Solving Dynamic Models with Heterogeneous Agents and Aggregate Uncertainty with Dynare or Dynare++

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Abstract

This paper shows how models with heterogeneous agents and aggregate uncertainty can be solved using Dynare or Dynare++ software that implements a perturbation approach. When the explicit aggregation algorithm (XPA) is used to obtain aggregate laws of motion, this can be accomplished by combining a Dynare program with a very simple Matlab program. When the Krusell-Smith algorithm is used, then the Matlab program needed is somewhat more involved, but still relatively simple. We calculate and compare 1^{st} and 2^{nd}-order numerical solutions using both algorithms. These numerical procedures are also compared with the algorithm that solves the individual policy rules with a projection instead of a perturbation procedure. Finally, we discuss a procedure that efficiently chooses which cross-sectional moments to include as aggregate state variables when nonlinearities are important and the mean is not a sufficient statistic.

Key Words: numerical solutions, projection methods
JEL Classification: C63, D52

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Introduction

The most populair model in macroeconomics is the New Keynesian model. There are many quite sophisticated versions of this model, but most of them rely on the representative agent paradigm. If there are heterogeneous agents in the model, then the degree of heterogeneity is limited, for example, by just having two representative agents, like a patient and an impatient agent. The assumptions needed to justify a representative agent framework, like perfect insurance of idiosyncratic risk, are believed to be unrealistic. Even if there are representative agent models that can describe the aggregate economy well, they obviously cannot explain any distributional aspects, although these are of interest to many economists. Fortunately, there are now numerous papers in which heterogeneity and idiosyncratic risk play a key role. There are even quite a few papers in which there is both heterogeneity and aggregate risk, a combination that makes it more cumbersome to obtain a numerical solution. Nevertheless, the overwhelming majority of papers still relies on models with a representative agent.

An important reason for the ongoing dominance of models with a representative agent is without doubt the availability of user friendly software, like Dynare, to solve and estimate these types of models. In principle Dynare can handle many state variables and, thus, should be able to solve models with multiple agents. The problem is that in macroeconomic models we would like to have millions of agents and such a large number of agents is problematic, even for Dynare.[1] The standard assumption is to approximate these millions of agents with a continuum of agents, but Dynare requires a finite number.

In this paper, we show how to use Dynare to solve models with (i) a continuum of heterogeneous agents and (ii) aggregate risk. Two things are needed. First, a "mother" program is needed to solve for the laws of motion that describe the aggregate variables taking as given the solution of the individual policy rules. Second, a Dynare program is needed to solve for the individual policy rules taking as given the aggregate law of motion.

[1] An interesting question is whether using a relatively small number, say 100, is enough to approximate a macroeconomy with millions of agents. It would be cumbersome to use Dynare to solve models with 100 agents, but this is likely to be feasible at least for some models and for lower-order perturbation solutions.

This Dynare program, the *.mod file, needs to be able to read the coefficients of the aggregate law of motion. We show that this can be accomplished with an easy procedure for a Dynare program and with a somewhat more cumbersome procedure for a Dynare++ program.

Preston and Roca (2006) were the first to realize that one can use perturbation techniques to solve models with a continuum of heterogeneous agents and aggregate risk. den Haan and Rendahl (2009) develop the XPA algorithm with which aggregate laws of motion are obtained by explicitly aggregating the individual policy rules, even when the individual policy rules are not linear. In the algorithm of den Haan and Rendahl (2009), the individual policy rules can be obtained with any algorithm. In this paper, we show that if the individual policy rules are obtained with perturbation techniques, that the XPA solution is identical to the one obtained by the algorithm of Preston and Roca (2006). The advantage of the XPA algorithm is the simplicity and transparency and this approach is used here to write the programs.

What is required to be able to implement XPA with Dynare? The Dynare program itself, that solves for the individual laws of motion would be almost a standard Dynare program. However it differs from a standard Dynare program in two aspects. First, as explained above it needs to be able to incorporate the values of the coefficients of the aggregate laws of motion. The second modification is related to the feature of XPA that to be able to explicitly aggregate higher-order individual policy functions you need auxiliary policy rules. For example, if the model has a policy rule for the capital choice, k, then one needs an auxiliary policy rule for k^2 if a second-order perturbation solution is used. But this would just boil down to adding one variable and one equation to the standard Dynare file. The mother program would be very simple. There are two parts to the mother program. First, it would contain a few steps of algebra to get the coefficients of the aggregate laws of motion from the individual laws of motion. Second, it has to contain a procedure to make the laws of motion of the aggregate variables consistent with the laws of motion of the individual variables. For example, one can start with an aggregate policy rule, solve for the individual policy rule using perturbation techniques, obtain the aggregate law of motion by explicit aggregation of the individual policy rule, and iterate until the aggregate law of motion has converged. But there are better, more efficient, ways of obtaining the consistency and we'll discuss how to write the mother program to implement them.

We use this algorithm to generate first and second-order solutions. We assess the accuracy and compare the solutions with those obtained with alternative algorithms. The first alternative algorithm uses the same Dynare program to solve for the individual policy rules, but obtains the aggregate law using the simulation procedure proposed by Krusell and Smith (1998) instead of explicit aggregation. The mother program is then a bit more complicated, because it has to include a simulation procedure, but the structure of the program remains the same. We also consider an algorithm in which the aggregate laws of motion are still solved by explicit aggregation, but the individual policy rules are obtained using a projection method. Given that there is no standard software (yet) to implement projection methods, this procedure is obviously much more involved than any of the others considered.

The main contributions of this paper are (*i*) to establish the link between the algorithm of Preston and Roca (2006) and the XPA algorithm of den Haan and Rendahl (2009), (*ii*) describe how Dynare and Dynare++ can be used to implement these algorithms, and (*iii*) compare the different numerical solutions and access their accuracy. Finally, there is one more important issue that is addressed in this paper and that describing a procedure to limit the number of state variables. In several models considered so far, the computational burden turns out to be low, because only a small set of moments, in fact typically just the mean, is sufficient to capture the predictive information in the distribution, a property referred to as *approximate aggregation*.[2] This is not likely to be a universal property. Key for approximate aggregation to hold is that the individual policy functions are close to being linear in the relevant part of the state space.[3] Linear policy rules are unlikely to be a realistic property in general and it is (hopefully) only a matter of time in which we will

[2]See Krusell and Smith (2006).

[3]For example, nonlinearities do not matter for the very poor agents because their behavior is not important for the aggregate anyway.

use models in which nonlinearities matter for aggregation. But if nonlinearities matter, then additional moments have to be used as state variables. Strictly following the logic of the XPA algorithm implies that each non-linear basis term used in the individual policy function gives rise to an additional cross-sectional moment in the set of state variables. We show how to reduce the dimension of this set by using combinations of these state variables. This contribution is likely to be more important when the individual policy rules are solved with a projection procedure than with a perturbation procedure given that it is tricky to keep the cost of projection methods low with a large number of state variables.

1 The Model

In this section we describe the model which is a standard heterogeneous agents model with aggregate productivity shocks similar to the one discussed in Aiyagari (1994), Bewley (undated) and Huggett (1993). It describes a simple exchange economy with incomplete markets, aggregate uncertainty and an infinite number of agents. The source of heterogeneity comes from the assumption that there are idiosyncratic income shocks, which partially uninsurable.

Problem for the individual agent The economy is represented by a stochastic growth model with a continuum of individuals indexed by $i \in [0, 1]$. The individual agents are characterized by facing an idiosyncratic unemployment risk. All agents are ex ante identical, however there is ex post heterogeneity due to incomplete insurance markets and borrowing constraints. Every quarter individuals differ from each other through their asset holdings and employment opportunities. In order to transfer their ressources over time, agents can only control their capital holdings. We simplify the model with respect to the one described in Preston and Roca (2006), and we assume that an employed agent earns a wage rate of w, while an unemployed agent has no income. However agents can insure themselves, at least partially, against employment risks by building up a capital stock. To insure satisfaction of intertemporal budget constraints, capital holdings are restricted by a borrowing limit $b \geq 0$, ensuring the repayment of loans and the absence of Ponzi schemes.

Agent i's maximization problem is given by

$$\max_{\{c_{it}, a_{i,t+1}\}} E_t \sum_{t=0}^{\infty} \beta^t \frac{c_{it}^{1-\gamma} - 1}{1-\gamma}$$
$$\text{s.t. } c_{it} + a_{i,t+1} = r(k_t, l_t, z_t)a_{it} + w(k_t, l_t, z_t)e_{it}\bar{l} + (1-\delta)a_{it} \tag{1}$$
$$a_{it} \geq b$$

The utility function is characterized by a standard constant relative risk aversion (CRRA) utility function with risk aversion parameter $\gamma > 0$. This function is twice continuously differentiable, increasing, and concave in the level of consumption of household i, c_{it}. Let a_{it} be the agent's beginning-of-period asset holdings, $a_{i,t+1}$ is the next period level of asset constrained, $\bar{l}$ is the time endowment, β the subjective discount rate, $0 < \delta < 1$ is the depreciation rate. r_t and w_t is the interest rate and wage rate, respectively.

Each agent faces partially insurable labor market income risk and is endowed with one unit of time. This endowment is transformed into labor input according to $l_{it} = e_{it}\bar{l}$. The stochastic employment opportunity e_{it} follows an autoregressive process of the form

$$e_{i,t+1} = (1-\rho_e)\mu_e + \rho_e e_{it} + \varepsilon^e_{i,t+1} \text{ , } \varepsilon^e_t \sim \mathcal{N}(0, \sigma^2_e) \tag{2}$$

where $0 < \rho_e < 1$, $\mu_e > 0$ and $\varepsilon^e_{i,t+1}$ a bounded *i.i.d.* disturbance with mean and variance $(0, \sigma^2_e)$.

Problem for the firm Markets are competitive and the production technology of the firm is characterized by a Cobb-Douglas production function. Let k_t and l_t stand for per capita capital and the employment rate, respectively. Per capita aggregate output takes as inputs the aggregate capital stock and labor supply

$$y_t = z_t k_t^{\alpha} l_t^{1-\alpha}$$

In the original model, as described in Krusell and Smith (1998), aggregate productivity, z_t, is an exogeneous stochastic process that can take on two values, which are interpreted as the good and bad state of the economy. For reasons that will be described later, we transform the discrete space to a continuous support by assuming that the aggregate technology shock z_t , common to all households, satisfies

$$z_{t+1} = (1-\rho_z)\mu_z + \rho_z z_t + \varepsilon^z_{t+1} \tag{3}$$

where $0 < \rho_z < 1$, $\mu_z > 0$ and ε^z_{t+1} a bounded *i.i.d.* disturbance with mean and variance $(0, \sigma^2_z)$. We assume that firms rent their factors of production from households in competitive spot markets. These aggregate inputs imply market interest and wage rates equal to

$$r(k_t, l_t, z_t) = \alpha z_t \left(\frac{k_t}{l_t}\right)^{\alpha-1} \tag{4}$$

$$w(k_t, l_t, z_t) = (1-\alpha) z_t \left(\frac{k_t}{l_t}\right)^{\alpha} \tag{5}$$

In order to solve the optimization program given by expression (1) agents must forecast future prices. Under the maintained assumptions (l_t, z_t) follow an exogeneous stochastic process. Therefore in order to forecast future wage and rental rates, agents must know the stochastic process that describes the evolution of the aggregate capital stock. However, the stochastic properties of the aggregate capital stock depend on the distribution of capital holdings in the population. As a consequence, the whole capital distribution becomes a state variable. In a setup with a continuum of agents, capital distribution is a function, which cannot be used as an argument of the individual policy rules. Krusell and Smith (1998) propose to summarize this distribution by a discrete and finite set of moments. For instance, if we take into consideration only the first order moments, the law of motion for aggregate capital, k_{t+1}, is given by

$$k_{t+1} = \zeta_0(s) + \sum_{i=1}^{I} \zeta_i(s) M(i) + \zeta_{I+1} z_t \tag{6}$$

where $M(i)$ is the cross-sectional average of a^i with $k = M(1)$, and s is a vector containing the aggregate state variables.

Equilibrium An equilibrium for this benchmark model then consists of the following

- *Optimality :* given (4), (5), and (6) the household decision rules solve the maximization problems given in expression (1).

- *Factor prices :* wage and rental rates are factor marginal productivities and are determined by expressions (4) and (5), respectively.

- *Aggregation :* Factor inputs are generated by aggregation over agents with $k_t \equiv \int a_{it} di$ and $l_t \equiv \int l_{it} di$. A transition law for the cross-sectional distributions of capital, that is consistent with the investment policy function. Let k_t represent the beginning-of-period aggregate capital with the following transition law

$$k_{t+1} = \zeta_0(s) + \sum_{i=1}^{I} \zeta_i(s) M(i) + \zeta_{I+1} z_t$$

This law of motion reveals an advantage of working with a continuum of agents. Because we apply a law of large numbers, we know that conditional on z_t, there is no uncertainty in determining k_{t+1}

Penalty Function Perturbation methods cannot directly be applied to models with occasionally-binding inequality constraints. In order to deal with the nonnegativity constraint for asset holdings, we replace the borrowing constraint by a penalty function (Judd (1998)). The basic idea of this approach is that we allow anything to be feasible, but we change the objective function such that it has undesirable consequences if the constraints are violated. By using this approach we have converted the original problem, given in expression (1) into an optimization problem with only equality constraints, which allows us to apply a standard perturbation method. Recent contributions that use this approach for heterogeneous agents models include Kim, Kollmann, and Kim (2009) and Preston and Roca (2006). There are many ways to represent the penalty function. We focus on three specifications from the recent literature on heterogeneous agents models. We will use the specification described in de Wind (2008)

$$\mathbf{P}^{dW}(a_{i,t+1}) = \frac{\eta_1}{\eta_0} \exp\big(- \eta_0 (a_{i,t+1} + b)\big) - \eta_2 (a_{i,t+1} + b) \tag{7}$$

Alternative specification can be found in Preston and Roca (2006)

$$\mathbf{P}^{PR}(a_{i,t+1}) = \frac{1}{(a_{i,t+1}+b)^2} - \eta(a_{i,t+1}+b) \tag{8}$$

or in Kim, Kollmann, and Kim (2009)

$$\mathbf{P}^{KKK}(a_{i,t+1}) = \log\left(\frac{a_{it+1}+b}{\bar{a}+b}\right) - \frac{a_{it+1}-\bar{a}}{\bar{a}} \tag{9}$$

where $\bar{a}$ is the steady state value of a_{it}.

These functions have the property that as individual asset holdings approach the borrowing constraint b the interior function approaches infinity.

The first derivative with respect to $a_{i,t+1}$ of these functions are given by

$$p^{dW}(a_{i,t+1}) = -\eta_1 \exp\big(-\eta_0(a_{i,t+1}+b)\big) - \eta_2 \tag{10}$$

$$p^{PR}(a_{i,t+1}) = -\frac{2}{(a_{i,t+1}+b)^3} - \eta \tag{11}$$

$$p^{KKK}(a_{i,t+1}) = \frac{\bar{a}+b}{a_{i,t+1}+b} - \frac{1}{\bar{a}} \tag{12}$$

The modified model In order to solve this model by perturbation, we need to transform the optimization problem, given in expression (1), such that we get rid of the borrowing constraint. In fact the borrowing constraint is the reason of non-differentiability. One way of avoiding this problem is to replace the borrowing constraint with a penalty function, such that the individual problem will look like

$$\max_{\{c_{it},a_{i,t+1}\}} E_t \sum_{t=0}^{\infty} \beta^t \frac{c_{it}^{1-\gamma}-1}{1-\gamma} - \phi\mathbf{P}(a_{i,t+1})$$

$$\text{s.t. } c_{it} + a_{i,t+1} = r(k_t, l_t, z_t)a_{it} + w(k_t, l_t, z_t)e_{it}\bar{l} + (1-\delta)a_{it} \tag{13}$$

This maximization problem faced by an individual in the economy can be represented as a dynamic programming problem in which a_{it}, e_{it}, Γ_t and z_t are the state variables and

c_{it}, a_{it+1} are the decision variables. The optimality equation for this model is given by

$$\begin{aligned}
V(a_i, e_i; z, \Gamma) \quad &= \quad \max_{\{c_i, a_i'\}} \left\{ \frac{c_i^{1-\gamma} - 1}{1-\gamma} + \beta E\big[V(a_i', e_i'; z', \Gamma') - \phi \mathbf{P}(a_i')\big] \right\} \\
\text{s.t} \quad & (1-\delta)a_i + r(k,l,z)a_i + w(k,l,z)e_i\bar{l} - c_i - a_i' \geq 0 \\
& z' = (1-\rho_z)\mu_z + \rho_z z + \varepsilon'^z \\
& e_i' = (1-\rho_e)\mu_e + \rho_e e_i + \varepsilon_i'^e \\
& \Gamma' = H(\Gamma, z)
\end{aligned} \tag{14}$$

where function $V(\cdot)$ is the value function of a type i household, r and w are functions that describe the factor prices, and Γ denotes the distribution of capital holdings in the population.

When we use a first order approximation, the problem can be represented as follows[4]

$$\begin{aligned}
V(a_i, e_i; z, k) \quad &= \quad \max_{\{c_i, a_i'\}} \left\{ \frac{c_i^{1-\gamma} - 1}{1-\gamma} + \beta E\big[V(a_i', e_i'; z', k') - \phi \mathbf{P}(a_i')\big] \right\} \\
\text{s.t} \quad & (1-\delta)a_i + r(k,l,z)a_i + w(k,l,z)e_i\bar{l} - c_i - a_i' \geq 0 \\
& z' = (1-\rho_z)\mu_z + \rho_z z + \varepsilon'^z \\
& e_i' = (1-\rho_e)\mu_e + \rho_e e_i + \varepsilon_i'^e \\
& k' = \zeta_0 + \zeta_1 k + \zeta_2 z
\end{aligned} \tag{15}$$

[4]We have to keep in mind that simply substituting Γ by k is an abuse of notation, since the real model always depends on the evolution of the whole distribution Γ.

Similarly, the optimality equation for the second order approximated model is

$$
\begin{aligned}
V(a_i, e_i; z, k, M_{ae}, M_{a^2}) \quad &= \quad \max_{\{c_i, a_i'\}} \left\{ \frac{c_i^{1-\gamma} - 1}{1-\gamma} + \beta E\big[V(a_i', e_i'; z', k', M_{ae}', M_{a^2}') - \phi \mathbf{P}(a_i')\big] \right\} \\
\text{s.t} \quad & (1-\delta)a_i + r(k, l, z)a_i + w(k, l, z)e_i\bar{l} - c_i - a_i' \geq 0 \\
& z' = (1-\rho_z)\mu_z + \rho_z z + \varepsilon'^z \\
& e_i' = (1-\rho_e)\mu_e + \rho_e e_i + \varepsilon_i'^e \\
& k' = \zeta_0 + \zeta_1 k + \zeta_2 z + \zeta_3 M_{ae} \\
& \qquad + \zeta_4 M_{a^2} + \zeta_5 k^2 + \zeta_6 z^2 + \zeta_7 kz \\
& M_{a^2}' = \bar{\zeta}_0 + \bar{\zeta}_1 k + \bar{\zeta}_2 z + \bar{\zeta}_3 M_{ae} \\
& \qquad + \bar{\zeta}_4 M_{a^2} + \bar{\zeta}_5 k^2 + \bar{\zeta}_6 z^2 + \bar{\zeta}_7 kz \\
& M_{ae}' = \tilde{\zeta}_0 + \tilde{\zeta}_1 k + \tilde{\zeta}_2 z + \tilde{\zeta}_3 M_{ae} \\
& \qquad + \tilde{\zeta}_4 M_{a^2} + \tilde{\zeta}_5 k^2 + \tilde{\zeta}_6 z^2 + \tilde{\zeta}_7 kz
\end{aligned}
\tag{16}
$$

2 Parameterization

The parameterization of the benchmark model is standard but we will briefly remind it in this section. The time period is one quarter. The intertemporal discount factor β is set equal to 0.99 and the depreciation rate δ to 0.1. Most studies estimate the relative risk aversion parameter γ to be between one and two. We choose for the benchmark model a value for γ equal to 1. For this reason we include different values within this range in our sensitivity analysis. The share of capital α is 0.33. The normalizing constant $\bar{l}$ is set equal to 1. The aggregate technology shock is specified by $\mu^z = 1$, $\rho^z = 0.9$ and $\sigma^z = 0.01$ to correspond the two state Markov process close to the one adopted in Krusell and Smith (1998). As it was mentionned earlier the law of motion for individual's employment status is transformed to continuous support, as shown in equation (2). This allows for labor market conditions to depend on the aggregate state. The individual's employment status is specified as $\mu^e = 1$, $\rho^e = 0.95$ and $\sigma^e = 0.005$.

The analysis assumes agents are constraint to hold positive quantities of assets, such that the borrowing limit is set to $b = 0$. In other words, the credit limit is set at 0, which means that agents are not allowed to hold net debt. The parameter ϕ controls the sensitivity to the borrowing constraint in the modified utility function and it is set to $\phi = 0.05$. This ensures that no agent violates the borrowing constraint.

The parameterization given in Table 4 is used to check whether the individual problem, which has been modified with a penality function, behaves in the same way as the original model.

3 Solution Methods

Since there is no analytical solution to this model, we have to rely on numerical approximation methods. We give a general overview of the algorithm used for solving the present model. We start by assuming that the agents only use the first I moments of the wealth distribution in order to perceive current and future prices. This means that agents have access to the law of motion for the capital stock. Given this law of motion, each agent can compute its optimal choice. The iterative procedure used to approximate this aggregate law of motion will be decribed in the next section.

The general algorithm can be described as follows: (1) Select the order I of the moment, (2) choose a functional form for the aggregate law of motion, (3) solve the individual problem, (4) use the individual decision problem to update the aggregate law of motion given in step (2), (5) iterate until convergence.

In order to study the sensitivity of the algorithm to the choice of various methods, we solve the individual problem using two methods. Furthermore we compare the updating procedure of the aggregate law of motion using two different approaches. These procedures will be explained in more detail in the following sections.

The overall algorithm is summarized in Figure 1 and the estimated parameters, the ζ's, are presented in Table 12.

3.1 Solving the individual problem

Projection Method In the present version of the model, agents differ only in the realizations of the shocks that determine their employment status. As a consequence the policy rules for each agent will be the same. The first step in the projection method is to define a bounded state space. Let $\mathcal{P}_n(x)$ denote a polynomial of degree n on the vector x. In the projection method we replace next periods asset holding, a_{it+1}, by a function of the state variables of the agent that is, $\mathcal{P}_n(a, e, z, k; \Theta_n)$. We will choose $\mathcal{P}_n(\cdot)$, and Θ_n, the vector of parameters, to make the marginal utility of consumption $c(a, e, z, k; \Theta_n)^{-\gamma}$ as close as possible to the conditional expectation. Note that $\mathcal{P}_n(a, e, z, k; \Theta_n)$ has been used to compute $c(a, e, z, k; \Theta_n)^{-\gamma}$. Since the objective of this paper is to compare two numerical approximation methods up to order $n = 2$, we can use ordinary polynomials.

After fixing n, we need to find Θ_n. The algorithm starts with an initial guess of Θ_n, Θ_n^0. Starting from Θ_n^0, we choose a new Θ_n in order to minimize the residual function

$$\begin{aligned} R(\cdot, \Theta_n) \quad \equiv \quad & c(a, e, z, k; \Theta_n)^{-\gamma} \\ & -\beta E_t\left[c'(a', e', z', k'; \Theta_n)^{-\gamma}\left(r'(a', e', z', k'; \Theta_n) + 1 - \delta\right) + p(a')\right] \end{aligned} \tag{17}$$

The expectation is computed using a deterministic integration method, more specifically the Gauss-Hermite method Abramowitz and Stegun (1964). In general the algorithm starts with a first-order polynomial and then adds higher-order terms until the results do not change anymore. However we have decided to stop at order $n = 2$, as the current version of Dynare is able to do second order approximations only[5] .

Perturbation Method In the perturbation method we compute the deterministic steady state of the model and then we use an nth order Taylor expansion around this value.

Similarly, perturbation methods are typically performed on the optimality conditions, which are given by

$$E_t f(y_{t+1}, y_t, y_{t-1}, \epsilon_t) = 0$$

where $\epsilon_t = \sigma e_t$ and $E(\epsilon_t) = 0$, $E(\epsilon_t \epsilon_t') = \sigma^2 \Sigma_e$, and $E(\epsilon_t \epsilon_\tau') = 0$ for $t \neq \tau$. The policy function of this model is given by

$$y_t = h(y_{t-1}, \epsilon_t, \sigma)$$

[5]For higher-order perturbation methods, we could use Dynare++

The principle differs with respect to the projection method in several points. For instance first order perturbation, linearizes the system around a point of the state space such as the steady state of the economy. Next we take first order Taylor expansions around this steady state value, which requires calculations of the first derivatives of the functions h and f. First order perturbation methods have been widely used since the work of Blanchard and Kahn (1980). The method is described in King and Watson (1998), Klein (2000) and Sims (2002).

First order perturbation methods are fast and widely used. However, this approximation may be insufficient when analyzing the impact risk has on the behaviour of the economy.

For this reason we may want to solve the model using second order approximations. This method may produce locally accurate approximations for capturing the dynamics of the model without ignoring the nonlinear features of it. In this case we compute a second order Taylor expansion of the model. The method is described in Schmitt-Grohé and Uribe (2004)

First and second order perturbation methods are implemented in the open-source software Dynare. For higher order perturbation methods, Dynare++ is available.

3.2 Updating the Law of Motion

Krusell and Smith (1998): Simulation and regression The basic idea of the Krusell and Smith (1998) algorithm relies in summarizing the cross-sectional distribution of capital and employment status with a limited set of moments. The algorithm specifies a law of motion for these moments and finds the approximating function using a simulation procedure. Given a set of individual policy rules, a time series of cross-sectional moments is generated and new laws of motion for the aggregate moments are estimated using the simulated data.

However, this iterative procedure relies on simulation which generates two types of sampling variation. The first is due to the fact of using a finite instead of a continuum of agents. The second one is due to the aggregate shock.

Furthermore, since the method relies on simulated data to obtain numerical solutions, it has two disadvantages. First, by introducing sampling noise the policy functions themselves become stochastic. This effect can be reduced by using long time series, but sampling noise disappears at a slow rate. Second, the values of the state variables used to find the best fit for the aggregate law of motion are endogeneous and are typically clustered around their mean.

den Haan and Rendahl (2009): Explicit aggregation An alternative approach is to obtain the law of motion, which describes aggregate behaviour, by explicitly aggregating the individual policy rule. This approach is simpler than the methods that rely on parameterization of the cross-sectional distribution and it is not computationally intensive as the methods that rely on simulation and regression.

In order to present the algorithm, we consider, for the moment, the first order approximation of a model with individual and aggregate uncertainty. Consider the following simple model in which all agents are identical except of their initial capital stock, a_{i0}, and their employment status, e_{it}. We parameterize the individual policy function as

$$a_{i,t+1} = \theta_0 + \theta_1 a_{it} + \theta_2 e_{it} + \theta_3 z_t + \theta_4 k_t \tag{18}$$

From equation (18) it can be seen that the individual policy rule is assumed to be a polynomial of order 1 in a, e, z and k. Using monomials as in equation (18) simplifies the exposition, den Haan and Rendahl (2009) show that other basis functions can be used as well, such as orthogonal polynomials or B-splines.

The key step in heterogeneous agents is to establish a law of motion for aggregate capital, k. Given the expression in equation (18), this aggregate law of motion for k follows directly from aggregating the individual policy rule. That is,

$$\begin{aligned} k_{t+1} &= \theta_0 + \theta_1 \int a_{it} di + \theta_2 \int e_{it} di + \theta_3 z_t + \theta_4 k_t \\ &= (\theta_0 + \theta_2) + (\theta_1 + \theta_4) k_t + \theta_3 z_t \end{aligned} \tag{19}$$

Note that, in order to apply this method, we need an expression for the average level of the capital stock and not, for example, for the average of the log capital stock. Consequently, the left-hand side of equation (18) has to be equal the level of a_{it+1}, as it is the case in our example.

Equation (19) is obtained in a pretty straightforward way, since we have only first order terms in equation (18). As soon as we have higher order polynomials in equation (18), we will have higher order cross-sectional moments in equation (19). In other words, we need to include higher order terms as inputs for predicting k_{t+1}.

However this last issue raises additional problems, since it implies that including higher-order cross-sectional moments requires additional aggregate laws of motions to predict those. This is due to the fact that they will appear as arguments in next period's policy function. In order to illustrate this, let us take the second order approximation of the individual policy function, which is given by

$$\begin{aligned} a_{it+1} &= \mathcal{P}_2(a_{it}, e_{it}, z_t, k_t, M_{ae}, M_{a^2}; \Theta) \\ &= \theta_0 + \theta_1 e_{it} + \theta_3 a_{it} + \theta_4 a_{it} e_{it} + \theta_5 a_{it}^2 + \theta_6 z_t + \theta_7 e_{it} z_t + \theta_8 a_{it} z_t \\ &\quad + \theta_9 z_t^2 + \theta_{10} k_t + \theta_{11} e_{it} k_t \theta_{12} a_{it} a_{it} k_t + \theta_{13} z_t k_t + \theta_{14} k_t^2 \\ &\quad + \theta_{15} M_{ae,t} + \theta_{16} M_{a^2,t} \end{aligned} \tag{20}$$

In equation (20), we see the appearance of second order terms. For this particular model, we are especially interested in a_{it}^2 and $a_{it}e_{it}$. As it was noted previously, the presence of these terms requires aggregate laws of motions to predict these moments.

One way to get a policy rule for a_{it+1}^2 is to use the one that is implied by the approximation of a_{it+1} given in equation (18). However by taking the square of equation (18), we will end up with a polynomial of order higher than 2, which means that additional moments would have to be added. Then additional policy rules would be needed to predict these additional moments, which in turn would introduce more state variables. Without modification, a solution based on explicit aggregation requires including an infinite number of moments as state variables whenever the order of approximation is higher than one.

The key approximating step of the algorithm described in den Haan and Rendahl (2009) is to break this infinite regress problem and to construct separate approximations to the policy rules for a_{it+1}^2 and $a_{it+1}e_{it+1}$.

For a_{it+1}^2, we use an approximation, which has the same form as equation (20)

$$a_{it+1}^2 = \mathcal{P}_2(a_{it}, e_{it}, z_t, k_t, M_{ae}, M_{a^2}; \bar{\Theta}) \tag{21}$$

For $a_{it+1}e_{it+1}$, we have to be more careful

$$\begin{aligned} a_{it+1}e_{it+1} &= a_{it+1}\big((1-\rho_e) + \rho_e e_{it} + \varepsilon_{it+1}\big) \\ &= (1-\rho_e)a_{it+1} + \rho_e a_{it+1}e_{it} + a_{it+1}\varepsilon_{it+1} \end{aligned} \tag{22}$$

In order to compute the last expression, we need to approximate $a_{it+1}e_{it}$, and for doing this we use again a similar approximation as in equation (18)

$$a_{it+1}e_{it} = \mathcal{P}_2(a_{it}, e_{it}, z_t, k_t, M_{ae}, M_{a^2}, k_{t+1}; \tilde{\Theta}) \tag{23}$$

The coefficients of the approximating functions in equations (20), (21) and (23) can now be solved for using projection methods or perturbation methods. Once we obtain Θ, $\bar{\Theta}$ and $\tilde{\Theta}$, we can deduce, by explicit aggregation, the laws of motions for k_{t+1}, $M_{a^2,t+1}$ and $M_{ae,t+1}$.

4 Accuracy Checks

The sensitivity of the results to the number of Chebyshev nodes, and for the Gauss-Hermite nodes used in the numerical integration is examined by increasing the limit from 10 to 20, and 10 to 50, respectively. These changes had a negligeable effect on the results, and in order to keep the computational time low, we chose 10 nodes for each method.

In order to check for the accuracy of the numerical approximation, we compute the Euler equation errors, as described in Judd (1998)

$$E(a, e, z, k; \Theta_n) = \frac{R(a, e, z, k; \Theta_n)}{\hat{c}(a, e, z, k; \Theta_n)^{-\gamma}}$$

This term is a dimension-free quantity that expresses optimization error as a fraction of current consumption. In economic terms, it tells us how irrational agents would be in using the approximating rule. For instance, in our benchmark model the maximum value of the error is found to be 0.0046, meaning that this approximation implies that agents make

0.46 percent errors in their period-to-period consumption decisions. This value is very low (Table 5) and we can conclude that the approximation is very good. This is not surprising as the calibration of our benchmark model implies low uncertainty and quasi-linear policy functions.

Table 5 shows the log Euler equation errors, where $\log ||E||_\infty$ and $\log ||E||_1$ represent respectively the maximum error and the average error in the bounded state space $[a_{min}, a_{max}] \times [e_{min}, e_{max}] \times [z_{min}, z_{max}] \times [k_{min}, k_{max}]$. Finally the Euler equation errors and the least squares projection method is computed in a 4-D state space with 10000 nodes.

5 Results

This section discusses selected quantitative results for the heterogeneous agents model. We focus on the dynamics of aggregate capital for studying model properties. Our comparison exercise has been done in various dimensions.

We first used linear approximation methods to solve the model. For this we used four alternative algorithms obtained from the combination perturbation/projection and simulation-based/explicit aggregation. In this particular framework, we find that explicit aggregation produces results that are very close to those obtained in Krusell and Smith (1998) methodology that relies on simulation and aggregation. However in practice we find that the algorithm which uses explicit aggregation is much faster and requires a very small amount of RAM.

When confronting the projection method to the perturbation method, the results diverge as soon as we make the agents more risk-averse and include additional idiosyncratic uncertainty. This is not surprising, since the projection method approximates the model for a larger portion of the state space, while the perturbation methods use local approximations around the steady state. In this case a high value for the risk aversion parameter and a larger deviation from the steady state may be ill-suited for the perturbation method.

In order to capture nonlinearities, and in order to study the impact of uncertainty on aggregate wealth, we solve the model using second order perturbation and compute the aggregate capital of the economy. In Table 8 we compare these results and find that, the mean of aggregate capital decreases as idiosyncratic risk and risk aversion increases.

Our finding so far shows that in our benchmark heterogeneous agent model, local and global solutions are very similar. This is not surprising as our benchmark parameterization implies quasi-linear policy function. In order to investigate the behaviour for an economy with higher uncertainty and increased risk-aversion, we approximate the model for various structural parameter values.

When comparing the results, we meet our expectations. Let us first start with the comparison between the simulation-based approach with the explicit aggregation. As it is shown in Tables 9 to 11, both methods yield results that are very close. For the results obtained with perturbation method we find that there is no difference whether we update the aggregate law of motion by using simulation/regression or by explicit aggregation.

However we observe minor differences when we increase the risk and uncertainty (see for example the case where $\gamma = 1.5$ and $\sigma_e = 0.05$ or the case $\gamma = 1.5$ and $\sigma_e = 0.1$). The figures in Table 8 show aggregate capital obtained through linear approximation methods. Table 8 shows the impact on aggregate capital if we use second order perturbation methods.

When comparing the results obtained from perturbation with those obtained from projection, we observe that the results differ as we increase the idiosyncratic uncertainty of the model. Although that this result is not new, since projection methods cover a large part of the state space and hence takes into account nonlinearities, when we move away from the steady state.

However we see the impact of high uncertainty on the determination of the aggregate law of motion. We find that by increasing the risk aversion coefficient, γ, from 1 to 2, it increases the average asset holdings from around 5.49 to 5.59, which is small (1.83%).

The same exercise has been done by keeping the level of idiosyncratic risk fixed, but by increasing the risk aversion parameter γ up to 3.5. All approximations show a decrease in the level of aggregate capital when γ increases. When combining the global solution method with explicit aggregation, it can be seen that the mean of aggregate capital decreases much faster when γ increases (Table 6).

These results shed a light on the approximation accuracy of local methods in heterogeneous agents models.

Conclusion

This paper has shown how to apply a perturbation method to solve dynamic models with heterogeneous agents and aggregate uncertainty. Solving these type of models is difficult as the equilibrium depends on the wealth distribution in the economy. In order to make the problem amendable to a perturbation method, we substitute the borrowing constraint with a penalty function, which transforms the original problem into a form with only equality constraints.

Furthermore, we compare the results obtained through the perturbation methods to those computed through projection methods. The key benefits of the method presented here is that it is much easier to use, since it is possible to solve the individual problem with Dynare. It is also very fast, such that it can be used for estimating the model.

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Figures and Tables : Solving Dynamic Models with Heterogeneous Agents and Aggregate Uncertainty with Dynare or Dynare++

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Key Words : Incomplete markets, numerical solutions, projection methods, perturbation methods
JEL Classification : C63, D52

Abstract

This papers show how models with heterogeneous agents and aggregate uncertainty can be solved using Dynare or Dynare ++ software that implements a perturbation approach. When the explicit aggregation algorithm (XPA) is used to obtain aggregate laws of motion, this can be accomplished by combining a Dynare program with a very simple Matlab program. When the Krusell-Smith algorithm is used, then the Matlab program needed is somewhat more involved, but still relatively simple. We calculate and compare 1^{st} and 2^{nd}-order numerical solutions using both algorithms. These numerical procedures are also compared with the algorithm that solves the individual policy rules with a projection instead of a perturbation procedure. Finally, we discuss a procedure that efficiently chooses which cross-sectional moments to include as aggregate state variables when nonlinearities are important and the mean is not a sufficient statistic.

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Figure 1: Approximation Aggregation - First Order Approximation

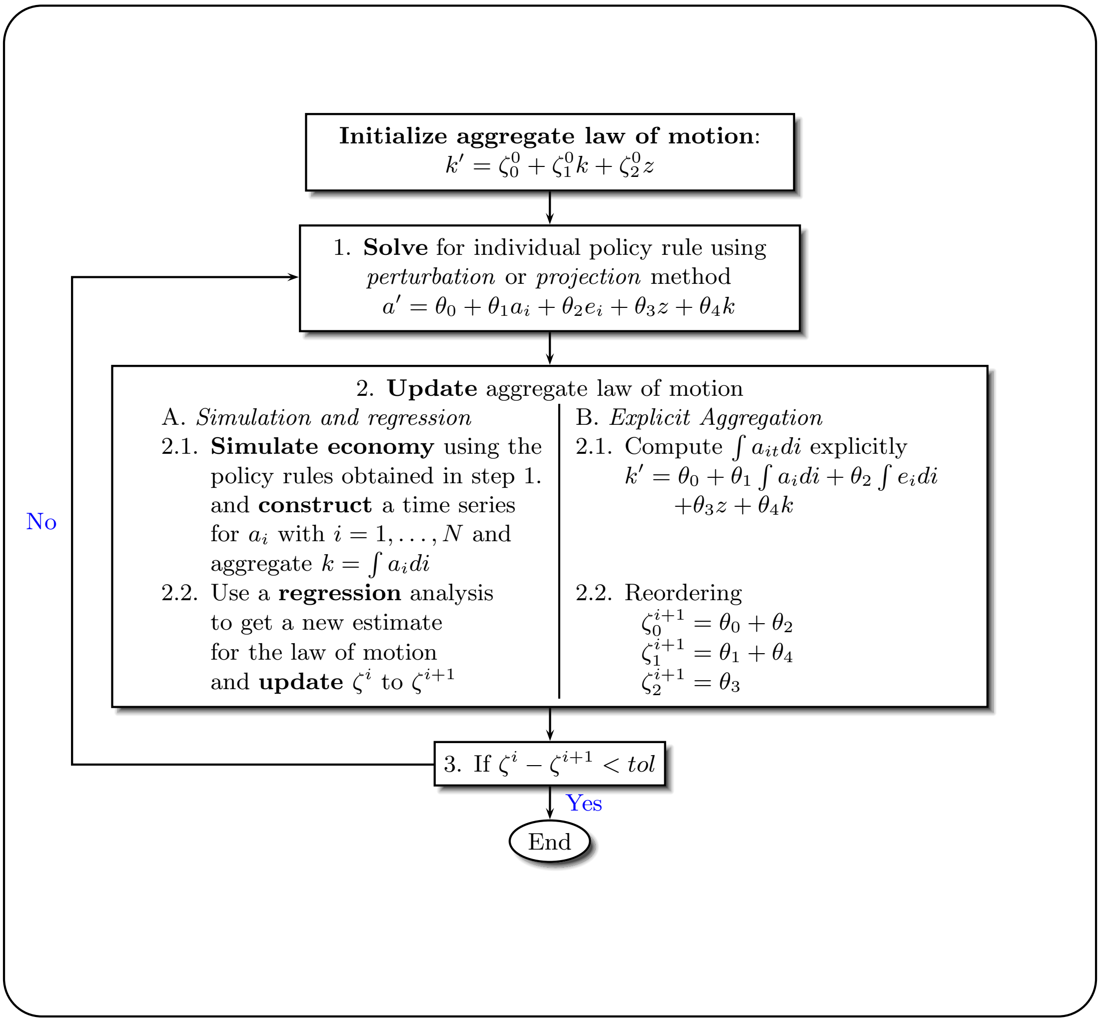
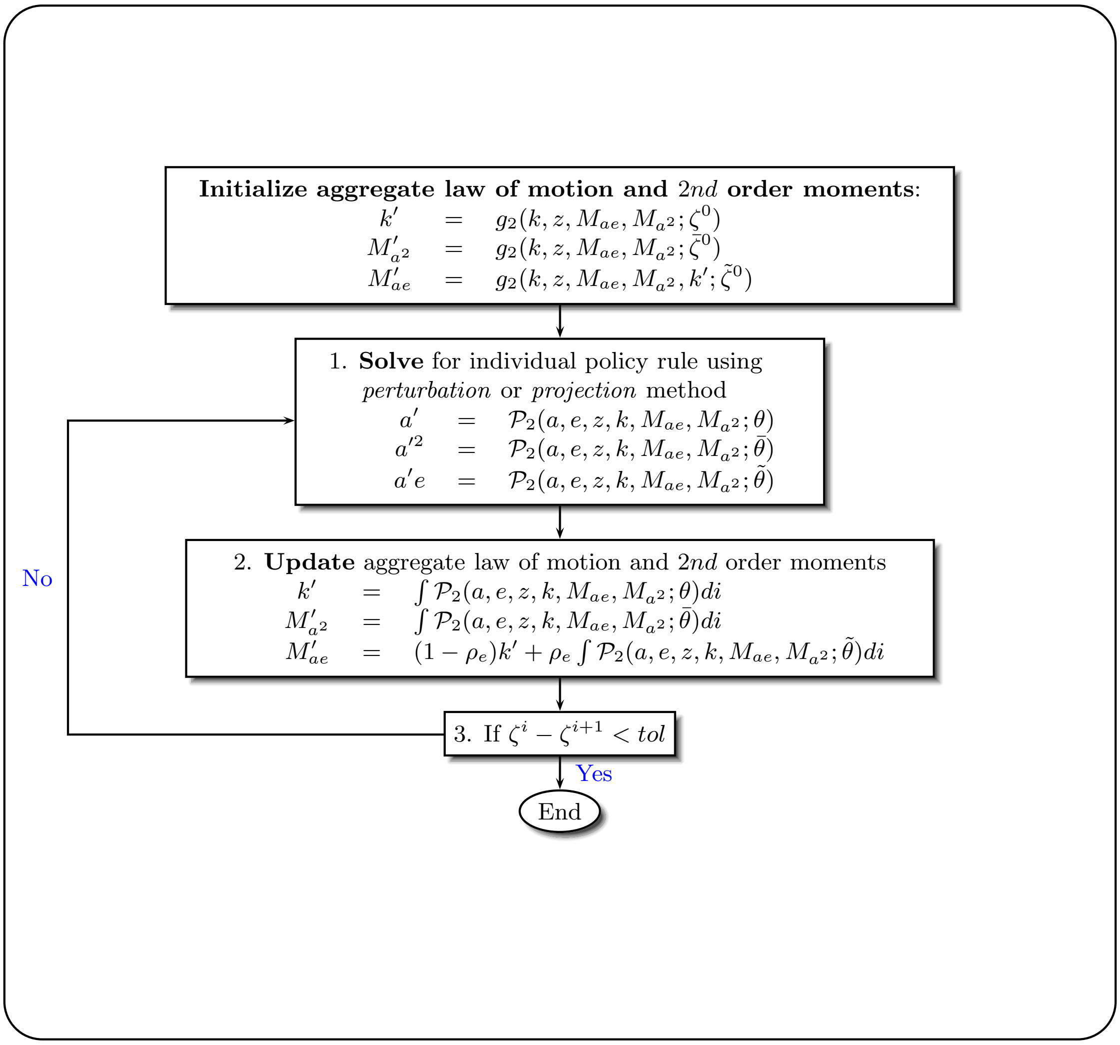


Figure 2: Approximation Aggregation - Second Order Approximation



Matlab and Dynare Code

The **Matlab** code:

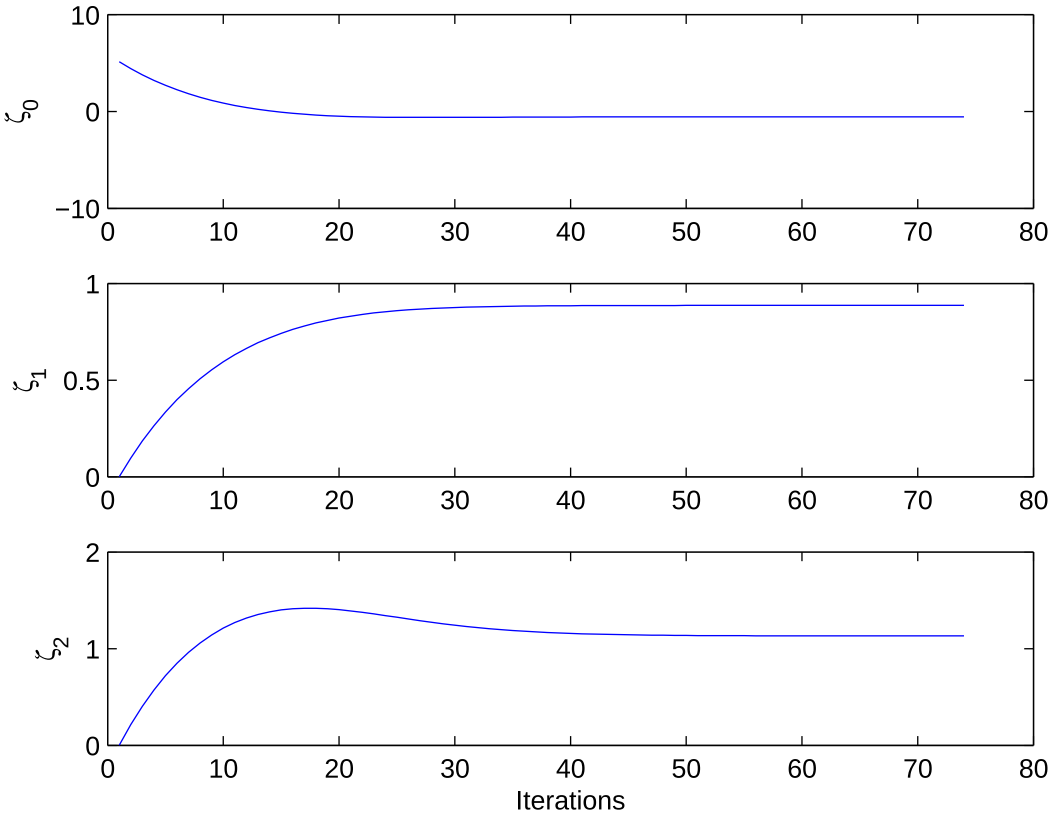
```
for idxIter = 1:iIter

 % Solve the model for given coeffs. of the aggregate law of motion (vZetaOld)
 dynare Dimension4PF noclearall ;

 vZetaNew(1) = vTheta(1) + vTheta(3);
 vZetaNew(2) = vTheta(2) + vTheta(5);
 vZetaNew(3) = vTheta(4);

 % Check convergence of coefficients
 dConv = fnConvergence(vZetaNew,vZetaOld,iTol);
 if dConv == 1
     break;
 end

 vZetaOld = dLambda * vZetaNew + (1-dLambda) * vZetaOld;

 pZeta0 = vZetaOld(1);
 pZeta1 = vZetaOld(2);
 pZeta2 = vZetaOld(3);
 delete InitParams.mat;
 save InitParams.mat pZeta0 pZeta1 pZeta2;

end
```

The **Dynare** mod-file:

```
var ... ; // declare endogeneous variables
varexo ...; // declare exogeneous variables
parameters  ... ; // declare parameters

load InitParams; // load coefficients for ALM
set_param_value('pZeta0',pZeta0);
set_param_value('pZeta1',pZeta1);
set_param_value('pZeta2',pZeta2);

load StructParams; // load structural parameters (sensitivity analysis)
set_param_value('pGamma',pGamma);
set_param_value('pSigmae',pSigmae);

model; ... end;  // model block

initval; ... end; // initial values for solver

shocks; ... end; // declare shocks

stoch_simul(order=1,nocorr,noprint,nomoments,IRF=0);
```

```
// Collecting parameters

mPolicy = [oo_.dr.ys'; oo_.dr.ghx'; oo_.dr.ghu']; // read coefficients of policy
  // functions
mPolA = mPolicy(:,2);
// Rearrange parameters
dTheta0 = mPolA(1)-mPolA(2)*mPolA(1)-mPolA(6)-mPolA(7)-mPolA(5)*mPolicy(1,5);
dTheta1 = mPolA(2);
dTheta2 = mPolA(6);
dTheta3 = mPolA(7);
dTheta4 = mPolA(5);

vTheta = [dTheta0 dTheta1 dTheta2 dTheta3 dTheta4];
```

Table 1: Convergence of the ALM coefficients during the updating process.



(a) $\gamma = 1.00$

Table 2: Approximating Functions: Family of Monomials (2nd order projection method)

	cst	a	e	z	k	M_{ae}	M_{a^2}
cst	cst	a	e	z	k	M_{ae}	M_{a^2}
a	a	a^2	ae	az	ak	aM_{ae}	aM_{a^2}
e	e	ea	e^2	ez	ek	eM_{ae}	eM_{a^2}
z	z	za	ze	z^2	zk	zM_{ae}	zM_{a^2}
k	k	ka	ke	kz	k^2	kM_{ae}	kM_{a^2}
M_{ae}	M_{ae}	$M_{ae}a$	$M_{ae}e$	$M_{ae}z$	$M_{ae}k$	$M_{ae}M_{ae}$	$M_{ae}M_{a^2}$
M_{a^2}	M_{a^2}	$M_{a^2}a$	$M_{a^2}e$	$M_{a^2}z$	$M_{a^2}k$	$M_{a^2}M_{ae}$	$M_{a^2}^2$

Table 3: Approximating Functions: Family of Monomials (2nd order perturbation method)

	cst	a	e_{-1}	z_{-1}	k	M_{ae}	M_{a^2}	ε^e	ε^z
cst	cst	a	e_{-1}	z_{-1}	k	M_{ae}	M_{a^2}	ε^e	ε^z
a	a	a^2	ae_{-1}	az_{-1}	ak	aM_{ae}	aM_{a^2}	$a\varepsilon^e$	$a\varepsilon^z$
e_{-1}	e_{-1}	$e_{-1}a$	e_{-1}^2	$e_{-1}z_{-1}$	$e_{-1}k$	$e_{-1}M_{ae}$	$e_{-1}M_{a^2}$	$e_{-1}\varepsilon^e$	$e_{-1}\varepsilon^z$
z_{-1}	z_{-1}	$z_{-1}a$	$z_{-1}e_{-1}$	z_{-1}^2	$z_{-1}k$	$z_{-1}M_{ae}$	$z_{-1}M_{a^2}$	$z_{-1}\varepsilon^e$	$z_{-1}\varepsilon^z$
k	k	ka	ke_{-1}	kz_{-1}	k^2	kM_{ae}	kM_{a^2}	$k\varepsilon^e$	$k\varepsilon^z$
M_{ae}	M_{ae}	$M_{ae}a$	$M_{ae}e_{-1}$	$M_{ae}z_{-1}$	$M_{ae}k$	$M_{ae}M_{ae}$	$M_{ae}M_{a^2}$	$M_{ae}\varepsilon^e$	$M_{ae}\varepsilon^z$
M_{a^2}	M_{a^2}	$M_{a^2}a$	$M_{a^2}e_{-1}$	$M_{a^2}z_{-1}$	$M_{a^2}k$	$M_{a^2}M_{ae}$	$M_{a^2}^2$	$M_{a^2}\varepsilon^e$	$M_{a^2}\varepsilon^z$
ε^e	ε^e	$\varepsilon^e a$	$\varepsilon^e e_{-1}$	$\varepsilon^e z_{-1}$	$\varepsilon^e k$	$\varepsilon^e M_{ae}$	$\varepsilon^e M_{a^2}$	$\varepsilon^e\varepsilon^e$	$\varepsilon^e\varepsilon^z$
ε^z	ε^z	$\varepsilon^z a$	$\varepsilon^z e_{-1}$	$\varepsilon^z z_{-1}$	$\varepsilon^z k$	$\varepsilon^z M_{ae}$	$\varepsilon^z M_{a^2}$	$\varepsilon^z\varepsilon^e$	$\varepsilon^z\varepsilon^z$

Table 4: Parameterization of the Benchmark Economy

Parameter	α	β	δ	γ	μ^e	ρ^e	σ^e	μ^z	ρ^z	σ^z	$\bar{l}$	η_0	η_1	η_2
	.33	.99	.1	1	1	.95	.005	1	.9	.01	1	.1	.1	0

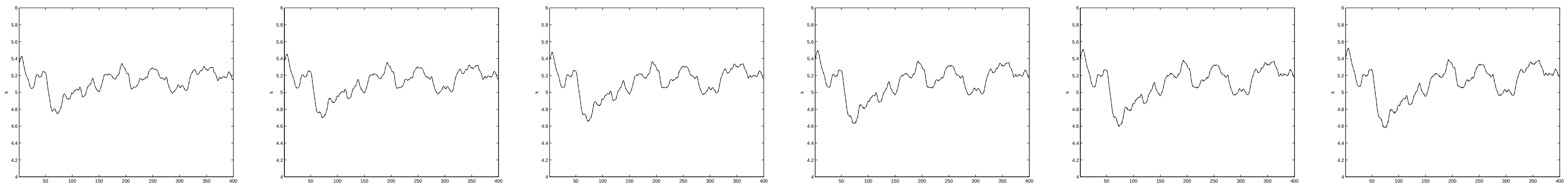
Table 5: log Euler equation errors

		First Order Approximation			
		Projection		Perturbation	
γ	σ_e	$\|\|E\|\|_\infty$	$\|\|E\|\|_1$	$\|\|E\|\|_\infty$	$\|\|E\|\|_1$
1.0	0.005	-5.2119	-7.1647	-5.2734	-7.5850
	0.050	-5.1441	-7.1013	-4.9779	-6.6844
	0.100	-4.8109	-6.0265	-4.4006	-5.1563
1.5	0.005	-5.2679	-7.4768	-5.3530	-7.6463
	0.050	-5.3538	-7.1730	-4.9949	-6.5249
	0.100	-4.7907	-5.7823	-4.2882	-4.9477
2.0	0.005	-5.2555	-7.5171	-5.3215	-7.6430
	0.050	-5.1424	-7.0488	-4.9148	-6.3372
	0.100	-4.5529	-5.6149	-4.0674	-4.7447

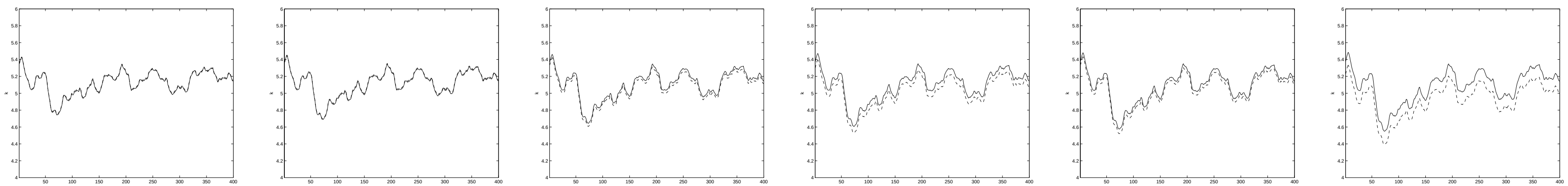
Table 6: Sample mean and standard deviation for aggregate capital ($T = 400$)

Method	Perturbation				Projection			
Order	1		2		1		2	
γ	Mean	Std.dev.	Mean	Std.dev.	Mean	Std.dev.	Mean	Std.dev.
1.00	5.1283	0.1330	5.1284	0.1337	5.1244	0.1340	5.1339	0.1333
1.50	5.1255	0.1496	5.1256	0.1506	5.1154	0.1505	5.1260	0.1501
2.00	5.1231	0.1638	5.1236	0.1652	5.1062	0.1645	5.0732	0.1643
2.50	5.1209	0.1759	5.1218	0.1779	5.0973	0.1762	5.0271	0.1762
3.00	5.1190	0.1862	5.1204	0.1888	5.0889	0.1859	5.0455	0.1877
3.50	5.1174	0.1950	5.1191	0.1983	5.0811	0.1938	4.9366	0.1937

Table 7: Aggregate law of motion for capital, k, approximated with **perturbation method** (first row) and **projection method** (second row) with $\sigma_e = 0.005$. First order approximations are represented through continuous line, while second order approximations are plotted in dashed line.

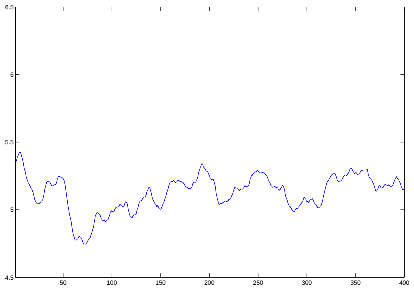


(a) $\gamma = 1.00$ (b) $\gamma = 1.50$ (c) $\gamma = 2.00$ (d) $\gamma = 2.50$ (e) $\gamma = 3.00$ (f) $\gamma = 3.50$

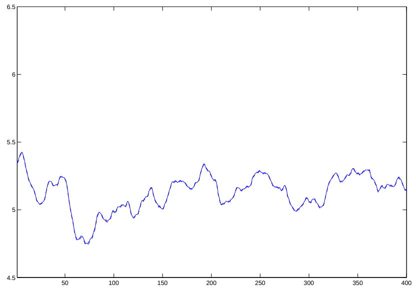


(g) $\gamma = 1.00$ (h) $\gamma = 1.50$ (i) $\gamma = 2.00$ (j) $\gamma = 2.50$ (k) $\gamma = 3.00$ (l) $\gamma = 3.50$

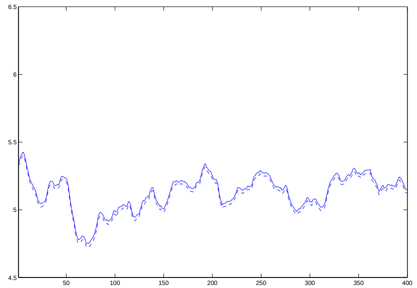
Table 8: Aggregate law of motion for capital, k, approximated with **first order perturbation method** (*continuous line*) and **second order perturbation method** (*dashed line*)



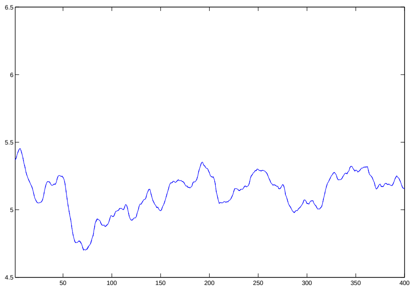
(a) $\gamma = 1.0$, $\sigma_e = 0.005$



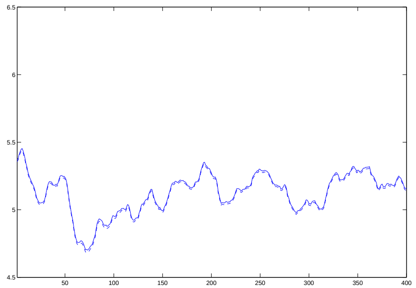
(b) $\gamma = 1.0$, $\sigma_e = 0.05$



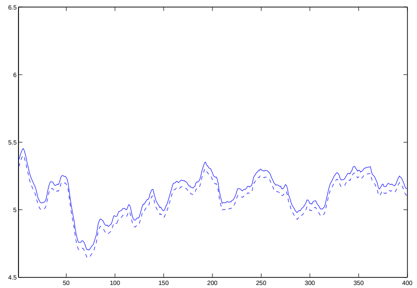
(c) $\gamma = 1.0$, $\sigma_e = 0.1$



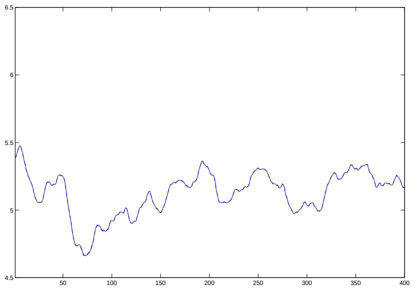
(d) $\gamma = 1.5$, $\sigma_e = 0.005$



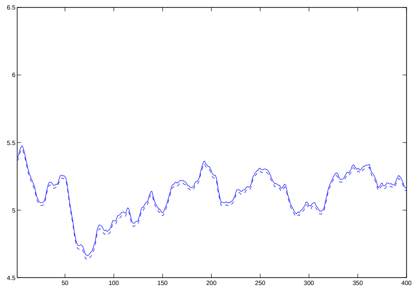
(e) $\gamma = 1.5$, $\sigma_e = 0.05$



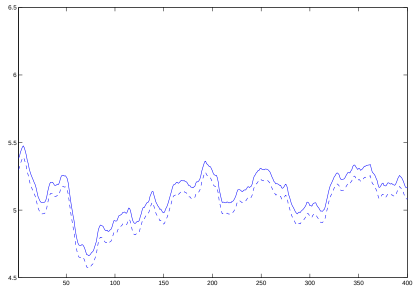
(f) $\gamma = 1.5$, $\sigma_e = 0.1$



(g) $\gamma = 2.0$, $\sigma_e = 0.005$



(h) $\gamma = 2.0$, $\sigma_e = 0.05$



(i) $\gamma = 2.0$, $\sigma_e = 0.1$

Table 9: Results from First Order Approximation for $\gamma = 1.0$

		Method for ALM		Individual Policy Function					Coefficients for the ALM		
				θ_0	θ_1	θ_2	θ_3	θ_4	ζ_0	ζ_1	ζ_2
γ	σ_e			*const*	a	e	z	k	*const*	k	z
1.0	0.005	*Simulation*	Perturbation	-0.67965	0.91636	0.37787	1.03880	-0.05953	-0.30165	0.85681	1.03873
			Projection	-0.69286	0.91831	0.38486	1.04203	-0.06096	-0.30813	0.85735	1.04214
		Xpa	Perturbation	-0.67961	0.91636	0.37787	1.03877	-0.05953	-0.30175	0.85683	1.03878
			Projection	-0.69772	0.91869	0.38446	1.04509	-0.06091	-0.31062	0.85763	1.04315
	0.05	*Simulation*	Perturbation	-0.67981	0.91636	0.37785	1.03914	-0.05956	-0.30086	0.85690	1.03746
			Projection	-0.69204	0.91879	0.37657	1.05128	-0.06042	-0.31710	0.85860	1.05165
		Xpa	Perturbation	-0.67961	0.91636	0.37787	1.03877	-0.05953	-0.30175	0.85683	1.03878
			Projection	-0.69612	0.91897	0.37788	1.05298	-0.06040	-0.31850	0.85868	1.05269
	0.1	*Simulation*	Perturbation	-0.67947	0.91636	0.37784	1.03965	-0.05973	-0.30001	0.85740	1.03401
			Projection	-0.68543	0.92041	0.34434	1.08375	-0.05843	-0.34011	0.86216	1.08183
		Xpa	Perturbation	-0.67961	0.91636	0.37787	1.03877	-0.05953	-0.30175	0.85683	1.03878
			Projection	-0.68900	0.92054	0.34559	1.08491	-0.05836	-0.34479	0.86245	1.08491

Table 10: Results from First Order Approximation for $\gamma = 1.5$

		Method for ALM		Individual Policy Function					Coefficients for the ALM		
				θ_0	θ_1	θ_2	θ_3	θ_4	ζ_0	ζ_1	ζ_2
γ	σ_e			*const*	*a*	*e*	*z*	*k*	*const*	*k*	*z*
1.5	0.005	*Simulation*	Perturbation	-0.81453	0.92867	0.41797	1.01232	-0.04828	-0.39642	0.88037	1.01224
			Projection	-0.82294	0.93028	0.42339	1.01331	-0.04969	-0.39961	0.88059	1.01334
		Xpa	Perturbation	-0.81449	0.92867	0.41797	1.01229	-0.04829	-0.39652	0.88038	1.01230
			Projection	-0.81655	0.93019	0.42395	1.00935	-0.05019	-0.40450	0.88102	1.01602
	0.05	*Simulation*	Perturbation	-0.81464	0.92867	0.41795	1.01265	-0.04832	-0.39544	0.88048	1.01066
			Projection	-0.82802	0.93121	0.41417	1.02880	-0.04897	-0.41170	0.88229	1.02634
		Xpa	Perturbation	-0.81449	0.92867	0.41797	1.01229	-0.04829	-0.39652	0.88038	1.01230
			Projection	-0.84207	0.93202	0.41792	1.03330	-0.04865	-0.41798	0.88267	1.03070
	0.1	*Simulation*	Perturbation	-0.81422	0.92867	0.41793	1.01310	-0.04849	-0.39422	0.88103	1.00662
			Projection	-0.83385	0.93494	0.37473	1.08155	-0.04673	-0.46210	0.88960	1.07679
		Xpa	Perturbation	-0.81449	0.92867	0.41797	1.01229	-0.04829	-0.39652	0.88038	1.01230
			Projection	-0.82472	0.93437	0.37466	1.07504	-0.04675	-0.46713	0.88910	1.08486

Table 11: Results from First Order Approximation for $\gamma = 2.0$

		Method for ALM		Individual Policy Function					Coefficients for the ALM		
				θ_0	θ_1	θ_2	θ_3	θ_4	ζ_0	ζ_1	ζ_2
γ	σ_e			*const*	a	e	z	k	*const*	k	z
2.0	0.005	*Simulation*	Perturbation	-0.90549	0.93550	0.44355	1.00576	-0.04114	-0.46179	0.89435	1.00567
			Projection	-0.90979	0.93666	0.44729	1.00523	-0.04238	-0.46250	0.89426	1.00530
		Xpa	Perturbation	-0.90545	0.93550	0.44355	1.00573	-0.04114	-0.46190	0.89436	1.00574
			Projection	-0.90958	0.93672	0.44777	1.00413	-0.04236	-0.46848	0.89480	1.00850
	0.05	*Simulation*	Perturbation	-0.90556	0.93550	0.44353	1.00607	-0.04118	-0.46068	0.89447	1.00389
			Projection	-0.92205	0.93822	0.43716	1.02745	-0.04148	-0.48468	0.89665	1.02768
		Xpa	Perturbation	-0.90545	0.93550	0.44355	1.00573	-0.04114	-0.46190	0.89436	1.00574
			Projection	-0.93137	0.93873	0.44007	1.03040	-0.04133	-0.48934	0.89719	1.02956
	0.1	*Simulation*	Perturbation	-0.90512	0.93550	0.44350	1.00646	-0.04133	-0.45916	0.89502	0.99951
			Projection	-0.95692	0.94548	0.39286	1.10999	-0.03885	-0.56627	0.90757	1.10671
		Xpa	Perturbation	-0.90545	0.93550	0.44355	1.00573	-0.04114	-0.46190	0.89436	1.00574
			Projection	-0.94895	0.94498	0.39268	1.10484	-0.03896	-0.57097	0.90732	1.11321

Table 12: Coefficients of the aggregate law of motion after convergence

	Simulation		Xpa	
	Perturbation	Projection	Perturbation	Projection
ζ_0	-0.30165	-0.30813	-0.30175	-0.31326
ζ_1	0.85681	0.85735	0.85683	0.85777
ζ_2	1.03873	1.04214	1.03878	1.04510
Iteration	-	-	-	-
Dampening λ	0.5	0.5	0.5	0.5
T periods	21000	21000	-	-
Burn-in	10000	10000	-	-
N agents	10000	10000	-	-
tol	10^{-5}	10^{-5}	10^{-5}	10^{-5}