}| < |\frac{df_j(W; M)}{dW}|$$

c) Suppose $M_i < W^s$ for all $i < n' + 1$. Then there exists an increasing vector of wages $(W^1, W^2, ..., W^{n'})$ (not neccessarily in Ω) such that if $W \in (W^i, W^{i+1})$ then $f(W; M) = f_i(W; M)$. If $W \in (y/r, W^1)$ then $f(W; M) = f_0(W; M)$. If $W > W^n$ then $f(W; M) = f_n(W; M)$.

d) $f(\theta)$ is discontinuous at $M_{n'+1}$, as $\lim_{W \to M^-_{n'+1}} f(W; M) > 0$ while $\lim_{W \to M^+_{n'+1}} f(W, M) = 0$. For all other values of W, f is continuous, and strictly decreasing in W for all $W < M_{n'+1}$. f is continuously differentiable except at the points $(W^1, W^2, ..., W^{n'}, M_{n+1})$. At this points, $\lim_{W \to W^{i-}} df(W; M)/dW > \lim_{W \to W^{i+}} df(W; M)/dW$.

From d) it follows that the function f is not globally convex.

3.2 Firms

It follows that at any point in time, a firm of type j maximizes the value of search given by[3]

$$\pi_j = -c(v) + v_j q(\theta)[M_j - W_j]. \tag{5}$$

The first part is the flow cost of posting vacancies while the second part is the gain from search. The NPV to the firm of finding a worker is the difference between the joint income and the promised value to the worker, taking into account the probability q that each vacancy v_i is matched. When setting wages, the firm takes into account the relationship between the NPV wage W the firm sets and the arrival rate of workers to the vacancy. In competitive search equilibrium, the perceived relationship between the posted wage W and the arrival rate of workers can be written as $q = q(p)$ and $p = f(W; M)$. To avoid uninteresting details we assume that the firm attaches the same wage to each of its vacancies. (Note though that we allow identical firms to set different wages.) The firm's maximization problem can thus be defined as follows:

Definition 1 *The maximization problem of a firm of type j reads*

$$\max_{W.v} -c(v) + vq[M_j - W].$$

subject to

$$q = q(f(\theta))$$

[3]The profit of a firm of type j, Π_j, can be written as

$$\Pi_j(t) = \int_t^{\infty} [(y_j - w_j(t))N_j - c(v)]e^{-(r+\delta)}t$$

At any point in time, the firm decides on the number of vacancies to be posted and the wages attached to them. This only influences profits through future hirings, and is independent of the stock of existing workers.

Denote the vector of solutions to firm $j's$ maximization problem by $Wj = (W_{1j}, .., W_{i_jj})$. Below we show that W_j is finite. The number of vacancies posted by firm j is unique and denoted v_j.

Finally, the expected profit of a firm of type j entering the market can be written as

$$\Pi_j = \frac{\pi_j}{r+\delta} \tag{6}$$

3.3 Aggregate consistency

Let $N = (N_0, N_1, ...N_j...)$, where N_j denote the measure of workers in type j firms. Let τ denote a matrix of fractions, where τ_{jk} denote the fraction of firms of type j that search in submarket k. Finally, let κ denote a matrix of fractions, where κ_{ik} the fraction of workers hired in a type i firms searching in submarket k.

$$\sum_k \tau_{jk} = \sum_k^n \kappa_{jk} = 1 \text{ for all } j \tag{7}$$

In steady state, inflow of workers into type j firms has to be equal to outflow, hence

$$\sum_k N_j \tau_{jk} q(p_k) = [s + \delta + \sum_k p_{jk}\kappa_{jk}]N_j \tag{8}$$

for all j. The labor market tightness in each submarket is denoted by θ_{ij}. It follows that

$$\theta_k = k\frac{\sum_j \alpha_j \tau_{jk} v_{jk}}{\sum_j \kappa_{jk} N_j} \tag{9}$$

3.4 General equilibrium

We are now in a position to define the general equilibrium.

Definition 2 *General equilibrium is defined as a vector of asset values M^* and of employment stocks N^*, wages W^*, labor market tightnesses $\overline{\theta}^*$, matrices κ^* and τ, and a number k such that*

1. *For all $W_j \in W^*$, W_j solves firm type j s maximization problem for some firm type $\dot{j}$*
2. *If $\kappa_{ik} > 0$, then M_i is obtained by workers of type i by searching in submarket k.*
3. *If $\tau_{jk} > 0$, then W_k solves the firm's maximization problem*
4. *The expected profit of entering the market is equal to the entry cost K, i.e.,*

$$E\Pi_j = K$$

5. *The flow equations (7), (8) and (9) are all satisfied.*

In addition we make the following *equilibrium refinement*: if the equilibrium conditions 1-5 have more than one solution, the equilibria where M_0 is maximized will be chosen. Otherwise, a market maker (as in Moen 1997) can make a profitable deviation: Suppose there exist two equilibrium constellations E^1 and E^2 such that $M_0^2 > M_0^1$. Suppose we are in the E^1 equilibrium. Then a market maker can construct a deviation consisting of a small measure δ of firms of each type. In the deviation, let all agents play exactly the same strategy as in E^2. Higher type firms are recruited into the group as the measure of workers to search for builds up. In the E^1 and the E^2 equilibrium, expected profit of entering is the same and equal to K. The average profit to the deviating group would have been equal to K if M_0^1 was equal to M_0^2. However, since $M_0^1 < M_0^2$, the deviatior can cash in the difference, and this makes the deviation profitable.

4 Characterizing equilibrium and equilibrium properties

In order to characterize the equilibrium of the market, the following result is useful (recall that $\eta = -q'(\theta)\theta/q$):

Lemma 2 *Let $W > M_0$ be an NPV wage that is advertised in equilibrium. In the corresponding submarket, there is exactly one type of firms, and all the workers in the submarket are of the same type (working in the same type of firms). Furthermore, if a submarket with firms of type j and workers of type i opens up, the wage W_{ij} in this submarket is uniquely determined by the first order condition*

$$\frac{\eta}{1-\eta} = \frac{W_{ij} - M_i}{M_j - W_{ij}} \tag{10}$$

The lemma simplifies characterization of equilibrium. Each worker-firm combination leads to at most one operating submarket, and each submarket can be attributed to exactly one worker-firm combination. Let submarket ij denote a market in which workers currently employed in firms of type i and firms of type j search for each-other. Hence we can describe the matrix κ as an nxn matrix, where κ_{ij} gives the fraction of workers employed in firms of type i that search in the ij submarket. Note that $\kappa_{ij} = 0$ for all $j < i$. Similarly, let τ_{ij} denote the fraction of firms of type j searching in the ij submarket. Then $\tau_{ij} = 0$ if $i \geq j$. The first order condition for v_{ij} writes (where θ_{ij} is the labor market tightness in the market)

$$c'(v) = (M_j - W_{ij})q(\theta_{ij}) \tag{11}$$

By slightly rearranging the first order conditions we obtain from 1

$$\begin{aligned} rM_i &= y_i + (s+\delta)(M_0 - M_i) + \max_j p(\theta_{ij})\eta[M_j - M_i] && (12) \\ W_{ij} &= M_i + \eta(M_j - M_i) && (13) \\ c'(v) &= (1-\eta)(M_j - M_i)q(\theta_{ij}) && (14) \end{aligned}$$

In addition we require that firms only attend submarkets where they maximize profits $\pi_{ij} = \pi_j$ (maximum profit). The first conditions defines joint income and ensures efficient on-the-job search. The second equation defines the traditional efficient rent sharing in competitive search equilibrium, the Hosios condition. The third equation equates the marginal cost of vacancy posting to its expected benefit. To summarize, firms in this submarket post a measure of vacancies v_1 independently of their employment status. This feature follows directly from the constant returns to scale assumptions. Note that also in steady state the firm is characterized by continuous job turnover, even though employment does not grow. The wage contract posted by the firm is also constant throughout the life of the firm and does not feature any transitional dynamics.

Remark 1 *Note that as long as $y_i > y_0$, firms of type i are active in equilibrium. Since workers search equally well on and off jobs, the joint income of a worker and a firm of type i is then strictly greater than M_0. The firm will thus offer a wage $W = M_0 + (1-\beta)(M - M_0)$ and attract some workers.*

The net present value of profits Π_j is then given by (6). With quadratic costs, the NPV profit reads

$$\Pi_j = \frac{[(1-\beta)(M_j - M_i)q(\theta_{ij})] - c(v_{ij})}{r + s + \delta}$$

Proposition 1 *The equilibrium exists*

We are not able to show that the equilibrium is unique. However, there are complementarities between the layers, an increase in the number of firms searching for a given firm type increases the joint income of that firm type. Hence we cannot rule out that the economy is trapped in a local maximum. However, with our purification of the equilibrium concept, the following holds

Proposition 2 *The equilibrium is efficient*

Our next lemma characterizes wage distributions and search behavior of workers and firms

Proposition 3 *Maximum separation:*

a) Let $i < j$. Then workers in a firm of type j always search for jobs with strictly higher wages than workers employed in firms of type $l < j$. Firms of type j always offer a strictly higher wage than firms of type i.

b) Let I_k denote the set of worker types searching for firms of type k. Consider I_k and I_l, $k > l$. Then all elements in I_k are greater than or equal to all elements in I_l. Hence I_k and I_l have at most one common element.

It follows that the market, to the largest extent possible, separates workers and firms so that the low-type workers search for the low-type firms. Note the similarity with the non-assortative matching results in the search literature (Shimer and Smith (2001), Eeckout and Kirkcher (2008). If the production technology is linear in the productivities of the worker and the firm, it is optimal that the high-type firms match with the low-type workers and vice versa. Similarly, in our model it is optimal that the workers in a firm with a high current productivity search for vacancies with high productivity, and vice versa.

From an efficiency point of view, the result can be understood as follows:, recall that if vacancies are filled quickly that requires long worker queues, and the flip-side of the coin is that workers find jobs slowly. It is therefore optimal that the most "patient" workers, i.e., the workers with the highest current wage, search for the most "impatient" firms, the firms with the highest productivity. It is also trivial to extend the efficiency result above to the n-firm case.

Note that the growth rate of a firm of a given type depends on the wage that it offers. Thus, firms of different productivities may offer different wages and attract workers at different speeds, as an efficient response to search frictions. Furthermore, the size of a firm in a given market converges to a steady state level. Thus, firms do not grow indefinitely. In order to obtain indefinite growth, one may alter the cost function of posting wages. This will be done in a later.

Let us turn to uniqueness. We have the following conjecture:

Conjecture 4 *The equilibrium vector M^* is unique*

In the general case we are only able to show that M_0 and M_n are unique. That M_0 is unique follows directly from our equilibrium refinement, and from the definition of M_n it then follows that M_n is unique.

In the $n = 2$ case we can show that this implies that M_1 is unique. Since M_0 is unique, it follows that M_2 is unique. Hence we must have that $p^*_{02}(M_2 - M_0) = p^*_{01}(M_1 - M_0)$, which pins down p_{02}. Suppose there M^*_1 is an equilibrium value of M_1. Suppose $M_1 < M^*_1$. We want to show that M_1 cannot be an equilibrium. Suppose it is. Then we must have that $p_{01} > p^*_{01}$, and hence $q_{01} < q^*_{01}$ and $v_1 < v^*_1$. Thus, $N_1 < N^*_1$. From the equal profit condition it follows that $q_{01} > q^*_{01}$. Since $N_1 < N^*_1$ it follows that $p_{12} > p^*_{12}$. But then $M_1 > M^*_1$, a contradiction.

4.1

5 Large number of firm types

In this section we let the number of firm types go to infinity. To simplify the exposition we assume that the distribution of productivities converges to a uniform distribution on

$[y_0, y^{\max}]$. We construct a sequence of distributions, with an increasing number of productivity types. Thus, at distribution number n, the mass is equally distributed at n mass points, and where

$$\begin{aligned} y_1 &= y_0 + \frac{y^{\max} - y_0}{n+1} \\ y_i &= y_0 + i\frac{y^{\max} - y_0}{n+1} \end{aligned}$$

It is convenient to work with a subsequence of the $n$sequence of equilibria. Denote this sequence with σ. The first element in σ is obtained by setting $n = 1$ (i.e., with y_1 in the middle). The next element is obtained when the higher and the lower interval both are divided in two, and so forth. Thus

$$\begin{aligned} t &= 1 \text{ corresponds to } n = 1 \\ \text{if } t &= n, \text{ then } t + 1 = 2(n-1) + 1 \end{aligned}$$

Thus, if one type of firms has output y' for some t, then some type of firms will have output y' for any $t' > t$.

Recall that I_k denotes the set of firm types a worker in a type k firm searches for. Suppose there exists a firm type with productivity exactly equal to y' for some t', and hence for all $t > t'$. Let $I_{y,t}$ denote the set of firms workers in a firm with productivity y applies to in economy σ_t. Let $\Delta_t^w y$ denote the different between the highest and lowest productivity of the firms in $I_{y,t}$. Analogously, let $J_{y,k}$ denote the set of workers a firm with productivity y attracts, and let $\Delta_t^f y$ denote the difference between the highest and the lowest productivity in $J_{y,k}$.

For all t, we know that the equilibrium exists. Furthermore, without showing that the sequence of equilibria converges, we are still able to show the following:

Proposition 5 *As t increases towards infinity, the following is true*

a) There is an interval $(y_0, y_{un}]$ so that all unemployed workers search for firms with productivity in the interval $(y_0, y_{un}]$, while no employed workers search for these jobs.

b) Suppose y^i is in the support of productivities for some t'. Then $\lim_{t\to 0} \Delta_t^w y' = \lim_{t\to 0} \Delta_t^f y' = 0$.

Proof: a) The unemployment rate is strictly positive and bounded away from zero and from one for all t. Hence, unemployed workers must search for a strictly positive measure of firms, bounded away from zero. Furthermore, due to maximum separation, unemployed workers will search for firms with lower productivity than employed workers, and the sets I_0 and I_k, $k > 0$ have at most one element in common. This shows a)

b) Let $I'_{y^i,t}$ denote the firm types that only workers of type y^i searches for. Clearly, due to the maximum separation result, this includes all elements of $I'_{y^i,t}$ except possibly the end points. Define $\Delta_t'^w y^i$ and $\Delta_t'^f y^i$ accordingly. $\lim_{t\to 0} \Delta_t^w y' = \lim_{t\to 0} \Delta_t'^w y' \lim_{t\to 0} \Delta_t^f y' = 0$. Suppose $\Delta_t'^w y'$ does not converge to zero. Then the rate at which the searching firms find workers converges to zero. Hence it would be more profitable for these firms to search for unemployed workers, a contradiction. The claim that $\lim_{t\to 0}\Delta_t^f y' = 0$ is proved in an analogous way.

Thus, as the number of firms grows, the search pattern of the workers get close to a pure job ladder. Unemployed workers randomize over a set of firms with low productivity. Workers employed in firms with productivity y applies to firms with productivity y' for some $y' > y$.

We have already seen that the equilibrium maximizes M_0 given the zero profit constraint of the firm. It follows that M_{0t} is increasing in t (a planner could choose exactly the same hiring strategies at $t+1$ as at t, just by treating close firms as identical. This would give exactly the same value of M_{0t} and M_{0t+1}. When the planner chooses another strategy, this must imply that $M_{0,t} \leq M_{0,t+1}$.) Since M_{0t} is bounded above it follows that the sequence M_{0t} converges. Unfortunately, we have not yet a complete proof for convergence of the other variables, and hence state our next claim as a conjecture.

Conjecture 6 *The sequence of equilibria converges to a limit equilibrium, given by a function of joint incomes $M(y)$, a function of labor market tightnesses $\theta(y)$. In addition, the search behavior can be described as follows:*

a) Unemployed workers search for jobs with output in the interval (y_0, y^z) for some $y^z \in (y_0, y^{\max})$.

b) Workers hired in firms of type $y > y_0$ searches for a job in firms of type $f(y) > y$, where f is continuos, strictly increasing and $\lim_{y\to y^{\max}} f(y) = y^{\max}$.

From the envelope theorem it follows that $M'(y) = \frac{1}{r+s+p+\delta}$. Furthermore, $M(y) = y^{\max}/(r+s+\delta)$. It follows that we can calculate $M_{p()}(y)$. From the first order condition we know that

$$W(F(y)) - M_{p()}(y) = \eta(M_{p()}(f(y_1) - M_{p()}(y))$$

It follows that

$$(r+s+\delta)M_{p()}(y) = y_1 + sM_0 + p(y)\eta[M_{p()}(F(y)) - M_{p()}(y)]$$

6 Linear vacancy costs and convex hiring costs

In this section we first study the equilibrium when the vacancy cost function is almost linear. Then we look at the effects of hiring costs on the equilibrium of the model.

6.1 Almost linear vacancy costs

In this section we characterize the equilibrium of the model when the degree of convexity of $c(v)$ vanishes. To this end, write the vacancy cost function as

$$c(v) = c_0 v + cv^2/2$$

where c_0 and c are constants.

Proposition 7 *For sufficiently low values of c, only type n firms post vacancies. Hence there is no on-the-job search*

Consider a firm of the highest type. Suppose this firm posts all its vacancies in submarket i. The number of vacancies posted and the profit of the firm is then given by

$$\begin{aligned} v_{ni} &= \frac{(1-\eta)q_{ni}(M_n - M_i) - c_0}{c} \\ \pi_{ni} &= \frac{[(1-\eta)q_{ni}(M_n - M_i) - c_0]^2}{2c} \end{aligned}$$

where $q_{ni} = q((1-\eta)(M_n - M_i, M)$. It follows that $q_{ni}(M_n - M_i) - c_0$ converges to zero as c_0 converges to zero, otherwise profits explode.

Since $c_0 > 0$, $M_n - M_{n-1}$ does not go to zero as c vanishes. Let ΔM denote a lower bound on $M_n - M_{n-1}$. Suppose firm $n-1$ enters submarket i. Since $M_{n-1} < M_n$, it follows that $q_{n,i} > q_{n-1,i}$. Since $q_{ni}(M_n - M_i) - c_0$ converges to zero as c_0 converges to zero, it follows that $q_{n-1,i}(M_{n-1} - M_i) - c_0$ becomes negative for sufficiently small values of c. Thus firm of type $n-1$ does not post any vacancies, $N_1 = N_2 = ... = N_{n-1} = 0$. By definition, there cannot be any on-the-job search.

The proposition shows that convex hiring costs is neccessary in order to obtain on-the-job search.

6.2 Hiring costs

In this subsection we introduce convex hiring costs, as in Sargent (1987). More specifically, we include a cost term $\gamma(h)$, where $h = qv$, and assume that $\gamma(0) = \gamma'(0) = 0$ and that $\gamma'()$ and $\gamma''()$ are strictly positive for $qv > 0$. To simplify we set $\gamma(h) = \gamma(h)^2/2$. In addition we include linear hiring costs.

Consider a firm of type j that searches for workers and offers a wage W_j. The profit flow from hiring can then be written as

$$\begin{aligned} \pi_j &= -c_0 v - \gamma h^2/2 + h(M_j - W_j) \\ &= hM_j - \gamma h^2/2 - c_0 v - hW_j \end{aligned}$$

It is instructive to divide the firms' maximization problem into two steps:

- Optimal hiring channels: For a given h, minimize $c_0 v + hW_j$ subject to $h = q(f(W_j))v$, and denote the minimum by $C(h)$
- Optimal hiring level: Find the optimal h, i.e., the h that solves $hM_j - C(h) - \gamma h^2/2$

The Lagrangian associated with the first problem writes

$$L = c_0 v + hW_j - \lambda[q(f(W_j))v - h]$$

The first order conditions for W is given by the ugly expression

$$W = \frac{c_0 f_W(W) + \eta el_W f(W) q^2}{q^2 - q f_W}$$

(provided that our calculations are correct).

The important issue is that W is independent of both h and M_j. The hiring technology is such that by doubling the the number of vacancies the firm posts, it doubles both the number of hirings and the cost $c_0 v + hW$. The optimal wage is thus independent of scale, as the scaling is done through the number of vacancies posted. We denote by C the unit cost of hiring. The first order condition for optimal scale is thus given by

$$\begin{aligned} h &= \frac{M_j - C}{\gamma} \\ \pi_j &= \frac{(M_j - C)^2}{2\gamma} \end{aligned}$$

It follows that if the entry cost is sufficiently high, C is very small, and all firm types will operate in equilibrium.

Efficient on-the-job search implies that workers will only search for jobs if $W > M_j$. From the definition of W it follows that $W > C$. Since firms will only attract workers in the first place if $M_j > C$, it follows that no on-the-job search takes place in equilibrium

Proposition 8 *Suppose the costs of posting vacancies is linear. In addition, suppose there are costs associated with hiring, and that these costs are convex in the hiring rate $h = vq$. Then the following is true*

a) Firms with different productivities may coexist and post vacancies in equilibrium
b) There is no on-the-job search
c) The optimal wage is independent of the productivity of the firm.

Thus, convex hiring cost may explain why firms of different productivities coexist in equilibrium. However, convex can neither explain on-the-job search nor the observed relationship between wages and firm productivity.

7 Search costs decreasing in firm size

The model presented above implies a constant gross hiring rate of workers. As the firms grow, so does the number of separations, and as a result there exists a steady state level of employment. Thus, the growth rate of any given firm is decreasing with firm age and size, and is hence violating Gibraths law. We also keep our assumption that the matching function is Cobb-Douglas (this can easily be generalized).

The dynamic properties of the firms depend crucially on the functional form of the cost of hiring vacancies. To see this, suppose the cost of posting vacancies can be written as

$$c = Nc(\frac{v}{N})$$

The cost function can be rationalized as follows: Suppose $c(v_i)$ is the individual effort cost of assisting in the hiring process by exerting v_i units of effort. Since $c()$ is convex, it is optimal to distribute effort equally over the work force so that each worker contributes $v_i = v/N$ units, where v is the total effort level of the firm, i.e., the number of vacancies. Total effort cost is then $Nc(\frac{v}{N})$. In what follows we assume that the effort cost is quadratic, hence we can write

$$c(v, N) = \frac{v^2}{2cN}$$

In all other aspects the model proceeds exactly as before. For a given NPV income the workers' on-the-job search behavior can be described in exactly the same way as above. In particular, there exists a function $\theta = f(W; \widetilde{M}_0, ..., \widetilde{M}_n)$ showing the labor market tightness as a function of wages and joint incomes. Furthermore, by applying the same arguments as above it follows that the labor market endogenously separates into submarkets as described above.

Let $\widetilde{v} = v/N$ denote the vacancy rate as a fraction of employment. However, the joint expected income of a match is different.

Lemma 3 *Consider a firm of type j that searches for workers hired in type i, searching as if her NPV income is $\widetilde{M}_i$. Let an asterix denote optimal values. Then the following holds:*

a) The relevant joint income for the worker and the firm writes

$$\widetilde{M}_j = \frac{y_j + \max_k p_{jk}(W_{jk} - \widetilde{M}_{ij}) + (s+\delta)\widetilde{M}_0 + [\widetilde{v}_j^* q_{ij}^*(1-\beta)(\widetilde{M}_{ij} - \widetilde{M}_i) - \widetilde{v}_{ij}^{*2}/2c]}{r+s+\delta} \tag{15}$$

Proposition 9 *b) The optimal wage reads*

$$W_{ij}^* = \beta(\widetilde{M}_j - \widetilde{M}_i) \tag{16}$$

and $q^* = q(\theta(W_{ij}^*, \overline{M}_i))$.

c)The optimal fraction of vacancies to workers, $\widetilde{v}$, reads

$$\widetilde{v}_{ij}^* = (1-\beta)q_{ij}^*(\widetilde{M}_{jj} - \widetilde{M}_i)c \tag{17}$$

Note that the only difference between M_j defined by (1) and $\widetilde{M_j}$ defined by (15) is the last term in the nominator. This term reflects what we refer to as *breeding.* More workers imply that hiring will be more easy in the future, and this increases the value of a match. For the problem to be well defined, the effect of breeding must not be too large, since the value of a job then will be infinite. Note also that in the first order conditions (16) and (17) $\widetilde{M}j$ is taken as given. We are now able to show the following result

Proposition 10 *Suppose $rM_i > y_i - s(M_i - M_0)$ (some on-the-job search takes place). Then the firms' maximization problem is well defined provided that c is sufficiently small (the search cost is sufficiently high). Furthermore, given that the maximization problem is well defined it has a unique solution $\widetilde{v}_{ij}^*$,W_{ij}^*.*

Thus, it follows that the fraction of vacancies to employees is constant. The growth rate of the firm (conditioned on survival) is then $v_{ij}^* q_{ij}^* - (s + p_{jk})$, independently on firm size. Note that the growth rate may be positive or negative. Note also that the wage rate is independent of firm size, while profits from future hirings is proportional to firm size

Define $\Pi_j = \max_{i<j} \Pi_{ij}$. In equilibrium submarket ij is open if and only if

$$\Pi_{ij} = \Pi_j$$

Finally, free entry of firms implies that

$$E^j \Pi_j = K$$

It can be shown that the equilibrium exists and is efficient.

8 Endogenizing productivity differences

Suppose the firms can choose between (y_1, K_1) and (y_2, K_2), $y_1 < y_2$, $K_1 < K_2$. Furthermore, suppose that with no on-the-job search, the parameters are such that all firms choose the lowest investments.

Conjecture 11 *Suppose we allow for on-the-job search. Then the resulting equilibrium is a pure job-ladder equilibrium, determined by the zero profit conditions*

$$\begin{aligned}\Pi_1 &= K_1 \\ \Pi_{12} &= K_2\end{aligned}$$

cetch of proof: First note that the 2-market cannot be empty. Suppose it is. Then a firm that opens up can fill its vacancies infinitely quickly, and thus as long as $y_2 > y_1$ can obtain unbound profit. Furthermore, by assumption it will not be profitable for firm 2 to search for unemployed workers, since in this market they will be dominated by type-1 firms.

Suppose the firms could choose investment-output combinations from a menu defined by the function $y = F(K)$, where $F(K)$ is increasing and concave and bounded above by $\overline{y}$. Then we conjecture that the equilibrium is a pure job ladder equilibrium with infinitely many steps.

9 Equilibrium with two types of firms

Consider the special case with two types of firms. We also assume that the matching function is Cobb-Douglas, $x(u,v) = Au^{\beta}v^{1-\beta}$. In this case we can get some more structure and therefore also some more results regarding the opening and closing of submarkets.

Our first observation is that the $_{12}$ market is always open. Suppose not. Then a high-type firm that opens vacancies with a wage slightly above y_1 would attract applications for all workers employed in type 1 firms. The firm would thus obtain an infinitely high arrival rate q of job offers, and would make infinitely high profit. A deviation from equilibrium would thus surely be profitable.

The next question is whether the $0,2$ market will open up (stairways to heaven). If not, we say that we have a pure job ladder. Whether or not we have a pure job ladder depends on parameter values. However, with very mild restrictions on $c(v)$ (that $\lim_{v\to\infty} c'(v)/v = \infty$) we can show the following proposition:

Proposition 12 *a) Suppose K is high, so that few firms enter the market. Then high-type firms search both for unemployed and employed workers.*

b) Suppose there exists a pure job ladder for some values of K. Provided that the number of vacancies is not too flexible (c'' is sufficiently large around the equilibrium point), then there exists a K^* such that there is a pure job ladder for $K < K^*$ while both the $1,2$ and the $0,2$ market open up if $K > K^*$.

We need the qualifier in order to ensure that the measure of vacancies in the economy goes to zero when the measure of firms goes to zero. The proposition thus states that the pure job ladder equilibrium only prevails if the frictions in the market are sufficiently low. Thus, contrary to what one may expect, a pure job ladder (if it emerges at all) emerges when there are many jobs relative to workers and hence the unemployment rate is low.

Our second question regards the relationship between the share of high-type firms in the equilibrium. Let a balanced increase in α_2 denote an increase in α_2 where other variables (for instance the entry cost K) is adjusted so that the number of firms k is kept constant. We are able to show the following result:

Proposition 13 *a) For high values of α_2, both the $0,2$ and the $1,2$ submarkets are active. For low values of α_2, only the 12 market is active.*

b) Consider balanced changes in α_2. Suppose $c''(e)$ is large. Then there exists a unique $\alpha = \alpha^*$ such that the $_{02}$ market is open if and only if $\alpha_2 > \alpha^*$.

Two remarks regarding b) is warranted. The first regards the fact that we are only considering balanced changes. The reason is the following. Suppose that $\alpha_2 = \alpha^*$, so that there is a pure job ladder and the 2-firms are just indifferent by entering the $_{12}$ and the $_{02}$ market. Consider an increase in α_2. This has two effects on equilibrium. First it becomes more crowded in the $_{12}$ market, and this favours the $_{02}$ market. However, if we let k vary, it follows that k will increase, and as seen in proposition (12) this favours the pure job ladder equilibrium. In general we are not able to show which force is the stronger, and hence cannot guarantee that there is a unique switching point. However, with balanced changes we can.

The second comment regards the requirements on $c''(e)$. When α_2 increases, the direct effect is that p_{01} decreases, as there are fewer low-type firms. Our proof of uniqueness of the switching point depends on p_{01} being decreasing in α_2. However, low-type firms may post more vacancies, and in principle this may imply that p_{01} increases in α_2. As we have not been able to rule this out, we instead put restrictions on $c''(e)$ so that the flexibility of e is not too big.

9.1 Basic Calibration and Comparative Static

Table 1 and 2 report the basic parameter values for our calibration. The calibration is based on quarterly statistics and the pure interest rate is 1 percent. The productivity level in low type firms is set to a baseline reference value of $y_1 = 1$, while the premium for the high type is 15 percent. The flow value of unemployment z is 0.55, a value far the replacement rate observed in real life labour markets. The matching function is Cobb Douglas with an elasticity β equal to 0.5. The parameter of the search cost is 0.15, while the entry cost k is 5, a value roughly equal to five times times the output produced by a low productivity job.The sum of the separation s and the firm death rate is 0.06. The proportion of low productivity firms is 0.155. In the specification of the model presented in this section, we assume that the convexity of the vacancy is extreme so that each firm can post at most a maximum number of vacancies $\overline{v} = 0.15$. The rest of the parameters are reported in 1

The baseline equilibrium features an unemployment rate equals to 7.7 percent and a job finding probability equal to 0.7, in line with the basic quarterly statistics in advanced economies. Unemployment flows are 5.5 percent, consistent with the quarterly job creation rate in the US manufacturing sector compiled by Davis and Haltiwanger. Job to job mobility is slightly below 5 percent. In Table 1 most of the unemployed workers search for low productivity firms, as indicated by $k_{01} = 0.99$. Similarly, high productivity firms search mainly among the employed sector, as indicated by the fraction of firms hiring from the employment pool ($\tau = 0.99$) The equilibrium allocation is described in the central part of Table 1. The job finding rate for unemployed workers p_{01} is the largest among the various job finding rates, but the bulk of workers in the labor market is employed in type 2 firms. Indeed, type 2 firms absorb 68 percent of the total workforce. As a result, the submarket 02, albeit significant, represents a fringe of the entire economy.

The idea of the baseline simulation from Table 1 to Table 2 is to show that a small decrease in the share of high productivity firms α lead the economy to move toward a pure job ladder equilibrium. Indeed, the only parameter that changes between Tables and 1 and

2 is α. Recall that in the baseline specification of Table 1 the equilibrium value of τ is very close to one and as a result the submarket 02 is very small. A small decrease α, similarly to that experienced from Table 1 to Table i2 leads to an equilibrium value of $\tau > 1$, a value that is not consistent with all three submarkets being operative. In other words, as α falls with respect to the value assigned in Table 1, the economy moves to a *pure job ladder equilibrium*. In moving from Table 1 to Table 2 α falls from 0.155 to 0.154, suggesting that α^* in our numerical example is inside this small interval. The economy described in Table 2 does look very similar to that described in Table 1, even though two only submarkets are operative. Note also that the equilibrium value of unemployment M_0^* does slightly fall as α falls. This is not surprising since in a pure job ladder equilibrium the share of high productivity firms is higher. Workers start out in low productivity firms and eventually graduate to high type jobs through on the job search. Eventually, firm and match specific shocks at rate δ and s induce another round of job ladder. The bottom part of the Table 1 features also an important relationship between firm size and firm wages, where the latter are measured in terms of PDV wages. Clearly, high type firms are larger in size and pay higher wage.

9.2 Comparative Static

The main object of the simulations carried out in this section is to show the mechanics of the model for different values of α in the baseline simulation provided above. As $\overline{y} = (1-\alpha)y_1 + \alpha y_2$, an increase in α is akin to an increase in average productivity. The basic charts of the simulations are provided in Figure 1 and 2. First note that when $\alpha = 0$ or 1, the model collapses to the traditional matching model without on-the-job search (Pissarides 2000). As expected, the transition rate from unemployment to employment is higher and unemployment lower when $\alpha = 1$ than when $\alpha = 0$. (In Figure 1 unemployment falls from 0.0968 to 0.083 as α increases from 0 to 1). We refer to this as a pure *productivity effect*, and it is caused by higher entry of firms and a higher f when output per firm is high.

For interior values, an increase in α comes along with important *composition effects*. While the value functions increases smoothly as the economy becomes more productivity (top left panel in Figure 2), the increase in the job finding rate p_{01} in the pure job ladder is humped-shaped. For a fixed number of firms, an increase in α reduces the number of jobs available to the unemployed (which are hired in firm of type 1), and increase the jobs available to the employed (which are hired in firm of type 2). This composition effect tend to reduce the job finding rate p_{01}. The productivity effect increases the number of firms, and hence work in the opposite direction, but in the pure job ladder equilibrium it only dominates the composition effect for very low values of α. Note also that job-to-job movements, by definition equal to zero at the extremes, tends naturally to grow as the economy operates into a pure job ladder equilibrium.

For higher values of α, the mixed job ladder equilibrium emerges, with a different type of composition effects. In particular, the $_{02}$ submarket is characterized by lower job-finding rates. A higher α on some intervals imply larger variations in the queue lengths among unemployed workers, and this tends to increase unemployment. For relatively low levels of α this effect dominates the productivity effects. Eventually, as the share of high productivity firms increases toward 1, the pure productivity effects emerges and unemployment falls.

Finally, the non monotonic behavior of entry deserves few comments. When α is low, the value of a high-type firm (given by 6) is extremely high since they grow so quickly. This explains the hump-shaped form of f, the number of firms in the economy.

10 Conclusion

We have developed a competitive flavored matching model where on-the-job search is an optimal response to productivity differences between firms and costly search. In the resulting equilibrium, workers hired in firms with different productivities, and firms with different productivities search in separated job markets, in such a way unemployed workers search for low-type firms, while employed workers search for jobs in firms that are more productive. The equilibrium thus features a job ladder, where workers gradually moves to jobs with higher wages. The model can thus explain the following stylized facts about the labor market:

Figure 1: Increase in Average Productivity, Stocks and Job Finding Rates

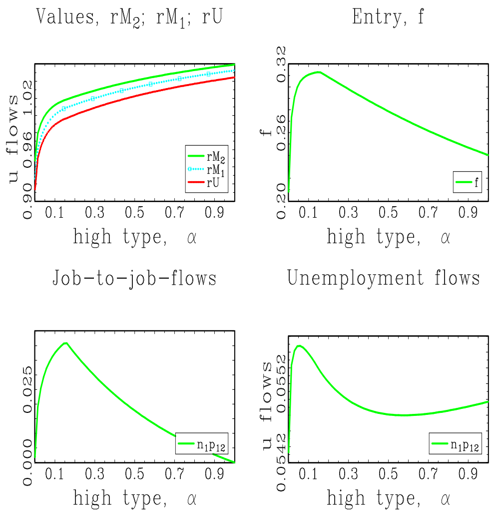

Figure 2: Increase in Average Productivity, Value Functions, Flows and Entry

1) firms with different productivities coexist in the labor market, 2) On-the-job search is prevalent and worker flows between firms are large, and 3) more productive firms are larger and pay higher wages than less productive firms. The equilibrium is efficient.

Interestingly, we find that if the costs of maintaining vacancies are close to being linear, only the most productive firms open vacancies, and hence there will be no on-the-job search. If we include hiring costs (convex in the flow qv of new hires), more than one firm type may open vacancies. However, in this case wages will be independent of productivity and there will be no on-the-job search. Hence convex hiring costs seem to be neccessary in order to explain the stylized facts of on-the-job search.

11 APPENDIX:

11.1 Proof of lemma 1

Proof: a) An unemployed worker obtains y_0/r, hence $y_0/r \leq M_0$. A type n worker cannot gain from on-the-job search since $W \leq y_n$, and hence $M_n = y_n/(r+s+\delta)$. A worker and a firm of type j (hereafter referred to as a worker of type j) can always obtain a strictly higher joint income than a worker of type $i < j$ by following exactly the same search strategy as the type i worker, hence $M_j > M_i$. Analogously, a worker of type i can always mimic the search strategy of a worker of type $j > j$. Let the associated joint income be denoted M_i'. From (1),

$$(r+s+\delta)(M_j - M_i') = y_j - y_i$$

or

$$(r+s+\delta)M_j - y_j = (r+s+\delta)M_i' - y_i$$

Since $M_i' \leq M_i$ it follows that $(r+s+\delta)M_j - y_j \leq (r+s+\delta)M_i - y_i$.

b) We want to show that the indifference curve has the following single crossing property: Suppose $i < j \leq n'$. Then there exists a wage W' such that $f_i(W', M) = f_j(W', M)$, $f_i(W, M) < f_j(W, M)$ for all $W < W'$, and $f_i(W, M) > f_j(W, M)$ for all $W > W'$.

Note that $\lim_{W \to M_j^+} f_j = \infty$ while $f_i(M_j, M) < \infty$. Thus, for W close to but above M_j we have that $f_i(W, M) < f_j(W, M)$. The ratio of $p(f_i)$ to $p(f_j)$ reads

$$\begin{aligned} \frac{p(f_i)}{p(f_j)} &= \frac{(r+s+\delta)M_i - y_i - (s+\delta)M_0}{(r+s+\delta)M_j - y_j - (s+\delta)M_0} \frac{W - M_j}{W - M_i} \\ \lim_{W \to \infty} \frac{p(f_i)}{p(f_j)} &= \frac{(r+s+\delta)M_i - y_i - (s+\delta)M_0}{(r+s+\delta)M_j - y_j - (s+\delta)M_0} < 1 \end{aligned}$$

(from a in this lemma). Thus, for sufficiently large values of W, $p(\, f_i(W; M)) < p(f_j(W, M))$, and hence $f_i(W; M) < f_j(W, M)$. Since f_i and f_j are continuous it follows that there exists a value W' such that $f_i(W'; M) = f_j(W', M)$.

c) Suppose now that $i < j < k$. Let W^{ik} be the unique solution to $f_i(W; M) = f_k(W; M)$. Suppose $W^{ik} < W^{ij}$. From the single-crossing property just arrived it follows that $f_i < f_j$

for all $W < W^{ij}$ and that $f_k < f_j$ for all $W > W^{ik}$. Since, by assumption,$W^{ik} < W^{ij}$, it follows that hence $f_j(W;M) > f(W,M)$. Since all workers types at or below except the highest search this is a contradiction.

For wages below M_1, only type zero workers will apply, hence $f(W,M) = f_0(W;M)$. At $W^1 = W^{01}$, $f_0(W^1;M) = f_1(W;M)$. Let $W^2 = W^{12}$. Then $f(W) = f_2(W;M)$ for $W \in [W^1, W^2]$ (which may have zero measure) and so forth. Furthermore, by definition type n' gains from search, and hence searches for a job with a wage $W \leq W^s$. Hence there must be an interval $[W, W^s]$ at which $f_{n'} = f_n$

d) For all $i \leq n'$, each of the functions $f_i(W;M)$ is continuously differentiable for $W > M_i$. It follows that $f(W;M) = \min_i f_i(W;M)$ is continuos and piecewise differentiable for all $W < M_{n'+1}$. However, as the nominator in (3) is zero for $i = n' + 1$, it follows that $f_{n'+1}$ is zero for any $W > M_{n'+1}$. Hence f is discontinuous at $M_{n'+1}$,as it jumps down to zero at this point.

11.2 Proof of lemma 2

Suppose a submarket attracts $i-$workers and j-workers, $i < j$. Denote the npv wage in the submarket by W'. Then $f_i(W',M) = f_j(W',M) = f(W',M)$. From lemma (1) this can only be the case if $j = i + 1$, that is, if $W' = W^i$.

From lemma 1 it follows that

$$\begin{aligned}
\lim_{W \to W^{i-}} \frac{\partial q(\theta(W^i))}{\partial W} &= \lim_{W \to W^{i-}} q'(\theta(W_i)) \frac{\partial \theta(W^i)}{\partial W} \\
&< \lim_{W \to W^{i-}} q'(\theta(W_i)) \frac{\partial \theta(W^i)}{\partial W} \\
&= \lim_{W \to W^{i-}} \frac{\partial q(\theta(W^i))}{\partial W}
\end{aligned}$$

It follows that $W = W^i$ cannot be a solution to any firm's maximization problem and hence cannot be an equilibrium wage. Hence a submarket cannot contain two different worker types.

Suppose then that two firm types i and j, $i < j$ offer the wage W'. The optimal wage for firm j solves

$$\max_{v,W} -c(v) + vq(f(W))[M_j - W]$$

with first order condition for W given by

$$\frac{q'(f(W))f'(W)}{q} = \frac{1}{M_j - W} \tag{18}$$

The left-hand side is independent of j, while the right hand side is increasing in j. It follows that the first order conditions cannot be satisfied for two different firm types simultaneously.

In order to derive (10), first note that

$$\frac{dp^{-1}(\theta)}{d\theta} = \frac{1}{p'(\theta)} = \frac{1}{q + q'(\theta)}$$

(since $p(\theta) = \theta q(\theta)$). From (3) it thus follows that

$$f'(W) = -\frac{1}{q + q'(\theta)} \frac{\theta q(\theta)}{W - M_i} \tag{19}$$

which inserted into (**??**) gives

$$-\frac{q'(\theta)}{q + q'(\theta)} \frac{\theta q(\theta)}{W - M_i} = \frac{q}{M_j - W}$$

Inserting $\eta = -q'(\theta)\theta / q(\theta)$ and reorganizing slightly gives

$$\frac{\eta}{1 - \eta} = \frac{W - M_i}{M_j - W}$$

By assumption, the left-hand side is decreasing and the right-hand side is increasing in W. Thus, for given M_i and M_j the equation uniquely defines W. Trivially, the first order condition for vacancies is given by (11)

11.3 Proof of existence

The strategy for the proof is to construct a mapping for which the equilibrium of the model is a fixed point, and then apply Brouwer's fixed point theorem. We assume that the matching function is Cobb-Douglas, $x(u, v) = Au^{\beta}v^{1-\beta}$ (generalizing the proof is simple).

To this end, let $\overline{\kappa}$ denote a matrix describing submarket choices of workers κ_{ij}, $\kappa_{ij} = 0$ if $i \geq j$, and $\sum_{j=i+1}^{n} \kappa_{ij} = 1$ for all i. Similarly, let $\overline{\theta}$ denote a matrix of labor market tightnesses θ_{ij}, $\theta_{ij} = 0$ if $i \geq j$. We require that $0 \leq \theta_{ij} \leq \theta^{\max}$ for all $i < j$, where $\theta^{\max}$ will be defined below. Finally, let the real number k denote the measure of firms in the economy. We require that $k \leq k^{\max}$. It follows that the set $D^n \in R^{2(n+1)^2+1}$ of allowed vectors $(\overline{\kappa}, \overline{\theta}, k)$ is closed and convex.

We want to construct a continuos mapping $\Gamma : D^n \to D^n$, and proceed as follows: Let $\overline{p}$ denote the matrix of transition probabilities $p_{ij} = A\theta_{ij}^{1-\beta}$. Analogous with (**??**), define

$$(r + s + \delta)M_{ij} = y_i + (s + \delta)M_0 + p_{ij}\beta(M_j - M_{ij}) \tag{20}$$

(where M_0 is replaced with M_0). Let $\overline{M}$ denote the matrix of values M_{ij}. Given the matrix $\overline{p}(\overline{\theta})$, the matrix $\overline{M}$ is uniquely defined as a continuos function of $\overline{\theta}$, $\overline{M}(\overline{\theta})$. To see this, first note that M_n is independent of $\overline{\theta}$. Suppose M_{ij} and $M_i = \max_j M_{ij}$ are uniquely defined as continuos functions of $\overline{\theta}$ for all $i, j > i$, for all $i > k$. It then follows from equation (20) and the definition of M_i that this also holds for M_{kj}, $j > k$, and M_k. Thus it holds for all i, j such that $j > i$.

The gross income flow of a firm of type j of posting a vacancy in submarket i is given by $\rho_{ij} = q(\theta_{ij})(1-\beta)(M_j - M_{ij})$. Define $\rho_j = \max_i \rho_{ij}$. Now define θ^a_{ij} implicitly by the function

$$q(\theta^a_{ij})(1-\beta)(M_j - M_{ij}) = \rho_j$$

The equation thus shows the values of θ_{ij} such that the firm of type j is indifferent between searching in submarket i and in the best submarket given $\overline{\theta}$. Finally, let v^a_j be defined by the equation $c'(v^a_j) = \rho_j$ (the optimal number of vacancies given ρ_j). It follows that both $\overline{\theta}^a$ and v^a_i are continuos functions of $\overline{\theta}$.

Given the initial vector $\overline{\theta}$, equation (8) uniquely defines $N_0, N_1, ...N_n$ as continuous functions of $\overline{\theta}$. In each submarket, aggregate consistency requires that

$$N_i \kappa_{ij} \theta_{ij} = k\alpha_i \tau_{ij} v_{ij} \tag{21}$$

Sum over i. This gives

$$\sum_{i<j} N_i \kappa_{ij} \theta_{ij} = \sum_{i<j} k\alpha_i \tau_{ij} v_{ij}$$

We now insert $v_{ij} = v^a_j$ into this equation. Define the constant ζ_j by the expression

$$\sum_{i<j} N_i \kappa_{ij} \theta_{ij} = \zeta_j k \alpha_i v^a_j \tag{22}$$

Finally, define

$$\widehat{\theta}_{ij} = \zeta_j(\widehat{\theta})\theta^a_{ij}(\widehat{\theta})$$

This is our updating rule for rule for θ unless the upper bound $\theta^{\max}$ binds, in which case $\widehat{\theta}_{ij} = \theta^{\max}$.

Consider the searching workers. Suppose M_i is obtained for $j \in J_i$. For all $j \notin J_i$, define

$$\widehat{\kappa}_{ij} = \frac{M_{ij}}{M_i}\kappa_{ij}$$

Note that $\widehat{\kappa}_{ij}$ is continuous in $\overline{\theta}$ and κ_{ij}. Define the constant η_i by the expression

$$\eta_i \sum_{j \in J_i} \kappa_{ij} + \sum_{j \notin J_i} \widehat{\kappa}_{ij} = 1$$

For all $j \in J_i$, the updating rule reads

$$\widehat{\kappa}_{ij} = \eta_i \kappa_{ij}$$

Finally, the expected profit of a firm of type i entering the market and searching for a firm $j \in J_i$ reads

$$\Pi_i = \frac{1}{r+\delta}\left\{v^a_i q(\theta_{ij})(M_{ij} - M_i)(1-\beta) - c(v^a_i)\right\}$$

The expected profit of entering, given the initial parameter values, is

$$E\Pi = \sum_i \alpha_i \Pi_i$$

The updating rule for k reads

$$\widehat{k} = k\frac{E\Pi}{K}$$

unless the upper bound $k^{\max}$ binds, in which case $\widehat{k} = k^{\max}$.

We have thus constructed a mapping $\Gamma : D^n \to D^n$, which by construction is continuos. It follows from Brouwers fixed point theorem that the mapping has a fixed point.

Our next step is to show that a fixed point of Γ is an equilibrium of our model. Denote the fixed point by D^*. First, given the asset value matrix M_{ij}, the firm sets the optimal sharing rule by construction. Furthermore, by the very definition of $\widehat{\theta}$ it follows that the all firm are indifferent by entering any submarket $_{ij}$. Thus, the firms' search behavior is optimal.

Second, from the updating rule for κ it follows that if $\kappa_{ij}^* > 0$, then it is optimal workers in firm j to search for a position in firm j.

Third, we have to show that the model is internally consistent, and satisfies (21). At the fixed point, $\zeta_j = 1$ for all j. Hence (22) is satisfied. However, this means that the weights τ_{ij} give us enough degrees of freedom to satisfy (21).

By construction, the labor market tightness θ_{ij}^* is defined even in submarkets where $\kappa_{ij} = 0$, i.e., even in empty submarkets. We have thus ruled out the situations where no agents enter a submarket which potentially may be active because no-one else enter the market.

Finally, we characterize the bounds. Consider the equilibrium with $\alpha_n = 1$ (only firms of the highest productivity). Then the model collapses to the standard search model, and it is trivial to show that this equilibrium exists. Define k^1 and θ^1 as the equilibrium values of k and θ in this equilibrium, respectively. Define $k^{\max} = k^1/\alpha_n$. With this number of firms, we know that $M^n \leq M^{'}$, where the latter is the joint income of a worker and a firm (type n) in equilibrium when $\alpha_n = 1$. It follows that the expected income of entering when $\alpha_n < 1$ is lower than K. Hence $k^* < k^{\max}$. Finally, let W_{01}' and θ_{01}' denote the solution to the maximization problem of a firm of type 1 in the $\alpha_n = 1$ equilibrium. By construction, $\theta_{ij}^* < \theta_{01}' = \theta^{\max}$. (Note that we could have worked with queue length $1/\theta$ instead, in which case defining an upper bound for θ would be easier).

11.4 Proof of efficiency

We assume that the matching function is Cobb-Douglas, $x(u, v) = Au^\beta v^{1-\beta}$ (generalizing the proof is simple). The welfare function is defined as

$$W = \int_0^\infty [\sum_{j=0}^n N_j y_j - \sum_{i=0}^n \sum_{j=i+1}^n \alpha_j k c(v_{ij}) - aK] e^{-rt} dt$$

The law of motions are

$$
\begin{aligned}
\dot{N}_j &= \sum_{i=0}^{j-1} x(\kappa_{ij}N_i, \alpha_j k\tau_{ij}v_{ij}) - \sum_{i=j+1}^{n} x(\kappa_{ji}N_j, \alpha_i k\tau_{ji}v_{ji}) - (s+\delta)N_j \\
\dot{k} &= a - \delta k
\end{aligned}
$$

The initial conditions take care of the requirement that $\sum_i N_i = 1$. The controls are a, κ_{ij}, τ_{ij} and v_{ij}. All κ_{ij}, τ_{ij} have to be between zero and 1, this will be discussed later. The current-value Hamiltonian reads

$$
\begin{aligned}
H &= \sum_{j=0}^{n} N_j y_j - \sum_{i=0}^{n}\sum_{j=i+1}^{n} \alpha_j k c(v_{ij}) - aK \\
&\quad + \sum_{j=0}^{n} \lambda_j [\sum_{i=0}^{j-1} x(\kappa_{ij}N_i, \alpha_j k\tau_{ij}v_{ij}) - \sum_{i=j+1}^{n} x(\kappa_{ji}N_j, \alpha_i k\tau_{ji}v_{ji}) - (s+\delta)N_j] \\
&\quad + A[a - \delta f]
\end{aligned}
$$

The controls are chosen so as to maximize H. Note that $x(u,v) = Au^{\beta}v^{(1-\beta)}$ it follows that $x_v = (1-\beta)Au^{\beta}v^{-\beta} = (1-\beta)q(\theta)$. We thus get the following first order conditions for vacancy creation:

$$c'(v_{ij}) = (1-\beta)q(\theta_{ij})[\lambda_j - \lambda_i] \tag{23}$$

The first order conditions for the other controls read

$$
\begin{aligned}
A &= K \qquad (24)\\
p_{ij}(\lambda_j - \lambda_i) &= \max_k p_{ik}(\lambda_i - \lambda_k) \text{ if } \kappa_{ij} > 0 \qquad (25)\\
q_{ij}(\lambda_j - \lambda_i) &= \max_k q_{kj}(\lambda_j - \lambda_k) \text{ if } \tau_{ij} > 0. \qquad (26)
\end{aligned}
$$

Equation (26) and equation (23) implies that $v_{ij} = v_{kj} = v_j$ in all submarkets where firm j is active. Finally, the value functions for the adjoint variables are given by (in steady state)

$$
\begin{aligned}
(r+s+\delta)\lambda_j &= y_j + \beta \max_{k>j} p_{jk}(\lambda_k - \lambda_j) + (s+\delta)\lambda_u \qquad (27)\\
(r+\delta)A &= \sum_{j=1}^{n} \alpha_j [(1-\beta)v_j \max_k q_{jk}(\lambda_k - \lambda_j) - \frac{v_1^2}{2c}]
\end{aligned}
$$

It follows that the first order conditions of the planner is exactly equal to the market solution. More than that, the maximization problem for the controls is exactly equal to the maximization problem of the firm. Thus, the planner's solution and the decentralized solution is the same.

11.5 Proof of proposition 3

a) Let $j > i$, and suppose a firm of type j advertise a wage W^l with job finding rate q^l, while the firm of type i advertise a wage W^h with job finding rate q^h. Profit maximization then implies that

$$\begin{aligned} q^h(M_i - W^h) &\geq q^l(M_i - W^l) \\ q^l(M_j - W^l) &\geq q^h(M_j - W^h) \end{aligned}$$

Combining the two gives

$$q^h(M_i - W^h) - q^h(M_j - W^h) \geq q^l(M_i - W^l) - q^l(M_j - W^h)$$

or

$$(q^h - q^l)(M_i - M_j) \geq 0$$

Since $M_i < M_j$, this is a contradiction.

From lemma 1 we know that $f_i(W) = f(W)$ at the interval $[W^{i-1}, W^i]$. Furthermore, from the proof of lemma 2 we know that W^i cannot be an equilibrium point. It follows that a worker of type j always searches for higher wages than a worker of type $i < j$.

b) Suppose to the contrary that I_l has an element, say i, that is strictly greater than one element in I_k, say j. From a) it follows that worker i searches for strictly higher wages than worker j. Hence firm l advertises a wage that is strictly higher than a wage advertising by firm k, $l < k$. We know from a) that this is a contradiction.

11.6 Proof of proposition 12

Consider a situation where K is arbitrarily high. For firms to be willing to enter, it follows that Π_2 must be arbitrarily high. However, this can only be the case if the arrival rate of jobs in the market the firm searches for is arbitrarily high, i.e., k must be arbitrarily small.

It follows that p_{01} is arbitrarily low, and thus also that N_1 is arbitrarily low. Suppose all high-type firms were searching for employed 27workers. Since the ratio of high-to low type firms is bounded, the labor market tightness in this market would be bounded. Hence the firms could not obtain an arbitrarily high profit, and we cannot be in equilibrium. In the unemployed submarkets, the labor market tightness by contrast is arbitrarily small and the firms get an arbitrarily high profit.

Suppose then that for some K', there exists a pure job ladder. We want to show that then there is a pure job ladder for all $K < K'$. First we keep the number of vacancies per firm constant. By a revealed preference type of argument one cans show that k is decreasing in K.

We first want to show that $el_k p_{01} > el_k p_{02}$. First note that

$$\begin{aligned} el_k p_{01} &= el_k(\frac{\alpha_1 k}{u})^{1-\beta} = (1-\beta)(1 - el_k u) \\ el_k p_{12} &= el_k(\frac{\alpha_2 k}{N_1})^{1-\beta} = (1-\beta)(1 - el_k N_1) \end{aligned}$$

Now

$$N_1 = \frac{p_{01}u}{s+\delta+p_{12}} \quad (28)$$
$$u = \frac{s+\delta}{s+\delta+p_{01}} \quad (29)$$

For given stocks u and N_1, $el_k p_{01} = el_k p_{12}$. From (28) and (29) it then follows that $el_k N_1 > el_k u$ (since $p_{01}u = (s+\delta)(1-u)$ is increasing in k). It thus follows that $el_k p_{01} > el_k p_{12}$.

Note that $el_k(\lambda_2 - \lambda_0) < el_k(\lambda_1 - \lambda_0)$ (since λ_1 is increasing in k). From (25) it follows that $el_k p_{02} > el_k p01$. From (27) it follows that we can write

$$\begin{aligned} \lambda_2 - \lambda_0 &= \frac{y_2 - y_0}{r+s+(1-\beta)p_{02}} \\ \lambda_2 - \lambda_1 &= \frac{y_2 - y_1}{r+s+(1-\beta)p_{12}} \end{aligned}$$

Taking elasticities give

$$\begin{aligned} el_k(\lambda_2 - \lambda_0) &= el_p \frac{y_2 - y_0}{r+s+(1-\beta)p_{02}} el_k p_{02} \\ el_k(\lambda_2 - \lambda_1) &= el_p \frac{y_2 - y_1}{r+s+(1-\beta)p_{12}} el_k p_{12} \end{aligned}$$

Now $el_x \frac{1}{a+x} = -\frac{x}{a+x}$ (for any constant a) which is decreasing in k. Furthermore, we have seen that $el_k p_{02} > el_k p_{12}$. It thus follows that

$$el_k(\lambda_2 - \lambda_1) > el_k(\lambda_2 - \lambda_0)$$

Finally, from (26) it follows that the result holds if

$$el_k q_{02} + el_k(\lambda_2 - \lambda_0) < el_k q_{12} + el_k(\lambda_1 - \lambda_0)$$

Since $el_k p_{02} > el_k p_{12}$ it follows that $el_k p_{02} < el_k p_{12}$, and we have just shown that $el_k(\lambda_2 - \lambda_1) > el_k(\lambda_2 - \lambda_0)$. The result thus follows.

11.7 Proof of claim related to α^*

a) Sketch of proof: For any number $\varepsilon > 0$. Consider a high-type firm that sets $w = y_1 + \varepsilon$. As $\alpha_0 \to 1$, the arrival rate of workers to this firms goes to infinity, independently of which wage $w \in (y_1, y_2)$ the other high-type firms choose. Thus profits go to infinity. If a high-type firm searches for unemployed workers, the arrival rate of workers to the firm will be bounded, and hence also profit. The claim thus follows. By a similar argument, it also follows that at least some high-type firms searches for employed workers as long as $\alpha > 0$.

b) We want to show the following claim: For a given number of firms k, there exists a unique α^* with the following property: If $\alpha > \alpha^*$ there exists a pure job ladder. If $\alpha < \alpha^*$

some high-type firms search for unemployed workers. We start by assuming that the number of vacancies per firm is constant.

Consider first the case where $\alpha \to 1$. Note that λ_{12} is limited above. We want to show that $\lim_{\alpha\to 1} q_{12} = \infty$. Suppose not, and suppose instead that q is bounded by $\overline{q}$. Since λ_{12} is limited above by $\overline{\lambda} = y_2/(r+s+\delta)$ it follows that v is limited above by $\frac{\overline{\lambda}\overline{q}}{c}$.

Let $\overline{N_1}$ denote the value of N_1 in the limit as $\alpha \to 1$. Clearly $\overline{N_1} > 0$ and $rM_0 > z$. It follows that

$$\begin{aligned}\lim_{\alpha\to 1} q_{12} &= \lim_{\alpha\to 1} A[\frac{(1-\alpha)kv}{N_1}]^{-\beta} \\ &\leq \lim_{\alpha\to 1} A[\frac{(1-\alpha)k\overline{v}}{N_1}]^{-\beta} \\ &= \infty\end{aligned}$$

Hence q cannot be limited above. But then it follows that the profit of searching for employed workers goes to infinity as α goes to zero.

Consider then the profitability of a high-type firm searching for unemployed workers. Since λ_{02} is bounded above by $\overline{\lambda} = y_2/(r+s+\delta)$, the profit can only goes to infinity if q_{12} does. Suppose it does. Then workers applying to this job has a job finding rate of $p=0$ and thus receives $rM_0 = z$. However, the workers would then prefer to search for the low-type firm and we cannot be in equilibrium. It follows that it is more profitable to search for employed than for unemployed workers if α is sufficiently close to 1.

Suppose then $a \to 0$. It follows that $N_1 \to 0$. We want to show that the proportion of high-type firms searching for employed workers goes to 0. Suppose not, and suppose the share is bounded below by $\tau^{\min} > 0$. Suppose that in the limit, $v_{12} > 0$. It follows that

$$\begin{aligned}\lim_{\alpha\to 0} q_{12} &= \lim_{\alpha\to 1} A[\frac{(1-\alpha)k\tau v_{12}}{N_1}]^{-\beta} \\ &\leq \lim_{\alpha\to 1} A[\frac{(1-\alpha)k\tau^{\min} v_{12}}{N_1}]^{-\beta} = 0\end{aligned}$$

Note also that $v_{12} = 0$ if and only if $q_{12}\lambda_{12} = 0$. Thus both if $v_{12} = 0$ in the limit and when it is not the assumption that $\tau^{\min} > 0$ is inconsistent with (26).

Finally we want to show that for any $\alpha > 0$, $\tau > 0$. Suppose not. Then there exists an $\alpha > 0$ such that $\tau = 0$. If (26) is satisfied we must have that $v_{12} < \infty$. But then it follows that $q_{12} = \infty$, hence (26) cannot be satisfied. Again we have derived a contradiction.

Finally we want to show that there exists a unique α^* as described above. That there exists a α^* such that (26) is satisfied with equality for $\tau = 1$ follows from continuity and the results just laid out. What is left is to show that this α^* is unique. To this end it is sufficient to show that if (26) is satisfied with equality for $\tau = 1$, then an decrease in α implies that the right-hand side of (26) is strictly greater than the left-hand side for $\tau = 1$.

In what follows we will work with α_2 rather than α, the fraction of high-type firms. We want to show that an increase in α_2 for a given k, and given that $\tau = 1$ implies that searching for unemployed workers become strictly more profitable than searching for employed workers. (from 26)

$$q_{12}(\lambda_2 - \lambda_1) < q_{02}(\lambda_2 - \lambda_0) \tag{30}$$

for α_2' marginally greater than α_2^*.

First note that an increase in α increases λ_1. Suppose λ_0 decreases. From p_{01} (27) it follows that decreases. From (27) it also follows that

$$\lambda_2 - \lambda_0 = \frac{y_2 - r\lambda_0}{r + s + \delta}$$

which thus increases. From (25) and the fact that p_{01} decreases, it follows that p_{02} decreases. From the matching function it follows that that q_{02} increases. Thus the right-hand side of 30 increases. An increase in α_2 increases p_2, and from (**??**) it follows that $(\lambda_2 - \lambda_1)$ decreases and q_{12} decreases. Thus the left-hand side of 30 decreases. Hence we are done in this case.

Suppose then that λ_0 is increasing in α_2 (which indeed seems likely). In what follows we re-scale the model by setting $z = 0$. Clearly this can be done without loss of generality, as the maximization problem is unchanged if all flows z, y_1, y_2 are reduced equally much. It follows that we can write

$$\lambda_0 = \frac{p_{01}}{r + p_{01}}\lambda_1$$

Thus, from (27)

$$\lambda_0(1 - \frac{s + \delta}{r + s + \delta})\frac{r + p_{01}}{p_{01}} = \frac{y_1 + p_{12}(\lambda_2 - \lambda_1)}{r + s + \delta}$$

Taking elasticities wrt α_2 gives

$$el\lambda_0 + X < elp_{12} + el(\lambda_2 - \lambda_1)$$

where $X = el\frac{r+p_{01}}{p_{01}} > 0$. An increase in α follows that

$$el\lambda_0 < elp_{12} + el(\lambda_2 - \lambda_1)$$

From (27) it follows that $r\lambda_0 = p_{02}(\lambda_2 - \lambda_0)$. Taking elasticities and using the above equation give

$$elp_{02} + el(\lambda_2 - \lambda_0) < elp_{12} + el(\lambda_2 - \lambda_1)$$

or

$$elp_{02} < elp_{12} + el(\lambda_2 - \lambda_1) - el(\lambda_2 - \lambda_0) \tag{31}$$

As $\frac{\delta\lambda_1}{\delta\alpha_2} > \frac{\delta\lambda_0}{\delta\alpha_2}$ and $\lambda_2 - \lambda_1 < \lambda_2 - \lambda_0$ it follows that $0 > el(\lambda_2 - \lambda_0) > el(\lambda_2 - \lambda_1)$. From (31) it thus follows that $elp_{02} < elp_{12}$ and thus that $elq_{02} > elq_{12}$. Together this implies that (30) is satisfied.

11.8 Proof of lemma 3.

Consider a firm of type j that searches for workers employed at level i having an NPV wage of $\widetilde{M}_i$. Furthermore, assume that the workers in that firm searches for jobs in firms of type k which pay W_{jk} and which they obtain at rate p_{jk} (still we supress the dependence of k in the expressions below). Since the agents are risk neutral and use the same discount factor, the timing of the payment to the worker is irrelevant, and for notational convenicence we assume that the worker is paid the entire NPV wage W_{ij} upfront. The net present value of profit of a firm with initial labor stock N_0 can be written as (we index state variables by j and choice variables and the adjungated variable by ij)

$$\begin{aligned}
\Pi_{ij} &= \int_0^\infty [N_j[y_j + sM_0 + p_{jk}W_{jk}] - \frac{\widetilde{v}_{ij}N_j}{2} - N_j\widetilde{v}_{ij}W_{ij}q(W_{ij}))]e^{-(r+\delta)t}dt \\
s.t.N_j(0) &= N_0 \\
\dot{N}_j &= \widetilde{v}_{ij}q(W_{ij})N_j - (s + p_{jk})N_j
\end{aligned}$$

where $\widetilde{v}_{ij} = v_{ij}/N_j$ and where $q(W_{ij}) \equiv q(p_i(W_{ij}, M_i)$, The Hamiltonian reads

$$H = N_j[y_j + sM_0 + p_{jk}W_{jk} - N_j\widetilde{v}_{ij}W_{ij}q(p(W_{ij})))] + \lambda_{ij}[\widetilde{v}_{ij}q(W_{ij})N_j - (\delta + p_{jk})N_j].$$

First order conditions for W reads (after some manipulation)

$$\begin{aligned}
W_{ij} &= (\lambda_{ij} - M_i)\beta \\
\widetilde{v}_{ij} &= (\lambda_{ij} - M_i)(1-\beta)q \\
(r+\delta)\lambda_{ij} &= y_j + sM_0 + p_{jk}(W_{jk} - \lambda_{ij}) + [(\lambda - W)\widetilde{v}^* q(W^*) - \frac{\widetilde{v}_{ij}}{2c}]
\end{aligned}$$

Which gives us the conditions in the lemma (with $\widetilde{M}_{ij}$ substituted in for λ_{ij}). (Have to say something about the max, that is postponed).

11.9 Proof of proposition 10

In order to show that the problem is well defined it is sufficient to show that $\widetilde{M}_{ij}$ is bounded for all $\widetilde{v_{ij}}$ and all W_{ij}. We will show that this is always the case for sufficiently high search costs, i.e., for sufficiently low values of c. By assumption, $(r+s)M_i > y_i + s\widetilde{M}_0$, hence $q_{ij}(f(W_{ij}, M))$ is finite for any finite W_{ij}. Define

$$\overline{W}_j = \frac{y_j + \max_{k>j} p_{jk}W_{jk} + (s+\delta)\widetilde{M}_0}{r+s+\delta}$$

and define $\overline{q}_j = q_{ij}(\overline{W}_j)$. Note that $\overline{W}_j$ strictly exceeds the NPV of the income a worker generates, hence by paying $\overline{W}_j$ to all the workers the firm surely obtains a negative profit. Rewrite (15) to

$$
\begin{aligned}
\widetilde{M}_{ij} &= \frac{y_j + \max_k p_{jk}(W_{jk} - \widetilde{M}_{ij}) + (s+\delta)\widetilde{M}_0 - \widetilde{v}_{ij}^* q_{ij}^*(1-\beta)\widetilde{M}_i - \widetilde{v}_{ij}^{*2}/2c]}{r+s+\delta - \widetilde{v}_{ij}^* q_{ij}^*(1-\beta)} \\
&< \frac{y_j + \max_k p_{jk}(W_{jk} - \widetilde{M}_{ij}) + (s+\delta)\widetilde{M}_0 - \widetilde{v}_{ij}^* \overline{q}_j(1-\beta)\widetilde{M}_i) - \widetilde{v}_{ij}^{*2}/2c}{r+s+\delta - \widetilde{v}_{ij}^* \overline{q}(1-\beta)}
\end{aligned}
$$

In the second expression we have substituted in $\overline{q}_j > q_{ij}$. For sufficiently high values of $\widetilde{v}_{ij}$, the denominator is negative. Define $\widetilde{v}^0$ as the value of $\widetilde{v}_{ij}$ that makes the denominator equal to zero. It follows that

$$\widetilde{v}^0 = \frac{r+s+\delta}{(1-\beta)\overline{q}}$$

It is sufficient to show that the nominator is negative for $v = \widetilde{v}^0$, i.e., that

$$y_j + \max_k p_{jk}(W_{jk} - \widetilde{M}_{ij}) + (s+\delta)\widetilde{M}_0 - \widetilde{v}^0 \overline{q}(1-\beta)\widetilde{M}_i) - (\widetilde{v}^0)^2/2c < 0$$

which is trivially satisfied for sufficiently low values of c.

Then we turn to uniqueness. Note that if W_{ij} and v_{ij} satisfies (16) and (17), then this is a local maximum for $\widetilde{M}_{ij}$. Suppose W'_{ij} and v'_{ij} constitute a local but not a global maximum for joint income, and let $\widetilde{M}'_{ij}$ denote the corresponding joint income. Then it follows that for $\widetilde{M}'_{ij}$, the first order conditions (16) and (17) have at least two solutions. However, given $\widetilde{M}'_{ij}$ the firm's maximization problem is exactly as in the previous section

11.10 Computation of the General Equilibrium

11.11

To solve the model one needs to set following 10 parameters: r, s, δ, y_1 and y_2, z, β c, k, α. In addition, the matching function we use is cobb douglas with share parameter β and with constant A.

The procedure to compute the equilibrium is as follows. First, the procedure tries to solve for the model with three submarkets. If this fails the procedure switches to the pure job ladder equilibrium. The solution is basically computed in four steps. The first steps (step i) solves for the asset equations in the general model, the second steps (step ii) computes τ, the proportion of good firms that hire directly from the unemployment pool and the final steps solves for the stock. Step three (step iii) is reached only if the proportion of firms that hires directly from the unemployed is less than one. In case this proportion τ is greater than one, the procedure goes to the step four (step iv) and solves for the pure job ladder equilibrium.

11.11.1 Step i): Solving for the Asset values in the general model

The procedure starts from assigning an arbitrary initial guess value of $M_1 = M_1^{'}$ and $rM_0 = rM_0^{'}$. Given the initial guess, one can compute recursively $M_2^{'}, p_{01}^{'}, p_{02}^{'}, p_{12}^{'}$

$$
\begin{aligned}
M_2^{'} &= \frac{y_2 + (r+\delta)M_0^{'}}{r+\delta+s}; \qquad \text{using } M_2 = \frac{y_2 + (r+\delta)M_0}{r+\delta+s} \\
p_{12}^{'} &= \frac{(r+\delta+s)M_1^{'} - y_1 + rM_0^{'}}{\beta(M_2^{'} - M_1^{'})} \qquad \text{using } M_1 = \frac{y_1 + (s+\delta)M_0 + p_{12}\beta(M_2 - M_1)}{r+\delta+s} \\
p_{01}^{'} &= \frac{rM_0^{'} - z}{\beta(M_1^{'} - M_0^{'})} \qquad \text{using } rM_0 = z + \beta p_{01}(M_1 - M_0) \\
p_{02}^{'} &= \frac{rM_0^{'} - z}{\beta(M_2^{'} - M_0^{'})} \qquad \text{using } rM_0 = z + \beta p_{02}(M_2 - M_0)
\end{aligned}
$$

Given these values we define the function $d(M_1^{'}, M_0^{'})$ as the difference in profits across high type firms so that

$$
\begin{aligned}
d(\Pi) &= \Pi_{12}() - \Pi_{02}() \\
d &= \frac{[(M_2^{'} - M_0^{'})(1-\beta)p_{12}^{'}]^2}{2} - \frac{[(M_2^{'} - M_1^{'})(1-\beta)p_{02}^{'-\frac{\beta}{1-\beta}}]^2}{2}
\end{aligned}
$$

For given value of $M_0^{'}$, the procedure updates the value of $M_1^{'}$ so that

$$M_1^{''} = M_1^{'} - \lambda d(\Pi)$$

where $\lambda > 0$ is an adjustment parameter. In other, words we reduce the value $M_1^{''}$ as long as $d()$ is positive. Given $M_1^{''}$ and holding fixed $M_0^{'}$ update $M_2^{''}, p_{01}^{''}, p_{02}^{''}, p_{12}^{''}$ using $M_2^{''}$ and proceed further until

$$d(\Pi) \simeq 0$$

Given M'' expected profits at entry are

$$dE\Pi = \alpha\Pi_{01} + \Pi_{12} - k$$

and update the value of $M_0^{'}$ so that

$$M_0^{''} = M_0^{'} + \lambda_1 dE\Pi$$

Given $M_0^{''}$, update the asset values and redo the procedure for finding $d(\Pi) \simeq 0$, and calculating $M_0^{'''}$. The equilibrium in the first step is obtained for a couple M_1^{*} and M_0^{*} so that

$$
\begin{aligned}
d(\Pi) &\simeq 0 \\
dE\Pi(\Pi) &\simeq 0
\end{aligned}
$$

11.11.2 Step ii): Obtaining the fraction of firms τ that hire directly from the employed

The first step of the model has solved for M_1, rM_0 $M, p_{01}, p_{12}, p_{02}$. The rest of the equations are obtained from

$$
\begin{aligned}
(p_{01})^{\frac{1}{1-\beta}} &= \frac{(1-\alpha)kv_1(0)}{k_{01}n_0} \\
(p_{12})^{\frac{1}{1-\beta}} &= \frac{\tau\alpha kv_2(1)}{n_1}^{1-\beta} \\
(p_{02})^{\frac{1}{1-\beta}} &= \frac{(1-\tau)\alpha kv_2(0)}{(1-k_{01})n_0}
\end{aligned}
$$

and the flows conditions

$$
\begin{aligned}
p_{02}k_{02}n_0 + p_{12}k_{01}n_1 &= (\delta+s)n_2 \\
p_{01}k_{01}n_0 &= (\delta+s+p_{12})n_1 \\
n_0+n_1+n_2 &= 1 \\
k_{01}+k_{02} &= 1
\end{aligned}
$$

Since $\frac{n_1}{k_{01}n_0} = \frac{p_{01}}{\delta+s+p_{12}}$ dividing the equation for $\theta_{01} = (p_{01})^{\frac{1}{1-\beta}}$ and $\theta_{12} = (p_{12})^{\frac{1}{1-\beta}}$one obtains immediately and expression for τ as

$$
\tau^* = \frac{\theta_{12}}{\theta_{01}} \frac{\alpha v_1(0)}{(1-\alpha)v_2(1)} \frac{p_{01}}{\delta+s+p_{12}}
$$

where $v_i = c(M_i - M_0)_i q(p_i)$ $i = 1, 2$. If $\tau^* < 1$ the equilibrium with all submarket is consistent and steps iii can be completed. Conversely, if $\tau^* > 1$ the routine solves for the pure job ladder equilibrium.

11.12 Step iii): Obtaining stocks in the general model

Assume $k = k'$ and $k_{01} = k'_{01}$ and obtain recursively

$$
\begin{aligned}
n_0^{'} &= \frac{\delta+s}{\delta+s+p_{01}k'_{01}+p_{12}(1-k'_{01})} \\
n_1^{'} &= \frac{p_{01}k_{01}n_0}{\delta+s+p_{12}} \\
n_2^{'} &= 1-n_0^{'}-n_1^{'}
\end{aligned}
$$

Given these values obtain the function dk as

$$
dk = (1-k_{01})\theta(p_{02})n_0 - (1-\tau)(1-\alpha)k'v_2(0)
$$

and update

$$k'' = k' + \lambda dk$$

Continue the procedure as long as k'' is such that

$$dk \simeq 0$$

With the completion of step iii the general equilibrium is fully solved.

Given k'' obtain the function dk

$$dk = k - \frac{n_1\theta_{12}(p_{12})}{\tau * (1-\alpha) * v_2(1)}$$

and update the value of k' so that

$$k'' = k' - \lambda_1 dk$$

Given k'' , update the stocks and redo the procedure for finding $d(k) \simeq 0$, and calculating k'''. The equilibrium in the first step is obtained for a couple k^* and k^* so that

$$\begin{aligned} d(k) &\simeq 0 \\ dE\Pi(k) &\simeq 0 \end{aligned}$$

11.12.1 Step iv. Solve for the pure job ladder equilibrium

The step iv is reached only if the routine finds a value of $\tau > 1$ in step ii. The procedure starts from an arbitrary initial guess value of $M_1 = M_1'$ and $rM_0 = rM_0'$. Given the initial guess, it computes recursively $M_2', p_{01}', p_{02}', p_{12}'$

$$\begin{aligned}
M_2' &= \frac{y_2 + (r+\delta)M_0'}{r+\delta+s}; \qquad \text{using } M_2 = \frac{y_2 + (r+\delta)M_0}{r+\delta+s} \\
p_{12}' &= \frac{(r+\delta+s)M_1' - y_1 + rM_0'}{\beta(M_2' - M_1')} \qquad \text{using } M_1 = \frac{y_1 + (s+\delta)M_0 + p_{12}\beta(M_2 - M_1)}{r+\delta+s} \\
p_{01}' &= \frac{rM_0' - z}{\beta(M_1' - M_0')} \qquad \text{using } rM_0 = z + \beta p_{01}(M_1 - M_0) \\
n_0' &= \frac{\delta+s}{\delta+s+p_{01}'} \\
n_1' &= \frac{p_{01}'(\delta+s)}{(\delta+s+p_{01}')(\delta+s+p_{12}')} \\
n_2' &= 1 - n_1' - u_1' \\
k' &= \frac{n_1'\theta_2(p_{12}')}{(1-\alpha)v_2(1)}
\end{aligned}$$

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Table 1: Baseline Calibration with three submarkets

Parameter	Notation	Value
Pure Discount Rate	r	0.010
Separation Rate	s	0.040
Firm Bankruptcy Rate	δ	0.020
Bargaining Share	β	0.500
entry cost	k	5.000
low type proportion	α	0.1550
high type productivity	y_1	1.000
low type productivity	y_2	1.150
unemployed income	z	0.550
search cost parameter	c	0.150
Maximum vacancies per firm	$v*$	0.150
matching function parameter	A	1.000
matching function elasticity	β	0.500
Equilibrium Values		
Joint Income 1	rM_1	1.0089
Joint Income 2	rM_2	1.0184
unemployment flow value	rU	0.9965
unempl. job finding rate in low type	p_{01}	0.7177
on the job finding rate	p_{12}	0.1762
unempl. job finding rate directly to high type	p_{02}	0.4072
Equilibrium Quantities		
Unemployment	n_0	0.0772
Employment in Low productivity type	n_1	0.2343
Employment in High productivity type	n_2	0.6885
Proportion of unemployed in submkt 01	k_{01}	0.9990
Number of Firms	f	0.3133
Proportion of high type firms in submarket 12	τ	0.9982
Worker Flows		
Unemployment Flows	$n_0 * (p_{01} + p_{02})$	0.0554
Job to Job Flows	$n_1 * p_{12}$	0.0413
Firm Size, PDV Wages and Profits		
Profits in submarket 01	Π_{01}	3.5846
Profits in submarket 02	Π_{02}	12.7165
Profits in submarket 12	Π_{12}	12.7165
Firm Size in submarket 01	N_{01}	0.8851
Firm Size in submarket 02	N_{02}	6.1402
Firm Size in submarket 12	N_{12}	14.1918
Wages in submarket 01	rW_{01}	1.0027
Wages in submarket 02	rW_{02}	1.0074
Wages in submarket 12	rW_{12}	1.0107
Source: Authors' calculation		

Table 2: Baseline Calibration with two submarkets

Parameter	Notation	Value
Pure Discount Rate	r	0.010
Separation Rate	s	0.040
Firm Bankruptcy Rate	δ	0.020
Bargaining Share	β	0.500
entry cost	k	5.000
low type proportion	α	0.1540
high type productivity	y_1	1.000
low type productivity	y_2	1.150
unemployed income	z	0.550
search cost parameter	c	0.150
Maximum vacancies per firm	$v*$	0.150
matching function parameter	A	1.000
matching function elasticity	β	0.500
Equilibrium Values		
Joint Income 1	rM_1	1.0088
Joint Income 2	rM_2	1.0183
unemployment flow value	rU	0.9964
unempl. job finding rate in low type	p_{01}	0.7180
on the job finding rate	p_{12}	0.1754
unempl. job finding rate directly to high type	p_{02}	0.0000
Equilibrium Quantities		
Unemployment	n_0	0.0771
Employment in Low productivity type	n_1	0.2352
Employment in High productivity type	n_2	0.6877
Proportion of unemployed in submkt 01	k_{01}	1.0000
Number of Firms	f	0.3133
Proportion of high type firms in submarket 12	τ	1.0000
Worker Flows		
Unemployment Flows	$n_0 * (p_{01} + p_{02})$	0.0554
Job to Job Flows	$n_1 * p_{12}$	0.0413
Firm Size, PDV Wages and Profits		
Profits in submarket 01	Π_{01}	3.5790
Profits in submarket 12	Π_{12}	12.8061
Firm Size in submarket 01	N_{01}	0.8874
Firm Size in submarket 12	N_{12}	14.2524
Wages in submarket 01	rW_{01}	1.0026
Wages in submarket 12	rW_{12}	1.0107
Source: Authors' calculation		