

The Long and the Short of Asset Prices: Using Long Run Consumption-Return Correlations to Test Asset Pricing Models*

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Abstract

This paper examines a new set of implications of existing asset pricing models for the correlation between returns and consumption growth over the short and the long run. The findings suggest that external habit formation models face a challenge in producing two robust facts in aggregate data, namely, that stock market returns lead consumption growth, and that the correlation between returns and consumption growth is higher at low frequencies than it is at high frequencies. To reconcile these facts with a consumption-based model, I show that one needs to focus on models that contain a "forward looking" consumption component, i.e., models that allow for both trend and cyclical fluctuations in consumption, and that link expected returns to the cyclical fluctuations in consumption. The models by Bansal and Yaron (2004) and Panageas and Yu (2005) provide examples of such models.

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1 Introduction

Since the seminal paper of Mehra and Prescott (1985), it has been well understood that standard consumption-based models face serious challenges in accounting quantitatively for many asset pricing phenomena. For example, equity premia in the data are large, risk-free rates are low, and the volatility of asset prices is high when compared to the magnitudes predicted by the standard Lucas-tree type model. It has proved challenging to augment the standard model to account for these basic facts. Early attempts were able to match some of the moments in the data, but at the cost of failing in other dimensions (especially at the cost of producing volatile interest rates). In recent years, however, there has been progress in constructing models that can account for all asset pricing phenomena simultaneously.

The model by Campbell and Cochrane (1999) stands out as the first paper to engineer a consumption-based pricing kernel that can account for most well-established asset pricing puzzles at the same time. The key mechanism in Campbell and Cochrane (1999) is the presence of a slow moving habit level that makes agents' risk aversion depend on *past* consumption shocks. These persistent variations in risk aversion make the pricing kernel volatile and can account for the high equity premia that are observed in reality.

Bansal and Yaron (2004) demonstrate an alternative way to account for most well-known asset pricing puzzles. They assume the presence of a slowly moving predictable component in consumption growth along with countercyclical volatility in consumption and Epstein-Zin (1989) preference specification. These assumptions together imply that risk premia depend on the predictable component in consumption growth, and hence on whether consumption is above or below its stochastic trend line. Thus, in contrast to Campbell and Cochrane (1999), in the Bansal and Yaron (2004) model risk premia are dependent on *anticipated* (i.e., forward-looking) changes in consumption. In a similar vein Panageas and Yu (2005) propose a production-based asset pricing model whereby variations in risk premia are caused by sluggish adjustments to technological innovations. In the first stages of a technological innovation, growth options are abundant. Because of their riskiness, they drive excess returns up. This phenomenon is reversed in the latter stages of the cycle, when growth options become depleted. The Panageas and Yu (2005) model resembles Bansal and Yaron (2004) in that it links excess returns to anticipated changes in consumption. Because both of these models are observationally similar for the purposes of this paper, I refer to both of them as “trend-cycle” models, in order to capture the dependence of excess returns on the cyclical component of consumption.

Even though the above models are able to match many asset pricing moments simultaneously, they operate through different mechanisms and constitute fundamentally different views of the sources and the pricing of risk. In this paper, I propose a simple and intuitive way to distinguish between the mechanisms underlying the sources of risks. In particular, I develop tests based on the implications of the two models for the joint time-series properties of consumption and returns.

Specifically, I show that reasonably calibrated versions of the Campbell and Cochrane (1999) model have strong implications for the correlations between consumption and excess returns over different horizons. The presence of a habit level that depends on past consumption growth implies that excess returns and consumption growth should exhibit a stronger correlation at short horizons than at long horizons. This is in contrast to models that feature predictable consumption growth together with countercyclical variation in risk premia.

The intuition for this finding is simple. In external habit models, negative past consumption shocks should predict high future excess returns, because of the negative dependence of risk aversion on past consumption innovations. Given a long enough horizon this negative covariation between *expected* excess returns and past consumption growth will drive correlations down, potentially even below zero. By contrast, in models with predictable consumption growth and countercyclical excess returns, a high expected excess return is associated with the anticipation of positive consumption growth over the long run. Hence, as the horizon increases the correlation between consumption and returns also increases.

In the data, I find that the correlation between consumption and returns increases as the horizon increases, consistent with trend-cycle type models and contrary to external habit formation models. Accordingly, I conclude that models that derive variations in expected returns from expectations of future consumption growth are better suited to match this pattern in the data, as has been recognized by previous literature (e.g., Parker and Julliard (2005)).

This basic message is reinforced by examining lead-lag relationships between consumption and returns. Using both Granger causality tests and spectral analysis, I show that asset pricing returns lead consumption growth. However, the strong reliance of the Campbell and Cochrane (1999) model on past consumption innovations is inconsistent with this observation. Interestingly, this remains the case even when I allow consumption to have predictable components inside a Campbell and Cochrane (1999) model. This is because the driving force in the model is still the surplus ratio. Expected returns are mostly driven by this variable, which is the average of the past consumption innovations. Hence, consumption still leads

returns in external habit models even in the presence of predictable consumption growth. By contrast, trend-cycle type models can account for this pattern in the data since expected excess returns anticipate consumption growth.

To sharpen the results even further, in addition to performing the analysis in the time domain, I also consider the joint time-series properties of consumption and returns in the frequency domain. Besides allowing me to examine a richer set of time-series implications, the frequency domain facilitates the derivation of analytical results for the models under consideration. This allows me to show more clearly, how the presence of a persistent habit process leads to an attenuated correlation between consumption and returns at low frequencies. Specifically, through a log-linearization argument, I show that as long as the external habit model produces a countercyclical risk premium or a procyclical price dividend ratio, the model implies that more covariation between consumption and returns comes from high frequency components, while in the data, the opposite obtains.

To account for the possibility that the patterns I observe are driven by small sample effects, I construct test statistics that depend on the difference between low-frequency (i.e., long horizon) and high-frequency (i.e., short horizon) correlations between consumption and returns. By repeatedly simulating small sample data from various versions of Campbell and Cochrane (1999) type models and trend-cycle models, I argue that my results are not driven by finite sample effects, and that there is enough power in the data to tell the models apart.

Note that consumption CAPM focuses on how asset prices respond to shocks in consumption, and how small consumption shocks can result in big movements in asset prices. Hence, I analyze the relation between consumption and asset prices, and in particular I focus my analysis on the relation between consumption and returns. I could also analyze the relation between consumption and price dividend ratios. However, given the issue of dividends measurement (Bansal and Yaron (2006)), I concentrate on the relation between consumption and returns in this paper.

A more general conclusion from this paper is the following. If an asset pricing model implies that the expected returns negatively depend on the past consumption growth and if this dependence is strong enough, then the model would produce the opposite long run relation between consumption and returns and the wrong lead-lag relation. On the other hand, if a model implies that expected returns positively depend on anticipated future consumption growth, then the model is likely to match these relations between consumption and returns.

Related Literature: Otrok, Ravikumar, and Whiteman (2002) argue that habit

agents are much more averse to high-frequency fluctuations than to low-frequency fluctuations, and the size of the equity premium is determined by a relatively insignificant amount of high-frequency volatility in U.S. consumption. Daniel and Marshall (1999) show that the performance of asset pricing models improves significantly at the two-year horizon since the correlation between consumption and returns is higher at longer horizons. Parker and Julliard (2005) show that the standard consumption-based CAPM can explain the size and value premium much better in long horizons since long horizon covariance between consumption and asset returns appears to be a better measure of risk. Although Daniel and Marshall (1999) and Parker and Julliard (2005) show that by using the long-run correlation between consumption and returns, the consumption-based CAPM performs much better both in the time-series and the cross-section, I show that these long horizon covariances are inconsistent with external habit formation.

The remainder of the paper is organized as follows. In Section 2, I analyze an external habit formation model with *i.i.d.* consumption growth. Section 3 presents the trend-cycle models. In Section 4, I examine a general external habit formation model with predictable consumption growth. Section 5 provides robustness checks, and Section 6 investigates the cross-sectional implications of the trend-cycle model and the habit formation model. Section 7 concludes the paper. All the technical derivations appear in the Appendix.

2 External Habit Formation Model

Most of the key results for the external habit persistence model crucially depend on the slow-moving surplus ratio. Further, this slow-moving feature of the model has clear implications for the long run. Therefore, it is worthwhile to explore the low-frequency properties of the model.

I begin by setting up a standard external habit formation model following Campbell and Cochrane (1999) with an *i.i.d.* consumption growth rate. I also incorporate a cointegration constraint between dividends and consumption because the focus of this paper is on the long horizon implications of different models. Hence, this cointegration constraint could potentially play an important role¹.

To begin, I present a log-linear solution of the model, and I derive analytically its long

¹A number of recent papers, including Bansal, Dittmar, and Lundblad (2001), Hansen, Heaton, and Li (2005), and Bansal, Dittmar, and Kiku (2007) suggest that dividends and consumption are stochastically cointegrated, and that this cointegration is important for understanding asset pricing.

run implications under the log-linear approximation. I show that in order for the habit model to match the first two moments of the consumption and asset market data, the model must counterfactually produce larger correlations between consumption growth and asset returns at short horizons than at longer horizons. Also the correlation at long horizons (or low frequencies) is negative for reasonable parametrization. Furthermore, consumption leads asset returns in this external habit model, while the opposite is true in the data. This lead-lag relation result is not surprising at all given that consumption growth is *i.i.d.*. However, as I show in Section 4, the lead-lag relation is still contrary to the data, even with a predictable $ARMA(2, 2)$ consumption growth process.

2.1 External Habit Formation Model with *i.i.d.* Growth Rate

In this section, I set up an external habit persistence model that closely follows the specification of Campbell and Cochrane (1999). Let $c_t = \log(C_t)$ and $d_t = \log(D_t)$ denote log real per capita values of consumption and the stock dividend. The consumption growth rate $g_{c,t} = c_t - c_{t-1}$ is generated according to

$$g_{c,t} = \mu_c + \epsilon_{c,t}, \quad (2.1)$$

where $\epsilon_{c,t}$ is an *i.i.d.* normal with standard error σ_c . The cointegrating constraint is that $d_t - c_t$ is a stationary process, which evolves as follows:²

$$\begin{aligned} d_t &= \mu_{dc} + c_t + \delta_t \\ \delta_t &= \rho_\delta \delta_{t-1} + \epsilon_{\delta,t}, \end{aligned}$$

where $\epsilon_{\delta,t}$ is an *i.i.d.* normal with standard error σ_δ and $\rho_{c\delta}$ is the correlation between $\epsilon_{c,t}$ and $\epsilon_{\delta,t}$. This model assumes that $0 \leq \rho_\delta \leq 1$. It follows that the dividend growth $g_{d,t}$ is generated as

$$\begin{aligned} g_{d,t} &= d_t - d_{t-1} = g_{c,t} + \delta_t - \delta_{t-1} \\ &= \mu_c + \epsilon_{c,t} + (\rho_\delta - 1) \delta_{t-1} + \epsilon_{\delta,t}. \end{aligned}$$

This setup of the dynamics of consumption and dividends is a direct extension of Campbell and Cochrane (1999). Here, c_t and d_t are each $I(1)$, and these two series are cointegrated except for when $\rho_\delta = 1$, in which case the model reduces to that of Campbell and Cochrane

²I could assume that $d_t - \lambda_{cd}c_t$ is a stationary process and λ_{cd} can be estimated from the data. This generalization does not affect the results qualitatively. It only changes the quantitative results slightly.

(1999) and the dividends can wander arbitrarily far from consumption as time passes. The agent is assumed to maximize the lifetime utility

$$E_t \sum_{k=0}^{\infty} \beta^k \frac{(C_{t+k} - X_{t+k})^{1-\gamma} - 1}{1-\gamma},$$

where C_t is real consumption, X_t is the agent's habit level at time t , γ is the risk-aversion coefficient, and β is the time discount factor of the agent. The surplus ratio is defined as $S_t = \frac{C_t - X_t}{C_t}$, and $s_t = \log(S_t)$. The dynamics of the log surplus ratio s_t are given by

$$s_{t+1} = (1 - \phi) \bar{s} + \phi s_t + \lambda(s_t) \epsilon_{c,t+1}, \quad (2.2)$$

where $\bar{s}$ is the steady state of the log surplus ratio, ϕ determines the persistence of the surplus ratio (which also largely determines the persistence of the price dividend ratio), and the sensitivity function $\lambda(s)$ is given by

$$\lambda(s_t) = \begin{cases} \frac{1}{\bar{s}} \sqrt{1 - 2(s_t - \bar{s})} - 1, & s_t \leq s_{\max} \\ 0, & s_t \geq s_{\max} \end{cases}$$

with

$$s_{\max} = \bar{s} + \frac{1}{2} (1 - \bar{S}^2), \quad \bar{S} = \sigma_c \sqrt{\frac{\gamma}{1 - \phi}}.$$

In the continuous time limit, $s_{\max}$ is the upper bound on s_t . The implication of the above specification is that the risk-free rate is a constant and habit comoves (positively) with consumption. Under the assumption of external habit, the pricing kernel M_t satisfies

$$\begin{aligned} M_{t+1} &= \beta \left(\frac{S_{t+1} C_{t+1}}{S_t C_t} \right)^{-\gamma} \\ &= \beta \exp \{ -\gamma [(\phi - 1)(s_t - \bar{s}) + [1 + \lambda(s_t)] \epsilon_{c,t+1} + \mu_c] \}. \end{aligned}$$

Hence, by the Euler equation, the functional equation for the price dividend ratio $Z_t = P_{d,t}/D_t$ for an asset that pays dividend D_t is

$$Z_t = E_t \left[M_{t+1} (Z_{t+1} + 1) \frac{D_{t+1}}{D_t} \right].$$

Therefore, the price dividend ratio Z_t is a function of the state variables (s_t, δ_t) , and can be obtained as the solution to the following functional equation:

$$Z(s_t, \delta_t) = \beta E_t \left[\frac{\exp \{ -\gamma [(\phi - 1)(s_t - \bar{s}) + [1 + \lambda(s_t)] \epsilon_{c,t+1} + \mu_c] \}}{(Z(s_{t+1}, \delta_{t+1}) + 1) \cdot \exp(\mu_c + \epsilon_{c,t+1} + (\rho_\delta - 1) \delta_t + \epsilon_{\delta,t+1})} \right]. \quad (2.3)$$

The above setup is the standard external habit formation model except for the cointegration constraint. To further explore the long-run implications of the model, in the following section I provide a log-linear approximation of the model and I derive analytically some qualitative features of the model in the long run.

2.2 Log-Linear Solution of the Model

Before solving the functional equation (2.3) numerically, it is worthwhile to explore the log-linear approximation of the model to gain better intuition. Although the first-order approximation is not numerically very accurate given the highly nonlinear nature of the model, it provides correct intuition. Assume that the log price dividend ratio $z_t = \log(Z_t)$ can be approximated by a linear function of the state variables (s_t, δ_t) , that is,

$$z_t \approx a_0 + a_1 s_t + a_2 \delta_t, \quad (2.4)$$

where the constant coefficients a_0 , a_1 , and a_2 are to be determined. Furthermore, approximate the nonlinear sensitivity function $\lambda(s)$ by the linear function³

$$\lambda(s) \approx -\alpha_\lambda (s - s_{\max}),$$

where α_λ is a constant to closely approximate the sensitivity function. The results that will obtain are not sensitive to the choice of α_λ . In the Appendix, the coefficients a_0 , a_1 , and a_2 are solved in closed-form. Hence, a linear approximation of the log price dividend ratio can be obtained. Now, substituting equation (2.4) and equation (2.2) back into Campbell-Shiller's (e.g. Campbell and Shiller (1988)) log-linear return approximation gives⁴

$$\begin{aligned} r_{t+1} &\approx \kappa_0 + g_{d,t+1} + \kappa_1 z_{t+1} - z_t \\ &\approx \alpha - \beta_S S_t + \left[1 + a_1 \kappa_1 \frac{1 - \bar{S}}{\bar{S}} \right] \epsilon_{c,t+1} + [1 + a_2 \kappa_1] \epsilon_{\delta,t+1}, \end{aligned} \quad (2.5)$$

where $\beta_S = \frac{a_1(1-\kappa_1\phi)}{\bar{S}}$ and the constant α is given by equation (7.4) in the Appendix. The coefficient β_S is positive if and only if a_1 is positive. Hence, as long as the price dividend ratio is procyclical, β_S is positive, and hence the risk premium is countercyclical. The parameters κ_1 and κ_0 are determined endogenously as

$$\begin{aligned} \kappa_1 &= \frac{\exp(E[z_t])}{1 + \exp(E[z_t])} \\ \kappa_0 &= -\log \kappa_1 - (1 - \kappa_1) \log \left(\frac{1}{\kappa_1} - 1 \right). \end{aligned}$$

The habit level X_t can be further approximated as an exponentially weighted average of past consumption,

$$X_t \approx \frac{1 - \phi}{\phi} \sum_{k=1}^{\infty} \phi^k C_{t-k}, \quad (2.6)$$

³Another linear approximation around the steady state $\bar{s}$, $\lambda(s) \approx \frac{1}{\bar{S}} - 1 + \frac{1}{\bar{S}}(s - \bar{s})$, is also used and the results are almost identical.

⁴Since the risk-free rate is a constant in this model, the returns are equivalent to excess returns.

where ϕ is the measure of habit persistence. Equation (2.6) implies that the habit level X_t and the consumption level are cointegrated⁵. Substitute equation (2.6) back into the definition of the surplus ratio, approximate to the first order, and simplify to obtain

$$S_t \approx \tilde{S}_t \equiv \sum_{j=1}^{\infty} \phi^{j-1} g_{t+1-j}. \quad (2.7)$$

Thus, the asset returns can be approximated by

$$r_{t+1} \approx \alpha - \beta_S \sum_{j=1}^{\infty} \phi^{j-1} g_{t+1-j} + \left[1 + a_1 \kappa_1 \frac{1 - \bar{S}}{\bar{S}} \right] \epsilon_{c,t+1} + [1 + a_2 \kappa_1] \epsilon_{\delta,t+1}. \quad (2.8)$$

With the above approximation of returns, some long run properties of the model can be analytically derived. The K -horizon covariance between asset returns and consumption is (see the Appendix for the detailed calculations)

$$\begin{aligned} cov \left(\sum_{j=1}^K r_{t+j}, \sum_{j=1}^K g_{c,t+j} \right) &= \frac{\beta_S \sigma_c^2}{1 - \phi} + \frac{\phi (1 - \phi^{K-1}) \beta_S \sigma_c^2}{(1 - \phi)^2} \\ &\quad + \left[\left(1 - a_1 \kappa_1 - \frac{a_1 (1 - \kappa_1)}{\bar{S} (1 - \phi)} \right) \sigma_c^2 + (1 + a_2 \kappa_1) \sigma_{c\delta} \right] \cdot K. \end{aligned}$$

When horizon K is sufficiently large, the sign of the correlation at very long horizons will be determined by the coefficient on K in the above equation. Hence, the model implies a negative long horizon correlation if and only if

$$1 - a_1 \kappa_1 - \frac{a_1 (1 - \kappa_1)}{\bar{S} (1 - \phi)} + (1 + a_2 \kappa_1) \frac{\sigma_{c\delta}}{\sigma_c^2} < 0. \quad (2.9)$$

Furthermore, the correlation between consumption growth and asset returns is decreasing as the horizon increases. To see this, first write down the long horizon asset returns

$$\sum_{j=1}^K r_{t+j} \approx \alpha K - \beta_S \sum_{j=1}^K \tilde{S}_{t+j-1} + \left(1 + a_1 \kappa_1 \frac{1 - \bar{S}}{\bar{S}} \right) \sum_{j=1}^K \epsilon_{c,t+j} + (1 + a_2 \kappa_1) \sum_{j=1}^K \epsilon_{\delta,t+j}. \quad (2.10)$$

The long horizon correlation between asset returns and the growth rate comes from the last three terms in the above equation. Notice that the surplus ratio $\tilde{S}_{t+j-1}$ is a smoothed average of the past consumption growth rate. As the horizon K increases, more negative correlation results from the second term since $\beta_S > 0$ while the correlation from the last two terms stays constant. Hence, the correlation between consumption growth and asset returns decreases as horizon K increases.

⁵Because the weight $\frac{1-\phi}{\phi} \sum_{k=1}^{\infty} \phi^k = 1$.

The above approximation analysis provides good intuition on how the model works and the qualitative features of the model in the long run. To obtain the quantitative implications of the model, I further solve this model numerically by assuming that the log price dividend ratio is a quadratic function of the state variables⁶. Using the linear approximation as the initial value, the algorithm converges very fast. The parameter values are chosen close to Campbell and Cochrane (1999) as in Table 1. Since the cointegration is incorporated into the model, the persistence parameter ρ_δ for the difference between log dividends and log consumption needs to be chosen. This parameter is taken from Bansal, Gallant, and Tauchen (2007)⁷. I then simulate 48,000 quarters of artificial data to calculate population values for a variety of statistics. Table 2 shows the summary statistics of the equity premium, risk-free rate, and price dividend ratio from the simulated model. To facilitate comparison with Campbell and Cochrane (1999), I report the simulated moments of consumption and asset returns together with those of both the post-war sample and the long sample from Table 2 of Campbell and Cochrane (1999). As in Campbell and Cochrane (1999), the external habit formation model matches these moments well.

To see whether the implied long run features of the model are consistent with the data, I plot the correlations between consumption growth and asset returns at different horizons in Figure 1. In the data, the relation is upward sloping. However, for the habit formation model, the correlation is monotonically decreasing in horizon as predicted by the long-linear approximation⁸. I consider three variants of the external habit model. Habit model 1 is the calibrated model with parameter values given by Table 1 where consumption and dividends are cointegrated. Habit model 2 is the same model without cointegration (i.e. $\rho_\delta = 1$). Habit Model 3 is the habit model with a lower correlation between consumption and dividends. When consumption and dividends are not cointegrated as in Campbell and Cochrane (1999), the correlation between consumption and dividends is indeed lower as shown in the graph. The correlation between consumption growth and asset returns is too large in the model. This correlation level can be lowered by a different parameter combination as shown by habit model 3 or by time aggregation as in Campbell and Cochrane (1999). However, the

⁶As in Tallarini and Zhang (2005) and Bansal, Gallant and Tauchen (2007), a quadratic polynomial approximation works well.

⁷Lettau and Wachter (2007) use $\rho_\delta = 0.91$ for the annual frequency, or equivalently, $\rho_\delta = 0.9922$ for the monthly frequency. The results remain the same if ρ_δ is set to 0.9922.

⁸If the model is simulated at the monthly frequency, and time-averaged to the quarterly frequency, the correlation could increase from the first quarter to the second quarter, and then decrease monotonically as the horizon increases.

decreasing pattern in the correlation over the long horizon remains⁹.

2.3 Frequency Domain Analysis of the Model

Since the focus of this paper is on the long horizon (low-frequency) implication of different models, the most convenient way to proceed is to use frequency-domain analysis. More importantly, frequency-domain analysis allows a set of analytical results to be derived, and allows me to analyze the correlations at different frequencies and the lead-lag relation between consumption and asset returns in a unified framework. Before proceeding, I give an intuitive explanation of coherence, cospectrum, and phase that will prove useful for the rest of the analysis (see Brockwell and Davis (1991) for details).

The following exposition of the concept follows closely the exposition in Daniel and Marshall (1999). The coherence of the consumption growth rate and stock market returns at frequency λ measures the correlation between the consumption growth rate and returns at frequency λ . Essentially, coherence analysis splits each of the two series into a set of periodic components at different frequencies, and then determines the correlation of a set of periodic components for the two series around each frequency. When the frequency is λ , the corresponding length of the cycle is $1/\lambda$ quarters. Hence, when $\lambda = 0.5$, the corresponding cycle is two quarters. Since coherence is always positive, the sign of the correlation at different frequencies can't be determined from the coherence spectrum. To identify the sign of the correlation, the cospectrum needs to be examined. The cospectrum at frequency λ can be interpreted as the portion of the covariance between consumption growth and asset returns that is attributable to cycles with frequency λ . Since the covariance can be positive or negative, the cospectrum can also be positive or negative. The slope of the phase spectrum at any frequency λ is the *group delay* at frequency λ , and precisely measures the number of leads or lags between consumption growth and asset returns. When this slope is positive, consumption leads the market return. On the other hand, when this slope is negative, asset market returns lead consumption growth.

In the Appendix, I show that the cross-spectrum between consumption growth and asset returns in the external habit model is given by

$$f_{12}(\lambda) = \frac{1}{2\pi} \left(\frac{\beta_S (\phi - e^{i\lambda})}{1 + \phi^2 - 2\phi \cos(\lambda)} + 1 + a_1 \kappa_1 \frac{1 - \bar{S}}{\bar{S}} \right) \sigma_c^2 + \frac{1}{2\pi} [1 + a_2 \kappa_1] \sigma_{c\delta}. \quad (2.11)$$

⁹As we increase horizon further, the cumulative correlations start to slowly decline in the data which is the same with the habit model. However, the standard error is large as horizons increase.

Hence, the cospectrum $C_{sp}(\lambda)$ (the real part of the cross-spectrum $f_{12}(\lambda)$) can be given by

$$C_{sp}(\lambda) = \left(\frac{\beta_S (\phi - \cos(\lambda))}{1 + \phi^2 - 2\phi \cos(\lambda)} + 1 + a_1 \kappa_1 \frac{1 - \bar{S}}{\bar{S}} \right) \frac{\sigma_c^2}{2\pi} + (1 + a_2 \kappa_1) \frac{\sigma_{c\delta}}{2\pi}.$$

Taking the derivative of the above equation yields

$$C'_{sp}(\lambda) = \frac{\beta_S \sin(\lambda)}{2\pi (1 + \phi^2 - 2\phi \cos(\lambda))^2} (1 - \phi^2),$$

which is positive as long as $\beta_S > 0$. Hence, the portion of the covariance contributed by components at frequency λ is increasing as the frequency λ is increased when $\beta_S > 0$. This partially restates the early result that the correlation between consumption growth and asset returns decreases as the horizon increases.

Another way to show the negative correlations at long horizons is to examine the sign of the cross-spectrum between consumption growth and asset returns at the frequency $\lambda = 0$. The cross-spectrum at frequency zero is

$$f_{12}(0) = \frac{1}{2\pi} \left(-\frac{a_1 (1 - \kappa_1)}{\bar{S} (1 - \phi)} + 1 - a_1 \kappa_1 \right) \sigma_c^2 + \frac{1}{2\pi} (1 + a_2 \kappa_1) \sigma_{c\delta},$$

which is exactly the left hand side of equation (2.9) divided by 2π . Below I show that equation (2.9) is typically satisfied in models that can match the first two moments of the aggregate data. When equation (2.9) holds, the low-frequency correlations between consumption growth rate and asset returns are negative (since the function $f_{12}(\lambda)$ is continuous in λ), which is not supported by the data as I show later. Therefore, the sign on the correlation at frequency $\lambda = 0$ is the same as the sign on the long-horizon correlation, which is not unexpected.

From the expression for the cross-spectrum in equation (2.11), the phase spectrum $\phi(\lambda)$ can be calculated as follows

$$\tan(\phi(\lambda)) = \frac{-\beta_S \sin(\lambda) \sigma_c^2}{\beta_S (\phi - \cos(\lambda)) \sigma_c^2 + \left[\left(1 + a_1 \kappa_1 \frac{1 - \bar{S}}{\bar{S}} \right) \sigma_c^2 + (1 + a_2 \kappa_1) \sigma_{c\delta} \right] (1 + \phi^2 - 2\phi \cos(\lambda))}. \quad (2.12)$$

To investigate the lead-lag relation between consumption growth and asset returns, I need to examine the sign of the slope of the phase spectrum by differentiating equation (2.12). If the correlation between consumption innovations and return innovations is positive, that is,

$$\left(1 + a_1 \kappa_1 \frac{1 - \bar{S}}{\bar{S}} \right) \sigma_c^2 + (1 + a_2 \kappa_1) \sigma_{c\delta} \geq 0, \quad (2.13)$$

I show in the Appendix that

$$\phi'(\lambda) > 0.$$

Note that a positive slope at frequency λ implies that consumption growth leads asset returns at frequency λ . Hence, when equation (2.9) holds and the correlation between the innovations in consumption and innovations in returns is positive, consumption growth leads asset market returns in the external habit formation model. The above discussion leads to the following two propositions.

Proposition 1: *If equation (2.9) holds, that is,*

$$1 - a_1\kappa_1 - a_1 \frac{(1 - \kappa_1)}{\bar{S}(1 - \phi)} + (1 + a_2\kappa_1) \frac{\sigma_{c\delta}}{\sigma_c^2} < 0,$$

then there exists a frequency λ^ such that, for $\lambda < \lambda^*$, the correlation between the consumption growth rate and asset returns at frequency λ is negative. If, in addition, equation (2.13) holds, the slope of the phase spectrum between consumption growth and asset returns is positive. Hence, consumption growth leads asset returns.*

Proposition 2: *Under the external habit formation model, the analytical approximation shows that when*

$$\beta_S = \frac{a_1(1 - \kappa_1\phi)}{\bar{S}} > 0,$$

the cospectrum between consumption growth and asset returns is increasing in the frequency. That is, the portion of the covariance between consumption growth and asset returns that is attributable to cycles with frequency λ is increasing with the frequency λ . Hence, the high frequency cycles contribute more to the covariance between consumption growth and asset returns.

It is very natural for consumption to lead returns in this model since the expected returns depend on the surplus ratio, which is a smoothed average of the *past* consumption innovations. Given its central importance in Proposition 1, it is also interesting to examine when equation (2.9) is satisfied¹⁰. Equation (2.9) holds as long as a_1 is not too small. Since a_1 is the exposure of the log price dividend ratio to the surplus ratio, if a_1 is too small the model can't produce quantitative results for the first two moments of the aggregate data.

¹⁰Notice that $\delta_t = d_t - c_t$; hence, it is reasonable to assume that $\sigma_{c\delta} \leq 0$. Note also that $-1 \leq a_2\kappa_1 = \frac{\rho_\delta - 1}{1 - \rho_\delta} \kappa_1 \leq 0$, and hence, $(1 + a_2\kappa_1) \frac{\sigma_{c\delta}}{\sigma_c^2} \leq 0$. Therefore, for equation (2.9) to hold, we only need the condition $1 - a_1\kappa_1 - a_1 \frac{(1 - \kappa_1)}{\bar{S}(1 - \phi)} < 0$. Furthermore, since a_1 usually ranges from 0.5 to 1.5, and $\bar{S}$ is usually less than 0.1 to produce a high equity premium, the condition $1 - a_1\kappa_1 - a_1 \frac{(1 - \kappa_1)}{\bar{S}(1 - \phi)} < 0$ can be easily satisfied. Therefore, equation (2.9) typically holds.

Hence, for the model to make quantitative sense, a_1 can't be too small, and the condition in Proposition 1 is typically satisfied.

If $\beta_S > 0$, then the expected asset returns are high when the surplus ratio is low, and thus, the equity premium is countercyclical. Therefore, a positive β_S is a very reasonable assumption. Indeed, as I show in the Appendix, under very mild conditions β_S is positive. For example, when the correlation between consumption growth and dividend growth is positive, β_S is positive. Also notice that $\beta_S > 0$ if and only if $a_1 > 0$. A positive a_1 implies a procyclical price dividend ratio. Therefore, as long as the external habit persistence model produces a procyclical price dividend ratio, the cospectrum between consumption growth and asset returns is an increasing function of the frequency λ . As I show later in this section, this is at odds with the data.

Proposition 1 implies that the low-frequency correlation between consumption growth and asset returns is typically negative for a external habit formation model. At first glance, this seems to contradict the cointegration constraint between dividends and consumption. However, the low-frequency correlation between consumption growth and asset returns is not necessarily positive. To see this, note that the Campbell-Shiller decomposition of returns implies that cumulative returns can be written as

$$\begin{aligned} \sum_{j=1}^K r_{t+j} &\approx K\kappa_0 + \sum_{j=1}^K g_{d,t+j} + \kappa_1 \sum_{j=1}^K z_{t+j} - \sum_{j=1}^K z_{t+j-1} \\ &= K\kappa_0 + \sum_{j=1}^K g_{d,t+j} + (\kappa_1 - 1) \sum_{j=1}^{K-1} z_{t+j} + \kappa_1 z_{t+K} - z_t. \end{aligned}$$

Since the log price dividend ratio z_t is stationary, the correlation between long-run returns and long-run consumption resulting from the term $\kappa_1 z_{t+K} - z_t$ is negligible. In the long run, $\sum_{j=1}^K g_{d,t+j}$ and $\sum_{j=1}^K g_{c,t+j}$ are perfectly correlated. However, the term $(\kappa_1 - 1) \sum_{j=1}^{K-1} z_{t+j}$ is negatively correlated with $\sum_{j=1}^K g_{c,t+j}$ because $\kappa_1 - 1$ is negative and the price dividend ratio is positively correlated with the surplus ratio ($z_t \approx a_0 + a_1 s_t + a_2 \delta_t$). Since each z_{t+j} includes a smoothed average of past consumption growth, $\sum_{j=1}^{K-1} z_{t+j}$ is a *double* summation of past consumption innovations. Consequently, the covariance between $(1 - \kappa_1) \sum_{j=1}^{K-1} z_{t+j}$ and $\sum_{j=1}^K g_{c,t+j}$ will typically be higher than the covariance between $\sum_{j=1}^{K-1} g_{d,t+j}$ and $\sum_{j=1}^K g_{c,t+j}$ if the horizon K is big enough. When the negative effect between $(\kappa_1 - 1) \sum_{j=1}^{K-1} z_{t+j}$ and $\sum_{j=1}^K g_{c,t+j}$ dominates, the long run correlation between consumption growth and asset returns could be negative.

Proposition 1 and Proposition 2 provide the qualitative features of the cross-spectrum

between consumption and asset returns by a log-linear approximation. The exact cross-spectrum can be obtained from 48,000 quarters of artificial data with the parameter values given by Table 1. The top panel of Figure 2 plots the coherence between consumption growth and asset returns from the model simulation, and the middle panel plots the cospectrum. The cospectrum is normalized so that the area under the curve is one. The solid line in Figure 2 shows that in the simulated model, the cospectrum is increasing. The dotted line is the cospectrum from the analytical approximation¹¹. The bottom panel is the phase spectrum, which is increasing. It can be seen that the exact solution and the analytic approximation are extremely close for the phase spectrum. Since the phase spectrum determines the sign of the cospectrum, the claim about the sign of correlations based on analytical approximations is also valid under the exact solution.

Next, I shall document a few stylized facts in the data, and compare these facts with the above implications of the external habit models. For the real data, the top panel of Figure 3 confirms Daniel and Marshall's (1999) finding that the coherence between the quarterly consumption growth rate and quarterly market excess returns is much higher at low frequencies (around 0.5) than at high frequencies (around 0.1). Therefore, most of the correlation between consumption growth and asset market returns comes from the comovement at low frequencies. The middle panel also shows that most of the covariance comes from the low-frequency covariation. The 95% confidence interval is also given by the dotted line, and the confidence interval for the cospectrum is above zero at frequency $\lambda = 0$, while the cospectrum at frequency $\lambda = 0$ is negative for the simulated model. The high-frequency cospectrum is close to zero. The bottom panel of Figure 3 shows that the phase spectrum is nearly monotonically decreasing. For most frequencies, in this phase spectrum, the slope is negative. Hence, market returns lead consumption growth¹². Figure 4 plots the coherence, cospectrum, and phase spectrum for both the model and the data together. From this graph, it can be seen that *the coherence, cospectrum, and phase spectrum are all declining in the data, while they are all increasing in the external habit formation model*. In Section 5, I shall show that the decreasing pattern in the data is robust across different samples and

¹¹The approximation is quite accurate in general. Given the high nonlinearity of the model, the difference between the linear approximation and the exact solution is not negligible for some regions. However, the shape of the spectrum is very similar.

¹²Notice that the confidence interval for the phase spectrum is very wide for high frequencies. This is because the coherence between consumption and asset returns is very small for high frequencies. It is hard to determine which is leading when their comovement is very small.

econometric methods. In other words, in the data, asset market returns lead consumption and more covariation between consumption and returns comes from low frequencies, while it is the opposite for the external habit model.

To account for small sample issues, I also run 1,000 Monte Carlo experiments, each with 100 years of observations. I use a band-pass filter (e.g. Baxter and King (1999)) to calculate the low-frequency (with cycle longer than three years) and high-frequency (with cycle between 0.5 and three years) correlations between consumption and asset returns in each Monte Carlo experiment. Then, the difference between the low-frequency correlation and high-frequency correlation is obtained for each experiment. In the data, the difference between low-frequency correlations and high frequency correlations is about 23%. None of the 1,000 Monte Carlo experiments for the external habit model can produce such a big difference. Hence, it can be safely claimed that the model can't produce the same long horizon feature as that in the data. Figure 5 plots the histogram for these 1,000 differences between the low-frequency correlation and high-frequency correlation between consumption and asset returns from the 1,000 Monte Carlo experiments. Indeed, all the differences are less than the difference from the data, and most of these differences are negative as predicted by the model.

In conclusion, I have shown that the external habit formation model cannot match the long run features of the data.

3 Trend-Cycle Models

The previous section showed that the standard external habit formation model has difficulty matching the long run relation between consumption and asset returns. Hence, the question is what kind of model can produce the correct long run correlation and lead-lag relation between consumption growth and asset returns. In the standard Lucas tree model (Lucas (1978)), where *i.i.d.* consumption growth and CRRA preferences are usually assumed, the coherence, cospectrum, and phase spectrum are all flat. To obtain a decreasing coherence, cospectrum, and phase spectrum, it is necessary to modify either the preferences or the consumption dynamics. It is difficult to match the first two moments of the equity premium and the risk-free rate by modifying the consumption dynamics alone¹³. If consumption is

¹³If the CRRA preferences are maintained, but consumption growth is a predictable process (for example, AR(1)), and if a large risk aversion coefficient is assumed to generate enough of an equity premium, then the model could generate decreasing coherence and cospectrum. However, the phase spectrum would be

i.i.d., it is impossible to obtain the lead-lag effect, although it is possible to get the long-run correlation right. Therefore, to match the long run correlation, the lead-lag effect, and the equity premium, it appears necessary to modify both preferences and consumption dynamics for an endowment economy.

In the following, I give a sketch of a structural trend-cycle model to provide the motivation for the consumption dynamics and the link between expected returns and consumption. Then, through a reduced-form model, I illustrate the intuition on how this type of model can produce the right patterns in the cross-spectrum. Finally, I simulate two structural models to show that these models can indeed generate the desired long run features.

3.1 A Structural Trend-Cycle Model

In this section, I give a sketch of the trend-cycle model in Panageas and Yu (2005). There exists a continuum of firms indexed by $j \in [0, 1]$. Each firm owns a collection of trees that have been planted in different technological epochs N_t . Epoches arrive at the Poisson rate. Total earnings is just the sum of the earnings produced by the trees it owns. Each tree in turn produces earnings that are the product of three components: a) a vintage-specific component that is common across all trees of the same technological epoch, b) a time-invariant tree-specific component, and c) an aggregate productivity shock θ_t which follows a geometric Brownian motion. Any given firm determines the time at which it plants a tree in an optimal manner. Taken directly from Panageas and Yu (2005), the aggregate log consumption (which equals to aggregate log output) can be written as

$$c_t = \log(\theta_t) + N_t \log(\bar{A}) + x_t, \quad (3.1)$$

where the first two terms are the stochastic trend of the economy and the last term

$$x_t = \log \left[\sum_{n=-\infty..N-1} \bar{A}^{(n-N)} F(K_{n,\bar{\tau}_n}) + F(K_{N,t}) \right], \quad (3.2)$$

is a geometrically declining average of the random terms $F(K_{n,\bar{\tau}_n})$ which measures the tree-specific productivity of all the trees planted in technological epoch n . This means that x_t behaves like an autoregressive process (across epochs). Hence, the model is able to produce endogenous cycles, on top of the pure random walk stochastic trend $\log(\theta_t) + N_t \log(\bar{A})$ that we assumed at the outset.

increasing in this case since expected returns depend on the state variable, past consumption growth. This dependence is especially strong when the risk-aversion coefficient is large.

The expected excess return on the market is a weighted average of the returns on assets in place and the returns on the options to adopt new technologies. The expected returns on options are higher than those on assets in place. When the current level of consumption is below its stochastic trend, this implies that there is a large number of unexploited investment opportunities for firms. Accordingly, the relative weight of growth options will be substantial. Hence, up to a first-order approximation, the expected excess return in the Panageas and Yu (2005) model is given by

$$\mu_t - r \approx \alpha_0 - \beta_C x_t. \quad (3.3)$$

The Bansal and Yaron's model produces the same implication if there is stochastic volatility in consumption and this volatility is countercyclical. In a nutshell, this model implies that consumption consists of a random walk and an autoregressive cycle, and that the expected excess return is approximately a linear function of the cyclical component of consumption. To see how the trend-cycle models can produce the correct pattern of cross-spectrum, it is easiest to first work on a reduced-form model. I then simulate two structural models: Bansal and Yaron (2004) and Panageas and Yu (2005). Since a predictable consumption growth rate is key for trend-cycle models, I first fit an *ARMA* process to the quarterly consumption data, that I then input into the reduced form models.

3.2 The Estimation of Consumption Dynamics

The estimation results show that indeed a good description for log consumption is a stochastic trend plus an *AR*(2) cycle, which is equivalent to an *ARIMA*(2, 1, 2) process¹⁴. For an *ARIMA*(2, 1, 2) log consumption c_t , the consumption growth rate $g_{c,t}$ has the dynamics

$$g_{c,t} - \mu_c = \rho_{c,1}(g_{c,t-1} - \mu_c) + \rho_{c,2}(g_{c,t-2} - \mu_c) + \epsilon_{c,t} + \theta_{c,1}\epsilon_{c,t-1} + \theta_{c,2}\epsilon_{c,t-2}, \quad (3.4)$$

¹⁴As in Morley, Nelson, and Zivot (2003), there is a one-to-one correspondence between an *ARIMA*(2, 1, 2) and a trend-cycle decomposition with an *AR*(2) cycle component for the log consumption level. Furthermore, the *AR*(2) cyclic component is the simplest cycle dynamic such that all the parameters in the trend-cycle model are identifiable. Below, I will assume that log consumption follows a trend-cycle process that is equivalent to the current *ARIMA*(2, 1, 2).

where $\epsilon_{c,t} \sim WN(0, \sigma_c^2)$. This $ARIMA(2, 1, 2)$ process has the following equivalent trend-cycle representation for log consumption:

$$\begin{aligned} c_t &= T_t + x_t \\ T_t &= T_{t-1} + \mu_c + \xi_t \\ x_t &= \rho_{x,1}x_{t-1} + \rho_{x,2}x_{t-2} + \epsilon_{x,t}, \end{aligned} \tag{3.5}$$

where T_t is the stochastic trend, x_t is the cyclical component of log consumption, $\epsilon_{x,t} \sim WN(0, \sigma_{\epsilon_x}^2)$, $\xi_t \sim WN(0, \sigma_\xi^2)$, and $corr(\xi_t, \epsilon_{x,t}) = \rho_{\xi, \epsilon_x}$. Table 3 reports the estimates for the consumption process. All coefficients of the $ARIMA(2, 1, 2)$ are significant at the 5% level. Moreover, the implied correlation between the trend innovations and the cycle innovations is highly negative at $\rho_{\xi, \epsilon_x} = -0.9569$. This negative correlation is consistent with the implication of Panageas and Yu (2005), in which investment and consumption experience a delay when a new round of technological advancement arrives. Morley, Nelson, and Zivot (2003) also find a large negative correlation coefficient between the innovations in the trend and cycle components of GDP. A positive productivity shock (e.g., the invention of the internet) will immediately shift the long run path of output upwards, leaving actual output below the trend until it catches up.

I then use Kalman filter to extract the cyclical consumption component x_t . By regressing excess market returns on this cyclical component, I can estimate parameter β_C , which is around two in the data. The estimation of the latent variable x_t is not very accurate, so is the estimation of β_C . Hence, in the following analysis, I try different values of β_C and show that the results are not sensitive to the value of β_C .

3.3 A Reduced-Form Trend-Cycle Model

In this section, a reduced-form trend-cycle risk model is analyzed to provide intuition on why this class of models can produce the desired pattern in the cross-spectrum. I assume that the log consumption c_t consists of a stochastic trend component T_t plus an $AR(2)$ cycle component x_t as in equation (3.5), which is equivalent to the $ARIMA(2, 1, 2)$ process for log consumption. Hence, the consumption growth rate is given by $g_{c,t} = x_t - x_{t-1} + \xi_t$. In accordance with Bansal and Yaron (2004) and Panageas and Yu (2005), the expected excess return is further assumed to be negatively correlated with current cycle component x_t in the

following way¹⁵:

$$E_t(r_{t+1}) = \alpha_0 - \beta_C x_t, \quad \text{where } \beta_C > 0. \quad (3.6)$$

In a reduced-form model without a cointegration constraint, the realized excess return can be written as

$$r_t = \alpha_0 - \beta_C x_{t-1} + u_t,$$

where the innovation u_t is normally distributed with mean zero and standard error σ_u . If the dividends (in logs) are assumed to be cointegrated with consumption (in logs), then

$$\delta_t \equiv d_t - c_t = \mu_{dc} + \sum_{k=0}^{\infty} \psi_k \epsilon_{\delta,t-k}, \quad (3.7)$$

where $\sum_{k=1}^{\infty} |\psi_k| < \infty$. As I show in the Appendix, plugging equation (3.6) and equation (3.7) into Campbell-Shiller's log-linear approximation on returns implies that

$$r_t \approx \alpha_0 - \beta_C x_{t-1} + \epsilon_{\delta,t} \bar{\psi} + \epsilon_{x,t} \cdot (\rho^* + \beta_C \kappa_1 \bar{\rho}) + \xi_t, \quad (3.8)$$

where $\bar{\psi}$, ρ^* , and $\bar{\rho}$ are constants defined by equation (7.6) in the Appendix. Therefore, the cointegration constraint simply adds a restriction on the innovations on returns:

$$u_t \equiv \epsilon_{\delta,t} \bar{\psi} + \epsilon_{x,t} \cdot (\rho^* + \beta_C \kappa_1 \bar{\rho}) + \xi_t.$$

To determine the cross-spectrum between consumption growth and asset returns, we only need to know the correlation between the innovations on returns u_t and the innovations on the growth rate $(\xi_t, \epsilon_{x,t})$. Instead of estimating the parameters ψ_k , $\rho_{\delta,\xi} \equiv \text{corr}(\epsilon_{\delta,t}, \xi_t)$, $\rho_{\delta,\epsilon_x} \equiv \text{corr}(\epsilon_{\delta,t}, \epsilon_{x,t})$ (which might have substantial errors) and taking care of the internal link between the parameter κ_1 , the price dividend ratio, and returns, here I just fix the correlations $\rho_{u,\xi} \equiv \text{corr}(u_t, \xi_t)$ and $\rho_{u,\epsilon_x} \equiv \text{corr}(u_t, \epsilon_{x,t})$ at different values and plot the cross-spectrum under different scenarios. This approach allows me to examine the sensitivity of the cross-spectrum to the underlying parameters.

Figure 6 through Figure 8 plot the coherence, cospectrum, and phase spectrum under different parameter values. I fix the values for the parameters on the consumption dynamics (estimated in Section 3.2) and change the values of β_C , $\rho_{u,\xi}$, and ρ_{u,ϵ_x} . Moreover, σ_u is fixed at 0.08 to match the market volatility. In Figure 6, the parameter values are $\beta_C = 2$,

¹⁵Since the cyclical component x_t is assumed to be an $AR(2)$ process, it is more reasonable to assume that expected returns also depend on the lagged cyclical component. Here, for simplicity, I ignore the lagged cyclical component. The results are robust if the lagged cyclical component is included in the expected returns.

$\rho_{u,\xi} = 0$, and $\rho_{u,\epsilon_x} = 0$. In the data, these correlations are indeed very small. It can be seen that all the of them are downward sloping. When the values on the correlation are changed to $\rho_{u,\xi} = 0.2$, $\rho_{u,\epsilon_x} = -0.2$, the coherence increases and the slope becomes steeper. However, this decreasing pattern remains. As the predictability of returns is increased to $\beta_C = 5$, the results are similar. The economic intuition is as follows. To see the lead-lag relation, assume that the realized asset return is high in current period. It is likely that the past cycle level x_{t-1} is low. Since the cycle component x_t is mean-reverting, the future consumption growth is expected to be high. Thus, high asset returns predict a high consumption growth rate. To see the long horizon correlations, assume that consumption is well below its stochastic trend, then the expected return and the expected consumption growth are both high. In the short run, the randomness could weaken the relation between realized returns and realized consumption growth. However, consumption will eventually catch up, and the randomness will be cancelled away over longer horizons. Thus, the comovement between consumption growth and asset returns is stronger over longer horizons.

To mathematically see why this reduced-form model can produce the desired pattern in the spectrum, first examine the long horizon correlation between the consumption growth rate and asset returns. For simplicity, assume x_t is an AR(1) process. Then

$$\sum_{j=1}^K g_{t+j} = \sum_{j=1}^K \xi_{t+j} + \sum_{j=1}^K \rho_{x,1}^{K-j} \epsilon_{x,t+j} + (\rho_{x,1}^K - 1) x_t \quad (3.9)$$

$$\sum_{j=1}^K r_{t+j} = \sum_{j=1}^K u_{t+j} - \frac{\beta_C}{1 - \rho_{x,1}} \sum_{j=1}^K \epsilon_{x,t+j} + \frac{\beta_C}{1 - \rho_{x,1}} \sum_{j=1}^K \rho_{x,1}^{K-j} \epsilon_{x,t+j} - \beta_C \frac{\rho_{x,1}^K - 1}{\rho_{x,1} - 1} x_t \quad (3.10)$$

Notice that the first term on the right hand side of equation (3.9) and the first two terms on the right hand side of equation (3.10) are random walks, and the rest of terms are stationary. Hence, when K is big, the correlations among the random walks determine the correlation between cumulative growth rate and cumulative asset returns. Also notice that the covariances among stationary terms themselves and the covariances between the stationary terms and random walks are $cov\left(\sum_{j=1}^K \rho_{x,1}^{K-j} \epsilon_{x,t+j}, \frac{-\beta_C}{1 - \rho_{x,1}} \sum_{j=1}^K \left(1 - \rho_{x,1}^{K-j}\right) \epsilon_{x,t+j}\right) < 0$, $cov\left(\sum_{j=1}^K \rho_{x,1}^{K-j} \epsilon_{x,t+j}, \sum_{j=1}^K u_{t+j}\right)$, $cov\left(\sum_{j=1}^K \xi_{t+j}, \frac{\beta_C}{1 - \rho_{x,1}} \sum_{j=1}^K \rho_{x,1}^{K-j} \epsilon_{x,t+j}\right) < 0$, and $cov\left((\rho_{x,1}^K - 1) x_t, -\beta_C \frac{\rho_{x,1}^K - 1}{\rho_{x,1} - 1} x_t\right) > 0$. As long as the covariance $cov(u_t, \epsilon_{x,t})$ and $\sigma_{\epsilon_x}^2$ are not too big compared with $-cov(\xi_t, \epsilon_{x,t})$ (In the data, $cov(u_t, \epsilon_{x,t})$ is very close to zero, and negative.)¹⁶, the stationary terms contribute negatively to the correlation between cumulative consumption and asset returns. As a result, as the horizon K increases, this negative

¹⁶Actually, the exact condition is $(1 - \rho_{x,1}^2) \sigma_{\epsilon_x, u} + (1 - \rho_{x,1}) \beta_C \sigma_{\epsilon_x}^2 + (1 + \rho) \beta_C \sigma_{\xi, \epsilon_x} < 0$. This condition

effects from the stationary terms diminish. Therefore, the long horizon correlation between consumption and returns increases with horizon K .

3.4 Simulation From Structural Models

I have shown through a reduced-form trend-cycle model that this class of models can produce the desired relationship between consumption growth and asset returns at long horizons. In this section, based on a calibrated structural model in Panageas and Yu (2005)¹⁷, I simulate 48,000 quarters of excess returns and the consumption growth rate and I plot the cross-spectra of these two simulated series. Figure 9 plots the coherence, cospectrum, and phase spectrum for the simulated data. These spectra are indeed all decreasing. Moreover, the correlation between consumption growth and asset returns increases with horizons which is the same with the pattern as in the data.

It is important to remark here that the Bansal and Yaron (2004) model has similar long run implications. Following Bansal and Yaron (2004), the dynamics of consumption and dividends are assumed to be

$$\begin{aligned}\tilde{x}_{t+1} &= \rho_x \tilde{x}_t + \varphi_e \sigma_t e_{t+1} \\ g_{c,t+1} &= \mu + \tilde{x}_t + \sigma_t \eta_{t+1} \\ g_{d,t+1} &= \mu_d + \phi \tilde{x}_t + \varphi_d \sigma_t u_{t+1} \\ \sigma_{t+1}^2 &= \sigma^2 + \nu_1 (\sigma_t^2 - \sigma^2) + \sigma_w w_{t+1},\end{aligned}\tag{3.11}$$

where the innovations e_t , η_t , u_t , and w_t are *i.i.d.* $N(0, 1)$, and $\tilde{x}_t$ is the long run component in consumption. It follows from equation (A12) and equation (A14) of Bansal and Yaron

is typically satisfied when calibrated to the data since $\rho_{x,1}$ is close to one, and $\sigma_{\epsilon_x}^2$ is very small. Furthermore, when horizon K goes to infinity, the asymptotic correlation between cumulative consumption growth and asset returns is $\frac{\left[\sigma_{\xi,u} - \frac{\beta_C \sigma_{\xi,\epsilon_x}}{1-\rho}\right]}{\sigma_{\xi} \sqrt{\left[\sigma_u^2 + \frac{\beta_C^2 \sigma_{\epsilon_x}^2}{(1-\rho)^2} - 2\beta_C \frac{1}{1-\rho} \sigma_{\epsilon_x,u}\right]}}$, which is typically positive when using the estimated parameter values.

¹⁷In Panageas and Yu (2005), their model features two types of shocks: "small", frequent, and disembodied shocks to productivity and "large" technological innovations, which are embodied into new vintages of the capital stock. The latter type of shocks affect the economy with lags, since firms need to invest before they can take advantage of the new technology. This delayed reaction of consumption to large technological innovation helps them explain why the short run correlation between consumption and asset returns is weaker than its long run counterparts.

(2004) that the excess returns can be approximated by

$$\begin{aligned}
r_{t+1} \approx & \beta_{m,w} \lambda_{m,w} \sigma_w^2 - 0.5 \beta_{m,w}^2 \sigma_w^2 + (\beta_{m,e} \lambda_{m,e} - 0.5 \beta_{m,e}^2 - 0.5 \varphi_d^2) \sigma_t^2 \\
& + \kappa_{1,m} A_{1,m} \varphi_e \sigma_t e_{t+1} + \kappa_{1,m} A_{2,m} \sigma_w w_{t+1} + \varphi_d \sigma_t u_{t+1},
\end{aligned} \tag{3.12}$$

where all the constants are defined in the Appendix of Bansal and Yaron (2004). Taking the parameter values from the calibrated model of Bansal and Yaron (2004), the model can match the first two moments of the equity premium, risk-free rate, and consumption growth. With the above return and consumption dynamics, I simulate 48,000 quarters of artificial data. The resulting coherence, cospectrum, and phase spectrum are all decreasing as shown in Figure 10. Therefore, the long run risk model produces a higher comovement between consumption and returns over longer horizons and the model also implies that asset returns lead consumption. Notice that the phase spectrum at high frequencies are very volatile. This is because the coherence is very close to zero for high frequencies. When the comovement is weak, it is hard to determine which series are leading, and which series are lagging.

As a robustness check, I also run 1,000 Monte Carlo experiments, each with 100 years of observations, as in Section 2. A band-pass filter is used to obtain the difference between the low-frequency correlation and high-frequency correlation for each experiment. The Monte Carlo results show that the 95th quantile of the differences is 0.34, while this difference in the data is about 23%. Hence, the 95% confidence interval from the model includes the corresponding value from the data. The same is true for the model in Panageas and Yu (2005), with a 95th quantile of the differences equal 0.50.

4 Habit Formation Model With Predictable Consumption Growth

It has been shown that the external habit model with *i.i.d.* consumption growth cannot produce a consistent cross-spectrum between consumption and asset returns as seen in the data. It has also been shown that when consumption is assumed to have a cyclical component, the long-run risk and trend cycle models can produce the desired relation between consumption growth and asset returns. As a robustness check, I use the same $ARMA(2,2)$ consumption growth for the external habit formation model (as estimated in Section 3.2) and assume that

consumption is cointegrated with dividends as follows:

$$\begin{aligned} g_{c,t} &= \mu_c (1 - \rho_{c,1} - \rho_{c,2}) + \rho_{c,1}g_{c,t-1} + \rho_{c,2}g_{c,t-2} + \epsilon_{c,t} + \theta_{c,1}\epsilon_{c,t-1} + \theta_{c,2}\epsilon_{c,t-2} \\ \delta_t &\equiv d_t - c_t - \mu_{dc} = \rho_\delta \delta_{t-1} + \epsilon_{\delta,t}. \end{aligned}$$

Hence, dividend growth is given by

$$\begin{aligned} g_{d,t} &= \mu_c (1 - \rho_{c1} - \rho_{c2}) + \rho_{c,1}g_{c,t-1} + \rho_{c,2}g_{c,t-2} \\ &\quad + \epsilon_{c,t} + \theta_{c,1}\epsilon_{c,t-1} + \theta_{c,2}\epsilon_{c,t-2} + (\rho_\delta - 1) \delta_{t-1} + \epsilon_{\delta,t}. \end{aligned}$$

Here, I assume the same dynamics for the log surplus ratio s_t as in equation (2.2). The state variables in this economy are thus $(\delta_t, g_t, g_{t-1}, s_t, \epsilon_{c,t}, \epsilon_{c,t-1})$. In this model, the risk-free rate is no longer a constant; instead, it depends on the state variables. However, its variation is still very small. In fact, the risk-free rate follows

$$\begin{aligned} r_t^f &= -\log(E_t M_{t+1}) \\ &= \gamma \left[\begin{aligned} &(\phi - 1)(s_t - \bar{s}) + \mu_c(1 - \rho_{c,1} - \rho_{c,2}) \\ &+ \rho_{c,1}g_{c,t} + \rho_{c,2}g_{c,t-1} + \theta_{c,1}\epsilon_{c,t} + \theta_{c,2}\epsilon_{c,t-1} \end{aligned} \right] - 0.5\gamma^2 [1 + \lambda(s_t)]^2 \sigma_c^2 - \log(\beta) \end{aligned} \quad (4.1)$$

To solve for the price dividend ratio, a log-linear approximation to the log price dividend ratio is derived the same way as in the *i.i.d.* case. This linear approximated function is then used as the initial point to numerically solve for the exact price dividend ratio. This approach stabilizes the numerical solution. Table 4 lists the parameter values used in the simulation. The parameters for consumption dynamics are taken from the estimation results in Section 3, Table 3. Table 5 reports the summary statistics of the equity premium, risk-free rate, and price dividend ratio from the simulated data. As in the *i.i.d.* case, the model can match both the equity premium and risk-free rate.

Table 6 shows the correlation between consumption and asset returns at different horizons. The correlation is decreasing for the simulated model as the horizon increases. Figure 11 plots the coherence, cospectrum, and phase spectrum in this generalized model. It can be seen that the long run correlation between consumption growth and asset returns is still negative as in Section 2. Furthermore, from the phase spectrum, consumption still leads asset returns, as in Section 2. Although there is a hump-shaped cospectrum that is similar to that in the data, the cospectrum in the model is very large at high frequencies compared with these quantities in the data. Hence, the main message in the *i.i.d.* case remains true even if the consumption growth rate is assumed to be an *ARMA*(2, 2) process in the external habit formation model.

As in the case of *i.i.d.* consumption growth, I run 1,000 Monte Carlo experiments, each with 100 years of observations. A band-pass filter is used to obtain the difference between the low-frequency correlation and high-frequency correlation for each experiment. The Monte Carlo results show that the 95th quantile of the differences is 4%, while this difference in the data is about 23%. Furthermore, none of these 1,000 differences in the simulation is as big as 23%. Therefore, even with predictable consumption the model cannot generate such a big difference between low-frequency correlation and high-frequency correlation as in the data.

To understand why the results still hold in the case with predictable consumption growth, a simplified model with $AR(1)$ consumption growth is considered in the following. Let $\rho_{c,2} = 0$, $\theta_{c,1} = 0$, and $\theta_{c,2} = 0$. The previous analysis shows that the cointegration constraint doesn't play a significant role. Therefore, to simplify the model further, assume $\rho_\delta = 1$. Under this set of simplified assumptions, it is shown in the Appendix that the excess returns can be given by

$$r_{t+1} \approx \alpha_1 + \frac{(\phi\kappa_1 - 1)a_1}{\bar{S}}\tilde{S}_t + \left(\kappa_1 a_3 + 1 + \kappa_1 a_1 \frac{1 - \bar{S}}{\bar{S}}\right)\epsilon_{c,t+1} + \epsilon_{\delta,t+1}, \quad (4.2)$$

where α_1 is some constant. Notice that the expected excess return only depends on the surplus ratio, although the expected gross return indeed depends on the other state variable, lagged growth rate. It follows from the same argument following equation (2.10) in Section 2 that the correlations between consumption growth and asset returns are decreasing as horizons increase. Again, since the surplus ratio is approximately weighted average of past consumption growth, equation (4.2) implies that consumption growth leads asset returns. Accordingly, the external habit persistence model with predictable consumption growth cannot produce desired lead-lag relation and a higher low-frequency correlation between consumption and returns because the main driving force is still the surplus ratio.

5 Robustness Checks

At the end of Section 2, a spectral analysis shows that the coherence, cospectrum, and phase spectrum between quarterly consumption growth and quarterly asset returns are decreasing. The purpose of this section is to show that these features in the data are robust across different data samples and econometric methods. Furthermore, a band-pass filter analysis and Granger causality test are applied to the simulated data from different models. The results corroborate the model implications from previous spectral analysis.

5.1 Band-Pass Filter Analysis and Granger Causality Test

In this section, I perform a band-pass filter analysis and Granger causality test on both real data and artificial data simulated from different models. The band-pass filter is used to extract the low-frequency and high-frequency components of consumption and asset returns. The resulting correlations between the consumption growth rate and market returns at different frequencies are then calculated. For real data, the correlation is 0.114 for higher frequencies (with cycles between 2 and 12 quarters) and 0.342 for lower frequencies (with cycles longer than 12 quarters). This confirms the earlier result from spectral analysis that the comovement between consumption and asset returns is stronger over longer horizons. The first two rows of Table 7 list the low-frequency and high-frequency correlations for different models and the data. It can be seen that the external habit formation models produce higher correlations at high frequencies, while the long run risk model, trend-cycle model and real data generate higher correlations at lower frequencies. This confirms the earlier coherence and cospectrum results. To get rid of the small sample problem, the last row of Table 7 reports the 95th quantile of the difference between the low-frequency correlation and high-frequency correlation between consumption and returns. It is seen that the 95th quantiles from the external habit models are much smaller than the value in the data. However, the 95th quantiles for the trend-cycle models are higher than the value from the data. Hence, the trend-cycle models are consistent with the data, while the external habit formation models are not.

The phase spectrum analysis in Section 2 shows that stock market returns lead consumption growth. Here, I conduct a formal Granger causality test. To implement this test, I assume an autoregressive lag length of two and estimate the following equation by OLS:

$$r_t = c_1 + \alpha_1 r_{t-1} + \alpha_2 r_{t-2} + \beta_1 g_{c,t-1} + \beta_2 g_{c,t-2} + u_{r,t},$$

where r_t is the quarterly market excess return, and $g_{c,t}$ is the quarterly consumption growth rate. I then conduct an F test of the following null hypothesis:

$$H_0 : \beta_1 = \beta_2 = 0.$$

Similarly, I can estimate the OLS

$$g_{c,t} = c_2 + \gamma_1 g_{c,t-1} + \gamma_2 g_{c,t-2} + \eta_1 r_{t-1} + \eta_2 r_{t-2} + u_{c,t},$$

and then conduct an F test of the null hypothesis

$$H_0 : \eta_1 = \eta_2 = 0.$$

The p -value of the Granger causality test of consumption Granger-causing returns is 0.4482, while the p -value of the Granger causality test of returns Granger-causing consumption is 4.3770×10^{-4} . Hence, the statistical test indicates that stock market returns do Granger-cause consumption, while consumption does not Granger-cause stock market returns. Therefore, the Granger causality test confirms our phase spectrum results. For the annual data, the results are stronger.

Tables 8 and 9 report the Granger causality test results for the Fama-French 25 portfolios and the consumption growth rate. Table 8 gives the p -value for the test of consumption growth Granger-causing asset returns. All these p -values are large, so consumption growth does not Granger-cause asset returns. Table 9 gives the p -value for the test of asset returns Granger-causing consumption growth. All of these p -values are very small. Hence, the Fama-French 25 portfolio returns do Granger-cause consumption growth. This confirms our results for the aggregate market data.

Table 10 presents the p -values from Granger causality tests for different models and real data. The results show that for the external habit formation models, the consumption growth Granger-causes asset returns. However, it is asset returns that Granger-cause consumption for the long-run risk model, trend-cycle model, and real data. Hence, the band-pass filter analysis and the Granger causality tests both confirm the earlier results from spectral analysis.

5.2 Parametric Estimation for Cross-Spectrum and Spectral Analysis for Annual Data

To reconfirm the results from the spectral analysis for the real data in Section 2, I first divide the whole sample into two subsamples, the resulting graphs continue to display the same decreasing pattern as the graphs in Figure 3.

A parametric method can also be used to estimate the cross-spectrum. First, I estimate a $VAR(2)$ for consumption growth and market excess returns. Then, by using the estimated parameter values, the cross-spectrum between consumption growth and asset returns can be obtained analytically as plotted in Figure 12 (detailed calculations are given in the Appendix). It can be seen that the decreasing pattern remains. The phase spectrum is increasing for very high frequencies. However, it is decreasing for horizons longer than one year. Since most covariation between consumption and returns comes from low-frequency cycles, overall, asset returns still lead consumption. It is also worth noting that the phase

spectrum at very high frequencies are sensitive to different estimation methods and subsamples. In particular, the confidence intervals at high frequencies are very wide because the coherence is small at high frequencies. However, all of the other decreasing patterns are very robust to different estimation approaches and subsamples. Since there might be serious measurement errors in the pre-war consumption data, I mainly focus on post-war quarterly consumption data. Furthermore, quarterly data have more observations, so the power of the statistical inference is larger. However, as a robustness check, I also plot the cross-spectrum for annual data in Figure 13. The observed patterns are the same as those in quarterly data.

The higher correlation at long horizons could result from frictions such as delayed consumption because of inattention. However, it is hard to believe that these frictions can affect the correlation at horizons longer than one year. Therefore, the higher correlation between consumption growth and asset returns must originate from more fundamental economic reasons. This paper does not investigate the origin of these economic forces.

5.3 Parametric Estimation for Cross-Spectrum with Cointegration

Last section provides a parametric estimation of the cross-spectrum between consumption and returns through a standard VAR system. In this section, I will show that the results from last section are robust if the cointegration constraint between consumption and dividends is incorporated into the VAR system. I first estimate the cointegrating relation between dividends and consumption by taking the difference between log dividends and log consumption¹⁸. Using this cointegrating residual, δ_t , I model its dynamics jointly with consumption growth and market excess returns, via the following first-order error-correction VAR (EC-VAR) structure:

$$\begin{bmatrix} g_t \\ \delta_t \\ r_t \end{bmatrix} = \begin{bmatrix} \rho_c & a_{c\delta} & a_{cr} \\ a_{\delta c} & \rho_\delta & a_{\delta r} \\ a_{rc} & a_{r\delta} & \rho_r \end{bmatrix} \begin{bmatrix} g_{t-1} \\ \delta_{t-1} \\ r_{t-1} \end{bmatrix} + \begin{bmatrix} \epsilon_{c,t} \\ \epsilon_{\delta,t} \\ \epsilon_{r,t} \end{bmatrix}. \quad (5.1)$$

Notice that this EC-VAR specification reduces to the standard VAR as in previous section if the error-correction term, δ_t , is excluded from the EC-VAR. Without cointegration constraint, consumption and dividends can arbitrarily drift far away from each other. Cointegration ties the long run dynamics of the two series together. As a result, the error-correction

¹⁸I also estimate this cointegrating residual by regressing the de-trended dividends onto the stochastic trend in consumption. The results for the following spectral analysis are almost identical.

term may be important for predicting future growth rate and returns, and its inclusion could potentially alter the dynamics between consumption growth and returns (See Bansal, Dittmar, and Kiku (2007), Bansal and Yaron (2006))). In a nutshell, there could be large differences between the standard VAR in previous section and the EC-VAR in this section for the implications on the relation between consumption and returns. After the EC-VAR is estimated from the data, the coherence, cospectrum, and phase spectrum between consumption growth and asset returns can be derived analytically. They are plotted in Figure 14. As we can see, the decreasing pattern in these spectra remains. Hence, even with the cointegration constraint, the data still indicate that more covariation between consumption and returns comes from low-frequency components, and asset returns lead consumption.

Since the consumption data is time aggregated, it is worthy to emphasize that the pattern in spectrum remains even when only the odd (or even) observations are used in the analysis. Hence, the decreasing pattern in the cross-spectrum is not a result of time-aggregation in the data.

5.4 Other Forms of Habit Formation Model

In Section 2, I have shown that the external habit formation model with difference utility form cannot match the long-run features of the data. If I change the specific sensitivity function $\lambda(s_t)$ as in Campbell and Cochrane (1999), the interest rate would change a lot. However, the equity premium would remain very similar. Hence, the results obtained previously are valid even with a different sensitivity function. Abel (1990) proposes a ratio form of external habit formation (Abel calls this catching up with the Joneses). Under Abel's model, it can be shown that both coherence and cospectrum between consumption growth and gross equity returns are increasing, as in the difference form of external habit formation models¹⁹. Even with predictable consumption growth, the above results are still true if the risk-aversion coefficient is large enough to produce a reasonable equity premium. Hence, for the ratio-form of external habit model, it also produces higher covariations between consumption and returns at higher frequencies, the same as the difference-form of external habit model.

The reason why external habit models produce the opposite pattern of the data is that the expected returns negatively depend on the past consumption growth. For any model with

¹⁹Notice that under the case of *i.i.d.* consumption growth, the coherence and cospectrum between consumption and excess returns are constant. Whereas in the data the coherence and cospectrum between consumption growth and gross returns (and excess returns) are decreasing.

such implications on expected returns, it is likely for that model to generate the opposite results as the data. For internal habit models, the marginal rate of substitution depends on both current surplus ratio and expected future consumption growth. As a result, it is less of a problem for internal habit models.

6 Conclusions

In this paper, I analyze asset pricing models by focusing on their low-frequency implications. I argue that the standard external habit formation model faces a challenge in generating the same long run correlation and lead-lag relation between consumption and market returns as in the data. However, models such as Bansal and Yaron (2004) and Panageas and Yu (2005) can generate the same pattern as the one found in the data. In these models log consumption includes both a stochastic trend and a cyclical component. Moreover, the expected return depends negatively on the cyclical consumption component. I conclude that forward-looking consumption terms in the pricing of risk are important to match these features of the data.

In future research, the methodology in this paper could also be applied to cross-sectional analysis such as value premium. Preliminary analysis shows that the slope of the coherence and cospectrum are steeper for value portfolios than growth portfolios, and that the low-frequency correlation between consumption and value stocks is higher than the low-frequency correlation between consumption and growth stocks. This property in the data could be used to evaluate the performance of different asset pricing models that can produce the value premium. For instance, the analysis could be applied to the cash-flow sensitivity model by Bansal, Dittmar, and Lundblad (2001) and the duration model by Lettau and Wachter (2007).

7 Appendix

Log-Linear Approximation to Price Dividend Ratio and Returns:

To derive the log-linear approximation to the price dividend ratio and asset returns, let $m_t = \log(M_t)$ be the log IMRS. Plugging the log-linear approximation to returns $r_{t+1} \approx k_0 + g_{d,t+1} + \kappa_1 z_{t+1} - z_t$ and the linear approximation to the log price dividend ratio $z_t \approx a_0 + a_1 s_t + a_2 \delta_t$ into the Euler equation, I obtain

$$1 = E_t(\exp(m_{t+1} + r_{t+1}))$$

$$= \exp \left(\begin{array}{l} \left[\begin{array}{l} \gamma(\phi - 1)\bar{s} + \log(\beta) + k_0 + \mu_c(1 - \gamma) \\ + a_0\kappa_1 - a_0 + a_1\kappa_1(1 - \phi)\bar{s} + 0.5[1 + a_2\kappa_1]^2\sigma_\delta^2 \end{array} \right] \\ + [-\gamma(\phi - 1) - a_1 + a_1\kappa_1\phi]s_t \\ + [a_2\kappa_1\rho_\delta - a_2 + (\rho_\delta - 1)]\delta_t \\ + 0.5[1 + a_1\kappa_1\lambda(s_t) - \gamma[1 + \lambda(s_t)]]^2\sigma_c^2 \\ + [1 + a_1\kappa_1\lambda(s_t) - \gamma[1 + \lambda(s_t)]] [1 + a_2\kappa_1]\sigma_{c\delta} \end{array} \right).$$

Replacing the sensitivity function $\lambda(s)$ with its linear approximation $\lambda(s) \approx -a_\lambda(s - s_{\max})$ in the above equation, and, setting the coefficients in front of the state variables to zero, it follows that

$$a_2 = \frac{\rho_\delta - 1}{1 - \kappa_1\rho_\delta}. \quad (7.1)$$

a_1 can be determined as the unique positive root of the following quadratic equation if consumption growth is positively correlated with dividend growth²⁰:

$$0 = \left[2a_\lambda\kappa_1^2 - \frac{2\kappa_1^2}{\bar{S}^2} \right] a_1^2 + \left[\frac{\kappa_1\phi - 1}{0.5\sigma_c^2} - \frac{1}{0.5\sigma_c^2}a_\lambda[1 + a_2\kappa_1]\sigma_{c,\delta}\kappa_1 - 2a_\lambda\kappa_1(1 + \gamma) + 4\gamma\kappa_1\frac{1}{\bar{S}^2} \right] a_1$$

$$+ \left[\frac{1}{0.5\sigma_c^2}a_\lambda[1 + a_2\kappa_1]\sigma_{c,\delta}\gamma + 2a_\lambda\gamma - \frac{\gamma(\phi - 1)}{0.5\sigma_c^2} - \frac{2\gamma^2}{\bar{S}^2} \right]. \quad (7.2)$$

²⁰Notice that $\lambda'(s) \leq \frac{1}{\bar{S}^2}$, hence, it is natural to choose $0 < a_\lambda < \frac{1}{\bar{S}^2}$ to approximate the sensitivity function. Hence, the coefficient $2a_\lambda\kappa_1^2 - \frac{2\kappa_1^2}{\bar{S}^2}$ in the quadratic equation (7.2) is negative. Further, the constant term in the quadratic equation satisfies

$$\frac{1}{0.5\sigma_c^2}a_\lambda[1 + a_2\kappa_1]\sigma_{c,\delta}\gamma + 2a_\lambda\gamma - \frac{\gamma(\phi - 1)}{0.5\sigma_c^2} - \frac{2\gamma^2}{\bar{S}^2}$$

$$= \left[\left(\frac{1 - \kappa_1}{1 - \kappa_1\rho_\delta} \right) \frac{\sigma_{c,\delta}}{\sigma_c^2} + 1 \right] 2a_\lambda\gamma$$

If we assume $\frac{\sigma_{c,\delta}}{\sigma_c^2} \geq -1$, then the constant term in the quadratic equation is positive. Hence, there is a unique positive root for equation (7.2). Notice that $\epsilon_{c,t}$ is the innovation in c_t , and $\epsilon_{\delta,t}$ is the innovation in d_t minus the innovation in c_t . As long as the innovations in c_t and the innovations d_t are positively correlated (i.e. the consumption growth and dividend growth are positively correlated), $\frac{\sigma_{c,\delta}}{\sigma_c^2} > -1$ holds.

a_0 can be determined by the equation

$$a_0 = \frac{1}{1 - \kappa_1} \left[\begin{aligned} & \gamma (\phi - 1) \bar{s} + \log(\beta) + k_0 + \mu_c (1 - \gamma) \\ & + a_1 \rho (1 - \phi) \bar{s} + 0.5 [1 + a_2 \kappa_1]^2 \sigma_\delta^2 \\ & + 0.5 \sigma_c^2 \left[\begin{aligned} & (a_1 \kappa_1 - \gamma)^2 \frac{1}{\bar{S}^2} (1 + 2\bar{s}) + 2 (1 - a_1 \kappa_1) (a_1 \kappa_1 - \gamma) (1 + a_\lambda s_{\max}) \\ & + (1 - a_1 \kappa_1)^2 + [1 - \gamma + (a_1 \kappa_1 - \gamma) a_\lambda s_{\max}] \cdot [1 + a_2 \kappa_1] \frac{\sigma_{c,\delta}}{0.5 \sigma_c^2} \end{aligned} \right] \end{aligned} \right]. \quad (7.3)$$

Plug this linear approximation on the price dividend ratio back into Campbell-Shiller log-linear approximation to returns to obtain

$$\begin{aligned} r_{t+1} &\approx [\kappa_0 + \mu_c - a_0 + a_0 \kappa_1 + a_1 \kappa_1 (1 - \phi) \bar{s}] + [a_1 \kappa_1 \phi - a_1] s_t + [(\rho_\delta - 1) - a_2 + a_2 \kappa_1 \rho_\delta] \delta_t \\ &\quad + [1 + a_1 \kappa_1 \lambda(s_t)] \epsilon_{c,t+1} + [1 + a_2 \kappa_1] \epsilon_{\delta,t+1} \\ &\approx [\kappa_0 + \mu_c - a_0 + a_0 \kappa_1 + a_1 \kappa_1 (1 - \phi) \bar{s}] + [a_1 \kappa_1 \phi - a_1] \left(\frac{S_t}{\bar{S}} + \log(\bar{S}) - 1 \right) \\ &\quad + [(\rho_\delta - 1) - a_2 + a_2 \kappa_1 \rho_\delta] \delta_t + [1 + a_1 \kappa_1 \lambda(s_t)] \epsilon_{c,t+1} + [1 + a_2 \kappa_1] \epsilon_{\delta,t+1} \\ &\approx [\kappa_0 + \mu_c - a_0 + a_0 \kappa_1 + a_1 \kappa_1 (1 - \phi) \bar{s} + (a_1 \kappa_1 \phi - a_1) (\log(\bar{S}) - 1)] + \frac{a_1 \kappa_1 \phi - a_1}{\bar{S}} S_t \\ &\quad + [1 + a_1 \kappa_1 \lambda(s_t)] \epsilon_{c,t+1} + [1 + a_2 \kappa_1] \epsilon_{\delta,t+1}. \end{aligned}$$

Letting

$$\alpha = \kappa_0 + \mu_c - a_0 + a_0 \kappa_1 + a_1 \kappa_1 (1 - \phi) \bar{s} + (a_1 \kappa_1 \phi - a_1) (\log(\bar{S}) - 1) \quad (7.4)$$

$$\beta_S = \frac{a_1 (1 - \kappa_1 \phi)}{\bar{S}}, \quad (7.5)$$

and further approximating $\lambda(s_t)$ with $\lambda(\bar{s}) = \frac{1 - \bar{S}}{\bar{S}}$, it follows that

$$r_{t+1} \approx \alpha - \beta_S S_t + \left[1 + a_1 \kappa_1 \frac{1 - \bar{S}}{\bar{S}} \right] \epsilon_{c,t+1} + [1 + a_2 \kappa_1] \epsilon_{\delta,t+1}.$$

Proof of Propositions 1 & 2:

First, the surplus ratio S_t can be approximated by a smoothed average of past consumption innovations as in equation (2.7). Notice that the habit level can be approximated by $X_t \approx \frac{1 - \phi}{\phi} \sum_{k=1}^{\infty} \phi^k C_{t-k}$. Plug this equality back into the definition of the surplus ratio S_t to obtain

$$\begin{aligned} S_t &= 1 - \frac{1 - \phi}{\phi} \sum_{k=1}^{\infty} \phi^k \exp \left(- \sum_{j=t-k+1}^t g_j \right) \\ &\approx 1 - \frac{1 - \phi}{\phi} \sum_{k=1}^{\infty} \phi^k \left(1 - \sum_{j=t-k+1}^t g_j \right) \\ &= \sum_{j=1}^{\infty} \phi^{j-1} g_{t+1-j}. \end{aligned}$$

Hence, equation (2.7) follows. Now, substituting equation (2.7) back into equation (2.5), then Replacing each term in equations (2.5) and (2.1) by its spectral representation, and noting that the resulting equations are valid for all t , I obtain the following equations for the orthogonal increment processes Z_{g_c} , Z_r , Z_{ϵ_c} , and Z_{ϵ_δ} in the spectral representations of $\{g_t\}$, $\{r_t\}$, $\{\epsilon_{c,t}\}$, and $\{\epsilon_{\delta,t}\}$:

$$\begin{aligned} dZ_{g_c}(\lambda) &= dZ_{\epsilon_c}(\lambda) \\ dZ_r(\lambda) &= -\beta_S \sum_{j=1}^{\infty} \phi^{j-1} e^{-ij\lambda} \cdot dZ_{g_c}(\lambda) + \left[1 + a_1 \kappa_1 \frac{1 - \bar{S}}{\bar{S}}\right] dZ_{\epsilon_c}(\lambda) + [1 + a_2 \kappa_1] dZ_{\epsilon_\delta}(\lambda). \end{aligned}$$

Notice that $\sum_{j=1}^{\infty} \phi^{j-1} e^{-ij\lambda} = \frac{e^{-i\lambda}}{1 - \phi e^{-i\lambda}}$. Solve for $dZ_{g_c}(\lambda)$ and $dZ_r(\lambda)$ to obtain

$$\begin{aligned} dZ_{g_c}(\lambda) &= dZ_{\epsilon_c}(\lambda) \\ dZ_r(\lambda) &= \left[\frac{-\beta_S e^{-i\lambda}}{1 - \phi e^{-i\lambda}} + 1 + a_1 \kappa_1 \frac{1 - \bar{S}}{\bar{S}} \right] dZ_{\epsilon_c}(\lambda) + [1 + a_2 \kappa_1] dZ_{\epsilon_\delta}(\lambda). \end{aligned}$$

It follows that the multivariate spectrum is given by

$$\begin{aligned} 2\pi f_{11}(\lambda) &= \sigma_c^2 \\ 2\pi f_{22}(\lambda) &= \left| \frac{-\beta_S e^{-i\lambda}}{1 - \phi e^{-i\lambda}} + 1 + a_1 \kappa_1 \frac{1 - \bar{S}}{\bar{S}} \right|^2 \sigma_c^2 + [1 + a_2 \kappa_1]^2 \sigma_\delta^2 \\ &\quad + 2\text{Re} \left(\kappa_1 \frac{e^{-i\lambda}}{1 - \phi e^{-i\lambda}} + 1 + a_1 \kappa_1 \frac{1 - \bar{S}}{\bar{S}} \right) [1 + a_2 \kappa_1] \sigma_{c\delta} \\ 2\pi f_{12}(\lambda) &= \left(\frac{-\beta_S e^{-i\lambda}}{1 - \phi e^{-i\lambda}} + 1 + a_1 \kappa_1 \frac{1 - \bar{S}}{\bar{S}} \right)' \sigma_c^2 + [1 + a_2 \kappa_1] \sigma_{c\delta} \\ &= \left(\frac{\beta_S (\phi - e^{i\lambda})}{1 + \phi^2 - 2\phi \cos(\lambda)} + 1 + a_1 \kappa_1 \frac{1 - \bar{S}}{\bar{S}} \right) \sigma_c^2 + [1 + a_2 \kappa_1] \sigma_{c\delta}. \end{aligned}$$

For the cospectrum $C_{sp}(\lambda)$, the real part of the cross-spectrum $f_{12}(\lambda)$,

$$C_{sp}(\lambda) = \left(\frac{\beta_S (\phi - \cos(\lambda))}{1 + \phi^2 - 2\phi \cos(\lambda)} + 1 + a_1 \kappa_1 \frac{1 - \bar{S}}{\bar{S}} \right) \sigma_c^2 + [1 + a_2 \kappa_1] \sigma_{c\delta}.$$

Therefore, the derivative of the cospectrum is

$$\begin{aligned} C'_{sp}(\lambda) &= \beta_S \frac{\sin(\lambda) (1 + \phi^2 - 2\phi \cos(\lambda)) + 2\phi \sin(\lambda) (\cos(\lambda) - \phi)}{(1 + \phi^2 - 2\phi \cos(\lambda))^2} \\ &= \frac{\beta_S \sin(\lambda)}{(1 + \phi^2 - 2\phi \cos(\lambda))^2} [1 - \phi^2] \geq 0, \end{aligned}$$

and the portion of covariance contributed by components at frequency λ is increasing in the frequency λ . By definition, the coherence and the phase are, respectively,

$$h(\lambda) = \frac{|f_{12}|}{\sqrt{f_{11}f_{22}}}$$

$$\tan(\phi(\lambda)) = \frac{\frac{\beta_S \sin(\lambda)}{1+\phi^2-2\phi \cos(\lambda)} \sigma_\epsilon^2}{\left(\frac{\beta_S(\phi-\cos(\lambda))}{1+\phi^2-2\phi \cos(\lambda)} + 1 + a_1\kappa_1 \frac{1-\bar{S}}{\bar{S}} \right) \sigma_c^2 + [1 + a_2\kappa_1] \sigma_{cd}}.$$

At the frequency $\lambda = 0$, the cross-spectrum is

$$f_{12}(0) = \left(-\beta_S \frac{1-\phi}{1+\phi^2-2\phi} + 1 + a_1\kappa_1 \frac{1-\bar{S}}{\bar{S}} \right) \sigma_c^2 + [1 + a_2\kappa_1] \sigma_{cd}$$

$$= \left(-\frac{a_1(1-\kappa_1)}{\bar{S}(1-\phi)} + 1 - a_1\kappa_1 \right) \sigma_c^2 + [1 + a_2\kappa_1] \sigma_{cd}.$$

Therefore, the low-frequency correlation between consumption growth and asset returns is negative if and only if $\left(-\frac{a_1(1-\kappa_1)}{\bar{S}(1-\phi)} + 1 - a_1\kappa_1 \right) \sigma_c^2 + [1 + a_2\kappa_1] \sigma_{cd} < 0$. To find the conditions for a negative correlation at long horizons, first write down the long horizon return and the long horizon consumption growth rate

$$\sum_{j=1}^K r_{t+j} \approx \sum_{j=1}^K \alpha - \beta_S \tilde{S}_{t+j-1} + \left(1 + a_1\kappa_1 \frac{1-\bar{S}}{\bar{S}} \right) \epsilon_{c,t+j} + (1 + a_2\kappa_1) \epsilon_{\delta,t+j}$$

$$\approx K\alpha - \beta_S \sum_{j=1}^K \sum_{k=1}^{\infty} \phi^{k-1} \epsilon_{c,t+j-k} + \left(1 + a_1\kappa_1 \frac{1-\bar{S}}{\bar{S}} \right) \sum_{j=1}^K \epsilon_{c,t+j} + (1 + a_2\kappa_1) \sum_{j=1}^K \epsilon_{\delta,t+j}$$

$$\sum_{j=1}^K g_{c,t+j} = K\mu_c + \sum_{j=1}^K \epsilon_{c,t+j}.$$

Then, the long horizon covariances between return and consumption are

$$\text{cov} \left(\sum_{j=1}^K r_{t+j}, \sum_{j=1}^K g_{c,t+j} \right)$$

$$= \beta_S \sigma_c^2 \frac{1}{1-\phi} + \beta_S \sigma_c^2 \frac{\phi}{1-\phi} \frac{1-\phi^{K-1}}{1-\phi} + K \left(\sigma_c^2 \frac{\beta_S}{1-\phi} + \left[1 + a_1\kappa_1 \frac{1-\bar{S}}{\bar{S}} \right] \sigma_c^2 + [1 + a_2\kappa_1] \sigma_{cd} \right).$$

Notice that the long run variances are

$$\text{var} \left(\sum_{j=1}^K r_{t+j} \right) = \left[(K-1) \left(\frac{-\beta_S}{1-\phi} + 1 + a_1\kappa_1 \frac{1-\bar{S}}{\bar{S}} \right)^2 + \left(\frac{\beta_S}{1-\phi} \right)^2 \frac{\phi^2 \frac{1-\phi^{2(K-1)}}{1-\phi^2}}{1-\phi} \right] \sigma_c^2$$

$$+ \left[\frac{1-\phi^K}{1-\phi} \right]^2 \sigma_c^2 \frac{\beta_S^2}{1-\phi^2} + \left(1 + a_1\kappa_1 \frac{1-\bar{S}}{\bar{S}} \right)^2 \sigma_c^2 + K (1 + a_2\kappa_1)^2 \sigma_\delta^2$$

$$+ \left[(1-K) \frac{\beta_S}{1-\phi} + K \left(1 + a_1\kappa_1 \frac{1-\bar{S}}{\bar{S}} \right) + \frac{\beta_S \phi}{1-\phi} \frac{1-\phi^{K-1}}{1-\phi} \right] (1 + a_2\kappa_1) \sigma_{cd}.$$

Hence, the correlation at horizon K is just $\sqrt{\frac{cov(\sum_{j=1}^K r_{t+j}, \sum_{j=1}^K g_{c,t+j})}{var(\sum_{j=1}^K r_{t+j})var(\sum_{j=1}^K g_{c,t+j})}}$. When the horizon K is sufficiently large, the following quantity determines the sign of the correlation at very long horizons:

$$\begin{aligned} & \sigma_c^2 \frac{\beta_S}{\phi - 1} + \left[1 + a_1 \kappa_1 \frac{1 - \bar{S}}{\bar{S}} \right] \sigma_c^2 + [1 + a_2 \kappa_1] \sigma_{c\delta} \\ &= \left[1 - a_1 \kappa_1 - \frac{a_1 (1 - \kappa_1)}{\bar{S} (1 - \phi)} \right] \sigma_c^2 + (1 + a_2 \kappa_1) \sigma_{c\delta}. \end{aligned}$$

It can be seen that the sign of long-run correlation is the same as the sign of the correlation at frequency $\lambda = 0$.

By differentiating equation (2.12), the sign of the slope of the phase spectrum can be examined. To see this, start with

$$\begin{aligned} \phi'(\lambda) \propto & - \left\{ \beta_S (\phi - \cos(\lambda)) \sigma_c^2 + \left[\left(1 + a_1 \kappa_1 \frac{1 - \bar{S}}{\bar{S}} \right) \sigma_c^2 + [1 + a_2 \kappa_1] \sigma_{c\delta} \right] (1 + \phi^2 - 2\phi \cos(\lambda)) \right\} \\ & \cdot \beta_S \cos(\lambda) + \beta_S \sin(\lambda) \left\{ \beta_S \sin(\lambda) \sigma_c^2 + 2\phi \left[\left(1 + a_1 \kappa_1 \frac{1 - \bar{S}}{\bar{S}} \right) \sigma_c^2 + [1 + a_2 \kappa_1] \sigma_{c\delta} \right] \sin(\lambda) \right\}. \end{aligned}$$

Rearrange and simplify to obtain

$$\begin{aligned} \phi'(\lambda) \propto & - \frac{a_1 (\kappa_1 \phi - 1)}{\bar{S}} + 2\phi \left(1 + a_1 \kappa_1 \frac{1 - \bar{S}}{\bar{S}} \right) + 2\phi [1 + a_2 \kappa_1] \frac{\sigma_{c\delta}}{\sigma_c^2} \\ & - \left\{ 1 + a_1 \kappa_1 \frac{1 - \bar{S}}{\bar{S}} + (1 + a_2 \kappa_1) \frac{\sigma_{c\delta}}{\sigma_c^2} + \phi^2 + \frac{-a_1 \kappa_1 \bar{S} \phi^2}{\bar{S}} + \frac{a_1 \phi}{\bar{S}} + (1 + a_2 \kappa_1) \phi^2 \frac{\sigma_{c\delta}}{\sigma_c^2} \right\} \cos(\lambda) \\ \geq & a_1 \frac{(1 - \kappa_1) (1 - \phi) + \kappa_1 \bar{S} (1 - \phi)^2}{\bar{S}} - (1 - \phi)^2 - (1 + a_2 \kappa_1) \frac{\sigma_{c\delta}}{\sigma_c^2} (1 - \phi)^2 \\ = & - \left[1 - a_1 \frac{1 - \kappa_1 + \kappa_1 \bar{S} (1 - \phi)}{\bar{S} (1 - \phi)} + (1 + a_2 \kappa_1) \frac{\sigma_{c\delta}}{\sigma_c^2} \right] (1 - \phi)^2. \end{aligned}$$

The inequality above requires the assumption

$$(1 - a_1 \kappa_1) (1 + \phi^2) + \frac{a_1 \kappa_1}{\bar{S}} + \frac{a_1 \phi}{\bar{S}} + (1 + a_2 \kappa_1) (1 + \phi^2) \frac{\sigma_{c\delta}}{\sigma_c^2} > 0,$$

which is true if the correlation between the innovations of return and consumption is positive.

Log-Linear Approximation For the Habit Model With $ARMA(2, 2)$ Consumption Growth:

First, I assume a linear approximation of the price dividend ratio,

$$z_t = a_0 + a_1 s_t + a_2 \delta_t + a_3 g_{c,t} + a_4 g_{c,t-1} + a_5 \epsilon_{c,t} + a_6 \epsilon_{c,t-1}.$$

Plugging the above equation back into the Euler equation, replacing the sensitivity function $\lambda(s)$ with its linear approximation as before, and matching the coefficients, I obtain

$$\begin{aligned} a_2 &= \frac{\rho_\delta - 1}{1 - \kappa_1 \rho_\delta}, \\ a_3 &= -\frac{(\rho_{c,1} - \gamma \rho_{c,1}) + (\rho_{c,2} - \gamma \rho_{c,2}) \kappa_1}{(\kappa_1 \rho_{c,1} - 1) + \kappa_1^2 \rho_{c,2}}, \\ a_4 &= -\frac{(\rho_{c,1} - \gamma \rho_{c,1}) \kappa_1 \rho_{c,2} - (\rho_{c,2} - \gamma \rho_{c,2}) (\kappa_1 \rho_{c,1} - 1)}{(\kappa_1 \rho_{c,1} - 1) + \kappa_1^2 \rho_{c,2}}, \\ a_6 &= \theta_{c,2} - \gamma \theta_{c,2} + a_3 \kappa_1 \theta_{c,2}, \end{aligned}$$

and

$$a_5 = \theta_{c,1} - \gamma \theta_{c,1} + a_6 \kappa_1 + a_3 \kappa_1 \theta_{c,1}.$$

Furthermore, a_1 can be found as the positive root of the following quadratic equation:

$$\begin{aligned} 0 &= \left[a_\lambda \sigma_c^2 \kappa_1^2 - \sigma_c^2 \frac{\kappa_1^2}{\bar{S}^2} \right] a_1^2 \\ &+ \left[\sigma_c^2 \frac{2\kappa_1 \gamma}{\bar{S}^2} - a_\lambda \sigma_c^2 \kappa_1 (1 + a_3 \kappa_1 + a_5 \kappa_1 + \gamma) + (\kappa_1 \phi - 1) - a_\lambda [1 + a_2 \kappa_1] \sigma_{c\delta} \kappa_1 \right] a_1 \\ &+ a_\lambda \sigma_c^2 (1 + a_3 \kappa_1 + a_5 \kappa_1) \gamma - \sigma_c^2 \frac{\gamma^2}{\bar{S}^2} - \gamma (\phi - 1) + a_\lambda [1 + a_2 \kappa_1] \gamma \sigma_{c\delta}, \end{aligned}$$

and

$$a_0 = \frac{1}{1 - \kappa_1} \left[\begin{aligned} &\gamma (\phi - 1) \bar{s} + \log(\beta) + k_0 + \mu_c (1 - \rho_{c1} - \rho_{c2}) (1 - \gamma) \\ &+ a_3 \kappa_1 \mu_c (1 - \rho_{c,1} - \rho_{c,2}) + a_1 \kappa_1 (1 - \phi) \bar{s} + 0.5 [1 + a_2 \kappa_1]^2 \sigma_\delta^2 \\ &+ 0.5 \sigma_c^2 \left[\begin{aligned} &(a_1 \kappa_1 - \gamma)^2 \frac{1}{\bar{S}^2} (1 + 2\bar{s}) + (1 - a_1 \kappa_1 + a_3 \kappa_1 + a_5 \kappa_1)^2 \\ &+ 2(a_1 \kappa_1 - \gamma) (1 + a_\lambda s_{\max}) (1 - a_1 \kappa_1 + a_3 \kappa_1 + a_5 \kappa_1) \\ &+ [1 - \gamma + a_3 \kappa_1 + a_5 \kappa_1 + a_\lambda (a_1 \kappa_1 - \gamma) s_{\max}] [1 + a_2 \kappa_1] \frac{\sigma_{c\delta}}{0.5 \sigma_c^2} \end{aligned} \right] \end{aligned} \right].$$

Hence, a linear approximation of the log price dividend ratio is obtained. The approximated return is thus

$$\begin{aligned} r_{t+1} &\approx \alpha - \beta_S S_t + \beta_\delta \delta_t + \beta_{g,1} g_{c,t} + \beta_{g,2} g_{c,t-1} + \beta_{\epsilon,1} \epsilon_{c,t} + \beta_{\epsilon,2} \epsilon_{c,t-1} \\ &+ (1 + a_5 \rho + a_3 \rho + a_1 \rho \lambda(s_t)) \epsilon_{c,t+1} + (a_2 \rho + 1) \epsilon_{\delta,t+1}, \end{aligned}$$

where

$$\begin{aligned}
\alpha &= \kappa_0 + \mu_c (1 - \rho_{c1} - \rho_{c2}) - a_0 + \kappa_1 a_0 + a_1 \kappa_1 (1 - \phi) \bar{S} + a_3 \kappa_1 \mu_c (1 - \rho_{c,1} - \rho_{c,2}) \\
\beta_S &\approx \frac{a_1 (1 - \kappa_1 \phi)}{\bar{S}} \\
\beta_\delta &= \rho_\delta - 1 + a_2 \kappa_1 \rho_\delta - a_2 \equiv 0 \\
\beta_{g,1} &= \rho_{c,1} + a_3 \kappa_1 \rho_{c,1} + a_4 \kappa_1 - a_3 \\
\beta_{g,2} &= \rho_{c,2} + a_3 \kappa_1 \rho_{c,2} - a_4 \\
\beta_{\epsilon,1} &= \theta_{c,1} + a_3 \kappa_1 \theta_{c,1} + a_6 \kappa_1 - a_5 \\
\beta_{\epsilon,2} &= \theta_{c,2} + a_3 \kappa_1 \theta_{c,2} - a_6.
\end{aligned}$$

Especially, when $\rho_{c,2} = 0$, $\theta_{c,1} = 0$, $\theta_{c,2} = 0$, and $\rho_\delta = 1$. the total returns can be given by

$$\begin{aligned}
r_{t+1} \approx & \alpha_0 + \left(\kappa_1 a_3 + 1 + \kappa_1 a_1 \frac{1 - \bar{S}}{\bar{S}} \right) \epsilon_{c,t+1} + \epsilon_{\delta,t+1} \\
& + \gamma \rho_{c,1} g_{c,t} + \frac{(\phi \kappa_1 - 1) a_1}{\bar{S}} \tilde{S}_t.
\end{aligned}$$

Combining this with equation (4.1), equation (4.2) follows.

The Reduced-Form Trend-Cycle Model With Cointegration:

In the following, I derive equation (3.8) in Section (3). First, notice that

$$(1 - \rho_{x,1}L - \rho_{x,2}L^2)^{-1} = (1 - \hat{\rho}_1 L)^{-1} (1 - \hat{\rho}_2 L)^{-1} = \sum_{k=0}^{\infty} \tilde{\rho}_k L^k,$$

where

$$\hat{\rho}_i = \frac{-2\rho_{x,2}}{\rho_{x,1} \pm \sqrt{\rho_{x,1}^2 + 4\rho_{x,2}}}, \text{ and } \tilde{\rho}_k = \sum_{j=0}^k \hat{\rho}_1^j \hat{\rho}_2^{k-j}.$$

Then $x_t = \sum_{k=0}^{\infty} \tilde{\rho}_k \epsilon_{x,t-k}$. Rewrite the dynamics of consumption growth and dividend growth as follows

$$\begin{aligned}
g_{c,t} &= \sum_{k=0}^{\infty} \tilde{\rho}_k \epsilon_{x,t-k} - \sum_{k=0}^{\infty} \tilde{\rho}_k \epsilon_{x,t-1-k} + \xi_t \equiv \sum_{k=0}^{\infty} \tilde{\rho}_k^* \epsilon_{x,t-k} + \xi_t \\
\Delta d_t &= \sum_{k=0}^{\infty} \tilde{\rho}_k^* \epsilon_{x,t-k} + \xi_t + \Delta \delta_t,
\end{aligned}$$

where

$$\tilde{\rho}_k^* = \tilde{\rho}_k + \tilde{\rho}_{k-1}, \text{ and } \tilde{\rho}_{-1} \equiv 0.$$

Substituting the above equations into the log-linearized equation in asset returns r_t (e.g. Campbell (1991)), it follows that²¹

$$\begin{aligned}
r_t - E_{t-1}r_t &= (E_t - E_{t-1}) \left[\sum_{j=0}^{\infty} \kappa_1^j \Delta d_{t+j} - \sum_{j=1}^{\infty} \kappa_1^j r_{t+j} \right] \\
&= - \left\{ \sum_{j=0}^{\infty} \kappa_1^j E_{t-1} [\Delta \delta_{t+j}] - \sum_{j=0}^{\infty} \kappa_1^j E_t [\Delta \delta_{t+j}] \right\} + \xi_t \\
&\quad + \left\{ \sum_{j=0}^{\infty} \kappa_1^j E_t \left[\sum_{k=0}^{\infty} \tilde{\rho}_k^* \epsilon_{x,t+j-k} \right] - \sum_{j=0}^{\infty} \kappa_1^j E_{t-1} \left[\sum_{k=0}^{\infty} \tilde{\rho}_k^* \epsilon_{x,t+j-k} \right] \right\} \\
&\quad - \left\{ \sum_{j=1}^{\infty} \kappa_1^j E_{t-1} \left(\beta_C \sum_{k=0}^{\infty} \tilde{\rho}_k \epsilon_{x,t+j-1-k} \right) + \sum_{j=1}^{\infty} \kappa_1^j E_t \left(\beta_C \sum_{k=0}^{\infty} \tilde{\rho}_k \epsilon_{x,t+j-1-k} \right) \right\}.
\end{aligned}$$

I simplify each of the three terms in curly brackets in turn. For the first term,

$$\begin{aligned}
&\sum_{j=0}^{\infty} \kappa_1^j E_{t-1} [\Delta \delta_{t+j}] - \sum_{j=0}^{\infty} \kappa_1^j E_t [\Delta \delta_{t+j}] \\
&= \sum_{j=0}^{\infty} \kappa_1^j \left[\sum_{k=j+1}^{\infty} \psi_k \epsilon_{\delta,t+j-k} - \sum_{k=j}^{\infty} \psi_k \epsilon_{\delta,t+j-1-k} \right] - \sum_{j=0}^{\infty} \kappa_1^j \left[\sum_{k=j}^{\infty} \psi_k \epsilon_{\delta,t+j-k} - \sum_{k=\max(0,j-1)}^{\infty} \psi_k \epsilon_{\delta,t+j-1-k} \right] \\
&= \epsilon_{\delta,t} (\kappa_1 - 1) \sum_{j=0}^{\infty} \kappa_1^j \psi_j,
\end{aligned}$$

where ψ_{-1} is defined to be zero. For the second term,

$$\begin{aligned}
&\sum_{j=0}^{\infty} \kappa_1^j E_t \left[\sum_{k=0}^{\infty} \tilde{\rho}_k^* \epsilon_{x,t+j-k} \right] - \sum_{j=0}^{\infty} \kappa_1^j E_{t-1} \left[\sum_{k=0}^{\infty} \tilde{\rho}_k^* \epsilon_{x,t+j-k} \right] \\
&= \sum_{j=0}^{\infty} \kappa_1^j E_t \left[\sum_{k=j}^{\infty} \tilde{\rho}_k^* \epsilon_{x,t+j-k} \right] - \sum_{j=0}^{\infty} \kappa_1^j E_{t-1} \left[\sum_{k=j+1}^{\infty} \tilde{\rho}_k^* \epsilon_{x,t+j-k} \right] = \epsilon_{x,t} \sum_{j=0}^{\infty} \kappa_1^j \tilde{\rho}_j^*,
\end{aligned}$$

and for the last term,

$$\begin{aligned}
&-\sum_{j=1}^{\infty} \kappa_1^j E_{t-1} \left(\beta_C \sum_{k=0}^{\infty} \tilde{\rho}_k \epsilon_{x,t+j-1-k} \right) + \sum_{j=1}^{\infty} \kappa_1^j E_t \left(\beta_C \sum_{k=0}^{\infty} \tilde{\rho}_k \epsilon_{x,t+j-1-k} \right) \\
&= -\beta_C \sum_{j=1}^{\infty} \kappa_1^j \left[\sum_{k=j}^{\infty} \tilde{\rho}_k \epsilon_{x,t+j-1-k} - \sum_{k=j-1}^{\infty} \tilde{\rho}_k \epsilon_{x,t+j-1-k} \right] = \beta_C \epsilon_{x,t} \sum_{j=1}^{\infty} \kappa_1^j \tilde{\rho}_{j-1}.
\end{aligned}$$

²¹Here, the interest rate is assumed to be constant. The same calculation goes through, if the interest rate is a linear function of the state variable.

It therefore follows that

$$\begin{aligned} r_t &= a_0 + \epsilon_{\delta,t}\bar{\psi} + \epsilon_{x,t} \sum_{j=0}^{\infty} \kappa_1^j \tilde{\rho}_j^* + \beta_C \epsilon_{x,t} \sum_{j=1}^{\infty} \kappa_1^j \tilde{\rho}_{j-1} + \xi_t + \beta_C x_{t-1} \\ &= a_0 + \epsilon_{\delta,t}\bar{\psi} + \epsilon_{x,t} \cdot (\rho^* + \beta_C \kappa_1 \bar{\rho}) + \xi_t + \beta_C x_{t-1}, \end{aligned}$$

where

$$\bar{\psi} = (1 - \kappa_1) \sum_{j=0}^{\infty} \kappa_1^j \psi_j; \quad \rho^* = \sum_{j=0}^{\infty} \kappa_1^j \tilde{\rho}_j^*, \quad \bar{\rho} = \sum_{j=0}^{\infty} \kappa_1^j \tilde{\rho}_j. \quad (7.6)$$

Based on the derived dynamics on consumption and asset returns, the equations for the orthogonal increment processes in the spectral representations can be derived as before, and then the expressions for the coherence, cospectrum, and phase can be analytically obtained.

Parametric Estimation of the Cross-Spectrum for the Data:

I first estimate the following $VAR(2)$ for consumption growth and asset returns:

$$\begin{aligned} r_t &= c_1 + \alpha_1 r_{t-1} + \alpha_2 r_{t-2} + \beta_1 g_{t-1} + \beta_2 g_{t-2} + u_t \\ g_{c,t} &= c_2 + \gamma_1 g_{c,t-1} + \gamma_2 g_{c,t-2} + \eta_1 r_{t-1} + \eta_2 r_{t-2} + \epsilon_{c,t}. \end{aligned}$$

After the parameters are estimated, the cross-spectrum can be found in closed-form by the following argument. First, write the equations for the orthogonal increment processes Z_{g_c} , Z_r , Z_{ϵ_c} , and Z_u in the spectral representations of $\{g_{c,t}\}$, $\{r_t\}$, $\{\epsilon_{c,t}\}$, and $\{u_t\}$:

$$\begin{aligned} dZ_r &= (\alpha_1 e^{-i\lambda} + \alpha_2 e^{-2i\lambda}) dZ_r + (\beta_1 e^{-i\lambda} + \beta_2 e^{-2i\lambda}) dZ_g + dZ_u \\ dZ_{g_c} &= (\gamma_1 e^{-i\lambda} + \gamma_2 e^{-2i\lambda}) dZ_{g_c} + (\eta_1 e^{-i\lambda} + \eta_2 e^{-2i\lambda}) dZ_r + dZ_{\epsilon_c}. \end{aligned}$$

Rearrange to obtain

$$\begin{aligned} dZ_{g_c} &= \frac{\bar{\eta}}{D_e} dZ_u + \frac{(1 - \bar{\alpha})}{D_e} dZ_{\epsilon_c} \\ dZ_r &= \frac{1 - \bar{\gamma}}{D_e} dZ_u + \frac{\bar{\beta}}{D_e} dZ_{\epsilon_c}, \end{aligned}$$

where

$$\bar{X} = X_1 e^{-i\lambda} + X_2 e^{-2i\lambda} \text{ for } X = \alpha, \beta, \gamma, \text{ and } \eta,$$

and

$$D_e = (1 - \bar{\alpha})(1 - \bar{\gamma}) - \bar{\beta}\bar{\eta}.$$

Hence, the cross-spectrum can be obtained as

$$\begin{aligned}
2\pi f_{11} &= \left| \frac{\bar{\eta}}{D_e} \right|^2 \sigma_u^2 + \left| \frac{(1 - \bar{\alpha})}{D_e} \right|^2 \sigma_{\epsilon_c}^2 + 2\text{real} \left(\left[\frac{\bar{\eta}}{D_e} \right] \left[\frac{(1 - \bar{\alpha})}{D_e} \right]' \right) \sigma_{\epsilon_c, u} \\
2\pi f_{22} &= \left| \frac{1 - \bar{\gamma}}{D_e} \right|^2 \sigma_u^2 + \left| \frac{\bar{\beta}}{D_e} \right|^2 \sigma_{\epsilon_c}^2 + 2\text{real} \left(\left[\frac{(1 - \bar{\gamma})}{D_e} \right] \left[\frac{\bar{\beta}}{D_e} \right]' \right) \sigma_{\epsilon_c, u} \\
2\pi f_{12} &= \frac{\bar{\eta}}{D_e} \left(\frac{1 - \bar{\gamma}}{D_e} \right)' \sigma_u^2 + \frac{(1 - \bar{\alpha})}{D_e} \left(\frac{\bar{\beta}}{D_e} \right)' \sigma_{\epsilon_c}^2 + \left[\frac{\bar{\eta}}{D_e} \left(\frac{\bar{\beta}}{D_e} \right)' + \frac{(1 - \bar{\alpha})}{D_e} \left(\frac{1 - \bar{\gamma}}{D_e} \right)' \right] \sigma_{\epsilon_c, u}.
\end{aligned}$$

The parametric estimation of the cross-spectrum with EC-VAR can be obtained in a similar manner.

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Table 1: Parameter choices for the external habit formation model with *i.i.d.* consumption. All the parameter values are annualized.

Statistics	Variable	Value
Mean consumption growth (%)	g_c	1.89
Standard deviation of consumption growth (%)	σ_c	1.22
Log risk-free rate (%)	r^f	0.94
Persistence coefficient in habit	ϕ	0.87
Persistence coefficient in δ_t	ρ_δ	0.89
Standard deviation of the innovation in δ_t	σ_δ	0.112
Risk aversion coefficient	γ	2
Correlation between innovation in consumption and δ_t	$\rho_{c,\delta}$	-0.1
Subjective discount factor	β	0.89

Table 2: Summary statistics of simulated data for external habit formation model with *i.i.d.* consumption growth, and cointegrated consumption and dividends. All the quantities are annualized.

Statistics	Equity	Postwar Sample	Long Sample
$E(g_c)$	1.90	1.89	1.72
$\sigma(g_c)$	1.22	1.22	3.32
$E(r^f)$	0.94	0.94	2.92
$E(r - r^f)$	6.71	6.69	3.90
$\sigma(r - r^f)$	15.34	15.7	18.0
$\exp[E(p - d)]$	18.2987	24.7	21.1
$\sigma(p - d)$	0.3136	0.26	0.27
$AC_1(p - d)$	0.8432	0.87	0.78

Table 3: Estimation for consumption dynamics based on quarterly consumption data.

$ARIMA(2, 1, 2)$	μ_c	$\rho_{c,1}$	$\rho_{c,2}$	$\theta_{c,1}$	$\theta_{c,2}$	σ_c
<i>Estimate</i>	0.0055	1.3040	-0.5535	-1.0288	0.4359	0.0042
<i>Standard Error</i>	0.0005	0.3756	0.2388	0.3661	0.1489	0.0002
$Trend + AR(2)$	μ_c	$\rho_{x,1}$	$\rho_{x,2}$	σ_x	σ_ξ	ρ_{ξ, ϵ_x}
<i>Implied Value</i>	0.0055	1.3040	-0.5535	0.0050	0.0068	-0.9569

The following consumption dynamics are estimated:

$$g_{c,t} - \mu_c = \rho_{c,1}(g_{c,t-1} - \mu_c) + \rho_{c,2}(g_{c,t-2} - \mu_c) + \epsilon_{c,t} + \theta_{c,1}\epsilon_{c,t-1} + \theta_{c,2}\epsilon_{c,t-2}, \quad (7.7)$$

where $\epsilon_{c,t} \sim WN(0, \sigma_c^2)$. This $ARIMA(2, 1, 2)$ process has the following equivalent trend-cycle representation for log consumption:

$$\begin{aligned} c_t &= T_t + x_t \\ T_t &= T_{t-1} + \mu_c + \xi_t \\ x_t &= \rho_{x,1}x_{t-1} + \rho_{x,2}x_{t-2} + \epsilon_{x,t}, \end{aligned} \quad (7.8)$$

where T_t is the stochastic trend, x_t is the cyclical component in the log consumption, $\epsilon_{x,t} \sim WN(0, \sigma_{\epsilon_x}^2)$, $\xi_t \sim WN(0, \sigma_\xi^2)$, and $\text{corr}(\xi_t, \epsilon_{x,t}) = \rho_{\xi, \epsilon_x}$.

Table 4: Parameter choices for the external habit formation model with $ARMA(2, 2)$ consumption growth. All parameter values are in quarterly frequency.

Statistics	Variable	Value
Mean consumption growth (%)	g_c	0.5458
Standard deviation of the innovation in consumption (%)	σ_c	0.4158
Persistence coefficient in habit	ϕ	0.9658
$AR(1)$ coefficient of consumption growth	ρ_{c1}	1.3034
$AR(2)$ coefficient of consumption growth	ρ_{c2}	-0.5535
$MA(1)$ coefficient of consumption growth	θ_{c1}	-1.0288
$MA(2)$ coefficient of consumption growth	θ_{c2}	0.4359
Persistence coefficient in δ_t	ρ_δ	0.9719
Standard deviation of the innovation in δ_t	σ_δ	0.056
Risk aversion coefficient	γ	2
Correlation between innovation in consumption and δ_t	$\rho_{c,\delta}$	-0.1
Subjective discount factor	β	0.9740

Table 5: Summary statistics of simulated data from the external habit formation model with $ARMA(2, 2)$ consumption growth. All the quantities are annualized.

Statistics	Equity
$E(g_c)$	2.18
$\sigma(g_c)$	0.90
$E(r^f)$	1.21
$\sigma(r^f)$	0.68
$E(r - r^f)$	6.26
$\sigma(r - r^f)$	17.73
$exp[E(p - d)]$	16.72
$\sigma(p - d)$	0.37
$AC_1(p - d)$	0.84

Table 6: Long horizon correlations for the external habit formation model with $ARMA(2, 2)$ consumption growth. The calculations are based on quarterly real data and quarterly simulated data.

horizon (in quarters)	Data	Habit Formation
1	0.1561	0.7680
2	0.2041	0.7214
3	0.2466	0.6987
4	0.2702	0.6886
5	0.2779	0.6813
6	0.2844	0.6727
7	0.2812	0.6619
8	0.2769	0.6486

Table 7: Band-pass filter analysis: Band-pass filter analysis for the real data and artificial data simulated from different models. Here, C-C is the Campbell and Cochrane (1999) model, IID is the external habit formation model with *i.i.d.* consumption growth and cointegrated consumption and dividends. ARMA is the external habit formation model with $ARMA(2,2)$ consumption growth and cointegrated consumption and dividends. B-Y is the calibrated model from Bansal and Yaron (2004); P-Y is the calibrated model from Panageas and Yu (2005). High-frequency includes components with a cycle of less than three years. Low-frequency includes components with a cycle of more than three years. The first two rows are calculated from 48,000 quarters of simulated data. The third row is the 95th quantile of the differences between low-frequency correlation and high-frequency correlation, obtained from 1,000 Monte Carlo experiments each with 100 years of simulated data for different models. For real data, the third row is simply the difference between low-frequency correlation and high-frequency correlation.

	C-C	IID	ARMA	B-Y	P-Y	data
Low-Frequency Correlation	0.6523	0.7856	0.7283	0.1501	0.6090	0.3417
High-Frequency Correlation	0.7479	0.8870	0.8091	-0.0377	0.2460	0.1138
95% Diff. from Monte Carlo	0.02	0.02	0.04	0.34	0.50	0.23

Table 8: Granger causality test: p -value of the test of consumption growth Granger-causing asset returns based on quarterly data.

	$BM1$	$MB2$	$BM3$	$BM4$	$BM5$
$S1$	0.2061	0.2255	0.2815	0.4396	0.5542
$S2$	0.1138	0.2249	0.5238	0.3315	0.6353
$S3$	0.0849	0.2774	0.4562	0.5686	0.3665
$S4$	0.1155	0.2585	0.3853	0.8124	0.8279
$S5$	0.3041	0.3664	0.6938	0.7115	0.6400

Table 9: Granger causality test: p -value of the test of asset returns Granger-causing consumption growth based on quarterly data.

	$BM1$	$MB2$	$BM3$	$BM4$	$BM5$
$S1$	0.0016	0.0045	0.0035	0.0081	0.0032
$S2$	0.0034	0.0071	0.0044	0.0048	0.0026
$S3$	0.0024	0.0013	0.0017	0.0014	0.0026
$S4$	0.0051	0.0137	0.0018	0.0004	0.0174
$S5$	0.0008	0.0137	0.0443	0.0027	0.0136

Table 10: Granger causality test: p -values of Granger causality tests for real data and artificial data simulated from different models. C-C is the Campbell and Cochrane (1999) model, IID is the external habit formation model with *i.i.d.* consumption growth and cointegrated consumption and dividends. ARMA is the external habit formation model with ARMA(2,2) consumption growth and cointegrated consumption and dividends. B-Y is the calibrated model from Bansal and Yaron (2004); P-Y is the calibrated model from Panageas and Yu (2006). All the data are at quarterly frequency.

	C-C	IID	ARMA	B-Y	P-Y	data
Consumption Causes Returns	0	0	0	0.7740	0.7900	0.4482
Returns Cause Consumption	0.2361	0.2249	0.2015	0	0	4.3770×10^{-4}

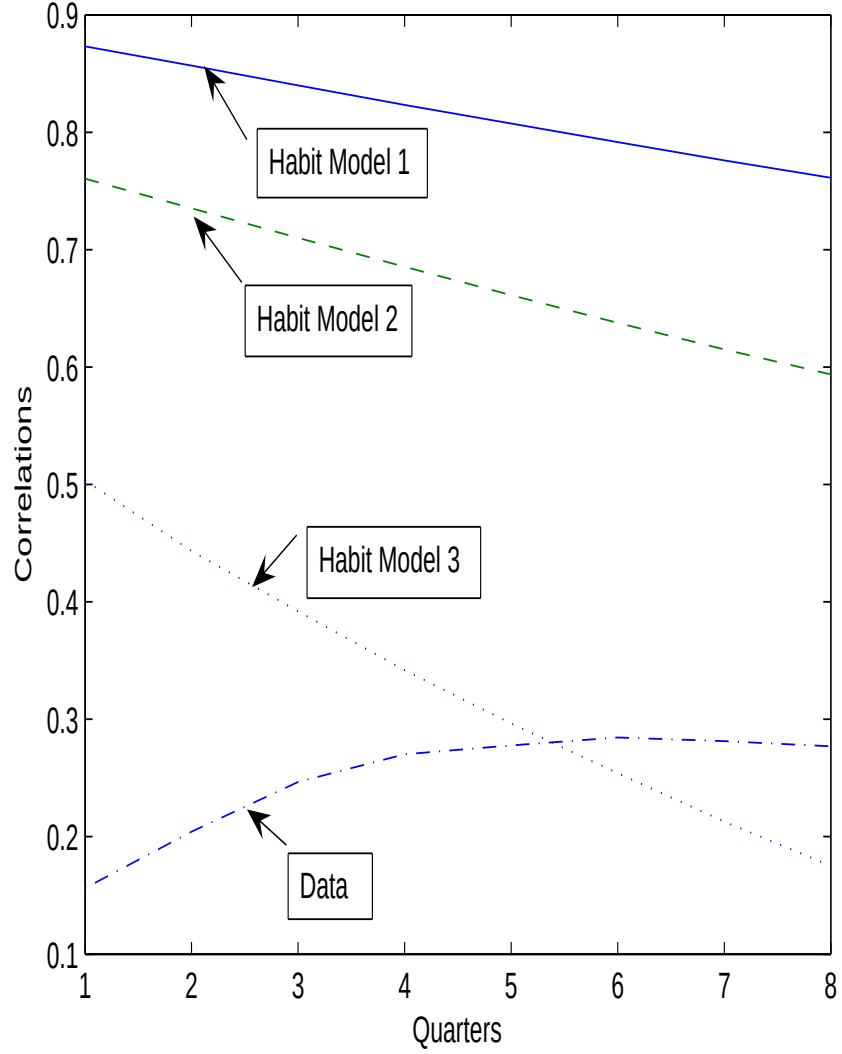


Figure 1: The correlations between consumption growth and returns at different horizons. Habit model 1 is the calibrated model with parameter value in Table 1. Habit model 2 is the model without cointegration between consumption and dividends. Habit Model 3 is the habit model with a lower correlation between consumption and dividends. Data line is for the correlation between consumption and returns at different horizons for the data.

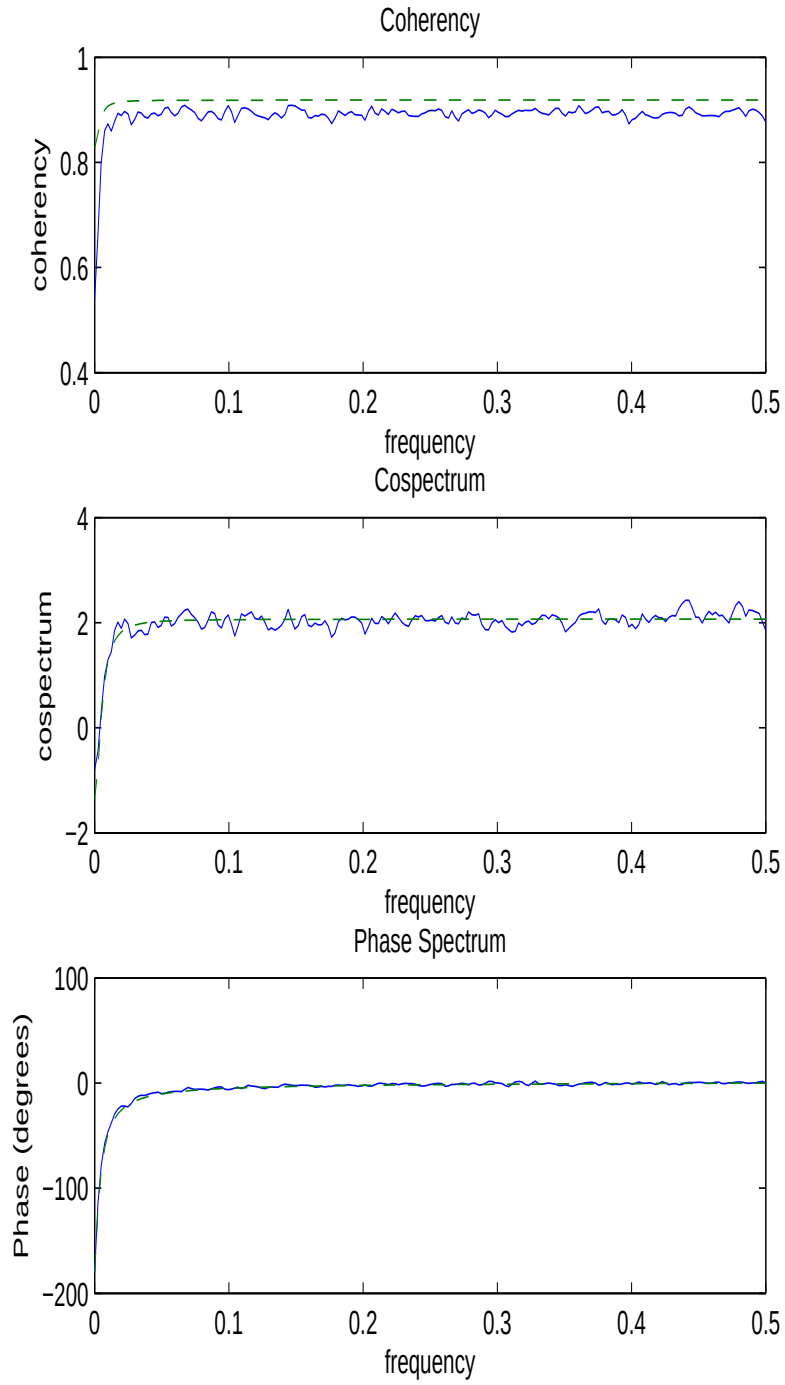


Figure 2: The three panels are the coherence, cospectrum, and phase spectrum between consumption growth and stock market excess returns in the simulated model. The solid line is calculated from the analytical approximation, and the dotted line is calculated from the 48,000 quarters of simulated data.

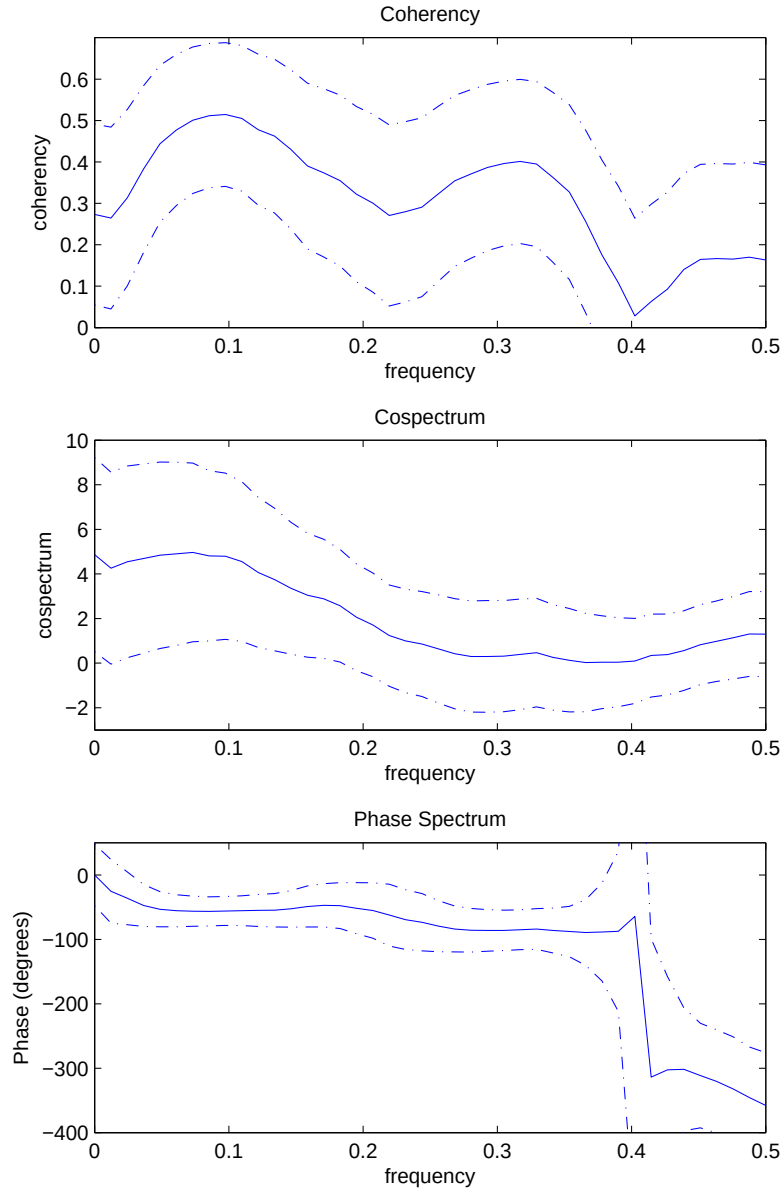


Figure 3: The three panels are the nonparametric estimation of the coherence, cospectrum, and phase spectrum between the quarterly consumption growth rate and quarterly stock market excess returns for the data. The quarterly consumption data and population data over the period 1952Q1-2006Q4 are taken from Fed St Louis, and the quarterly excess market return is taken from CRSP VW index. The graph indicates that returns lead consumption and comovements between consumption and returns are stronger at low frequencies.

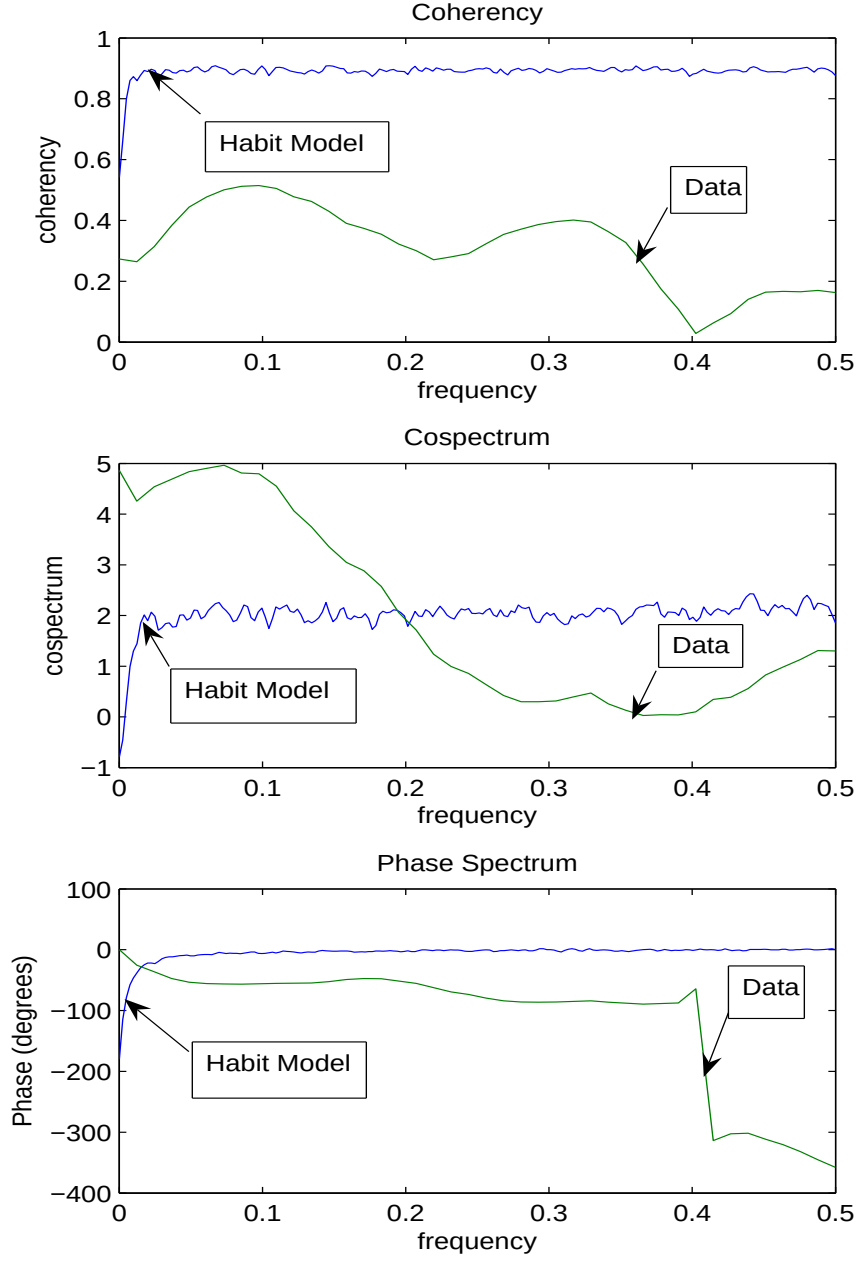


Figure 4: The three panels are the coherence, cospectrum, and phase spectrum between the quarterly consumption growth rate and quarterly stock market excess returns for real data, and for simulated habit model with *i.i.d.* consumption growth. The graph indicates that returns lead consumption and comovements between consumption and returns are stronger at low frequencies for the data, while it is the opposite for the model.

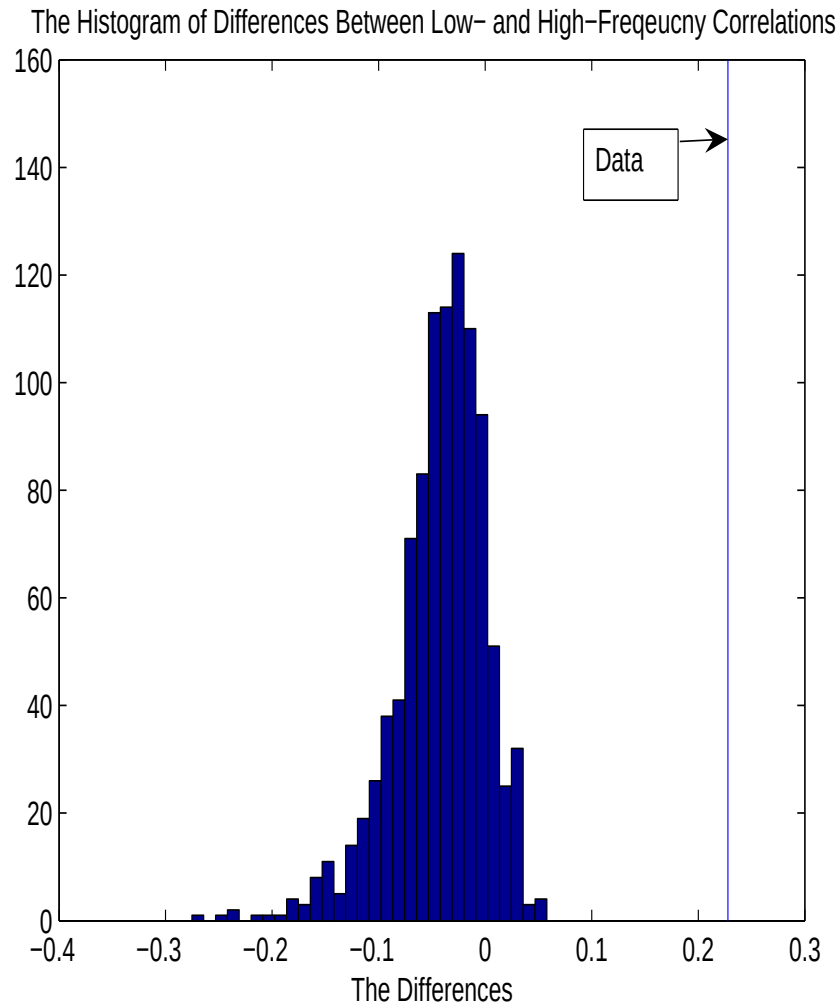


Figure 5: The histogram for the 1,000 differences between low-frequency correlations and high-frequency correlations between consumption growth and asset returns from the 1,000 Monte Carlo simulations for the external habit formation model.

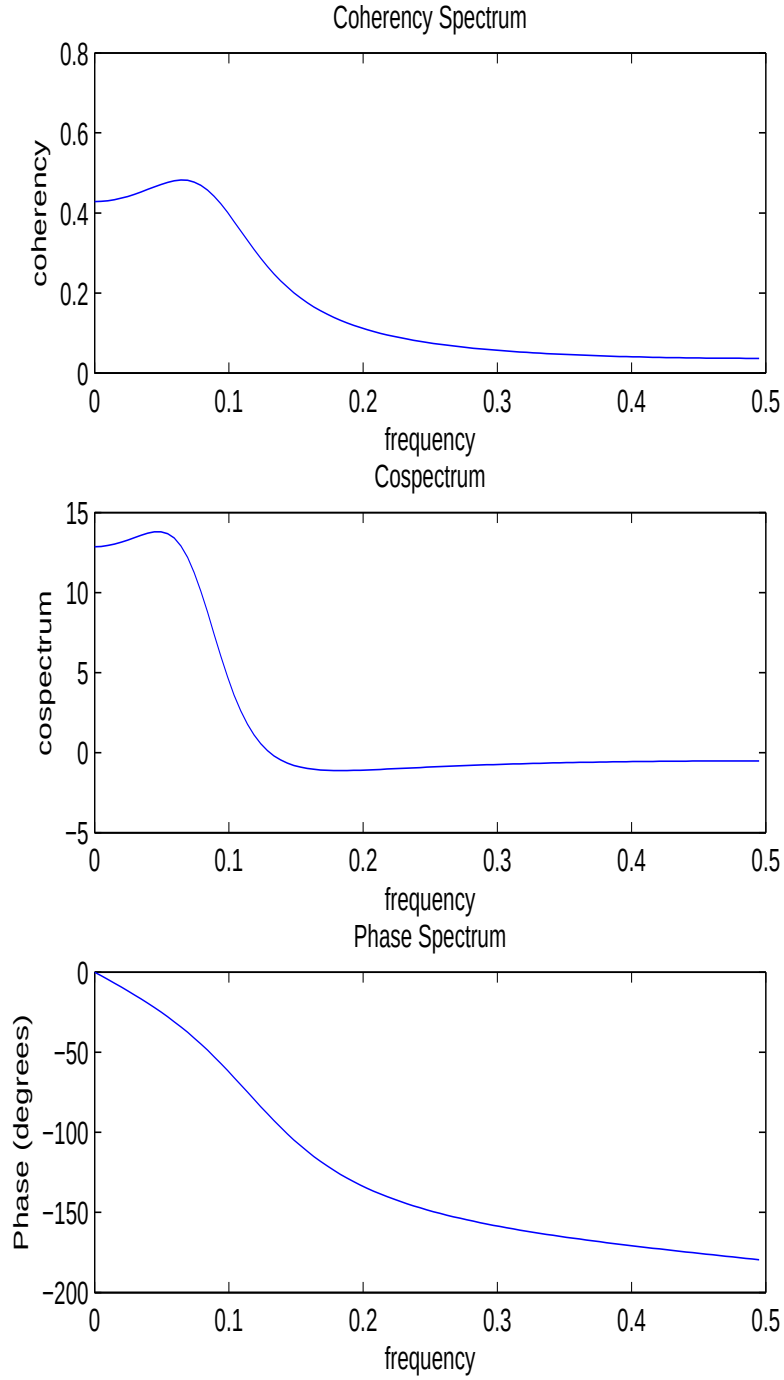


Figure 6: The three panels are the coherence, cospectrum, and phase spectrum between consumption growth rate and the stock market excess returns under the parameter values $\beta_C = 2$, $\rho_{u,\xi} = 0$ and $\rho_{u,\epsilon_x} = 0$ for the reduced-form trend-cycle model. The graph implies that returns lead consumption and comovements between consumption and returns are stronger at low frequencies.

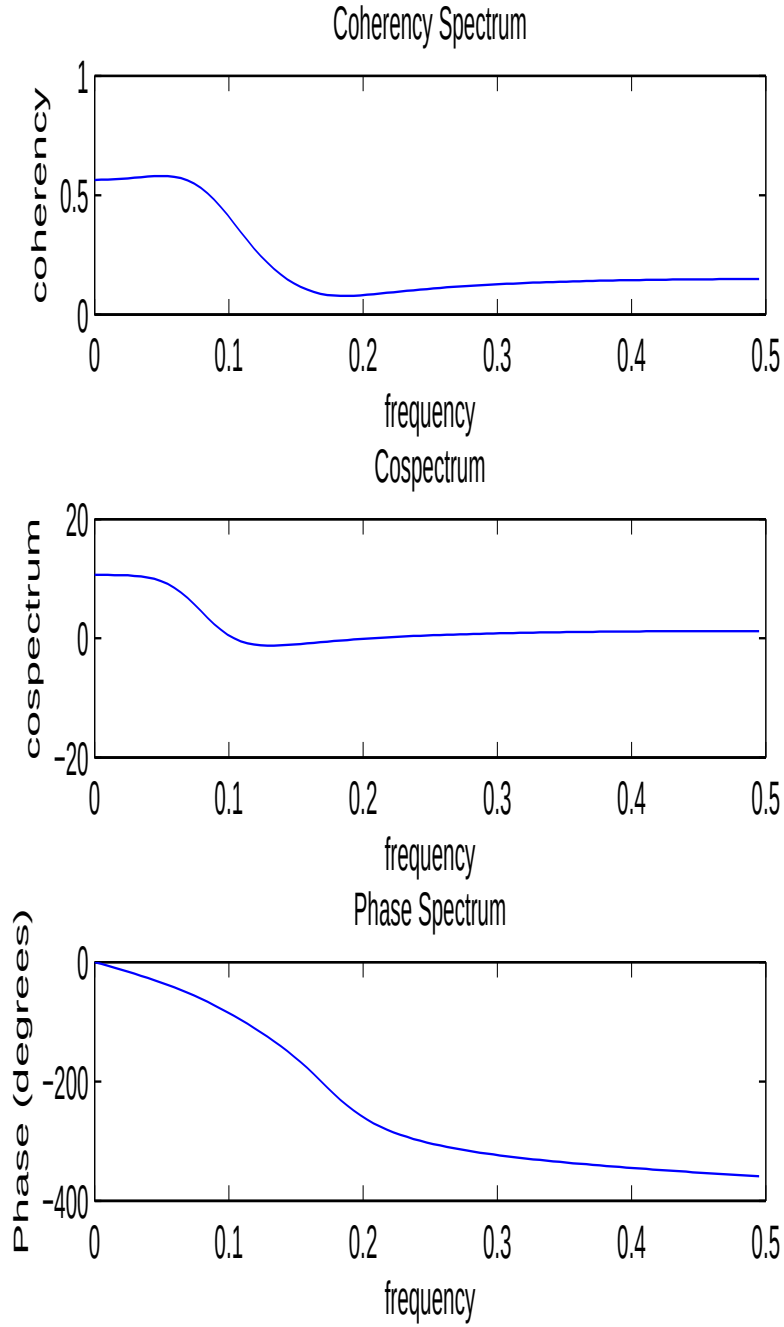


Figure 7: The three panels are the coherence, cospectrum, and phase spectrum between consumption growth rate and the stock market excess returns under the parameter values $\beta_C = 2$, $\rho_{u,\xi} = 0.2$ and $\rho_{u,\epsilon_x} = -0.2$ for the reduced-form trend-cycle model. The graph implies that returns lead consumption and comovements between consumption and returns are stronger at low frequencies.

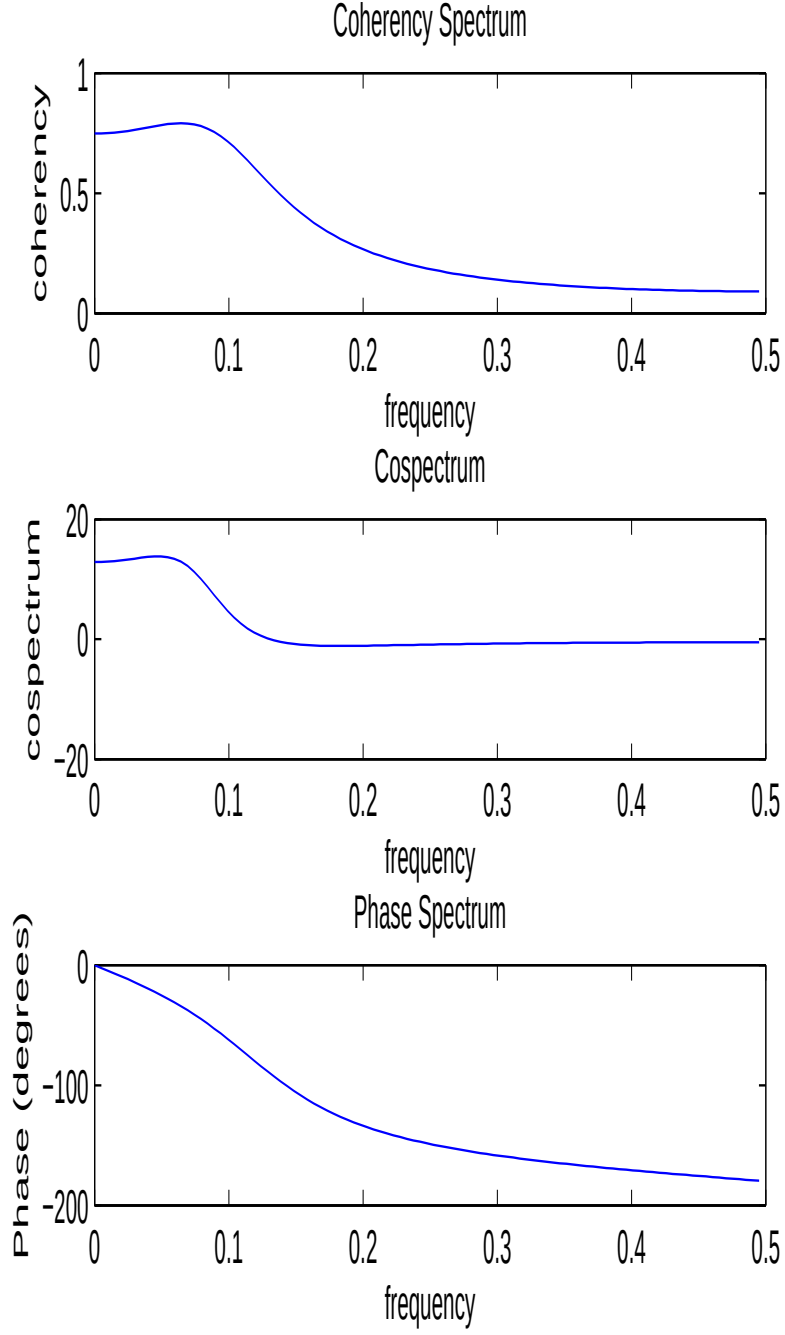


Figure 8: The three panels are the coherence, cospectrum, and phase spectrum between consumption growth rate and the stock market excess returns under the parameter values $\beta_C = 5$, $\rho_{u,\xi} = 0$ and $\rho_{u,\epsilon_x} = 0$ for the reduced-form trend-cycle model. The graph implies that returns lead consumption and comovements between consumption and returns are stronger at low frequencies.

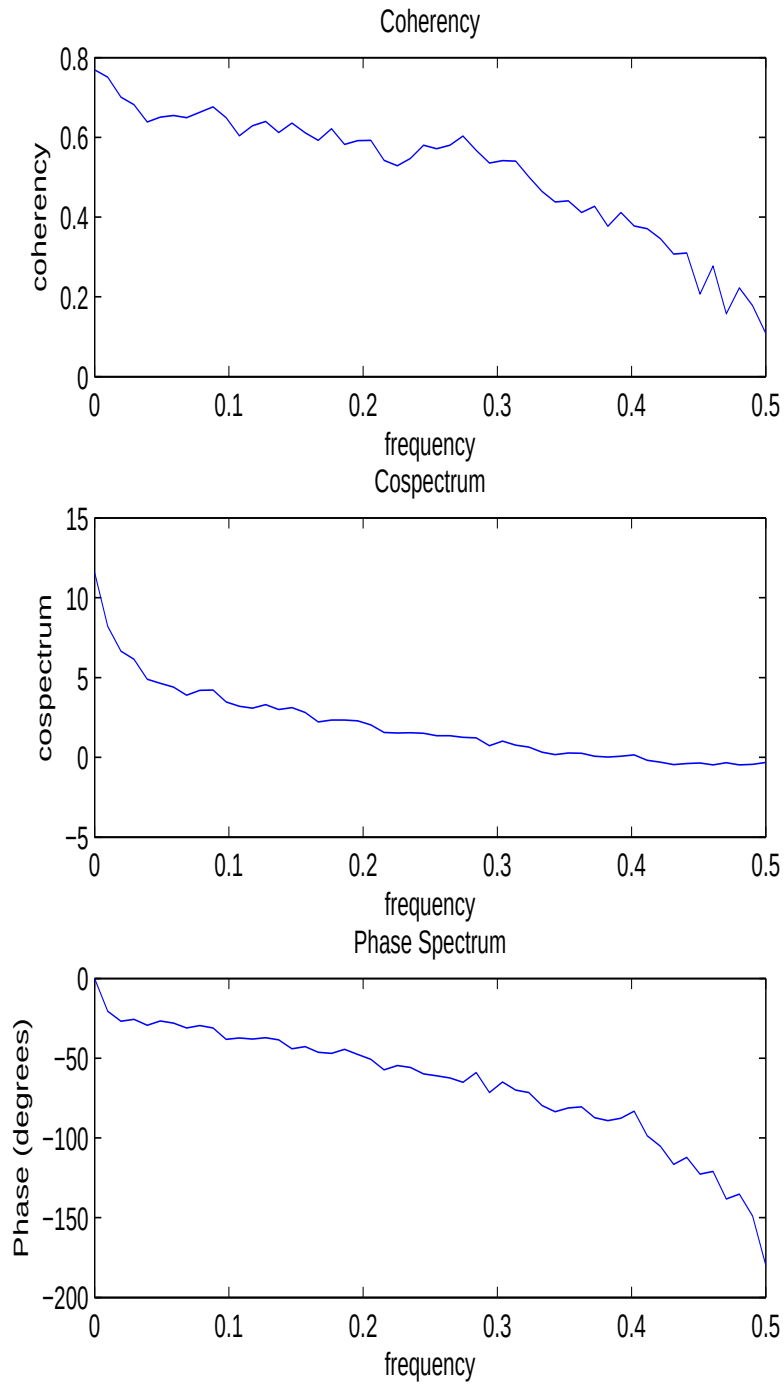


Figure 9: The three panels are the coherence, cospectrum, and phase spectrum between the quarterly consumption growth rate and the quarterly stock market excess returns for the simulated data in a calibrated model of Panageas and Yu (2005). The graph indicates that returns lead consumption and comovements between consumption and returns are stronger at low frequencies.

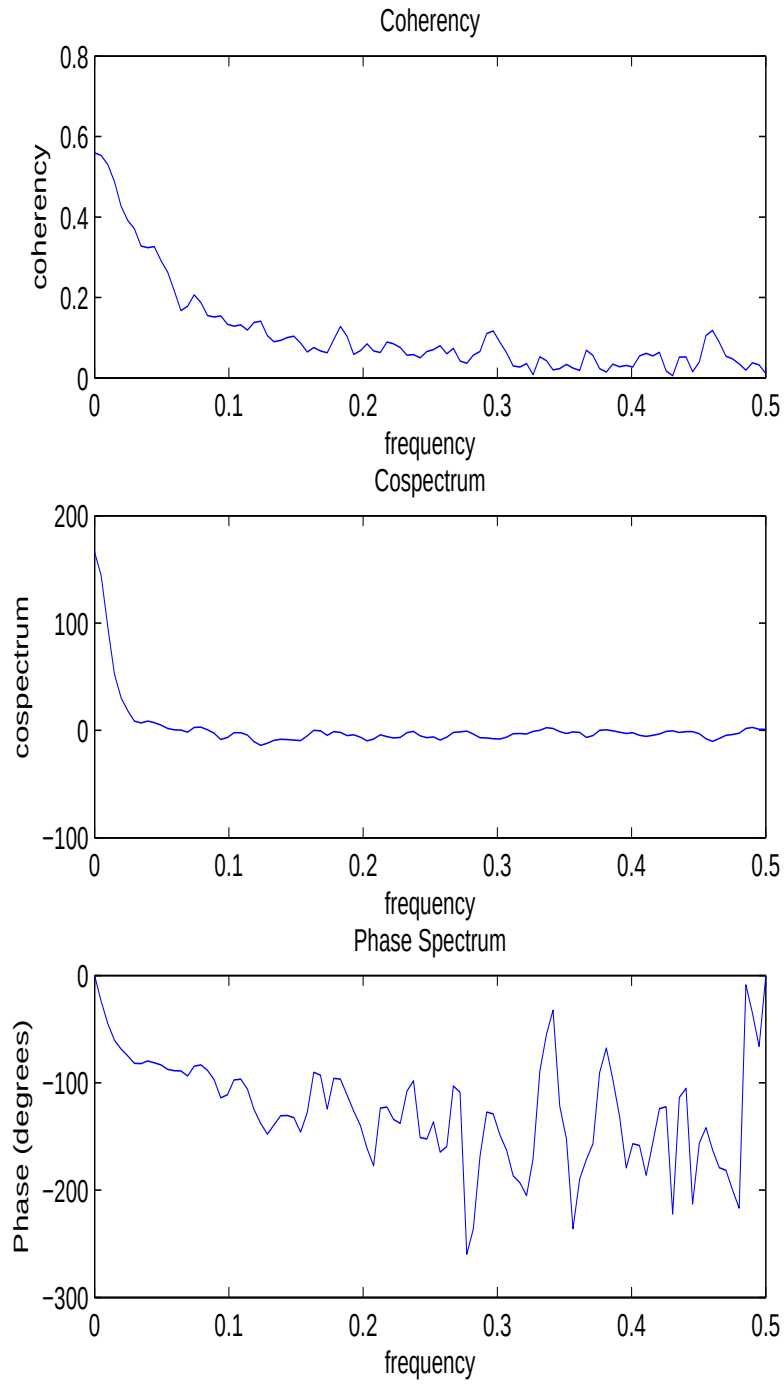


Figure 10: The three panels are the coherence, cospectrum, and phase spectrum between the quarterly consumption growth rate and the quarterly stock market excess returns for the simulated data in a calibrated model of Bansal and Yaron (2004). The graph indicates that returns lead consumption and comovements between consumption and returns are stronger at low frequencies.

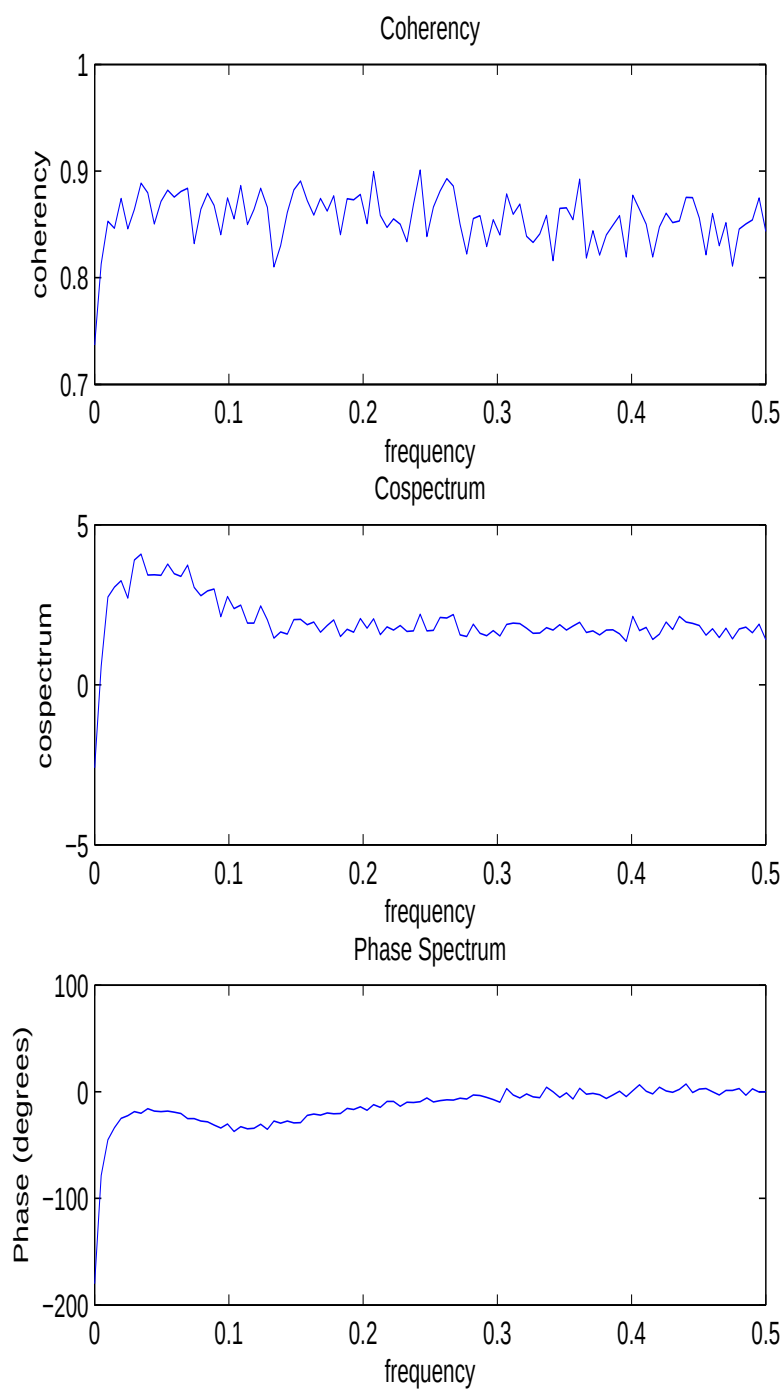


Figure 11: The three panels are the coherence, cospectrum, and phase spectrum between consumption growth rate and the stock market excess returns in the simulated external habit model with $ARMA(2,2)$ consumption growth rate. The graph indicates that returns lag consumption and comovements between consumption and returns are stronger at higher frequencies.

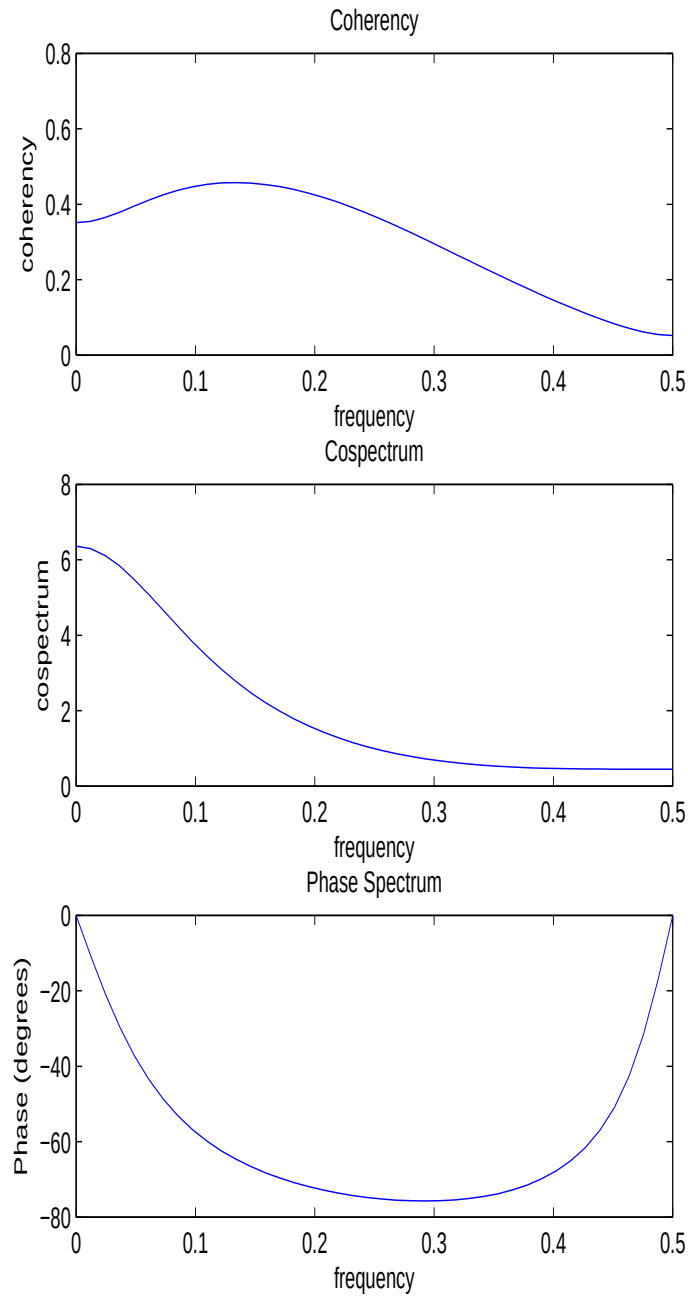


Figure 12: The three panedls are the parametric estimation of the coherence, cospectrum, and phase spectrum between consumption growth rate and the stock market excess return in the data. I first fit a VAR(2) on the consumption and returns, then obtain the analytical cross-spectrum by plugging in the estimated parameter values. The graph indicates that returns lead consumption and comovements between consumption and returns are stronger at low frequencies.

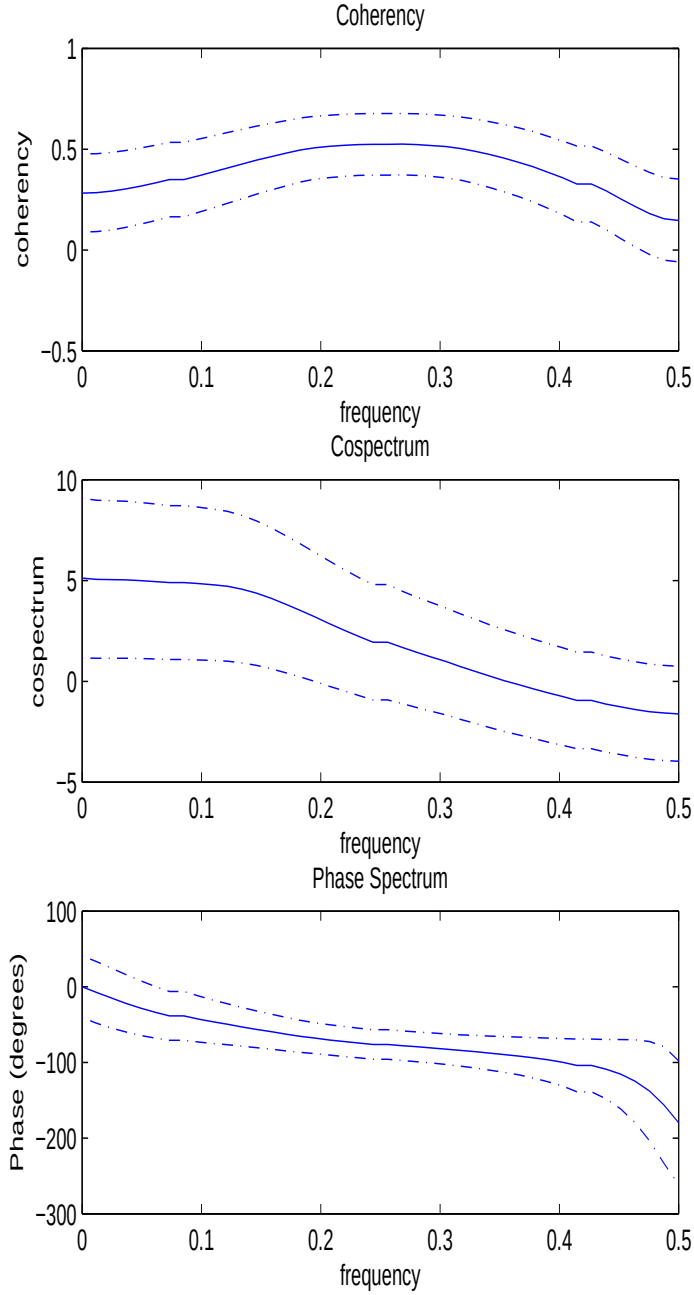


Figure 13: The nonparametric estimation of the coherence, cospectrum, and phase spectrum between annual consumption growth rate and the annual stock market excess returns in the data: The annual consumption data and population data over the period 1930-2006 are taken from BEA, and the annual excess market returns are calculated from CRSP VW index. The graph indicates that returns lead consumption and more covariation between consumption and returns comes from low frequencies.

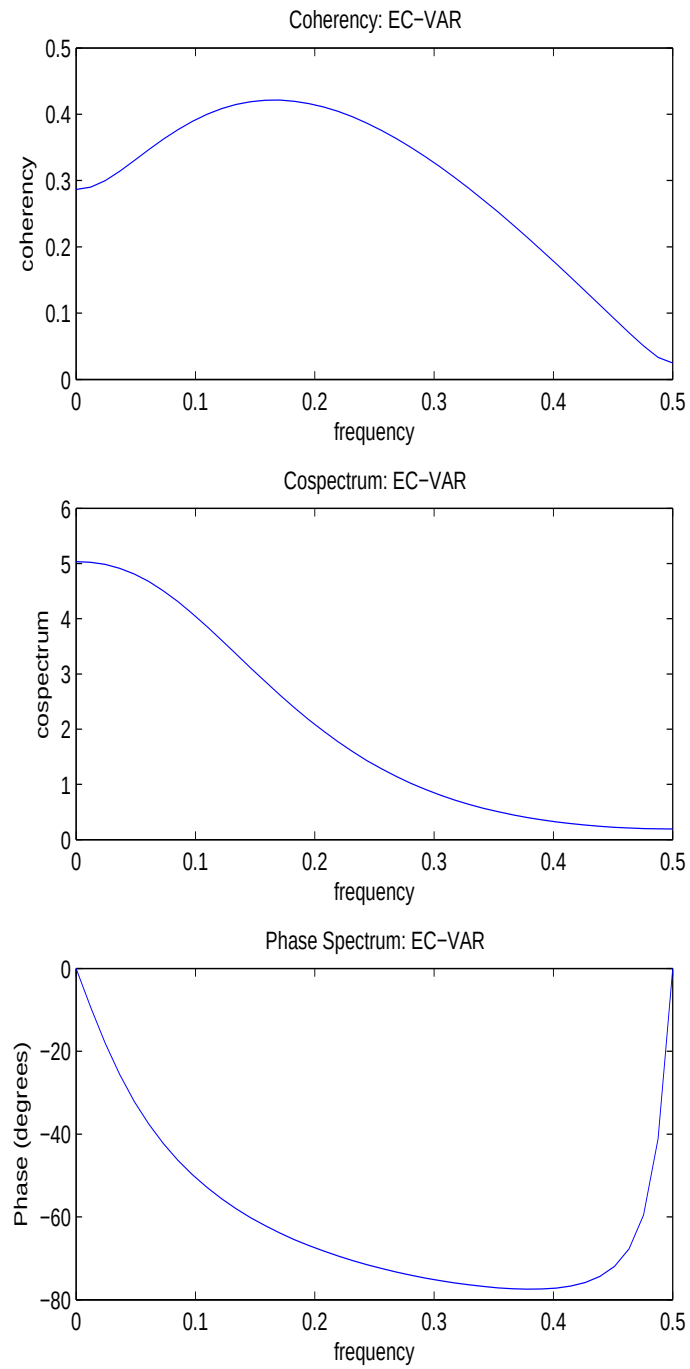


Figure 14: The three panels are the coherence, cospectrum, and phase spectrum between the quarterly consumption growth rate and the quarterly stock market excess returns implied by the EC-VAR model. The graph indicates that returns lead consumption and comovements between consumption and returns are stronger at low frequencies.