

Equilibrium Unemployment in a Generalized Search Model

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Abstract

We present a generalization of the standard Diamond-Mortensen-Pissarides undirected-search model of unemployment in which the hiring process is time-consuming as well as costly. We follow Stole and Zwiebel (1996a,b) and assume that wages are determined by continuous bargaining between the firm and its employees. This generates a non-trivial dispersion of firm sizes; when firms' production technologies exhibit decreasing returns to labor, it also generates wage dispersion, even though all firms and all workers are ex ante identical. We characterize the steady-state equilibrium of the model; some important special cases are characterized in closed form. We characterize the out-of-steady state dynamics of employment and wages of the economy in response to productivity shocks. A feature of the model is the ability of the economy to respond to shocks on both an intensive margin (a change in the intensity of vacancy posting of incumbent firms) as well as an extensive margin (a change in the number of active firms); we show that both margins, as well whether there are decreasing returns to labor at the firm level, are important for the qualitative behavior of the unemployment rate and of the distribution of employment and wages across firms.

1 Introduction

In this paper we study a model in which the process by which firms hire workers in a frictional labor market is both costly and time-consuming. Since firms wish to hire many workers in our model, the time-consuming nature of hiring endogenously generates dispersion in the distribution of firm sizes, as new firms are born and cannot grow immediately to their desired number of workers. Since we allow for firms' production functions to exhibit decreasing returns to labor, this heterogeneity in firm sizes corresponds also to dispersion in

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the marginal product of labor across firms. Since we model wage determination by bargaining – more precisely, since we use the generalization of Nash bargaining first investigated by Stole and Zwiebel (1996a,b) – this then accordingly generates dispersion in wages, even though, *ex ante*, all firms and all workers were identical.

The first part of our paper is devoted to solving for a steady-state economy. We study two variants of the model, differing only in the assumption by which we ensure that hiring is time-consuming for firms. In the first case, studied in Section 3, we assume that firms can post only a single vacancy at any one time. This has the disadvantage that the most productive firms (here, the smallest and youngest, since they remain small and have high marginal products of additional workers) cannot search more intensively for workers than can other firms; it has the advantage that we can characterize the solution of the model more nearly in closed form. Our second assumption concerning the hiring process is that posting additional vacancies, and therefore, in expectation, hiring workers faster, increases firms' vacancy posting costs, and more than proportionally. In Section 4 we study this, arguably more realistic case, the differential equations characterizing the firm's value function cannot be solved in closed form, but it is not difficult to solve them numerically.

We then move to studying the dynamics of the economy in response to productivity shocks. In Section 6, we solve the simplest case available to us, in which the production function of firms exhibits constant returns to scale in labor. Although this shuts down some of the most interesting features of the model, since there is now no dispersion in workers' marginal products and therefore no wage dispersion, this case has the advantage that the dynamics of the economy following the shock we consider can be solved for in closed form. Finally, in Section 7, we return to the case of decreasing returns to labor, and show that the dynamics of adjustment following a productivity shock are more complex in this case, with adjustment taking place over time and along multiple margins.

That firms have to pay some cost to post vacancies is of course a familiar feature of search models in the Mortensen-Pissarides tradition – in fact, it is necessary for the existence of a search friction. Models in which hiring of workers by large firms is time-consuming are rarer. In the benchmark Mortensen-Pissarides model, in the usual formulation in which firms wish to employ precisely one worker, a firm either has zero workers or one worker, and hiring is usually modeled as time-consuming (in the sense that a firm experiences a positive but finite Poisson rate at which it fills its unique vacancy); however, nothing about the model beyond the number of vacancies themselves would be altered if vacancy filling was immediate and the firm paid an entry cost equal to the expected present discounted value of the vacancy posting costs it incurs in the benchmark model. In this paper, by contrast, firms wish to hire more than a single worker, and so the fact that hiring is time-consuming generates frictional dispersion in firm sizes.

Two close cousins to the model we study occur in the work of Cahuc, Marque, and Wasmer (2008) and of Elsby and Michaels (2010). Both of these authors, as we do, use the bargain-

ing model of Stole and Zwiebel (1996a,b) as a basic assumption on how large firms might bargain wages with their workers in frictional labor markets. However, in both these papers, the possibility of firms being away from their target size is assumed away, directly so in the case of Cahuc, Marque, and Wasmer (2008), while Elsby and Michaels (2010) assume that the cost of posting additional vacancies exhibits constant returns to scale, under which assumption the optimal vacancy posting policy of a firm takes a ‘bang-bang’ form, in the sense that a new entrant firm posts an enormous number of vacancies for a vanishingly short period of time, growing immediately to its desired size, and then remaining there until the arrival of an idiosyncratic or aggregate shock. Hawkins (2010) also makes this assumption, but uses an alternate assumption on how bargaining between firms and workers occurs. The goal of this paper is to investigate explicitly the dispersion that arises from not allowing this degenerate firm growth pattern to occur.

2 Model

There is a unit measure of risk-neutral workers in the economy and a large measure of risk-neutral firms. Time is continuous; workers and firms discount the future at rate $r \geq 0$. Firms are either inactive or active. At any moment, any inactive firm can elect to become active by paying an entry cost of k units (all production and costs are measured in units of the single good produced in this economy). Active firms have the ability to operate a production technology which uses labor as the only input; the flow output of the final good produced by an active firm together with n workers is denoted $y(n)$. We assume that $y(n)$ is strictly increasing and (weakly) concave in n , and we normalize $y(0) = 0$.

The labor market is frictional; in order to hire workers, firms must post vacancies. We assume that to post v vacancies, firms pay a flow vacancy posting cost of $c(v)$. We assume that v must be chosen from some feasible set $S \subseteq [0, \infty)$. We consider two possible cases for S and $c(v)$. One possible assumption is that $S = [0, 1]$ and $c(v) = \gamma v$ where $\gamma > 0$ is a constant, and second, that $S = [0, \infty)$, with $c(v)$ increasing and strictly convex in v ; for this case a leading example is that $c(v) = \gamma v^{1+p}/(1+p)$ is a power function.

Matching between firms and workers is frictional. There is an aggregate matching function $M(u, \bar{v})$ that determines the flow rate of new meetings between firms and workers that are generated, as a function of the measure of unemployed workers, u and the total measure of vacancies posted by active firms, $\bar{v}$. Each unemployed worker is equally likely to meet some firm, while each firm meets a worker with rate proportional to the number of vacancies it posts. Which unemployed worker meets which vacancy is randomly determined, and does not depend on any other characteristics of worker or firm except for the number of vacancies posted by the firm. We assume that M exhibits constant returns to scale in $(u, \bar{v})$, and decreasing returns to scale in u or in $\bar{v}$ separately. Denote by θ the vacancy-to-unemployment ratio $\bar{v}/u$; then the Poisson rate at which a firm that posts v vacancies meets a firm is $vq(\theta)$,

where $q(\theta) \equiv M(u, \bar{v})/\bar{v}$. The rate with which an unemployed worker meets some firm is $\theta q(\theta) = M(u, \bar{v})/u$.

Wages paid by firms to workers are determined following Stole and Zwiebel (1996a,b) by assuming that firms and workers bargain over the marginal surplus generated by their employment relationship. To formulate this, denote the Hamilton-Jacobi-Bellman value of a firm with n employees at date t by $J(n, t)$, and the value of a worker employed at such a firm by $V(n, t)$. Denote the value of an unemployed worker by $V^u(t)$. Then we assume that wages are determined in such a way that

$$\phi J_n(n, t) = (1 - \phi) [V(n, t) - V^u(t)]. \quad (1)$$

Here the subscript $J_n(n, t)$ denotes partial differentiation with respect to the first argument. Note that by symmetry, a firm will pay the same wage to all its workers; denote by $w(n, t)$ the wage paid by a firm with n workers at time t .

Employment relationships are subject to two types of shocks. At Poisson rate $\delta > 0$, an active firm is destroyed; in this case, all its workers are returned to unemployment, and the firm is removed from the economy with zero scrapping value. At Poisson rate $s > 0$, each worker employed by the firm is separated from the firm; in this case, the firm continues in existence with all its other incumbent workers.

Unemployed workers generate unemployment income $b > 0$.

3 The case of a single vacancy: steady-state

In this section we solve for the case in which $S = [0, 1]$ (that is, firms may post up to a maximum of a single vacancy), and the cost of doing so is linear in the number of vacancies posted (that is, $c(v) = \gamma v$). We consider in this section only steady-state equilibria, so that all time arguments can be dropped.

3.1 Equilibrium Characterization

Assume for now that the value function of a firm is C^2 in its employment level n . In this case, the HJB equation for a firm may be written as¹

$$(r + \delta)J(n) = y(n) - nw(n) + -snJ'(n) + \max_{0 \leq v \leq 1} [-\gamma v + q(\theta)vJ'(n)]. \quad (2)$$

Note that the maximand $-\gamma v + vJ'(n)$ is linear in v , so that the optimal vacancy posting policy for a firm is to set $v = 1$ if $J'(n) > \gamma$, $v = 0$ if $J'(n) < \gamma$, and indeterminate if $J'(n) = \gamma$. We will search for threshold equilibria, in which there is some n^* , possibly

¹In an earlier version of this paper, Acemoglu and Hawkins (2007), we solved a version of the model with workers of positive size ε and derived the form of the HJB equation more formally by taking limits as $\varepsilon \rightarrow 0$. The reader is referred to that version of the paper for more detail.

infinite, such that the optimal vacancy posting policy is $v(n) = 1$ for $n < n^*$ and $v(n) = 0$ for $n > n^*$. In this case the HJB equation can be written

$$(r + \delta)J(n) = \begin{cases} y(n) - nw(n) + -\gamma + (q(\theta) - sn)J'(n) & n < n^* \\ y(n) - nw(n) - snJ'(n) & n \geq n^*. \end{cases} \quad (3)$$

The HJB equation for workers may be written as

$$rV(n) = w(n) + (s + \delta)[V(n) - V^u] + (q(\theta)v(n) - sn)V'(n), \quad (4)$$

where $v(n)$ is again the vacancy-posting policy of a firm with n workers. In a threshold equilibrium, this can be expressed as

$$rV(n) = \begin{cases} w(n) + (s + \delta)[V(n) - V^u] + (q(\theta) - sn)V'(n) & n < n^* \\ w(n) + (s + \delta)[V(n) - V^u] & n = n^* \\ w(n) + (s + \delta)[V(n) - V^u] - snV'(n) & n > n^* \end{cases} \quad (5)$$

Note that in a steady state equilibrium, there will not be any firms with employment level greater than n^* , so that we will treat the case $n \leq n^*$ in greater detail below.

If n^* is finite, the fact that the firm's hiring strategy changes discretely from $h(n) = 1$ for $n < n^*$ to $h(n) = 0$ for $n > n^*$ requires the following boundary condition to be satisfied:

$$-\gamma + q(\theta)J'(n^*) = 0. \quad (6)$$

This condition might be termed a smooth pasting condition, but it will be true whether or not the cutoff n^* that the firm chooses is in fact optimal.² It arises from the fact that when its employment reaches n^* , the firm stops paying the flow cost γ of posting a vacancy. Since this boundary condition is true for any cutoff n^* , it is not sufficient to characterize the solution to the differential equation (2). An additional boundary condition comes from a standard smooth pasting argument. Intuitively, notice that the firm is solving an optimal stopping problem—at what point to stop posting additional vacancies. Since the cost of posting a vacancy is a constant flow cost, it is intuitive that a ‘super-contact’ condition on the second derivative of $J(\cdot)$ will be required. Intuitively, the super-contact condition requires that n^* is an optimal stopping point for the firm, in the sense that a small change in n^* should have no impact on the value of the firm. This is equivalent to the second-order condition

$$J''(n^*) = 0. \quad (7)$$

²See Dixit (1993, p. 42) for a discussion of smooth pasting and super-contact conditions in related problems.

In addition, we have a free entry condition, which takes the form

$$J(0) \leq k \text{ and } \theta \geq 0, \text{ with complementary slackness,} \quad (8)$$

which requires that in order for there to be positive activity in equilibrium, the value of k needs to be sufficiently low for firms to find it attractive to gain access to the production technology.

The Stole-Zwiebel bargaining equation (1) can be specialized to this stationary case as

$$\phi J'(n) = (1 - \phi) [V(n) - V^u]. \quad (9)$$

For future reference, differentiate this equation to note that

$$\phi J''(n) = (1 - \phi) V'(n). \quad (10)$$

The equation for rV^u that arises from the Bellman equation for an unemployed worker closes the model. To write this equation, it is necessary to introduce additional notation for the steady-state distribution of firms as a function of the number of workers already employed by the firm; this differs slightly from the case with discrete worker size. As before, in this section we consider only steady states, so we do not need to index the distribution of firms by size according to time.

Denote the firm-size distribution by $G(n)$. First, suppose that there exists a steady-state for $G(\cdot)$ such that the density function $g(n)$ is continuous on $(0, n^*)$, together possibly with an atom of mass G^* at n^* . Such a distribution $G(\cdot)$ is a steady-state distribution if $g(\cdot)$ satisfies the steady state accounting equation equating, for each $n \in (0, n^*)$ and each $\varepsilon > 0$ sufficiently small, the flow of firms into and out of the interval $(n - \varepsilon/2, n + \varepsilon/2)$:

$$\left(\frac{q(\theta)}{\varepsilon} + \frac{sn}{\varepsilon} + \delta \right) g(n) = \frac{q(\theta)}{\varepsilon} g(n - \varepsilon) + \frac{s(n + \varepsilon)}{\varepsilon} g(n + \varepsilon) + O(\varepsilon). \quad (11)$$

Taking limits as $\varepsilon \rightarrow 0$, it follows that any differentiable solution to this equation must satisfy the following differential equation for $g(n)$:

$$\frac{g'(n)}{g(n)} = \frac{s - \delta}{q(\theta) - sn}. \quad (12)$$

Integrating (12) gives that the general solution (for $q(\theta) - sn \geq 0$) is given by

$$g(n) = A(q(\theta) - sn)^{\frac{\delta - s}{s}}. \quad (13)$$

There are two cases to consider: the case where $n^* \geq q(\theta)/s$ and the one where $n^* < q(\theta)/s$. The importance of this distinction is that when $n = q(\theta)/s$, then ignoring firm death, flows due to hiring and worker separation balance out; for n larger, there is a net loss of firms

of size n even disregarding firm death. Thus the firm size distribution is supported on $[0, \min\{n^*, q(\theta)/s\}]$. In the case of interest, in which $n^* < q(\theta)/s$, then even at $n = n^*$, there is a positive “flow” of firms at $n - \varepsilon$ for any $\varepsilon > 0$; this leads to an atom at n^* . To calculate the size of this atom, again equate the flow of firms into and out of n^* , to obtain:

$$(q(\theta) - sn^*)g(n^*) = \delta G^*. \quad (14)$$

Notice that (14) assumes the outflow of firms from the state n^* is given only by firm death. This is a consequence of the feature that $v(n^*) = sn^*/q(\theta)$ i.e., firms at n^* still post vacancies and are hiring continuously, but only at a rate that allows them to counteract their loss of workers to the separation shock. This hiring is what maintains the atom at G^* . A more intuitive explanation for this is that at n^* , the firm is indifferent between posting a vacancy or not. If it posted a vacancy for sure when at n^* , so that $v(n^*) = 1$, it would quickly hire workers faster than it loses them to separation, since $q(\theta) > sn^*$ by assumption; the firm would then move to having $n > n^*$ workers. However, as soon as this happens, it would then strictly prefer not to post a vacancy, and would therefore lose workers again until its workforce size falls to n^* . Conversely, if the firm does not post a vacancy at n^* , so that $v(n^*) = 0$, then it would quickly drop to $n < n^*$ and begin hiring again. The only possibility is that the firm must use a mixed strategy and post a vacancy with a flow probability of $v(n^*) = sn^*/q(\theta)$; this ensures that the rate at which the firm attracts new workers, $v(n^*)q(\theta)$, equals the rate at which it loses workers, sn^* .

Now combining (13) and (14), we obtain:

$$A(q - sn^*)^{\frac{\delta}{s}} = \delta G^*.$$

To solve for A , note that since $G(\cdot)$ is a probability distribution, we must have

$$G^* + \int_0^{n^*} g(n) dn = 1.$$

This implies $A = \delta q(\theta)^{-\delta/s}$, so that the steady-state firm size distribution is given by

$$g(n) = \frac{\delta}{q(\theta)} \left(1 - \frac{sn}{q(\theta)}\right)^{\frac{\delta-s}{s}} = \frac{\delta}{q(\theta)^{\frac{\delta}{s}}} (q(\theta) - sn)^{\frac{\delta-s}{s}} \quad \text{if } n \in [0, \min(n^*, \frac{q(\theta)}{s})] \quad (15)$$

with an atom of mass

$$G^* \equiv \left(1 - \frac{sn^*}{q(\theta)}\right)^{\frac{\delta}{s}} = \left(\frac{q(\theta) - sn^*}{q(\theta)}\right)^{\frac{\delta}{s}} \quad (16)$$

at n^* in the case where $n^* < \frac{q(\theta)}{s}$.

Figure 1 shows this distribution for a case in which $sn^* < q(\theta)$ and $\delta < s$, which is the

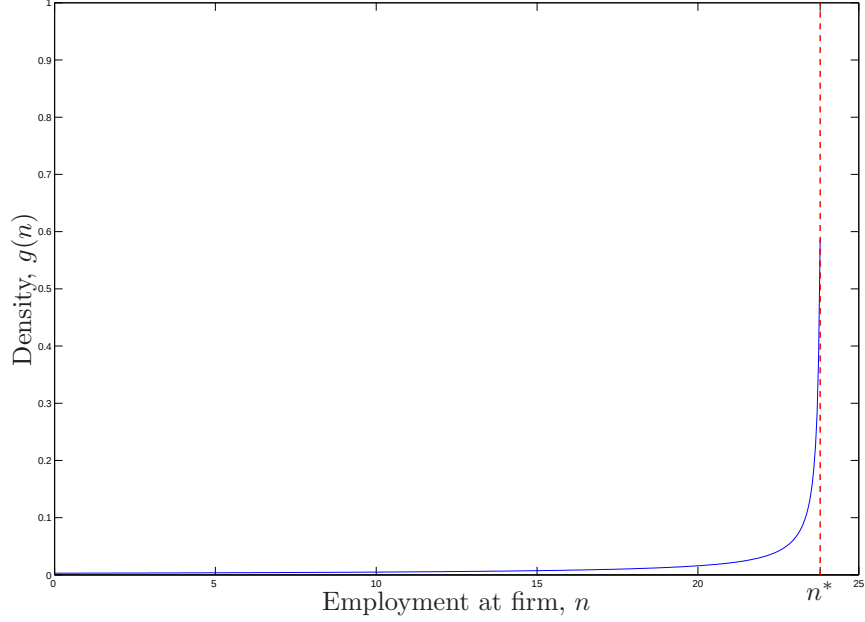


Figure 1: Steady state density function $g(n)$

empirically relevant case. The size G^* of the atom at n^* is not shown, although its location n^* is indicated with a vertical line.

Finally, observe that since the stochastic process for a firm's size satisfies an ergodicity condition, the steady state distribution $G(\cdot)$ is unique, so there was no loss of generality in solving only for a distribution in which $g(\cdot)$ is continuously differentiable on $(0, n^*)$.³

We can now write the Bellman equation for an unemployed worker. In the case where $q(\theta) > sn^*$ (i.e., when there is an atom of positive mass of firms at n^*), recall that we have $h(n^*) = \frac{sn^*}{q(\theta)}$; thus we have to incorporate the probability of an unemployed worker being hired by a firm of size n^* . Consequently, the Bellman equation for an unemployed worker is simply

$$rV^u = b + \theta q(\theta) \left[-V^u + \frac{\int_0^{n^*} V(n)g(n) dn + \frac{sn^*}{q(\theta)} V(n^*)G^*}{1 - \left(1 - \frac{sn^*}{q(\theta)}\right) G^*} \right] \quad (17)$$

The equations given above characterize almost completely any equilibrium of the model in the class we are considering (those that satisfy the differentiability assumptions and in which firms use symmetric cutoff hiring strategies). The only additional requirement is to check that firms and workers are behaving optimally.

This completes the characterization of an equilibrium, which we record as the following

³For the ergodicity argument it is convenient to think of firm death as a shock that changes the size of a firm to 0, rather than causing entry of a new firm; in this case the uniqueness of the invariant distribution is immediate from Theorem 11.9 of Stokey, Lucas, and Prescott (1989).

definition.

Definition 1. A tuple $\langle \theta, V^u, g(\cdot), J(\cdot), V(\cdot), v(\cdot), w(\cdot) \rangle$ is an *anonymous (threshold) steady-state equilibrium* if

- $J(m)$, $V(m)$, and $w(m)$ satisfy (3), (5), (6), (7), and (9).
- $g(m) \in \mathcal{G}$ satisfies (15) and (16).
- the value of an unemployed worker, V^u , satisfies (17).
- there is optimal vacancy posting, i.e.,

$$v(n) = \begin{cases} 1 & \text{if } -\gamma + q(\theta) J'(n) > 0 \\ 0 & \text{if } -\gamma + q(\theta) J'(n) < 0. \end{cases} \quad (18)$$

- there is free entry in the sense that equation (8) holds.

Notice that the steady-state equilibrium did not specify the unemployment rate u . This is because, as in the standard DMP model, the unemployment rate can be determined after the other endogenous variables. In particular, in steady state, a standard accounting argument implies that the u unemployed workers will be matched and thus hired at the flow rate $\theta q(\theta)$. On the other side, workers lose their job because of separations at the flow rate s and because of firm shutdowns at the flow rate δ . Consequently, the steady-state unemployment rate is given by equating flows into unemployment, $(1 - u)(s + \delta)$ with flows out of unemployment, $u\theta q(\theta)$, thus

$$u = \frac{s + \delta}{s + \delta + \theta q(\theta)}. \quad (19)$$

It is straightforward to verify that, as in the standard DMP model, u is a monotonically decreasing function of θ : steady-state unemployment is lower when the labor market is tighter.

3.2 Equilibrium Existence

To begin analysis of such equilibria, we first show that a simple expression for the wage function $w(\cdot)$ holds. To obtain this equation, first rearrange the worker HJB equation to observe that for $n < n^*$,

$$(r + \delta + s) [V(n) - V^u] = w(n) - rV^u + [qv(n) - sn] V'(n). \quad (20)$$

Use the bargaining equations (9) and (10) together with the firm's HJB equation (3) to observe immediately that

$$\phi [y'(n) - w(n) - nw'(n)] = (1 - \phi) [w(n) - rV^u], \quad (21)$$

or equivalently

$$w(n) + \phi nw'(n) = \phi y'(n) + (1 - \phi)rV^u. \quad (22)$$

Integrating this equation with respect to n implies that⁴

$$w(n) = (1 - \phi)rV^u + n^{-\frac{1}{\phi}} \left[c + \int_0^n \nu^{\frac{1-\phi}{\phi}} y'(\nu) d\nu \right], \quad (23)$$

where c is a constant of integration. Assuming that the wage bill for a small firm is finite, so that $nw(n)$ remains finite as $n \rightarrow 0^+$, it is immediate that the constant of integration c in (23) is zero, which establishes the following Lemma.

Lemma 1. *In any steady-state threshold equilibrium, wages satisfy*

$$w(n) = (1 - \phi)rV^u + n^{-\frac{1}{\phi}} \int_0^n \nu^{\frac{1-\phi}{\phi}} y'(\nu) d\nu. \quad (24)$$

Note that this wage equation takes the same form as in other papers using the SZ framework, such as Cahuc, Marque, and Wasmer (2008), and Elsbey and Michaels (2010). An alternative formulation of this equation that may be more intuitive is that

$$w(n) = (1 - \phi)rV^u + \phi \frac{\int_0^n \nu^{\frac{1-\phi}{\phi}} y'(\nu) d\nu}{\int_0^n \nu^{\frac{1-\phi}{\phi}} d\nu}; \quad (25)$$

that is, the wage of a worker at a firm employing n workers is a weighted average of the flow outside option, rV^u , and a term that is itself a weighted average of all the inframarginal products, $y'(\nu)$, for $\nu \in (0, n)$.

Generically equation (24) cannot be further simplified to give a closed form solution. Two notable cases where this is possible, however, are that if $y(n) = An^\alpha$ is Cobb-Douglas, then the wage takes the form

$$w(n) = (1 - \phi)rV^u + \frac{\alpha\phi}{1 - \phi + \alpha\phi} An^{\alpha-1}, \quad (26)$$

also if alternatively $\phi = \frac{1}{2}$, integration by parts establishes that

$$w(n) = \frac{1}{2}rV^u + \frac{1}{n} \left[y(n) - \frac{1}{n} \int_0^n y(\nu) d\nu \right]. \quad (27)$$

⁴We assume that the integral on the right side of (23) is finite. In the case that $y(n) = An^\alpha$ is Cobb-Douglas, a necessary and sufficient condition to ensure this is that $\frac{1}{\phi} + \alpha > 1$, so that $\nu^{\frac{1-\phi}{\phi}} y'(\nu) = A\nu^{\frac{1}{\phi} + \alpha - 2}$ is integrable near $\nu = 0$. Since $\phi < 1$ and $\alpha > 0$, the condition is always satisfied in this case.

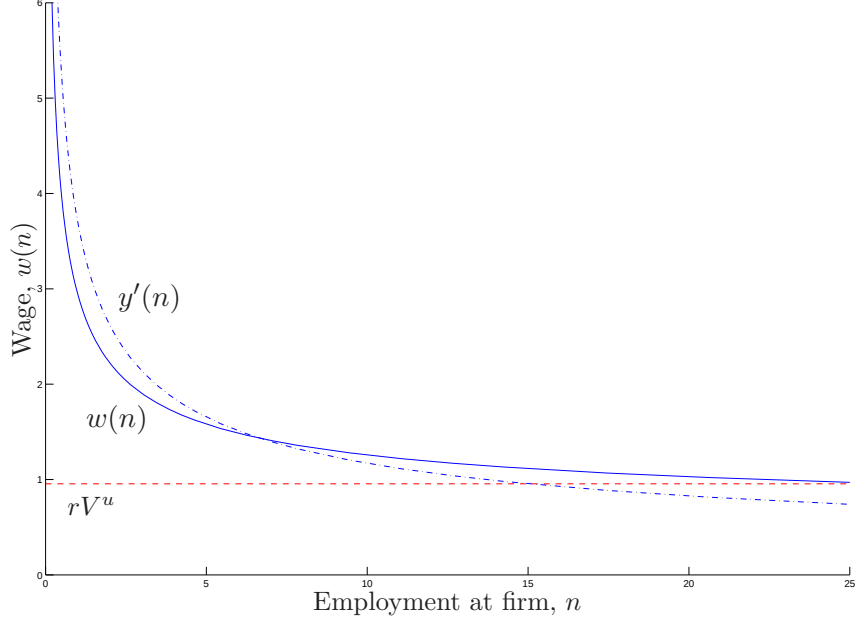


Figure 2: Wage, marginal product, and outside option

The lemma shows that despite the additional general equilibrium interactions, wages in this model take a form identical to those in Stole and Zwiebel (1996a,b) and Wolinsky (2000). The first term in (24) is the contribution of the (flow value of the) outside option of the worker to his wage. The second term is the worker's share of his contribution to the value of the firm, taking into account that if the worker were to quit, this would also influence the wages of other employees of the firm.

This explicit form characterization of wages will be important in further characterization and proving the existence of a steady-state equilibrium. A graphical representation of the dependence of wages on the number of workers employed at the firm is indicated in Figure 2. (This figure shows the wage function arising in the example of a Cobb-Douglas production function used in the calibrations in Section 7 below.) Also shown are a horizontal line indicating the flow value of the unemployed, rV^u , and the marginal product function, $y'(n)$.

Our next results show that, as depicted in Figure 2, wages and flow profits satisfy convenient boundary conditions.

Lemma 2. *In a steady-state equilibrium, wages are strictly positive, strictly decreasing with firm size, and, satisfy*

$$\lim_{n \rightarrow 0^+} w(n) = +\infty \text{ and } \lim_{n \rightarrow \infty} w(n) = (1 - \phi)rV^u.$$

Moreover, the flow profit $\pi(n) = y(n) - nw(n)$ of the firm is maximized at a unique $n \in (0, \infty)$ and satisfies

$$\lim_{n \rightarrow 0^+} \pi(n) = 0 \text{ and } \lim_{n \rightarrow \infty} \pi(n) = -\infty.$$

Proof. See Appendix. $\square$

The fact that wages at very small firms become very large arises from the Inada condition on the firm's production function, since the marginal product also becomes arbitrarily large as n decreases to 0.

The closed form equation for wages also allows the value functions for firms to be derived in closed form as shown in the following lemma.

Lemma 3. *In a steady-state equilibrium with labor market tightness θ , the firm's value function $J(\cdot)$ satisfies*

$$J(n) = (q(\theta) - sn)^{-\frac{r+\delta}{s}} \left[q(\theta)^{\frac{r+\delta}{s}} k + \int_0^n (q(\theta) - s\nu)^{\frac{r+\delta}{s}-1} (\gamma - \pi(n)) d\nu \right] \quad (28)$$

for all $0 < n < n^*$.

Proof. See Appendix. $\square$

Note that because the wage is known in closed form from (24) above, so too is the profit function $\pi(\cdot)$; thus (28) does indeed give a closed form solution for the firm's value function.

While the closed form solution for the firm's value function looks slightly complicated, it has a simple structure and enables us to further characterize the form of the equilibrium. In particular, differentiating this value function, $J(\cdot)$, with respect to n allows the boundary conditions at n^* also to be written in closed form. More specifically, equation (6) becomes

$$J(n^*) = \frac{1}{r+\delta} \left[\pi(n^*) - \frac{sn^*\gamma}{q(\theta)} \right], \quad (29)$$

while the smooth pasting condition (7) becomes

$$\frac{1-\phi}{\phi} n^{*-1/\phi} \int_0^{n^*} \nu^{\frac{1-\phi}{\phi}} y'(\nu) d\nu = (1-\phi)rV^u + \frac{(r+\delta+s)\gamma}{q(\theta)}. \quad (30)$$

It is interesting to observe that this latter can also be expressed as

$$w(n^*) = rV^u + \frac{(r+\delta+s)\gamma}{q(\theta)}. \quad (31)$$

Equation (31) is very intuitive and states that the firm continues to hire until the wage it pays equals the outside option of the worker, rV^u , plus a term that is proportional to the severity of the labor market friction (parameterized by the flow cost of posting an application, γ , divided by the productivity of that posting, $1/q(\theta)$). This wage equation is

therefore comparable to the result obtained in a static setting by Stole and Zwiebel (1996a) (see their Corollary 1 on page 396) and generalized to a dynamic setting by Wolinsky (2000). In these previous analyses, since there is no hiring margin (and no frictions), the second term is absent. Consequently, those models always imply “over-hiring” relative to a hypothetical competitive benchmark; firms will hire more than this competitive benchmark in order to reduce the marginal product of workers and thus their bargaining power according to the Shapley bargaining protocol (see Stole and Zwiebel, 1996a). Our analysis shows that this over-hiring result may or may not apply in general equilibrium; when γ is small, it will, but it could fail to do so when γ is large and $q(\theta)$ is relatively small. Equation (31) also corresponds to equations obtained by Cahuc, Marque, and Wasmer (2008) and Elsby and Michaels (2010), who considered cases where firms doing positive amounts of hiring are always at their target hiring level n^* .

Substituting the closed form expressions for $J(n^*)$ given by (29) into the formula for $J(\cdot)$ given by (28) and rearranging gives an expression which will be useful in characterizing equilibria.

$$k = J(0) = J(n^*) \left(\frac{q(\theta) - sn^*}{q(\theta)} \right)^{\frac{r+\delta}{s}} - q(\theta)^{-\frac{r+\delta}{s}} \int_0^{n^*} (q(\theta) - s\nu)^{\frac{r+\delta}{s}-1} [\gamma - \pi(\nu)] d\nu. \quad (32)$$

Equation (31), together with the worker’s Bellman equation (4) and the closed form equation for wages given by (24), allow the worker’s Bellman equation to be expressed more simply also.

Lemma 4. *For $0 < n < n^*$, the worker’s value function $V(\cdot)$ satisfies*

$$\begin{aligned} V(n) = V^u + \frac{\gamma}{q(\theta)} \left(\frac{q(\theta) - sn^*}{q(\theta) - sn} \right)^{\frac{r+\delta+s}{s}} \\ + (q(\theta) - sn)^{-\frac{r+\delta+s}{s}} \int_n^{n^*} (q(\theta) - s\nu)^{\frac{r+\delta}{s}} w(\nu) d\nu. \end{aligned} \quad (33)$$

Equivalently,

$$\begin{aligned} V(n) = V^u + \frac{\gamma}{q(\theta)} \left(\frac{q(\theta) - sn^*}{q(\theta) - sn} \right)^{\frac{r+\delta+s}{s}} + \frac{1 - \left(\frac{q(\theta) - sn^*}{q(\theta) - sn} \right)^{\frac{r+\delta+s}{s}}}{(r + \delta + s)} (1 - \phi) r V^u \\ + (q(\theta) - sn)^{-\frac{r+\delta+s}{s}} \int_n^{n^*} (q(\theta) - s\nu)^{\frac{r+\delta}{s}} \psi(\nu) d\nu. \end{aligned} \quad (34)$$

where

$$\psi(n) \equiv n^{-\frac{1}{\phi}} \int_0^n \nu^{\frac{1-\phi}{\phi}} y'(\nu) d\nu.$$

Proof. See Appendix. □

These equations now enable us to represent a steady-state equilibrium as the intersection

of two curves in $(q(\theta), rV^u)$ -space. In particular, suppose we know that in some equilibrium, the rate at which firms meet workers and the flow value of an unemployed worker equal $q(\theta)$ and rV^u respectively. Then firms take as given the path of wages that they will have to pay, $w(\cdot)$, given by (24). This means that their decision as to when to stop hiring, n^* , is given as the solution to an optimal stopping problem; n^* is then determined as the solution to (30), and the value of $J(n^*)$ can be deduced from (29). Denote this value by $n^*(q, rV^u)$. Next, solving the differential equation (28) with initial condition $(n^*(q, rV^u), J(n^*(q, rV^u)))$ allows us to solve for $J(0)$. The resulting expression is given in closed form by the right side of (32). If the free entry condition is satisfied, then the value of $J(0)$ so derived must equal the capital cost of entry k to ensure that (32) is satisfied. This provides one condition for (q, rV^u) to be part of an equilibrium. The remaining condition for (q, rV^u) to be part of an equilibrium is $(q, rV^u, n^*(q, rV^u))$ must satisfy (17), the Bellman equation of an unemployed worker. It remains to check that firm and worker behavior is optimal; for firms this is true by construction of $n^*(q, rV^u)$, while for workers, this follows since according to Lemma 2 and equation (31), the wage at any firm that is represented in equilibrium is strictly greater than rV^u , so that it is always optimal to accept any job offer. We record this conclusion as the following Proposition.

Proposition 1. *Let $q(\theta) > 0$ and $rV^u > 0$ be given. Then there is a steady-state equilibrium allocation with queue length $q(\theta)$ and value of an unemployed worker given by rV^u if and only if (17) and (32) are satisfied with n^* defined as the unique solution to (30) and $J(n^*)$ given by (29).*

Proof. Most of the proof is given in the discussion preceding the statement of the proposition. The uniqueness of $n^*(q, rV^u)$ follows from the proof of Lemma 2, together with (30). $\square$

We are now in a position to prove an existence theorem.

Theorem 1. *An steady-state equilibrium with cutoff hiring strategies exists.*

Proof. See Appendix. $\square$

The proof of the Theorem consists of showing that there exist (q, rV^u) satisfying the hypothesis of Proposition 1. Here, we present a diagrammatic exposition, emphasizing the intuition. The proof of Theorem 1 establishes that an equilibrium with positive activity exists if

$$k < \frac{1}{r + \delta} \max_{n > 0} \left\{ y(n) - n^{-\frac{1-\phi}{\phi}} \int_0^n \nu^{\frac{1-\phi}{\phi}} y'(\nu) d\nu - n(1 - \phi)b \right\},$$

where the existence of the maximum on the right side follows as in the proof of Lemma 2. In this case, Figure 3 shows a diagram depicting in (q, rV^u) -space the two curves described in the discussion preceding Proposition 1. The upward-sloping curve is the free-entry condition of firms, equation (32); it is upward-sloping since, all else equal, an increase in rV^u , must be

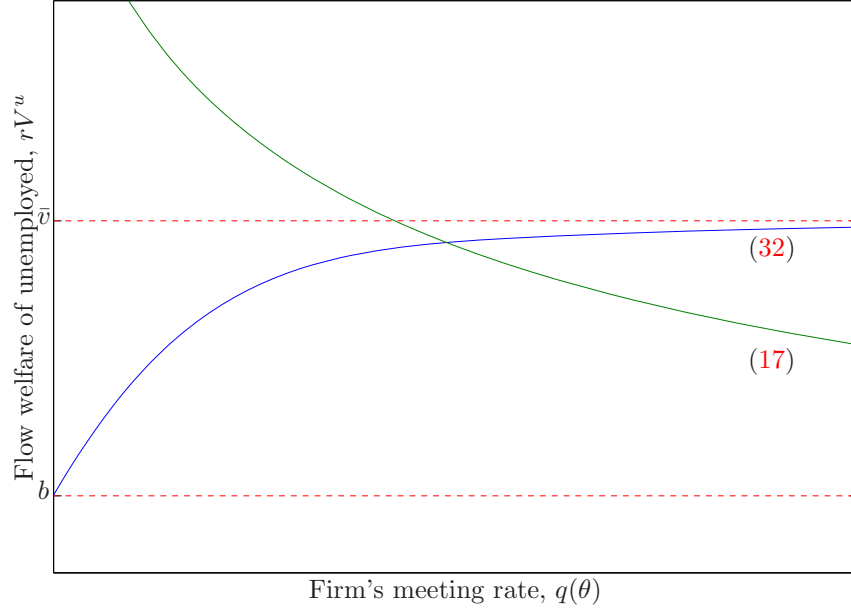


Figure 3: Equilibrium determination

compensated by an increase in q , which makes entry into the labor market more profitable for firms. This is because a higher rV^u translates into higher wages, so that the profit margins of firms decline. Zero profits can only be ensured by leaving vacancies unfilled for shorter durations, thus by an increase in q . The downward-sloping curve is the Bellman equation for unemployed workers. It is downward-sloping since an increase in rV^u on the right side of (17) corresponds to an increase in wages; to keep the flow value of an unemployed worker satisfying this equation, it must be that hiring is more rapid (that is, q is larger), so that when hired, the worker spends less time earning the high wage he receives when his firm is smaller.

Comparative statics of the response of the endogenous variables $q(\theta)$ and rV^u can now be obtained from the diagrammatic representation of the equilibrium. While general conclusions are difficult to draw, the general features of the comparative statics are quite clear. The movements of the free-entry condition, (32), are generally unambiguous. For example, in response to an increase in productivity, it moves upwards. This is because for a given (q, rV^u) , increased productivity increases flow profits for all firms, and so increases the implied value of entry, $J(0)$; to keep the free entry condition satisfied, rV^u must increase for each q . Similarly, the free-entry condition, (32) moves downwards in response to an increase in k . Since a change in k does not affect the other curve, it has unambiguous effects on the steady-state equilibrium in the situation depicted in the diagram in which the worker's Bellman equation is downward-sloping; an increase in k reduces rV^u and increases $q(\theta)$. This

also corresponds to a decline in θ and therefore, from (19), to an increase in the steady-state unemployment rate. In all calibrated examples we have investigated, the worker's Bellman equation has indeed been downward-sloping. We therefore conjecture that an increase in the cost of entry unambiguously reduces the tightness of the labor market, θ , and increases steady-state unemployment, u , but we do not at present have a proof of this assertion.

The impact of a productivity shock on equilibrium variables, on the other hand, is ambiguous because productivity shocks have a potentially ambiguous effect on the other curve. This is because the impact of a productivity increase on the optimal employment level of firms, $n^*(q, rV^u)$, is ambiguous. For a given (q, rV^u) , the wages paid at a firm with any fixed number n of workers, $w(n)$, increase; however, the increase in n^* means that more workers are employed at larger firms, which, all else equal, pay lower wages. In calibrated examples, the first effect tends to dominate, so that the curve moves upwards. An example where this is the case is shown in Figure 4. The dashed lines indicate the movement of the curves after a Hicks-neutral increase in productivity. In this case, the utility of workers increases unambiguously, but the response of the equilibrium job-finding rate for firms, $q(\theta)$, is ambiguous. Nevertheless, in many calibrated examples, including the example shown in Figure 4, $q(\theta)$ decreases in response to the increase in productivity, so that workers' job-finding rate rises and steady-state unemployment falls. Another interesting feature of this example is that n^* also falls in response to the positive productivity shock. This implies that in the new steady state firms are, on average, smaller. Consequently, much of the adjustment to the new steady-state takes place *at the extensive margin*, that is, by the entry of new firms, while existing firms in fact decline in size. This is a pattern we find consistently in the calibrations, and underlines the importance of considering separating the intensive and extensive margins of employment creation.

Although the model studied thus far is relatively tractable, it is notable that the distribution of firms by employment, as indicated by Figure 1 and equations (15) and (16), is difficult to match to data. First, the distribution of firm sizes is right-skewed in the empirically-realistic case that $s > \delta$. This is the realistic case to consider since the vast majority of separations do not occur at establishment shutdown.⁵ Second, the maximum firm size possible in the model is $q(\theta)/s$ (since a firm with that many workers loses workers to the separation shock as rapidly as it hires). Since $1/q(\theta)$ is the expected duration of a vacancy in the model, this suggests a discrepancy between the size of the largest establishments in the economy and the observed vacancy filling time, which is of the order of a couple of months (van Ours and Ridder, 1992). For example, if expected vacancy duration is 2 months, the worker separation rate to unemployment is 0.10 at quarterly frequencies (Shimer, 2005), of

⁵For example, Davis, Haltiwanger, and Schuh (1996, Figure 2.3, page 29) report for U.S. manufacturing plants that in quarterly data, 11.6% of job destruction occurs in plant shutdowns, while in annual data, 22.9% of job destruction occurs in plant shutdowns. The reasons for the discrepancy are that closure takes time and that transitory plant-level employment changes are more important in higher frequency data (fn. 9, p. 27).

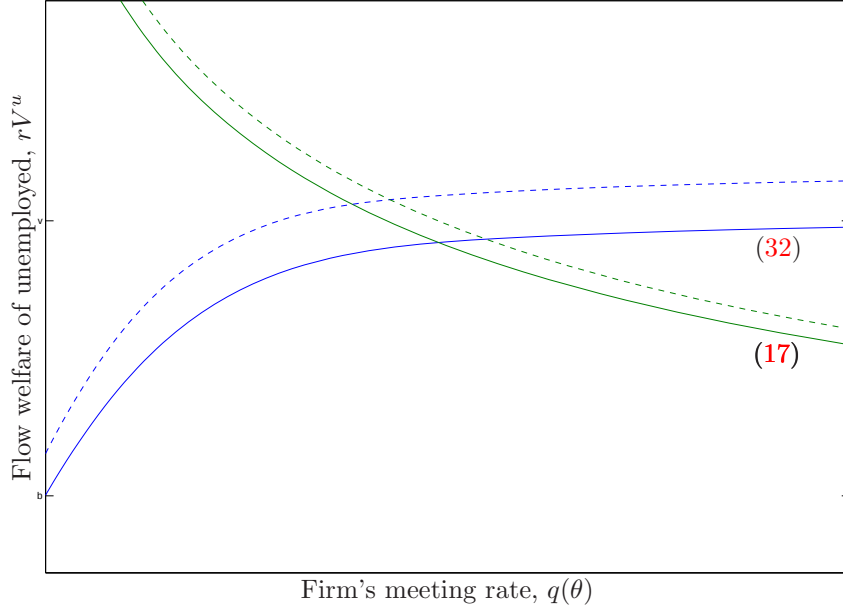


Figure 4: Response to a positive productivity shock

which (an upper bound) 25% arise from firm destruction, then $q(\theta) = 1.5$ and $s = 0.075$ at quarterly frequencies, giving a maximum firm size of 20. In the next section we therefore investigate a variant of the model in which firms can post multiple vacancies.

4 Intensive margin for hiring

In this section we consider the alternative formulation of the model in which firms may choose to post any non-negative number of vacancies (that is, $S = [0, \infty)$) at an increasing and convex cost $c(v)$. The characterization of this variant of the model is relatively similar to the case above, but it is less easy to solve in closed form; it is therefore presented only briefly.

With this modification of the model the HJB equation now takes the form

$$(r + \delta)J(n) = y(n) - nw(n) - snJ'(n) + \max_{v \geq 0} [-c(v) + q(\theta)vJ'(n)] . \quad (35)$$

(Compare (2) above.) The first-order condition for this problem is

$$c'(v) = qJ'(n); \quad (36)$$

denote the solution by $v(n)$. This solution is unique conditional on $J(\cdot)$ by the convexity of $c(\cdot)$, and does in fact define the optimum provided $J(\cdot)$ is weakly concave.

It turns out that wage determination is unchanged in this variant of the model from the basic model in which firms post a single vacancy, as described in Section 3 above. Equation (24) still holds.⁶

Further progress in closed form is not possible, since the differential equation (35) contains both a linear and a nonlinear term in $J'(n)$, the second arising from the dependence, via (36), of $v(n)$ on $J'(n)$. To see this, for the sake of concreteness, we assume that

$$c(v) = \frac{\gamma v^{1+p}}{1+p}. \quad (37)$$

One can then verify that

$$v(n) = \left(\frac{q}{\gamma} J'(n) \right)^{\frac{1}{p}}, \quad (38)$$

so that (35) becomes (where $\pi(n) \equiv y(n) - nw(n)$),

$$(r + \delta)J(n) = \pi(n) + \frac{p}{1+p} \frac{q^{\frac{1+p}{p}}}{\gamma^{\frac{1}{p}}} J'(n)^{\frac{1+p}{p}} - snJ'(n). \quad (39)$$

Although this differential equation cannot be solved in closed form, it can readily be solved numerically.

To find the boundary conditions, observe that the right limit is the (smallest) n^* such that $qv(n^*) = sn^*$, since at $n = n^*$, it follows that $\dot{n} = 0$. A firm at $n = n^*$ remains there forever until it is exogenously destroyed. We know from the Bellman equation that

$$(r + \delta)J(n^*) = \pi(n^*) - c(v(n^*)). \quad (40)$$

Using (37) above, one can verify that the condition $qv(n^*) = sn^*$ implies that

$$J'(n^*) = \frac{(sn^*)^p \gamma}{q^{1+p}}, \quad (41)$$

so that the boundary condition becomes

$$(r + \delta)J(n^*) = \pi(n^*) - \frac{(sn^*)^{1+p} \gamma}{(1+p)q^{1+p}}. \quad (42)$$

In addition, differentiating (39) and simplifying using (41) allows us also to conclude that

$$(r + \delta + s)J'(n^*) = \pi'(n^*). \quad (43)$$

Substituting (41) into (43) and using the functional form for wages from (24) allows determination of n^* ; (42) can then be solved to determine the value of $J(n^*)$. One can verify that the super-contact condition $J''(n^*) = 0$ continues to apply, although unlike in Section 3,

⁶A proof is available from the authors on request.

it does not need to be imposed separately since it is implied by the boundary conditions already given.

We also have a free entry condition, taking the form

$$J(0) = k. \quad (44)$$

Characterizing the remaining equilibrium objects proceeds analogously to Section 3 above. The firm size distribution $G(\cdot)$ has a continuous density on $[0, n^*]$ (and no atom at n^*); the density satisfies the differential equation

$$\frac{g'(n)}{g(n)} = \frac{s - \delta - qv'(n)}{qv(n) - sn}. \quad (45)$$

It follows that

$$g(n) = C \exp \left(\int_0^n \frac{s - \delta - qv'(\nu)}{qv(\nu) - s\nu} d\nu \right), \quad (46)$$

where the constant of integration C is chosen so that $g(\cdot)$ integrates to 1 on the region of integration $[0, n^*]$.

Finally, the HJB equation for an unemployed worker is given similarly to (17) by

$$rV^u = b + \theta q(\theta) \frac{\int_0^{n^*} g(n)v(n) [V(n) - V^u] dn}{\int_0^{n^*} g(n)v(n) dn}, \quad (47)$$

it's worth noting that using the bargaining equation (1), this can also be expressed as

$$rV^u = b + \frac{\phi}{1 - \phi} \theta q(\theta) \frac{\int_0^{n^*} g(n)v(n)J'(n) dn}{\int_0^{n^*} g(n)v(n) dn}, \quad (48)$$

or, using (38) to express $J'(n)$ in terms of $v(n)$, as

$$rV^u = b + \frac{\phi}{1 - \phi} \gamma \theta \frac{\int_0^{n^*} g(n)v(n)^{1+p} dn}{\int_0^{n^*} g(n)v(n) dn}. \quad (49)$$

Using a proof analogous to that of Theorem 1, one can prove that an equilibrium exists and is unique. (This proof is omitted in this version of the paper.)

On the issue that motivated adding an intensive margin of vacancy posting, the inability of the model to reproduce the empirical distribution of firm or establishment sizes, some improvement has been generated in that such sizes are no longer bounded above by $q(\theta)/sn$. However, because $J(n)$ is concave, $J'(n)$ is a decreasing function, so that according to (38), $v(n)$ is also a decreasing function of n . This implies that the firm size distribution is more right-skewed than it was before: intuitively, small firms, which have a high marginal profit from additional hiring, both because their marginal product of an additional worker is high and also because hiring an additional worker has a large effect on reducing the wages of

incumbent workers, are now able to hire more rapidly than in the absence of an intensive margin. Small firms now grow much more rapidly than before, leading to a more right-skewed firm size distribution.

The difficulty of the model in matching the empirical firm size distribution is reminiscent of the issue faced by the Burdett-Mortensen model in the same area. As those authors also do, adding productive heterogeneity of firms would be required to do a better job in this area.

5 Out-of-steady state dynamics

In this section, we study the out of steady-state behavior of our model. Recall from Section 2 that in an environment that changes over time, we need to add a time argument to endogenous variables (for example, $J(n, t)$ rather than $J(n)$), and we denote partial derivatives by subscripts (thus, $J_{nn}(n, t)$ denotes the second partial derivative of $J(n, t)$ with respect to n).

A dynamic equilibrium is defined similarly to a steady-state equilibrium, but is naturally more involved, since all objects are time-varying. Before defining such an equilibrium more formally, let us develop the equivalent of the steady-state Bellman equations. A standard derivation gives the firm's Hamilton-Jacobi-Bellman equation, which requires that the value of a firm with n workers at time t satisfy:

$$(r + \delta)J(n, t) = y(n, t) - nw(n, t) - snJ_n(n, t) + \max_{v \in S} \{-c(v) + q(\theta(t))J_n(n, t)\} + J_t(n, t), \quad (50)$$

where $\theta(t)$ is labor market tightness at time t , $y(n, t)$ is the output of a firm with n employees at time t , and $w(n, t)$ is the wage function at time t , which will again be determined in equilibrium. All the terms have similar interpretation to the Bellman equation (2) above, except that there is also the time derivative of the value function, $J_t(n, t)$ on the right hand side. Denote by $v(n, t)$ the optimal vacancy-posting strategy of the firm.

In addition, the HJB equation for the workers implies that the value function $V(n, t)$ must satisfy

$$(r + \delta + s)V(n, t) = w(n, t) + [v(n, t)q(\theta(t)) - sn]V_n(n, t) + (\delta + s)V^u(t) + V_t(n, t), \quad (51)$$

which again only differs from the state-state equation (4) because of the time derivative $V_t(n, t)$ on the right hand side.

Free entry still holds at all dates, so we also need to have

$$J(0, t) \leq k, \quad (52)$$

with equality whenever there is entry.

In addition, the analogs of the smooth pasting and optimal hiring (super-contact) conditions are given by

$$-\gamma + q(\theta(t))J_n(n^*(t), t), \quad (53)$$

and

$$J_{nn}(n^*(t), t) = 0, \quad (54)$$

where $n^*(t)$ is the “ideal size” of the firm which is now time varying, because productivity and labor market tightness vary over time. Recall that as in the steady-state analysis, $n^*(t)$ is such that in the threshold dynamic equilibrium we have $v(n, t) = 0$ for $n > n^*(t)$ and $v(n, t) = 1$ for $n < n^*(t)$.

Wages are again determined by bargaining à la Stole and Zwiebel, as in (1). The value of an unemployed worker, which is now time varying, $V^u(t)$, In particular, we have

$$rV^u(t) = b + \theta(t)q(\theta(t)) \left[-V^u(t) + \frac{\int_0^\infty V(m, t)g(m, t)h(m, t)dn}{\int_0^\infty g(m, t)h(m, t)dm} \right] + V_t^u(t). \quad (55)$$

To complete the description of the environment, we need to specify the distribution of firm sizes over time, represented by $g(n, t)$. To derive this distribution, let us reason as in the previous section and start with the case in which each worker is of size $\varepsilon > 0$. In this case, away from steady state, the rate of change in $g(n, t)$ over time, $g_t(n, t)$, is given by the difference between flows in and flows out of firms into the “state” of having n employees. Therefore, for $n < n^*(t)$,

$$g_t(n, t) = - \left(\frac{q(\theta(t))v(n, t)}{\varepsilon} + \frac{sn}{\varepsilon} + \delta \right) g(n, t) + \frac{q(\theta(t))v(n - \varepsilon, t)}{\varepsilon} g(n - \varepsilon, t) + \frac{s(n + \varepsilon)}{\varepsilon} g(n + \varepsilon, t) + O(\varepsilon).$$

Now taking the limit $\varepsilon \rightarrow 0$, we obtain the partial differential equation for $n < n^*(t)$:⁷

$$g_t(n, t) = -(\delta - s)g(n, t) + sng_n(n, t) - q(\theta(t)) \frac{\partial}{\partial n} [v(n)g(n)](n, t). \quad (56)$$

In addition, as in the steady state, there may be an atom at $n^*(t)$, the employment level beyond which firms do not hire at time t . Assuming that $n_t^*(t)$ always satisfies $-sn^*(t) < n_t^*(t) < q(\theta(t)) - sn^*(t)$, then firms with $n^*(t)$ workers will be able to choose a hiring policy $h \in [0, 1]$ such that they remain at the cutoff $n^*(t)$ as t varies. Also, assume that there are no firms with $n > n^*(t)$. In this case (which is the relevant one in our calibrations), the change in $G^*(t)$ over time arises from adding those firms near enough to $n^*(t)$ that by hiring

⁷For $n > n^*(t)$, we can derive a similar partial differential equation without the $q(\theta(t))$ terms. Nevertheless, if there are initially no firms above $n^*(t)$ and provided that $n_t^*(t) > -sn^*(t)$ at all t , there never will be any such firms, so that this partial differential equation is of limited interest for our focus.

with $h = 1$, they reach $n^*(t)$, and subtracting those firms at $n^*(t)$ that die exogenously.

$$G_t^*(t) = [q(\theta(t)) - sn^*(t) - n_t^*(t)]g(n^*, t) - \delta G^*(t). \quad (57)$$

Given these equations, we can define a dynamic equilibrium as follows:

Definition 2. A tuple $\langle \theta(t), V^u(t), g(n, t), J(n, t), V(n, t), h(n, t), w(n, t) \rangle$ is an *anonymous (threshold) dynamic equilibrium* if

- $J(n, t)$, $V(n, t)$ and $w(n, t)$ satisfy (1), (50), (51), (53), and (54).
- $g(n, t) \in \mathcal{G}$ satisfies (56) and (57).
- the value of an unemployed worker, $V^u(t)$, satisfies (55).
- there is optimal vacancy posting, i.e.,

$$v(n, t) = \arg \max_{v \in S} -c(v) + q(\theta(t))vJ_n(n, t). \quad (58)$$

- there is free entry in the sense that equation (52) holds.

As in the analysis of the steady-state equilibrium, we can differentiate the firm's Bellman equation with respect to n and compare with the worker's Bellman equation, which yields the following analog to (21)

$$\phi [y'(n) - w(n) - nw'(n)] = (1 - \phi) [w(n) - rV^u(t) + V_t^u(t)], \quad (59)$$

or equivalently

$$w(n) + \phi nw'(n) = \phi y'(n) + (1 - \phi) (rV^u(t) - V_t^u(t)). \quad (60)$$

This can be solved very similarly to (24) to establish that

$$w(n) = (1 - \phi) [rV^u(t) - V_t^u(t)] + n^{-\frac{1}{\phi}} \int_0^n \nu^{\frac{1-\phi}{\phi}} y'(\nu) d\nu. \quad (61)$$

The only difference between this equation and (24), which applied in the steady-state analysis, is the presence of $V_t^u(t)$ in the first term, which represents that the effect of the change in the outside option of the worker on his current wage.

Rather than proceed to further study of the general case of these equations, in the next section we specialize to study the special case where the production function $y(n)$ is linear. In this case, substantial simplification is possible. In Section 7 we return to a numerical examination of dynamics in the general case.

6 Constant returns to scale

In this section, we specialize to studying the case of a constant returns to labor production function. Although perhaps less realistic than the case of decreasing returns to scale studied in the benchmark model, the great advantage of this case is that, as in the Mortensen-Pissarides model, there are no dynamics in the vacancy-unemployment ratio which, following a shock, jumps immediately to its new steady-state value.

Therefore, in this section, we assume that the production function takes the linear form

$$y(n) = An \quad (62)$$

This makes possible some significant simplification in the equations determined in the more general case studied above. First, the wage equation (24) now takes the simple form

$$w(n, t) \equiv w(t) = (1 - \phi) [rV^u(t) - V_t^u(t)] + \phi A, \quad (63)$$

independent of n . This means that the at any time t , all employed workers in the economy are paid the same wage.

Rather than solve for a general productivity process, we will only study the dynamics of the economy after an unanticipated permanent positive productivity shock. We will look for an equilibrium in which $J(n, t)$, $V(n, t)$, $V^u(t)$, and $\theta(t)$ jump at the moment of the shock and are constant thereafter. (Note that this already allows a simplification of (63), since $V_t^u(t) \equiv 0$.) In this case, we can write the HJB equations for workers (either before the shock, or after the shock, the difference occurring only in the values of w and θ) simply as a function of whether they are unemployed (V^u) or employed (V^E):

$$rV^u = b + \theta q(\theta) [V^E - V^u] \quad (64)$$

$$rV^E = w + (s + \delta) [V^u - V^E] \quad (65)$$

Solving these equations shows that

$$\begin{aligned} rV^u &= \frac{(r + s + \delta)b + \theta q(\theta)w}{r + s + \delta + \theta q(\theta)} \\ &= \frac{(r + s + \delta)b + \phi \theta q(\theta)A}{r + s + \delta + \phi \theta q(\theta)}. \end{aligned} \quad (66)$$

and

$$w = \frac{(1 - \phi)(r + s + \delta)b + [\phi(r + s + \delta) + \phi \theta q(\theta)]A}{r + s + \delta + \phi \theta q(\theta)}. \quad (67)$$

Now, because the value of employment at a firm of size n , $V(n)$, is independent of n and equal to V^E , it follows from the wage bargaining equation (61) that also $J'(n)$ is

independent of n , so that $J(n)$ is linear. In particular,

$$J'(n) = \frac{1-\phi}{\phi} [V^E - V^u] = \frac{(1-\phi)(A-b)}{r+s+\delta+\theta q(\theta)}. \quad (68)$$

The constancy of $J'(n)$ in n then implies from (36) that vacancy posting is constant across all active firms. Denote, by abuse of notation, the equilibrium vacancy posting per firm also by v ; then (36) implies that

$$c'(v) = \frac{q(\theta)(1-\phi)(A-b)}{r+s+\delta+\theta q(\theta)}. \quad (69)$$

Substitute $n = 0$ into the Bellman equation (35) to see that the value of $J(0)$ satisfies:

$$(r+\delta)J(0) = \frac{q(\theta)(1-\phi)(A-b)}{r+s+\sigma+\theta q(\theta)}v - c(v). \quad (70)$$

Note that the right side of (70) is decreasing in $\theta q(\theta)$ (to see this, use the fact that $q(\theta)$ is decreasing in θ – or, equivalently, in $\theta q(\theta)$ – as is $(r+s+\delta+\theta q(\theta))^{-1}$, and apply the envelope theorem since v is chosen optimally). Thus together with the free entry condition $J(0) = k$, equation (70) uniquely determines θ ; moreover, it establishes that θ is a decreasing function of A . Using the power functional form for $c(\cdot)$ given by (37), it is immediate that

$$v = \left(\frac{q(\theta)(1-\phi)(A-b)}{\gamma(r+s+\delta+\theta q(\theta))} \right)^{\frac{1}{p}}, \quad (71)$$

$$c(v) = \frac{1}{1+p} \gamma^{-\frac{1}{p}} \left(\frac{q(\theta)(1-\phi)(A-b)}{r+s+\delta+\theta q(\theta)} \right)^{\frac{1+p}{p}}, \quad (72)$$

and

$$(r+\delta)J(0) = (r+\delta)k = \frac{p}{1+p} \gamma^{-\frac{1}{p}} \left(\frac{q(\theta)(1-\phi)(A-b)}{r+s+\delta+\theta q(\theta)} \right)^{\frac{1+p}{p}}. \quad (73)$$

Equation (73) determines θ .

Next, it's necessary to solve for the firm size distribution $g(\cdot)$. I assume that before the unanticipated productivity shock, the economy is in the steady state corresponding to the initial value of productivity. Specializing (45) and (46) to the case of constant v , as indicated by (71), it follows that the key differential equation for the firm size distribution is

$$\frac{g'(n)}{g(n)} = \frac{\delta - s}{sn - q(\theta)v}. \quad (74)$$

Integrating and choosing the constant of integration so that $g(\cdot)$ integrates to 1 on $[0, n^* = q(\theta)v/s]$ establishes that

$$g(n) = \left(\frac{\delta}{q(\theta)v} \right)^{\frac{\delta}{s}} (q(\theta)v - sn)^{\frac{\delta-s}{s}}. \quad (75)$$

This determines the relative number of firms at each $n \in [0, q(\theta)v/s]$. To determine the total mass F of active firms in the economy, note that the total number of employees is just

$$F \int_0^{\frac{q(\theta)v}{s}} n g(n) dn = \frac{F q(\theta)v}{\delta + s},$$

where the closed-form expression comes from substituting for $g(n)$ from above and integrating by parts. By definition, the value of labor market tightness θ is equal to the ratio of the total number of posted vacancies, vF , to the number of unemployed workers, which is the measure of all workers (1) less the measure of employed workers calculated above. That is, $\theta = vF/(1 - Fq(\theta)v/(\delta + s))$, or

$$F = \frac{\theta}{v \left(1 + \frac{\theta q(\theta)}{\delta + s}\right)}. \quad (76)$$

This completes the characterization of the steady state.

In the remainder of this section, I consider the response of the economy to a one-time unanticipated permanent positive increase in A . In the baseline case, suppose that free entry continues to hold after the arrival of the shock. Because firms' vacancy posting strategy depends only on constant parameters and on the market tightness, (71) guarantees that an equilibrium of the conjectured type does in fact exist. Given that the free entry holds, on the arrival of the unanticipated productivity shock, a discrete number of firms enter, and the labor market tightness θ jumps immediately to its new steady state level. The firm size distribution and the unemployment rate then converge slowly to the new steady state, but because of the block recursive structure of the model, the transition paths for these variables do not need to be known in order to calculate the dynamic behavior of other variables.

It is interesting to note that firms' optimal choice of how many vacancies to post, given by equation (71), together with the free entry condition (73), ensures that in fact there is no change in firms' vacancy posting after the arrival of the productivity shock, since these two equations can be combined to show that

$$v = \left(\frac{1+p}{p} \frac{(r+\delta)k}{\gamma} \right)^{\frac{1}{1+p}}; \quad (77)$$

since A does not occur on the right side of (77), v does not change with the increase in A .

An alternative assumption is that no entry is possible after the arrival of the shock. In this case, after the arrival of the shock, (73) does not hold after the arrival of the shock; instead, the number of firms F remains constant. In this case, θ again jumps immediately on the arrival of the shock and remains constant thereafter; the new steady state value can be determined by substituting from (71), which determines v as a function of θ , into (76). The resulting equation can then be solved for θ . In the calibrated example below, it can be

verified that the solution for θ is unique.

Since analytic results are not in general available, to discuss the effects of a productivity shock, it's useful to calibrate the model. We choose the unit of time to be quarterly. We assume that the matching function $M(u, \bar{v})$ takes the form $M(u, \bar{v}) = Zu^\eta \bar{v}^{1-\eta}$ where $Z > 0$ is a constant; this implies that $q(\theta) = Z\theta^{-\eta}$. For the sake of comparability with the literature, we borrow parameters as closely as possible from the benchmark paper in the quantitative study of fluctuations in the Mortensen-Pissarides family of models, which is Shimer (2005). Parameter values are reported in Table 1. Some values require some comment. First, we set the ratio of the unemployment income to match output, given by b/A , to 0.4; as is familiar from the work of Hagedorn and Manovskii (2008), increasing this value amplifies the effect of productivity shocks on the model, so we also consider the effect of a changing this value to 0.95. We set the scale parameter Z in the matching function to 1.355 and the elasticity η to 0.72, following Shimer, to match the mean job-finding rate and the slope of the Beveridge curve conditional on an initial steady-state value for θ of 1. This is sensible since vacancies are relatively poorly measured in the U.S., while the model admits a normalization: multiplying Z by $x^{\eta-1}$ and γ by $x^{-(1+p)\eta}$ changes the equilibrium only by multiplying θ by x , $q(\theta)$ by $x^{-\eta}$, v by x^η , so that $q(\theta)v$ and $\theta q(\theta)$, the matching rates for firms and workers, are unchanged. We set $\phi = 0.72$ for comparability with Shimer (one can verify that the analog of the Hosios condition also applies in the case of the linear production function studied here). We set $s + \delta = 0.10$ to match Shimer, and follow Davis, Haltiwanger, and Schuh (1996) so that one-sixth of job destruction is attributable to firm shut-down.

We somewhat arbitrarily impose in our baseline calibration that $p = 1$, so that the vacancy cost function is quadratic. (We therefore also consider the case of much more convex vacancy-posting costs, by investigating also the case $p = 4$.) Finally, we take from Davis, Haltiwanger, Jarmin, and Miranda (2006) that the average employment of U.S. firms (both publicly- and privately-held) is 23.8, and we calibrate so that the mean firm size matches this figure. Together with a steady-state unemployment rate of 6.87%, following Shimer, this requires that the value of F be $(1 - 0.0687)/23.8 = 0.0391$. According to (76), this in turn requires targeting $v = 1.76$, and from (71) we get that the vacancy posting cost $\gamma = 0.088$. The only remaining unknown parameter is the entry cost k ; (73) establishes that this must equal 4.75. To put these derived parameters in context, note that at the optimal level of v , the flow cost incurred by a firm of posting vacancies is $c(v) = 0.136$, or around 14% of the output of a single employee, and it takes an expected duration of $(q(\theta)v)^{-1} = 0.420$ quarters for a worker to be hired. That is, generating a new hire costs roughly 5.7% of one quarter's production by a single worker. (However, since wages are a large fraction of output, this corresponds to roughly 3.3 quarters of the flow profits accruing to the firm from a match.) The free entry cost corresponds to just over one year's output of a single worker, or around 69 worker-years of flow profit.

Variable	Value	Explanation
r	0.012	Annual rate of time preference 4.71%
δ	0.0167	1/6 of separations from firm closure
s	0.0833	Quarterly job destruction rate 0.1
A	1	Normalization
b	0.4	Shimer (2005)
Z	1.355	Shimer (2005); unemployment rate of 6.87%
η	0.72	Shimer (2005)
γ	0.088	See text
p	1	See text
k	4.75	See text
ϕ	0.72	Shimer (2005); Hosios condition

Table 1: Parameterization, Linear Model

In Table 2, we report the results the arrival of an unanticipated permanent 1% increase in A . The top panel of Table 2 reports on the case of our baseline calibration with quadratic vacancy-posting costs and the Shimer (2005) value of $b = 0.40$. In the second panel we modify this by changing b to the much higher value of 0.95. Finally, in the third panel, we report the results with the original value of b but much more convex vacancy-posting costs. Note that in each case we recalibrate so as to match the targets described in the discussion above. This is achieved by modifying the values of γ and k to be consistent with unchanged pre-shock values of F and θ . In the second panel, with high b , this requires setting $\gamma = 0.0074$ and $k = 0.3962$. In the third panel, with high p , this requires $\gamma = 0.0163$ and $k = 7.6062$.

First, in the low- b calibration, the elasticity of the response of market tightness and of unemployment are in fact less than in the benchmark MP model. Wages rise slightly less than one-for-one with the shock. Under free entry, consistently with (77), vacancies per firm do not change, so that the increase in entry is associated with a fall both in mean employment per firm, as well as in the size of the largest firms. If no firms can enter, vacancies per firm rise slightly, and the size of firms grows slightly. The effect on unemployment and the firm size distribution is significantly attenuated if adjustment must occur on the intensive margin of increased vacancy posting, as indicated by the third column.

The calibration of the model with the higher value of b shows a much more elastic response of the unemployment rate and of the size distribution of firms to the productivity shock. The elasticity of θ with respect to productivity increases to around 20; that of the unemployment rate to -0.47 , and the number of firms (not reported) now responds to the 1% productivity shock by increasing from 0.0391 to 0.0449, an increase of just under 15%. Correspondingly, mean firm size now shrinks by around 13%, as does the size of the largest firms.

Finally, the third panel shows that, the more convex the cost of increasing the intensity

$b = 0.40, p = 1$			
	Baseline	Shock, free entry	Shock, no entry
Market tightness, θ	1	1.017	1.010
Unemployment, u	6.87%	6.84%	6.86%
Wage, w	0.983	0.992	0.992
Vacancies per firm, v	1.756	1.756	1.769
Mean employment per firm	23.80	23.51	23.81
Maximum employment per firm	28.56	28.21	28.57
$b = 0.95, p = 1$			
Market tightness, θ	1	1.205	1.112
Unemployment, u	6.87%	6.55%	6.69%
Wage, w	0.999	1.008	1.008
Vacancies per firm, v	1.756	1.756	1.900
Mean employment per firm	23.80	20.81	23.85
Maximum employment per firm	28.56	24.98	28.62
$b = 0.40, p = 4$			
Market tightness, θ	1	1.017	1.004
Unemployment, u	6.87%	6.84%	6.87%
Wage, w	0.983	0.992	0.993
Vacancies per firm, v	1.756	1.756	1.762
Mean employment per firm	23.80	23.51	23.80
Maximum employment per firm	28.56	28.21	28.56

Table 2: Comparison of Steady States, Linear Model

with which a firm posts vacancies, the smaller the effect of the productivity shock on unemployment and on the firm size distribution; the effect on wages is virtually unchanged.

In addition to comparing steady states, it's possible to characterize the dynamics of unemployment and the firm size distribution. (Other variables, such as wages, jump immediately to their new steady-state values.) The differential equation indicating the evolution of unemployment over time is

$$u_t(t) = (s + \delta)(1 - u(t)) - \theta q(\theta)u(t);$$

this has solution

$$u(t) = u^1 + (u^0 - u^1) \exp(-(s + \delta + \theta q(\theta))t),$$

where u^0 and u^1 are respectively the pre- and post-shock steady state values of unemployment, given by (19) with the corresponding values of θ . Average employment per firm can then be determined by dividing the number of employed workers by the number of firms, $(1 - u(t))/F$. Note that this variable jumps downwards initially under free entry. One might also be interested in the employment of the largest firms in the economy. The differential

equation for the evolution of the size of any particular firm is

$$\dot{n}_t(t) = q(\theta)v - sn(t);$$

analogously, this has solution

$$n(t) = n^1 + (n^0 - n^1) \exp(-st),$$

where n^1 is the limiting value of the firm's size, given by $n^1 = q(\theta)v/s$, and n^0 is its size at the time of the productivity shock. Note that the convergence of firms to the new steady state size is much slower than that of the unemployment rate (the ratio $(s + \delta + \theta q(\theta))/s$ is around 17.5 in the calibration). Figure 5 shows this graphically.

Finally, it should be noted that the simple ‘bang-bang’ dynamics of the market tightness that were characterized in this section were a result of the assumption of a positive, permanent productivity shock. It was important that the shock be positive, so that firms would enter upon its arrival; had the shock been negative, the returns to entry would have become negative at the arrival of the shock, and there would have been no entry for some time, until the number of firms reduced, according to the exogenous process for firm destruction, to a level consistent with a zero value of entry. Because of the temporarily higher number of firms than in the new steady state, vacancy posting per firm would also be temporarily lower than in the new steady state. This generates dynamics in labor market tightness, making the model impossible to solve in closed form. Since the motivation for studying the linear case in this section was to study a case for which exact dynamics could be identified, we do not pursue this further. Instead, in the next section, we study the dynamics of adjustment to a positive productivity shock in the case of a production function with decreasing returns to labor.

7 Dynamics

In this section we study the version of the model in which the production function of firms exhibits decreasing returns to labor. Since we need to use numerical methods to solve the model in this case, we specialize to the case of a Cobb-Douglas production function. In this case, we have a closed-form solution for the wage; specializing (61) to the case of a Cobb-Douglas production function, $y(n) = An^\alpha$, establishes that

$$w(n) = (1 - \phi) [rV^u(t) - V_t^u(t)] + \frac{\alpha\phi}{1 - \phi + \alpha\phi} An^{\alpha-1}. \quad (78)$$

This implies that

$$\pi(n) \equiv y(n) - nw(n) = \frac{1 - \phi}{1 - \phi + \alpha\phi} An^\alpha - (1 - \phi)n [rV^u(t) - V_t^u(t)]. \quad (79)$$

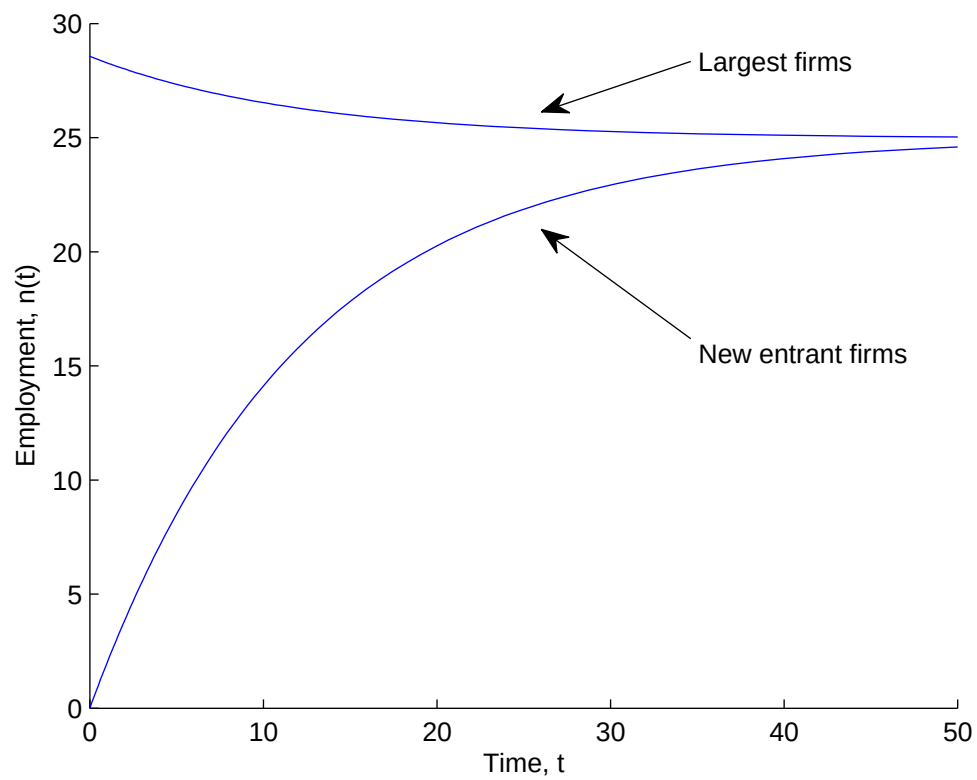
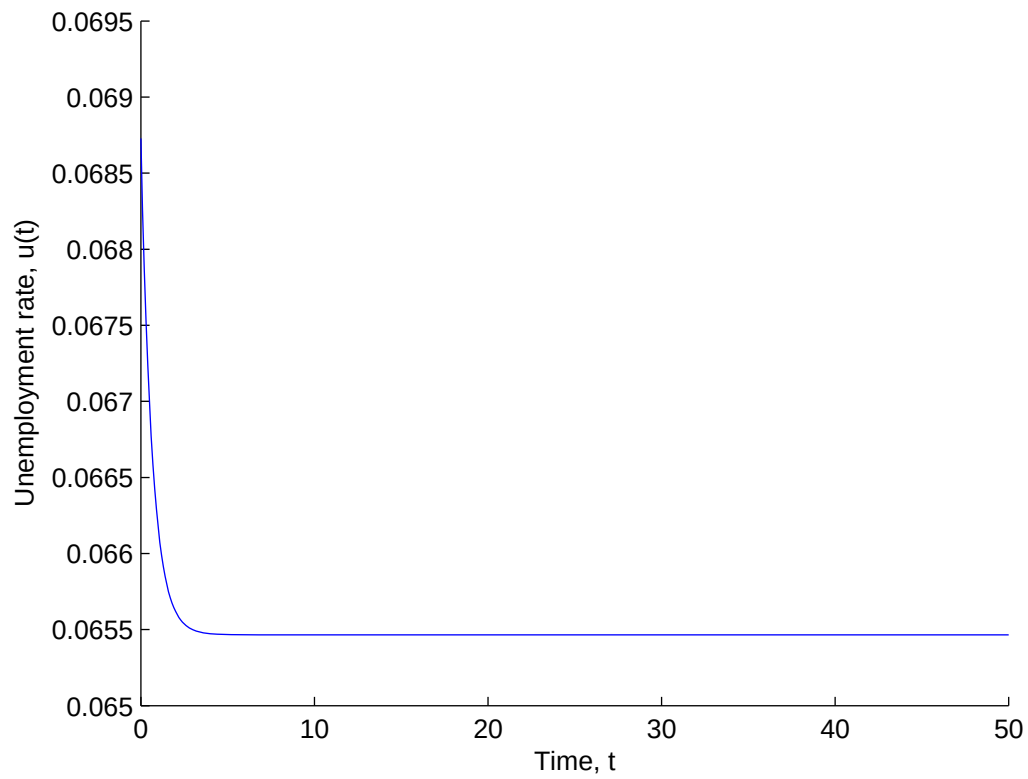


Figure 5: Evolution of unemployment rate and firm size

We also assume that the vacancy-posting cost takes the convex form given by (37); this implies that the differential equation defining the firm's value function $J(\cdot)$ in steady state is given by (39) with $\pi(\cdot)$ taking the functional form given by (79), so that

$$(r + \delta)J(n) = \frac{1 - \phi}{1 - \phi + \alpha\phi} An^\alpha - (1 - \phi)n[rV^u(t) - V_t^u(t)] + \frac{p}{1 + p} \frac{q^{\frac{1+p}{p}}}{\gamma^{\frac{1}{p}}} J'(n)^{\frac{1+p}{p}} - snJ'(n). \quad (80)$$

The boundary conditions for this differential equation are given by (41) and (42). In addition, firms' vacancy posting policy is given by (38), and the free entry condition (44) must also hold.

We first solve numerically for the steady state equilibrium. The equations characterizing the firm size distribution are given by (45) and (46), while the other key endogenous variable, rV^u , is given in terms of the function $J(\cdot)$ by (48). This allows us to solve numerically for the steady state. We first guess values of the two key endogenous variables, θ and rV^u and then solve the initial value problem implied by (80) with boundary conditions (41) and (42). This allows us to determine the vacancy posting policy of the firm via (38), and this in turn allows us to construct the firm size distribution according to (46). We can then check whether the free entry condition (44) and the Bellman equation for the unemployed worker (48), neither of which were used so far, hold; if not, we alter our initial guesses for θ and rV^u and repeat the process.

The resulting steady-state equilibrium is qualitatively similar to that calculated in Section 3 above, so we omit further characterization here.

In order to solve for the transitional dynamics following an unanticipated permanent productivity shock, analogously to the case considered in the case of a constant returns to labor production function in Section 6, a slight modification of the above procedure is required. We use the following algorithm. We take a discrete-time approximation to the model, as well as approximating the value function on a discrete state space for n (and using cubic splines to interpolate for other n).

In addition, we use (49) rather than (48).

This is necessary since numerically differentiating an incorrect guess for $J(\cdot)$ induces errors which are intractable numerically.

- Select a time T by which the transition will be largely complete, and impose that from time T onwards, the economy will be in the steady state corresponding to the new, higher productivity level.
- Guess time paths for $\{\theta(t)\}_{t=0}^{T-1}$ and $\{rV^u(t)\}_{t=0}^{T-1}$. If entry is allowed, guess also a time path for firm entry, $\{e(t)\}_{t=0}^{T-1}$.
- Solve for the initial steady state firm size distribution, $G(\cdot, 0)$.
- Solve for the final steady state value function, $J(\cdot, T)$.

- Solve recursively for the functions $\{J(\cdot, t)\}_{t=0}^{T-1}$, iterating backwards in time, and using the assumed time paths for $\theta(t)$ and $rV^u(t)$. In this process, calculate the optimal vacancy posting policies of firms, $v(n, t)$ for $t \in 0, 1, 2, \dots, T-1$.
- Using the guessed time paths of $\theta(\cdot)$ and $e(\cdot)$ and the calculated vacancy posting policies $v(\cdot, \cdot)$, simulate the evolution of the firm size distribution $G(\cdot, t)$ and of the unemployment rate $u(t)$.
- Use these, together with the appropriate out-of-steady-state modification of equation (49), to calculate the resulting time paths of $rV^u(t)$ and $\theta(t)$; denote these time paths by $r\hat{V}^u(t)$ and $\hat{\theta}(t)$. If the calculated time paths equal the guesses, stop. If not, update the guesses by selecting new guesses

$$V_{new}^u(t) = (1 - \lambda_V)V^u(t) + \lambda_V\hat{V}^u(t) \quad \text{and} \quad \theta_{new}(t) = (1 - \lambda_\theta)\theta(t) + \lambda_\theta\hat{\theta}(t),$$

where λ_V and λ_θ are constants. (In the case of free entry, check also whether the free entry condition holds for all $t = 0, 1, \dots, T-1$, and if not, reduce (respectively, increase) entry slightly at times when the calculated value of entry, $J(0, t)$, is less than (respectively, greater than) k .)

Table 3 gives the parameterization for the pre-shock steady state of the model. The parameters are as much as possible consistent with those chosen in Table 1 for the linear model. We somewhat choose the degree of decreasing returns to labor, α to be significantly less than 1, so as to demonstrate the effect of moving away from the constant returns case studied in Section 6. The normalization used in that section to set θ to 1 by altering the values of Z and γ is no longer available in the context of this section since this would also modify the distribution of vacancy posting across firms; we therefore need to choose Z and γ so as to be able to match an unemployment rate of 6.87% and a value of $b/(rV^u)$ around 0.40. None of the calibration targets mentioned so far pin down that level of productivity (we have targeted its ratio to other variables, but not the level); we choose it so as to normalize the value of rV^u to 1.

Figure 6 shows what happens after a 1% positive productivity shock when no free entry is allowed. The four panels are labeled. Note that θ rises slowly, whereas rV^u jumps immediately from its former steady state value of 1 almost to the new steady state value. Firm size grows slowly, slightly overshoots, and then declines.

With free entry, the numerical results are still somewhat preliminary; one can see from the sixth panel of Figure 7 that the free entry condition doesn't yet hold exactly. However, one can see in this case, for this calibration, the decline in the largest firms' size is monotonic; entry is mostly almost immediate, but not all occurs at the impact of the shock (if one imposes this, one can verify that the value of entry is less than the entry cost for the whole transition path). The unemployed gain more in steady state (wages rise more than without

Variable	Value	Explanation
r	0.012	Annual rate of time preference 4.71%
δ	0.0167	1/6 of separations from firm closure
s	0.0833	Quarterly job destruction rate 0.1
A	20.81	Normalization ($rV^u = 1$)
α	0.667	See text
b	0.4	Shimer (2005)
Z	0.701	Shimer (2005); unemployment rate of 6.87%
θ	0.400	See text
η	0.72	Shimer (2005)
γ	0.012	See text
p	1	See text
k	23,600	See text
ϕ	0.72	Shimer (2005); Hosios condition

Table 3: Parameterization, Decreasing Returns Model

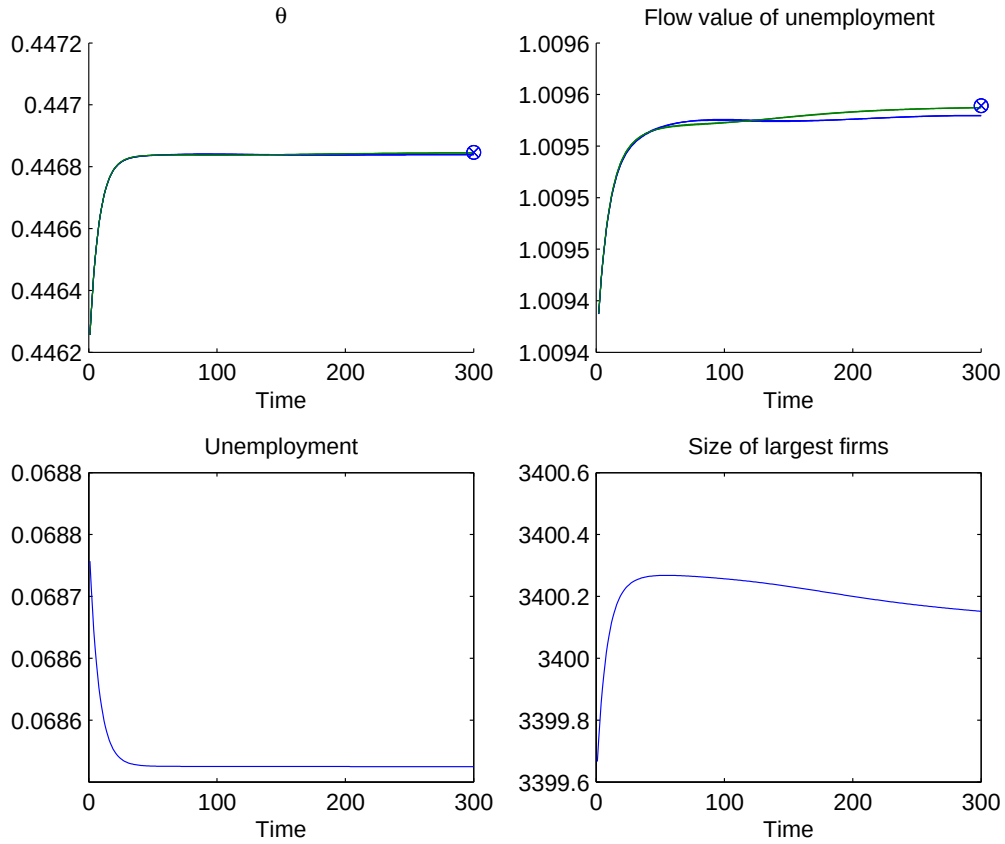


Figure 6: Effects of productivity shock: decreasing returns, no entry

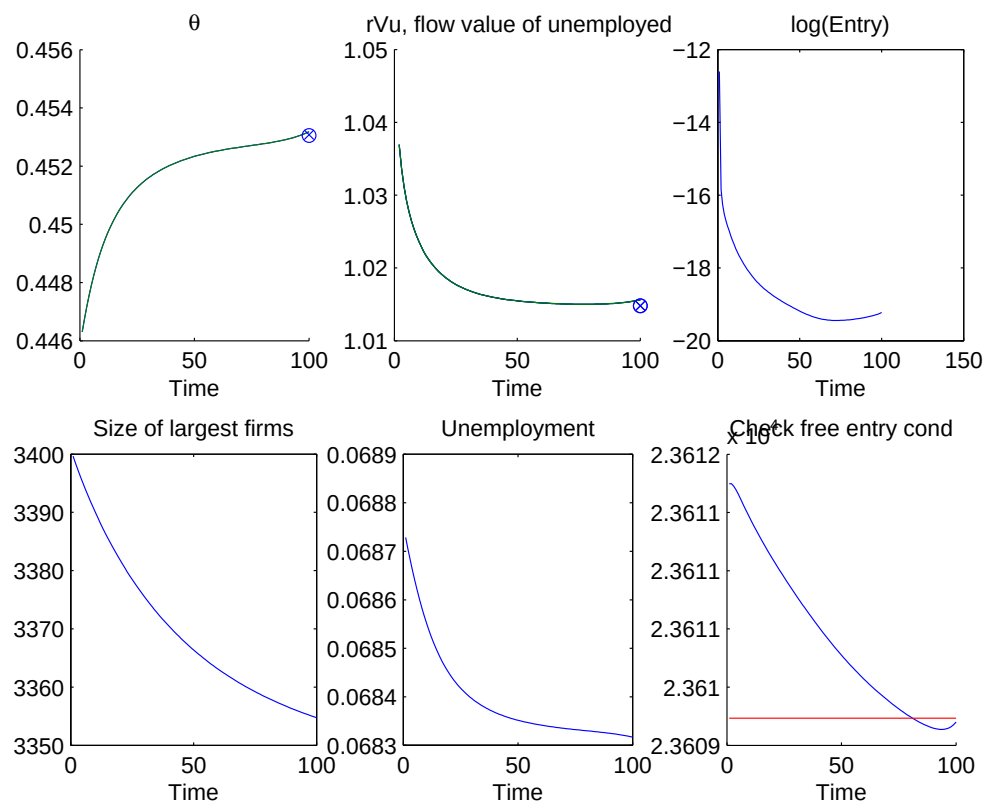


Figure 7: Effects of productivity shock: decreasing returns, free entry

free entry: note the steady state change in rV^u is more than 1.5%, rather than the less than 1% increase with no free entry), and even more in the short run, while there is a burst of entry of new firms that temporarily pay very high wages. Unemployment falls about twice as much as in the case without free entry, and θ rises by around ten times as much as in the case without free entry, although the elasticity is still small.

In summary, some of the qualitative features of the transitional dynamics here are very similar to those observed in the linear case of Section 6. (For example, the size of the largest firms under free entry falls monotonically towards its new steady-state level.) However, some important new features are observed. Most notable of these is the behavior of wages with and without entry. With our assumption of time-to-hire frictions, new entrant firms after the shock will remain small for some time. If there are decreasing returns to labor, this means a greater-than-usual number of high marginal product jobs will be temporarily available. Under the bargaining assumption we use, these jobs also pay high wages. This generates the possibility for the unemployed to earn high wages if they are lucky enough to match with such a firm, causing rV^u to jump above its new steady-state value on impact of the shock. In turn, the temporarily high values of θ cause some firms to delay entering until a finite time after the arrival of the productivity shock. None of these effects would be present without all three contributing factors, time-consuming hiring, bargained wages, and decreasing returns to scale production functions, so this is novel to our model.

It's also significant that labor-market tightness no longer jumps directly to the new steady state after the arrival of the shock. This is unlike the 'bang-bang' dynamics that occurred in the case of a constant returns production function in Section 6. The intuition here is that the effects mentioned in the previous paragraph cause firms to be relatively less willing to post many vacancies immediately after the shock, when wages are high because of the high value of rV^u . As rV^u , and therefore wages, fall, so too does the vacancy/unemployment ratio increase as it becomes more profitable to increase hiring.

In future work, we will modify the numerical exercise in this section to make the parameterization more closely comparable to that in Section 6.

8 Conclusion

In this paper we studied the implications of time-consuming hiring in a model in which firms wish to employ multiple workers. We showed that in steady state, this can generate both firm-size dispersion and wage dispersion. The model also provides multiple margins, via entry and by increased vacancy-posting of existing firms, for an economy to respond to productivity shocks; which of these are available, as well as the extent to which firms have decreasing returns to labor, are crucial for the qualitative and quantitative dynamics that result.

9 Appendix A: Omitted Proofs

Proof of Lemma 2. Define

$$\psi(n) = n^{-\frac{1}{\phi}} \int_0^n \nu^{\frac{1-\phi}{\phi}} y'(\nu) d\nu.,$$

Then $\psi(n) > 0$, $\psi'(n) < 0$, and $\lim_{n \rightarrow 0^+} \psi(n) = +\infty$ and $\lim_{n \rightarrow \infty} \psi(n) = 0$. The first of these claims is obvious; the second follows from writing

$$\begin{aligned} \psi'(n) &= -\frac{1}{\phi} n^{-\frac{1+\phi}{\phi}} \int_0^n \nu^{\frac{1-\phi}{\phi}} y'(\nu) d\nu + n^{-1} y'(n) \\ &= \frac{1}{\phi} n^{-\frac{1+\phi}{\phi}} \int_0^n \nu^{\frac{1-\phi}{\phi}} [y'(n) - y'(\nu)] d\nu \end{aligned}$$

which is strictly negative because of the strict concavity of $y(\cdot)$. The third claim follows because, according to (25), $\psi(n)$ is a weighted average of the values of $y'(\nu)$ on the interval $\nu \in (0, n)$, and $y(\cdot)$ satisfies an Inada condition. Finally, to see that $\psi(n) \rightarrow 0$ as $n \rightarrow \infty$, integrate by parts to obtain that

$$0 < \psi(n) = n^{-1} y(n) - \frac{1-\phi}{\phi} n^{-\frac{1}{\phi}} \int_0^n \nu^{\frac{1-2\phi}{\phi}} y(\nu) d\nu < n^{-1} y(n).$$

Since $n^{-1} y(n) \rightarrow 0$ as $n \rightarrow \infty$, the result follows by the squeeze principle.

To obtain the results concerning the profit function, note that by definition

$$\pi(n) = y(n) - n^{-\frac{1-\phi}{\phi}} \int_0^n \nu^{\frac{1-\phi}{\phi}} y'(\nu) d\nu - n(1-\phi)rV^u.$$

It follows that $\pi'(n) = \frac{1-\phi}{\phi} \psi(n) - (1-\phi)rV^u$, from which it is immediate that $\pi(\cdot)$ is strictly concave because $\psi(\cdot)$ is strictly decreasing. This shows that the maximizer of $\pi(\cdot)$ is unique in $[0, \infty)$. In fact it is strictly positive since as $n \rightarrow 0^+$, $\pi'(n) \rightarrow \frac{1-\phi}{\phi} \lim_{n \rightarrow 0^+} \psi(n) - (1-\phi)rV^u = +\infty$ according to a result already proved. To establish that $\pi(n) \rightarrow 0$ as $n \rightarrow 0^+$, use the last expression in the previous paragraph to note that

$$\pi(n) = \frac{1-\phi}{\phi} \frac{\int_0^n \nu^{\frac{1-2\phi}{\phi}} y(\nu) d\nu}{n^{\frac{1-\phi}{\phi}}} - n(1-\phi)rV^u$$

and apply L'Hôpital's rule to see the limit therefore has the same value as that of $\frac{(1-\phi)^2}{\phi^2} y(n) - n(1-\phi)rV^u$, which is zero since $y(0) = 0$. (Note that the limit of the integral in the numerator is indeed zero, so that L'Hôpital's rule can be applied, because of the Inada condition satisfied by $y(\cdot)$.) To establish that $\pi(n) \rightarrow \infty$ as $n \rightarrow \infty$, note that

$$\lim_{n \rightarrow \infty} \pi'(n) = \frac{1-\phi}{\phi} \lim_{n \rightarrow \infty} \psi(n) - (1-\phi)rV^u = -(1-\phi)rV^u < 0.$$

□

Proof of Lemma 3. First, substituting from (24) into (2) establishes that for $0 < n < n^*$, the firm's value function $J(\cdot)$ satisfies

$$(q(\theta) - sn)J'(n) = (r + \delta)J(n) + \gamma - \pi(n). \quad (81)$$

If $q(\theta) - sn^* > 0$, then the function $n \mapsto \frac{r+\delta}{q(\theta)-sn}$ satisfies a Lipschitz condition on $[0, n^*]$; it follows from Picard's theorem and associated results on ordinary differential equations that there is a unique solution to (81) on $(0, n^*)$ in this case. This solution is given by

$$J(n) = (q - sn)^{-\frac{r+\delta}{s}} \left[K + \int_0^n (q - sn)^{\frac{r+\delta}{s}-1} (\gamma - \pi(\nu)) d\nu \right] \quad (82)$$

where K is a constant of integration. It follows from the free entry condition (8) that, provided there is positive activity in the economy, $K = q^{\frac{r+\delta}{s}}k$. □

Proof of Lemma 4. The proof is analogous to that of Lemma 3. Write $T(n) = V(n) - V^u$. Then the worker's Bellman equation (4) is

$$(r + s + \delta)T(n) = w(n) + (q - sn)T'(n).$$

A boundary condition is given by the fact that at n^* , the worker is paid $w(n^*)$ until the job ends (with flow probability $\delta + s$); this implies that $V(n^*)$ satisfies

$$rV(n^*) = w(n^*) + (s + \delta)[V^u - V(n^*)],$$

or, since $w(n^*) = rV^u + (r + \delta + s)\gamma/q$,

$$T(n^*) = \frac{\gamma}{q}.$$

The usual integration argument shows that the unique closed form solution for $T(\cdot)$ is as given in the statement of the Lemma. □

Proof of Theorem 1. Define two functions $\chi, \omega : \mathbb{R}_+ \times \mathbb{R} \rightarrow \mathbb{R}$ by

$$\chi(q, rV^u) = J(n^*) \left(\frac{q - sn^*}{q} \right)^{\frac{r+\delta}{s}} - \frac{1}{q^{\frac{r+\delta}{s}}} \int_0^{n^*} (q - s\nu)^{\frac{r+\delta}{s}-1} [\gamma - \pi(\nu)] d\nu \quad (83)$$

$$\omega(q, rV^u) = rV^u - b - \theta q \left[-V^u + \frac{\int_0^{n^*} V(n)g(n) dn + \frac{sn^*}{q} V(n^*)G^*}{1 - \left(1 - \frac{sn^*}{q}\right) G^*} \right] \quad (84)$$

$$\begin{aligned} &= rV^u - b - \theta q \int_0^{n^*} \left\{ \frac{\gamma}{q} \left(\frac{q - sn^*}{q - sn} \right)^{\frac{r+\delta+s}{s}} + \frac{1 - \left(\frac{q - sn^*}{q - sn} \right)^{\frac{r+\delta+s}{s}}}{(r + \delta + s)} (1 - \phi) rV^u \right. \\ &\quad \left. + (q - sn)^{-\frac{r+\delta+s}{s}} \int_n^{n^*} (q - s\nu)^{\frac{r+\delta}{s}} \psi(\nu) d\nu \right\} d\hat{G}(n) \end{aligned} \quad (85)$$

where n^* and $J(n^*)$ are defined as in Proposition 1, according to (29) and (30), and where $\hat{G}(\cdot)$ is the probability measure with continuous density

$$\frac{g(n)}{1 - \left(1 - \frac{sn^*}{q}\right) G^*}$$

on $[0, n^*]$ and an atom of mass sn^*G^*/q at n^* . Note that $\chi(q, rV^u)$ represents the maximum value of an entrant firm that takes q and rV^u as given and expects to pay wages $w(\cdot)$ as given by (24); $rV^u - \omega(q, rV^u)$ is the value of an unemployed worker in an economy populated by such firms. As observed in the discussion preceding Proposition 1, (q, rV^u) is part of an equilibrium allocation iff

$$k = \chi(q, rV^u) \quad (86)$$

$$0 = \omega(q, rV^u). \quad (87)$$

We will show that such an intersection exists by first observing that (86) defines a continuous 1-manifold in $\mathbb{R}^+ \times \mathbb{R}$, then observing that ω restricted to this manifold defines a continuous function, showing that it takes positive and negative values, and applying the intermediate value theorem.

To see that (86) defines a continuous 1-manifold, first note that $\chi(q, rV^u)$ is nondecreasing in q and nonincreasing in rV^u , with both relationships being strict if there is positive activity (that is, if it is optimal for a firm with zero workers to hire). This follows immediately from the definition of $\chi(q, rV^u)$ as the maximal value of the problem for the firm as described. First, clearly if rV^u increases, then for any q , $\chi(q, rV^u)$ must decrease, since if the firm keeps the same hiring strategy as before, then it would increase the value of its program as $w(n)$ decreases for each n ; reoptimizing the hiring strategy can only increase this effect. Second, if q increases to $q' > q$, then the firm could keep the value of its program the same by keeping the same cutoff n^* but posting a vacancy with probability $q/q' < 1$ for

each $n < n^*$. Moreover, for each $q > 0$, it's clear that there is a unique $rV^u \in \mathbb{R}$ such that $\chi(q, rV^u) = k$. Finally, since $\chi(\cdot)$ is continuous, it follows that (86) defines a continuous curve in $\mathbb{R}^+ \times \mathbb{R}$. Call this curve C . Since it's clear that $\omega|_C$ is continuous (since ω itself is continuous), an equilibrium will exist iff we can find points $(q_1, v_1), (q_2, v_2) \in C$ such that $\omega(q_1, v_1)$ and $\omega(q_2, v_2)$ differ in sign.

To do this, first define $\bar{v}$ to solve

$$k = \frac{1}{r + \delta} \max_{n > 0} \left[y(n) - n^{-\frac{1-\phi}{\phi}} \int_0^n \nu^{\frac{1-\phi}{\phi}} y'(\nu) d\nu - n(1 - \phi)\bar{v} \right].$$

It therefore follows that $\lim_{q \rightarrow \infty} \chi(q, \bar{v}) = 0$. Also, for $v > \bar{v}$, $\chi(q, v) < k$ by construction. Now, if $q \rightarrow \infty$, then any firm will instantaneously hire n^* ; thus in the limit, $J(n) = J(n^*)$ for all $n \in [0, n^*]$. From the definition of $\omega(\cdot)$, it follows that in this case, $\omega(q, \bar{v}) = \bar{v} - b$.

In the other extreme, let $\hat{q} > 0$ satisfy $\chi(\hat{q}, b) = 0$; such a $\hat{q}$ will exist provided that $\bar{v} > b$. Suppose that this is true. Then it is clear from the definition of $\bar{v}$ that $n^*(\hat{q}, b) > 0$. Also, by definition,

$$\omega(\hat{q}, b) = -\theta \hat{q} \int_0^{n^*} [-V^u + V(n)] dG(n).$$

If q is finite, then $\theta q > 0$, and $G(\cdot)$ is a probability measure that places positive measure on every subset of $[0, n^*]$ of positive Lebesgue measure, while $V(n) - V^u > 0$ for each $n > 0$. Thus the only possibilities are that $q = +\infty$ (which is impossible since $\bar{v} \neq b$), or that $\omega(\hat{q}, b) < 0$. Thus if $\bar{v} - b > 0$, then C contains points at which ω takes values of opposite signs, which completes the proof of the existence of an equilibrium via the intermediate value theorem.

If $\bar{v} - b \leq 0$ then it is trivial to prove that there is an equilibrium in which no firm ever enters. $\square$

10 Appendix B: Heuristic Argument For the Smooth Pasting Condition

Here we present a brief heuristic argument for the form of the boundary condition for the optimal $n^*(t)$, the ‘super-contact’ condition $J_{nn}(n^*(t), t) = 0$.

Consider the case of a firm whose current employment level is n_0 , while the target level of employment at which the firm stops hiring is $n^*(t)$. Assume that $n^*(0) = n_0^*$ and that $\frac{dn^*(t)}{dt} = v + O(t)$, with $v > -s$. In this case, the optimum policy for a firm with $n_0 > n_0^*$ workers is not to hire at all and allow its workforce to fall by attrition at rate sn . (The assumption that $v > -s$ assures that the time for the workforce to fall to $n^*(t)$ occurs at some t such that the first-order approximation $n^*(t) \approx n_0^* + vt$ is valid.) Then in this case, we can deduce that since

$$n(t) = n_0 e^{-st}$$

it follows that the time taken for $n(t)$ to fall to $n^*(t)$ is the time τ satisfying that

$$n_0 e^{-s\tau} = n_0^* + v\tau$$

or

$$n_0^* + \delta n = (n_0^* + v\tau) e^{s\tau}$$

where $\delta n = n_0 - n_0^*$. Brute force calculation (optionally using the Lagrange inversion formula for power series) shows that

$$\tau = \frac{\delta n}{v + sn_0^*} - \frac{\delta n^2}{2} \frac{s^2 n_0^* + 2vs}{(v + sn_0^*)^3} + O(\delta n^3).$$

Now, in order to write an expression for $J(n, t)$ for such n , introduce the notation that the flow net profit of a firm, ignoring vacancy posting costs (that is, production less wages) is denoted $\phi(n)$. Then we can write

$$J(n_0, 0) = \int_0^\tau e^{-(r+\delta)t} \phi(n(t)) dt + e^{-(r+\delta)\tau} J(n^*(\tau), \tau),$$

while

$$J(n_0^*, 0) = \int_0^\tau e^{-(r+\delta)t} \left[\phi(n^*(t)) - \frac{\gamma}{q}(v + sn^*(t)) \right] dt + e^{-(r+\delta)\tau} J(n^*(\tau), \tau).$$

Subtracting gives that

$$\begin{aligned} J(n_0, 0) - J(n_0^*, 0) &= \int_0^\tau e^{-(r+\delta)t} \left[\phi(n(t)) - \phi(n^*(t)) + \frac{\gamma}{q}(v + sn^*(t)) \right] dt \\ &= \int_0^\tau e^{-(r+\delta)t} \left[\phi'(n) \delta n \left(1 - \frac{t}{\tau} \right) + \frac{\gamma}{q}(v + s(n_0^* + vt)) \right] dt + O(\tau^3) \\ &= \left[\delta n \phi'(n) + \frac{\gamma}{q}(v + sn_0^*) \right] \int_0^\tau e^{-(r+\delta)t} dt + \left[\frac{\gamma sv}{q} - \frac{\delta n \phi'(n)}{\tau} \right] \int_0^\tau t e^{-(r+\delta)t} dt + O(\tau^3) \\ &= \left[\delta n \phi'(n) + \frac{\gamma}{q}(v + sn_0^*) \right] \left[\tau - \frac{r+\delta}{2} \tau^2 \right] + \left[\frac{\gamma sv}{q} - \frac{\delta n \phi'(n)}{\tau} \right] \left[\frac{\tau^2}{2} - \frac{(r+\delta)\tau^3}{3} \right] + O(\tau^3) \\ &= \frac{\gamma}{q} \delta n - \frac{\delta n^2}{2(v + sn_0^*)} \left[\phi'(n) - \frac{\gamma}{q}(r + \delta + s) \right] + O(\delta n^3), \end{aligned}$$

where some algebra has been omitted, first substituting in for τ in terms of δn , and then simplifying in the last line. It follows that $J_n(n_0^*, 0) = \frac{\gamma}{q}$, the familiar condition that the firm should stop hiring once the marginal benefit of doing so equals the hiring cost. In addition, $J_{nn}(n_0^*, 0) = 0$ if and only if $\phi'(n) - \frac{\gamma}{q}(r + \delta + s) = 0$.

Next, let us consider whether a firm would like to deviate from this plan. Consider a firm that has a workforce of n_0^* workers at time 0. For n_0^* to be optimal, it must be that the firm wants to set its workforce equal to $n_*(t)$ at all $t \geq 0$ (note that this assumes that

$n^*(t)$ never moves fast enough that this is actually feasible for the firm, something that isn't obvious a priori, though it is true in the calibrated examples we consider). The obvious deviation to consider is that the firm increases its hiring by a discrete amount for a small unit of time τ , then allows its workforce to fall by attrition back to the optimal level at some $t > \tau$. Without loss of generality we can assume that the firm increases its hiring from the level required to keep on the path of $n^*(t)$, which is $v + sn^*(t)$, by some level $\delta h > 0$ such that $-sn^*(t) < v + sn^*(t) + \delta h \leq q$, for a duration $\tau > 0$. (Think of δh as 'large' and τ as 'small,' despite the notation. The inequality restriction on δh implies that hiring this amount is feasible for the firm.) Then, from time τ to time $\tau + \tau' > \tau$, the firm reduces its hiring by δh below the level required had it kept on the path of $n^*(t)$. (This will require that also $-sn^*(t) < v + sn^*(t) - \delta h \leq q$, at least for τ sufficiently small.)

To calculate the benefit to the firm of this strategy, we need first to calculate the path of the variable $\hat{n}(t)$, defined by $\hat{n}(t) = n(t) - n^*(t)$. We know that

$$\hat{n}_t(t) = \begin{cases} \delta h - s\hat{n} & t < \tau \\ -\delta h - s\hat{n} & \tau < t < \tau + \tau' \end{cases}$$

so that

$$\hat{n}(t) = \begin{cases} \frac{\delta h}{s} (1 - e^{-st}) & t < \tau \\ \frac{\delta h}{s} (-1 + (2 - e^{-s\tau})e^{-s(t-\tau)}) & \tau < t < \tau + \tau' \end{cases}$$

Since τ' is defined by $\hat{n}(\tau + \tau') = 0$, it follows that

$$e^{s\tau'} = 2 - e^{-s\tau}$$

or, solving up to second order,

$$\tau' = \tau - s\tau^2 + O(\tau^3).$$

Next, taking a first order Taylor expansion of $\phi(\cdot)$ about n_0^* , we can calculate the benefit to the firm from this strategy as equal to

$$\int_0^\tau e^{-(r+\delta)t} \left[\phi'(n_0^*)\hat{n}(t) - \frac{\gamma\delta h}{q} \right] dt + e^{-(r+\delta)\tau} \int_0^{\tau'} e^{-(r+\delta)s} \left[\phi'(n_0^*)\hat{n}(s + \tau) + \frac{\gamma\delta h}{q} \right] ds + O(\tau^3).$$

Substitute for

$$\hat{n}(t) = \begin{cases} \delta h t + O(\tau^2) & 0 < t < \tau \\ \delta h (\tau - (t - \tau)) + O(\tau^2) & \tau < t < \tau + \tau' \end{cases}$$

and write first-order expansions of the other terms under the integrals to get that the benefit

is equal to

$$\begin{aligned}
B &= \int_0^\tau (1 - (r + \delta)t) \left[\phi'(n_0^*)\delta h t - \frac{\gamma\delta h}{q} \right] dt \\
&\quad + (1 - (r + \delta)\tau) \int_0^{\tau'} (1 - (r + \delta)s) \left[\phi'(n_0^*)\delta h (\tau - s) + \frac{\gamma\delta h}{q} \right] ds + O(\tau^3) \\
&= \left[-\tau \frac{\gamma\delta h}{q} + \frac{1}{2}\phi'(n_0^*)\delta h \tau^2 + \frac{1}{2}(r + \delta) \frac{\gamma\delta h}{q} \tau^2 \right] \\
&\quad + \left[\tau \frac{\gamma\delta h}{q} + \tau^2 \delta h \left(\frac{1}{2}\phi'(n_0^*) - \frac{\gamma}{q} \left[s + \frac{3}{2}(r + \delta) \right] \right) \right] + O(\tau^3) \\
&= \delta h \tau^2 \left[\phi'(n_0^*) - \frac{\gamma}{q}(r + \delta + s) \right] + O(\tau^3).
\end{aligned}$$

Since δh can be positive or negative, it is therefore a necessary condition that the term in brackets equal zero. That is, $\phi'(n_0^*) = \frac{\gamma}{q}(r + \delta + s)$, or $J_{nn} = 0$.

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