

Endogenous matching predictions in a repeated partnership model with imperfect monitoring

(Preliminary and incomplete. Comments are welcome.)

Ayça Kaya
University of Iowa

Galina Vereschagina
Arizona State University

Abstract

This paper embeds a repeated partnership game with imperfect monitoring into a matching environment. We show that even though the underlying technology of production exhibits no complementarities with respect to types of the partners, the presence of imperfect monitoring leads to non-trivial matching predictions. In particular, if the agents' effort is complementary to their own and their partners' types (marginal products of effort are increasing in types), equilibrium matching structure is negative (i.e., the high-type agents are matched with the low-type partners). If, on the other hand, effort and type are (sufficiently) substitutable, the types are matched positively in the equilibrium.

1 Introduction

We study the model of a matching market, the participants of which are heterogeneous with respect to their levels of productivity. The productivity of each agent is public information. Once a match is formed, the partners repeatedly choose unobservable effort levels, which affect the probability of success in the current period. The outcome is publicly observable. After the outcome is observed, the partners have the option to make transfers to each other.

Since underlying production technology is assumed to exhibit no complementarities, the model delivers no matching predictions, when effort choices of partners are observable. Imperfect monitoring, however, leads to inefficient punishments, the size and frequency of which create non-trivial interaction between the types of the matched partners. This is because the equilibrium matchings maximize the total surplus and, hence, minimize inefficient punishments across matches. For this purpose, the types that lead to frequent punishments end up being matched with the types that require smaller size punishments.

Observe that in Pareto optimal equilibria punishments occur only after failure (low output). Therefore, it is always the case that more productive agents (the ones who are more likely to produce successful output) induce less frequent punishments. At the same time, the size of the punishment is related to how much the deviation by one of the partners affects the probability of success: if the change in this probability is small, the punishment should be big to deter the deviation. It is important to note that this change in the probability of success measures the marginal product of effort of the deviator. If it is increasing with the deviator's productivity (i.e. the agent's effort and productivity are complementary), the matches of more productive agents (who lead to smaller punishments) with the less productive ones (who punish more frequently) would decrease the loss in the total surplus due to imperfect monitoring. This mechanism, pushing towards negative assortative matching, would be reinforced further if the deviator's effort is also complementary to the partner's productivity. This is because the less productive partners, who, while playing the role of 'monitors', induce bigger punishments, should indeed be matched with more productive agents, who punish more frequently, in order to minimize inefficient punishments across the agents. Therefore, when agents' effort is complementary to their own as well as their partners' types, equilibrium matching is negative assortative. Notice that such complementarity would be a natural assumption if, for instance, 'type' stands for the capital stock that an agent brings to a partnership, and 'effort' is the amount of (unobservable) work hours put into operating the partnership's capital.

Alternatively, it is possible that effort can be a substitute for the agent's own as well as his partner's type. This could occur if, for example, the 'type' measures accumulated knowledge, effort helps accumulate more knowledge and the probability of success is a decreasing returns to scale production function of the cumulative knowledge. In this case, higher type agents induce larger size punishments (both as 'monitors' as well as potential deviators) and, therefore, matching them with other higher type agents who punish less frequently, would help reduce inefficient punishments across matches. On the other hand, matching two high type agents together would increase the size of the pair's punishment in case of failure, thus making positive assortative matching less likely. We show that, when the noise is sufficiently small, the second effect is weak¹, and substitutability between effort and type leads to positive assortative matching.

2 Model

We consider a one-to-one matching market for partnerships with N participants. The participants are heterogeneous with types from $\Theta \subset \mathbb{R}$. Types are public. Once a match is formed,

¹INTUITION NEEDED

the participants engage in a repeated partnership game which is described below. We assume that once a matching is formed it never breaks up, therefore the matching stage of the game is static. We consider the stable matchings of the market where the payoff possibilities of each match is given by the equilibrium payoffs of the ensuing partnership game. Formal description of the environment and definitions follow.

2.1 Technology

The partnership game played by a matched pair of types $n, m \in \Theta$ is the infinite repetition of the following stage game: at the beginning of each period each partner chooses an effort level $e_i \in \{0, 1\}$, $i = 1, 2$. Then an output $y \in Y = \{\bar{y}, 0\}$ is realized, with $\bar{y} > 0$. The choice of $e_i = 1$ entails a type-dependent cost c_n or c_m while choosing $e_i = 0$ is costless.

Effort choices and output probabilities are linked as follows:

$$(1) \quad Prob(y = \bar{y}_{nm}) = \begin{cases} p(n, m) & \text{if } e_1 = e_2 = 1 \\ q_1(m, n) & \text{if } e_1 = 1, e_2 = 0 \\ q_2(m, n) & \text{if } e_1 = 0, e_2 = 1 \\ 0 & \text{if } e_1 = e_2 = 0 \end{cases}$$

We assume $p(n, m) = p(m, n)$ and $q_1(m, n) = q_2(n, m)$. We also assume that $p(\cdot, \cdot)$, and $q_i(\cdot, \cdot)$ are twice continuously differentiable.

We make the following assumptions:

A 1 For any n and m , $(p(m, n) - q_2(m, n))\bar{y} > c_n$ and $(p(m, n) - q_1(m, n))\bar{y} > c_m$.

A 2 $p(n, m)$, $q_1(m, n)$ and $q_2(m, n)$ are (weakly) increasing in m and n and $c_m \leq c_n$ for any $m \geq n$.

A 3 $\frac{c_m}{p(m, n) - q_1(m, n)} + \frac{c_n}{p(m, n) - q_2(m, n)} > \bar{y}$.

Assumption A1 implies that exerting effort is socially optimal for each agent in any match. Assumption A2 says that an increase in the type of a partner raises the probability of high output for any combination of efforts and that high types have lower cost of exerting effort. Under assumption A 3 the output cannot be split in such a way that in the stage game it is incentive compatible for both partners to exert effort.

2.2 The partnership game

In the stage game, after the realization of the output $\bar{y}$, the partners make transfers $t_1, t_2 \in [0, \bar{y}]$. Here, t_i represents the net transfer that player i receives. We assume that no transfers are made when $y = 0$. This assumption is equivalent to limited liability: transfers cannot exceed the total output a partner is entitled to in one period. We assume that each partner is entitled to half the output. Therefore, when the output is $\bar{y}$, the total payment to player i is $\frac{\bar{y}}{2} + t_i$. We also assume that $t_1 + t_2 = 0$. That is, the sum of the ex-post payments to each of the players is equal to the total output. Throughout e refers to an effort profile (e_1, e_2) and t refers to a profile of contingent transfers t_1, t_2 .

In the repeated game, each player discounts the future with a common discount factor δ . Each period, the realized output y and the transfers t_i are publicly observable while the choice of effort by each player is not. The public outcome $h(\tau)$ in period τ consists of the realized output $y(\tau) \in Y$ and transfers $t_1(\tau), t_2(\tau) \in [-\frac{\bar{y}}{2}, \frac{\bar{y}}{2}] \times [-\frac{\bar{y}}{2}, \frac{\bar{y}}{2}]$. A public history of length τ is therefore $h^\tau = (h(1), \dots, h(\tau))$. Let $\mathcal{H}^\tau$ represent the set of all public histories of length τ and $\mathcal{H} = \bigcup_{\tau=1}^{\infty} \mathcal{H}^\tau \cup \{h_0\}$ represent the set of all public histories. Here h_0 is the null history. A pure public strategy for player i is a map $\sigma_i : \mathcal{H} \rightarrow \{0, 1\} \times \mathbb{R}^2$ that maps each public history to a stage game strategy of player i , consisting of effort choice and output-contingent transfers. We focus on the pure strategy public perfect equilibria (PSPPE) of this game.² A PSPPE is a public strategy profile $\sigma = (\sigma_1, \sigma_2)$ such that σ_1 is a best response to σ_2 and vice versa.

Let $W_{mn}(\delta)$ represent the set of PSPPE payoff vectors of the repeated partnership game for discount factor δ when the partners have types m and n , respectively. Also define the Pareto frontier of $W_{mn}(\delta)$ by

$$\mathcal{W}_{mn}^\delta(v) = \sup\{w | \exists v' \geq v \text{ such that } (v', w) \in W_{mn}(\delta)\}$$

Let $W_{mn} = \lim_{\delta \rightarrow 1} W_{mn}(\delta)$ where the limit is with respect to the Hausdorff distance. Finally, define the Pareto frontier of W_{mn} by

$$\mathcal{W}_{mn}(v) = \sup\{w | \exists v' \geq v \text{ such that } (v', w) \in W_{mn}\}$$

That is, $\mathcal{W}_{mn}(v)$ is the maximum payoff that player 2 can get among equilibria where player 1's payoff is at least v as $\delta \rightarrow 1$.

²DISCUSS THIS RESTRICTION

2.3 The matching game

The description of the matching game follows Legros and Newman [4]. As also described at the beginning of the section, the economy includes N agents who are heterogeneous with types from a compact set $\Theta \subset \mathbb{R}$. Let $\kappa : N \rightarrow \Theta$ be the type assignment; i.e. $\kappa(i)$ is the type of agent i . The type of each agent is publicly known. With an abuse of notation we use N to refer to the set of agents as well as the number of agents. In our context, a matching is a one-to-one map $M : N \rightarrow N$ such that for any $i, j \in N$, $i = M(j)$ if and only if $j = M(i)$. Each matching induces a “matching correspondence” $\mathcal{M} : \Theta \rightrightarrows \Theta$ defined by

$$\mathcal{M}(m) = \{n | \exists i, j \in N \text{ with } \kappa(i) = m, \kappa(j) = n, i \in M(j)\}$$

By positive assortative matching (PAM) we mean a matching M that induces a matching correspondence $\mathcal{M}$ that satisfies:

$$\forall m, n, m', n' : \text{ if } m > n; m' \in \mathcal{M}(m); \text{ and } n' \in \mathcal{M}(n) \text{ implies } m' \geq n'.$$

By negative assortative matching (NAM) we mean a matching M that induces a matching correspondence $\mathcal{M}$ that satisfies:

$$\forall m, n, m', n' : \text{ if } m > n; m' \in \mathcal{M}(m); \text{ and } n' \in \mathcal{M}(n) \text{ implies } m' \leq n'.$$

The achievable utility pairs of a match between two agents of type n and m , respectively, is described by the function $\mathcal{W}_{mn}(\cdot)$ introduced in the previous subsection. Therefore, the payoff pair (v, w) is feasible for a pair i, j if $w \leq \mathcal{W}_{\kappa(i)\kappa(j)}(v)$.

An equilibrium of the matching game is a matching M and a payoff assignment $v^* : N \rightarrow \mathbb{R}$ such that (1) for all i, j with $i \in M(j)$, $(v^*(i), v^*(j))$ is feasible given their types; and (2) the matching is stable: for any $i, j \in N$, there exists no feasible payoff vector w such that $w(i) > v^*(i)$ and $w(j) > v^*(j)$.

3 Characterization of equilibrium payoffs of the partnership game

Once a matching is formed, the partners in a match play a repeated partnership game described in the previous section. As is well-known in this setting, the set of equilibrium payoff vectors is difficult to characterize. However, it is possible to bound the equilibrium payoff set using techniques introduced in [3]. The following proposition is a direct application of [3]’s result to

our setting:

Proposition 1 *Define*

$$(2) \quad \eta(m, n) = \frac{c_m - (p(m, n) - q_2(m, n))\bar{y}/2}{p(m, n) - q_2(m, n)} + \frac{c_n - (p(m, n) - q_1(m, n))\bar{y}/2}{p(m, n) - q_1(m, n)}$$

For any $m \geq n$, let

$$(3) \quad \begin{aligned} \mathcal{W}_{mn}(v) = & -v + p(m, n)\bar{y} - c_m - c_n \\ & - \min\{ (1 - p(m, n))\eta(m, n) \ , \ (p(m, n) - q_1(m, n))\bar{y} - c_n \} \end{aligned}$$

For any $\varepsilon > 0$ there exists $\bar{\delta}_{mn}(\varepsilon) < 1$ such that for any $\delta > \bar{\delta}_{mn}(\varepsilon)$ and for all v :

$$\mathcal{W}_{mn}^\delta(v) \in (\mathcal{W}_{mn}(v) - \varepsilon, \mathcal{W}_{mn}(v)]$$

Proof: See appendix.

Equation (3) describes the Pareto frontier of a set that bounds the equilibrium payoff vectors. The rest of Proposition 1 states that the true Pareto frontier of the equilibrium payoff vectors converges to this bound as the discount factor δ approaches 1. The derivation of this frontier is included in the Appendix. The proof of the convergence result follows from [3] with minor modifications.

Note that for any pair of types m, n , the limit frontier $\mathcal{W}_{mn}$ has slope -1. Therefore, the sum of payoffs for the two players for any payoff vector on the Pareto frontier is fixed. For the rest of the analysis it is convenient to denote this sum by $S(m, n)$. That is

$$(4) \quad \begin{aligned} S(m, n) = & p(m, n)\bar{y} - c_m - c_n \\ & - \min\{ (1 - p(m, n))\eta(m, n) \ , \ (p(m, n) - q_1(m, n))\bar{y} - c_n \} \end{aligned}$$

This is the limit as $\delta \rightarrow 1$ of the maximum surplus obtainable in a match between two partners of types m and n .

For comparison, note that if there were no moral hazard (perfect monitoring) it would be possible to use trigger strategies to implement effort by both agents at no cost and therefore the total surplus would be given by:

$$(5) \quad S^*(m, n) = p(m, n)\bar{y} - c_m - c_n.$$

When monitoring is imperfect, implementing effort by both agents is costly. This cost is given by $(1-p(m,n))\eta(m,n)$. When this cost is too high it is too expensive to make both agents work and therefore the surplus is maximized by having only the most productive agent to exert effort. In this case the loss in surplus with respect to the first best is given by $(p(m,n)-q_1(m,n))\bar{y}-c_n$.

The expression $(1-p(m,n))\eta(m,n)$ which is the loss in total surplus when implementing effort by both agents has a very intuitive interpretation, as suggested by [1]. It is the “expected monitoring cost”: inefficient punishments of size $\eta(m,n)$ occur when low output is realized, i.e. with frequency $(1-p(m,n))$. The size of the punishment $\eta(m,n)$ is the sum of the punishments required to satisfy the incentive constraints of each agent. For instance, if there are no transfers, the size of the punishment for agent of type m is

$$\eta_1(m,n) = \frac{c_m - (p(m,n) - q_2(m,n))\bar{y}/2}{p(m,n) - q_2(m,n)}$$

The expression in the numerator is the instantaneous gain obtained by agent type m from deviating. On the other hand, the denominator measures by how much the frequency of punishment increases in case of deviation. Therefore, the optimal punishment is such that the expected cost of punishment is equalized with the expected gain.

4 Equilibrium matching patterns

It is well known that in a matching model with transferable utility positive (negative) assortative matching obtains in equilibrium when the surplus function has increasing (decreasing) differences. While the limit frontier $\mathcal{W}_{mn}(v)$ has slope -1 (and, therefore, utility is transferable), this is not necessarily true for the Pareto frontiers when $\delta < 1$. In this Section we first establish that when δ is sufficiently close to 1, for positive (negative) assortative matching to obtain in equilibrium it is enough that the limit frontier satisfies strict increasing (decreasing) differences. Then in the rest of the Section we derive sufficient conditions under which $\mathcal{W}_{mn}(v)$ has this property.

For the rest of the paper we make the following assumptions:

A 4 $\frac{\partial^2 p(m,n)}{\partial m \partial n} = 0$, $\frac{\partial^2 q_1(m,n)}{\partial m \partial n} = 0$ and $\frac{\partial^2 q_2(m,n)}{\partial m \partial n} = 0$

A 5 For any m , $\frac{\partial p(m,n)}{\partial n} - \frac{\partial q_1(m,n)}{\partial n} > 0$ and for any n , $\frac{\partial p(m,n)}{\partial m} - \frac{\partial q_1(m,n)}{\partial m} \geq 0$.

A 6 For any m , $\frac{\partial p(m,n)}{\partial n} - \frac{\partial q_1(m,n)}{\partial n} < 0$ and for any n , $\frac{\partial p(m,n)}{\partial m} - \frac{\partial q_1(m,n)}{\partial m} \leq 0$.

Assumption A4 guarantees that there is no inherent complementarity or anti-complementarity in the production technology. Recall that in the model with perfect monitoring the total surplus is given by (5). Therefore, Assumption A4 implies that the model with perfect monitoring would not generate any matching predictions.

Assumptions A5 and A6 describe the interaction between the agents' efforts and types. Recall that $p(m, n) - q_1(m, n)$ measures the decline in the probability of success when the second partner deviates and, therefore, is the marginal product of the deviating partner's effort. Then, under assumption A5, effort is complementary to type: it is more productive when the agent is of higher type or is matched with a higher type partner. In this case, one can interpret 'type' as capital that the agent brings to the match and effort is labor (where capital and labor have some degree of complementarity, as in Cobb-Douglas production function). In contrast, if assumption A6 holds, effort substitutes for the agent's type. This could happen, for instance, if 'type' represents the amount of accumulated knowledge, effort is exerted to generate more knowledge accumulation and the probability of success is a decreasing returns to scale function of total knowledge.

In the results below we will assume that either A5 or A6 holds. Note, however, that we do not allow the agent's effort to be complementary to his own type and substitutable for the partner's type (or vice versa).³

Definition 1 *A function $f : \Theta \times \Theta \rightarrow \mathbb{R}$ exhibits increasing (decreasing) differences if for all $m > m', n > n'$:*

$$f(m, n) + f(m', n') - f(m, n') - f(m', n) > (<) 0$$

Remark 1 [5] proves the following result: A twice continuously differentiable function $f : \Theta \times \Theta \rightarrow \mathbb{R}$ exhibits increasing (decreasing) differences if and only if $\frac{\partial^2 f(m, n)}{\partial m \partial n} > 0$ ($\frac{\partial^2 f(m, n)}{\partial m \partial n} < 0$).

Lemma 1 *Assume $S(m, n)$ exhibits strict increasing (decreasing) differences. Then there exists $\bar{\delta} < 1$ such that for all $\delta > \bar{\delta}$, all equilibrium matchings are positive (negative) assortative.*

Proof: Since for $\delta < 1$ the utility is not necessarily transferable, we use the sufficient conditions developed in [4] (namely, generalized increasing or decreasing differences) that guarantee positive (negative) assortative matching in equilibrium in non-transferable case.

Let $T = \{\kappa(i) | i \in N\}$. That is T is the set of types of the N agents in the economy. First

³If we choose to do this, we still can provide a set of results about the matching patterns, based on the same intuition.

assume increasing differences, i.e.

$$(S(m, n) - S(m, n')) - (S(m', n) - S(m', n')) > 0$$

Let $\varepsilon = \inf\{S(m, n) + S(m', n') - S(m, n') - S(m', n) | m > m', n > n' \in T\}$ and $\varepsilon^* = \frac{1}{4}\varepsilon$. And let $\bar{\delta} = \max\{\bar{\delta}_{mn}(\varepsilon^*) | m, n \in T\}$. Since T is finite, $\varepsilon^* > 0$ and $\bar{\delta} < 1$. First we show that for any $\delta > \bar{\delta}$ the following holds:⁴

For any $m > m', n > n'$ and v, v' :

$$\mathcal{W}_{m'n}^\delta(v) = \mathcal{W}_{m'n'}^\delta(v') \Rightarrow \mathcal{W}_{mn}^\delta(v) > \mathcal{W}_{mn'}^\delta(v')$$

To see this, take v, v' such that $\mathcal{W}_{m'n}^\delta(v) = \mathcal{W}_{m'n'}^\delta(v')$. Then by Proposition 1,

$$-(\mathcal{W}_{m'n}(v) - \mathcal{W}_{m'n'}(v')) = -(v - v' + S(m', n) - S(m', n')) < \frac{1}{2}\varepsilon$$

and

$$\mathcal{W}_{mn}(v) - \mathcal{W}_{mn'}(v') = v - v' + S(m, n) - S(m, n') < \frac{1}{2}\varepsilon + \mathcal{W}_{mn}^\delta(v) - \mathcal{W}_{mn'}^\delta(v')$$

Therefore,

$$\begin{aligned} \mathcal{W}_{mn}^\delta(v) - \mathcal{W}_{mn'}^\delta(v') &> -\frac{1}{2}\varepsilon + v - v' + S(m, n) - S(m, n') > \dots \\ \dots &> -(v - v' + S(m', n) - S(m', n')) + v - v' + S(m, n) - S(m, n') = \dots \\ \dots &= S(m, n) - S(m, n') - (S(m', n) - S(m', n')) > 0 \end{aligned}$$

where the last inequality is from increasing differences. This is the generalized increasing differences conditions introduced in [4]. And therefore the result follows. The argument for the case with decreasing differences is analogous. $\square$

In the light of this lemma, we need to analyze whether $S(m, n)$ has increasing or decreasing differences in order to understand the equilibrium matching patterns. Recall that when both agents are to exert effort $S(m, n)$ is equal to the first best surplus $S^*(m, n)$ net of the cost of incentive provision $(1 - p(m, n))\eta(m, n)$. Observe that the types of the partners affect both the frequency $(1 - p(m, n))$ and the size $\eta(m, n)$ of inefficient punishment. As we show below, this potentially creates non-trivial interaction between the types of the partners, thereby generating increasing or decreasing differences in the match's surplus $S(m, n)$. The following two Propositions establish that when both agents work, the equilibrium matching is positive (negative) assortative depending on whether the marginal product of agent's effort is complementary to his own and his partner's types or substitutes for them.

⁴This is the generalized increasing differences condition derived in [4].

Proposition 2 *Suppose that assumptions A1-A5 hold and $(p(m, n) - q_1(m, n))\bar{y} - c_n \geq (1 - p(m, n))\eta(m, n)$, for all m, n . Then, there exists $\bar{\delta} < 1$ such that for all $\delta > \bar{\delta}$, all equilibrium matchings are negative assortative.*

Proof: By Lemma 1, it suffices to show that $\mathcal{W}_{mn}(v)$ has strictly decreasing differences in (m, n) . First, recall that

$$\mathcal{W}_{mn}(v) = -v + p(m, n)\bar{y} - c_m - c_n - (1 - p(m, n))\eta(m, n)$$

whenever $(p(m, n) - q_1(m, n))\bar{y} - c_n < (1 - p(m, n))\eta(m, n)$. Thus, by Assumption A4, it suffices to verify that

$$(6) \quad \frac{\partial^2(1 - p(m, n))\eta(m, n)}{\partial m \partial n} > 0 \quad \text{for all } m \text{ and } n.$$

Recall that

$$\eta(m, n) = \frac{c(m) - (p(m, n) - q_2(m, n))\bar{y}/2}{p(m, n) - q_2(m, n)} + \frac{c(n) - (p(m, n) - q_1(m, n))\bar{y}/2}{p(m, n) - q_1(m, n)},$$

which can be simplified as

$$\eta(m, n) = \frac{c(m)}{p(m, n) - q_2(m, n)} + \frac{c(n)}{p(m, n) - q_1(m, n)} - \bar{y}$$

Since $p(m, n) = p(n, m)$ and $q_2(m, n) = q_1(n, m)$, (6) holds when $\frac{1-p(m, n)}{p(m, n)-q_1(m, n)}c(n)$ satisfies strict increasing differences. For brevity, drop the reference to m and n , the subindex in q_1 and denote the corresponding partial derivatives by p_m , p_n , q_m and q_n . Then

$$(7) \quad \begin{aligned} \frac{\partial^2 \frac{1-p}{p-q}c}{\partial m \partial n} &= \frac{1}{(p-q)^3} \left[p_m(p_n - q_n) + p_n(p_m - q_m) + 2(p_m - q_m)(p_n - q_n) \frac{1-p}{p-q} \right] c + \\ &+ \frac{1-p}{(p-q)^2} \left[q_m - p_m \frac{1-q}{1-p} \right] c'(n) \end{aligned}$$

The above expression is strictly positive since, $p_n - q_n > 0$ and $p_m - q_m \geq 0$ (by assumption A5), $c'(n) < 0$ (by assumption A2) and $p > q$ implying that $\frac{1-q}{1-p} > 1$. Hence, $\mathcal{W}_{mn}(v)$ has strictly decreasing differences in (m, n) . $\square$

The intuition behind Proposition 2 is as follows. Equilibrium matching pattern maximizes the total surplus. Hence, it minimizes the sum of the costs of incentive provision $(1 - p(n, m))\eta(m, n)$ across matched pairs. On the one hand, the frequency of inefficient punishments $1 - p(n, m)$ is decreasing in the types of both partners. On the other hand, the size of the punishment $\eta(m, n)$ is inversely related to $p(m, n) - q_1(m, n)$, which is increasing

in m and n by assumption A5. The monotonicity in n means that higher types are “better monitors” because the deviations of their partners have larger effects on the probability of success and, therefore, are more easily detectable. By the same token, the monotonicity of $p(m, n) - q_1(m, n)$ in m implies that higher types are easier to monitor.

Since the frequency and the size of the punishments are simultaneously influenced by the types of both partners, the effect of the interaction of the two types on $S(m, n)$ can be decomposed into four parts:

- First, the types that are harder to monitor (i.e. needing larger size punishments) should be matched with the types that lead to less frequent punishments. This is the effect of the monitored agent’s own type on $\eta(m, n)$ and his partner’s type on $(1 - p(m, n))$.
- Second, the types that are better monitors (i.e. who reduce the size of necessary inefficient punishment) should be matched with the types that lead to more frequent punishments. This is the effect of the monitored agent’s own type on $(1 - p(m, n))$ and his partner’s type on $\eta(m, n)$.
- Third, the types that are harder to monitor should be matched with better monitors. This is the effect of both types on $\eta(m, n)$.
- Fourth, the types that lead to more frequent punishments should be matched with types that lead to less frequent punishments. This is the effect of both types on $1 - p(m, n)$.

Under assumption A5, the first three effects simultaneously push the equilibrium matching towards negative assortative, because higher types are better monitors, are easier to monitor and lead to less frequent punishments. The fourth effect, however, does not play any role because, by assumption A4, there is no complementarity in $p(m, n)$.⁵

Notice that for the result in Proposition 2 to obtain it is crucial that higher types are better monitors and are easier to monitor. If, however, this assumption is reversed (i.e., A5 is replaced with A6), the first and second effects push towards positive assortative matching while the third effect continues to push towards negative assortative matching. Therefore, if the degree of substitutability between effort and type is sufficiently large, positive assortative matching could obtain in equilibrium. Proposition 3 gives sufficient conditions for this to happen.

⁵In the proof of Proposition 2 the first three effects correspond (in the same order) to the three terms in the right hand side of (7).

Proposition 3 *Suppose that assumption A1-A4 and A6 hold. Suppose also that $c_m = c > 0$ for all m . Then, there exists $\bar{\delta} < 1$ such that for all $\delta > \bar{\delta}$, all equilibrium matchings are positive assortative if and only if the following two conditions hold for any m and n :*

$$(i) \quad (p(m, n) - q_1(m, n))\bar{y} - c_n \geq (1 - p(m, n))\eta(m, n)$$

$$(ii) \quad 2 \frac{\partial(p(m, n) - q_1(m, n))}{\partial m} \frac{\partial(p(m, n) - q_1(m, n))}{\partial n} \frac{1 - p(m, n)}{p - q_1(m, n)} < - \frac{\partial p(m, n)}{\partial m} \frac{\partial(p(m, n) - q_1(m, n))}{\partial n} - \frac{\partial p(m, n)}{\partial n} \frac{\partial(p(m, n) - q_1(m, n))}{\partial m}$$

Proof: By Lemma 1, it suffices to verify that $\mathcal{W}_{mn}(v)$ has strictly increasing differences in (m, n) , which is true when $(1 - p(m, n))\eta(m, n)$ has strictly decreasing differences. Condition (ii) in the Proposition stipulates that $\frac{\partial^2 \frac{1-p}{p-q}}{\partial m \partial n}$ defined in (7) is strictly negative, implying strict decreasing differences of $(1 - p(m, n))\eta(m, n)$. $\square$

Observe that condition (ii) in Proposition 3 holds if the ratio $\frac{1-p(m, n)}{p-q_1(m, n)}$ is sufficiently close to 0, i.e. when the amount of noise is sufficiently small. Denote by $l(m, n) = \frac{1-q(m, n)}{1-p(m, n)}$ the likelihood ratio. Then the following Corollary follows directly from Proposition 3:

Corollary 1 *Suppose that assumption A1-A4, A6 hold and for, any m and n $(p(m, n) - q_1(m, n))\bar{y} - c_n \geq (1 - p(m, n))\eta(m, n)$. Then, there exist $\bar{\delta} < 1$ and $\underline{l} > 1$ such that for all $\delta > \bar{\delta}$, all equilibrium matchings are negative assortative provided that $l(m, n) \leq \underline{l}$ for all m and n (i.e. the noise is sufficiently small in all possible matches).*

Notice that the results in Propositions 2 and 3 hold only if $(p(m, n) - q_1(m, n))\bar{y} - c_n \geq (1 - p(m, n))\eta_{mn}$. That is, when the monitoring technology is good enough. In this case, implementing effort by both partners is not too costly. If, however, $(p(m, n) - q_1(m, n))\bar{y} - c_n < (1 - p(m, n))\eta_{mn}$, then it is too expensive to provide incentives for both partners to work. Instead, it becomes optimal to make only the more productive agent exert effort. In this case, positive assortative matching cannot be an equilibrium matching, because if a higher type agent is matched with another higher type agent, one of them will be idle. Negative assortative matching would be one of possible equilibrium outcomes. In fact any matching where the top half of the types are randomly matched with the bottom half of the types will be an equilibrium outcome.

Example: Linear Case

Assumption A4 implies that the functions $p(m, n)$, $q_1(m, n)$ and $q_2(m, n)$ must be separable in m and n . Suppose that they are linear in m and n , i.e.

$$(8) \quad \begin{aligned} p(m, n) &= p(m + n) + a \\ q_1(m, n) &= q_1 m + q_2 n + b, \end{aligned}$$

where $p, q_1, q_2 \geq 0$. Assume that m and n can take values in $[0, 1]$. Then $a \in [0, \frac{1}{2p})$ and $b \in [0, \frac{1}{q_1+q_2})$ to guarantee that the probability of success is between 0 and 1.

By assumption A5, it must be that $p > q_2$. By Proposition 2, for the equilibrium matching to be negative assortative it suffices that $p \geq q_1$. As an extreme example, consider the case with $q_1 = p$, $q_2 = 0$ and $a = b = 0$, i.e. the probabilities of high output are simply the sums of the individual contributions of the working agents. In this case the types are sorted negatively in the equilibrium.

On the other hand, if $p < q_1$, positive assortative matching may result in the equilibrium. One natural question is how large should q_1 be in order to lead to positive sorting in equilibrium. It is straightforward to verify that if $q_1 - p > p - q_2$, positive matching occurs. In this case, however, it is necessary that $a > b$ in order to guarantee that $p(m, n) > q_1(m, n)$.

5 Discussion

Different definitions of type: if cost of effort is the same across types, the results continue to hold. If productivities are the same and cost of effort is the only difference across types, then, not surprisingly, no matching predictions. Note that there would be complementarity even in the latter case, if we did not allow asymmetric punishments in the partnership game because it would be impossible to bind the incentive constraints of each of asymmetric partners.

Different supports of output: higher type may mean that “success”=higher output.

Non-transferability: we think that non-transferability will act in favor of positive matching. [Why?]

Break-ups??

The role of dynamics as opposed to money burning and commitment.

Substitutability between my effort and my type.

Symmetric punishments (PAM?).

Relation to a static model.

6 Appendix: Characterizing the Pareto frontier of the repeated partnership game payoffs

In what follows, for economy of notation we drop reference to types m, n when obvious from context.

6.1 Bounding the equilibrium payoff set: methodology

In the [3] characterization, the bounding set that is shown to be the limit of the set of equilibrium value vectors is the intersection of the largest half-spaces in each direction whose boundary values can be decomposed on these hyperplanes. A half space $H(\lambda, k)$ with direction $(\lambda, 1 - \lambda)$ and level k is the set $\{v \in \mathbb{R}^2 | \lambda v_1 + (1 - \lambda)v_2 \leq k\}$. Also for brevity, define $u_i(e, t) = -c_i e_i + E\{\frac{y}{2} + t_i(y) | e\}$ where $E\{\cdot | e\}$ represents the expectation taken with respect to y using the distribution induced by the effort profile e .

The following definition is adopted from [2] and [3] for our setting:

Definition 2 *A value vector $v = (v_1, v_2)$ is decomposable on a set $W \in \mathbb{R}^2$ if there exists an effort profile e , transfers t and continuation value vectors $\gamma(y) \in W$ for each $y \in \{\bar{y}, \underline{y}\}$ such that*

$$(PK) \quad v_i = (1 - \delta)[u_i(e, t) + \delta E\{\gamma(y) | e\}]$$

$$(IC) \quad v_i \geq (1 - \delta)[u_i(e'_i, e_{-i}, t) + \delta E\{\gamma_i(y) | e'_i, e_{-i}\}] \text{ for any } e'_i \in \{0, 1\}$$

The first condition is the promise keeping condition: it guarantees that the current payoff and the expected continuation payoff average to v . The second condition is the standard incentive compatibility condition. Strictly speaking, the definition should also include the conditions stipulating that the transfers are also incentive compatible. That is,

$$(9) \quad \forall y : (1 - \delta)t_i(y) + \delta \gamma_i(y) \geq 0$$

The constraint takes this form because the transfers are observable and deviations can be punished by switching to the worst equilibrium with payoffs $(0, 0)$. Notice that as $\delta \rightarrow 1$, (9) becomes $\gamma_i(y) \geq 0$. Since we are characterizing the limit case, in what follows, we ignore this constraint keeping in mind that $v, w \geq 0$.

The largest half-space in direction λ whose boundary values can be decomposed on itself by effort profile e and transfers t is $H(\lambda, k^*(\lambda, e, t))$ where $k^*(\lambda, e, t)$ is characterized by the following linear programming problem:

$$(10) \quad k^*(\lambda, e, t) = \max_x \lambda[u_1(e, t) + E_y\{x_1(y)|e\}] + (1 - \lambda)[u_1(e, t) + E_y\{x_2(y)|e\}]$$

subject to

$$u_1(e, t) + E_y\{x_1(y)|e\} \geq u_i(e'_i, e_{-i}, t) + E_y\{x_i(y)|e'_i, e_{-i}\}$$

$$\lambda x_1(y) + (1 - \lambda)x_2(y) \leq 0$$

This can be seen by noting that if the continuation values $x_i(y)$ are obtained via the normalization $x_i(y) = (\gamma_i(y) - v) \frac{\delta}{1-\delta}$ from unnormalized continuation values $\gamma_i(y)$, the first constraint is equivalent to the condition (IC) , the objective function is nothing but $\lambda v_1 + (1 - \lambda)v_2$ for some $v = (v_1, v_2)$ for which (PK) is satisfied. Finally, the last constraint guarantees that the unnormalized continuation values γ come from the hyperplane $H(\lambda, k^*(\lambda, e, t))$. Define

$$k^*(\lambda) = \max_{e, t} k^*(\lambda, e, t)$$

Therefore, $H(\lambda, k^*(\lambda))$ is the largest half space in direction λ . Now, define the set

$$W = \bigcap_{\lambda} H(\lambda, k^*(\lambda))$$

The rest of this section is devoted to characterizing the set W for our repeated partnership game.

6.2 Characterizing $k^*(\lambda, EE, t)$

The linear programming problem described in the previous section becomes:

$$\begin{aligned}
k^*(\lambda, EE, t) = & \max_{x_1, x_2} \lambda \left(\frac{p\bar{y}}{2} - c_1 + pt_1 + px_1(\bar{y}) + (1-p)x_1(\underline{y}) \right) + \dots \\
& \dots + (1-\lambda) \left(\frac{p\bar{y}}{2} - c_2 + pt_2 + px_2(\bar{y}) + (1-p)x_2(\underline{y}) \right) \\
\text{subject to: } & \frac{1}{p-q_2} \left(c_1 - \frac{(p-q_2)\bar{y}}{2} \right) - t_1 \leq x_1(\bar{y}) - x_1(\underline{y}) \\
& \frac{1}{p-q_1} \left(c_2 - \frac{(p-q_1)\bar{y}}{2} \right) - t_2 \leq x_2(\bar{y}) - x_2(\underline{y}) \\
& \lambda x_1(\underline{y}) + (1-\lambda)x_2(\underline{y}) \leq 0 \quad \text{for all } y \in Y
\end{aligned}$$

Denote the left hand side of the IC constraint for agent i by L_i . That is,

$$L_1 = \frac{1}{p-q_2} \left(c_1 - \frac{(p-q_2)\bar{y}}{2} \right) - t_1.$$

$$L_2 = \frac{1}{p-q_1} \left(c_2 - \frac{(p-q_1)\bar{y}}{2} \right) - t_2.$$

To characterize the solution to this problem, it is convenient to distinguish between two separate cases,

$$(11) \quad \lambda L_1 + (1-\lambda)L_2 \leq 0$$

and

$$(12) \quad \lambda L_1 + (1-\lambda)L_2 > 0.$$

If λ and t satisfy the first inequality then it is possible to choose $x_1(\cdot), x_2(\cdot)$ in such a way that $\lambda x_1(\bar{y}) + (1-\lambda)x_2(\bar{y}) = \lambda x_1(\underline{y}) + (1-\lambda)x_2(\underline{y}) = 0$ and incentive constraints are satisfied. If (11) holds with strict inequality, at least one of the incentive constraints will be slack. Therefore, in this case

$$(13) \quad k^*(\lambda, EE, t) = \lambda \left(\frac{p\bar{y}}{2} - c_1 + pt_1 \right) + (1-\lambda) \left(\frac{p\bar{y}}{2} - c_2 + pt_2 \right)$$

Thus (11) implies that orthogonal implementation is possible.

On the other hand, if λ and t are such that (12) holds, then

$$\lambda x_1(\underline{y}) + (1 - \lambda)x_2(\underline{y}) < \lambda x_1(\bar{y}) + (1 - \lambda)x_2(\bar{y})$$

and it would be optimal to choose a combination of $x_1(\bar{y})$ and $x_2(\bar{y})$ that satisfies $\lambda x_1(\bar{y}) + (1 - \lambda)x_2(\bar{y}) = 0$. Obviously, $\lambda x_1(\underline{y}) + (1 - \lambda)x_2(\underline{y}) = 0$ is not feasible any more, and in the optimal solution both incentive constraints must bind. Therefore

$$(14) \quad k^*(\lambda, EE, t) = \lambda \left[\frac{p\bar{y}}{2} - c_1 + pt_1 - (1 - p)L_1 \right] + (1 - \lambda) \left[\frac{p\bar{y}}{2} - c_2 + pt_2 - (1 - p)L_2 \right].$$

Note that if (11) holds with equality then both incentive constraints must bind and therefore equations (14) and (13) deliver the same value.

The next step is to choose the transfer profile $t^*(\lambda)$ that maximizes the level of the hyperplane in the direction λ when EE is the effort profile. This is done in the following Lemma.

Lemma 2 *Define*

$$(15) \quad t_1^*(y) = -t_2^* = \begin{cases} \frac{\bar{y}}{2} & \text{if } \lambda \geq \frac{1}{2} \\ -\frac{\bar{y}}{2} & \text{if } \lambda < \frac{1}{2} \end{cases}$$

Then $t^* \in \mathbf{argmax}_t k^*(\lambda, EE, t)$.

Proof: The result follows from observing that t^* defined in (2) maximizes (13) and minimizes the left hand side of (11). $\square$

This Lemma implies that for large δ and when v, w is on the frontier, one of the agents receives all of the current output if both agents work. Therefore, $k^*(\lambda, EE)$ can be expressed as follows:

$$(16) \quad k^*(\lambda, EE) = \begin{cases} \lambda(p\bar{y} - c_1) + (1 - \lambda)(-c_2) - (1 - p) \max\{0, \eta_1(\lambda)\} & \text{if } \lambda > \frac{1}{2} \\ \lambda(-c_1) + (1 - \lambda)(p\bar{y} - c_2) - (1 - p) \max\{0, \eta_2(\lambda)\} & \text{if } \lambda \leq \frac{1}{2} \end{cases}$$

where

$$\eta_1(\lambda) = \lambda \frac{c_1 - (p - q_2)\bar{y}}{p - q_2} + (1 - \lambda) \frac{c_2}{p - q_1}$$

and

$$\eta_2(\lambda) = \lambda \frac{c_1}{p - q_2} + (1 - \lambda) \frac{c_2 - (p - q_1)\bar{y}}{p - q_1}$$

Note that $\eta_1(\lambda)$ and $\eta_2(\lambda)$ are the values of $\lambda L_1 + (1 - \lambda)L_2$ evaluated at the corresponding

optimal t 's. The term $\max\{0, \eta_1(\lambda)\}$ in equation (16) becomes positive when the condition (12) holds: ICs are binding and orthogonal implementation is not possible.

Correspondingly, the hyperplane associated with the optimal transfer schedule for $\lambda \geq \frac{1}{2}$ [the hyperplane $\lambda v + (1 - \lambda)w = k^*(\lambda, EE)$] passes through either A_1 or B_1 defined below—namely the one which delivers a lower level in this direction λ .⁶

$$(17) \quad \begin{aligned} A_1 : & \quad (p\bar{y} - c_1, -c_2) \\ B_1 : & \quad \left(p\bar{y} - c_1 - \frac{1-p}{p-q_2}(c_1 - (p-q_2)\bar{y}), -c_2 - \frac{1-p}{p-q_1}c_2 \right) \end{aligned}$$

For $\lambda \leq \frac{1}{2}$ the corresponding hyperplane passes through the lower one of the following two points:

$$(18) \quad \begin{aligned} A_2 : & \quad (-c_1, p\bar{y} - c_2) \\ B_2 : & \quad \left(-c_1 - \frac{1-p}{p-q_2}c_1, p\bar{y} - c_2 - \frac{1-p}{p-q_1}(c_2 - (p-q_1)\bar{y}) \right) \end{aligned}$$

6.3 Characterizing $k^*(\lambda, ES)$ and $k^*(\lambda, SE)$

For the effort profile ES and transfers t , the linear programming problem (10) can be written as

$$(19) \quad \begin{aligned} k^*(\lambda, ES, t) = \mathbf{argmax}_x \quad & \lambda[q_1(\bar{y} + t_1) - c_1 + q_1x_1(\bar{y}) + (1 - q_1)x_1(\underline{y})] + \dots \\ & \dots(1 - \lambda)[q_1(\bar{y} + t_2) + (1 - q_1)t_2 + q_nx_2(\bar{y}) + (1 - q_n)x_2(\underline{y})] \end{aligned}$$

subject to

$$\begin{aligned} x_1(\bar{y}) - x_1(\underline{y}) & \geq \frac{1}{q_1} \left(c_1 - \frac{q_1\bar{y}}{2} \right) - t_1 \\ x_2(\bar{y}) - x_2(\underline{y}) & \leq \frac{1}{p - q_1} \left(c_1 - \frac{(p - q_1)\bar{y}}{2} \right) - t_2 \\ \lambda x_1(y) + (1 - \lambda)x_2(y) & \leq 0; \quad y \in \{\underline{y}, \bar{y}\} \end{aligned}$$

If there were no incentive constraints, it would always be possible to enforce (ES, t) orthogonally, which would deliver

$$(20) \quad k^*(\lambda, ES, t) = \lambda[-c_n + q_1(\bar{y} + t_1)] + (1 - \lambda)[q_1(\bar{y} + t_1)].$$

⁶By this we mean, the inner product $(\lambda, 1 - \lambda) \times (v, w)$ is minimized.

Since the incentive constraints in (19) bound $x_1(\bar{y}) - x_1(\underline{y})$ from below and $x_2(\bar{y}) - x_2(\underline{y})$ from above, it is always possible to choose such x that $\lambda x_1(y) + (1 - \lambda)x_2(y) = 0$ for any y and both incentive constraints are satisfied.⁷ Thus (20) is also a solution to the linear program (19).

To maximize $k^*(\lambda, ES, t)$ with respect to t we need to set $t_1 = -t_2 = -\frac{\bar{y}}{2}$ if $\lambda < \frac{1}{2}$ and $t_1 = -t_2 = \frac{\bar{y}}{2}$ otherwise. Therefore,

$$k^*(\lambda, ES) = \begin{cases} \lambda[q_1\bar{y} - c_1] & \text{if } \lambda \geq \frac{1}{2} \\ \lambda(-c_1) + (1 - \lambda)q_1\bar{y} & \text{otherwise} \end{cases}.$$

The level of the largest half-space in direction λ decomposed on itself by SE can be straightforwardly determined in a similar way:

$$k^*(\lambda, SE) = \begin{cases} \lambda q_2\bar{y} + (1 - \lambda)(-c_2) & \text{if } \lambda \geq \frac{1}{2} \\ (1 - \lambda)[q_2\bar{y} - c_2] & \text{otherwise} \end{cases}.$$

6.4 Characterizing $k^*(\lambda)$

For each λ the level of largest half space in the direction λ is found as

$$(21) \quad k^*(\lambda) = \max\{k^*(\lambda, EE), k^*(\lambda, ES), k^*(\lambda, SE)\}$$

It is convenient to first characterize $\max\{k^*(\lambda, ES), k^*(\lambda, SE)\}$ and then compare it with $k^*(\lambda, EE)$.

For $\lambda \geq \frac{1}{2}$ the hyperplane $\lambda v + (1 - \lambda)w = k^*(\lambda, ES)$ passes through D_1 and hyperplane $\lambda v + (1 - \lambda)w = k^*(\lambda, SE)$ passes through G_1 defined as follows:

$$(22) \quad \begin{aligned} D_1 : & \quad (q_1\bar{y} - c_1, 0) \\ G_1 : & \quad (q_2\bar{y}, -c_2) \end{aligned}$$

For $\lambda \leq \frac{1}{2}$ the corresponding hyperplanes pass through D_2 and G_2 :

$$(23) \quad \begin{aligned} D_2 : & \quad (-c_1, q_1\bar{y}) \\ G_2 : & \quad (0, q_2\bar{y} - c_2) \end{aligned}$$

It is easy to see that for $m > n$ the points G_1, G_2 lie below the line connecting the points D_1

⁷We only need to make sure that $x_1(\bar{y}) - x_1(\underline{y})$ and $x_2(\bar{y}) - x_2(\underline{y})$ are sufficiently far away from each other.

and D_2 .⁸ Therefore, for $\lambda \geq \frac{1}{2}$, the hyperplane passing through D_1 should be chosen and for $\lambda \leq \frac{1}{2}$ the hyperplane passing through D_2 should be chosen. This is intuitive because it implies that whenever only one of the agents works it is the more efficient one.

In the light of this discussion, we get that whenever $m > n$:

$$k^*(\lambda) = \max\{k^*(\lambda, EE), k^*(\lambda, ES)\}.$$

6.5 Characterization of $\mathcal{W}(\cdot)$

Recall that

$$\mathcal{W} = \bigcap_{\lambda} \{(v, w) | \lambda v + (1 - \lambda)w \leq k^*(\lambda)\}$$

Proposition 4 *Define*

$$\eta = \frac{c_1 - (p - q_2)\bar{y}/2}{p - q_2} + \frac{c_2 - (p - q_1)\bar{y}/2}{p - q_1}$$

Then,

$$\mathcal{W}(v) = -v + p\bar{y} - c_1 - c_2 - \min\{2(1 - p)\eta, (p - q_1)\bar{y} - c_2\}$$

Proof: First, notice that $\overline{A_1A_2}$, $\overline{B_1B_2}$ and $\overline{D_1D_2}$ all have slopes -1. Also, all points $A_1, A_2, B_1, B_2, D_1, D_2$ lie outside of the positive orthant. Next, observe that $\overline{A_1A_2}$ lies above $\overline{B_1B_2}$ by assumption A??, and by assumption A1, $\overline{A_1A_2}$ lies above $\overline{D_1D_2}$. Finally, observe that $p\bar{y} - c_m - c_n - 2\eta$ and $q_1\bar{y} - c_1$ are the levels of $\overline{B_1B_2}$ and $\overline{D_1D_2}$, respectively. $\square$

6.6 Proof of Proposition 1

The following proposition reproduces the result of FLM [3] that $\mathcal{W}$ is the limit of the equilibrium payoff set as δ converges to 1.

Proposition 5 (FLM [3]) *Let $V \subset \text{int}W$ be smooth⁹ and convex. Then there exists $\bar{\delta}$ such that for any $\delta > \bar{\delta}$, $V \subset W(\delta)$.*

⁸To see this note that the line connecting D_1 and D_2 has slope -1 and level $f(1, 0) - c_m$ while the lines through G_1 and G_2 with slope -1 have levels $f(0, 1) - c_n < f(1, 0) - c_m$.

⁹A smooth set is closed, with non-empty interior and its boundary is twice continuously differentiable??.

Proof: We note that the Pareto frontier of the repeated partnership game is obtained using $(t_1, t_2) \in \{(-\bar{y}/2, \bar{y}/2), (\bar{y}/2, -\bar{y}/2)\}$. That is, restricting attention to equilibria that use only these transfers does not shrink the equilibrium set. Therefore, the proof directly follows from [3] [Theorem ???]. $\square$

Corollary 2 (Proposition 1) *Define*

$$\eta(m, n) = \frac{c_m - (p(m, n) - q_2(m, n))\bar{y}/2}{p(m, n) - q_2(m, n)} + \frac{c_n - (p(m, n) - q_1(m, n))\bar{y}/2}{p(m, n) - q_1(m, n)}$$

For any $m \geq n$, let

$$\begin{aligned} \mathcal{W}_{mn}(v) = & -v + p(m, n)\bar{y} - c_m - c_n \\ & - \min\{ (1 - p(m, n))\eta(m, n) \ , \ (p(m, n) - q_1(m, n))\bar{y} - c_n \} \end{aligned}$$

For any $\varepsilon > 0$ there exists $\bar{\delta}_{mn}(\varepsilon) < 1$ such that for any $\delta > \bar{\delta}_{mn}(\varepsilon)$ and for all v :

$$\mathcal{W}_{mn}^\delta(v) \in (\mathcal{W}_{mn}(v) - \varepsilon, \mathcal{W}_{mn}(v)]$$

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