

The Social Cost of Near-Rational Investment:^{*}

Why we should worry about volatile stock markets

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Abstract

Excess volatility in stock returns may arise and drastically reduce welfare even if the stock market appears to be efficient and disconnected from the real economy. We solve a macroeconomic model in which information about fundamentals is dispersed and agents make small, correlated errors around their optimal investment policies. As information aggregates in the market, these errors amplify and result in large amounts of excess volatility in stock returns. The increase in volatility makes holding stocks unattractive and distorts the long-run level of capital accumulation. Through its effect on capital accumulation excess volatility causes costly (first-order) distortions in the long-run level of consumption.

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1 Introduction

An important function of stock prices is to aggregate information that is dispersed across market participants. Market prices reflect the information held by all investors and direct resources to their most efficient use. If stock prices reflect information, investors have an incentive to learn from the equilibrium price and to update their expectations accordingly. *Anything* that moves the market price has an impact on the expectations held by *all* market participants. We explore the general equilibrium implications of this dynamic in a world in which people are less than perfect - they make small mistakes when investing their wealth.

We solve a real business cycle model in which information is dispersed across market participants. Households observe the equilibrium stock price as well as a private signal about aggregate productivity in the next period. Based on this information they trade in stocks and bonds. As households place their trades, the equilibrium stock price aggregates the information in the market and becomes informative about future productivity.

The first main insight from our model is that if information is dispersed, large amounts of excess volatility in stock returns may arise in equilibrium although the individual behavior of all economic actors is almost fully characterized by the rational model. Because households optimize when they decide how to allocate their portfolios between stocks and bonds, small (potentially infinitesimal) deviations from their optimal policy have little impact on their individual welfare. However, if these deviations are correlated across households (say households are on average just a little bit too optimistic in some states of the world and a little bit too pessimistic in others), they affect the equilibrium price and hence may have a large external effect on the equilibrium expectations held by all market participants. Households' trades determine the equilibrium stock price and all households inform on this equilibrium price when forming their expectations. The average deviation that households make from their optimal policy amplifies as information aggregates in the market and leads to a much larger deviation in the market price of capital. If information is dispersed, small errors in households' investment decisions may therefore result in large amounts of excess volatility in equilibrium stock returns.

The second main insight from our model is that the equilibrium variance of stock returns influences the amount of capital that is accumulated in the economy. If the equilibrium variance of stock returns rises, stocks become a riskier asset to hold and households demand a higher risk premium for holding stocks rather than bonds. This risk premium determines the marginal product of capital in the long run (at the stochastic steady state). Changes in the variance of stock returns thus change the *level* of capital accumulation, output and consumption. Excess volatility in stock returns may therefore cause large aggregate welfare losses by distorting the level of consumption in the stochastic steady state. Interestingly, this is true even if the capital stock responds very little to any given change in stock returns and there is an observed disconnect

between the stock market and the real economy.

The combination of these two insights produces a surprising result: A model in which excess volatility in stock returns causes large aggregate welfare losses although there are no opportunities for earning abnormal returns in financial markets and all households are arbitrarily close to their rational behavior.

The Model Our model is a standard real business cycle model in which a consumption good is produced from capital and labor. Households supply labor to a representative firm and invest their wealth by trading claims to capital ('stocks') and bonds. The consumption good can be transformed into capital, and vice versa, by incurring a convex adjustment cost. The accumulation of capital is thus governed by its price relative to the consumption good (Tobin's Q). The only source of real risk in the economy are shocks to total factor productivity. We extend this standard setup by assuming that each household receives a private signal about productivity in the next period and solve for equilibrium expectations.

We first analyze the case in which all households are perfectly accurate in making their investment decisions (the rational expectations equilibrium). In this case, welfare increases monotonically with the amount of information contained in the equilibrium stock price: The more households can infer about future productivity, the less risk is associated with investing in the stock market and the higher is welfare. If the stock price is completely uninformative about future productivity the rational expectations equilibrium coincides with the standard real business cycle model.

We then show that the rational expectations equilibrium is very unstable in the sense that the economy behaves very differently if households make small, correlated errors around their optimal investment policy. We refer to this as the "near-rational expectations equilibrium" to emphasize that the expected utility cost accruing to an individual household due to deviations from its optimal policy must be below a specific threshold, say 0.1% of consumption.

When households form their expectations about tomorrow's productivity they inform on the equilibrium stock price. However, they cannot infer whether a given change in the stock price is attributable to information about productivity or to near-rational errors made by their peers. The average near-rational error thus feeds from the stock price into households' expectations and back into the stock price. The more dispersed information is across households the stronger is this feedback effect, because households rely more heavily on the stock price when they have less to learn from their private signal. In particular, we show that a given level of excess volatility in stock returns can be sustained by arbitrarily small near-rational errors if information is sufficiently disperse.¹

¹We define excess volatility as the difference in the standard deviation of stock returns in the rational vs the near-rational expectations equilibrium.

We remain agnostic about the exact mechanism prompting households to make small correlated errors in their investment decisions. One interpretation is that households falsely believe that an uninformative public signal contains a tiny amount of information about future productivity (Dumas et al. (2006)). Alternatively, we may think of a world in which there are two computer programs for pricing stocks; a free program which prices stocks with a small error and another version which is available at a menu cost and prices stocks accurately.² The point is that the private gain from avoiding near-rational errors is low, while the social gain from avoiding the resulting excess volatility in stock returns may be large.

This is easiest to see for the example of a small open economy in which households can borrow and lend at an exogenous international interest rate. Risk-averse investors demand a higher risk premium for holding stocks when returns are excessively volatile. The marginal unit of capital installed must therefore yield a higher expected return in order to compensate investors for the additional risk they are bearing. It follows that excess volatility depresses the equilibrium level of capital installed at the stochastic steady state and consequently lowers output and consumption.³ Moreover, returns to capital rise while wages fall.⁴

Because welfare losses are driven mainly by a distortion in the stochastic steady state rather than an intertemporal misallocation of capital, excess volatility in stock returns may cause large welfare losses even if the capital stock responds little to any given change in stock returns. In our model, the elasticity of the capital stock with respect to stock returns is therefore completely uninformative about the welfare consequences of excess volatility in stock returns. This contrasts with a widely held view among macroeconomists that pathologies in the stock market may not matter for the real economy if there is an observed disconnect between stock returns and changes in the capital stock (Morck et al. (1990)).

Calibration We quantify the aggregate welfare losses caused by excess volatility as the percentage rise in consumption that would make households indifferent between remaining in an equilibrium in which stock prices are excessively volatile and transitioning to the stochastic steady state of an economy in which all households behave fully rationally until the end of time. We calibrate our model to match the standard deviation of stock returns observed in the data. Our baseline results are for the case of a small open economy. If we attribute half of the observed volatility of stock returns to mispricings, welfare losses due to excess volatility amount

² A large literature in behavioral finance has developed mechanisms that prompt households to make correlated mistakes in their investment decisions. Some examples are Odean (1998); Odean (1999); Daniel, Hirshleifer, and Subrahmanyam (2001); Barberis, Shleifer, and Vishny (1998); Bikhchandani, Hirshleifer, and Welch (1998); Hong and Stein (1999) and Allen and Gale (2001).

³ The stochastic steady-state is the level of capital, bonds, and prices at which those quantities do not change in expectation.

⁴ In a closed economy the fact remains that any distortion in the level of output and consumption is associated with first-order welfare losses. However, the effects are slightly more complicated (due to the precautionary savings motive), such that excess volatility in stock returns may drive consumption at the stochastic steady state up or down.

to 4.1% of consumption. Out of this total loss, 3.6 percentage points are directly attributable to the depression in the steady state capital stock caused by higher risk premia. The welfare losses caused by changes in average capital adjustment costs and the variability of consumption together account only for the remaining 0.5 percentage points. The results for a closed economy are quantitatively and qualitatively similar.

Related Literature This paper is to our knowledge the first to address welfare effects of excess volatility in stock returns within a full-fledged dynamic stochastic general equilibrium model. In a related paper, Mertens (2009) derives policies which mitigate the welfare cost of excess volatility. He shows that the stabilization of asset prices enhances welfare and history-dependent policies may improve the information content of asset prices.

Our work relates to a literature that studies the welfare cost of excess volatility in stock returns, including Stein (1987) and Lansing (2008). Most closely related are DeLong, Shleifer, Summers, and Waldmann (1989) who analyze the general equilibrium effects of noise-trader risk in an overlapping generations model with endogenous capital accumulation. A large literature in macroeconomics and in corporate finance focuses on the sensitivity of firms' investment to a given mispricing in the stock market. Some representative papers in this area are Morck, Shleifer, and Vishny (1990); Blanchard, Rhee, and Summers (1993); Baker, Stein, and Wurgler (2003); Gilchrist, Himmelberg, and Huberman (2005); and Farhi and Panageas (2006).⁵ While most of these papers find that investment responds moderately to mispricings in the stock market, our model suggests that welfare losses due to excess volatility in stock returns may be large regardless of how responsive investment is to the stock market.

The notion of near-rationality is due to Akerlof and Yellen (1985) and Mankiw (1985). In their models near-rational behavior amplifies business cycles. Our application is closest to Cochrane (1989) and Chetty (2009) who use the utility cost of small deviations around an optimal policy to derive "economic standard errors". Other recent applications include Woodford (2005) and Dupor (2005).

The way in which we solve for equilibrium expectations is close to Hellwig (1980) and to an evolving strand of literature that studies the social value of information, including Morris and Shin (2002), Amador and Weill (2007), Angeletos et al. (2007), and Angeletos and La'O (2008).

A technical complication is that our model requires solving for equilibrium expectations under dispersed information in a non-linear (general equilibrium) framework. We are able to do so due to recent advances in computational economics. We follow the solution method in Mertens (2009) to solve for the equilibrium. This method builds on Judd (1998) and Judd and Guu (2000) in using a higher-order expansion in all state variables around the deterministic

⁵Also see Galeotti and Schiantarelli (1994); Polk and Sapienza (2003); Panageas (2005); and Chirinko and Schaller (2006)

steady state of the model with a nonlinear change of variables (Judd (2002)).⁶

In the main part of the paper we concentrate on the slightly more tractable small open economy version of the model (alternatively we may think of it as a closed economy in which households have access to a certain type of storage technology). After setting up the model we discuss equilibrium expectations and how excess volatility endogenously arises in the model. In section 4 we build intuition for the macroeconomic implications of excess volatility by presenting a simplified version of the model which allows us to show all the main results with pen and paper. In this simplified version of the model households consist of two specialized agents: a "capitalist" who has access to the stock and bond markets and a "worker" who provides labor services but is excluded from trading in financial markets. We then solve the full model computationally in section 6. Section 7 gives parallel results for the closed economy.

2 Setup of the Model

The model is a de-centralization of the standard Mendoza (1991) framework: A continuum of households work and trade in stocks and bonds. A representative firm produces a homogenous consumption good by renting capital and labor services from households. Total factor productivity is random in every period and the firm adjusts factor demand accordingly. An investment goods sector has the ability to transform units of the consumption good into units of capital, while incurring convex adjustment costs.⁷ All households and firms are price takers and plan for infinite horizons.

At the beginning of each period, households receive a private signal about productivity in the next period. Given this signal and their knowledge of prices and the state of the economy, they form expectations of future returns. Households make correlated near-rational mistakes when forming expectations about future productivity.

2.1 Economic Environment

Technology is characterized by a homogenous production function that uses capital, K_t , and labor, L as inputs

$$Y_t = e^{\eta_t} F(K_t, L), \quad (1)$$

where Y_t stands for output of the consumption good. Total factor productivity, η_t , is normally

⁶See Devereux and Sutherland (2006) and Tille and van Wincoop (2007) for other recent applications based on perturbation.

⁷The alternative to introducing an investment goods sector is to incorporate the investment decision into the firm's problem. The two modeling devices are equivalent as long as there are no frictions in contracting between management and shareholders.

distributed with a mean of $-\frac{1}{2}\sigma_\eta^2$ and a variance of σ_η^2 . The equation of motion of the capital stock is

$$K_{t+1} = K_t (1 - \delta) + I_t, \quad (2)$$

where I_t denotes aggregate investment and δ is the rate of depreciation. Furthermore, there are convex adjustment costs to capital,

$$AC = \frac{1}{2}\chi \frac{I_t^2}{K_t}, \quad (3)$$

where χ is a positive constant. There is costless trade in the consumption good at the world price, which we normalize to one. All households can borrow and lend abroad at rate r . Foreign direct investment and international contracts contingent on η are not permitted.

2.2 Households

There is continuum of identical households indexed by $i \in [0, 1]$. At the beginning of every period each household receives a private signal about tomorrow's real shock:

$$s_t(i) = \eta_{t+1} + \nu_t(i), \quad (4)$$

where $\nu_t(i)$ represents *i.i.d.* draws from a normal distribution with zero mean and variance σ_ν^2 . Given this information and their knowledge about the economy, households maximize lifetime utility by choosing an intertemporal allocation of consumption, $\{C_t(i)\}_{t=0}^\infty$, and by weighting their portfolios between stocks and bonds at every point in time, $\{\omega_t(i)\}_{t=0}^\infty$, where ω represents the share of equity in their portfolio. Formally, an individual household's problem is

$$\max_{\{C_t(i)\}_{t=0}^\infty, \{\omega_t(i)\}_{t=0}^\infty} U_t(i) = \mathcal{E}_{it} \left\{ \sum_{s=t}^\infty \beta^{s-t} \log(C_s(i)) \right\} \quad (5)$$

s.t.

$$W_{t+1}(i) = [(1 - \omega_t(i))(1 + r) + \omega_t(i)(1 + \tilde{r}_{t+1})](W_t(i) + w_t L - C_t(i)) \quad \forall t, \quad (6)$$

where $\mathcal{E}$ stands for the household i 's expectation operator, $W_t(i)$ stands for financial wealth of household i at time t and $\tilde{r}_{t+1}$ is the equilibrium return on stocks. We denote the market price of capital with Q_t and dividends with D_t :⁸

$$1 + \tilde{r}_{t+1} = \frac{Q_{t+1}(1 - \delta) + D_{t+1}}{Q_t}. \quad (7)$$

⁸Note that we implicitly assume here that stocks split proportionally to the percentage change in aggregate capital stock at the end of each period. The stock price is then always equal to the price of a claim to one unit of capital.

Finally, E_{it} denotes the rational expectations operator, conditional on all information available to household i at time t :

$$E_{it}(\cdot) = E(\cdot | Q_t, s_t(i), K_t, B_t, \eta_t). \quad (8)$$

The expectation operator $\mathcal{E}$ allows households to make small "mistakes" when forecasting future productivity. In particular, we assume that their expectation of tomorrow's productivity deviates from the rational expectation by a small error $\tilde{\epsilon}_t$:

$$E_{it}^b(1 + \eta_{t+1}) \equiv E_{it}(1 + \eta_{t+1}) e^{\tilde{\epsilon}_t}. \quad (9)$$

For simplicity we assume that all capitalists make the same small mistake. Alternatively, we may think of $\tilde{\epsilon}_t$ as the average mistake made by households trading in the stock market. The deviation caused by $\tilde{\epsilon}_t$ is zero in expectation and its variance, $\sigma_{\tilde{\epsilon}}^2$, is small enough such that the expected utility loss from making this mistake is below some threshold level.⁹ Our favorite interpretation of this error is that households observe an uninformative public signal and falsely believe that it contains a small amount of information about η_{t+1} (Dumas et al. (2006)). However, we may think of a number of other interpretations involving animal spirits, menu costs, behavioral biases, or even an evolutionary regime under which households invest by rules of thumb and change their rules only if others have experienced a significant utility gain from doing so.

Households can insure against idiosyncratic risk due to their private signal. They can buy contingent claims at the beginning of the period that pay off at the beginning of the next period. Contingent claims trading thus completes markets between periods and leads all households, in equilibrium, to hold the same amount of wealth.

2.3 Firms

A representative firm purchases capital and labor services from households. As it merely rents services from an existing capital stock, its maximization collapses to a period-by-period problem.¹⁰ The firm's problem is to

$$\max_{K_t^d, L_t^d} e^{\eta_t} F(K_t^d, L_t^d) - w_t L_t^d - D_t K_t, \quad (10)$$

where K_t^d and L_t^d denote factor demands for capital and labor respectively. First order conditions

⁹More precisely, $\tilde{\epsilon}_t(i)$ has a mean of $-\frac{1}{2}\sigma_{\tilde{\epsilon}}^2$ such that agents hold the correct expectation of log returns in expectation.

¹⁰Note that by choosing a structure in which firms rent capital services from households, we abstract from all principal agent problems between managers and stockholders that are sometimes debated in the literature. Managers therefore cannot prevent errors in stock prices from impacting investment decisions, as in Blanchard, Rhee, and Summers (1993). On the other hand, they do not amplify shocks or overinvest as in Albuquerque and Wang (2005).

with respect to capital and labor pin down the fair wage and the dividend. Both factors receive their marginal product:

$$e^{\eta_t} F_K \left(K_t^d, L_t^d \right) = D_t \quad (11)$$

and

$$e^{\eta_t} F_L \left(K_t^d, L_t^d \right) = w_t. \quad (12)$$

As the production function is linear homogenous, the representative firm makes zero economic profits.

2.4 Investment Goods Sector

The representative firm owns an investment goods sector which converts the consumption good into units of capital, while incurring adjustment costs. It takes the price of capital as given and then performs instant arbitrage:

$$\max_{I_t} Q_t I_t - I_t - \frac{1}{2} \chi \frac{I_t^2}{K_t}, \quad (13)$$

where the first term is the revenue from selling I_t units of capital and the second and third terms are the cost of acquiring the necessary units of consumption goods (recall the price of the consumption good is normalized to one) and the adjustment costs respectively. Since there are decreasing returns to scale in converting consumption goods to capital, the investment goods sector makes positive profits in each period. Profits are paid to shareholders as a part of dividends.

Taking the first order condition of (13), gives us equilibrium investment as a function of the market price of capital:

$$I_t = \frac{K_t}{\chi} (Q_t - 1) \quad (14)$$

Whenever the market price of capital is above one, investment is positive, raising the capital stock in the following period. When it is below one the investment goods sector buys units of capital and transforms them back into the consumption good. Note that the parameter χ scales the adjustment costs and can be used to calibrate the sensitivity of capital investment with respect to the stock price.

2.5 Definition of Equilibrium

Definition 2.1

Given a time path of shocks $\{\eta_t, \tilde{\epsilon}_t, \{\tilde{\nu}_t(i) : i \in [0, 1]\}\}_{t=0}^{\infty}$ an equilibrium in this economy is a time path of quantities $\{\{C_t(i), B_t(i), W_t(i), \omega_t(i) : i \in [0, 1]\}, C_t, B_t, W_t, \omega_t, K_t^d, L_t^d, Y_t, K_t$

, $I_t\}_{t=0}^\infty$, signals $\{s_t(i) : i \in [0, 1]\}_{t=0}^\infty$ and prices $\{Q_t, r, D_t, w_t\}_{t=0}^\infty$ with the following properties:

1. $\{C_t(i), \omega_t(i)\}_{t=0}^\infty$ solve the households' maximization problem (5) given the vector of prices, initial wealth, and the random sequences $\{\tilde{\epsilon}_t, \{\tilde{\nu}_t(i)\}_{t=0}^\infty$;
2. $\{K_t^d, L_t^d\}_{t=0}^\infty$ solve the representative firm's maximization problem (10) given the vector of prices;
3. $\{W_t(i)\}_{t=0}^\infty$ evolve according to the budget constraint (6);
4. $\{I_t\}_{t=0}^\infty$ is the investment goods sector's optimal policy (14) given the vector of prices;
5. $\{Y_t\}_{t=0}^\infty$ is determined by the production function (1);
6. $\{K_t\}_{t=0}^\infty$ evolves according to (2);
7. There is a perfectly elastic supply of the consumption good in world markets at the price of which is normalized to one;
8. $\{w_t\}_{t=0}^\infty$ clears the labor market such that $L = L^d$;
9. $\{Q_t\}_{t=0}^\infty$ clears the stock market;
10. There is a perfectly elastic supply of bonds in world markets at the rate r ;
11. $\{D_t\}_{t=0}^\infty$ clears the market for capital services such that $K_t = K_t^d$;
12. $\{s_t(i)\}_{t=0}^\infty$ is determined by (4);
13. $\{B_t(i), C_t, B_t, W_t, \omega_t\}_{t=0}^\infty$ are given by the identities

$$B_t(i) = (1 - \omega_t(i)) (W_t(i) - C_t(i)), \quad (15)$$

$$X_t = \int_0^1 X_t(i) di, \quad X = C, B, W \quad (16)$$

and

$$\omega_t = \frac{Q_t K_{t+1}}{W_t - C_t}. \quad (17)$$

The *rational expectations equilibrium* is the equilibrium for which the expectations operator $\mathcal{E}$ in equation (5) coincides with the rational expectation in (8). The *near-rational expectations equilibrium* posits that households make small errors around the rational expectation, as given in (9). The idea behind the near-rational expectations equilibrium is that small errors in

households' policies result in minor welfare losses for the individual household. The following definition formalizes what it means for near-rational households to suffer only "economically small" losses:

Definition 2.2

A near-rational expectations equilibrium is k -percent stable if the welfare gain to an individual household of obtaining rational expectations is less than $k\%$ of consumption.

3 Equilibrium Expectations

In this section we explore how small correlated mistakes in households' investment behavior may result in large errors in market expectations and in excess volatility in stock returns. To fix ideas, let us define the error in market expectations of η_{t+1} as the difference between the average expectation held by households in the near-rational expectations equilibrium and the average expectation they would hold if $\tilde{\varepsilon}_t$ happened to be zero in this period. We call the error in the market expectations

$$\varepsilon_t = \gamma \tilde{\varepsilon}_t$$

and solve for γ below. The main insight is that the multiplier γ may be very large. This amplification of errors is a result of households learning from equilibrium prices: a rise in prices causes households to revise their expectations upwards; and when households act on their revised expectations, the price rises further. The trades of each household represent an externality on other households' expectations.

3.1 Solving for Expectations in General Equilibrium

In order to say more about the relationship between $\tilde{\varepsilon}_t$ and ε_t we need to solve for equilibrium expectations. This is a challenge because our model is non-linear, and in particular because the market price of capital is a non-linear function of η_{t+1} . If the market for stocks is to clear, the value of stocks demanded by households must equal the value of capital supplied. While the supply of capital depends directly on Q from (14), the demand for stocks depends (among other things) on expected returns; and households' expectations of returns in turn depend on Q .

The goal is to obtain a solution for equilibrium expectations as a function of the underlying state variables and shocks. Mertens (2009) shows how to use perturbation methods in combination with a non-linear change of variables to transform the equilibrium conditions into a form which we can solve with standard techniques. He shows two important results: The first is that if we replace all equilibrium quantities by their power series, we can solve for the equilibrium values and see that they depend on the expectation of η_{t+1} . Second, if we apply an appropriate

nonlinear change of variables, we can compute the expectation over all terms in the (nonlinear) equilibrium conditions. The key insight here is that there is no disagreement among households about higher moments of the shocks.

Using these results we are able to obtain a transformation of the equilibrium stock price, $\hat{q}_t$ that has the same information content as the price (i.e. both variables span the same σ -algebra). Framed in terms of this variable, the equilibrium boils down to computing prices and expectations such that the following equation is satisfied.

$$\hat{q}_t [\eta_{t+1}, \tilde{\varepsilon}_t, K_t, B_t, \eta_t] = \int E(\eta_{t+1} | \hat{q}_t, s_t(i), K_t, B_t, \eta_t) di + \tilde{\varepsilon}_t, \quad (18)$$

where $\hat{q}_t$ is a function of the state variables and shocks. Equation (18) is the familiar linear equilibrium condition of a standard noisy rational expectations model. We can now apply the standard methods to solve for equilibrium expectations in terms of $\hat{q}_t$ (Hellwig (1980)) and then transform the system back to recover the equilibrium Q_t .

3.2 Amplification of Small Errors

We now to obtain equilibrium expectations by solving for $\hat{q}$. As it turns out we are able to show all the main qualitative results on the aggregation of information in this linear form. In section 6 we map the solution back into its original form to show the quantitative implications for the equilibrium stock price and for stock returns.

Since $\hat{q}$ equals the market expectation of η_{t+1} in (18), we may guess that the solution for $\hat{q}_t$ is some linear function of η_{t+1} and $\tilde{\varepsilon}_t$:

$$\hat{q}_t = \pi_0 + \pi_1 \eta_{t+1} + \gamma \tilde{\varepsilon}_t. \quad (19)$$

This guess formally defines the multiplier γ . Our task is to solve for the coefficients in this equation. Assuming that our guess for $\hat{q}_t$ is correct, the rational expectation of η_{t+1} given the private signal and $\hat{q}_t$ is

$$E_{it}(\eta_{t+1}) = A_0 + A_1 s_t(i) + A_2 \hat{q}_t, \quad (20)$$

where the constants A_0 , A_1 and A_2 are the weights that households give to the prior, the private signal and the market price of capital respectively. We get market expectations by adding the near-rational error and summing up across households. Combining this expression with our guess (19) yields

$$\int E(\eta_{t+1} | \hat{q}_t, s_t(i), K_t, B_t, \eta_t) di + \tilde{\varepsilon}_t = (A_0 + A_2 \pi_0) + (A_1 + A_2 \pi_1) \eta_{t+1} + A_2 \gamma \tilde{\varepsilon}_t + \tilde{\varepsilon}_t, \quad (21)$$

where we have used the fact that $\int s_t(i)di = \eta_{t+1}$. This expression reflects all the different ways in which $\tilde{\epsilon}$ affects market expectations: The last term on the right hand side is the direct effect of the near-rational error on individual expectations. If we introduced a fully rational household into the economy and gave it the same private signal as one of the near-rational households, the two households' expectations of η_{t+1} would differ exactly by $\tilde{\epsilon}_t$. The third term on the right hand side represents the deviation in market expectations that results from the fact that the market price transmits the average error as well as information about future fundamentals. The extent of this amplification depends on how much weight the market price has in the rational expectation (20) and on how much noise $\tilde{\epsilon}_t$ causes in $\hat{q}_t$. Finally, the second term on the right hand side tells us that the mere fact that households make near-rational errors may reduce the extent to which the market can predict η_{t+1} by changing the coefficients A_1 and A_2 .

Plugging (21) into (18) and matching coefficients with (18) allows us to solve for the amplification of $\tilde{\epsilon}_t$:

Proposition 3.1

Through its effect on the market price of capital, the near-rational error, $\tilde{\epsilon}$, feeds back into the rational expectation of future productivity. The larger the weight on the market price of capital in the rational expectation, the larger is the variance in ϵ_t relative to the variance in $\tilde{\epsilon}$. We have that

$$\gamma = \frac{1}{1 - A_2} \quad (22)$$

Proof. See appendix E. ■

The more weight households place on the market price of capital when forming their expectations about η_{t+1} , the larger is the error in market expectations relative to $\tilde{\epsilon}_t$. We can solve for A_2 and the other coefficients in (19) and (20) by applying the projection theorem. With explicit solutions in hand, we can show the following result:

Proposition 3.2

For any given level of $\sigma_{\tilde{\epsilon}}$, the noise to signal ratio in the market price of capital goes to infinity as the precision of the private signal goes to zero,

$$\lim_{\sigma_{\nu} \rightarrow \infty} \frac{\text{var}(\gamma \tilde{\epsilon}_t)}{\text{var}(\pi_1 \eta_{t+1})} = \infty.$$

Proof. See appendix E. ■

As information becomes more dispersed across households, the private signal becomes less informative relative to the stock price. Households adjust by paying relatively more attention to the public signal. This has two effects. First, if households put less weight on their private signal, less information enters the equilibrium price. Second, the more attention they pay to the

market price, the larger is the amplification of ε_t . Both effects result in a rising noise to signal ratio in equilibrium stock prices. The implication of this finding is that if the private signal received by households is sufficiently noisy, *any* ratio of noise to fundamental volatility in stock returns can be supported in equilibrium, while households have *arbitrarily little* to gain from changing their behavior.

While the model can generate an arbitrarily large noise to signal ratio, there is an upper bound for the amount of overall stock market volatility in the model: As the noise in the private signal increases, households put more and more weight on the stock price when forming their expectations. This leads to a larger and larger amplification of a given amount of near-rational errors and hence a larger and larger amount of noise in the stock price. As both the private signal and the stock price become less informative and households begin to rely more on their priors, the overall amount of volatility in market expectations peaks and then eventually decreases.

Figure 1 illustrates this point. It plots the standard deviation of the error in market expectations of η_{t+1} over the standard deviation of near-rational errors and the noise in the private signal (all standard deviations are normalized with the standard deviation of productivity shocks). As σ_ν rises, σ_ε rises, then peaks and slowly begins to decrease. However, the absolute amount of information in market expectations falls at a faster rate than the absolute amount of noise, such that the noise to signal ratio in Figure 2 continues to rise monotonically.¹¹

This pattern highlights a second channel through which near-rational errors affect the aggregation of information in the stock market:

Proposition 3.3

The absolute amount of information aggregated in the stock price decreases with σ_ε ,

$$\frac{\partial \pi_1}{\partial \sigma_\varepsilon} < 0$$

Proof. See appendix E. ■

While near-rational errors amplify and lead to large amounts of noise in the stock price, they simultaneously hamper the capacity of the stock market to transmit and aggregate information. The conditional variance of η_{t+1} in the near-rational expectations equilibrium therefore exceeds the conditional variance in the rational expectations equilibrium for two reasons: First, because the stock price becomes more variable and second because it contains less information.

Since we know that $\hat{q}$ and Q have the same information content, we can make a parallel statement about the conditional variance of stock returns, which leads us to the following

¹¹Note that while the relationship depicted in the figure is independent of the parameters of the model, the extent to which errors in market expectations of η_{t+1} translate into volatility in stock returns does depend on parameters and the vector of state variables.

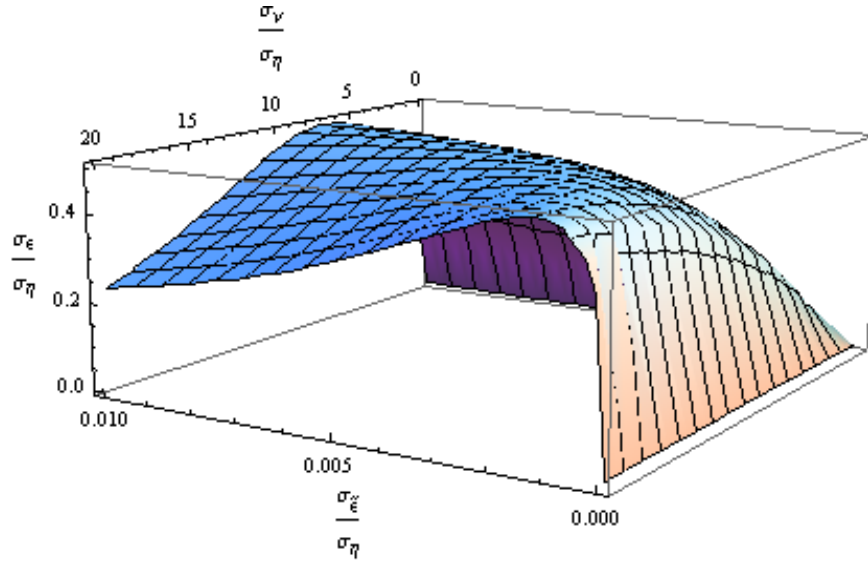


Figure 1: Standard deviation of the error in market expectations of η_{t+1} plotted over the standard deviation of the near-rational error and the standard deviation of noise in the private signal. All values are normalized with the standard deviation of the productivity shock.

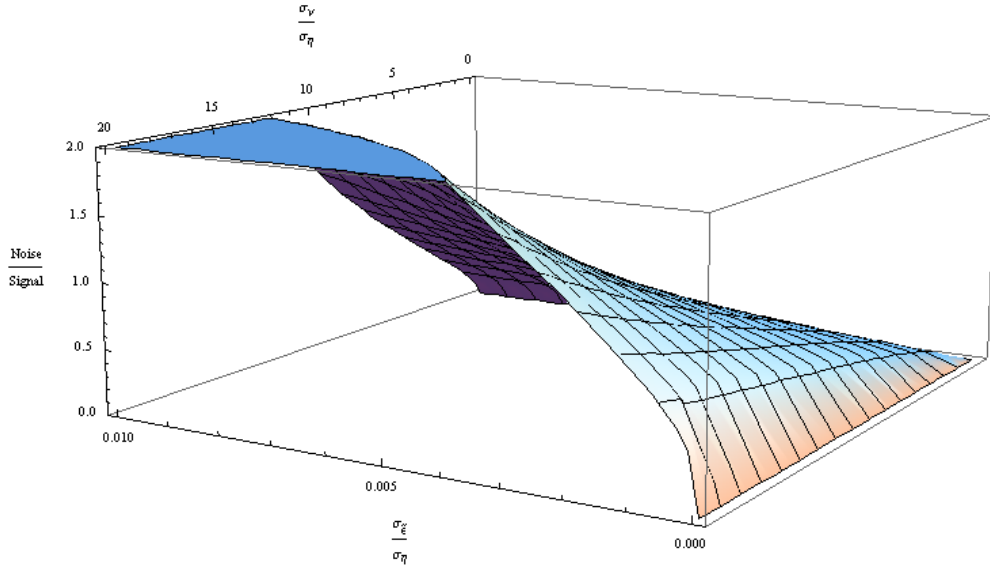


Figure 2: Noise to signal ratio in stock prices plotted over the standard deviation of the near-rational error and the standard deviation of noise in the private signal. All values are normalized with the standard deviation of the productivity shock.

definition:

Definition 3.4

Excess volatility in stock returns is the percentage amount by which the conditional standard deviation of stock returns in the near-rational expectations equilibrium, σ , exceeds the conditional standard deviation of stock returns in the rational expectations equilibrium, σ^ ,*

$$\frac{\sigma - \sigma^*}{\sigma} 100.$$

4 Intuition: The Macroeconomic Effects of Volatility

Now that we have an intuition for how excess volatility in stock returns endogenously arises in our model we turn to the effect it has on the macroeconomic equilibrium. In order to provide a maximum of intuition for the mechanisms at work this section focuses on a simplified version of the model for which we are able to derive the main results analytically. In section 6 we show computationally that the relevant implications of the simplified model carry over to the full model.

Assume that households consist of two specialized individuals, a "capitalist" who trades in the stock and bond markets and a "worker" who provides labor services, receives wages and the profits from the investment goods sector, but is excluded from trading in financial markets. This division eliminates non-tradable income from the capitalist's portfolio problem such that we can solve it with pen and paper. A capitalist's budget constraint is

$$W_{t+1}(i) = ((1 - \omega_t(i))(1 + r) + \omega_t(i)(1 + \tilde{r}_{t+1}))(W_t(i) - C_t(i)) \quad \forall t \quad (23)$$

Taking as given that the distribution of equilibrium asset returns is approximately log-normal (which turns out to be true in equilibrium)¹², we can solve for the capitalist's optimal consumption and portfolio allocation:

Lemma 4.1

Capitalists' optimal consumption is a constant fraction of financial wealth

$$C_t(i) = (1 - \beta)W_t(i) \quad (24)$$

and the optimal portfolio share of stocks is the expected excess return divided by the conditional

¹²This claim is verified in appendix D. We require approximate log-normality for the analytical solution below but not for the computational results.

variance of stock returns, σ^2

$$\omega_t(i) = \frac{E_{it}(1 + \tilde{r}_{t+1}) - (1 + r)}{\sigma^2}. \quad (25)$$

Proof. Appendix A gives a detailed derivation which proceeds analogous to Samuelson (1969). ■

Where of course the capitalist, rather than the entire household makes small mistakes when investing in the stock market, (9). The stock market clears when the value of shares demanded equals the value of shares in circulation:

$$\int_0^1 \beta \frac{E_{it}^b(1 + \tilde{r}_{t+1}) - (1 + r)}{\sigma^2} W_t(i) di = Q_t K_{t+1}. \quad (26)$$

It is this condition that links the stock market to the real economy. We can apply the definition (7), as well as (24) to get

$$E_t^b \left(\frac{Q_{t+1}(1 - \delta) + D_{t+1}}{Q_t} \right) = 1 + r + \omega_t \sigma^2, \quad (27)$$

where ω_t is defined in equation (17) and represents the aggregate degree of leverage required in order to finance the domestic capital stock. In equilibrium, the average capitalist holds a share ω_t of her wealth in stocks. The left hand side of (27) is the market expectation of stock returns; the right hand side is the required return that investors demand given the risk that they are exposed to. The equity premium, $\omega_t \sigma^2$, rises with the variance of stock returns and with the amount of leverage required to hold the domestic capital stock.

Any error in aggregate expectations has two important channels through which it affects the real side of the model. First, it causes a temporary misallocation of capital by distorting Q_t and aggregate investment (14). Second, the rise in the conditional variance of returns implied by excess volatility raises the equity premium and with it the expected dividend demanded by capitalists in general equilibrium.¹³ While the former channel mainly influences the dynamics of the model, the latter channel has a direct effect on the stochastic steady state. We discuss each in turn.

¹³Note that by assuming the real shock and the aggregate error in expectations to be i.i.d. and log-normal, we have implicitly assumed that capitalists' near-rational heuristics automatically create excess volatility in asset prices. This need not be the case in general. If the two shocks are strongly negatively correlated, it is possible that heuristic pricing rules would dampen the volatility of asset prices. For an example, see Grinblatt and Han (2002).

4.1 Distortion of Capital Accumulation

In the simplified version of the model we are able to obtain a closed form solution for the stochastic steady state and thus analytically show the following result:

Definition 4.2

The stochastic steady-state is the level of capital, bonds, and prices at which those quantities do not change in expectation.

Proposition 4.3

The equilibrium has a unique stochastic steady state iff $\beta \leq \frac{1}{1+r}$. At the stochastic steady state the aggregate degree of leverage is

$$\omega_o = \sqrt{\frac{1}{\sigma^2} \left(\frac{1-\beta}{\beta} - r \right)}; \quad (28)$$

and the stochastic steady state capital stock is characterized by

$$(1 + \delta\chi) (r + \omega_o\sigma^2 + \delta) = F_K(K_o, L). \quad (29)$$

Proof. See Appendix B. ■

The intuition for the first result is simple: If the time discount factor is larger than $\frac{1}{1+r}$, investors are so patient that even those holding a perfectly riskless portfolio containing only bonds would accumulate wealth indefinitely. In that case, no stochastic steady state can exist. However, if $\beta \leq \frac{1}{1+r}$, there exists a unique value ω_o at which the average capitalist has an expected portfolio return that exactly matches his time discount factor: $\beta = (1 + r + \omega_o^2\sigma^2)^{-1}$. At this value, there is no expected growth in consumption and the economy is at its steady state.¹⁴

The second result, (29), follows directly from applying the steady state to equation (27). On the left hand side, $1 + \delta\chi$ is the market price of a unit of capital at the stochastic steady state. This is multiplied with the required return to capital: the risk free rate plus the risk premium and the rate of depreciation. At the stochastic steady state, the required return on one unit of capital must equal the expected divided, which is precisely the expected marginal product of capital (on the right hand side of the equation). This brings us to the one of the main results of this paper:

Proposition 4.4

A rise in the variance of stock returns, and hence excess volatility, unambiguously depresses the

¹⁴Conversely we can determine the stochastic steady state wealth of our economy relative to the value of its capital stock by choosing an appropriate time discount factor. We shall make use of this feature when we calibrate the model in section 5.

stochastic steady state level of capital stock and output.

$$\frac{\partial K_o}{\partial \sigma} < 0$$

Proof. We use (28) to eliminate ω_o in (29) and take the total differential, see Appendix B for details. ■

The higher the risk of investing in stocks, the higher is the risk premium demanded by capitalists. A higher risk premium implies higher dividends at the stochastic steady state and, with a neoclassical production function, a lower level of capital stock. Less installed capital in turn implies lower production. The variance of stock returns thus has a *level* effect on the stochastic steady state level of capital. It follows immediately that excess volatility in the stock market depresses output at the stochastic steady state.

Interestingly, this level effect may operate even if the stock market seems to have little influence on the allocation of capital in the economy:

Corollary 4.5

A rise in the variance stock returns depresses the stochastic steady state level of output even if the sensitivity of physical investment with respect to stock prices is low.

Proof. From (14) we have that $\frac{\partial(I_t/K_t)}{\partial Q_t} = \frac{1}{\chi}$. The sensitivity of physical investment as a share of the existing capital stock with respect to the stock price is fully determined by the adjustment cost parameter χ . From (29) and (28) we have that $\frac{\partial^2 F_K(K_o, L)}{\partial \sigma^2 \partial \chi} = \delta \sqrt{\frac{1}{\sigma^2} \left(\frac{1-\beta}{\beta} - r \right)} > 0$. ■

If the adjustment cost parameter χ is sufficiently large, the stock market in this economy may appear as a “sideshow” (Morck, Shleifer, and Vishny (1990)) in the sense that a given change in the stock price has little influence on investment. To the casual observer it may therefore seem as though excess volatility in the stock market has no influence on the real economy. However, a low responsiveness of physical investment to the stock price is uninformative about the impact that excess volatility has on the stochastic steady state. Excess volatility may cause a large depression of output at the stochastic steady state while leaving virtually no evidence in the physical investment data. Since our model does not exempt replacement investments from capital adjustment costs, the impact of an incremental rise in stock market volatility on the stochastic steady state level of capital actually *rises* with χ , implying that excess volatility may actually have a larger effect on the stochastic steady state in economies in which the stock market appears to be a “sideshow”.

Finally, the volatility of stock returns has an important implication for the distribution of income in the economy:

Corollary 4.6

A rise in the standard deviation of stock returns unambiguously lowers wages and raises dividends in the stochastic steady state.

Proof. The result follows directly from (11), (12) and proposition 4.4. ■

Excess volatility may paradoxically raise the incomes of stock market investors: At lower levels of K , dividends rise relative to wages, increasing the return to each unit of capital. Over some range, such a rationing always raises the total payments to capital. As the variability of pricing errors rises, it pushes the economy towards higher dividends, compensating capital for the loss of aggregate output at the expense of payments to labor. In the simplified version of our model, excess volatility in the stock market may work like a coordination device that allows capitalists to ration the capital stock and thereby earn monopoly rents on their assets.

4.2 Dynamics of the Model

We can best understand the dynamic effects of the aggregate error in market expectations, ϵ_t , by comparing the near-rational expectations equilibrium with the rational expectations equilibrium in which all capitalists behave fully rationally. There are two reasons why the conditional variance of stock returns is lower in the rational expectations equilibrium. First, because it has no noise in the equilibrium price; and second because the absence of noise in the equilibrium price enhances the stock market's ability to aggregate information,

$$\sigma^* < \sigma,$$

where we denote the variables pertaining to the rational expectations equilibrium with an asterisk. From Proposition 4.4, it follows that the stochastic steady state level of capital, output, and consumption is higher in the rational expectations equilibrium.

Solving the dynamics of the model requires a computational algorithm that we discuss in section 5. However, we can gain some intuition from the simplified version of our model. Equations (27), (2), (14), (11), and a standard transversality condition jointly determine the market price of capital. Every vector of state variables and shocks is therefore associated with a unique stock price. In the rational expectations equilibrium, the market price of capital equals its fundamental value. In the near-rational expectations equilibrium near-rational errors cause large departures of the market price of capital from its fundamental value. Through the arbitrage performed by the investment goods sector, the error then passes into physical investment, causing a temporary misallocation of capital. It follows that the volatility of physical investment is higher in the near-rational expectations equilibrium than in the rational expectations equilibrium.

Regardless of initial conditions and of whether capitalists behave near-rationally or not, the

economy transitions to its stochastic steady state in expectation. To understand this, imagine an economy that is at its stochastic steady state and receives a positive shock. Capitalists will save a fraction of the rise in dividends and are now on average richer than they were before. This implies that the aggregate portfolio share required to finance the domestic capital stock in the following period falls, $\omega_{o+1} < \omega_o$. As capitalists are now less leveraged, they require a lower risk premium in the next period. Expected returns therefore tend to be lower following a positive shock and higher following a negative shock. Equilibrium returns thus exhibit negative autocorrelation and thereby generate stationary dynamics.¹⁵

To summarize, the near-rational expectations equilibrium of the simplified version of our model exhibits a higher volatility of returns and investment around a lower stochastic steady state level of capital and output. In unconditional expectation, the returns to capital are higher and wages are lower than in the (fully rational) benchmark case. As we show below, all of these conclusions carry over to the full version of the model.

5 Quantifying Welfare Cost

In this section we return to the full version of our model and quantify the welfare cost of the near-rational errors made by households. To this end, we first derive a standard welfare metric, based on a simple experiment in which near-rational behavior is purged from financial markets and the economy transitions to the stochastic steady state of the rational expectations equilibrium. We then briefly describe the computational algorithm used to solve this problem and calibrate the model to the data.

5.1 Welfare Calculations

Consider an economy that is at the stochastic steady state of the near-rational expectations equilibrium and suppose that at time 0, there is a credible announcement that all households henceforth commit to fully rational behavior until the end of time. Immediately after the announcement, the conditional variance of stock returns falls and households require a lower risk-premium for holding stocks. The stochastic steady state levels of capital and output rise. Although the economy does not jump to the new stochastic steady state immediately, it accumulates capital over time and converges to it in expectation. Over the adjustment process, output rises, wages rise and returns to capital fall. Aggregate consumption increases not only

¹⁵There is a large body of literature discussing the non-stationarity of small open economy models (see for example Schmitt-Grohé and Uribe (2003)). The issue of non-stationarity is, however, a consequence of the linearization techniques typically employed to solve these models and *not* an inherent feature of the small open economy setup. Since we solve our model using higher order expansions we obtain stationary dynamics.

due to the rise in output, but may increase further due to a fall in capital adjustment costs incurred. Finally, households now enjoy smoother consumption due to the reduced volatility of the capital stock.¹⁶

Formally, we ask by what fractions λ , we would have to raise the average household's consumption in order to make it indifferent between remaining in the near-rational expectations equilibrium and transitioning to the stochastic steady state of the rational expectations equilibrium. λ then indicates the magnitude of the welfare loss attributable to excess volatility as a fraction of total consumption. λ is defined as follows:

$$\int_0^1 E_{i0} \sum_{t=0}^{\infty} \beta^t \log((1 + \lambda)C_t(i)) di \equiv \int_0^1 E_{i0} \sum_{t=0}^{\infty} \beta^t \log(C_t^*(i)) di. \quad (30)$$

From (30) we can see that welfare losses may result either from a lower *level* of consumption or from a higher *volatility* of consumption. Given our previous discussion, we can identify three channels through which excess volatility can affect welfare: (1) A drop in the level of consumption due to the distortion in the stochastic steady state level of capital; (2) A drop in the level of consumption due to incurring excess capital adjustment costs; (3) A rise in the volatility of consumption. In appendix C, we derive fractions λ^Δ , λ^χ and λ^σ which quantify the relevance of each of these channels respectively. We have that $1 + \lambda = (1 + \lambda^\Delta)(1 + \lambda^\chi)(1 + \lambda^\sigma)$.

5.2 Numerical Solution (under revision)

As with many general equilibrium models, our specification does not allow for an analytical solution. Since our results crucially depend on the correct location of the stochastic steady state, we cannot resort to the standard practice of approximating the model (log-)linearly around the deterministic steady state. However, since we are able to analytically determine the optimal policy and the location of the stochastic steady state, we can obtain an accurate solution of the model. We first present an algorithm for calculating the saddle manifold, before introducing our simulation methods and results.

For the purposes of our simulations, we assume Cobb-Douglas technology, for which the closed-form solution for the stochastic steady-state level of capital follows from (29). We further

¹⁶Because near-rational errors are the only noise in the market price of capital, our model in its strictest interpretation yields an additional source of welfare losses due to excess volatility, which is that in the benchmark case, agents can perfectly infer the fundamental component of future returns. In the interest of preserving tractability we refrain from adding any additional frictions (such as limiting the number of signals available to a finite number, introducing liquidity trades, etc.) to eliminate this prediction. When performing our welfare calculations we hold the conditional variance of stock returns due to fundamental risk constant between the near-rational competitive equilibrium and the benchmark case. In this sense our estimates may be interpreted as a lower bound of total welfare losses.

know that the shadow price of capital at the steady state must be $q_o = 1 + \delta\chi$. Perturbing the steady state, we use a manipulation of equation (27) to link the shadow price today with the expected shadow price tomorrow. Iterating the equation backwards, we can obtain the shadow price as a function of K , ω , and ε . It turns out, however, that we can write the function in terms of two variables, K and the left-hand side of equation (27), only. In order to obtain a good perturbation from the steady-state, we analytically compute the derivative of the saddle manifold at the steady state. We follow the dynamics backwards and thus follow the true surface. Tests show that our estimate for $q(K, \omega)$ converges to the steady-state as desired in a range of accuracy of 10^{-7} . When simulating, we interpolate the grid points using cubic splines exploiting the fact that the manifold must be smooth. The interval was chosen to be big enough such that all simulations stay within the range.

Having solved for the shadow price of capital, we simulate the model using the analytically derived optimal policies. We first draw the productivity shocks and errors in aggregate expectations and truncate the distributions at their upper and lower tails. We then find the unique combination of (Q, ω) that clears the market and is consistent with the optimal policy. Using these results, we can solve for investment, consumption, and stock returns. We do not allow workers smooth consumption in our simulations, but the welfare losses from random consumption paths are, as we show, negligible. In the simulations, some economies that receive a combination of large negative productivity shocks and highly positive errors in aggregate expectations find themselves unable to repay all debt. We subsidize those economies to avoid bankruptcy but exclude them from the computation of our results. We simulate 20000 economies for 100 periods in order to compute all net present values.

5.3 Calibration

In all of our calibrations, we match the standard deviation of stock returns to 0.18 which conforms to long-run international data (Campbell (2003)). Of this value, we attribute the fraction $\frac{\sigma_\varepsilon}{\sigma}$ to excess volatility. Unfortunately, more than two decades of literature following the original volatility tests by Shiller (1981) and LeRoy and Porter (1981) have not yielded a close range of estimates for this parameter. We therefore experiment with a range of 0.3 to 0.7 and choose 0.5 for our baseline calibration, which is towards the lower end of estimates available and still implies a counterfactually large standard deviation of output.¹⁷ We choose an adjustment cost parameter $\chi = 2$, implying adjustment costs of 10% of total investment when changing the capital stock by 10%. The risk free rate on the world market is chosen as $r = 0.03$ and we set the rate of depreciation $\delta = 0.1$. We pick the time discount factor β such that the entire capital stock is owned by domestic households at the stochastic steady state, $\omega_o = 1$. Finally,

¹⁷If we calibrate our model to match the observed standard deviation of output we get a fraction $\frac{\sigma_\varepsilon}{\sigma}$ of around .95.

we choose a Cobb-Douglas production technology with a capital share of $\frac{1}{3}$. Since our economy is scale-independent, we can normalize labor supply to one without loss of generality.

6 Results

In this section we give numerical results for our model. The following section focuses on the simplified version of the model in which households consist of capitalists and workers. We then discuss the results for the full model.

6.1 Simplified Model

The most striking result from our simulations is that the welfare cost of near-rational investor behavior is very large, even for conservative choices of parameter values. Figure 3 plots the compensating variation for both capitalists and workers over the fraction of volatility of stock returns attributed to excess volatility. A positive λ corresponds to a welfare loss from excess volatility while a negative λ indicates a welfare gain. Note that over the entire range of $\frac{\sigma_\epsilon}{\sigma}$, workers incur significant losses, while capitalists actually profit from excessively volatile stock returns. These gains are attributable to the rise in the returns to capital that compensates capitalists for the additional risk they bear. However, the gains enjoyed by capitalists are of course insufficient to make up for the losses suffered by workers. The dotted line in the middle of the graphs plots the aggregate change in consumption corresponding to a situation in which costless compensation payments between capitalists and workers are possible (recall that we have chosen a capital share of $1/3$, which is why there are aggregate welfare losses although the solid line is far below zero). Even if only a fraction of .3 of the observed standard deviation of stock returns are attributed to excess volatility, aggregate losses amount to roughly 3% of consumption. At our baseline specification of .5, aggregate losses are 4.1% and they rise up to 5.1% at 0.7.

Even our lowest estimates for the welfare cost of excess volatility in stock markets are in the percentage range and are therefore substantially higher than any common estimate of the cost of business cycles, for which we can take 0.6% as an upper bound (Alvarez and Jermann (2005)). We can gain some intuition of why this is the case from figure 4. It plots the time path of two economies that start at the stochastic steady state level of capital of the near-rational expectations equilibrium corresponding to our baseline specification. The solid line gives the evolution of the capital stock of an economy that remains in the equilibrium in which households make near-rational errors when predicting future returns. The dashed line does the same for the benchmark case in which from time 0 onwards, all investors in the stock market credibly commit to be have fully rationally. A standard cost of business cycles calculation is equivalent

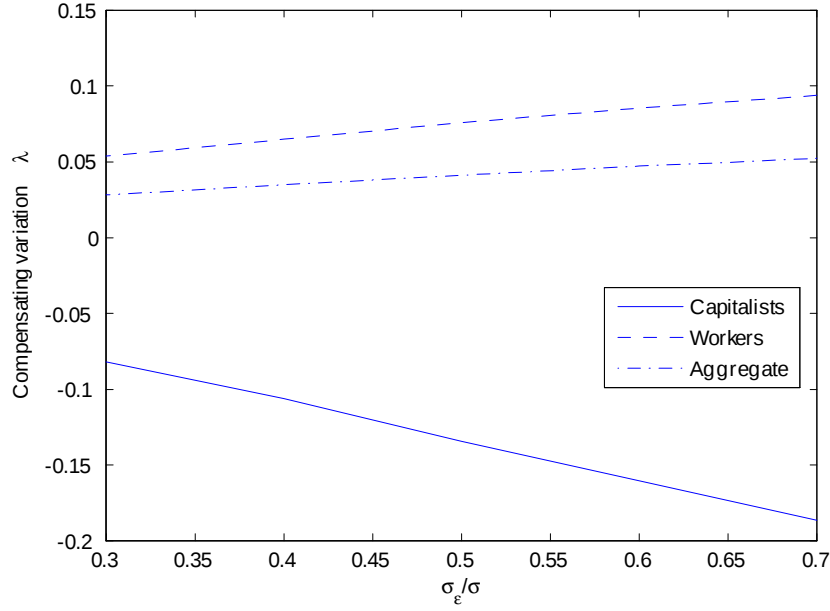


Figure 3: Compensating variation for eliminating all present and future near-rational behavior and transitioning to the steady state of an economy in which stocks are priced according to fundamentals plotted over the fraction of standard deviation of stock returns attributed to mispricings. The dashed line gives compensating variation for workers and the solid line for capitalists. The dotted line gives compensating variation in terms of aggregate consumption for the case in which costless compensation payments between the two classes of agents are admitted.

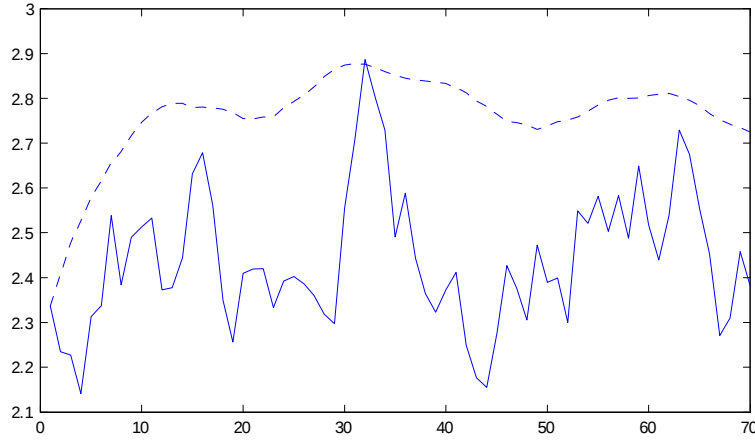


Figure 4: Time paths of the capital stock for an economy remaining in the near-rational expectations equilibrium (solid line) and an economy in which all households behave fully rationally (dashed line).

to calculating the gain from putting a straight line through the oscillations in the near-rational expectations equilibrium. However, the economy in the rational expectations equilibrium does not merely have a lower variance in its capital stock but it also converges to a higher steady state level of output and consumption.

Table 1 decomposes overall welfare losses for our baseline specification by type of agent and channel. We can loosely refer to the welfare cost of the larger oscillations of the solid line versus the dashed line in figure 4 as the sum of the losses attributable either to the higher volatility of consumption, λ^σ , or to excess adjustment costs, λ^χ . These are given in the columns 3 and 4 of the table respectively, while column 2 gives the losses attributable to the depressed stochastic steady state level of capital, λ^Δ . An increase of consumption of only 0.34% would be sufficient compensate the households in the near-rational expectations equilibrium for the losses they suffer from larger oscillations. Note that this number is actually significantly exaggerated as in our computation we do not allow workers smooth their consumption. If we did, λ^σ would be even smaller. The critical difference between the two economies therefore is clearly the depressed stochastic steady state level of capital due to higher risk premia, which accounts for the lion share of the losses incurred from excess volatility. Compensating households for the losses suffered through this channel would require a rise in their average consumption of 3.75%. At the same time, the losses accruing to individual capitalists due to their near-rational behavior may be arbitrarily small, depending on the precision of the private signal they receive. For our standard specification we get a 0.5% stable near-rational expectations equilibrium for all $\sigma_v > 0.55$.

	(1)	(2)	(3)	(4)
	Total, λ	Distortion of capital accumulation λ^Δ	Volatility of consumption λ^σ	Excess capital adjustment costs λ^χ
Aggregate	4.10	3.75	0.24	0.10
Capitalists	-13.45	-13.91	0.00	0.54
Workers	7.60	7.36	0.23	

Table 1: Decomposition of welfare losses in percent of consumption by channel and type of agent.

Figure 5 plots the comparative statics of the compensating variation with respect to the adjustment cost parameter χ . At our baseline specification with $\chi = 2$, the covariance of stock returns with investment is 0.011. At higher levels of χ , the sensitivity of investment to any given change in stock returns decreases and vice versa. At $\chi = 3$ the covariance is at 0.007 and at $\chi = 1$ it is 0.022. By varying the adjustment cost parameter we can therefore compare the welfare losses caused by excess volatility at different levels of observed co-movement between the stock market and investment. Strikingly, welfare losses are larger when the covariance between stock returns and physical investment is lower. The reason for this is that at higher levels of χ the stochastic steady state level of capital is lower, and any given increase in the standard deviation of stock returns has a larger effect. The extent to which the stock market and investment co-move therefore gives very little indication of the magnitude of welfare losses caused by excess volatility.

6.2 Full Model (under revision)

In the full model, households maximize (5) subject to (6). Since households' income has two non-tradable components, labor income, wL , and the profits accruing in the investment goods sector, θ , it is impossible to solve this problem in closed form. However, we may build some intuition as to how the model changes when households have labor income by looking at an approximate solution suggested by Viceira (2001):

The approximate optimal consumption rule is

$$C_t(i) = e^{b_{0t}} W_t(i)^{b_1} (w_t L + \theta_t)^{1-b_1} \quad (31)$$

and the approximate optimal portfolio share is given by

$$\omega_t(i) = \frac{1}{b_1 \sigma^2} (E_{it} (1 + \tilde{r}_{t+1}) - (1 + r) - (1 - b_1) \sigma_{\tilde{r}w}), \quad (32)$$

where b_1 is a constant and b_{0t} is a scaling-parameter which depends on the expected returns

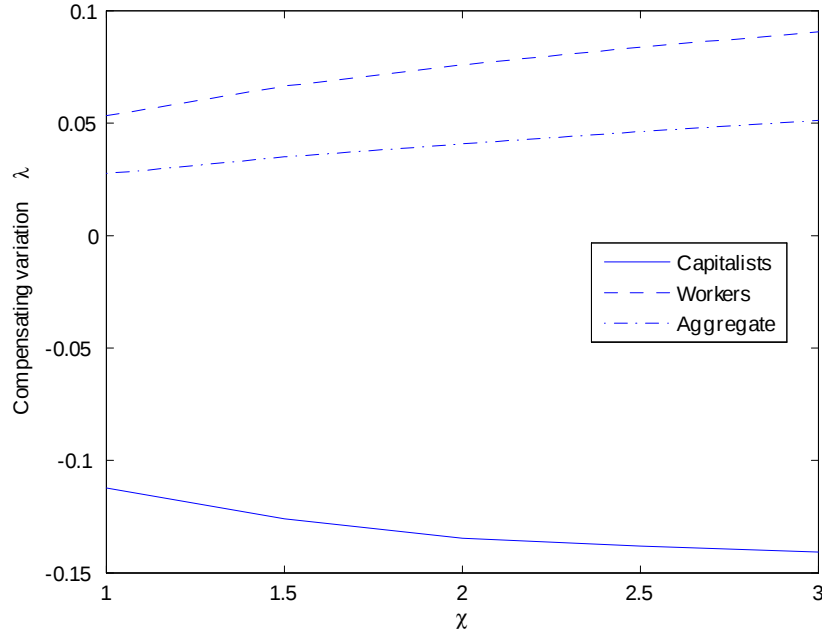


Figure 5: Compensating variation for eliminating all present and future near-rational behavior and transitioning to the steady state of an economy in which stocks are priced according to fundamentals plotted over the adjustment cost parameter χ . At $\chi = 1$ the covariance between stock returns and capital investment is 0.022. It falls as χ rises. At $\chi = 3$ the covariance is 0.007. The dashed line gives compensating variation for workers and the solid line for capitalists. The dotted line gives compensating variation in terms of aggregate consumption for the case in which costless compensation payments between the two classes of agents are admitted.

to capital and labor. Both expressions are given in appendix F. $\sigma_{\tilde{r}w}$ denotes the covariance between the non-tradable components of income and the returns to capital.

Proof. Appendix F gives a detailed proof which generalizes the result given in Viceira (2001) to a general equilibrium framework in which both returns to capital and labor income may be serially correlated. ■

Using these approximations to the optimal policy we can derive expressions similar to those in proposition 4.3 (see appendix F for details). The capital stock at the stochastic steady state is now pinned down by

$$(1 + \delta\chi) (r + b_1\omega_o\sigma^2 + (1 - b_1)\sigma_{\tilde{r}w} + \delta) = F_K(K_o, L). \quad (33)$$

The risk premium is now a weighted average between the familiar term $\omega_o\sigma^2$ and the covariance of stock-returns with the non-tradable components of income. The higher this covariance the higher is the risk premium required to hold stocks and the lower is K_o . This implies that the covariance of near-rational errors with fundamentals now matters for the location of the stochastic steady state.

We obtain a lower bound for the welfare losses attributable to excess volatility by maintaining the assumption that near-rational errors are uncorrelated with fundamentals. In our standard specification in which half of the observed standard deviation of stock returns are attributed to excess volatility, this lower bound amounts to 2.2% of household consumption. It remains large and of the same order of magnitude as the results discussed above for a broad range of parameters. We do not reproduce the comparative statics for households as they are very similar to the ones described in section 6.

7 Closed Economy (under revision)

8 Conclusion

We show that, excess volatility in stock returns may arise and drastically reduce welfare even if the stock market appears to be efficient and disconnected from the real economy. In our model, each household has some private information about future productivity. As individual investors trade in financial markets, prices come to reflect the information held by all market participants. If stock prices reflect information, investors have an incentive to learn from the equilibrium price and to update their expectations accordingly. But if investors watch the equilibrium price, then anything that moves the equilibrium price has an impact on the expectations held by all market participants.

In our model, stock market investors make small correlated errors when choosing their financial portfolio. These errors are amplified as households rationally inform on the equilibrium price when forming their expectations. If information is sufficiently disperse, arbitrarily small errors on the part of stock market investors may result in large amounts of excess volatility in stock returns. While individual investors suffer only small losses due to slight imbalances in their portfolios, the macroeconomic impact of the resulting excess volatility in stock returns may be large: Higher volatility in stock returns induces investors to demand higher risk premia for holding stocks. Higher risk premia in turn distort the level of capital installed at the stochastic steady state. Through its effect on capital accumulation, excess volatility in stock return causes costly (first-order) distortions in the level of consumption and large aggregate welfare losses.

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Appendix

A Capitalists' optimal policy

We first derive the optimal policy under full rationality. We can re-write (5) in Bellman form:

$$V(W_t(i), \pi_t(i)) = \max_{C_t(i), \omega_t(i)} \log(C_t(i)) + \beta E_{it} [V(W_{t+1}(i), \pi_{t+1}(i))],$$

where we abbreviate $\pi_t(i) = E_{it} (1 + \tilde{r}_{t+1}) - (1 + r)$. The conditions of optimality are:

$$\frac{1}{C_t(i)} = \beta E_{it} \left[R_{i,t+1}^p V'(W_{t+1}(i), \pi_{t+1}(i)) \right], \quad (34)$$

$$E_{it} \left((\tilde{r}_{t+1} - r) (W_t(i) - C_t(i)) V'(R_{i,t+1}^p (W_t(i) - C_t(i)), \pi_{t+1}(i)) \right) = 0, \quad (35)$$

and

$$V'(W_t(i), \pi_t(i)) = \beta E_{it} \left(R_{i,t+1}^p V'(W_{t+1}(i), \pi_{t+1}(i)) \right),$$

where $R_{i,t+1}^p \equiv ((1 - \omega_t(i))(1 + r) + \omega_t(i)(1 + \tilde{r}_{t+1}))$ and V' denotes $\frac{\partial V}{\partial W}$. It follows immediately that

$$\frac{1}{C_t(i)} = V'(W_t(i)). \quad (36)$$

Guess the value function:

$$V_t(W_t(i)) = \kappa_1 \log(W_t(i)) + \kappa_2 (\pi_t(i)) + \kappa_3 \quad (37)$$

Verification yields:

$$\begin{aligned} \kappa_1 &= \frac{1}{1 - \beta} \\ \kappa_2 &= \frac{1}{1 - \beta} E_{it} \left\{ \sum_{s=1}^{\infty} \beta^s \log(R_{t+s}^{p*}(i)) \right\} \\ \kappa_3 &= \frac{1}{1 - \beta} \log(1 - \beta) + \frac{\beta}{(1 - \beta)^2} \log(\beta), \end{aligned}$$

where R_t^{p*} is the optimized portfolio return. Furthermore, the transversality condition has to hold:

$$\lim_{s \rightarrow \infty} \beta^s \kappa_2 (R_{t+s}^{p*}(i)) = 0$$

The first result in Proposition 4.1 follows directly from the combination of (36) and (37). For the second result, combine (35) with (37) to obtain

$$(1+r)E_{it} \left(R_{t+1}^p(i) \right)^{-1} = E_{it} (1 + \tilde{r}_{t+1}) \left(R_{t+1}^p(i) \right)^{-1},$$

take logs on both sides and use the fact that

$$\log E(\cdot) = E \log(\cdot) + \frac{1}{2} \text{var}(\log(\cdot)).$$

B Solving for the stochastic steady state

If at any time o the economy is at steady state, we can write $E_o B_{o+1} = B_o$ and $I_o = \delta K_o$. From equation (14) it immediately follows that $Q_o = 1 + \delta\chi$. We first calculate the steady state dividend, from which we then back out the steady state capital stock. Finally we derive the steady state value of ω .

From equation (11),

$$D_{t+1} = e^{\eta_{t+1}} F_K(K_{t+1}, L),$$

At the steady state:

$$\bar{E}_o D_{o+1} = F_K(K_o, L)$$

Plug this into (27) to obtain

$$r + \omega_o \sigma^2 = -\delta + \frac{1}{1 + \delta\chi} (F_K(K_o, L))$$

$$(1 + \delta\chi) (r + \omega_o \sigma^2 + \delta) = F_K(K_o, L)$$

This proves the second statement in Proposition 4.3.

With a Cobb-Douglas specification and a capital share of α we can further write

$$\left(\frac{(1 + \delta\chi) (r + \omega_o \sigma^2 + \delta)}{\alpha L^{1-\alpha}} \right)^{\frac{1}{\alpha-1}} = K_o. \quad (38)$$

We now turn to solving for ω_o . The first step is to derive the equilibrium resource constraint for capitalists from (2), (23), (14), (11) and (17): From (17) we get that $W_t - C_t = Q_t K_{t+1} + B_t$ plugging this into (23) yields

$$Q_t K_{t+1} + B_t + C_t = (1+r)B_{t-1} + (Q_t(1-\delta) + D_t) K_t.$$

Now we can use (2) to eliminate K_{t+1} :

$$Q_t(1 - \delta)K_t + Q_t I_t + B_t + C_t = (1 + r)B_{t-1} + (Q_t(1 - \delta) + D_t)K_t.$$

This simplifies to

$$Q_t I_t + B_t + C_t = (1 + r)B_{t-1} + D_t K_t. \quad (39)$$

The next step is to re-write (39) in terms of K_o and ω_o . For this purpose note that

$$C_o = (1 - \beta)W_o,$$

$$\beta W_o = K_o(1 + \delta\chi) + B_o,$$

and

$$B_o = \beta W_o(1 - \omega_o)$$

$$(1 + \delta\chi)K_o = \beta W_o \omega_o$$

$$\rightarrow B_o = \frac{1 - \omega_o}{\omega_o}(1 + \delta\chi)K_o$$

Plugging these conditions into (39) and simplifying yields

$$(1 + \delta\chi) \left(\delta + \frac{1 - \beta}{\beta} + \frac{1 - \omega_o}{\omega_o} \left(\frac{1 - \beta}{\beta} - r \right) \right) = F_K(K_o, L) \quad (40)$$

We can eliminate K_o from this equation by substituting in (29). Some manipulations yield

$$\omega_o = \sqrt{\frac{1}{\sigma^2} \left(\frac{1 - \beta}{\beta} - r \right)},$$

proving the first statement in Proposition 4.3.

Combining (28) and (29) and taking the total differential gives

$$\frac{dK_o}{d\sigma} = \frac{1 + \delta\chi}{F_{KK}(K_o, L)} \left(\frac{1 - \beta}{\beta} - r \right)^{.5}.$$

Proposition 4.3 states that a stochastic steady state exists iff $\beta \leq \frac{1}{1+r}$. Proposition 4.4 then follows directly from the fact that $F_{KK}(K_t, L) < 0$.

C Decomposition of welfare losses

This section decomposes agents' total welfare losses into components attributable to additional variability of consumption, excess adjustment costs, and the lower steady state capital stock.

Given the parameters of the model and initial conditions K_0, ω_0 , define the expected utility level of the average capitalist in the near-rational competitive equilibrium E_0U as

$$E_0U = \int_0^1 \sum_{t=0}^{\infty} \beta^t \log(C_t(i)) di.$$

Similarly, given the same parameters and initial conditions define the expected utility level E_0U^* in the (fully rational) benchmark case as

$$E_0U^* = \int_0^1 \sum_{t=0}^{\infty} \beta^t \log(C_t^*(i)) di.$$

We can solve (30) for λ to obtain

$$1 + \lambda = \exp[(E_0U^* - E_0U)(1 - \beta)]. \quad (41)$$

Now define two new levels of utility: First, one at which consumers get compensated for the additional variability of their consumption in the stochastic steady state versus the benchmark case

$$E_0U^\sigma = \sum_{t=0}^{\infty} \beta^t E_0 \log(C_t) + \frac{1}{2(1 - \beta)} [\text{var}(\log(C)) - \text{var}(\log(C^*))],$$

where we have abstracted from any predictable variation in the conditional variance of consumption. The second level of utility we introduce furthermore compensates for the higher average adjustment cost incurred in the near-rational competitive equilibrium:

$$\begin{aligned} E_0U^{\sigma, \chi} = & E_0 \sum_{t=0}^{\infty} \beta^t \log \left(C_t + \left(\lim_{T \rightarrow \infty} \sum_{\tau=0}^T \frac{\chi}{2T} \left(\frac{I_\tau^2}{K_\tau} - \frac{I_\tau^{*2}}{K_\tau^*} \right) \right) \right) \\ & + \frac{1}{2(1 - \beta)} [\text{var}(\log(C)) - \text{var}(\log(C^*))]. \end{aligned}$$

We know from the discussion in the text that the remainder of the difference in average expected utility between the benchmark case and the near-rational competitive equilibrium must be due to the lower steady state level of capital in the presence of excess volatility.¹⁸ We can write

$$E_0U^\Delta = E_0U^* - E_0U^{\sigma, \chi}$$

All of the definitions above transfer to the case of workers analogously. However, since Workers are not concerned with excess adjustment costs we have that $E_0U^{w, \sigma} = E_0U^{w, \sigma \chi}$.

¹⁸We subsume the second order effect due to the variability of the capital stock in this category.

We can now apply these definitions in (30):

$$1 + \lambda = \exp [(E_0 U^* - E_0 U^{\sigma, \chi} + E_0 U^{\sigma, \chi} - E_0 U^\sigma + E_0 U^\sigma - E_0 U) (1 - \beta)]$$

and

$$\begin{aligned} 1 + \lambda = & \exp [(E_0 U^* - E_0 U^{\sigma, \chi}) (1 - \beta)] \cdot \exp [(E_0 U^{\sigma, \chi} - E_0 U^\sigma) (1 - \beta)] \\ & \cdot \exp [(E_0 U^\sigma - E_0 U) (1 - \beta)]. \end{aligned}$$

This implies that

$$1 + \lambda = (1 + \lambda^\Delta) (1 + \lambda^\chi) (1 + \lambda^\sigma).$$

D Distribution of equilibrium returns

Lemma D.1

If η and ϵ are conditionally normal, the distribution of stock returns is approximately conditionally log-normal around the stochastic steady state.

Proof. Applying the Campbell and Shiller (1988) log-approximation to (7) yields

$$\tilde{r}_{t+1} \approx \rho_1 \log(Q_{t+1}) + \rho_2 \log(D_{t+1}) - \rho_3 \log(Q_t), \quad (42)$$

where ρ_1 , ρ_2 , and ρ_3 are positive constants. Since $\log(D_{t+1})$ is normally distributed by assumption, we merely need to show that $\log(Q_{t+1}) \sim N$.

Taking logs of (17) and applying (2) and (14) yields

$$\log(\omega_{t+1}) = \log(Q_{t+1}) + \log(K_{t+1}) + \log\left((1 - \delta) + \frac{1}{\chi}(Q_{t+1} - 1)\right) - \log \beta - \log(R_{t+1}^p) - \log(W_t) \quad (43)$$

We can now use (??) to re-write the approximate aggregate portfolio share on the left hand side in terms $\log(Q_{t+1})$. Using the same log-linearization, we can replace the third term on the right hand side with $\frac{1}{\chi} \log(Q_{t+1})$. Finally, we substitute for the last term on the right hand side with the standard portfolio return approximation (Campbell and Viceira, 1998).

$$\rho_4 \log(Q_{t+1}) = \log(Q_{t+1}) + \log(K_{t+1}) + \frac{1}{\chi} \log(Q_{t+1}) - \log \beta - \omega_t \tilde{r}_{t+1} + (1 - \omega_t)r + \frac{1}{2} \omega_t (1 - \omega_t) \sigma^2, \quad (44)$$

where $\rho_4 = \frac{\partial \omega_t}{\partial Q_t} |_{Q_t=Q_o} (1 + \delta \chi)$. Note that K_{t+1} and ω_t are known at time t . To a first order approximation, $\log(Q_{t+1})$ therefore inherits the normal distribution of $\log(D_{t+1})$. ■

E Equilibrium Expectations (under revision)

Before we can prove Propositions 3.1 and 3.2

E.1 Proof of proposition 3.1

The immediate consequence of the introduction of decentralized information about future returns is that rational arbitrageurs who know the state of the economy and their private signal cannot directly infer ϵ_t . Instead they have to update their priors about the rationally predictable stochastic components of (??), ϵ_t and ζ_{t+1} given $s_t(i)$ and Q_t .

We guess that in equilibrium,

$$\log(Q_t) = \pi_0 + \pi_1 s_{at} + \gamma \tilde{\epsilon}_t \quad (45)$$

The rational expectation of (??) is

$$E_{it}(\log(Q_{t+1}(1 - \delta) + D_{t+1}) | Q_t, s_t(i)) = r + \sigma^2 + \frac{1}{f_1} \log(Q_t) + A_0 + A_1 s_t(i) + A_2 (\log(Q_t)),$$

where the constants A_0 , A_1 and A_2 are the weights given to the prior, the private signal and the market price of capital respectively in the formation of the rational expectation of the stochastic element of future returns.¹⁹ We can obtain the rational expectation of the average agent by defining the average signal received by agents as

$$s_{at} = \int_0^1 s_t(i) \frac{W_t(i)}{W_t} di$$

and substituting in $s_t(i) = s_{at}$. Using (45) can write

$$\begin{aligned} \bar{E}^r(\log(Q_{t+1}(1 - \delta) + D_{t+1}) | Q_t, s_t(i) = s_{at}) = & r + \sigma^2 + A_0 + \left(A_2 + \frac{1}{f_1}\right) \pi_0 \\ & + \left(A_1 + \pi_1 \left(A_2 + \frac{1}{f_1}\right)\right) s_{at} + \gamma \left(A_2 + \frac{1}{f_1}\right) \tilde{\epsilon}_t. \end{aligned} \quad (46)$$

Finally, combine this expression with (??) and plug both into (??) to obtain

$$\log(Q_t) = \left[A'_0 + \left(A_1 + \pi_1 \left(A_2 + \frac{1}{f_1} \right) \right) s_{at} + \left(1 + \gamma \left(A_2 + \frac{1}{f_1} \right) \right) \tilde{\epsilon}_t \right] f_1, \quad (47)$$

¹⁹The constants given in the text are defined as $a_0 = A_0 + r + \sigma^2$, $a_s = A_1$, and $a_Q = \frac{1}{f_1} + A_2$, respectively.

where A'_0 is a positive constant. We can now match coefficients with (45) to get

$$\pi_1 = \frac{-A_1}{A_2} \quad (48)$$

and

$$\gamma = \frac{-1}{A_2}. \quad (49)$$

Given (??) we can furthermore write:

$$\epsilon_t = \frac{-1}{A_2 f_1} \tilde{\epsilon}_t, \quad (50)$$

which proves Proposition 3.1. The constants given in the text are defined as $a_0 = A_0 + r + \sigma^2$, $a_s = A_1$, and $a_Q = \frac{1}{f_1} + A_2$, respectively.

E.2 Proof of proposition 3.2

In order to prove existence of a k -percent equilibrium, we first prove the following lemma:

Lemma E.1

For any given k and σ_ϵ , there exists a critical $\hat{a}$ such that there is a near-rational k -percent equilibrium if $a_Q \in [1 - \hat{a}, 1 + \hat{a}]$.

Proof. We can rewrite expected discounted utility as

$$\begin{aligned} E_0 \left[\sum_{t=0}^{\infty} \beta^t \log(C_t) \right] &= E_0 \left[\sum_{t=0}^{\infty} \beta^t \log((1 - \beta)W_t) \right] \\ &= \frac{\log(1 - \beta)}{1 - \beta} + \frac{\beta \log \beta}{1 - \beta} + \frac{1}{1 - \beta} \log(W_0) + E_0 \left[\sum_{t=0}^{\infty} \beta^t \log\left(\prod_{s=t}^{\infty} R_s^p\right) \right] \\ &= \frac{\log(1 - \beta)}{1 - \beta} + \frac{\beta \log \beta}{1 - \beta} + \frac{r}{(1 - \beta)^2} + \frac{1}{1 - \beta} \log(W_0) \\ &\quad + \frac{1}{1 - \beta} \sum_{t=0}^{\infty} \beta^t E_0 \left[\omega_t \sigma^2 + \frac{1}{2} \omega_t^2 \sigma^2 \right] \end{aligned}$$

Using the this expression, we can now solve for the percentage rise in consumption that an individual near-rational agent would require in order to be indifferent between her near-rational heuristics and fully rational behavior. Optimal rules under fully rational behavior are denoted by a superscript r ; and superscript b refers to near-rational behavior. We have that

$$E_0 \left[\sum_{t=0}^{\infty} \beta^t \log(C_t^r) \right] < E_0 \left[\sum_{t=0}^{\infty} \beta^t \log((1 + k)C_t^b) \right]$$

$$\begin{aligned}
&\Longleftrightarrow \frac{1}{1-\beta} \log(1+k) > E_0 \left[\sum_{t=0}^{\infty} \beta^t \log(C_t^r) \right] - E_0 \left[\sum_{t=0}^{\infty} \beta^t \log(C_t^b) \right] \\
&\Longleftrightarrow \log(1+k) > \frac{1}{2} \sigma^2 E_0 \left[\sum_{t=0}^{\infty} \beta^t \left(\omega_t^{r2} - \omega_t^{b2} \right) \right] \\
&\Longleftrightarrow \log(1+k) > \frac{1}{2\sigma^2} E_0 \left[\sum_{t=0}^{\infty} \beta^t E_t^r(R_t) (1 - e^{\tilde{\varepsilon}})^2 \right] \\
&\Longleftrightarrow k > \exp \left(\frac{1}{2} \frac{1}{1-\beta} \frac{\sigma_{\tilde{\varepsilon}}^2}{\sigma^2} (1+r+\pi) \right) - 1;
\end{aligned}$$

and using equation (22) we can write

$$\hat{a} = \sqrt{2(1-\beta) \frac{\sigma_{\tilde{\varepsilon}}^2 \log(1+k)}{\sigma_{\tilde{\varepsilon}}^2 (1+r+\pi)}}.$$

■

Given this result we can now proceed to proving proposition 3.2. We apply the projection theorem to obtain explicit solutions for A_1 and A_2 . The random triple

$$\begin{bmatrix} \zeta_{t+1} - \epsilon_t + f_1(1-\delta)\epsilon_{t+1} \\ \log Q_t \\ s_t(i) \end{bmatrix} \sim N(\mu, \Sigma),$$

where

$$\mu = \begin{bmatrix} -\frac{1}{2}(\sigma^2 + \sigma_{\epsilon}^2) \\ f_1(A'_0 - \frac{1}{2}\sigma_{\epsilon}^2) \\ 0 \end{bmatrix}$$

and

$$\Sigma = \begin{bmatrix} \sigma_{\zeta}^2 + (1 + f_1^2(1-\delta)^2) \sigma_{\epsilon}^2 & -\frac{A_1}{A_2} \sigma_{\zeta}^2 - f_1 \sigma_{\epsilon}^2 & \sigma_{\zeta}^2 \\ -\frac{A_1}{A_2} \sigma_{\zeta}^2 - f_1 \sigma_{\epsilon}^2 & \left(\frac{A_1}{A_2}\right)^2 \sigma_{\zeta}^2 + f_1^2 \sigma_{\epsilon}^2 & -\frac{A_1}{A_2} \sigma_{\epsilon}^2 \\ \sigma_{\zeta}^2 & -\frac{A_1}{A_2} \sigma_{\epsilon}^2 & \sigma_{\zeta}^2 + \sigma_{\nu}^2 \end{bmatrix},$$

where $\sigma_{\zeta}^2 = \sigma^2 - f_1^2(1-\delta)^2 \sigma_{\epsilon}^2$. The projection theorem states that

$$\begin{aligned}
\begin{bmatrix} A_1 \\ A_2 \end{bmatrix} &= \Sigma_{[1,2-3]} \Sigma_{[2-3,2-3]}^{-1} \\
&= \begin{bmatrix} \frac{\left(\frac{A_1}{A_2}\right)^2 (\sigma_\zeta^4 - \sigma_\zeta^2 \sigma_\epsilon^2) - f_1 \frac{A_1}{A_2} \sigma_\epsilon^4 + f_1^2 \sigma_\zeta^2 \sigma_\epsilon^2}{\left(\left(\frac{A_1}{A_2}\right)^2 \sigma_\zeta^2 + f_1^2 \sigma_\epsilon^2\right) (\sigma_\zeta^2 + \sigma_\nu^2) - \left(\frac{A_1}{A_2}\right)^2 (\sigma_\epsilon^2)^2} \\ \frac{-\frac{A_1}{A_2} \sigma_\zeta^2 (\sigma_\zeta^2 + \sigma_\nu^2 - \sigma_\epsilon^2) - f_1 \sigma_\epsilon^2 (\sigma_\zeta^2 + \sigma_\nu^2)}{\left(\left(\frac{A_1}{A_2}\right)^2 \sigma_\zeta^2 + f_1^2 \sigma_\epsilon^2\right) (\sigma_\zeta^2 + \sigma_\nu^2) - \left(\frac{A_1}{A_2}\right)^2 (\sigma_\epsilon^2)^2} \end{bmatrix}
\end{aligned}$$

This system of equations has two solutions in $\frac{A_1}{A_2}$,

$$\frac{A_1}{A_2} = \frac{1}{2} \frac{f_1 \sigma_\epsilon^2 (\sigma_\epsilon^2 - \sigma_\zeta^2 - \sigma_\nu^2)}{\sigma_\zeta^2 (2\sigma_\zeta^2 - 2\sigma_\epsilon^2 + \sigma_\nu^2)} \pm \left(\frac{1}{4} \left(\frac{f_1 \sigma_\epsilon^2 (\sigma_\epsilon^2 - \sigma_\zeta^2 - \sigma_\nu^2)}{\sigma_\zeta^2 (2\sigma_\zeta^2 - 2\sigma_\epsilon^2 + \sigma_\nu^2)} \right)^2 - \frac{f_1^2 \sigma_\epsilon^2}{2\sigma_\zeta^2 - 2\sigma_\epsilon^2 + \sigma_\nu^2} \right)^{\frac{1}{2}}$$

Both solutions are valid and we can solve for the corresponding A_2 .

Using the previous lemma, we get the critical $\hat{a}$. We now plug in the two solutions for A_2^1 and A_2^2 . It turns out that if $\sigma_\nu > \hat{\sigma}_\nu$, $a_Q^1 \in [1 - \hat{a}, 1 + \hat{a}]$ and $a_Q^2 \notin [1 - \hat{a}, 1 + \hat{a}]$. Hence, we have shown uniqueness of a near-rational k-percent equilibrium.

F Approximate Solution for Full Model (under revision)

For convenience of notation, we abbreviate

$$R_{t+1}^p = \omega_t(1 + \tilde{r}_{t+1}) + (1 - \omega_t)(1 + r). \quad (51)$$

and omit the indicator i . Since households receive both financial and labor income equation (23) becomes

$$W_{t+1} = R_{t+1}^p(W_t - C_t + w_t L_t + \theta_t) \quad (52)$$

Maximizing (5) subject to (6) yields

$$1 = E_t \left[\beta \left(\frac{C_{t+1}}{C_t} \right)^{-1} (1 + \tilde{r}_{t+1}) \right] \quad (53)$$

and

$$1 = E_t \left[\beta \left(\frac{C_{t+1}}{C_t} \right)^{-1} \right] (1 + r). \quad (54)$$

In order to give analytically tractable solutions, we now approximate the intertemporal budget constraint and the first-order conditions using a Taylor series approach.

- The log-linearization of the intertemporal budget constraint turns out to be

$$\varpi_{t+1} - l_{t+1} \approx \vartheta + \rho_{\varpi}(\varpi_t - l_t) - \rho_c(c_t - l_t) - \Delta l_{t+1} + r_{t+1}^p \quad (55)$$

where lowercase letters denote variables in logs, Δ the first difference operator, and ϑ and ρ are constants from the log-linearization. In order to avoid confusion, the log of wealth and of non-tradable income are denoted as $\varpi = \log(W)$ and $l = \log(wL + \theta)$ respectively. (See F.1 for the derivation of (55).)

- The portfolio return can be approximated using the standard Campbell and Shiller (1988) approach

$$r_{t+1}^p = \omega_t(\tilde{r}_{t+1} - r) + r + \frac{1}{2}\omega_t(1 - \omega_t)\sigma^2. \quad (56)$$

- A log-approximation of the Euler equations, yields

$$0 = \log \beta - E_t[c_{t+1} - c_t] + E_t[\tilde{r}_{t+1}] + \frac{1}{2}Var_t[\tilde{r}_{t+1} - (c_{t+1} - c_t)]. \quad (57)$$

and

$$0 = \log \beta - E_t[c_{t+1} - c_t] + r + \frac{1}{2}Var_t[c_{t+1} - c_t]. \quad (58)$$

(See F.2 for the derivation.)

We start out by guessing the optimal policies for consumption

$$c_t - l_t = b_{0t} + b_1(\varpi_t - l_t). \quad (59)$$

From the difference of the approximate Euler equations (57) and (58), we get

$$E_t\tilde{r}_{t+1} - r + \frac{1}{2}Var_t(\tilde{r}_{t+1}) = Cov_t(\Delta c_{t+1}, \tilde{r}_{t+1}). \quad (60)$$

We use the trivial equality

$$\Delta c_{t+1} = (c_{t+1} - l_{t+1}) - (c_t - l_t) + (\Delta l_{t+1}) \quad (61)$$

together with the guess for optimal consumption and the log-linear intertemporal budget constraint and obtain

$$Cov_t(\Delta c_{t+1}, \tilde{r}_{t+1}) = b_1\omega_t\sigma^2 + (1 - b_1)\sigma_{\tilde{r}w} \quad (62)$$

under the assumption that $b_{0,t+1}$ is known at t . Substituting back into equation (60) gives us

$$E_t\tilde{r}_{t+1} - r + \frac{1}{2}\sigma^2 = b_1\omega_t\sigma^2 + (1 - b_1)\sigma_{\tilde{r}w}. \quad (63)$$

Solving for the portfolio share proves the first part of the proposition.

In order to prove optimality of the consumption rule, we start from the log-linear Euler equation for $i = p$ and equation (61) which combined yield

$$E_t [c_{t+1} - l_{t+1}] = c_t - l_t + \log \beta + E_t r_{t+1}^p - E_t \Delta l_{t+1} + \frac{1}{2} \text{Var}_t(r_{t+1}^p - \Delta c_{t+1}) \quad (64)$$

and summarize the last terms as Υ_t to write

$$E_t [c_{t+1} - l_{t+1}] = c_t - l_t + \Upsilon_t \quad (65)$$

We can simplify the variance term

$$\begin{aligned} \text{Var}_t(r_{t+1}^p - \Delta c_{t+1}) &= \text{Var}_t(\omega_t \tilde{r}_{t+1} - \Delta c_{t+1}) \\ &= \omega_t^2 \sigma^2 + \text{Var}_t(\Delta c_{t+1}) - \omega_t(c_{t+1} - c_t, \tilde{r}_{t+1}) \end{aligned} \quad (66)$$

The last term, we computed in equation (62) and the variance of log consumption growth is

$$\begin{aligned} \text{Var}_t(\Delta c_{t+1}) &= \text{Var}_t(\Delta l_{t+1} + (c_{t+1} - l_{t+1})) \\ &= \text{Var}_t((1 - b_1)\Delta l_{t+1} + \omega_t \tilde{r}_{t+1}) \end{aligned} \quad (67)$$

where we used equation (61) and again assumed that $b_{0,t+1}$ is known at time t . Plugging our guess once more in equation (65), yields

$$b_{0,t+1} + b_1 E_t [\varpi_{t+1} - l_{t+1}] = \Upsilon_t + b_{0t} + b_1(\varpi_t - l_t) \quad (68)$$

or (together with the approximation to the intertemporal budget constraint)

$$b_{0,t+1} + b_1 \vartheta + b_1 \rho_\varpi (\varpi_t - l_t) - b_1 \rho_c b_{0,t} - \rho_c b_1^2 (\varpi_t - l_t) - b_1 E_t \Delta l_{t+1} + b_1 E_t r_{t+1}^p = \Upsilon_t + b_{0t} + b_1(\varpi_t - l_t). \quad (69)$$

Matching coefficients leads to

$$b_1 = \frac{\rho_\varpi - 1}{\rho_c} \quad (70)$$

and

$$b_{0,t+1} = \Upsilon_t + b_{0,t} - b_1 E_t r_{t+1}^p + b_1 E_t \Delta l_{t+1} + b_1 \rho_c b_{0,t} - b_1 \vartheta. \quad (71)$$

The steady-state level of b_0 is thus

$$b_o = -\frac{1}{\rho_c b_1} \left[(1 - b_1) E_t r_{t+1}^p + \log \beta + \frac{1}{2} V^* - (1 - b_1) E_t \Delta l_{t+1} - b_1 \vartheta \right], \quad (72)$$

where

$$V^* = \text{Var}_t[r_{t+1}^p - \Delta c_{t+1}] \quad (73)$$

$$= (1 - b_1)^2 (\omega_t^2 \sigma^2 - 2\omega_t \sigma_{\tilde{r}w} + \text{var}_t(l_{t+1})). \quad (74)$$

This concludes the proof of proposition ??.

F.1 Derivation of (55):

We rewrite the budget constraint as

$$\frac{W_{t+1}}{(w_{t+1}L + \theta_{t+1})} = \left(1 + \frac{W_t}{(w_tL + \theta_t)} - \frac{C_t}{(w_tL + \theta_t)}\right) \frac{(w_tL + \theta_t)}{(w_{t+1}L + \theta_{t+1})} R_{t+1}^p \quad (75)$$

and take logs on both sides.

$$\varpi_{t+1} - l_{t+1} = \log(1 + \exp(\varpi_t - l_t) - \exp(c_t - l_t)) - \Delta l_{t+1} + r_{t+1}^p \quad (76)$$

A Taylor expansion gives us

$$\begin{aligned} \varpi_{t+1} - l_{t+1} &= \log(1 + \exp(E_0(\varpi_t - l_t)) - \exp(E_0(c_t - l_t))) \\ &\quad + \frac{\exp(E_0(\varpi_t - l_t))}{1 + \exp(E_0(\varpi_t - l_t)) - \exp(E_0(c_t - l_t))} (\varpi_t - l_t - (E_0(\varpi_t - l_t))) \\ &\quad - \frac{\exp(E_0(c_t - l_t))}{1 + \exp(E_0(\varpi_t - l_t)) - \exp(E_0(c_t - l_t))} (c_t - l_t - (E_0(c_t - l_t))) \\ &\quad - \Delta l_{t+1} + r_{t+1}^p \\ &= \underbrace{-\log(1 - \rho_\varpi - \rho_c)(1 - \rho_\varpi + \rho_c) - \rho_\varpi \log(\rho_\varpi) + \rho_c \log(\rho_c)}_{\equiv \vartheta} \\ &\quad + \rho_\varpi(\varpi_t - l_t) - \rho_c(c_t - l_t) - \Delta l_{t+1} + r_{t+1}^p \end{aligned} \quad (77)$$

F.2 Derivation of (57):

We start manipulate (57) to get

$$1 = E_t[\exp(\log \beta - \gamma(\Delta c_{t+1})) + \tilde{r}_{t+1}] \equiv E_t[\exp(x_{t+1})].$$

We take a Taylor expansion around an arbitrary point $\bar{x}$ to get

$$1 \approx E_t \left[\exp(\bar{x}) \left(1 + (x_{t+1} - \bar{x}) + \frac{1}{2} (x_{t+1} - \bar{x})^2 \right) \right] \quad (78)$$

Using another approximation for the exponential function, we arrive at

$$1 \approx E_t \left[1 + \bar{x} + \frac{1}{2} \text{Var}_t(x_{t+1}) \right] \quad (79)$$

(analogously for (58)).

F.3 Steady state

We can use the approximations to the optimal policy derived above in order to characterize the steady state of the economy. The market clearing condition (26) becomes

$$\int_0^1 \frac{1}{b_1} \left(\frac{E_{it}^b(1 + \tilde{r}_{t+1}) - (1 + r)}{\sigma^2} - (1 - b_1) \frac{\sigma \tilde{r}_w}{\sigma^2} \right) (W_t(i) + w_t L + \theta_t - C_t(i)) di = Q_t K_{t+1}. \quad (80)$$

Applying the definitions (7) and (??) we can write

$$\bar{E}_t \left(\frac{Q_{t+1}(1 - \delta) + D_{t+1}}{Q_t} \right) e^{\epsilon_t} = 1 + r + b_1 \omega_t \sigma^2 + (1 - b_1) \sigma \tilde{r}_w, \quad (81)$$

If at any time o the economy is at steady state, we can furthermore write $E_o B_{o+1} = B_o$ and $I_o = \delta K_o$. From equation (14) it immediately follows that $Q_o = 1 + \delta \chi$. We first calculate the steady state dividend, from which we then back out the steady state capital stock. From equation (11),

$$D_{t+1} = e^{\eta_{t+1}} F_K(K_{t+1}, L),$$

At the steady state:

$$\bar{E}_o D_{o+1} = F_K(K_o, L)$$

Plug this into (81) to obtain

$$r + b_1 \omega_o \sigma^2 + (1 - b_1) \sigma \tilde{r}_w = -\delta + \frac{1}{1 + \delta \chi} (F_K(K_o, L))$$

$$(1 + \delta \chi) (r + b_1 \omega_o \sigma^2 + (1 - b_1) \sigma \tilde{r}_w + \delta) = F_K(K_o, L)$$

This is the expression given in the text.

With a Cobb-Douglas specification and a capital share of α we can further write

$$\left(\frac{(1 + \delta \chi) (r + b_1 \omega_o \sigma^2 + (1 - b_1) \sigma \tilde{r}_w + \delta)}{\alpha L^{1-\alpha}} \right)^{\frac{1}{\alpha-1}} = K_o. \quad (82)$$

We now characterize ω_o , for which we can no longer solve in closed form. We first integrate

(6) and apply $W_t + w_t L + \theta_t - C_t = Q_t K_{t+1} + B_t$ to obtain

$$Q_t K_{t+1} + B_t + C_t = (1+r)B_{t-1} + (Q_t(1-\delta) + D_t)K_t + w_t L + \theta_t.$$

Now we can use (2) to eliminate K_{t+1} :

$$Q_t(1-\delta)K_t + Q_t I_t + B_t + C_t = (1+r)B_{t-1} + (Q_t(1-\delta) + D_t)K_t + w_t L + \theta_t.$$

This simplifies to

$$Q_t I_t + B_t + C_t = (1+r)B_{t-1} + D_t K_t + w_t L + \theta_t. \quad (83)$$

Recall that at the steady state,

$$Q_o = 1 + \delta\chi,$$

$$I_o = \delta K_o,$$

$$\theta_o = \frac{1}{2}\chi\delta^2 K_o,$$

$$B_o = \frac{1-\omega_o}{\omega_o}(1+\delta\chi)K_o,$$

and

$$D_o = F_K(K_o, L).$$

Plugging these conditions into (83) yields

$$(1+\delta\chi)\delta K_o + C_o = r(1+\delta\chi)\frac{1-\omega_o}{\omega_o}K_o + F_K(K_o, L)K_o + F_L(K_o, L)L + \frac{1}{2}\chi\delta^2 K_o \quad (84)$$

Together, equations (82), (84), (31), (70) and (71) define a non-linear system of equations that determine the unique steady state values K_o , ω_o , C_o and the parameters b_o and b_1 . We use these conditions in our numerical solution of the problem.