Aggregate uncertainty and learning in a search model*

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Abstract

We present an equilibrium search model in which buyers learn about the aggregate market conditions. In a steady-state, dynamic matching and bargaining game, the inflow of new buyers ("Demand") can be either high or low, depending on the unkown state of the world. Buyers participate in sealed bid second price auctions. Although buyers do not know the state, they learn about the state through unsuccessful bids. Beliefs depend on the bidding history, inducing endogenous heterogeneity into a population of ex ante identical agents.

We show that there exists a unique equilibrium in monotone strategies. We characterize equilibrium behavior for given levels of frictions. We show that new buyers experiment with low bids initially. Over time, the fact that the buyer is loosing provides bad news about the state of the world, revealing that there are many competing buyers. Consequently, the buyer becomes more pessimistic and bids more aggressively. The resulting equilibrium bid pattern is monotone increasing in the buyer's search duration. Different from most existing models of search with learning, in our model, the price distribution (the distribution of competing bids) is endogeneous and depends on the level of frictions. Interestingly, lower frictions are not necessarily beneficial for buyers. While lower search frictions make search less costly, the price distribution can be much worse with lower frictions. We show that, indeed, expected equilibrium payoffs may be non-monotone in the level of search frictions. Finally we show that, as search frictions vanish, the equilibrium allocation converges to the competitive allocation. Thus, the decentralized search market aggregates dispersed information and the resulting equilibrium trading price reveals the state.

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1 Introduction

In most markets agents learn about the market environment as they interact with other agents. Important aggregate variables like market tightness and the size of search frictions may be not completely known to the participating agents. Michael Rothschield observed decades ago that equilibrium search theory, while successful, suffers from "the untenable assumption that searchers know the probability distribution from which they are searching." Citing empirical evidence about unstable and highly variable wage distributions, he writes that "it seems absurd to suppose that consumers know them with any reasonable degree of accuracy." These observations might be called the "second Rothschield critique." His first critique was, of course, directed against models in which the price distribution are assumed exogeneously. Incidentally, up until today, the main contributions to search theory with learning are Rothschield (1974) and Burdett and Vishwanath (1988), provide only partial equilibrium results, assuming the distribution to be fixed exogeneously.

In the current paper, we develop an equilibrium search theory with learning that addresses both of Rothschield's critiques, allowing for aggregate uncertainty about the state of the market and deriving the price distribution as an endogenous object. In doing so, we face the challenges that have made progress in this area difficult: As an agent learns new information about market conditions he changes his behavior, but so do other agents. Therefore, upon receiving new information an agent also updates his belief about how the others behave. This feature of learning introduces endogenous dispersion in beliefs about the state of the market, and also about how the other agents behave. This dispersion then translates to an endogenous dispersion of transaction prices for agents with different beliefs.

Our equilibrium model can shed light on phenomena that are not possible to study if one takes the price (offer) distribution as given. For example, with a given offer distribution an agent is strictly better off if he is able to sample more often, i.e., if the search frictions decrease. However, in our general equilibrium approach this may not be the case. On the one hand, as frictions vanish buyers are able to sample more often, which allows them to experiment and obtain a better price. On the other hand, if all buyers follow the same strategy and sample more often, then there are more buyers on the market, increasing competition between them and reducing their equilibrium utility. In equilibrium, the second effect can dominate as frictions vanish. Although buyers can sample more often from any given price distribution, the equilibrium price distribution is getting worse as frictions vanish through buyer competition. Thus, intuition gained from a partial equilibrium search model with a fixed price distribution can be incomplete.

For our analysis, we build on the dynamic matching and bargaining model by Satterthwaite and Shneyerov (2008).[1] In particular, there is an infinite sequence of (discrete) time periods, and in

[1] We also adopt some simplifying assumptions like no reserve price, no buyer heterogeneity in valuations and second price auction as opposed to a first price auction studied there.

everyone of them a continuum of buyers and buyers arrive in the market. All buyers are randomly matched to one of the sellers and each seller conducts a sealed bid auction. After the auction she gives notice to the winner and tells her how much to pay. However, no buyer observes the bids or even the number of her respective opponents. At the end of each round all agents who transacted leave the economy. An unsuccessful agent stays for the next period with probability, and has to leave the market without trading with probability $1 - \delta$.

The defining feature of our model is that the mass of incoming buyers can be either large or small; it does not change over time, though. Competition among buyers is more intense in the high state, so the continuation value from participating in future auctions is lower. Therefore, the buyers would like to bid more in the high state than in the low one. They are, however, assumed not to know the state in which they are. Upon birth, every buyer forms an initial belief about the state of the world. Moreover, after every auction the losing buyers obtain additional information regarding the state, because they are able to make some inference about the bidding behavior of their opponents. Thus, the buyers' beliefs are endogenous and evolve over time.

As argued before, continuation values, and, thus, equilibrium bids may be non-monotone in the level of frictions δ. When frictions are very high ($\delta \to 0$) the market interaction becomes segmented over time, and thus bidding the true valuation for the object is optimal, since future market possibilities are limited. When there are many bidders in a given state compared to the number of sellers, and frictions vanish ($\delta \to 1$) then, as we argued above, competition among the bidders becomes very intense and their continuation values vanish, causing them to bid close to their use value for the object as we argue below. For intermediate values of δ neither of those pressures are present, and thus the bidders place a bid strictly less than their use value implying that equilibrium bids are not monotone in δ.

To account for benefits from participating in future auctions, the buyers' optimal strategy in our setting is to shade their bids, i.e. to bid less than they would do in the absence of any future buying opportunities. More precisely, the buyers should simply subtract the opportunity cost of winning from their valuation. In the case of a second-price sealed-bid auction, the optimal bidding function then takes the form $\beta = v - \delta V$, where V denotes the value of bidding in future auctions. We study a monotone equilibrium where buyers with more pessimistic beliefs bid more aggressively. The mechanics of the learning process in such an equilibrium is simple: if a buyer has lost many times, then he holds it more likely that there are many other buyers and is inclined to bid higher, because his continuation utility V is low. This effect is similar to the discouragement effect of Burdett and Vishwanath (1988). They establish that as the agent obtains unfavorable news over time (i.e. losing an auction), he learns that the state of the world is more likely to be unfavorable. The agent is thus discouraged from future search, preferring to accept an offer that he would have rejected earlier on.

The above bidding behavior is similar to the equilibrium of a static common value (second price) auction where the value of winning depends on the bidder's signal, but also on the signals (beliefs) of the other bidders. However, matters are more complicated, because the continuation value of a bidder depends on his future strategy, which in turn is affected by the information available to the bidder after losing the current auction. Therefore, a seller can influence the bidding strategies of agents by changing the information revealed after the auction is concluded. An important question is then how the revenue of a seller would change if she revealed the bids or the number of buyers after her auction (and the buyers knew this beforehand). We show that, for any given distribution of buyers' types, announcing the winning bid has an adverse effect on revenue, unlike in a standard static common value model with affiliated signals. The intuition behind this result is that any (ex-post) revelation of information only serves to increase the buyers' continuation payoff, thereby lowering all bids. This result is different from the linkage principle that states that in a (static) common value auction the auctioneer should reveal all that he knows to maximize his revenue. The key difference from static common value auctions is that here information revelation after the auction is important as it influences continuation values and thus current bids, an effect absent in a static auction.

We also prove that as the frictions vanish, the market allocation converges to the competitive allocation. In particular, if there are more (less) buyers than sellers, then the expected transaction prices converges to $v(0)$, which is the competitive market clearing price. It is also interesting to discuss the time pattern of an individual bidder as frictions vanish. Since the market allocation becomes competitive in the limit it is natural to think that agents bid very close to the competitive price. We show that this is not the case in equilibrium, and when one state (the high state) has more buyers than sellers while the other state (the low state) vice versa, then agents keep bidding very low for many periods, even though they learn it very quickly (upon a few losses) that the state is very likely to be high. In other words, an agent keeps experimenting with low bids for several periods hoping to obtain a bargain even though he may already know that the economy is very likely to be in the high state. Experimentation is very effective when frictions are very low ($\delta \to 1$), yielding a very fast rate of learning, and thus seemingly very little experimentation should be expected to happen in equilibrium. However, experimentation also becomes exceedingly cheap, and this second effect dominates the first yielding a long string of non-competitive bids (i.e. low bids when an agent knows that it is the high state with a very high probability) as it is shown is Section 4.1. While this exact equilibrium bid pattern is specific to our model, our result shows that once both learning and price formation occur endogenously, it is very hard to predict how agents behave over time, *even if* the equilibrium allocation is known to converge to the full information competitive equilibrium.

To establish the existence of a monotone stationary equilibrium, we establish that the bid distribution in the high state stochastically dominates the bid distribution in the low state, if the other buyers use monotone strategies. More precisely, we show that the two distributions satisfy the monotone likelihood ratio property. This implies that the continuation value of a given buyer is lower when the state is high, making him bid more aggressively if he believes that the state is more likely to be high. Using this key insight, one can use the contraction mapping theorem on value functions to prove existence of a monotone equilibrium. However, establishing that the monotone likelihood ratio property holds is non-trivial. We show it in Appendix B that the first order statistics of a random number of i.i.d. random variables may not satisfy the MLRP property even if the original random variable does. Therefore, we cannot rely on existing results and have to establish the property directly in the context of our model.

An interesting feature of our setup is that while the value function, as in other models with learning, is convex, but proving that a unique interior best reply exists is much simpler than in Gonzalez and Shi (2008). The key idea is that while the continuation value is convex, but the actual bid function is such that the utility achieved in the current period is concave making the entire objective function (quasi-)concave in the maximization variable. It is an open question whether in other equilibrium search models with learning this insight could be used to simplify analysis. Our set-up is related to Duffie and Manso (2007) who assume that all the information about current bids (and thus beliefs) are revealed and study a model of information percolation. As long as there are trading frictions, their model also generates dispersion in beliefs and trading prices, since a buyer only learns the information that other buyers have who participate in the same auction as he does. Naturally, the speed of learning is much faster under their assumptions.

The paper is organized as follows: in Section 2 we formalize the ideas laid out in this introduction by setting up a model of sequential auctions with uncertainty about the ratio of buyers to sellers in the market ("aggregate demand uncertainty"). In Section 3 we conduct preliminary analysis, and Section 4 shows that the model has a unique steady state equilibrium and characterize some of its properties. Section 5 asks whether sellers could increase their revenue by employing a more transparent informational regime than the one assumed in Section 3. Finally, a discussion of the model and conclusions are presented in Section 6.

2 Model and equilibrium

2.1 Setup

We provide a dynamic model of a decentralized trading market that is set in discrete time where agents learn about aggregate market conditions over time. We study the steady state equilibrium of this model. There are a continuum of buyers and a continuum of sellers present in the market. In periods ($t = ..., -1, 0, 1, ...$) the sellers and buyers trade an indivisible, homogeneous good. The sellers do not exhibit any strategic behavior (they do not take any decision, like setting a reserve price) in the baseline model.[2] The buyers are heterogeneous with respect to their beliefs about the state of the economy; each buyer is identified by his belief $\theta \in [0,1]$ with an interpretation that is discussed below. It is important to note that the belief of a given buyer is not fixed, but is evolving over time in an endogenous manner.

There are two states of the world, $w \in \{H, L\}$. In the low state the mass of buyers born each period is d^L, while in the high state it is d^H greater than d^L. In both states of the world there is an equal number of sellers born each period, which we normalize to 1. Each buyer enters the market with a belief $\theta_0 \in [\underline{\theta}, \overline{\theta}]$ with $1/2 \leq \underline{\theta} < \overline{\theta} < 1$.[3] The assumption that $\underline{\theta} > 1/2$ is while restrictive, still quite natural. The mere fact of being born is bad news because there are more buyers in the high state. Thus, if buyers get little additional information (i.e. the difference between $\underline{\theta}$ and $\overline{\theta}$ is not too large) and the ex-ante probability of the two states are equal, then their priors are above 1/2. This is, in effect, what we assume. We also assume that the beliefs of the entering buyers in state $\{H, L\}$ are distributed according to the commonly known atomless distribution functions G^H, G^L respectively with corresponding densities g^H, g^L. It is useful to note, that the no-introspection condition implies that upon knowing ones own type a buyer would not want to update his belief, which places a restriction on functions g^H, g^L. Using Bayes rule, this restriction can be written $\frac{g^H(\theta)}{g^H(\theta)+g^L(\theta)} = \theta$.

Each period consists of several steps:

1. Entry occurs as described in the previous paragraph.

2. Each buyer is randomly matched with one seller. In other words, a seller may match with any buyer with the same probability. The probability that a seller is matched with $k = 0, 1, 2, ...$ buyers is equal to $e^{-\mu}\mu^k/k!$, where μ is the endogenous ratio of the mass of buyers to the mass of sellers.[4] Intuitively, if larger the value of market tightness μ, the larger the expected number of bidders at each seller is. No buyer observes how many other buyers are matched with the same seller.

3. Each seller runs a sealed bid second price auction with no reserve price, so each period each buyer submits a bid to the seller he is matched with. Each period any buyer only observes his own bid, and whether he managed to trade or not.

[2]This simplifying assumption is made to keep tractability. We discuss the problem of adding a reservation price after the model description. In Section 5 we consider whether the sellers have any incentive to (unilaterally) change the amount of information revealed in the auction process. We show that any individual seller has no incentive to do so lending credibility to our starting assumptions.

[3]The spread of the support can be arbitrarily small. In fact, in a previous version of the model we set it degenerate (i.e. $\underline{\theta} = \overline{\theta}$) and our results were very similar qualitatively.

[4]This follows, because for large markets the binomial distribution approximates the Poisson distribution, as it is stated by Poisson's theorem.

4. A seller leaves the market if he sold his object, otherwise he stays with probability $1-\delta$ to offer his object in the next period. The winning buyer pays, obtains the object and leaves the market. The losers stay in the pool with probability $\delta \in [0,1)$ and are matched with a new seller in the next period. Those who do not stay leave the market permanently, obtaining a utility of zero in this period and forever after.

5. Upon losing all buyers update their beliefs each period. The motion of beliefs of the buyers is formally described in the next Section. At this point it is sufficient to note that when a buyer loses he updates his belief using Bayes rule: his starting belief in a given period is his prior, and he conditions only on having had a rival bidder who bid more than him.[5]

6. All remaining agents (not exiting upon trading or with the exogenous probability $1-\delta$ in the previous period) stay on the market for the next period.

It is worthwhile to discuss the objective function of the buyers. As we stated earlier, each buyer is risk neutral and maximizes the expected surplus with no discounting. However, the exogenous leaving probability $1-\delta$ takes several roles, including discounting. On the individual level of the agents, the exit rate δ acts similar to a discount rate: not trading today creates a risk of losing trading opportunities altogether with probability $1-\delta$. On the aggregate level, the exit rate ensures that a steady state exists for all strategy profiles.

Finally, let us discuss our choice of modeling the sellers as non-strategic agents. If the sellers were able to choose an open reserve price, then they would be able to signal their beliefs and there would be multiple equilibrium induced by off equilibrium beliefs. If the sellers chose a secret reserve price, i.e. decide ex post whether to trade, then the entire bid distribution would influence the decision of the seller and the analysis would also become intractable. Therefore, to simplify matters we choose to completely ignore the incentives of the seller, and focus only on the other side of the market, still achieving endogenous price formation through buyer competition.

2.2 Steady State Equilibrium and Some Preliminary Results

We restrict attention to equilibria in stationary strategies where the distribution of bids depends only on the state, not on (calendar) time. An immediate consequence is that in any period the set of optimal bids of a buyer depends only on her current belief about the likelihood of being in the high state; we denote her belief by $\theta \in [0,1]$.

For tractability, we restrict attention to symmetric pure strategy equilibria in which the bid is a strictly increasing function of the belief of the buyer. A symmetric steady state equilibrium

[5]Since losing is more likely in the high state when there are more rivals on average, therefore one expects that the posterior upon losing attaches a higher probability to the high state than the prior. This feature will hold in the equilibrium we concentrate on.

is a vector $(\Gamma^H, \Gamma^L, S^H, D^H, S^L, D^L, \beta, \theta^+)$. We now describe each of these components. First, in state $w \in \{H, L\}$ the distribution of beliefs is Γ^w. We restrict attention to the case where Γ^w is atomless.[6] The masses of buyers and sellers present in any given period are S^w and D^w. Second, the bidding strategy β maps a belief from $[0, 1]$ to a non-negative bid where β is a strictly increasing function. Define the generalized inverse of β as $\beta^{-1}(b) = \inf\{\theta|\beta(\theta) \geq b\}$[7]. Finally, function $\theta^+(\theta, x)$ describes the posterior of a buyer with initial belief θ conditional on losing against buyers with beliefs above x. We characterize the equilibrium requirements for these objects; a formal definition of equilibrium follows at the end of this Section.

We first derive the posterior function θ^+. The fact of loosing implies that the buyer was in a match where the bidder with the the highest belief θ had belief above x. Let $\theta_{(1)}$ denote the first order statistics among beliefs in any given match, and let $\Gamma^w_{(1)}$ denote its cdf in state $w \in \{H, L\}$, so $\Gamma^w_{(1)}(x)$ is the probability (in state w) that the highest belief in the auction is below x. A convenient implication of Poisson matching is that this is also the probability that the competitors of a given bidder all have beliefs below x. Thus, Bayes rule (if it applies) requires that the posterior θ^+ is

$$\theta^+(\theta, x) = \frac{\theta\left(1 - \Gamma^H_{(1)}(x)\right)}{\theta\left(1 - \Gamma^H_{(1)}(x)\right) + (1 - \theta)(1 - \Gamma^L_{(1)}(x))}. \tag{1}$$

The posterior defines an *endogenous* transition function: given a type θ_n and an action x today, tomorrow's belief upon losing will be $\theta_{n+1} = \theta^+(\theta_n, x)$.

The event in which the all buyers have belief below x includes the event in which there is no buyer at all in the match. Hence, $\Gamma^w_{(1)}(0)$ is the probability that there is no bidder in a given auction[8] and $\Gamma^w_{(1)}(0)$ is the probability that a given bidder does not have any competitor. Let $\mu^w = \frac{D^w}{S^w}$ measure market tightness in state w, and note, that the fact that Γ^w has the Poisson distribution implies that $\Gamma^w_{(1)}(0) = e^{-\mu^w}$. Since types are distributed without atoms by assumption, Poisson matching implies that for all θ

$$\Gamma^w_{(1)}(\theta) = e^{-\mu^w(1-\Gamma^w(\theta))}.$$

To see this, note that the number of buyers beliefs higher than θ has a Poisson distribution; and the expected number of such buyers is equal to the ratio of buyers with belief higher than θ to the sellers, which is equal to $\mu^w(1 - \Gamma^w(\theta))$. Then the probability that there is no such buyer visiting

[6] This restriction is without loss of generality in the sense that no equilibrium exists where Γ^w has atoms. Therefore, to ease exposition we do not formally discuss the case with atoms. The straightforward intuition for why Γ^w is atomless in equilibrium is as follows. Since the entering cohort has an atomless distribution, and bidding and updating are strictly monotone, therefore no atoms can exist in Γ^w, unless there was a belief $\theta^* < 1$ such that a bidder with such a belief would keep his belief upon losing. We show that no such θ^* exist in our existence proof.

[7] We adopt the convention that $\beta^{-1}(b) = 1$ if $\beta(\theta) < b$ for all θ.

[8] In equilibrium, there is no mass point at any belief θ.

(denoted by $\Gamma^w_{(1)}(\theta)$ above) is indeed equal to $e^{-\mu^w(1-\Gamma^w(\theta))}$. If the probability of the high state is θ, then the expected probability that the highest belief is below x is given by

$$\Gamma^\theta_{(1)}(x) = \theta\Gamma^H_{(1)}(x) + (1-\theta)\Gamma^L_{(1)}(x).$$

To derive the steady state condition for the population suppose that the size of the population of sellers is S^w today; tomorrow's population of sellers will consists of those sellers who did not get matched with a buyer and the newly entering sellers. In steady state, these two populations must be identical,

$$S^w = 1 + \delta\Gamma^w_{(1)}(0)\,S^w. \tag{2}$$

Denote the inflow of buyers with type less than θ as $d^w(\theta) = d^w G^w(\theta)$. Then $d^w(\theta) = 0$ if $\theta \le \underline{\theta}$ and $d^w(\theta) = d^w$ if $\theta \ge \overline{\theta}$. The stationarity condition can be written as

$$D^w\Gamma^w(\theta) = d^w(\theta) + \delta D^w \int_{\{\theta':\theta^+(\theta',\beta(\theta'))\le\theta\}} \left(1-\Gamma^w_{(1)}(\theta')\right) d\Gamma^w(\theta'). \tag{3}$$

Given a population of buyers today, the steady state mass of buyers with type below θ is equal to $D^w\Gamma^w(\theta)$, hence the left hand side of (3). This mass has to be equal to the inflow with type less than θ ($d^w(\theta)$) plus the mass of buyers who lose, survive, and update to some type less than θ (the second term on the right hand side). In a symmetric equilibrium with increasing bidding strategies, the probability that type θ' loses is $1-\Gamma^w_{(1)}(\theta')$, since she loses if and only if there is another bidder with a higher type (and therefore, a higher bid).

Given that strategy β is used by all other buyers, the value function must satisfy

$$V(\theta) = \max_b v\Gamma^\theta_{(1)}(0) + \int_{\{t:\beta(t)\le b\}} (v-\beta(t))\,d\Gamma^\theta_{(1)}(t) + \delta\left(1-\Gamma^\theta_{(1)}\left(\beta^{-1}(b)\right)\right) V\left(\theta^+\left(\theta,\beta^{-1}(b)\right)\right). \tag{4}$$

Also, the equilibrium bidding strategy β must satisfy the Bellman condition for optimality given V

$$\beta(\theta) \in \arg\max_b\ v\Gamma^\theta_{(1)}(0) + \int_{\{t:\beta(t)\le b\}} (v-\beta(t))\,d\Gamma^\theta_{(1)}(t) + \delta\left(1-\Gamma^\theta_{(1)}\left(\beta^{-1}(b)\right)\right) V\left(\theta^+\left(\theta,\beta^{-1}(b)\right)\right). \tag{5}$$

A *steady state equilibrium in symmetric, strictly increasing bidding strategies* (an equilibrium from now on) consists of masses of buyers and sellers, S^H, D^H, S^L, D^L and distribution functions Γ^H, Γ^L such that the steady state conditions (2) and (3) hold for all θ with the additional requirements that $\Gamma^H(0) = \Gamma^L(0) = 0$ and $\Gamma^H(1) = \Gamma^L(1) = 1$; transition functions θ^+ that is consistent with Bayes' rule whenever applicable (i.e. (1) holds); and a strictly increasing bidding function β that is optimal, i.e. (5) holds where the value function V satisfies (4).

To study payoff consequences of deviations, it is useful to introduce further notation and some characterization results. Let $EU(\theta,\beta|w)$ denote the utility of a bidder with belief θ who follows strategy β in state $w = H, L$ if all other buyers follow strategy β:

$$EU(\theta,\beta|w) = v\Gamma^{w}_{(1)}(0) + \int_{\{t:t\leq\theta\}} (v-\beta(t))\, d\Gamma^{w}_{(1)}(t) + \delta\left(1-\Gamma^{w}_{(1)}(\theta)\right) EU(\theta^{+}(\theta,\theta),\beta|w),$$

and let

$$EU\left(\theta,\beta|\hat{\theta}\right) = \hat{\theta}EU(\theta,\beta|H) + (1-\hat{\theta})EU(\theta,\beta|L).$$

Note, that $EU(\theta,\beta|\cdot)$ is linear in $\hat{\theta}$ by construction. Also, by construction $V(\theta) = EU(\theta,\beta|\theta)$ and $EU(\theta,\beta|\theta) \geq EU\left(\theta',\beta|\theta\right)$ for all θ' by optimality, i.e., $EU(\cdot,\beta|\cdot)$ is a family of linear functions that support $V(\theta)$. We will use these properties extensively below.

Next, we show that the value function V is convex, which shows that individual buyers value information. This is stated in the next Lemma together with a useful envelope formula. The first equation states that payoffs depend on the belief only through the direct effect. The second equation uses the linearity of expected payoffs in beliefs. The envelope formula is immediate and the proof is omitted:

Lemma 1 *(Envelope Formula) Given any increasing bidding strategy β used by the other bidders, the value function $V(\theta)$ of a bidder is convex. If β is an optimal strategy when the other bidders use β, then at all differentiability points of V,*

$$\begin{aligned} V'(\theta) &= \frac{\partial}{\partial\hat{\theta}} EU\left(\theta,\beta|\hat{\theta}\right)|_{\hat{\theta}=\theta} \\ EU\left(\theta,\beta|\hat{\theta}\right) &= V(\theta) + \left(\hat{\theta}-\theta\right)V'(\theta) \end{aligned}$$

Proof: The value function is convex, because with $\theta^{\alpha} = \left(\alpha\theta + (1-\alpha)\hat{\theta}\right)$

$$\begin{aligned} V\left(\alpha\theta + (1-\alpha)\hat{\theta}\right) &= \alpha EU(\theta^{\alpha},\beta|\theta) + (1-\alpha)EU\left(\theta^{\alpha},\beta|\hat{\theta}\right) \\ &\leq \alpha EU(\theta,\beta|\theta) + (1-\alpha)EU\left(\hat{\theta},\beta|\hat{\theta}\right) \\ &= \alpha V(\theta) + (1-\alpha)V\left(\hat{\theta}\right), \end{aligned}$$

where the equalities follow by construction, and the inequality follows by the remark after the definition of function EU. QED

Thus, the derivative of the value function at θ allows us to derive the expected payoff from following the bid sequence used by type θ, given any probability $\hat{\theta}$, not necessarily equal to the actual belief θ. The usefulness of this will be clear as we proceed with our analysis. Figure 1 (see the last page) illustrates what we have discussed above.

3 Characterization and existence of equilibrium

3.1 Necessary conditions for equilibrium

In this Section we describe properties of equilibrium. Let us start with a short summary of observations; we prove these observations in a sequence of Lemmas below.

(i) The market is "tighter" in the high state, $\Phi^H(1)/S^H > \Phi^L(1)/S^L$. Thus, the expected number of bidders at any given seller is higher in the high state.

(ii) Loosing is bad news. Having lost, a buyer updates his beliefs and becomes more pessimistic. Coupled with our assumption that bids are strictly increasing, this implies that a buyer's bid is increasing over time. Moreover, having lost, a buyer "regrets" his bid; the optimal bid condition conditional on losing is strictly higher.

(iii) Buyers stay longer in the market in the high state and the expected price they pay is higher. The higher expected price in the high state is the consequence of more intense competition.

The observations are not immediate. The first Lemma which characterizes the steady state mass of agents on the market and the steady state belief distribution of the agents (Lemma 2). The second Lemma (Lemma 3) aids proving the monotonicity results of point (ii). Finally, we provide a formula relating the bid function with the value function (Lemma 1).

We start the analysis by showing that the total mass of buyers and sellers, and the equilibrium belief distributions are uniquely determined in equilibrium. The proof is in (online) Appendix 1:

Lemma 2 *The following results hold:*

i)There is a unique mass of buyers D^w and sellers S^w in equilibrium. The market is tighter in the high state, $\frac{D^H}{S^H} > \frac{D^L}{S^L}$.

ii) If bidding strategies are strictly increasing, then there exists a unique distribution Γ^w that satisfies the steady state conditions. Γ^w is twice continuously differentiable almost everywhere.

The next Lemma provides useful characterization results that help in showing that a buyer with higher prior belief has also a higher posterior, and that a buyer with a higher belief bids higher:

Lemma 3 *i) The distributions $\Gamma^H_{(1)}, \Gamma^L_{(1)}$ and their densities (where they exist) satisfy the Monotone Likelihood Ratio Property: $\Gamma^H_{(1)}/\Gamma^L_{(1)}$ and $\gamma^H_{(1)}/\gamma^L_{(1)}$ are increasing in θ.*

ii) On the support of Γ^w, the posterior $\theta^+(\theta, \theta')$ is strictly increasing in θ' and in θ for all $\theta' \geq \underline{\theta}$.

The proof of the results concerning equilibrium masses $\{D^w, S^w\}$ utilizes the fact that, within each match of a seller with $N \geq 1$ buyers, one buyer will win, independent of the equilibrium bid functions and equilibrium type distributions. This allows us to simplify the analysis and describe the steady state masses of buyers and sellers by two simple equilibrium conditions. It is also intuitive that in the high state the market is tighter for the sellers than in the low state. Algebraically this follows from the fact that a higher inflow in the high state implies that more buyers stay on the market, because each buyer has a lower chance to transact. Moreover, each seller has a higher chance to transact in the high state, so they leave the market quicker, and thus there are less sellers on the market in the high state.

The existence, uniqueness and differentiability of the type (belief) distributions Γ^H, Γ^L are proved in a constructive manner. It is useful here to highlight the intuition behind the monotonicity results. The distributions of beliefs Γ^H, Γ^L (and their densities) must necessarily have the monotone likelihood ratio.[9] The fact of losing an auction entails bad news; and the fact of loosing an auction with a higher bid entails even worse news, if the first order statistic $\Gamma^w_{(1)}(\theta)$ has the monotone likelihood property. But, since the number of bidders is random, $\Gamma^w_{(1)}(\theta)$ is the first order statistic of a *random* number of random variables and, which each of these random variable satisfies the MLRP, $\Gamma^w_{(1)}(\theta)$ itself does not need to,[10] see the counterexample in Appendix 2. However, the Lemma above shows that under the assumption that $\underline{\theta} \geq 1/2$ the monotone likelihood ratio property holds and not only for the posteriors upon losing, but also for the posteriors upon tieing (at the top spot) that is relevant for bidding incentives as we show below in Lemma 1. Our proof in the Appendix depends on the assumption that $\underline{\theta} \geq 1/2$, but one can show that the equilibrium losing probability is monotone even without this assumption. However, the posterior upon tieing is monotone only under this assumption. After the formal proof in the Appendix, we discuss the reasons for this.

Given uniqueness of Γ^w, from now on we may restrict attention to the unique equilibrium distribution and the equilibrium posteriors. Therefore, to characterize the equilibrium we only need to characterize equilibrium bid incentives. Before stating the formal characterization result, we define the posterior of type θ after tieing with a buyer with belief x at the top spot as

$$\theta^0(\theta, x) = \frac{\theta\left(1 - \gamma^H_{(1)}(x)\right)}{\theta\left(1 - \gamma^H_{(1)}(x)\right) + (1-\theta)(1 - \gamma^L_{(1)}(x))}, \tag{6}$$

[9] Given a uniform prior, the belief of a type θ can be calculated from Bayes rule as

$$\theta = \frac{\gamma^H(\theta)}{\gamma^H(\theta) + \gamma^L(\theta)}.$$

Rewriting implies that $\frac{\gamma^H}{\gamma^L}$ increases in θ. If the density of θ has the MLRP, it follows from standard arguments that the cdf of θ has the MLRP property as well.

[10] The first order statistics of a *fixed* number of random variables, each of which having the MLRP, has the MLRP as well.

where $\gamma^{w}_{(1)}$ is density function corresponding to $\Gamma^{w}_{(1)}$, where it exist. Then in equilibrium the posterior of a bidder with belief θ upon tieing is $\theta^{0}(\theta,\theta)$. In the proof of Lemma 2 we discuss the tieing posterior $\theta^{0}(\theta,\theta)$ has the same monotonicity properties that were stated for $\theta^{+}(\theta,\theta)$ there. The following Lemma provides a characterization for the equilibrium bid function:

Lemma 4 *Suppose the bidding strategy β is strictly increasing and part of an equilibrium. Then β satisfies the following equation for (Γ^{w}) almost all θ :*

$$\beta(\theta) = v - \delta V\left(\theta^{+}(\theta,\theta)\right) + \delta V'\left(\theta^{+}(\theta,\theta)\right)(\theta^{+}(\theta,\theta) - \theta^{0}(\theta,\theta)).$$

Proof: Our starting point for characterizing the equilibrium bid function is formula (5). The derivative of the function to be maximized in that formula at $b = \beta(x)$ is

$$\beta^{-1\prime}(\beta(x))[\gamma^{\theta}_{(1)}(x)\{v - \beta(x) - \delta V\left(\theta^{+}(\theta,x)\right)\} + \delta\left(1 - \Gamma^{\theta}_{(1)}(x)\right)V'\left(\theta^{+}(\theta,x)\right)\frac{\partial}{\partial x}\theta^{+}(\theta,x)]. \quad (7)$$

Using the definition of θ^{+} implies that

$$\frac{\partial}{\partial x}\theta^{+}(\theta,x) = \frac{-\theta\gamma^{H}_{(1)}(x)\left(1 - \Gamma^{\theta}_{(1)}(x)\right) + \gamma^{\theta}_{(1)}(x)\,\theta\left(1 - \Gamma^{H}_{(1)}(x)\right)}{\left(1 - \Gamma^{\theta}_{(1)}(x)\right)^{2}},$$

and thus

$$\begin{aligned}\left(1 - \Gamma^{\theta}_{(1)}(x)\right)\frac{\partial}{\partial x}\theta^{+}(\theta,x) &= \frac{\gamma^{\theta}_{(1)}(x)\,\theta\left(1 - \Gamma^{H}_{(1)}(x)\right)}{1 - \Gamma^{\theta}_{(1)}(x)} - \theta\gamma^{H}_{(1)}(x) = \\ &= \gamma^{\theta}_{(1)}(x)(\theta^{+}(\theta,x) - \theta^{0}(\theta,x)),\end{aligned} \quad (8)$$

where the last step uses that $\theta^{0}(\theta,x) = \frac{\theta\gamma^{H}_{(1)}(x)}{\gamma^{\theta}_{(1)}(x)}$. Using (7), the relevant first order condition indicating that $b = \beta(\theta)$ is optimal for type θ can be written as

$$\begin{aligned}\beta(\theta) &= v - \delta V\left(\theta^{+}(\theta,\theta)\right) + \delta\left(1 - \Gamma^{\theta}_{(1)}(\theta)\right)V'\left(\theta^{+}(\theta,\theta)\right)\frac{\frac{\partial}{\partial x}|_{x=\theta}\,\theta^{+}(\theta,x)}{\gamma^{\theta}_{(1)}(\theta)} = \\ \beta(\theta) &= v - \delta V\left(\theta^{+}(\theta,\theta)\right) + \delta V'\left(\theta^{+}(\theta,\theta)\right)(\theta^{+}(\theta,\theta) - \theta^{0}(\theta,\theta)),\end{aligned} \quad (9)$$

where the second step uses (8). QED

The formula (9) expresses a necessary condition for an equilibrium bid function. To provide an intuition, note that in a standard second price auction the optimal bid is equal to the value of winning minus the value of the outside option. But in our dynamic model, the bid function cannot be expressed as the difference between v and the continuation value upon losing $\delta V\left(\theta^{+}(\theta,\theta)\right)$.

To investigate why the bid function in (9) takes a more complex form, we combine Lemma 1, and Lemma 4 to provide another, perhaps more intuitive, characterization of the equilibrium bid function:

Corollary 1 *Suppose the bidding strategy β is strictly increasing and part of an equilibrium and let θ^+ and θ^0 be defined as above. Then, for (Γ^w) almost all types θ it holds that*

$$\beta(\theta) = v - \delta EU\left(\theta^+(\theta, \theta), \beta | \theta^0(\theta, \theta)\right).$$

The following two observations provide intuition for the form of the equilibrium bid function as characterized in Corollary 1. First, the bid must be equal to the valuation minus the relevant continuation payoff from tomorrow in a dynamic second price auction. Second, to describe the relevant continuation value, note that the strategy adopted from tomorrow on is the optimal strategy given the updated belief, which belief can be written as $\theta^+(\theta, \theta)$. However, the utility stream accruing from this strategy is evaluated using the posterior conditional on being tied, because changing one's bid affects one's profits only when tieing. Therefore, the expected continuation payoff is calculated by evaluating the utility derived from the bidding sequence of a bidder with belief $\theta^+(\theta, \theta)$, but assuming that the probability of the high state is equal to $\theta^0(\theta, \theta)$.

3.2 Existence and Uniqueness of Equilibrium

In this Section we prove that the exogenous parameters - δ, d^H, d^L - determine the market outcome in an essentially unique way. We have already established that the equilibrium belief distributions (Γ^H, Γ^L) are unique. In the rest of the Section we establish existence and uniqueness of the equilibrium bid function:

Proposition 1 *There exists an almost everywhere unique symmetric equilibrium in strictly increasing bidding strategies; payoffs $V(\theta)$ are unique and for almost all θ,*

$$\beta = v - \delta EU\left(\theta^+(\theta, \theta), \beta | \theta^0(\theta, \theta)\right). \tag{10}$$

The proposition is proven through a sequence of Lemmas which are combined at the end of this Section. We show that if function EU satisfies certain minimally necessary conditions, then the second order condition of bidder optimization holds, and one can express the optimal bid β as a simple function of EU (Lemma 5). Then we define a mapping T that describes the implied function of continuation utilities if all other bidders use the optimal bid function β given continuation value EU. We show that there is a unique fixed point of mapping T (Lemma 6), which is then shown to imply that there exists a unique equilibrium bid function.

Using convexity of the value function V, we first characterize the utility from bidding in a certain way at a given value of the beliefs. In other papers of search models with learning it has been shown that convexity of value function may create problems for proving existence of equilibrium, since it implies that a first order approach may not be used. However, in our model this property turns out to be helpful and key in the analysis that follows below. As we have seen in Lemma 1, in equilibrium the envelope formula implies a close connection between the value function $V(\theta)$ and the payoffs of any type $\hat{\theta}$ from following the strategy of type θ, which was denoted by $EU\left(\theta,\beta|\hat{\theta}\right)$. To exploit this connection we introduce a set of functions that can play the role of expected payoffs $EU\left(\theta,\beta|\hat{\theta}\right)$ in equilibrium. A function $W\left(\theta,\hat{\theta}\right)$ is called **regular** if θ parameterizes a family of nonincreasing, linear functions $W(\theta,\cdot)$, which are the tangents of a nonincreasing convex functions, i.e. for all $\theta,\hat{\theta}$ it holds that $W(\theta,\theta)\geq W(\hat{\theta},\theta)$ and $\widetilde{V}(\theta)=W(\theta,\theta)$ for some convex non-increasing function $\widetilde{V}:[0,1]\rightarrow[0,1]$. The properties of regular functions are described formally in Appendix 3. (Recall that these are exactly the properties of $EU\left(\theta,\beta|\hat{\theta}\right)$.)

The following important Lemma shows that, given the continuation utility function W, the necessary condition for optimal bidding (see formula (10)) is also sufficient if the presumed value function W satisfies the relevant regularity conditions:

Lemma 5 *If function $\beta=v-\delta W(\theta^{+}(\theta,\theta),\theta^{0}(\theta,\theta))$ is strictly increasing, then it is a best reply for each bidder to use bid function β, if all the other bidders use β and the continuation values are given by $\widetilde{V}(\theta)=W(\theta,\theta)$, where W is a regular function.*

The rest of the proof establishes that there exists a function W^{*} such that the Bellman equation (4) is satisfied when one defines function V by equation $V(\theta)=W^{*}(\theta,\theta)$.[11] This is done by defining a map T on regular continuation utility functions whose fixed point satisfies this consistency requirement. More precisely, given a regular function W, let $TW\left(\theta,\hat{\theta}\right)$ be the implied expected payoff of a type $\hat{\theta}$ who follows the bidding sequence of type θ now and forever in the future. To calculate the implied function TW, take a regular function W and the corresponding candidate bidding function $\beta=v-\delta W\left(\theta^{+}(\theta,\theta),\theta^{0}(\theta,\theta)\right)$. Note, that conditional on losing with a bid $\beta(\theta)$, and having an initial type $\hat{\theta}$, the induced belief is $\theta^{+}\left(\theta,\hat{\theta}\right)$. Therefore, conditional on losing with a strategy that follows the bidding sequence of type θ now and forever in the future, the continuation payoff can be written as $W\left(\theta^{+}(\theta,\theta),\theta^{+}(\theta,\hat{\theta})\right)$.Then the implied utility $TW\left(\theta,\hat{\theta}\right)$ can be calculated as follows:

$$TW\left(\theta,\hat{\theta}\right)=$$

$$v\Gamma^{\hat{\theta}}(0)+\int_{\theta_0}^{\theta}(v-\beta(t))\,d\Gamma_{(1)}^{\hat{\theta}}(t)+\delta\left(1-\Gamma_{(1)}^{\hat{\theta}}(\theta)\right)W\left(\theta^{+}(\theta,\theta),\theta^{+}\left(\theta,\hat{\theta}\right)\right).$$

[11] We prove a slightly different version of the Bellman equation, since we use function W instead of V as the main tool of the analysis. It is easy to see that the version proven below implies (4) after setting $\theta=\hat{\theta}$.

The next Lemma shows that there exists a unique regular solution of equation $TW = W$. The Lemma follows after checking that T satisfies Blackwell's sufficient condition for a contraction and that if W is regular, then TW is regular.

Lemma 6 *There is a unique regular function such that $W^* = TW^*$ and the induced bid function $\beta^* = v - \delta W^* \left(\theta^+ (\theta, \theta), \theta^0 (\theta, \theta)\right)$ is strictly increasing in θ.*

Proof of Proposition 1:

Using the above results, we turn to proving existence of an equilibrium. By Lemma 6, the candidate bidding function $\beta^* = v - \delta W^* \left(\theta^+ (\theta, \theta), \theta^0 (\theta, \theta)\right)$ is strictly increasing in θ. Lemma 5 implies that β^* is a best response for a bidder, if all other bidders follow the same strategy and the continuation values are given by function W^*, hence (5) holds after substituting $\theta = \hat{\theta}$ and letting $V(\theta) = W^*(\theta, \theta)$. Finally, by being a fixed point of operator T, W^* is indeed the value function if all bidders employ strategy β; in other words (4) holds after substituting $\theta = \hat{\theta}$ and using that $V(\theta) = W^*(\theta, \theta)$. The other conditions for equilibrium hold by the way $\Gamma^H, \Gamma^L, S^H, S^L, D^H, D^L$ were constructed (see Lemma 2). Therefore, an equilibrium with strictly increasing bidding strategies exists. Uniqueness of the equilibrium bid function β follows directly from the uniqueness result of Lemma 6, and the fact that the equilibrium value function W is regular, and thus $W = W^*$ has to hold. QED

4 Effect of market frictions on behavior and allocation

In this Section we analyze the case where the level search frictions vanish in a series of markets indexed by k, i.e. it holds that $\lim_{k \to \infty} \delta_k = 1$. To facilitate comparative statics analysis, let us introduce another interpretation for δ. Let Δ denote the length between time periods, and d be the (fixed) probability of participation in the *next* period. Letting $\delta = 1 - \Delta d$, one can interpret a change in δ as a change in Δ indicating how much time it takes to come back to participate one more time. In this interpretation, the market friction $1 - \delta$ arises because it takes time to come back to the market after a loss. As this time lag vanishes, market frictions vanish.

We first ask the question whether the expected price converges to the competitive price (Section 4.1), and then investigate the time pattern of the bids of an individual buyer as search frictions vanish (Section 4.2). In the end of this Section we combine our results to discuss how market allocations, behavior and the value of information change when the level of market frictions changes.

4.1 The convergence of the expected trading price

The main result of the Section states that as market frictions vanish, the equilibrium transaction price converges to the competitive price in both states:

Proposition 2 *Consider the case $d^H > 1$ and $d^L < 1$. For any type $\theta \in [\underline{\theta}, \overline{\theta}]$ the expected price conditional on winning in the high state converges to v, and the expected price conditional on winning in the low state converges to 0. If $d^H, d^L > 1$, then for any type $\theta \in [\underline{\theta}, \overline{\theta}]$ the expected price conditional on winning converges to v in both states, and if $d^H, d^L < 1$, then for any type $\theta \in [\underline{\theta}, \overline{\theta}]$ the expected price conditional on winning converges to 0 in both states.*

The result is proven in the online Appendix. We describe the two most important intermediate results here, and the main difficulties the proof involves. The first important result shows that as market frictions vanish, the market imbalance between the two sides explodes:

Lemma 7 *Suppose that $\left(\beta_k, D_k^w, S_k^w, \theta_k^+\right)$ is a sequence of equilibria with $\lim_{k\to\infty} \delta_k = 1$. Then*

$$\lim_{k\to\infty} \mu_k^w = \lim_{k\to\infty} \frac{D_k^w}{S_k^w} = \begin{cases} 0 & \text{if } d^w < 1 \\ \infty & \text{if } d^w > 1 \end{cases}.$$

If $d^w < 1$, the probability of being the sole bidder $e^{-D_k^w/S_k^w} \to 1$. If $d^w > 1$, the probability of loosing $\left(1 - e^{-D_k^w/S_k^w}\right) \to 1$. Moreover, the probability of being the sole bidder converges to zero faster than the exit probability, $\frac{1-\delta_k}{e^{-\mu_k^w}} \to \infty$.

All the results, except for the last, of Lemma 7 follow from standard arguments: in a dynamic market interaction agents on the long side of the market keep accumulating over time, which implies that the initial market imbalance is magnified when market interaction is repeated frequently. The last result implies that if $d^w > 1$, then although the exogenous probability of leaving converges to zero, but the probability of being a sole bidder (or in fact being one of n bidders for any finite n) converges to zero even faster. This implies that even though there is an ample time to experiment before an exogenous exit occurs, but the odds of having only a few opponents in at least *one* of the auctions in the lifetime of a bidder still converges to zero.

The previous Lemma suggests that the short side of the market captures the entire surplus from trading, and this is what the second important intermediate result indeed establishes:

Lemma 8 *Suppose $\left(\beta_k, D_k^w, S_k^w, \theta_k^+\right)$ is a sequence of equilibria for $\delta_k \to 1$. Then, for all $0 < \theta < 1$,*

$$\lim_{k\to\infty} V_k(\theta) = \begin{cases} 0 & \text{if } d^H > 1 \text{ and } d^L > 1 \\ v & \text{if } d^H < 1 \text{ and } d^L < 1 \\ \theta 0 + (1-\theta) v & \text{if } d^H > 1 \text{ and } d^L < 1 \end{cases}.$$

This result has two important aspects. First, in both states the short side of the market obtains all the rents. Second, the state pins down continuation values in the limit, namely the actions taken by a buyer does not influence his rent in either state in the limit. The result is fairly straightforward for the low state: in the limit the expected number of rival buyers converges to zero, so any buyer is able to obtain the object for free in the limit. In the high state the expected number of competing buyers at the same seller converges to infinity and thus it is not surprising that the expected utility in this state converges to zero. However, to formally establish this result one must calculate the bid functions, and since each buyer conditions his bid on the event of tieing (and not on losing), it is not clear that how each individual buyer behaves. Indeed, the next Section shows that many buyers with beliefs arbitrarily close to 1 bid arbitrarily close to zero in the limit. The formal analysis in the online Appendix shows notwithstanding these difficulties that the bidders earn zero rent in the limit, and also concludes that the transaction price converges to v in the high state.

4.2 Bid pattern of an individual over time

Now, we are ready to derive the limiting bidding behavior of the agents. The case where $d^H, d^L < 1$ simply yields that since the value function converges to v, therefore the bid of a bidder with *any* belief $\theta \in [0,1]$ converges to zero as $\delta \to 1$, and he wins with probability one in the limit with his very first bid. When $d^H, d^L > 1$ the value function converges to zero, and thus already the first bid (and then all the subsequent bids) converge to v.

The interesting case is thus where $d^L < 1 < d^H$. As we argue below, the posterior upon losing even once converges to one, and thus one may expect that already the second bid should converge to v, since the bidders get very pessimistic after one loss already. However, the bid is conditioned on having the belief of being tied at the top spot, and not upon losing, and thus this intuition may be misleading. Indeed, we show it below that the intuition fails and the bid after any finite number of losses converges to zero. We also provide an upper bound to how many times the bidders may bid less than v as $\delta \to 1$ and frictions vanish.

To state our results formally we introduce further notation. Suppose that an agent starts with a belief $\theta \in [\underline{\theta}, \overline{\theta}]$. The distribution of beliefs has an associated nicely behaved density function almost everywhere, so pick any point where derivatives exist. Recall that $\Gamma^w_{1,k}, \Gamma^w_k$ denotes the distribution functions of beliefs and first order statistics, and let the lower case letter γ denote the corresponding densities. Let $\widetilde{\theta}^t_k$ denote the posterior upon losing t times if the agent is born with belief θ. Formally, let $\widetilde{\theta}^0_k = \theta$ and for all $t \geq 1$ let

$$\widetilde{\theta}^t_k = \theta^+_k(\widetilde{\theta}^{t-1}_k, \widetilde{\theta}^{t-1}_k) = \frac{\widetilde{\theta}^{t-1}_k \left(1 - \Gamma^H_{1,k}\left(\widetilde{\theta}^{t-1}_k\right)\right)}{\widetilde{\theta}^{t-1}_k \left(1 - \Gamma^H_{1,k}\left(\widetilde{\theta}^{t-1}_k\right)\right) + (1 - \widetilde{\theta}^{t-1}_k)(1 - \Gamma^L_{1,k}\left(\widetilde{\theta}^{t-1}_k\right))}.$$

Also, let η_k^t denote the tieing posterior upon losing t times and then tieing:

$$\eta_k^t = \theta_k^0(\widetilde{\theta}_k^t, \widetilde{\theta}_k^t) = \frac{\widetilde{\theta}_k^{t-1} \gamma_{1,k}^H \left(\widetilde{\theta}_k^{t-1}\right)}{\widetilde{\theta}_k^{t-1} \gamma_{1,k}^H \left(\widetilde{\theta}_k^{t-1}\right) + (1-\widetilde{\theta}_k^{t-1})\gamma_{1,k}^L \left(\widetilde{\theta}_k^{t-1}\right)}.$$

It follows from Proposition 1 that

$$\lim_{k\to\infty} \beta_k(\widetilde{\theta}_k^t) = \lim_{k\to\infty} (v - \delta_k[\eta_k^t V_k^H(\widetilde{\theta}_k^{t+1}) + (1-\eta_k^t)V_k^L(\widetilde{\theta}_k^{t+1})]) =$$

$$= \lim_{k\to\infty} \eta_k^t (v - V_k^H(\widetilde{\theta}_k^{t+1})),$$

where the last step follows from the fact that in state L the number of other bidders is zero in expected value.

The next Proposition states that the tieing posterior η_k^t converges to zero for any fixed t, and thus fixing the number of losses at a finite t the bidder with that many losses bids zero in the limit. The Proposition also provides an upper bound as to how many losses does a bidder need to go through before his bid converges to v reflecting that the high state is very likely after a high number of losses. More precisely, the Proposition states that after (at most) $\sqrt{1/(1-\delta_k)}$ periods the posterior upon tieing converges to 1, and thus bidding becomes competitive.

Proposition 3 *Take any fixed* t*, and* $\theta \in [\underline{\theta}, \overline{\theta}]$*. Then* $\lim_{k\to\infty} \eta_k^t = 0$*, and consequently* $\lim_{k\to\infty} \beta_k(\widetilde{\theta}_k^t) = 0$*. Let* $t_k = \sqrt{1/(1-\delta_k)}$ *and* $\theta \in [\underline{\theta}, \overline{\theta}]$*. Then* $\lim_{k\to\infty} \beta_k(\widetilde{\theta}_k^{t_k}) = v$*.*

Proof: See the online Appendix.

This time pattern of the equilibrium bid function yields an important deviation from the results of the previous literature: even though as search frictions vanish the agents learn very quickly, but the equilibrium actions follow learning only with a substantial time lag. More specifically, an agent keeps experimenting with low bids for several periods hoping to obtain a bargain even though he may already know that the economy is very likely to be in the high state. Experimentation is very effective when frictions are very low ($\delta \to 1$), yielding a very fast rate of learning, and thus seemingly very little experimentation should be expected to happen in equilibrium. However, experimentation also becomes exceedingly cheap, and this second effect dominates the first yielding a long string of non-competitive bids (i.e. low bids when an agent knows that it is the high state with a very high probability) as it is shown is Section 4.1. While this exact equilibrium bid pattern is specific to our model, our result shows that once both learning and price formation occur endogenously, it is very hard to predict how agents behave over time, *even if* the equilibrium allocation is known to converge to the full information competitive equilibrium.

4.3 Discussion: The effect of market frictions

Our results confirm some of the findings of the literature on the effect of private information on trading in a dynamic environment with search frictions. Most of the papers in the literature provide valuable insight about the nature information dissemination when prices and allocations are given, but our focus is on the case when learning and endogenous price formation goes hand in hand. Golosov et al (2009) provide such a dynamic trading model of asset markets, so we compare our results with that paper. Just as in Golosov et al (2009) we show that as frictions vanish the equilibrium price converges to the fully revealing equilibrium of a competitive model. For any positive level of search frictions private information yields a positive rent for an agent. However, we show that this value converges to zero in the limit, since experimentation becomes very cheap and thus agents can learn from previous market outcomes, and thus how informed an agent initially becomes less important.

The model has important implications for the behavior of the value of information when search frictions change. First, when market frictions are prohibitive and thus continuation is very unlikely ($\delta \to 0$) each bidder bids his valuation v, since the outside option converges to 0. Therefore, knowing the state would not affect bids, and thus the value of information is zero in the limit. Second, as we remarked above, when market frictions vanish ($\delta \to 1$) information is not valuable either. Third, for any intermediate value of δ the optimal action of a bidder depends on his belief, so information has positive value. This is also expressed by the fact that the value function V is strictly convex for any $\delta \in (0,1)$. Therefore, the value of information is highest for intermediate values of δ, and is equal to zero at either end of the $[0,1]$ interval[12]. Thus in our model the value of information is highest when the market is somewhat, but not fully fragmented, and thus the future has some but not full power over the decisions of the agents.

5 The seller's optimal information policy

In the previous Section we have made the assumption that a buyer who unsuccessfully participates in an auction does not learn anything about her competitors, except the fact that one of them bid higher than she did. The sellers had no chance to provide buyers with any additional information. In this Section we relax this assumption by letting any individual seller to choose between conducting his auction in a nontransparent way (as in the previous Section) or reveal the winning bid after

[12] Some of these results depend on the exact mechanism the bidders participate in. For example, if the auctioneers used first price auctions and one assumed that δ was close to zero, then it is no longer the case that the optimal bid does not depend on the belief about the state. However, when δ is close to 1, the logic behind our reasoning for the case of the second price auction seems to carry through to other mechanisms. To see this, note that when δ is close to 1, then the imbalance between the two sides of the market becomes very large, and thus one would expect that the rent of the buyers do not depend on the exact mechanism played.

the auction to the respective participants.[13] The question we ask is whether a single seller has a unilateral incentive to announce the winning bid or the state (assuming he knows the state).

5.1 Winning bid revealed

In this Section we study how a single auctioneer's revenue is affected, if he revealed some information about the auction process beyond the identity of the winner. We focus on the case where the auctioneer considers revealing the winning bid. To formulate this question, we analyze a game where the bidders find themselves in a second price auction with no reserve price and with the following features. First, it is commonly known that the winning bid is announced after the auction is run. Second, all bidders take the continuation value function V as given, and equal to the equilibrium continuation value function derived in Sections 2 and 3, i.e. under the assumption that the other sellers do not reveal any information. Third, the bidders also take the distribution of the number of (other) bidders in each state as given, postulating a Poisson-distribution with parameters μ^H and μ^L in the two states. Finally, they hold the belief that the beliefs of the other bidders are distributed i.i.d. conditional on the state according to the distribution functions Γ^H, Γ^L that were derived in the previous Section. This also implies that the highest other type (the first order statistics) is distributed according to distribution functions Γ_1^H, Γ_1^L (as defined in the previous Section as well).

Before describing the payoffs of a generic bidder i we need to derive the posterior belief of i upon losing. Suppose that i loses with type (prior belief) θ and observes that the winning bid was b. Concentrating on an equilibrium with a bid function β^{rev} that is strictly increasing and symmetric among the bidders, bidder i can deduce that the type of the winning bidder is the value θ^1 that solves $\beta^{rev}(\theta^1) = b$. Then the posterior of i upon losing can be written as

$$\Pr(H \mid \theta, \theta^1) = \frac{\theta\gamma_1^H(\theta^1)}{\theta\gamma_1^H(\theta^1) + (1-\theta)\gamma_1^L(\theta^1)} = \theta^0(\theta^1, \theta),$$

i.e. the posterior upon losing if the winning bid revealed can be expressed as a posterior upon tieing if the winning bid not revealed, the case studied as our baseline model.

Given this, the utility of bidder i is determined by the highest other type θ^1, the type of bidder i, θ_i and the bid of bidder i. Letting β^{rev} denote the purported equilibrium bid function, suppose that bidder i places a bid $\beta^{rev}(x)$ and faces a highest other type equal to θ^1. Then his utility is $v - \beta^{rev}(\theta^1)$ if $x \geq \theta^1$ and it is equal to $\delta V(\theta^0(\theta^1, \theta))$ if $x < \theta^1$. Therefore, the expected utility from

[13] Note that since we consider second-price auctions, the winning bid is not identical to the price. Assuming that the latter is revealed would lead to asymmetric learning on the part of the losing bidders, because each losing bidder would learn something different given their own bids.

bidding $\beta^{rev}(x)$ when the actual belief is θ can be written as (for any $\theta_0 \in (0, \underline{\theta})$)

$$u_i(\theta, x) = \Gamma_1^{\theta}(0)v + \int_{\theta_0}^{x} (v - \beta^{rev}(t))d\Gamma_1^{\theta}(t) + \delta \int_{x}^{1} V(\theta^0(t,\theta))d\Gamma_1^{\theta}(t).$$

Definition 1 *A symmetric monotone Bayesian equilibrium is a strictly increasing bid function β^{rev} such that for all θ it holds that $\theta \in \arg\max_x u_i(\theta, x)$.*

We have the following Theorem describing such equilibria:

Theorem 1 *There is an essentially unique symmetric monotone Bayesian equilibrium in the game defined above, and it holds that*

$$\beta^{rev}(\theta) = v - \delta V(\theta^0(\theta, \theta)). \tag{11}$$

Proof. Since the function u_i is almost everywhere differentiable in x, therefore the necessary first order condition implies (11) almost everywhere.[14] To prove sufficiency, first note that arguments from the previous Section imply that $\widetilde{\theta}^0(\theta, \theta)$ is increasing in θ, and thus given that V is strictly decreasing, therefore function β^{rev} is strictly increasing. The second order conditions of optimization can be established with minor modifications of the analysis in the previous Section. ■

Now, we are ready to compare the expected revenues of the seller when the winning bid is revealed and when it is not. Since the number of bidders, and their belief distributions are independent of the auction format, we can compare the revenues following a case by case analysis. The interesting case for comparison is when there are at least two bidders, otherwise the revenue is 0 in both auctions.

Corollary 2 *For any realization of types the revenue of the seller is greater in the case when no information is revealed and there are at least two bidders.*

Proof. Take any realized values of the second largest type θ. Recall the notation from the previous Section for $\theta^+(\theta, \theta)$ and $\theta^0(\theta, \theta)$ as denoting the probability of the high state conditional on losing and tieing. Then the revenue in the case with no information revelation is $v - \delta W(\theta^+(\theta, \theta), \theta^0(\theta, \theta))$, while the revenue in the case of revealing the winning bid is $v - \delta V(\theta^0(\theta, \theta))$, using that by definition $\theta^0(\theta, \theta) = \widetilde{\theta}^0(\theta, \theta)$ for all θ. By Property R3 of function W, it holds that

$$V(\theta^0(\theta, \theta)) = W(\theta^0(\theta, \theta), \theta^0(\theta, \theta)) \geq W(\theta^+(\theta, \theta), \theta^0(\theta, \theta))$$

for all θ, which completes the proof. (The comparison is strict for any θ in the interior of the support of the type distribution.) ■

[14] More precisely, just as in the previous section without bid revelation, this bidding formula must hold everywhere on the (common) support of Γ_H, Γ_L.

5.2 Revealing the state

To highlight the intuition behind our approach we also study the case where the seller knows the state of the world and can commit to announce it in both states or not announce it at all. Suppose that the state is announced after the auction is conducted. Then one can show that the expected revenue of the seller (taking the state as unknown, i.e. behind the veil of ignorance) is even lower than if the winning bid was announced only. Therefore, announcing the state after the auction is run hurts the seller.

The intuition for this result builds on how continuation values change when extra information is revealed in the auction process. Naturally, the more information is revealed in the auction process, the better decisions the losing bidders can make in future auctions, and thus the continuation values increase. Higher continuation values for the losers constitute better outside options, depressing current bids and thus hurting the revenues of the seller. This logic implies that any information revealed after the auction is not a policy that is advantageous for the seller.

The case where the seller announces the state *before* the auction is run is more complicated as it involves two opposing effects. In this case, in state $w \in \{L, H\}$ a bidder bids $v - \delta V^w$ where $V^H = V(1)$ and $V^L = V(0)$ are the continuation values in the two states. The intuition for such bidding is clear; since there is no relevant private information for the bidders, therefore they all bid their valuations for the object minus their known (and common) outside options. In other words we obtain a fierce Bertrand like competition in both states. The revenue is then 0 if at most one shows, and $v - \delta V^w$ if at least two show. The expected revenue of the seller is just the unweighted average of the expected revenues in the two states, given that each state is equal likely ex-ante (behind the veil of ignorance). Building on the intuition of the linkage principle, revealing the state before the auction is run may increase the revenues of the seller, because it fosters a fiercer competition between the buyers. However, the indirect effect still arises: announcing information leads to higher continuation values which depresses bids. Therefore, it is unclear whether revealing the state before the auction (or not revealing any information) increases the expected revenues or not. The balance may well depend on the parameter values of the model.

To conduct such a comparison we study a simplified model numerically in Appendix 6, an online Appendix. We study the case where each period a bidder has to pay a cost c to submit a bid, and if he does not submit a bid he obtains utility 0. We calibrate the model in a way that bidders stay for two or three periods, depending on their initial beliefs. Some bidders drop out after making two unsuccessful bids, while others place also a third one. One can then explicitly calculate equilibrium bid functions, expected revenues and obtain a revenue comparison between revealing the state before the auction and not revealing any information. This numerical exercise shows that the trade-off is indeed there: for some parameter values revealing the state increases expected revenues, but for others it decreases. Therefore, it is fair to say that the linkage principle effect is still relevant in dynamic auctions where the outside options have common value components, but a new effect that emanates from the endogeneity of continuation values can dominate the original insight of the linkage principle.

6 Conclusions

To be added.

7 Appendix

7.1 Appendix 2: MLRP

7.1.1 Counterexample without assumption $\underline{\theta} \geq 0.5$

Suppose $\mu^H = \mu^L = \mu$ (so the expected number of bidders is state independent) and suppose for some θ^*, the densities are constant for $\theta \leq \theta^*$ and for $\theta > \theta^*$, with $\gamma^L(\theta) = C > \gamma^H(\theta) = c$ if $\theta \leq \theta^*$ and $\gamma^L(\theta) = c < \gamma^H(\theta) = C$. These densities satisfy the MLRP. However, for $\theta < \theta^*$,

$$\frac{\gamma_1^H(\theta)'}{\gamma_1^H(\theta)} = 0 + \mu c < 0 + \mu C = \frac{\gamma_1^L(\theta)'}{\gamma_1^L(\theta)}$$

and hence, the first order statistic does not satisfy the MLRP.

7.1.2 Discussion of the MLRP for the tieing posterior

We describe the problem that leads to nonmonotonicity of the tieing posterior $\frac{\theta\gamma_1^H(\theta)}{(1-\theta)\gamma_1^L(\theta)}$ when $\underline{\theta} < 0.5$. The densities (where the derivatives γ^H or γ^L exist) satisfy

$$\gamma_1^H = \mu_H \gamma^H \Gamma_1^H \tag{12}$$

and

$$\gamma_1^L = \mu_L \gamma^L \Gamma_1^L. \tag{13}$$

Consider the no-introspection formula

$$\theta = \frac{D^H\gamma^H(\theta)}{D^H\gamma^H(\theta) + D^L\gamma^L(\theta)}, \tag{14}$$

where γ^H, γ^L are the derivatives of Γ^H, Γ^L. The three numbered formulas from above imply that

$$\frac{\gamma_1^H}{\gamma_1^L} = \frac{\theta}{1-\theta}\frac{S^L}{S^H}\frac{\Gamma_1^H}{\Gamma_1^L}. \tag{15}$$

Therefore, (15) implies that

$$\frac{\theta}{1-\theta}\frac{\gamma_1^H}{\gamma_1^L} = (\frac{\theta}{1-\theta})^2\frac{S^L}{S^H}\frac{\Gamma_1^H}{\Gamma_1^L},$$

and thus

$$(\frac{\theta}{1-\theta}\frac{\gamma_1^H}{\gamma_1^L})' = \frac{S^L}{S^H}(2\frac{\theta}{(1-\theta)^3}\frac{\Gamma_1^H}{\Gamma_1^L} + (\frac{\theta}{1-\theta})^2(\frac{\Gamma_1^H}{\Gamma_1^L})').$$

Then (using (13) and (15) again)

$$\frac{(\frac{\Gamma_1^H}{\Gamma_1^L})'}{\frac{\Gamma_1^H}{\Gamma_1^L}} = \frac{\frac{\gamma_1^L}{\Gamma_1^L}(\frac{\gamma_1^H}{\gamma_1^L} - \frac{\Gamma_1^H}{\Gamma_1^L})}{\frac{\Gamma_1^H}{\Gamma_1^L}} = \mu^L\gamma^L(\frac{\theta}{1-\theta}\frac{S^L}{S^H} - 1).$$

Thus $(\frac{\theta}{1-\theta}\frac{\gamma_1^H}{\gamma_1^L})' \geq 0$ if and only if

$$\frac{2}{\theta(1-\theta)} + \mu^L\gamma^L(\frac{\theta}{1-\theta}\frac{S^L}{S^H} - 1) \geq 0.$$

Now, take any θ such that $0 < \frac{\theta}{1-\theta}\frac{S^L}{S^H} < 1$ and make γ^L large. Then the right hand side becomes negative for a large enough γ^L. But nothing prevents those choices to be feasible as long as $\frac{\theta}{1-\underline{\theta}}\frac{S^L}{S^H} < 1$. In this case then for some beliefs the posterior upon tieing is not monotone increasing. (Of course, $\theta \geq 1/2$ implies that $\frac{\theta}{1-\theta}\frac{S^L}{S^H} - 1 \geq 0$ and this problem is avoided.)

Non-monotonicity occurs, because for such low beliefs there are many more buyers in the low state to tie with. More precisely, there are many buyers with exactly such beliefs compared to lower beliefs in the low state (compared to the high state) and thus MLRP fails. Formally, this is captured by (15), and the fact that for low enough beliefs $\frac{\theta}{1-\theta}\frac{S^L}{S^H} < 1$. This problem does not arise for the losing posterior, because upon losing the critical event is stacked more against the low state, than upon tieing and thus the MLRP is maintained for any parameter values.

7.2 Appendix 3

7.2.1 Properties of regular functions

A function $W\left(\theta, \hat{\theta}\right)$ is called **regular** if θ parameterizes a family of nonincreasing, linear functions $W(\theta, \cdot)$ which are the tangents of a nonincreasing convex functions, i.e.,

R1. $W\left(\theta, \hat{\theta}\right) = \hat{\theta}W^H(\theta) + (1-\hat{\theta})W^L(\theta)$ for some functions W^H, W^L

R2. for all $\hat{\theta}$, θ it holds that $W(\hat{\theta},\hat{\theta}) \geq W(\theta,\hat{\theta})$

R3. $W(\theta,\theta)$ is nonincreasing in θ

Observations about a regular function $W\left(\theta,\hat{\theta}\right)$ are that

R4 the function $W\left(\theta,\hat{\theta}\right)$ is (weakly) decreasing in $\hat{\theta}$ for all θ

R5 function W^H is (weakly) increasing, while W^L is (weakly) decreasing in θ.

R6 $W(\theta_h,\theta_{hh}) \leq W(\theta_l,\theta_{ll})$ when $\theta_h \geq \theta_l$, $\theta_{hh} \geq \theta_{ll}$ and $\theta_{hh} \leq \theta_h$ and $\theta_{ll} \leq \theta_l$.

R4 and R5 are immediate, so we only establish R6. (Figure 2 illustrates the main idea behind property R6.)

Property R6 is immediate for $\theta_h \geq \theta_{hh} \geq \theta_l$, since $W(\theta_l,\theta_{ll}) \geq W(\theta_l,\theta_l) \geq W(\theta_{hh},\theta_{hh}) \geq W(\theta_h,\theta_{hh})$.

Suppose $\theta_{hh} < \theta_l$, then, by $W\left(\theta,\theta'\right) = W(\theta,\theta) + \left(\theta'-\theta\right)\left(W^H(\theta) - W^L(\theta)\right)$, and

$$W(\theta_l,\theta_l) \geq W(\theta_h,\theta_l) = W(\theta_h,\theta_h) + (\theta_l-\theta_h)\left(W^H(\theta_h) - W^L(\theta_h)\right)$$

and by

$$W(\theta_h,\theta_h) \geq W(\theta_l,\theta_h) = W(\theta_l,\theta_l) + (\theta_h-\theta_l)\left(W^H(\theta_l) - W^L(\theta_l)\right)$$

and we obtain that $W(\theta_l,\theta_l) \geq W(\theta_h,\theta_h)$implies

$$\begin{aligned}
(\theta_l-\theta_h)\left(W^H(\theta_h) - W^L(\theta_h)\right) &\geq (\theta_h-\theta_l)\left(W^H(\theta_l) - W^L(\theta_l)\right) \\
(\theta_h-\theta_l)\left(W^H(\theta_h) - W^L(\theta_h)\right) &\leq (\theta_h-\theta_l)\left(W^H(\theta_l) - W^L(\theta_l)\right) \\
\left(W^H(\theta_h) - W^L(\theta_h)\right) &\leq \left(W^H(\theta_l) - W^L(\theta_l)\right)
\end{aligned}$$

and thus,

$$\begin{aligned}
W(\theta_l,\theta_{ll}) &= W(\theta_l,\theta_l) + (\theta_{ll}-\theta_l)\left(W^H(\theta_l) - W^L(\theta_l)\right) \\
&\geq W(\theta_h,\theta_l) + (\theta_{ll}-\theta_l)\left(W^H(\theta_l) - W^L(\theta_l)\right) \\
&\geq W(\theta_h,\theta_l) + (\theta_{ll}-\theta_l)\left(W^H(\theta_h) - W^L(\theta_h)\right) \\
&= W(\theta_h,\theta_h) + (\theta_l-\theta_h)\left(W^H(\theta_h) - W^L(\theta_h)\right) + (\theta_{ll}-\theta_l)\left(W^H(\theta_h) - W^L(\theta_h)\right) \\
&= W(\theta_h,\theta_{ll}) \geq W(\theta_h,\theta_{hh}),
\end{aligned}$$

where the last inequality follows using R4.

7.2.2 Proof of Lemma 5

Given W, let $U\left(\theta',\theta\right)$ denote the payoff of type θ from winning against all types less than θ' today and following the equilibrium strategy in the future given the updated belief upon losing. For convenience let us define θ_0 as being an arbitrary type below the lowest type in the support, $(0,\theta_0)$, and above zero. With this notation in place, one can write the expected utility $U\left(\theta',\theta\right)$ as

$$U\left(\theta',\theta\right) \equiv v\Gamma^{\theta}\left(0\right) + \int_{\{t:\theta_0<t\leq\theta'\}} \left(v-\beta\left(t\right)\right) d\Gamma^{\theta}_{(1)}\left(t\right) t + \delta\left(1-\Gamma^{\theta}_{(1)}\left(\theta'\right)\right) W\left(\theta^{+}(\theta',\theta),\theta^{+}(\theta',\theta)\right).$$

With this notation in hand, proving Lemma 5 is equivalent to showing that $\theta \in \arg\max_{\theta'} U\left(\theta',\theta\right)$ under the conditions stated. Formally it is sufficient for us to show that

Lemma 9 *Let W be regular and suppose that $\beta(\theta)=v-\delta W\left(\theta^{+}\left(\theta,\theta\right),\theta^{0}\left(\theta,\theta\right)\right)$ for almost all θ. Then $\theta \in \arg\max_{\theta'} U\left(\theta',\theta\right)$.*

First, $U\left(\theta',\theta\right)$ is continuous in both variables and payoffs are differentiable in the first variable for almost all θ. Let $U^{(1)}$ the derivative with respect to the first variable whenever it exists. Let W be a shorthand for $W(\theta^{+}\left(\theta',\theta\right),\theta^{+}\left(\theta',\theta\right))$ for the calculations below, and let W_1, W_2 denote the partial derivatives of W with respect to the first and second variables at $\left(\theta',\theta\right)$. Similarly, we use the shorthand notations $\theta^{+}=\theta^{+}\left(\theta',\theta\right)$, $\theta^{0}=\theta^{0}\left(\theta',\theta\right)$, $W^{H}=W^{H}(\theta^{+}\left(\theta',\theta\right))$, $W^{L}=W^{L}(\theta^{+}\left(\theta',\theta\right))$. We can write the derivative of the payoffs with respect to the first variable as

$$\begin{aligned}
U^{(1)}\left(\theta',\theta\right) &= \left(v-\beta\left(\theta'\right)\right)\gamma^{\theta}_{(1)}\left(\theta'\right) - \delta\gamma^{\theta}_{(1)}\left(\theta'\right)W + \delta\left(1-\Gamma^{\theta}_{(1)}\left(\theta'\right)\right)\frac{\partial\theta^{+}\left(\theta',\theta\right)}{\partial\theta'}\left(W_1+W_2\right) \\
&= \left(v-\beta\left(\theta'\right)\right)\gamma^{\theta}_{(1)}\left(\theta'\right) - \delta\gamma^{\theta}_{(1)}\left(\theta'\right)W + \delta\left(1-\Gamma^{\theta}_{(1)}\left(\theta'\right)\right)\frac{\partial\theta^{+}\left(\theta',\theta\right)}{\partial\theta'}\left(0+\left(W^{H}-W^{L}\right)\right) \\
&= \left(v-\beta\left(\theta'\right)\right)\gamma^{\theta}_{(1)}\left(\theta'\right) - \delta\gamma^{\theta}_{(1)}\left(\theta'\right)W + \delta\gamma^{\theta}_{(1)}\left(\theta'\right)\left(\theta^{+}-\theta^{0}\right)\left(W^{H}-W^{L}\right) \\
&= \left(v-\beta\left(\theta'\right)\right)\gamma^{\theta}_{(1)}\left(\theta'\right) - \delta\gamma^{\theta}_{(1)}\left(\theta'\right)\left(\theta^{+}W^{H}+\left(1-\theta^{+}\right)W^{L}\right) \\
&\quad +\delta\gamma^{\theta}_{(1)}\left(\theta'\right)\left(\theta^{+}-\theta^{0}\right)\left(W^{H}-W^{L}\right) \\
&= \gamma^{\theta}_{(1)}\left(\theta'\right)\left(v-\beta\left(\theta'\right)-\delta\left(\theta^{0}W^{H}+\left(1-\theta^{0}\right)W^{L}\right)\right) \\
&= \gamma^{\theta}_{(1)}\left(\theta'\right)\left(v-\beta\left(\theta'\right)-\delta W\left(\theta^{+},\theta^{0}\right)\right) \\
&= \gamma^{\theta}_{(1)}\left(\theta'\right)\delta\left(W\left(\theta^{+}\left(\theta',\theta'\right),\theta^{0}\left(\theta',\theta'\right)\right)-W\left(\theta^{+}\left(\theta',\theta\right),\theta^{0}\left(\theta',\theta\right)\right)\right),
\end{aligned}$$

with $U^{(1)}\left(\theta',\theta\right)=0$ at all θ' not in the support of Γ^{w}. On the support, both expression in the bracket are continuous and γ^{w} is a continuous density. The support consists of a countable number of intervals (generations, see Lemma 2 and it's proof), so that the derivative is discontinuous only at the boundaries of the support and it exists almost everywhere.

Now, we establish that $U^{(1)}\left(\theta',\theta\right)\leq 0$ for almost all $\theta'>\theta$, which implies that for all $\theta'>\theta$, $U\left(\theta',\theta\right)\leq U\left(\theta,\theta\right)$ follows. By definition of function β, we obtain that $U^{(1)}\left(\theta,\theta\right)=0$. At any $\theta'>\theta$,

$$W\left(\theta^{+}\left(\theta',\theta'\right),\theta^{0}\left(\theta',\theta'\right)\right)-W\left(\theta^{+}\left(\theta',\theta'\right),\theta^{0}\left(\theta',\theta\right)\right)\leq 0$$

by property *R4* of a regular function, $W\left(\theta,\hat{\theta}\right)$ being nonincreasing in $\hat{\theta}$. Furthermore,

$$W\left(\theta^{+}\left(\theta',\theta'\right),\theta^{0}\left(\theta',\theta\right)\right)-W\left(\theta^{+}\left(\theta',\theta\right),\theta^{0}\left(\theta',\theta\right)\right)\leq 0$$

by property *R2*, $W\left(\hat{\theta},\hat{\theta}\right)\geq W\left(\theta,\hat{\theta}\right)$ and thus

$$W\left(\theta^{+}\left(\theta',\theta'\right),\theta^{0}\left(\theta',\theta'\right)\right)-W\left(\theta^{+}\left(\theta',\theta\right),\theta^{0}\left(\theta',\theta\right)\right)\leq 0.$$

Therefore, $U^{(1)}\left(\theta',\theta\right)\leq U^{(1)}\left(\theta,\theta\right)=0$ for all $\theta'>\theta$. A similar argument establishes $U^{(1)}\left(\theta',\theta\right)\geq U^{(1)}\left(\theta,\theta\right)=0$ at all $\theta'<\theta$. Thus, $U\left(\theta,\theta\right)\geq U\left(\theta',\theta\right)$ for all θ', which concludes the proof. QED

7.2.3 Proof of Lemma 6

TW is defined on the set of regular functions. The set of regular functions is complete in the sup norm. Below we will show that the operator T maps regular functions into regular functions. Existence and uniqueness of a solution W^{*} follows then from Blackwell's sufficient conditions. Recall

$$TW\left(\theta,\hat{\theta}\right)=$$

$$=v\Gamma_{1}^{\hat{\theta}}\left(0\right)+\int_{\theta_{0}}^{\theta}\left(v-\beta\left(t\right)\right)d\Gamma_{1}^{\hat{\theta}}(t)+\delta\left(1-\Gamma_{1}^{\hat{\theta}}\left(\theta\right)\right)W\left(\theta^{+}\left(\theta,\theta\right),\theta^{+}\left(\theta,\hat{\theta}\right)\right).$$

and thus, if $W''\left(\theta,\hat{\theta}\right)>W'\left(\theta,\hat{\theta}\right)$ for all $\left(\theta,\hat{\theta}\right)$, then $TW''\left(\theta,\hat{\theta}\right)>TW'\left(\theta,\hat{\theta}\right)$ (by inspection). Furthermore, $T\left(W+a\right)\leq TW+\delta a$, again, by inspection. Hence, TW satisfies the two sufficient conditions for a contraction in the sup norm, see Stokey and Lucas (with Prescott) (1989) and a unique solution exists.

We show that TW is regular if W is regular. We first establish that property *R2* holds for function TW. To see this, note that for all $\theta,\hat{\theta}$ it holds that

$$TW\left(\hat{\theta},\hat{\theta}\right)\geq$$

$$\geq v\Gamma_{1}^{\hat{\theta}}\left(0\right)+\int_{\theta_{0}}^{\theta}\left(v-\beta\left(t\right)\right)d\Gamma_{1}^{\hat{\theta}}(t)+\delta\left(1-\Gamma_{1}^{\hat{\theta}}\left(\theta\right)\right)W\left(\theta^{+}\left(\theta,\hat{\theta}\right),\theta^{+}\left(\theta,\hat{\theta}\right)\right)$$

$$\geq v\Gamma_1^{\hat{\theta}}(0) + \int_{\theta_0}^{\theta} (v - \beta(t))\, d\Gamma_1^{\hat{\theta}}(t) + \delta \left(1 - \Gamma_1^{\hat{\theta}}(\theta)\right) W\left(\theta^+(\theta,\theta), \theta^+\left(\theta,\hat{\theta}\right)\right)$$

$$= TW\left(\theta, \hat{\theta}\right).$$

The first inequality follows by Lemma 9 that implies that the choice of $\theta' = \theta$ maximizes the payoff of type θ, and the second inequality follows from the fact that W satisfies property *R2*.

To show property *R1* (linearity) of TW, define

$$\begin{aligned} W_T^H(\theta) &\equiv \int_0^{\theta} (v - \beta(t))\, d\Gamma_1^H(t) + (1 - \Gamma_1^H(\theta)) W^H\left(\theta^+(\theta,\theta)\right), \\ W_T^L(\theta) &\equiv \int_0^{\theta} (v - \beta(t))\, d\Gamma_1^L(t) + (1 - \Gamma_1^L(\theta)) W^L\left(\theta^+(\theta,\theta)\right), \end{aligned}$$

where W^H, W^L are given by the linearity property (R1) of the regular function W. Noting that

$$\theta^+\left(\theta, \hat{\theta}\right) = \frac{\hat{\theta}(1 - \Gamma_1^H(\theta))}{1 - \Gamma_1^{\hat{\theta}}(\theta)},$$

we can rewrite the third term in definition of operator T as

$$\delta\left(1 - \Gamma_1^{\hat{\theta}}(\theta)\right) W\left(\theta^+(\theta,\theta), \theta^+\left(\theta,\hat{\theta}\right)\right) =$$

$$= \delta[\hat{\theta}(1 - \Gamma_1^H(\theta)) W^H\left(\theta^+(\theta,\theta)\right) + \left(1 - \hat{\theta}\right)\left(1 - \Gamma_1^L(\theta)\right) W^L\left(\theta^+(\theta,\theta)\right)],$$

and hence

$$TW\left(\theta, \hat{\theta}\right) = \hat{\theta} W_T^H(\theta) + \left(1 - \hat{\theta}\right) W_T^L(\theta).$$

Property *R3* for function TW: First, we show that $\beta = v - \delta W\left(\theta^+(\theta,\theta), \theta^0(\theta,\theta)\right)$ is nondecreasing in θ. Let $\theta_h > \theta_l$, $\boldsymbol{\theta}_{\mathbf{l}}^+ = \theta^+(\theta_l,\theta_l)$, $\boldsymbol{\theta}_{\mathbf{h}}^+ = \theta^+(\theta_h,\theta_h)$, $\boldsymbol{\theta}_{\mathbf{l}}^0 = \theta^0(\theta_l,\theta_l)$, $\boldsymbol{\theta}_{\mathbf{h}}^0 = \theta^0(\theta_h,\theta_h)$. Then Property *R6* for function W implies that

$$W\left(\boldsymbol{\theta}_{\mathbf{l}}^+, \boldsymbol{\theta}_{\mathbf{l}}^0\right) \geq W\left(\boldsymbol{\theta}_{\mathbf{h}}^+, \boldsymbol{\theta}_{\mathbf{h}}^0\right).$$

Next, we show that function W_T^H is increasing, while function W_T^L is decreasing. Using property *R2,* we obtain that for all $\hat{\theta}, \theta$ it holds that

$$TW\left(\hat{\theta}, \hat{\theta}\right) = \hat{\theta} W_T^H\left(\hat{\theta}\right) + \left(1 - \hat{\theta}\right) W_T^L\left(\hat{\theta}\right) \geq \tag{16}$$

$$\geq TW\left(\theta, \hat{\theta}\right) = \hat{\theta} W_T^H(\theta) + \left(1 - \hat{\theta}\right) W_T^L(\theta).$$

and

$$TW(\theta,\theta) = \theta W_T^H(\theta) + (1-\theta) W_T^L(\theta) \geq \tag{17}$$
$$\geq TW\left(\hat{\theta},\theta\right) = \theta W_T^H\left(\hat{\theta}\right) + (1-\theta) W_T^L\left(\hat{\theta}\right).$$

Using the linearity property, *R1* and after some algebra we obtain

$$(\hat{\theta}-\theta)[(W_T^H\left(\hat{\theta}\right) - W_T^L\left(\hat{\theta}\right)) - (W_T^H(\theta) - W_T^L(\theta))] \geq 0.$$

Let w.l.o.g. assume that $\hat{\theta} > \theta$ and thus

$$W_T^H\left(\hat{\theta}\right) - W_T^L\left(\hat{\theta}\right) \geq W_T^H(\theta) - W_T^L(\theta).$$

Suppose that $W_T^H(\hat{\theta}) < W_T^H(\theta)$ and thus by the last formula $W_T^L(\hat{\theta}) < W_T^L(\theta)$. Combining these two strict inequalities yields a contradiction to (16) and thus $W_T^H(\hat{\theta}) \geq W_T^H(\theta)$ holds. A similar argument (but now using (17)) yields that $W_T^L(\hat{\theta}) \leq W_T^L(\theta)$.

Note, that $W_T^H(1) < W_T^L(1)$, because

$$W_T^H(1) \equiv \int_{[0,1]} (v - \beta(t))\, d\Gamma_{(1)}^H(t)$$

and

$$W_T^L(1) \equiv \int_{[0,1]} (v - \beta(t))\, d\Gamma_{(1)}^L(t),$$

and $\Gamma_{(1)}^H$ first order stochastically dominates $\Gamma_{(1)}^L$ and function β is increasing as we showed above. Then monotonicity of functions W_T^H and W_T^L implies that for all θ it holds that

$$W_T^H(\theta) \leq W_T^H(1) < W_T^L(1) \leq W_T^L(\theta). \tag{18}$$

Therefore, for all $\hat{\theta} > \theta$

$$TW\left(\hat{\theta},\hat{\theta}\right) = \hat{\theta} W_T^H\left(\hat{\theta}\right) + \left(1-\hat{\theta}\right) W_T^L\left(\hat{\theta}\right) <$$
$$< \theta W_T^H\left(\hat{\theta}\right) + (1-\theta) W_T^L\left(\hat{\theta}\right) = TW\left(\hat{\theta},\theta\right) \leq TW(\theta,\theta), \tag{19}$$

where the strict inequality follows from (18), and the weak follows from property *R2* of function TW.

Thus TW satisfies property *R3*, and a unique fixed point of map T exists in the space of regular functions. Let W^* denote this fixed point. Since $W^* = TW^*$ is a regular function and TW^* is strictly decreasing in the second variable, for all $\theta_h > \theta_l$ it holds (using property *R6*) that

$$W^*\left(\boldsymbol{\theta}_{\mathbf{l}}^{+}, \boldsymbol{\theta}_{\mathbf{l}}^{0}\right) \geq W^*\left(\boldsymbol{\theta}_{\mathbf{h}}^{+}, \boldsymbol{\theta}_{\mathbf{l}}^{0}\right) > W^*\left(\boldsymbol{\theta}_{\mathbf{h}}^{+}, \boldsymbol{\theta}_{\mathbf{h}}^{0}\right).$$

Therefore,

$$\beta^*(\theta_l) = v - \delta W^*\left(\boldsymbol{\theta}_{\mathbf{l}}^{+}, \boldsymbol{\theta}_{\mathbf{l}}^{0}\right) < v - \delta W^*\left(\boldsymbol{\theta}_{\mathbf{h}}^{+}, \boldsymbol{\theta}_{\mathbf{h}}^{0}\right) = \beta^*(\theta_h),$$

and β^* is indeed strictly increasing. **QED**

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ONLINE APPENDIX

7.3 Appendix 1: Proof of Lemmas 2 and 3

Proof: In this Appendix we prove that there exists a unique steady state stock of buyers and sellers, and a unique belief distribution, and the other properties stated in Lemma 2. We start our analysis with the stocks D^w, S^w. The steady state conditions for the stocks can be written as:

$$d^w = (1-\delta) D^w + \delta S^w \left(1 - e^{-D^w/S^w}\right) \tag{20}$$

$$1 = (1-\delta) S^w + \delta S^w \left(1 - e^{-D^w/S^w}\right). \tag{21}$$

These conditions have a simple interpretation: the left hand side is the (exogenous) inflow for each market side. The right hand side is outflow from each market side, the sum of the number of traders who exit through discouragement and the number of traders who exit through trade. The number of agents who exit through trade is $\delta S^w \left(1 - e^{-D^w/S^w}\right)$, which is equal for both market sides. Inspecting these conditions proves existence and uniqueness of the total steady state masses:

Lemma 10 *There exist unique total masses $D^w \equiv H^w(1)$ and S^w that solve the steady state conditions. The masses $D^H > D^L$ and $S^H < S^L$.*

Proof: Every steady state must solve

$$\begin{aligned}
D^w &= d^w + \delta^w \int_0^1 \left(1 - e^{-(D^w - H^w(\theta))/S^w}\right) dH^w(\theta) \\
&= d^w + \delta D^w - \delta \int_0^1 e^{-(D^w - H^w(\theta))/S^w} dH^w(\theta) \\
&= d^w + \delta D^w - \delta \int_0^1 \frac{\partial}{\partial \theta} \left(S^w e^{-(D^w - H^w(\theta))/S^w}\right) d\theta \\
&= d^w + \delta D^w - \delta S^w \left(1 - e^{-D^w/S^w}\right)
\end{aligned}$$

and $S^w = 1 + \delta S^w e^{-D^w/S^w}$. Rewriting yields (20) and (21).

A solution exists and the solution is unique. The difference of (20) and (21) is

$$d^w - 1 = (1-\delta)(D^w - S^w) \tag{22}$$

defines D^w as a function of S^w, $J(S^w)$, and we can rewrite (21) as a function of S^w only,

$$1 = (1-\delta) S^w + \delta S^w \left(1 - e^{-J(S^w)/S^w}\right).$$

This equation has a solution by the intermediate value theorem. At $S^w \to 0$, the right hand side becomes zero, while for $S^w \to \infty$, the right hand side becomes infinite. A solution exists in between.

The solution is unique. Let S'^w, D'^w and S''^w and D''^w be two solutions and suppose $D''^w \geq D'^w$. Then, by (22), $S''^w \geq S'^w$. Inspection of (21) shows that $S''^w > S'^w$ leads to a contradiction (and, hence, it must be that $S''^w = S'^w$, which implies $D''^w = D'^w$ by (22)): The first term is trivially strictly increasing in S^w. Also, the second term is increasing in S^w and in D^w, which can be seen by taking the derivatives,[15]

$$\begin{aligned} \frac{\partial}{\partial S^w}\left(S^w\left(1-e^{-D^w/S^w}\right)\right) &= \left(1-e^{-D^w/S^w}\right) - \frac{D^w}{S^w}e^{-D^w/S^w} \geq 0 \\ \frac{\partial}{\partial D^w}\left(S^w\left(1-e^{-D^w/S^w}\right)\right) &= e^{-D^w/S^w} > 0. \end{aligned}$$

Thus, if (21) holds for S'^w, it cannot also hold for $S''^w > S'^w$. (Intuitively, if the number of buyers and sellers is higher, then a) more sellers exit due to discouragement (the first term) and b) more sellers trade (the second term) because there are simply more sellers (the first derivative is positive) and, in addition, the number of buyers increases and so less sellers have no bidders (the second derivative is positive).)

By assumption, $d^H > d^L$. We show that this implies $D^H > D^L$ and $S^L < S^H$. First, it cannot be that both, the number of sellers and the number of buyers increases or decreases when the state is changed from L to H. By our earlier observation, if both, the number of sellers and buyers increase, then the right hand side of (21) would strictly increase, leading to a failure of the equation. Similarly, the right hand side of (21) would strictly decrease if both market side shrink. Inspection of (22) shows that the difference $\left(D^H - S^H\right) > \left(D^L - S^L\right)$. Hence, it cannot be that the buyers' market side weakly decreases while the sellers' market side weakly increases. Therefore, $D^H > D^L$ and $S^L < S^H$, as claimed. **QED**

We are ready to describe the steady state distribution of beliefs, Γ^w. Let $H^w(\theta) = D^w\Gamma^w(\theta)$ denote the mass of the stock of buyers with belief lower than θ, and let $F^w(\theta) = d^w G^w(\theta)$ denote the flow of entering buyers with belief less than θ. Let us denote the corresponding density functions as h^w and f^w. Note, that $D^w = H^w(1)$ and $d^w = F^w(1)$. The steady state conditions can be written as

$$H^w(\theta) = F^w(\theta) + \delta \int_{\tau:\theta^+(\tau,\tau)\leq\theta} \left(1 - e^{-(D^w - H^w(\tau))/S^w}\right) dH^w(\tau).$$

[15] Note that $\frac{1}{x}\left(1-e^{-x}\right) \geq e^{-x}$ holds if $e^x - 1 \geq x$. The latter inequality is true , since, at zero, $e^0 - 1 \geq 0$ and $(e^x - 1 - x)' = e^x - 1 \geq 0$.

We say H^w satisfies the no-introspection property at θ if

$$\frac{\theta}{1-\theta}=\lim_{\varepsilon\to 0}\frac{H^H(\theta+\varepsilon)-H^H(\theta)}{H^L(\theta+\varepsilon)-H^L(\theta)}\equiv\frac{dH^H(\theta)}{dH^L(\theta)},$$

if $H^L(\theta+\varepsilon)-H^L(\theta)>0$ for all $\varepsilon>0$.[16] The no introspection property of the distribution of beliefs is a consistency check of our model; it implies that knowing ones own belief and the equilibrium belief distribution does not allow one to update his belief. Also, for the uniqueness of the steady state belief distribution, an important step is to verify that every steady state distribution satisfies the no introspection property. Together with Lemma 12 the property also implies that, in steady state, updating is monotone as we discuss later.

Lemma 11 *If a family H^w satisfies the steady state condition, then it has the no introspection property.*

Proof: Let $\theta^m(\theta)$ denote the posterior of θ after losing m times, with $\theta^1(\theta)=\theta^+(\theta,\theta)$, $\theta^2=\theta^+\left(\theta^1(\theta),\theta^1(\theta)\right)$,... . The steady state conditions

$$\begin{aligned}
H^w(\theta) &= F^w(\theta)+\delta\int_{\tau:\theta^1(\tau)\le\theta}\xi^w(\tau)\,dH^w(\tau)\\
&= F^w(\theta)+\delta\left(\int_{\tau:\theta^1(\tau)\le\theta}\xi^w(\tau)\,dF^w(\tau)+\delta\int_{\tau:\theta^2(\tau)\le\theta}\xi^w(\tau)\,dH^w(\tau)\right)\\
&= F^w(\theta)+\delta\int_{\tau:\theta^1(\tau)\le\theta}\xi^w(\tau)\,dF^w(\tau)+\delta^2\int_{\tau:\theta^2(\tau)\ \le\theta}\xi^w\left(\theta^1(\tau)\right)\xi^w(\tau)\,dF^w(\tau)+....
\end{aligned}$$

By assumption, the distribution of types in the inflow satisfies the no introspection property,

$$\frac{\theta}{1-\theta}=\frac{f^H(\theta)}{f^L(\theta)}$$

and, by definition of the posterior after losing,

$$\begin{aligned}
\frac{\theta}{1-\theta} &= \frac{\xi^H\left(\left(\theta^1\right)^{-1}(\theta)\right)}{\xi^L\left(\left(\theta^1\right)^{-1}(\theta)\right)}\frac{f^H\left(\left(\theta^1\right)^{-1}(\theta)\right)}{f^L\left(\left(\theta^1\right)^{-1}(\theta)\right)}\\
\frac{\theta}{1-\theta} &= \frac{\xi^H\left(\left(\theta^2\right)^{-1}(\theta)\right)}{\xi^L\left(\left(\theta^2\right)^{-1}(\theta)\right)}\frac{\xi^H\left(\left(\theta^1\right)^{-1}(\theta)\right)}{\xi^L\left(\left(\theta^1\right)^{-1}(\theta)\right)}\frac{f^H\left(\left(\theta^2\right)^{-1}(\theta)\right)}{f^L\left(\left(\theta^2\right)^{-1}(\theta)\right)}
\end{aligned}$$

[16]Here and in the following, statements about ratios are made conditional on the denominator being non-zero unless stated otherwise.

and so on. From the rewritten steady state definition

$$\begin{aligned}
&\frac{H^H(\theta+\varepsilon)-H^H(\theta)}{H^L(\theta+\varepsilon)-H^L(\theta)} \\
=\;&\frac{F^H(\theta+\varepsilon)-F^H(\theta)+\delta\int_{\tau:\theta^1(\tau)\in[\theta,\theta+\varepsilon]}\xi^H(\tau)\,dF^H(\tau)+\delta^2\int_{\tau:\theta^2(\tau)\in[\theta,\theta+\varepsilon]}\xi^L\left(\theta^1(\tau)\right)\xi^H(\tau)\,dF^H(\tau)+...}{F^H(\theta+\varepsilon)-F^L(\theta)+\delta\int_{\tau:\theta^1(\tau)\in[\theta,\theta+\varepsilon]}\xi^L(\tau)\,dF^L(\tau)+\delta^2\int_{\tau:\theta^2(\tau)\in[\theta,\theta+\varepsilon]}\xi^L\left(\theta^1(\tau)\right)\xi^L(\tau)\,dF^L(\tau)+...}
\end{aligned}$$

Taking limits with respect to ε proves the claim. **QED**

The distributions satisfy the MLRP for $\theta''' > \theta'' > \theta'$ if[17]

$$\frac{H^H\left(\theta'''\right)-H^H\left(\theta''\right)}{H^L\left(\theta'''\right)-H^L\left(\theta''\right)} \geq \frac{H^H\left(\theta''\right)-H^H\left(\theta'\right)}{H^L\left(\theta''\right)-H^L\left(\theta'\right)}$$

and they satisfy the strong MLRP if these ratios are larger than one,

$$\frac{H^H\left(\theta'''\right)-H^H\left(\theta''\right)}{H^L\left(\theta'''\right)-H^L\left(\theta''\right)} \geq \frac{H^H\left(\theta''\right)-H^H\left(\theta'\right)}{H^L\left(\theta''\right)-H^L\left(\theta'\right)} \geq 1 \tag{23}$$

It is easy to show that the no introspection condition implies that the steady state distributions H^w satisfy this strong monotone likelihood ratio property under our assumption that the support contains only types with beliefs above 0.5, $H^w(0.5)=0$. To see this note that the no introspection property implies that $\frac{D^H\gamma^H}{D^L\gamma^L}=\frac{h^H}{h^L}=\frac{\theta}{1-\theta}$ is increasing in θ, and thus the first inequality in (23) follows after an integration. The second inequality in that formula follows from the fact that already at $\theta=\underline{\theta}$, it holds that $\frac{h^H}{h^L}=\frac{\theta}{1-\underline{\theta}}\geq 1$, since $\underline{\theta}\geq 0.5$.

Before proceeding, it is useful to point out that the ratio of losing probabilities in the two states, the likelihood ratio upon losing, can be written as

$$\frac{1-e^{-\left(D^H-H^H(\theta)\right)/S^H}}{1-e^{-\left(D^L-H^L(\theta)\right)/S^L}}.$$

The next Lemma shows that the strong MLRP implies that the likelihood ratio upon losing is monotone.

Lemma 12 *Suppose H^H and H^L are weakly increasing functions and S^H and S^L are positive numbers $S^H \leq S^L$ such that the strong monotone likelihood ratio property holds. Then the following inequality holds for all $\theta'' > \theta'$*

$$\frac{1-e^{-\left(D^H-H^H(\theta'')\right)/S^H}}{1-e^{-\left(D^L-H^L(\theta'')\right)/S^L}} \geq \frac{1-e^{-\left(D^H-H^H(\theta')\right)/S^H}}{1-e^{-\left(D^L-H^L(\theta')\right)/S^L}}. \tag{24}$$

[17] As mentioned earlier, this and the following statements about ratios are made conditional on the denominator being non-zero unless stated otherwise.

Proof: It is sufficient to show that

$$\frac{1-e^{-\left(D^H-H^H(\theta'')\right)/S^H}}{1-e^{-(D^L-H^L(\theta''))/S^L}} \geq \frac{e^{-\left(D^H-H^H(\theta'')\right)/S^H}-e^{-\left(D^H-H^H(\theta')\right)/S^H}}{e^{-(D^L-H^L(\theta''))/S^L}-e^{-(D^L-H^L(\theta'))/S^L}} = \frac{e^{-D^H}\left(e^{H^H(\theta'')/S^H}-e^{H^H(\theta')/S^H}\right)}{e^{-D^L}\left(e^{H^L(\theta'')/S^L}-e^{H^L(\theta')/S^L}\right)} \tag{25}$$

We prove a stronger statement with lighter notation: Suppose $\frac{(q-x)}{(r-z)} \geq \frac{x-a}{z-D} \geq 1$ and $q \geq x$ and $x \geq a$. We show that

$$\frac{1-e^{-(q-x)}}{1-e^{-(r-z)}} \geq \frac{(q-x)}{(r-z)}\frac{e^{-(q-x)}}{e^{-(r-z)}} \tag{26}$$

and

$$\frac{e^{-(q-x)}-e^{-(q-a)}}{e^{-(r-z)}-e^{-(r-D)}} \leq \frac{(x-a)}{(z-D)}\frac{e^{-(q-x)}}{e^{-(r-z)}}. \tag{27}$$

Inequalities (26) and (27) imply $\frac{1-e^{-(q-x)}}{1-e^{-(r-z)}} \geq \frac{e^{-(q-x)}-e^{-(q-a)}}{e^{-(r-z)}-e^{-(r-D)}}$. (We renamed $q = 1/S^H \quad r = 1/S^L$ and $x = H^H\left(\theta''\right)/S^H$, $z = H^L\left(\theta''\right)/S^L$, $a = H^H\left(\theta'\right)/S^H$, $D = H^L\left(\theta''\right)/S^L$.)

Inequality (26) is equivalent to

$$\frac{e^q - e^x}{e^r - e^z} \geq \frac{(q-x)}{(r-z)}\frac{e^x}{e^z}.$$

Let f be some linear function such that $f(x) = z$ and $f(q) = r$, i.e.,

$$f(\tau) = z + \frac{r-z}{q-x}(\tau - x)$$

Note that $e^q = e^x + \int_x^q e^\tau d\tau$ and $e^r = e^z + \int_x^q f' e^{f(\tau)} d\tau$. The ratio $\frac{e^\tau}{f' e^{f(\tau)}} = \frac{q-x}{r-z}e^{\tau - f(\tau)}$ is increasing in τ if $\frac{(q-x)}{(r-z)} \geq 1$, noting that $\frac{\partial}{\partial\tau}(\tau - f(t)) = ((q-x)-(r-z))/(q-x)$. Hence

$$\begin{aligned}\frac{e^q - e^x}{e^r - e^z} &= \frac{\int_x^q e^\tau}{e^r - e^z} d\tau = \int_x^q \frac{e^\tau}{f' e^{f(\tau)}} \frac{f' e^{f(\tau)}}{e^r - e^z} d\tau \\ &\geq \frac{e^x}{f' e^{f(x)}} \int_x^q \frac{f' e^{f(\tau)}}{e^r - e^z} d\tau = \frac{q-x}{r-z}\frac{e^x}{e^z}.\end{aligned}$$

Inequality (27) is equivalent to

$$\frac{e^x - e^a}{e^z - e^D} \leq \frac{(x-a)}{(z-D)}\frac{e^x}{e^z}$$

with a similar trick: $e^x - e^a = \int_a^x e^{\tau} d\tau$. Let $g(\tau)$ be such that $g(a) = D$ and $g(x) = z$, with

$$g(\tau) = D + \frac{z-D}{x-a}(\tau - a).$$

Then, $e^z - e^D = \int_a^x g' e^{g(\tau)} d\tau$. And since $e^{\tau - g(\tau)}$ is increasing in τ if $\frac{(x-a)-(z-D)}{x-a} \geq 0$,

$$\frac{e^x - e^a}{e^z - e^D} = \int_a^x \frac{e^{\tau}}{g' e^{g(\tau)}} \frac{g' e^{g(\tau)}}{e^z - e^D} d\tau \leq \frac{x-a}{z-D} \frac{e^x}{e^z}. \qquad \textbf{QED}$$

To understand the importance of this result, let us define the posterior of a bidder θ who lost with an equilibrium bid. Given a family H^w, this posterior $\theta^+(\theta, \theta)$ can be written as

$$\frac{\theta^+(\theta, \theta)}{1 - \theta^+(\theta, \theta)} = \frac{\theta}{1-\theta} \frac{\left(1 - e^{-\left(D^H - H^H(\theta)\right)}\right)}{\left(1 - e^{-\left(D^L - H^L(\theta)\right)}\right)}$$

whenever $1 - e^{-\left(D^L - H^L(\theta)\right)} > 0$ and let $\theta^+(\theta, \theta) = 1$ otherwise. Then the above Lemma (Lemma 12) implies that θ^+ is increasing, i.e. updating upon losing is monotone in θ.

The proof also implies that the densities conditional on losing are ordered in the sense of MLRP too. That holds, because the central inequality (25) implies that, for all $\theta''' > \theta'' > \theta' > \theta$, the ratio of the losing densities are ordered as well

$$\frac{e^{-\left(H^H(1) - H^H(\theta''')\right)/S^H} - e^{-\left(H^H(1) - H^H(\theta'')\right)/S^H}}{e^{-\left(H^L(1) - H^L(\theta''')\right)/S^L} - e^{-\left(H^L(1) - H^L(\theta'')\right)/S^L}} \geq \frac{e^{-\left(H^H(1) - H^H(\theta')\right)/S^H} - e^{-\left(H^H(1) - H^H(\theta)\right)/S^H}}{e^{-\left(H^L(1) - H^L(\theta')\right)/S^L} - e^{-\left(H^L(1) - H^L(\theta)\right)/S^L}}. \tag{28}$$

Now, let $\theta^0(\theta, \theta) = \theta^0(\theta, \beta(\theta))$ denote the posterior upon tieing at the position with a prior θ, and

bid $\beta(\theta)$. Then

$$\theta^0(\theta, \theta) = \lim_{\theta' \to \theta} \frac{\theta}{1-\theta} \frac{e^{-\left(H^H(1) - H^H(\theta')\right)/S^H} - e^{-\left(H^H(1) - H^H(\theta)\right)/S^H}}{e^{-\left(H^L(1) - H^L(\theta')\right)/S^L} - e^{-\left(H^L(1) - H^L(\theta)\right)/S^L}}$$

is increasing after using the differential version of (28). This shows that the equilibrium is such that both the losing and the tieing posterior are increasing in the prior θ.

We are ready to construct the steady state distributions "from the bottom up." However, when doing so, we have little information about the upper tail. In particular, even if we can show that the ratio of the densities is increasing on some interval $\left[0, \hat{\theta}\right]$, this information alone would not necessarily be sufficient to conclude that updating is monotone, since we need to have information on the behavior of the ratio $\left(H^H(1) - H^H(\theta)\right) / \left(H^L(1) - H^L(\theta)\right)$. The next Lemma shows how to construct the equilibrium belief distribution explicitly, using that we can set $H^w(1) = D^w$, where D^w is the unique solution to the steady state conditions for the total masses.

Introducing the generalized inverse $\theta^{-1}\left(\theta'\right) = \inf\left\{\theta | \theta^+\left(\theta, \theta\right) \geq \theta'\right\}$, the steady state condition holds at θ, given a monotone inverse θ^{-1} and a family H^w if

$$H^w\left(\theta\right) = F^w\left(\theta\right) - \int_0^{\theta^{-1}(\theta)} \left(1 - e^{-(D^w - H^w(\tau))}\right) dH^w\left(\tau\right). \tag{29}$$

Given some family H^w with inverse θ^{-1} and some cutoff $\hat{\theta}$, we define the operator

$$T_{\hat{\theta}} H^w\left(\theta\right) = F^w\left(\theta\right) + \delta \int_0^{\theta^{-1}(\theta)} \left(1 - e^{-(D^w - H^w(\tau))}\right) dH^w\left(\tau\right)$$

for all $\theta \leq \hat{\theta}$. For $[\hat{\theta}, 1]$, $T_{\hat{\theta}} H^w$ is the linear extension, with $T_{\hat{\theta}} H^w\left(\hat{\theta}\right)$ as defined before and $T_{\hat{\theta}} H^w(1) = H^w(1) = D^w$ (and so the linear part of $T_{\hat{\theta}} H$ has a slope $(D^w - T_{\hat{\theta}} H^w(\hat{\theta}))/(1 - \hat{\theta})$). Note, that by construction it holds that if H^w satisfies the steady state conditions up to belief level θ, then $T_{\hat{\theta}} H^w$ satisfies these conditions up to $\widetilde{\theta}^+(\theta) > \theta$. The next Lemma provides a result that allows us to construct the steady state distribution in an iterative, constructive manner. (See Lemma 14 below.)

Lemma 13 *Suppose there is some family of continuous and monotone increasing functions H^w with a monotone updating rule θ^+ such that (i) the steady state conditions (29) hold on some interval $\left[0, \hat{\theta}\right]$ and such that (ii) H^w solves the steady state conditions for the total masses, i.e., $H^w(1) = D^w$. If (iii) the no-introspection property holds for all $\theta \leq \hat{\theta}$ and (iv) the strong monotone likelihood ratio property (23) holds everywhere for H^w, then for all $\theta''' > \theta'' > \theta'$*

$$\frac{T_{\hat{\theta}} H^H\left(\theta'''\right) - T_{\hat{\theta}} H^H\left(\theta''\right)}{T_{\hat{\theta}} H^L\left(\theta'''\right) - T_{\hat{\theta}} H^L\left(\theta''\right)} \geq \frac{T_{\hat{\theta}} H^H\left(\theta''\right) - T_{\hat{\theta}} H^H\left(\theta'\right)}{T_{\hat{\theta}} H^L\left(\theta''\right) - T_{\hat{\theta}} H^L\left(\theta'\right)} \geq 1. \tag{30}$$

Moreover, $T_{\hat{\theta}} H^w$ satisfies the no introspection condition on $\left[0, \theta^+\left(\hat{\theta}, \hat{\theta}\right)\right]$.

Proof: We drop the subscript $\hat{\theta}$ and write $TH^w\left(\theta\right)$. It is sufficient for (30) to show that for every $\hat{\theta} \geq \theta''' > \theta'$,

$$\frac{TH^H(1) - TH^H\left(\theta'''\right)}{TH^L(1) - TH^L\left(\theta'''\right)} \geq \frac{TH^H\left(\theta'''\right) - TH^H\left(\theta'\right)}{TH^L\left(\theta'''\right) - TH^L\left(\theta'\right)} \geq 1. \qquad 13.1$$

Sufficiency follows from linearity of TH^w on $\left[\hat{\theta}, 1\right]$ and standard reasoning about the ordering of ratios of sums. Sufficiency of (13.1) is immediate if $\hat{\theta} > \theta'''$. If $\theta''' \geq \theta'' > \hat{\theta} > \theta'$, then (by linearity)

$$\frac{TH^H(\theta''') - TH^H(\theta'')}{TH^L(\theta''') - TH^L(\theta'')} = \frac{TD^H - TH^H\left(\hat{\theta}\right)}{TD^L - TH^L\left(\hat{\theta}\right)}$$

and if (13.1) holds, then

$$\begin{aligned}
\frac{TH^H(1) - TH^H\left(\hat{\theta}\right)}{TH^L(1) - TH^L\left(\hat{\theta}\right)} &\geq \frac{TH^H(1) - TH^H\left(\hat{\theta}\right) + TH^H\left(\hat{\theta}\right) - TH^H(\theta')}{TH^L(1) - TH^L\left(\hat{\theta}\right) + TH^L\left(\hat{\theta}\right) - TH^L(\theta')} \\
&\geq \frac{TH^H(\theta'') - TH^H\left(\hat{\theta}\right) + TH^H\left(\hat{\theta}\right) - TH^H(\theta')}{TH^L(\theta'') - TH^H\left(\hat{\theta}\right) + TH^L\left(\hat{\theta}\right) - TH^L(\theta')} \\
&= \frac{TH^H(\theta'') - TH^H(\theta')}{TH^L(\theta'') - TH^L(\theta')}.
\end{aligned}$$

The last two inequalities imply the desired inequality (30). Similar reasoning establishes sufficiency for the remaining case, $\theta''' \geq \hat{\theta} \geq \theta'' > \theta'$.

To establish (13.1), note that (23) holds by hypothesis and, therefore, by Lemma 12 the updating rule θ^+ is strictly increasing. By assumption, $D^w = H^w(1)$ i.e.,

$$\begin{aligned}
D^w &= d^w + \delta D^w - \delta S^w\left(1 - e^{-D^w/S^w}\right) \\
&= TD^w
\end{aligned}$$

and rewriting

$$\begin{aligned}
D^w &= d^w + \delta \int_0^1 dH^w(\theta)\, d\theta - \delta \int_0^1 \frac{\partial}{\partial\theta}\left(S^w\left(1 - e^{-D^w/S^w}\right)\right) dH^w(\theta)\, d\theta \\
&= d^w + \delta \int_0^1 \left(1 - e^{-(D^w - H^w(\theta))/S^w}\right) dH^w(\theta)\, d\theta
\end{aligned}$$

Hence, we can rewrite the ratios using the definition of TH^w

$$\begin{aligned}
& \frac{TD^H - TH^H\left(\theta'''\right)}{TD^L - TH^L\left(\theta'''\right)} \\
= \quad & \frac{D^H - F^H\left(\theta'''\right) - \delta \int_0^{\theta^{-1}(\theta''')} \left(1 - e^{-\left(D^H - H^H(\tau)\right)/S^H}\right) dH^H\left(\tau\right)}{D^L - F^L\left(\theta'''\right) - \delta \int_0^{\theta^{-1}(\theta''')} \left(1 - e^{-(D^L - H^L(\tau))/S^L}\right) dH^L\left(\tau\right)} \\
= \quad & \frac{d^H - F^H\left(\theta'''\right) + \delta \int_{\theta^{-1}(\theta''')}^1 \left(1 - e^{-\left(D^H - H^H(\tau)\right)/S^H}\right) dH^H\left(\tau\right)}{d^L - F^L\left(\theta'''\right) + \delta \int_{\theta^{-1}(\theta''')}^1 \left(1 - e^{-(D^L - H^L(\tau))/S^L}\right) dH^L\left(\tau\right)}
\end{aligned}$$

Since F^w satisfies no introspection

$$\frac{d^H - F^H\left(\theta'''\right)}{d^L - F^L\left(\theta'''\right)} \geq \frac{dF^H\left(\theta'''\right)}{dF^L\left(\theta'''\right)} = \frac{\theta'''}{1 - \theta'''}$$

By Lemma 12, for all $\tau \geq \theta^{-1}\left(\theta'''\right)$,

$$\frac{\left(1 - e^{-\left(D^H - H^H(\tau)\right)/S^H}\right)}{\left(1 - e^{-(D^L - H^L(\tau))/S^L}\right)} \geq \frac{\left(1 - e^{-\left(D^H - H^H\left(\theta^{-1}(\theta''')\right)\right)}\right)}{\left(1 - e^{-\left(D^L - H^L\left(\theta^{-1}(\theta''')\right)\right)}\right)}$$

and so

$$\begin{aligned}
\frac{\int_{\theta^{-1}(\theta''')}^1 \left(1 - e^{-\left(D^H - H^H(\tau)\right)/S^H}\right) dH^H\left(\tau\right)}{\int_{\theta^{-1}(\theta''')}^1 \left(1 - e^{-(D^L - H^L(\tau))/S^L}\right) dH^L\left(\tau\right)} \quad \geq \quad & \frac{\left(1 - e^{-\left(D^H - H^H\left(\theta^{-1}(\theta''')\right)\right)}\right)}{\left(1 - e^{-\left(D^L - H^L\left(\theta^{-1}(\theta''')\right)\right)}\right)} \frac{D^H - H^H\left(\theta^{-1}\left(\theta'''\right)\right)}{D^L - H^L\left(\theta^{-1}\left(\theta'''\right)\right)} \\
\geq \quad & \frac{\left(1 - e^{-\left(D^H - H^H\left(\theta^{-1}(\theta''')\right)\right)}\right)}{\left(1 - e^{-\left(D^L - H^L\left(\theta^{-1}(\theta''')\right)\right)}\right)} \frac{dH^H\left(\theta^{-1}\left(\theta'''\right)\right)}{dH^L\left(\theta^{-1}\left(\theta'''\right)\right)} \\
= \quad & \frac{\theta'''}{1 - \theta'''}
\end{aligned}$$

where the second inequality follows from the monotone likelihood ratio property of H^w and the last equality follows from the definition of the updating rule θ^+. Hence,

$$\frac{d^H - F^H\left(\theta'''\right) + \delta \int_{\theta^{-1}(\theta''')}^1 \left(1 - e^{-\left(D^H - H^H(\tau)\right)/S^H}\right) dH^H\left(\tau\right)}{d^L - F^L\left(\theta'''\right) + \delta \int_{\theta^{-1}(\theta''')}^1 \left(1 - e^{-(D^L - H^L(\tau))/S^L}\right) dH^L\left(\tau\right)} \geq \frac{\theta'''}{1 - \theta'''}$$

We can rewrite the right hand side of (13.1) in a similar fashion. First, note that

$$\frac{F^H(\theta''') - F^H(\theta')}{F^L(\theta''') - F^L(\theta')} \leq \frac{dF^H(\theta''')}{dF^L(\theta''')} = \frac{\theta'''}{1-\theta'''}$$

and, by the same argument as before

$$\frac{\int_{\theta^{-1}(\theta')}^{\theta^{-1}(\theta''')} \left(1 - e^{-\left(D^H - H^H(\tau)\right)}\right) dH^H(\tau)}{\int_{\theta^{-1}(\theta')}^{\theta^{-1}(\theta''')} \left(1 - e^{-\left(D^L - H^L(\tau)\right)}\right) dH^L(\tau)} \leq \frac{\left(1 - e^{-\left(D^H - H^H\left(\theta^{-1}(\theta''')\right)\right)}\right) dH^H\left(\theta^{-1}(\theta''')\right)}{\left(1 - e^{-\left(D^L - H^L\left(\theta^{-1}(\theta''')\right)\right)}\right) dH^L\left(\theta^{-1}(\theta''')\right)} = \frac{\theta'''}{1-\theta'''}$$

and, hence, the right hand side is bounded from above. Similarly, for all θ',

$$\frac{\int_{\theta^{-1}(\theta')}^{\theta^{-1}(\theta''')} \left(1 - e^{-\left(D^H - H^H(\tau)\right)}\right) dH^H(\tau)}{\int_{\theta^{-1}(\theta')}^{\theta^{-1}(\theta''')} \left(1 - e^{-\left(D^L - H^L(\tau)\right)}\right) dH^L(\tau)} \geq \frac{\left(1 - e^{-\left(D^H - H^H\left(\theta^{-1}(\theta')\right)\right)}\right) dH^H\left(\theta^{-1}(\theta')\right)}{\left(1 - e^{-\left(D^L - H^L\left(\theta^{-1}(\theta')\right)\right)}\right) dH^L\left(\theta^{-1}(\theta')\right)} \geq \frac{\theta'}{1-\theta'} \geq 1.$$

(Note that for types θ' that are below the most optimistic type $\underline{\theta}$, monotonicity of the updating rule together with the steady state conditions require that $H^w(\theta') = 0$ and so $TH^L(\theta''') - TH^L(\theta') = TH^L(\theta''') - TH^L(\underline{\theta})$. By $\underline{\theta} \geq 0.5$ the claim follows.)

Together with the earlier lower bound on the left hand side,

$$\begin{aligned}
& \frac{TH^H(1) - TH^H(\theta''')}{TH^L(1) - TH^L(\theta''')} \\
= \; & \frac{d^H - F^H(\theta''') + \delta \int_{\theta^{-1}(\theta''')}^{1} \left(1 - e^{-\left(D^H - H^H(\tau)\right)/S^H}\right) dH^H(\tau)}{d^L - F^L(\theta''') + \delta \int_{\theta^{-1}(\theta''')}^{1} \left(1 - e^{-\left(D^L - H^L(\tau)\right)/S^L}\right) dH^L(\tau)} \\
\geq \; & \frac{\theta'''}{1-\theta'''} \\
\geq \; & \frac{F^H(\theta''') - F^H(\theta') + \int_{\theta^{-1}(\theta')}^{\theta^{-1}(\theta''')} \left(1 - e^{-\left(D^H - H^H(\tau)\right)}\right) dH^H(\tau)}{F^L(\theta''') - F^L(\theta') + \int_{\theta^{-1}(\theta')}^{\theta^{-1}(\theta''')} \left(1 - e^{-\left(D^L - H^L(\tau)\right)}\right) dH^L(\tau)} \\
= \; & \frac{TH^H(\theta''') - TH^H(\theta')}{TH^L(\theta''') - TH^L(\theta')} \geq 1
\end{aligned}$$

Finally, for all $\theta \in \left[0, \theta^+\left(\theta_n^l, \theta_n^l\right)\right]$ the no introspection condition holds (if $dH^w > 0$)

$$\lim_{\varepsilon \to 0} \frac{TH^H(\theta + \varepsilon) - TH^H(\theta)}{TH^L(\theta + \varepsilon) - TH^L(\theta)} = \frac{dF^H(\theta) + \left(1 - e^{-\left(D^H - H^H\left(\theta^{-1}(\theta)\right)\right)}\right) dH^H\left(\theta^{-1}(\theta)\right)}{dF^L(\theta) + \left(1 - e^{-\left(D^L - H^L\left(\theta^{-1}(\theta)\right)\right)}\right) dH^L\left(\theta^{-1}(\theta)\right)} = \frac{\theta}{1-\theta}.$$ **QED**

Lemma 14 *A steady state exists.*

Proof: Using induction, we construct a sequence of families of functions H_n^w and a sequence of implied posteriors of the most optimistic type $\theta_0^l = \underline{\theta}$, $\{\theta_n^l\}_{n=0}^{\infty}$ such that, for every n, the implied updating rule θ_n is monotone and the steady state conditions hold on $\left[0, \theta_n^l\right]$.

Base Case. Let $\theta_0^l = \underline{\theta}$ and let H_0^w be constant and equal to zero on $\left[0, \theta_0^l\right]$ and linear on $\left[\theta_0^l, 1\right]$ with slope $D^w / (1 - \underline{\theta})$, so that $H_0^w(1) = D^w$. The first element H_0^w is continuous and trivially satisfies conditions (i)-(iv) of Lemma 13 with cutoff θ_0^l. By construction, $H_0^w\left(\theta_0^l\right) = 0 < D^w$ and $\theta_0^l < 1$.

Inductive Step. Suppose there is a family of continuous functions H_n^w with updating rule θ_n such that conditions (i)-(iv) of Lemma 13 hold with some cutoff θ_n^l such that $H_n^w\left(\theta_n^l\right) < D^w$ and $\theta_n^l < 1$. Define the next element by

$$H_{n+1}^w \equiv T_{\theta_n^l} H_n^w.$$

Moreover, H_{n+1}^w is continuous and satisfies the conditions (i)-(iv) of Lemma 13 with cutoff $\theta_{n+1}^l = \theta^+\left(\theta_n^l, \theta_n^l\right)$ and $H_{n+1}^w\left(\theta_{n+1}^l\right) < D^w$ and $\theta_{n+1}^l < 1$, as we show now. The monotone likelihood ratio property everywhere and the no-introspection condition on $\left[0, \theta_{n+1}^l\right]$ follow by Lemma 13, while the steady state conditions hold on $\left[0, \theta_{n+1}^l\right]$ and at 1 by construction of the operator. (See the remark about the property of the T operator after the definition of that operator.) The new element $H_{n+1}^w = H_n^w$ on $\left[0, \theta_n^l\right]$, since H_n^w satisfies the steady state conditions by hypothesis, which implies

$$\begin{aligned}
H_n^w(\theta) &= F^w(\theta) + \delta \int_0^{\theta_n^{-1}(\theta)} \left(1 - e^{-\left(H_n^w(1) - H_n^w(\tau)\right)}\right) dH_n^w(\tau) \\
&= TH_n^w(\theta) = H_{n+1}^w(\theta).
\end{aligned}$$

Since $H_n^w\left(\theta_n^l\right) < D^w$, the updated type $\theta_{n+1}^l < 1$. Furthermore, the mass at the cutoff θ_{n+1}^l is smaller than D^w,

$$\begin{aligned}
D^w - H_{n+1}^w\left(\theta_{n+1}^l\right) &\geq \delta \int_{\theta_n^l}^1 \left(1 - e^{-\left(H_n^H(1) - H_n^H(\tau)\right)/S^H}\right) dH_n^H(\tau) \\
&> 0
\end{aligned}$$

where the first inequality follows from algebraic manipulations as in proof of Lemma 13 and the second inequality follows from $H_n^w\left(\theta_n^l\right) < H_n^w(1)$. Thus, $H_{n+1}^w\left(\theta_{n+1}^l\right) < H_{n+1}^w(1) = D^w$. Together, the new family H_{n+1}^w has all the listed properties.

Proceeding by induction allows us to construct a sequence $\{H_n^w\}$ such that the steady state conditions hold on $\left[0, \theta_n^l\right]$. We now show that $\theta_n^l \to 1$ and $H_n^w\left(\theta_n^l\right) \to D^w$.

First, by construction, the ratio of the losing probabilities at θ_n^l is increasing in n (for every n, the ratio of the losing probabilities is increasing in θ by Lemma 12) and, hence, for all n

$$\frac{1-e^{-\left(D^H-H_n^H\left(\theta_n^l\right)\right)/S^H}}{1-e^{-\left(D^L-H_n^L\left(\theta_n^l\right)\right)/S^L}} \geq \frac{1-e^{-\left(D^H-H_n^H\left(\theta_0^l\right)\right)/S^H}}{1-e^{-\left(D^L-H_n^L\left(\theta_0^l\right)\right)/S^L}} \equiv \bar{x} > 1.$$

(Recall that $D^L - H^L\left(\theta_n^l\right) > 0$ for all θ_n^l.) Hence, for all n

$$\frac{\theta_{n+1}^l}{1-\theta_{n+1}^l} = \frac{\theta_n^l}{1-\theta_n^l}\frac{1-e^{-\left(D^H-H_n^H\left(\theta_n^l\right)\right)/S^H}}{1-e^{-\left(D^L-H_n^L\left(\theta_n^l\right)\right)/S^L}} \geq \frac{\theta_n^l}{1-\theta_n^l}\bar{x}$$

which implies that $\ln\frac{\theta_{n+1}^l}{1-\theta_{n+1}^l} = \ln\frac{\theta_n^l}{1-\theta_n^l} + \ln\bar{x}$. Since $\ln\bar{x} > 0$, the ratio $\frac{\theta_{n+1}^l}{1-\theta_{n+1}^l} \to \infty$ when $n \to \infty$, which implies $\theta_{n+1}^l \to 1$.

Finally, $H_n^w\left(\theta_n^l\right) \to D^w$ follows from the observation that, for every n,

$$H_n^w(1) - H_n^w\left(\theta_n^l\right) = d^w - F^w\left(\theta_n^l\right) + \delta\int_{\theta_n^{-1}\left(\theta_n^l\right)}^1 \left(1-e^{-\left(H_n^w(1)-H_n^w(\tau)\right)/S^w}\right)dH_n^w(\tau)$$

and, hence,

$$H_n^w(1) - H_n^w\left(\theta_n^l\right) \leq d^w - F^w\left(\theta_n^l\right) + \delta\left(H_n^w(1) - H_n^w\left(\theta_n^l\right)\right)$$

From $\theta_n^l \to 1$, continuity of F^w implies that $d^w - F^w\left(\theta_n^l\right) \to 0$. Hence,

$$\lim\left(H_n^w(1) - H_n^w\left(\theta_n^l\right)\right) \leq \lim\left(d^w - F^w\left(\theta_n^l\right)\right) + \lim\delta\left(H_n^w(1) - H_n^w\left(\theta_n^l\right)\right)$$

implies

$$\lim\left(H_n^w(1) - H_n^w\left(\theta_n^l\right)\right) \leq \delta\lim\left(H_n^w(1) - H_n^w\left(\theta_n^l\right)\right)$$

and by $\delta < 1$, $\lim\left(H_n^w(1) - H_n^w\left(\theta_n^l\right)\right) = 0$.

We construct the steady state distribution $H^w(\theta)$: At $\theta = 1$, set $H^w(\theta) = D^w$. At $\theta < 1$, choose some n such that $\theta_n^l > \theta$ and set $H^w(\theta) = H_n^w(\theta)$. **QED**

Let $\left\{\theta_n^l, \theta_n^h\right\}_{n=0}^\infty$ be the updated types of the boundaries of support of the inflow, $\theta_n^l = \theta^+\left(\theta_{n-1}^l, \theta_{n-1}^l\right)$ and $\theta_0^l = \underline{\theta}$ and likewise for $\theta_n^h = \theta^+\left(\theta_{n-1}^h, \theta_{n-1}^h\right)$ and $\theta_0^h = \overline{\theta}$.

Lemma 15 *The steady state distributions are unique. The distributions H^w are twice continuously differentiable except possibly at the updated types of the boundaries and at one.*

Proof: Let H^w be the steady state that results from the construction in the proof of Lemma 14 with updated types $\left\{\theta_n^l\right\}_{n=0}^{\infty}$. Let H_x^w be a possibly different steady state. We prove uniqueness by induction on n.

Base Case ($n = 0$). By monotonicity, for all $\theta \leq \theta_0^l = \underline{\theta}$, both steady states must be constant and equal to zero, $H_x^w(\theta) = 0 = H^w(\theta)$.

Inductive step. Suppose for some n, $H^w(\theta) = H_x^w(\theta)$ for all $\theta \leq \theta_n^l$. We now show that this implies $H^w(\theta) = H_x^w(\theta)$ for all $\theta \leq \theta_{n+1}^l$. By Lemma 11 and Lemma 12, updating is monotone in every steady state. Take any $\theta' \in \left[0, \theta_{n+1}^l\right]$ and by $\theta^{-1}(\theta') \in \left[0, \theta_n^l\right]$ and identity of the distributions on $\left[0, \theta_n^l\right]$, the updating rules are identical on the relevant interval and, hence, $\theta_x^+\left(\theta^{-1}(\theta'), \theta^{-1}(\theta')\right) = \theta'$, which implies $\theta_x^{-1}(\theta') = \theta^{-1}(\theta')$ and $\theta_x^{-1}(\theta') \leq \theta_n^l$. Thus,

$$\begin{aligned}
H_x^w(\theta') &= F^w(\theta') + \int_0^{\theta_x^{-1}(\theta')} \left(1 - e^{-(H_x^w(1) - H_x^w(\tau))/S^w}\right) dH_x^w(\tau) \\
&= F^w(\theta') + \int_0^{\theta^{-1}(\theta')} \left(1 - e^{-(D^w - H^w(\tau))/S^w}\right) dH^w(\tau) \\
&= H^w(\theta').
\end{aligned}$$

Uniqueness follows for all θ by induction on n and by the observation that $H_x^w(1) = D^w$.

Differentiability follows by inspection of the steady state conditions. The initial element H_1^w is twice continuously differentiable except for θ_0^l, θ_1^l and possibly $\theta_0^h = \overline{\theta}$, if $\overline{\theta} \in \left(\theta_0^l, \theta_1^l\right)$. Then, H_2^w is twice continuously differentiable except for $\left(\theta_0^l, \theta_1^l, \theta_2^l\right)$ and $\left(\theta_0^h, \theta_1^h, \theta_2^h\right)$. **QED**

7.4 Appendix 4: Proof of Proposition 2

We start by proving Lemma 7. Let us first reiterate some equations established in the main text:

$$S_k^w = D_k^w - \frac{d^w - 1}{1 - \delta_k} \tag{31}$$

and

$$D_k^w = -S_k^w \log \frac{S_k^w - 1}{\delta_k S_k^w} \tag{32}$$

for $w = L, H$.

First, assume that $d^w < 1$. Then the limits are as follows:

$$\lim_{k \to \infty} D_k^w = d^w$$

and thus from (1) it follows that

$$\lim_{k\to\infty}(1-\delta_k)S_k^w = 1-d^w. \tag{33}$$

Letting $\mu_k^w = D_k^w/S_k^w$, it follows from (2) and (3) that

$$\lim_{k\to\infty}\frac{\mu_k^w}{1-\delta_k} = \lim_{\delta_k\to 1}\frac{1-e^{-\mu_k^w}}{1-\delta_k} = \lim_{\delta_k\to 1}\frac{1-S_k^w(1-\delta_k)}{\delta_k S_k^w(1-\delta_k)} = \frac{d^w}{1-d^w}, \tag{34}$$

and thus $\lim\mu_k = \infty$ indeed holds.

Next assume that $d^w > 1$. Then from (31) and (32):

$$\lim_{k\to\infty} S_k^w = 1$$

and

$$\lim_{k\to\infty}(1-\delta_k)D_k^w = d^w - 1. \tag{35}$$

Using (2) and (5) imply that

$$\lim_{k\to\infty}\frac{e^{-\mu_k^w}}{S_k^w-1} = 1, \tag{36}$$

and similarly

$$\lim_{k\to\infty}\frac{D_k^w}{-\log(S_k^w-1)} = 1. \tag{37}$$

Using now (7) and (5) implies

$$\lim_{k\to\infty} -(1-\delta_k)\log(S_k^w-1) = d^w - 1$$

or

$$\lim_{k\to\infty}\left(\frac{1}{S_k^w-1}\right)^{1-\delta_k} = e^{d^w-1}. \tag{38}$$

Using, now (8) and (6) (raised to power $1-\delta_k$) implies

$$\lim_{k\to\infty}(e^{-\mu_k^w})^{1-\delta_k} = e^{-(d^w-1)}$$

or

$$\lim_{k\to\infty}(1-\delta_k)\mu_k^w = d^w - 1. \tag{39}$$

This last formula then implies the remaining two statements of Lemma 7. QED

Next, let $q_k^H(\theta)$ be the lifetime winning probability in state H. With $1-\xi_k(\theta)$ being the probability of winning in the current period and with $\theta_k^t(\theta)$ being the t'th update of the type θ. So, with the equilibrium updating function $\theta_k^+(\theta)$, we have $\theta_k^1(\theta) = \theta_k^+(\theta,\theta)$, $\theta_k^t(\theta) = \theta_k^+(\theta_k^{t-1}(\theta), \theta_k^{t-1}(\theta))$

for all $t \geq 2$. Then

$$q_k^H(\theta) = 1 - \xi_k^H(\theta) + \delta_k \xi_k^H(\theta)\, q_k^H(\theta_k^1(\theta)) =$$

$$= 1 - \xi_k^H(\theta) + \delta_k \xi_k^H(\theta)\left(1 - \xi_k^H\left(\theta_k^1(\theta)\right)\right) + \delta_k^2 \xi_k^H(\theta)\, \xi_k^H\left(\theta_k^1(\theta)\right)\left(1 - \xi_k^H\left(\theta_k^2(\theta)\right)\right) .... =$$

$$= \left(1 - \xi_k^H(\theta)\right) + \sum_{t=1}^{\infty}\left\{\delta_k^t\left(1 - \xi_k^H\left(\theta_k^t(\theta)\right)\right)\xi_k^H(\theta)\prod_{\tau=1}^{t-1}\xi_k^H\left(\theta_k^\tau(\theta)\right)\right\}. \tag{40}$$

We now show that the steady state conditions imply a bound on the trading probabilities (there is only a mass one of sellers in the inflow):

Lemma 16 *The lifetime trading probabilities*

$$d^H \int_{\underline{\theta}}^{\overline{\theta}} q_k^H(\theta)\, dG^H(\theta) \leq 1.$$

Moreover,

$$\lim_{k\to\infty} d^H \int_{\underline{\theta}}^{\overline{\theta}} q_k^H(\theta)\, dG^H(\theta) = 1$$

Proof: The steady state conditions (see the existence proof) require that

$$\begin{aligned}
D_k^w \Gamma_k^w(\theta) &= d^w G^w(\theta) + \delta_k D_k^w \int_0^{\left(\theta_k^1\right)^{-1}(\theta)} \xi_k^w(\tau)\, d\Gamma_k^w(\tau) = \\
&= d^w G^w(\theta) + \delta_k \left(d^w \int_0^{\left(\theta_k^1\right)^{-1}(\theta)} \xi_k^w(\tau)\, dG^w(\tau) + \delta_k D_k^w \int_0^{\left(\theta_k^2\right)^{-1}(\theta)} \xi_k^w(\tau)\, \xi_k^w\left(\theta_k^1(\tau)\right) d\Gamma_k^w(\tau)\right) = \\
&= d^w \Big(\int_0^{\theta} dG^w(\tau) + \delta_k \int_0^{\left(\theta_k^1\right)^{-1}(\theta)} \xi_k^w(\tau)\, dG^w(\tau) + \delta_k^2 \int_0^{\left(\theta_k^2\right)^{-1}(\theta)} \xi_k^w(\tau)\, \xi_k^w\left(\theta_k^1(\tau)\right) dG^w(\tau) + ...
\end{aligned}$$

where $\left(\theta_k^t\right)^{-1}(\theta)$ is the type whose posterior after t updates is below θ. Similarly,

$$D_k^w \int_0^{\theta} \left(1 - \xi_k^w(\tau)\right) d\Gamma_k^w(\tau) = d^w \Big(\int_0^{\theta} \left(1 - \xi_k^w(\tau)\right) dG^w(\tau) + \delta_k \int_0^{\left(\theta_k^1\right)^{-1}(\theta)} \left(1 - \xi_k^w(\tau)\right) \xi_k^w(\theta)\, dG^w(\tau) + ...$$

The fact that the mass of buyers who trade is equal to the mass of sellers who trade implies that

$$D_k^w \int_0^1 \left(1 - \xi_k^w(\tau)\right) d\Gamma_k^w(\tau) = S_k^w\left(1 - e^{-D_k^w / S_k^w}\right) \leq 1.$$

Hence,

$$\begin{aligned} d^H \int_{\underline{\theta}}^{\overline{\theta}} q_k^H(\theta)\, dG^H(\theta) &= d^H \Big(\int_{\underline{\theta}}^{\overline{\theta}} \left(1-\xi_k^H(\tau)\right) dG^H(\theta) + \int_{\underline{\theta}}^{\overline{\theta}} \delta_k \left(\xi_k^H(\tau)\right)\left(1-\xi_k^H\left(\theta_k^1(\tau)\right)\right) dG^H(\theta) + ... \\ &= D_k^H \int_0^1 \left(1-\xi_k^w(\tau)\right) d\Gamma_k^H(\tau) \\ &\leq 1, \end{aligned}$$

and

$$\lim_{k\to\infty} d^H \int_{\underline{\theta}}^{\overline{\theta}} q_k^H(\theta)\, dG^H(\theta) = \lim_{k\to\infty} D_k^H \int_0^1 \left(1-\xi_k^H(\tau)\right) d\Gamma_k^H = \lim_{k\to\infty} S_k^H \left(1-e^{-D_k^H(1)/S_k^H}\right) = 1. \text{ QED}$$

The following Lemma states a useful result:

Lemma 17 *Consider the case $d^H > 1$ and $d^L < 1$. Then for all $\theta \in [\underline{\theta}, \overline{\theta}]$ it holds that*

$$\lim_{k\to\infty} q_k^H(\theta) = 1/d^H.$$

Proof: The posterior of the most optimistic new buyer after loosing once becomes one, $\theta_k^1(\underline{\theta}) = \theta_k^+(\underline{\theta}, \underline{\theta}) \to 1$ since the likelihood ratio of loosing $\frac{1-e^{-D_k^H/S_k^H}}{1-e^{-D_k^L/S_k^L}} \to \infty$. By θ being monotone, the posteriors $\theta_k^1(\theta) = \theta_k^+(\theta,\theta) \to 1$ for all $\theta \in \left[\underline{\theta}, \overline{\theta}\right]$. This implies that $\theta_k^+(\underline{\theta}, \underline{\theta}) > \overline{\theta}$ and thus by monotonicity for all $\theta \in \left[\underline{\theta}, \overline{\theta}\right]$:

$$\lim_{k\to\infty} q_k^H(\theta_k^1(\underline{\theta})) \geq \lim_{k\to\infty} q_k^H(\theta) \geq \lim_{k\to\infty} q_k^H(\underline{\theta}). \tag{41}$$

We now show that

$$\lim_{k\to\infty} q_k^H(\theta_k^1(\underline{\theta})) = \lim_{k\to\infty} q_k^H(\underline{\theta}). \tag{42}$$

To see this, note that Lemma 1 implies that $\lim_{k\to\infty} \xi_k^H(\underline{\theta}) = \lim_{k\to\infty} e^{-\mu_k^H} = 1$, and thus (40) implies that $\lim_{k\to\infty} q_k^H(\theta_k^1(\underline{\theta})) - \lim_{k\to\infty} q_k^H(\underline{\theta}) = 0$ indeed holds, because $\lim_{k\to\infty} \xi_k^H(\underline{\theta}) = \lim_{k\to\infty} e^{-\mu_k^H} = 1$ and $\lim_{k\to\infty} \delta_k = 1$. Then (41) and (42) imply that for all $\theta \in \left[\underline{\theta}, \overline{\theta}\right]$ it holds that $\lim_{k\to\infty} q_k^H(\theta) = \lim_{k\to\infty} q_k^H(\underline{\theta})$. Therefore, using Lemma 16 implies the claim of Lemma 17. QED

Let $\overline{V} = \lim_{k\to\infty} V_k(1)$. We prove the following:

Lemma 18 *Consider the case $d^H > 1$ and $d^L < 1$. For any type $\theta \in [\underline{\theta}, \overline{\theta}]$ the expected price conditional on winning in the high state converges to*

$$v - \overline{V}.$$

Proof: Let $q_k(\theta, b')$ denote the probability that a type θ will end up trading at a price $p \leq b'$. Let $\xi_k^w(\theta')$ be the probability that the highest competing type is below $\theta_k = \theta_k^{-1}(b')$. Then

$$
\begin{aligned}
q_k(\theta, b') &= \xi_k^w(\min\{\theta_k, \theta\}) + \delta_k(1 - \xi_k^w(\theta))\,\xi_k^w\left(\min\left\{\theta_k, \theta_k^1(\theta)\right\}\right) && (43)\\
&\quad +\delta_k^2(1 - \xi_k^w(\theta))\left(1 - \xi_k^w\left(\theta_k^1(\theta)\right)\right)\xi_k^w\left(\min\left\{\theta_k, \theta_k^2(\theta)\right\}\right) && (44)\\
&\quad +...
\end{aligned}
$$

(we use $(\min\{\theta_k, \theta\})$ because $\beta_k(\theta)$ might be below b', i.e., $\theta < \theta_k$). Suppose that for some b' and $\varepsilon > 0$ there exists $\theta^* \in [\underline{\theta}, \overline{\theta}]$ such that

$$
d^H \lim_{k\to\infty} q_k^H(\theta^*, b') > \varepsilon,
$$

then (using that $\overline{\theta} < 1$)[18]

$$
\lim_{k\to\infty} d^H \int_{\underline{\theta}}^{\overline{\theta}} q_k^H(\theta, b')\, dG^H(\theta) \geq \varepsilon. \tag{45}
$$

Rewriting (43) yields

$$
d^H \int_{\underline{\theta}}^{\overline{\theta}} q_k^H(\theta, b')\, dG^H(\theta) \leq
$$

$$
\leq d^H \int_{\underline{\theta}}^{\overline{\theta}} \left(\xi_k^w(\theta_k) + \delta_k(1 - \xi_k^w(\theta))\,\xi_k^w(\theta_k) + \delta_k^2(1 - \xi_k^w(\theta))\left(1 - \xi_k^w\left(\theta_k^1(\theta)\right)\right)\xi_k^w(\theta_k) + ..\right) dG^H(\theta)
$$

$$
= \xi_k^w(\theta_k)\, d^H \int_{\underline{\theta}}^{\overline{\theta}} \left(1 + \delta_k(1 - \xi_k^w(\theta)) + \delta_k^2(1 - \xi_k^w(\theta))\left(1 - \xi_k^w\left(\theta_k^1(\theta)\right)\right) + ..\right) dG^H(\theta)
$$

$$
= \xi_k^w(\theta_k)\, D_k^H \int_0^1 (1 - \xi_k^w(\tau))\, d\Gamma_k^H(\tau).
$$

Recall $\xi_k^w(\theta_k) = e^{-D_k^H(1-\Gamma_k^H(\theta_k))/S_k^H}$. By hypothesis, there is a strictly positive number of buyers for which the highest competing buyer has a type below $\theta_k = \theta_k^{-1}(b')$,

$$
\varepsilon \leq d^h \lim_{k\to\infty} q_k^H(\theta, b') \leq \lim_{k\to\infty} \xi_k^w(\theta_k)\, D_k^H \int_0^1 (1 - \xi_k^w(\tau))\, d\Gamma_k^H(\tau),
$$

and hence

$$
\lim_{k\to\infty} D_k^H e^{-D_k^H(1-\Gamma_k^H(\theta_k)/S_k^H} \geq \frac{\varepsilon}{\int_0^1 (1 - \xi_k^w(\tau))\, d\Gamma_k^H(\tau)} \geq \varepsilon > 0. \tag{i}
$$

[18] This follows, because for all θ', θ'' it holds that

$$
\lim[q_k^H(\theta', b') - q_k^H(\theta'', b')] = 0,
$$

and the convergence is also uniform. The proof of this is similar to the proof of (42) in Lemma 17, and is omitted.

Now, we would like to prove the following statement: there is a positive number of buyers with type below θ_k for whom the highest competing buyer has a type below $\theta_k = \theta_k^{-1}(b')$ as well,

$$\lim_{k\to\infty} D_k^H \Gamma_k^H(\theta_k) e^{-D_k^H(1-\Gamma_k^H(\theta_k)/S_k^H} = k > 0. \qquad \text{(ii)}$$

Recall, $S_k^H \to 1$ and $D_k^H \Gamma_k^H(\theta_k) \to \infty$. Given (i) and that $D_k^H \Gamma_k^H(\theta_k) \to \infty$, it must hold that

$$\lim_{k\to\infty} \Gamma_k^H(\theta_k) = 1, \tag{46}$$

otherwise the limit in (i) would be zero. Comparing (46) and (i) implies that (ii) indeed holds with $k = \varepsilon$.

The posterior of θ_k conditional on tieing is $\theta_k^{tie} = \widetilde{\theta}^0(\theta_k)$, and it holds that

$$\begin{aligned}
\frac{\theta_k^{tie}}{1-\theta_k^{tie}} &= \frac{\theta_k}{1-\theta_k} \frac{\gamma_k^H(\theta_k)}{\gamma_k^L(\theta_k)} \frac{D_k^H}{D_k^L} \frac{S_k^L}{S_k^H} \frac{e^{-D_k^H(1-\Gamma_k^H(\theta_k)/S_k^H}}{e^{-\left(D_k^L(1)-D_k^L(\theta_k)\right)/S_k^L}} \\
&\geq \frac{\theta_k}{1-\theta_k} \frac{S_k^L}{S_k^H} \frac{\Gamma_k^H(\theta_k)}{\Gamma_k^L(\theta_k)} \frac{D_k^H}{D_k^L} \frac{e^{-D_k^H(1-\Gamma_k^H(\theta_k))/S_k^H}}{e^{-D_k^L(1-\Gamma_k^L(\theta_k))/S_k^L}}
\end{aligned}$$

from the MLRP of Γ^w, $\frac{\gamma_k^H(\theta_k)}{\gamma_k^L(\theta_k)} \geq \frac{\Gamma_k^H(\theta_k)}{\Gamma_k^L(\theta_k)}$ and from before

$$\lim_{k\to\infty} \frac{D_k^H \Gamma_k^H(\theta_k) e^{-D_k^H(1-\Gamma_k^H(\theta_k)/S_k^H}}{D_k^L \Gamma_k^L(\theta_k) e^{-D_k^L(1-\Gamma_k^L(\theta_k)/S_k^L}} \geq \frac{k}{1}$$

and, hence,

$$\lim_{k\to\infty} \frac{\theta_k^{tie}}{1-\theta_k^{tie}} \geq \frac{k}{1} \lim_{k\to\infty} \frac{\theta_k}{1-\theta_k} \frac{S_k^L}{S_k^H} = \infty.$$

This then implies that $\beta_k(\theta_k) \to v - \bar{V}$. To see this , note that

$$\beta_k(\theta_k) = v - \delta_k(\theta_k^{tie} V_k^H(\widetilde{\theta}(\theta_k)) + (1-\theta_k^{tie}) V_k^L(\widetilde{\theta}(\theta_k))).$$

The properties of the value function imply that

$$V_k(\widetilde{\theta}(\theta_k)) \leq \theta_k^{tie} V_k^H(\widetilde{\theta}(\theta_k)) + (1-\theta_k^{tie}) V_k^L(\widetilde{\theta}(\theta_k)) \leq V_k(\theta_k^{tie}).$$

Since $\widetilde{\theta}(\theta_k) \to 1$ and $\theta_k^{tie} \to 1$ by the above, it holds by (uniform) continuity of V_k that

$$\lim_{k\to\infty} \theta_k^{tie} V_k^H(\widetilde{\theta}(\theta_k)) + (1-\theta_k^{tie}) V_k^L(\widetilde{\theta}(\theta_k)) = \lim_{k\to\infty} V_k(1) = \overline{V}.$$

Therefore, it follows that indeed

$$\beta_k(\theta_k) \to v - \bar{V}.$$

Therefore, at any price b' for which there is a positive chance of trading at b' or lower in the limit, $b' = v - \bar{V}$ follows. Moreover, in the limit the probability that each type $\theta \in [\underline{\theta}, \overline{\theta}]$ wins converges to $1/d^H > 0$ by Lemma 17. These facts immediately imply that the expected price upon trading converges to $v - \bar{V}$ for all types $\theta \in [\underline{\theta}, \overline{\theta}]$. QED

We are now ready to prove Lemma 8 from the main text:

Proof: Consider $d^H < 1$ and $d^L < 1$. Then $e^{-D_k^w/S_k^w} \to 1$ implies $V_k(\theta) \to v$ by inspection of payoffs. Similarly, if $d^L < 1$, $\lim_{k\to\infty} EU_k[\beta_k, \theta|L] = v$.

Consider the case $d^L > 1$. Expected payoffs of a buyer who knows $\theta = 0$ in the low state converge to zero

$$\begin{aligned} EU_k[\beta_k \equiv 0, \theta = 0|L] &= e^{-D_k^L/S_k^L} v + \delta_k \left(1 - e^{-D_k^L/S_k^L}\right) EU_k[\beta_k \equiv 0, \theta = 0|L] \\ EU_k[\beta_k \equiv 0, \theta = 0|L] &= \frac{e^{-D_k^L/S_k^L}}{\frac{1-\delta_k}{e^{-D_k^L/S_k^L}} + \delta_k} v \to 0 \end{aligned}$$

since $\frac{1-\delta_k}{e^{-D_k^L/S_k^L}} \to \infty$ as observed before. Since $EU_k[\beta_k \equiv 0, \theta = 0|L] = V_k(0)$ and V_k is monotone decreasing, $V_k(0) \to 0$ implies $\lim_{k\to\infty} V_k(\theta) \equiv 0$ for all θ.

Consider the case $d^H > 1$ and $d^L < 1$: As we have already argued, the posterior of the most optimistic new buyer (the one with belief $\underline{\theta}$) after loosing once becomes one, $\theta_k^1(\underline{\theta}) = \theta_k^+(\underline{\theta}, \beta_k(\underline{\theta})) \to 1$. Since in the high state the first cohort has vanishing winning probability, and there is no exogenous exit in the limit either, therefore it follows that

$$\lim_{k\to\infty} V_k^H(\underline{\theta}) = \lim_{k\to\infty} V_k^H(\theta_k^1(\underline{\theta})) = \bar{V}.$$

Lemma 17 implies that the winning probability of type $\underline{\theta}$ is $1/d^H$ in the high state. Lemma 18 implies that the expected price to be paid is $v - \overline{V}$. Putting these two results together yields that

$$\lim_{k\to\infty} V_k^H(\underline{\theta}) = \frac{v - (v - \overline{V})}{d^H} = \frac{\overline{V}}{d^H}.$$

Since $d^H > 1$, therefore the last two displayed equation imply that $\overline{V} = 0$. Thus in the high state the utility (for all types at least $\underline{\theta}$) converges to $v - \overline{V} = v$. Using that in the low state the probability of having at least one opponent converges to zero, means that the continuation value in the low state converges to v, which concludes the proof of the Lemma.

QED

The proof the last Lemma implies that $\overline{V} = 0$, and this together with Lemma 18 yields our main convergence result, Proposition 2 for the case where $d^L < 1 < d^H$. The first part of the Proposition follows from $\overline{V} = 0$ and Lemma 18, while the second part follows from the fact that the number of rival buyers in the low state converges to zero in expected value as δ approaches 1.

In the case where $d^H, d^L < 1$ it holds that there are zero competing buyers in expectations in the limit, and thus the buyers obtain the object for free. Finally, when $d^H, d^L > 1$ the continuation value for any belief θ converges to 0 by Lemma 8. Therefore, the fact that $W(\theta^+(\theta,\theta), \theta^0(\theta,\theta)) \leq V(\theta^0(\theta,\theta))$ implies that for all θ it holds that $W(\theta^+(\theta,\theta), \theta^0(\theta,\theta)) \to 0$, and thus $\beta(\theta) \to v$ for all v. Now, since there is at least one buyer almost surely in the limit, therefore the expected transaction price must converge to zero. QED

7.5 Appendix 5: Time pattern of bids

Proof of Proposition 3:

First, the sequence of losing posteriors can be derived as

$$\frac{\widetilde{\theta}_k^t}{1-\widetilde{\theta}_k^t} = \frac{\widetilde{\theta}_k^{t-1}}{1-\widetilde{\theta}_k^{t-1}} \frac{1-\Gamma_{1,k}^H(\widetilde{\theta}_k^{t-1})}{1-\Gamma_{1,k}^L(\widetilde{\theta}_k^{t-1})},$$

and also we know that the sequence $\widetilde{\theta}_k^t$ is strictly increasing. Next we note that $\lim_{k\to\infty} \Gamma_{1,k}^H(x) = \lim_{k\to\infty} 1 - \Gamma_{1,k}^L(x) = 0$ for all $x \geq \underline{\theta}$. Therefore, $\lim_{k\to\infty} \widetilde{\theta}_k^1 = 1$, and thus by monotonicity

$$\lim_{k\to\infty} \widetilde{\theta}_k^t = 1 \tag{47}$$

for all $t \geq 1$. In other words the posterior of any loser converges to 1.

Next, let us calculate the relevant tieing posteriors:

$$\frac{\eta_k^t}{1-\eta_k^t} = \frac{\widetilde{\theta}_k^t}{1-\widetilde{\theta}_k^t} \frac{\gamma_{1,k}^H(\widetilde{\theta}_k^t)}{\gamma_{1,k}^L(\widetilde{\theta}_k^t)}. \tag{48}$$

We know from before that

$$\gamma_{1,k}^H = \mu_k^H \gamma_k^H \Gamma_{1,k}^H \tag{49}$$

and

$$\gamma_{1,k}^L = \mu_k^L \gamma_k^L \Gamma_{1,k}^L. \tag{50}$$

The no introspection condition implies that for all $x \in [0,1]$,

$$x = \frac{D_k^H \gamma_k^H(x)}{D_k^H \gamma_k^H(x) + D_k^L \gamma_k^L(x)}. \tag{51}$$

Combining the last three formulas yields

$$\frac{\gamma_{1,k}^H(x)}{\gamma_{1,k}^L(x)} = \frac{x}{1-x} \frac{S_k^L}{S_k^H} \frac{\Gamma_{1,k}^H(x)}{\Gamma_{1,k}^L(x)}. \tag{52}$$

Then after substituting iteratively we obtain that

$$\begin{aligned}
\frac{\eta_k^t}{1-\eta_k^t} &= \frac{\widetilde{\theta}_k^{t-1}}{1-\widetilde{\theta}_k^{t-1}} \frac{1-\Gamma_{1,k}^H(\widetilde{\theta}_k^{t-1})}{1-\Gamma_{1,k}^L(\widetilde{\theta}_k^{t-1})} \frac{\widetilde{\theta}_k^{t}}{1-\widetilde{\theta}_k^{t}} \frac{S_k^L}{S_k^H} \frac{\Gamma_{1,k}^H(\widetilde{\theta}_k^t)}{\Gamma_{1,k}^L(\widetilde{\theta}_k^t)} = \\
&= (\frac{\widetilde{\theta}_k^{t-1}}{1-\widetilde{\theta}_k^{t-1}} \frac{1-\Gamma_{1,k}^H(\widetilde{\theta}_k^{t-1})}{1-\Gamma_{1,k}^L(\widetilde{\theta}_k^{t-1})})^2 \frac{S_k^L}{S_k^H} \frac{\Gamma_{1,k}^H(\widetilde{\theta}_k^t)}{\Gamma_{1,k}^L(\widetilde{\theta}_k^t)} = \\
&= (\frac{\widetilde{\theta}_k^{t-2}}{1-\widetilde{\theta}_k^{t-2}} \frac{1-\Gamma_{1,k}^H(\widetilde{\theta}_k^{t-2})}{1-\Gamma_{1,k}^L(\widetilde{\theta}_k^{t-2})} \frac{1-\Gamma_{1,k}^H(\widetilde{\theta}_k^{t-1})}{1-\Gamma_{1,k}^L(\widetilde{\theta}_k^{t-1})})^2 \frac{S_k^L}{S_k^H} \frac{\Gamma_{1,k}^H(\widetilde{\theta}_k^t)}{\Gamma_{1,k}^L(\widetilde{\theta}_k^t)} = ... = \\
&= (\frac{\theta}{1-\theta} \frac{1-\Gamma_{1,k}^H(\theta)}{1-\Gamma_{1,k}^L(\theta)} ... \frac{1-\Gamma_{1,k}^H(\widetilde{\theta}_k^{t-2})}{1-\Gamma_{1,k}^L(\widetilde{\theta}_k^{t-2})} \frac{1-\Gamma_{1,k}^H(\widetilde{\theta}_k^{t-1})}{1-\Gamma_{1,k}^L(\widetilde{\theta}_k^{t-1})})^2 \frac{S_k^L}{S_k^H} \frac{\Gamma_{1,k}^H(\widetilde{\theta}_k^t)}{\Gamma_{1,k}^L(\widetilde{\theta}_k^t)}.
\end{aligned} \tag{53}$$

Next note that (47) implies that $\lim_{k\to\infty} \widetilde{\theta}_k^1 > \overline{\theta}$ regardless of θ, and thus monotonicity of beliefs implies that for any starting belief $\theta \in [\underline{\theta}, \overline{\theta}]$ it holds that

$$\lim_{k\to\infty} \eta_k^t(\underline{\theta}) > \lim_{k\to\infty} \eta_k^{t-1}(\theta).$$

Therefore, to prove that for any finite t the tieing probability converges to zero for any starting belief θ, it is sufficient to prove it for $\theta = \underline{\theta}$. The above remarked monotonicity results imply that type $\widetilde{\theta}_k^t$ (when starting with $\theta = \underline{\theta}$) wins if and only if all the competitors lost $t-1$ times or less. Let where $D_k^{w,q}$ is the mass of buyers who lost q times. Then (when starting with $\theta = \underline{\theta}$) it holds that

$$\Gamma_1^w(\widetilde{\theta}_k^t) = e^{-\mu_k^w (1 - \frac{D_k^{w,0}}{D_k^w} - ... - \frac{D_k^{w,t-1}}{D_k^w})}.$$

It of course holds by construction that $D_k^{w,0} = d^w$.

For the high state it holds for any fixed $q \leq t$ that

$$\lim_{k\to\infty} \frac{D_k^{H,q}}{d^H} = 1,$$

since the first q cohort wins with probability zero in the limit, and there is no exogenous exit in the limit either. Thus

$$\lim_{k\to\infty} \frac{D_k^{H,0}}{D_k^H} + ... + \frac{D_k^{H,t-1}}{D_k^H} = \lim_{k\to\infty} t\frac{d^H}{D_k^H} = 0,$$

and

$$\lim_{k\to\infty} \frac{\Gamma_{1,k}^H(\widetilde{\theta}_k^t)}{e^{-\mu_k^H}} = 1. \tag{54}$$

For the low state notice that

$$\lim_{k\to\infty} \frac{(1-\Gamma_{1,k}^L(\theta^0))}{1-\delta_k} = \lim_{k\to\infty} \frac{1-e^{-\mu_k^L}}{1-\delta_k} = \frac{d^L}{1-d^L}.$$

Next, $1-\Gamma_{1,k}^L(\theta^1) = 1-e^{-\mu_k^L(1-\frac{D_k^{L,0}}{D_k^L})} = 1-e^{-\mu_k^L(1-\frac{d^L}{D_k^L})} \approx 1-e^{-\mu_k^L K_0(1-\delta_k)} \approx 1-e^{-K(1-\delta_k)^2}$. This implies that

$$\lim_{k\to\infty} \frac{1-\Gamma_{1,k}^L(\theta^1)}{(1-\delta_k)^2} = K \in (0,\infty).$$

An iteration of this argument yields that

$$\lim_{k\to\infty} \frac{1-\Gamma_{1,k}^L(\theta^q)}{(1-\delta_k)^{q+1}} = K^q \in (0,\infty)$$

for all finite q.

We are ready to estimate the posterior upon tieing in formula (53). Ignoring terms that have finite (and non-zero) limits we obtain that

$$\lim_{k\to\infty} \frac{\eta_k^t(\underline{\theta})}{1-\eta_k^t(\underline{\theta})} = \lim_{k\to\infty} (\frac{1}{1-\Gamma_{1,k}^L(\theta^0)} ... \frac{1}{1-\Gamma_{1,k}^L(\widetilde{\theta}_k^{t-2})} \frac{1}{1-\Gamma_{1,k}^L(\widetilde{\theta}_k^{t-1})})^2 S_k^L \Gamma_{1,k}^H(\widetilde{\theta}_k^t) =$$

$$\leq \lim_{k\to\infty} \frac{\frac{1}{1-\delta_k} e^{-1/(1-\delta_k)}}{(1-\delta_k)(1-\delta_k)^2 ... (1-\delta_k)^t} = 0,$$

after taking limits. This implies that for any finite t it holds that the first t bids upon entering converges to as k becomes large. QED

Now, suppose that we take the bid after $t_k = 1/(1-\delta_k)$ losses. Then the relevant probability upon tieing can be estimated and it can be shown

$$\lim_{k\to\infty}\frac{\eta_k^{t_k}(\underline{\theta})}{1-\eta_k^t(\underline{\theta})}\geq\lim_{k\to\infty}\frac{e^{-1/(1-\delta_k)}}{(1-\delta_k)^{1/(1-\delta_k)}}=\infty,$$

which means that for all θ it holds that

$$\lim_{k\to\infty}\eta_k^{1/(1-\delta_k)}(\theta)=1.$$

This means that although upon entering there is a long streak of very low bids, but after (at most) $1/(1-\delta_k)$ periods of losses the bid becomes $v-\overline{V}_k\to v$. Of course, we already know it from the previous Section that the probability of winning in those periods where the bid does not converge to v vanishes in the limit.

This argument could be strengthened to show that after $\sqrt{1/(1-\delta_k)}$ losses the relevant tieing probability converges to 1 already, and thus the bid converges to v. We can show that for any q it holds that

$$\lim_{k\to\infty}\frac{1-\Gamma_{1,k}^L(\widetilde{\theta}_k^q)}{(1-\delta_k)^{q+1}}\leq K^q\in(0,\infty). \tag{55}$$

To see this, note that $1-\Gamma_{1,k}^L(\widetilde{\theta}_k^t)=1-e^{-\mu_k^L(1-\frac{D_k^{L,0}}{D_k^L}-\ldots-\frac{D_k^{L,t-1}}{D_k^L})}$. Moreover, the losing probability in a given period is less than $1-e^{-\mu_k^L}\approx(1-\delta_k)\frac{d^L}{1-d^L}$ for any type. Therefore, for any q it holds that $D_k^{L,q+1}/D_k^{L,q}\leq(1-\delta_k)\frac{d^L}{1-d^L}$. Therefore, (55) follows using basic properties of the geometric sequence.

Therefore, again ignoring constant terms, it follows that

$$\lim_{k\to\infty}\frac{\eta_k^{t_k}(\underline{\theta})}{1-\eta_k^{t_k}(\underline{\theta})}\geq\lim_{k\to\infty}\frac{\frac{1}{1-\delta_k}e^{-1/(1-\delta_k)}}{(1-\delta_k)(1-\delta_k)^2\ldots(1-\delta_k)^t}=\lim_{k\to\infty}\frac{\eta_k^t(\underline{\theta})}{1-\eta_k^t(\underline{\theta})}\geq\lim_{k\to\infty}\frac{e^{-1/(1-\delta_k)}}{(1-\delta_k)^{\frac{t(t+1)}{2}+1}}=\infty,$$

if $t_k\geq\sqrt{1/(1-\delta_k)}$ holds. As we have argued above it holds for any $\theta\in[\underline{\theta},\overline{\theta}]$ that $\lim_{k\to\infty}\eta_k^{t_k}=\lim_{k\to\infty}\widetilde{\theta}_k^{t_k}=1$ and thus

$$\lim_{k\to\infty}\beta_k(\widetilde{\theta}_k^{t_k})=v-\lim_{k\to\infty}\delta_kV_k^H(\widetilde{\theta}_k^{t_k})\geq v-\lim_{k\to\infty}\delta_kV_k^H=v-\overline{V}=v,$$

where the last equality was established in the proof of Lemma 8. QED

7.6 Appendix 6: Numerical calculations about information revealed

7.6.1 A three-period example

Here we provide an example with participation costs where the buyers bid three times, and then stay out if they lost three times. We calibrate this example such that the seller has no incentive to reveal information. Formally, we consider the case where each buyer has to pay a cost c each period he visits a seller. We also assume that each entering buyer has the same belief $\theta = \frac{d^H}{d^H + d^L}$, which corresponds to the case where each buyer has a prior belief 0.5, and upon entering he updates this belief upwards to reflect that it is more likely for any buyer to enter if the state is high. Finally, we allow that the sellers all post the same non-negative reserve price r, that is fixed at a given value.

With this specification one can establish that the entering buyers randomize, and then upon losing buyers use their initial bid to follow a deterministic strategy from then on. To formalize such an equilibrium let us endow the entering buyers with a payoff irrelevant signal y uniformly distributed on $[0, 1]$. The equilibrium we characterize is the monotone equilibrium where bids are strictly increasing in y, and also buyers with higher beliefs bid higher as we argue below. Upon losing the type of the buyer is updated to $y + 1$ in each period. The belief of a buyer with type y is $\theta(y)$. By construction $\theta(y) = \frac{d^H}{d^H + d^L}$ for all $y \in [0, 1)$. Moreover, as we argued in the main text, updating is always upwards, and monotone in the bid placed, and therefore $\theta(y)$ is weakly increasing in y, and is strictly increasing when $y \geq 1$.

We calibrate the model such that each buyer makes exactly three bids before dropping out. (The next Section considers the case where depending on the initial bid, some buyers make two, while others make three bids before dropping out.) First, we calculate the steady state number of agents. The steady state condition for the number of sellers in state $w \in \{L, H\}$ is as before (because sellers do not drop out):

$$S^w = 1 + \delta S^w e^{-D^w/S^w}. \tag{56}$$

The steady state number of buyers, D^w is equal to the sum of the first, second and third generation buyers. The mass of first generation buyers is D^w, while the mass of second generation buyers is δ times D^w minus the mass of first generation buyers who trade,and so on for the third generation buyers. The number of first generation buyers who trade is equal to the number of sellers trading with first generation buyers T_1, which is equal to the mass of sellers trading at all, $T = S^w(1 - e^{-D^w/S^w})$ minus the sum of the mass of sellers who trade with second and third generation buyers $T_2 + T_3$. Note, that a seller trades with a second or third generation buyer if there is a second or third generation buyer present at all, because such buyers outbid all first generation buyers because of the monotonicity of the bid functions. The sum of the mass of second and third generation buyers is equal to $D^w - D^w = \delta(D^w - T_1) + \delta(\delta(D^w - T_1) - T_2)$, and thus the probability that a seller does *not* have a second or third generation buyer present is $e^{-\frac{D^w - d^w}{S^w}}$. Therefore, the mass of trades with second or third generation buyers is

$$T_2 + T_3 = S^w(1 - e^{-\frac{D^w - d^w}{S^w}}),$$

and thus the number of trades with first generation buyers is equal to

$$T_1 = T - (T_2 + T_3) = S^w(e^{-\frac{D^w - d^w}{S^w}} - e^{-D^w/S^w}) = S^w e^{-D^w/S^w}(e^{\frac{d^w}{S^w}} - 1) =$$

$$T_1 = \frac{S^w - 1}{\delta}(e^{\frac{d^w}{S^w}} - 1),$$

just as in the two-period example. A similar reasoning yields that

$$T_2 = S^w(e^{-\frac{D^w - d^w - \delta(d^w - T_1)}{S^w}} - e^{-\frac{D^w - d^w}{S^w}}),$$

and

$$T_3 = S^w(1 - e^{-\frac{D^w - d^w - \delta(d^w - T_1)}{S^w}}).$$

Now, we are ready to study the steady state condition for the mass of buyers

$$D^w = d^w + \delta(d^w - T_1) + \delta(\delta(d^w - T_1) - T_2).$$

Using, the above formulas this last condition becomes

$$D^w = d^w(1 + \delta + \delta^2) - (S^w - 1)(e^{\frac{d^w}{S^w}} - 1)(1 + \delta) - (S^w - 1)e^{\frac{d^w}{S^w}}(e^{\frac{\delta d^w - (S^w - 1)(e^{\frac{d^w}{S^w}} - 1)}{S^w}} - 1). \quad (57)$$

The two numbered formulas above (uniquely) determine the steady state mass of buyers and sellers, D^w and S^w in state $w \in \{L, H\}$.

Let Y^w denote the distribution function of a buyers type y in state w and let $Y_1^w = e^{-\frac{D^w}{S^w}(1 - Y^w)}$ denote the probability that the highest type is less than y. Let y_1^w be the derivative of Y_1^w. It holds that $Y^w(0) = 0$ and $Y^w(3) = 1$ by construction. Also, for all $y \in (0, 1]$ it holds that $Y^w(y) = \frac{d^w y}{D^w}$. For all $y \in (1, 2]$ it holds that

$$Y^w(y) = \frac{d^w}{D^w} + \int_1^y \delta \frac{d^w}{D^w}(1 - Y_1^w(z - 1))dz = \frac{d^w}{D^w} + \int_1^y \delta \frac{d^w}{D^w}(1 - e^{-\frac{D^w}{S^w}(1 - Y^w(z-1))})dz.$$

Finally, for all $y \in (2, 3]$ it holds that

$$Y^w(y) = Y^w(2) + \int_2^y \delta Y^w(z - 1)(1 - Y_1^w(z - 1))dz,$$

which can be calculated using the above formulas iteratively. Let $\theta^+(y) = \frac{\theta(y)(1-Y_1^H(y))}{\theta(y)(1-Y_1^H(y))+(1-\theta(y))(1-Y_1^L(y))}$ denote the posterior belief of type y upon losing, and $\theta^0(y) = \frac{\theta(y)y_1^H(y)}{\theta(y)y_1^H(y)+(1-\theta(y))y_1^L(y)}$ the posterior of type y upon tieing at the top.

Now, we turn to describing the bidding behavior of the buyers. In our calibration all third generation buyers make their last bid before dropping out. Therefore, for all such buyers, i.e. all types $y \in [2,3]$ bid $\beta(y) = v$. To describe the bidding of second generation buyers (types such that $y < 1$), one can extend the analysis of the two-period case, since the bid of the second generation buyers here are very similar to the bid of the first period buyers in the two-period case. Therefore, the bid for all second generation buyers ($y \in (1,2)$) is described by

$$\beta(y) = \beta_2(y) = v - \delta W(\theta^+(y), \theta^0(y)) = v - \delta W(\theta(y+1), \theta^0(y)). \tag{58}$$

Let e_w denote the expected value of the second highest bid in state w. For all $y \in [1,2)$ it holds that type $y+1$ bids v and thus his utility can be calculated by assuming that he always wins and pays the second highest bid[19], so his expected utility upon losing in state w is $v - e_w$. Therefore, for all $y \in [1,2)$ it holds that $W(\theta(y+1), \theta^0(y)) = \theta^0(y)(v - e_H) + (1-\theta^0(y))(v - e_L) - c$, so for such types

$$\beta(y) = \beta_2(y) = v(1-\delta) + c\delta + \delta[\theta^0(y)e_H + (1-\theta^0(y))e_L]. \tag{59}$$

The bid of the first generation buyers ($y < 1$) in the three-period case is more complicated. As it is true for any type y:

$$\beta(y) = \beta_1(y) = v - \delta W(\theta^+(y), \theta^0(y)).$$

The relevant continuation value can be written as

$$W(\theta^0(y), y+1) = \theta^0(y)(Y_1^H(0)(v-r) + \int_0^{y+1} y_1^H(z)(v-\beta(z))dz + \delta(1-Y_1^H(y+1))(v-e_H))+$$

$$+(1-\theta^0(y))(Y_1^L(0)(v-r) + \int_0^{y+1} y_1^L(z)(v-\beta(z))dz + \delta(1-Y_1^L(y+1))(v-e_L)) =$$

$$= \theta^0(y)(Y_1^H(0)(v-r) + \int_0^1 y_1^H(z)(v-\beta_1(z))dz+$$

$$+\int_1^{y+1} y_1^H(z)(v-\beta_2(z))dz + \delta(1-Y_1^H(y+1))(v-e_H))+$$

$$+(1-\theta^0(y))(Y_1^L(0)(v-r) + \int_0^1 y_1^L(z)(v-\beta_1(z))dz+$$

[19] In case multiple buyers bid v they all make zero profits, so it does not matter which buyer actually wins to object, so we can assume w.l.o.g. that the buyer in question wins the object and pays the second highest bid, which is v in this case.

$$+\int_1^{y+1} y_1^L(z)(v-\beta_2(z))dz+\delta(1-Y_1^L(y+1))(v-e_L))-c.$$

The key for understanding this formula is that in case of losing in the next period with bid $\beta(y+1)$, then in the third period he will win for sure with bid v, and in the second period the bid is $\beta_2(y)$. Using formula (59), and defining $A_H=\int_0^1 y_1^H(z)\beta_1(z)dz$, $A_L=\int_0^1 y_1^L(z)\beta_1(z)dz$ yields that

$$W(\theta^0(y),y+1)=\theta^0(y)(Y_1^H(0)(v-r)+\int_0^{y+1} y_1^H(z)vdz-A_H-$$

$$-\int_1^{y+1} y_1^H(z)(v(1-\delta)+c\delta+\delta[\nu(z)e_H+(1-\nu(z))e_L])dz+\delta(1-Y_1^H(y+1))(v-e_H))+$$

$$+(1-\theta^0(y))(Y_1^L(0)(v-r)+\int_0^{y+1} y_1^L(z)vdz-A_L-$$

$$-\int_1^{y+1} y_1^L(z)(v(1-\delta)+c\delta+\delta[\nu(z)e_H+(1-\nu(z))e_L])dz+\delta(1-Y_1^L(y+1))(v-e_L))-c.$$

There are four unknowns (A_H,A_L,e_H,e_L) in this equation thus one can rewrite it as $W(\theta^0(y),y+1)=\alpha(A_H,A_L,e_H,e_L)$ and thus

$$\beta_1(y)=v-\delta\alpha(A_H,A_L,e_H,e_L).$$

By definition

$$A_H=\int_0^1(v-\delta\alpha(A_H,A_L,e_H,e_L))y_1^H(y)dy \tag{60}$$

and

$$A_L=\int_0^1(v-\delta\alpha(A_H,A_L,e_H,e_L))y_1^L(y)dy. \tag{61}$$

Moreover, the expected payment when one bids v in state H is

$$e_H=Y_1^H(0)r+\int_0^1 y_1^H(z)\beta_1(z)dz+\int_1^2 y_1^H(z)\beta_2(z)dz+(1-Y_1^H(y+1))v=$$

$$=Y_1^H(0)r+A_H+\int_1^2 y_1^H(z)(v(1-\delta)+c\delta+\delta[\nu(z)e_H+(1-\nu(z))e_L])dz+(1-Y_1^H(y+1))v,$$

which can then also be written as a function of the same for unknowns:

$$e_H=\beta_H(A_H,A_L,e_H,e_L). \tag{62}$$

By a similar argument

$$e_L=\beta_L(A_H,A_L,e_H,e_L). \tag{63}$$

Then formulas (60), (61), (62) and (63) form a system of four equations in four unknowns, and one can show that a unique solution exists, which can be calculated numerically.

Upon substituting those values A_H, A_L, e_H, e_L, one obtains an explicit solution for the bid function by describing functions β_1 and β_2 by using the formulas above. Then using the density function for the second highest type ($\tau_w = -y_1^w \log Y_1^w$), one can calculate the (ex-ante expected) revenue when at least two buyers visit (otherwise it is the same regardless of the announcement policy) as follows:

$$Rn = 0.5(\int_0^1 \tau_H(y)\beta_1(y)dy + \int_1^2 \tau_H(y)\beta_2(y)dy + (1 - Y_1^H(2))v) +$$

$$+0.5(\int_0^1 \tau_L(y)\beta_1(y)dy + \int_1^2 \tau_L(y)\beta_2(y)dy + (1 - Y_1^L(2))v).$$

The revenue if the state is revealed and at least two buyers are present is v in state H, because if the state is known to be H then the buyers have a zero continuation value. If the state is L, then each buyer can calculate his continuation value by noting that he wins only if no other buyer visited the seller and pays r in this case. Therefore, the continuation value is

$$((v-r)e^{-\frac{D^L}{S^L}} - c) + \delta(1 - e^{-\frac{D^L}{S^L}})((v-r)e^{-\frac{D^L}{S^L}} - c) + \delta^2(1 - e^{-\frac{D^L}{S^L}})^2((v-r)e^{-\frac{D^L}{S^L}} - c) + ...$$

$$= \frac{(v-r)e^{-\frac{D^L}{S^L}} - c}{1 - \delta(1 - e^{-\frac{D^L}{S^L}})},$$

and thus the bid of each buyer is $\beta_L = v - \delta\frac{(v-r)e^{-\frac{D^L}{S^L}} - c}{1-\delta(1-e^{-\frac{D^L}{S^L}})}$. Therefore, the ex-ante expected revenue with information revelation (when at least two buyers visit) can be written as

$$Ry = 0.5(1 - Y_1^H(0)(1 + \frac{D^H}{S^H})) + 0.5(1 - Y_1^L(0)(1 + \frac{D^L}{S^L}))b_L.$$

For the incentive conditions (for participation), we need to check that any two-time loser wants to continue, but any three-time loser wants to stop. Specifically, we it is necessary and sufficient to check two conditions. First, we need to check that a first generation buyer who bid the lowest when he enters, would like to stay out if he lost three times. Letting B denote his belief after three losses[20] his incentive condition becomes

$$v - c - (Be_H + (1 - B)e_L) \leq 0.$$

[20] Let $B0$ denote his belief after one loss, and $B1$ denote his belief after two losses. Then $B0 = \frac{\theta(1-\Gamma_H(0))}{\theta(1-\Gamma_H(0))+(1-\theta)(1-\Gamma_L(0))}$ and $B1 = \frac{B0(1-\Gamma_H(1))}{B0(1-\Gamma_H(1))+(1-B0)(1-\Gamma_L(1))}$ and $B = \frac{B1(1-\Gamma_H(2))}{B1(1-\Gamma_H(2))+(1-B1)(1-\Gamma_L(2))}$.

Second, we need to check that a first generation buyer who bid the highest when entering would like to continue after two losses. Letting A denote his belief after two losses[21] his incentive condition becomes

$$v - c - (Ae_H + (1-A)e_L) \geq 0.$$

Upon calculating the above revenues numerically (and checking the incentive conditions) for values $v = 1, \delta = 0.6, c = 0.122, r = 0.45, d^L = 0.8, d^H = 1.3$ we obtain that $rn > ry$, i.e. revealing the state decreases the revenue of the seller.

7.6.2 A three-period example with cutpoints

We now modify the above example, so that we have that there is a type $\overline{y} \in (2,3)$ who is indifferent between staying in or not bidding any more. Therefore, all bidders with initial types $y \in [0, \overline{y}-2)$ bid three times as before, but types $[\overline{y}-2, 1]$ bid only twice before dropping out. This modification allows one to do richer comparative statics, since as the parameters (δ, dl, dh, c, r) change, all the relevant equilibrium variables change.[22]

The analysis is very similar to that of the previous Section, so we present it in less details. When characterizing the steady state masses equation (56) is still valid, since the sellers do not exit. However, equation (57) has to be modified to

$$D^w = d^w(1+\delta) - (S^w - 1)(e^{\frac{d^w}{S^w}} - 1)(1+\delta) + \tag{64}$$

$$+\delta^2 d^w \int_2^{\overline{y}} (1 - e^{-\frac{D^w}{S^w}(1 - \frac{d^w(z-2)}{D^w})})(1 - e^{-\frac{D^w}{S^w}(1 - \frac{d^w + \delta d^w \int_1^{z-1}(1 - e^{-\frac{D^w}{S^w}(1 - \frac{d^w(t-1)}{D^w})} dt}{D^w})})dz.$$

(One can show that this equation can be easily reduced to (57) after substituting from (56) in the original case where $\overline{y} = 3$ held. However, the reduction when $\overline{y} < 3$ is less helpful, and so is not considered.) Then equations (56) and (64) determine the two unknowns (D^w, S^w) in a unique manner. The description of the distribution of the types, $Y^w(y)$ is the same as before, but obviously $Y^w(\overline{y}) = 1$ holds now.

The analysis of the strategies are very similar to the one above except that now types larger than $\overline{y}-1$ bid v, types belonging to $[\overline{y}-2, \overline{y}-1)$ use function β_2 and types less than $\overline{y}-2$ use bid function β_1. Also, the definitions of some variables (e.g. A_L, A_H) change and whenever an integral was taken between 0 and 1 above, it is now taken only between 0 and $\overline{y}-2$. The modifications following from this are straightforward and thus are not dwelled upon. When calculating the revenue similar changes are made, and also instead of integrating on interval $[1,2]$, the interval

[21] Let $A0$ denote his belief after one loss. Then $A0 = \frac{\theta(1-\Gamma_H(1))}{\theta(1-\Gamma_H(1))+(1-\theta)(1-\Gamma_L(1))}$ and $A = \frac{A0(1-\Gamma_H(2))}{A0(1-\Gamma_H(2))+(1-A0)(1-\Gamma_L(2))}$.

[22] In the case where all types stay for three periods, a slight change of the parameters do not change this feature of the equilibrium, but they do change the exact bid functions. In the present case also the cutpoint $\overline{y}$ is sensitive to small changes in the parameters.

becomes $[\overline{y}-2, \overline{y}-1]$. Again, the changes are fairly straightforward. A more substantial change is that the type of $\overline{y}$ has to be indifferent between bidding or not. Denoting the belief of that type by A it follows that

$$v - c - (Ae_H + (1-A)e_L) = 0.$$

This is the single incentive constraint that needs to be checked, indeed this is the equation that pins down $\overline{y}$ given the parameters.

We numerically implement these calculations with Mathematica and obtain comparative statics results for the range of parameters checked:

1. Not surprisingly, $\overline{y}$ is decreasing in c: as the cost of bidding increases bidders get discouraged easier.

2. The benefit from concealing information tends to increase with c too.

3. $\overline{y}$ is decreasing in δ: as the search frictions decrease (δ increases) bidders get discouraged easier. This is a general equilibrium result: as the others are able to search better there is more crowd on the market and you yourself drop out earlier.

4. $\overline{y}$ is decreasing in r: as r increases, there is less incentive to bid and thus bidders get discouraged easier.

5. $\overline{y}$ is increasing in d^L: as d^L goes up losing is a weaker signal, and thus staying longer may be profitable

6. $\overline{y}$ is decreasing in d^H: as d^H increases, there is more congestion and thus bidders get discouraged easier