

Numerical Simulation of Nonoptimal Dynamic Equilibrium Models*

Zhigang Feng, Jianjun Miao, Adrian Peralta-Alva, and Manuel S. Santos[†]

Abstract

In this paper we present a recursive method for the computation of dynamic competitive equilibria in models with heterogeneous agents and market frictions. This method is based upon a convergent operator over an expanded set of state variables. The fixed point of this operator defines the set of all Markovian equilibria. We study approximation properties of the operator as well as the convergence of the moments of simulated sample paths. We apply our numerical algorithm to two stochastic growth economies, an overlapping generations economy with money, and an asset pricing model with financial frictions.

KEYWORDS: Heterogeneous agents, taxes, externalities, financial frictions, competitive equilibrium, computation, simulation.

*This is a preliminary and incomplete version. Comments are welcome. An earlier version of this paper was circulated as “Existence and Computation of Markov Equilibria for Dynamic Nonoptimal Economies” by J. Miao and M. Santos, March 2005.

[†]Z. Feng and M. Santos: Department of Economics, University of Miami, 5250 University Drive, Coral Gables, FL. J. Miao: Department of Economics, Boston University, 270 Bay State Road, Boston, MA. A. Peralta-Alva: Research Division, Federal Reserve Bank of Saint Louis. The usual disclaimer applies.

1 Introduction

In this paper we present a recursive method for the computation of sequential competitive equilibria for dynamic economic models in which the welfare theorems may fail to hold because of the presence of incomplete agents' participation, taxes, externalities, incomplete financial markets, and other financial frictions. These models have become central to analyze the effects of various macroeconomic policies, the evolution of wealth and income distribution, and the variability of asset prices.

Although these models are now prevalent in various areas of economic research, computation of their equilibrium solutions may be a formidable task. A main analytical problem is that dynamic programming arguments may fail to apply, and a continuous Markov equilibrium may not exist. More specifically, under regular assumptions the equilibrium correspondence is upper-semicontinuous, and may not admit an equilibrium selection that defines a Markov equilibrium. Therefore, existing numerical techniques for the computation of optimal solutions cannot be readily extended to non-optimal economies.

We shall address the following issues: *(i) Existence: Lack of Markov equilibria.* Even though the model may have a recursive structure, a Markovian equilibrium may not exist – or no Markov equilibrium may be continuous – over a natural space of state variables. *(ii) Computation: Non-convergence of the algorithm.* Backward iteration over a candidate equilibrium function may not reach a Markovian equilibrium solution. Contraction arguments underlying dynamic programming methods usually break down for non-optimal economies. *(iii) Approximation: Accuracy properties of the computed solution.* Approximation errors may cumulate over time. Consequently, as we refine the approximation we need to ensure that discretized versions of the algorithm approach an exact solution. Again, contraction arguments cannot be invoked to guarantee good approximation properties of the algorithm. *(iv) Simulation: Convergence of the moments from sample paths.* Standard laws of large numbers require certain regularity conditions – such as continuity of the law of motion – that would be rather imposing for the equilibria of these economies.

In dynamic competitive-markets economies with frictions the existence of Markovian equilibria has been well established under certain monotonicity properties on the equilibrium dynamics, and it remains largely unexplored in many other models in which these monotonicity conditions may not be satisfied. Regular examples of non-existence of Markovian equilibria have been found in

one-sector growth models with taxes and externalities [Santos (2002)], in exchange economies with incomplete financial markets [Krebs (2004) and Kubler and Schmedders (2002)], and in overlapping generations (OLG) economies [Kubler and Polemarchakis (2004)]. There are also positive results on existence of a continuous equilibrium over a minimal state space that rely on monotone equilibrium dynamics [e.g., see Bizer and Judd (1989), Coleman (1991), and Datta, Mirman and Reffett (2002)]. For the canonical one-sector growth model with taxes and externalities, monotonicity conditions follow from fairly mild restrictions on the primitives, but monotone dynamics are much harder to obtain in multi-sector models with heterogeneous agents and incomplete financial markets. Duffie *et al.* (1994) dispense with such monotonicity requirements by expanding the state space with endogenous variables such as asset prices and individual consumptions. By a suitable randomization of the equilibrium correspondence [Blume (1982)] they then prove the existence of an ergodic invariant distribution for a wide class of discrete-time infinite-horizon models with exogenous short-sale constraints on asset holdings. Building on these methods, Kubler and Schmedders (2003) prove the existence of a Markovian equilibrium for a class of financial economies with collateral requirements.

We extend these methods to various types of economies under an algorithm that seems more amenable to computation. We enlarge the natural state space to include agents' shadow values of investment. We prove that the set of all Markov equilibria can be characterized as the fixed-point solution of a convergent iterative procedure. (A key factor of convergence is that our operator is acting over candidate equilibrium sets on a compact domain.) Then, we develop a computable version of the theoretical algorithm. This numerical algorithm is shown to approximate the original fixed-point solution. Moreover, the moments derived from simulated paths of the computed solution converge to a set of moments of the invariant distributions of the model. We apply our methods to two growth economies, a stochastic OLG economy with money, and an asset pricing model with incomplete financial markets and heterogeneous agents. We illustrate the applicability of our algorithm by comparing our numerical solution with those generated from some other standard methods. These other methods may display low accuracy properties, fail to converge to the equilibrium solutions, or capture only one of the possible existing equilibria.

The computation of competitive equilibria for non-optimal economies has been of considerable interest in macroeconomics and finance [e.g., Castaneda, Diaz-Gimenez and Rios-Rull (2003), Krusell and Smith (1998), Heaton and Lucas (1998), Marcet and Singleton (1999), and Rios-Rull (1999)],

but most of this literature does not deal with the problem of existence of a Markovian equilibrium. Kubler and Schmedders (2003) refine the analysis of Duffie *et al.* (1994) and develop a computational algorithm over an expanded state space in which a Markov equilibrium does exist – although such equilibrium may not be continuous. A main problem with their numerical work is that they iterate over continuous equilibrium functions, and such iteration process does not guarantee convergence to a fixed-point solution. Also, their state space includes additional variables which seem hard to compute, and so their algorithm may not be computationally efficient.

The idea of enlarging the state space with the shadow values of investment was first suggested by Kydland and Prescott (1980) in their seminal study of time inconsistency. Abreu, Pierce and Stacchetti (APS, 1990) use a similar approach for the computation of sequential perfect equilibria in which they expand the state space with continuation utility values. The analyses of Kydland and Prescott and APS have been extended in several directions involving strategic decisions [e.g., Atkinson (1991), Chang (1998), Judd, Yeltekin and Conklin (2003), Marcet and Marimon (1998) and Phelan and Stacchetti (2001)], but none of these papers are concerned with the computation of sequential equilibria for competitive-market economies with heterogeneous agents. To the best of our knowledge, the only related paper is Miao (2003) who sets forth a recursive solution method for the model of Krusell and Smith (1998). However, as in the original APS approach Miao’s state space includes expected continuation utilities over the set of sequential competitive equilibria, and this choice of the state space does not seem operative for the computation of equilibrium solutions in the present framework.

Finally, for nonoptimal economies convergence properties of numerical algorithms and convergence of the simulated moments remain largely unexplored. As already discussed, Duffie *et al.* (1994) show existence of an ergodic distribution (which validates a law of large numbers for these economies). This result is not practical for computational purposes as it is usually hard to locate the ergodic set. In the absence of continuity of the equilibrium law of motion, other ways to validate laws of large numbers for these economies would be to resort to monotonicity assumptions on the equilibrium dynamics [Santos (2006)] or to artificial expansions of the noise space [Blume (1979)]. These latter approaches seem less attractive for these economies.

We proceed as follows. In Section 2 we present our general framework and lay out our theoretical algorithm. Section 3 studies the numerical implementation of our algorithm and its convergence

properties. Sections 4-6 explore the existence and computation of recursive equilibria for various families of models. We conclude in Section 7.

2 General Theory

In this section, we first set out a general analytic framework that encompasses various competitive equilibrium models. We then present our numerical approach and main results on existence and global convergence to the Markovian equilibrium correspondence.

2.1 The Analytical Framework

Time is discrete and denoted by $t = 0, 1, 2, \dots$. The state of the economy includes a state vector of endogenous variables x and vector of exogenous shocks z . Vector x belongs to a compact domain X and contains all predetermined variables, such as agents' holdings of physical capital, human capital, and financial assets. The exogenous state vector follows a Markov chain $(z_t)_{t \geq 0}$ over a finite set Z . This Markovian process is described by positive transition probabilities $\pi(z'|z)$ for all $z, z' \in Z$. The initial state, $z_0 \in Z$, is known to all agents in the economy. Then $z^t = (z_1, z_2, \dots, z_t) \in Z^t$ is a history of shocks, often called a date-event or node. Let y denote the vector of all other endogenous variables. These variables could be equilibrium prices or choice variables such as consumption and investment.

In all our models the dynamics of the state vector x is conformed by a system of non-linear equations:

$$\varphi(x_{t+1}, x_t, y_t, z_t) = 0. \quad (1)$$

Function φ incorporates technological constraints as well as individual budget constraints. For some models, such as the growth models analyzed in Section 3, function φ is known and we can explicitly solve for x_{t+1} as a function of (x_t, y_t, z_t) . In other applications such as in various models with adjustment costs, vector x_{t+1} may not admit an analytical representation.

Let m denote a vector of shadow values of the marginal investment return for all assets and all agents. This vector lies in a compact space M , and it will be a function of existing variables such

as prices, rates of interest, and marginal utilities and productivities:

$$m_t = h(x_t, y_t, z_t). \quad (2)$$

Let us assume that a competitive equilibrium exists and can be represented by a sequence $(x_t(z^t), y_t(z^t))_{t=0}^{\infty}$ satisfying (1), (2) and the additional system of equations

$$\Phi(x_t, y_t, z_t, E_t[m_{t+1}]) = 0, \quad (3)$$

where $E[m]$ is the expectations operator. Function Φ may describe individual optimality conditions (such as Euler equations), market-clearing conditions, various types of budget restrictions, and resource constraints. We assume that equations (1)-(3) fully characterize a dynamic competitive equilibrium, and that φ , h , and Φ are continuous functions.

2.2 Recursive Equilibrium Theory

In order to compute the set of dynamic equilibria for the model economy we define the equilibrium correspondence $V^*(x, z)$ containing all the equilibrium vectors m for any given state (x, z) . From this correspondence V^* we can generate recursively the set of dynamic equilibria as V^* is the fixed point of an operator $B : V \mapsto B(V)$ that links state variables to future equilibrium states. Operator B embodies all equilibrium conditions such as agents' optimization and market-clearing conditions from any initial node z to all immediate successor states z_+ . This operator is analogous to the expectations correspondence defined in Duffie *et al.* (1994), albeit it is defined over a smaller set of endogenous variables.

More precisely, let $B(V)(x, z)$ be the set of all values of m satisfying the following temporary equilibrium conditions: There exist y with $m = h(x, y, z)$ and $m_+(z_+) \in V(x_+, z_+)$ with $z_+ \in Z$ such that

$$\Phi(x, y, z, \sum_{z_+ \in Z} \pi(z_+|z) m_+(z_+)) = 0,$$

and

$$\varphi(x_+, x, y, z) = 0. \quad (4)$$

Note that operator B is well defined as a sequential competitive equilibrium is assumed to exist. Also, B is monotone in the sense that if $V \subset V'$ then $B(V) \subset B(V')$.¹ Moreover, if V has a closed graph then $B(V)$ has a closed graph since the above functions φ, h, Φ are all assumed to be continuous. Indeed, in all our models below operator B satisfies the following

Assumption 2.1 *Operator B preserves compactnes in the sense that if V is compact valued, then $B(V)$ is also compact valued.*

Using this assumption we can show existence of a fixed-point solution V^* which is globally convergent for every initial guess $V_0 \supset V^*$. Convergence, should be understood as pointwise convergence² in the Hausdorff metric [e.g., see Hildenbrand (1974)]. If V^* is a continuous correspondence then convergence will be uniform over all points (x, z) .

Theorem 2.1 (convergence) *Let V_0 be a compact-valued correspondence such that $V_0 \supset V^*$. Let $V_n = B(V_{n-1}), n \geq 1$. Then, $V_n \rightarrow V^*$ as $n \rightarrow \infty$. Moreover, V^* is the largest fixed point of the operator B , i.e., if $V = B(V)$, then $V \subset V^*$.*

Theorem 2.1 provides the theoretical foundations of our algorithm. The iterative process starts as follows: For all (x, z) , pick a sufficiently large compact set $V_0(x, z)$ such that $V_0(x, z) \supset V^*(x, z)$. Then apply operator B to V_0 and iterate until a desirable level of convergence is attained. This is possible since $\lim_{n \rightarrow \infty} V_n$ equals the equilibrium correspondence V^* . An important advantage of our approach is that if there are multiple equilibria, we can find all of them. Finally, by assumption 2.1 from operator B we can select a measurable policy function $y = g^y(x, z, m)$, and a transition function $m_+(z_+) = g^m(x, z, m; z_+)$, for all $z_+ \in Z$. These functions give a Markovian characterization of a dynamic equilibrium in the enlarged state space.

Note that the equilibrium shadow value correspondence V^* may not be single-valued, and hence there could be multiple equilibrium selections. Then, the usual recursive equilibrium may not exist.

¹For correspondences V, V' we say that $V \subset V'$ if $V(x, z) \subset V'(x, z)$ for all (x, z) .

²Later, we will establish uniform convergence of the simulated moments even though the equilibrium correspondence V^* is only upper semicontinuous.

That is, there may not exist an equilibrium function g such that $y = g(x, z)$. Kubler and Schmedders (2002) construct an example economy with multiple equilibria. They show that for $x_+ = f(x, y, z)$ the model does not admit a recursive equilibrium:

$$\Phi(x, g(x, z), z, \sum_{z_+ \in Z} \pi(z_+|z) h(f(x, g(x, z), z), g(f(x, g(x, z)), z_+))) = 0. \quad (5)$$

Kubler and Schmedders (2003) propose a computation procedure to recover such Markov equilibria numerically by a related expansion of the state space. But their computational algorithm relies on the assumption that the policy correspondence is a continuous function. Their algorithm may fail if there are multiple equilibria or if the policy function is not continuous. Our approach overcomes this problem as we illustrate by the various examples in the coming sections.

3 Numerical Implementation

Numerical implementation of our theoretical results requires the construction of a computable algorithm that approximates operator B . In this section we develop and study the properties of such an algorithm.

We first partition the state space into a finite set of simplices $\{X^j\}$ with non-empty interior and maximum diameter h . Over this partition we define a family of step correspondences that take discrete set-values. To obtain a computer representation of a step correspondence we resort to an outer approximation in which each set-value is defined by N elements. Using these two simplifications we get a discretized version of operator B , which we denote by $B^{h,N}$. By a suitable selection of an initial condition V_0 , the sequence $\{V_{n+1}^{h,N}\}$ defined recursively as $V_{n+1}^{h,N} = B^{h,N} V_n^{h,N}$ converges to a limit point $V^{*,h,N}$ containing the equilibrium correspondence V^* . Moreover, the sequence of fixed points $\{V^{*,h,N}\}$ approaches the equilibrium correspondence V^* as the accuracy of the discretizations goes to the limit. It should be understood that convergence is uniform in economies where the equilibrium correspondence is continuous. At a later stage, we address the issue of convergence of the moments obtained from simulations of our numerical approximations. This problem has been hardly addressed in the literature, and again it has to cope with the fact that the equilibrium correspondence may not be continuous.

3.1 The Numerical Algorithm

Let $\{X^j\}$ be a finite family of simplices with non-empty interior such that $\cup_j X^j = X$ and $\text{int}(X^j) \cap \text{int}(X^i)$ is empty for every pair X^i, X^j . Define the *mesh size* h of this discretization as

$$h = \max_j \text{diam} \{X^j\}.$$

Consider a correspondence $V : X \times Z \rightarrow 2^M$ that takes values in space M . Then, its step approximation V^h over the partition $\{X^j\}$ takes constant set-values $V^h(x, z)$ on each simplex X^j and is conformed by the union of sets $V(x, z)$ for $x \in X^j$ for given z . That is, for each z

$$V^h(x, z) = \cup_{x \in X^j} V(x, z). \quad (6)$$

Accordingly, we can define operator B^h that takes a correspondence V into the step correspondence $[B(V)]^h$. By similar arguments as above, we can prove that B^h has a fixed point solution V^{*h} . Moreover, we shall soon clarify the sense in which the correspondence V^{*h} constitutes an approximation to V^* .

As already mentioned, we also perform a discretization on the image space to obtain a computer representation of the step correspondence. We say that $\mathcal{C}^N(V(x, z)) \supseteq V(x, z)$ is an N -element outer approximation of $V(x, z)$ if $\mathcal{C}^N(V(x, z))$ can be generated by N elements. In what follows we assume that this approximation satisfies a strong uniform convergence property.³ Namely, for any $\varepsilon > 0$ there is $0 < N^* < \infty$ such that $d[\mathcal{C}^N(V(x, z)), V(x, z)] \leq \varepsilon$ for all $N > N^*$, and all $V(x, z)$. For instance, this latter property can be satisfied if the outer approximation is generated by convex combinations of N points as M is a compact set.

We are now ready to put forward a computable version of operator B . That is, we can define a new operator $B^{h,N}$ that sends a correspondence V to the step correspondence $[B(V)]^h$ and then each set-value is adjusted with the N -element outer approximation so as to get $\mathcal{C}^N([B(V)]^h)$. Sections 4 to 6 illustrate examples of this type of operators, and their application in different dynamic models. We now show that our discretized operator has good convergence properties: The fixed point of this operator $V^{*,h,N}$ contains the equilibrium correspondence V^* and it approaches V^* as we refine the discretizations. The proof of this result extends the convergence arguments of Beer (1980) to a

³Again, convergence should be understood in the Hausdorff metric d (see *opt. cit.*).

dynamic setting.

Theorem 3.1 *Suppose that $V_0 \supseteq V^*$ is an upper-semicontinuous correspondence. Consider the recursive sequence defined by $V_{n+1}^{h,N} = B^{h,N} V_n^{h,N}$ for given h and N and with initial condition V_0 . Then: (i) $V_n^{h,N} \supseteq V^*$ for all n ; (ii) $V_n^{h,N} \rightarrow V^{*,h,N}$ uniformly as $n \rightarrow \infty$; and (iii) $V^{*,h,N} \rightarrow V^*$ as $h \rightarrow 0$ and $N \rightarrow \infty$.*

The output of our numerical algorithm is summarized by the equilibrium correspondence $V_n^{*,h,N}$ from operator $B^{h,N}$. By Theorem 3.1, we have that $\text{graph}[\mathcal{C}^N([B(V_n^{h,N})]^h)]$ can be made arbitrarily close to $\text{graph}[B(V^*)]$ for appropriate choices of n , h , and N . As $\text{graph}[\mathcal{C}^N([B(V_n^{h,N})]^h)]$ is compact, by the measurable selection theorem [Hildenbrand (1974)] we can choose an approximate equilibrium selection $y = g_n^{y,h,N}(x, z, m)$, and a transition function $m_+(z_+) = g_n^{m,h,N}(x, z, m; z_+)$. From these functions we can generate approximate equilibrium paths $((x_t(z^t), y_t(z^t)))_{t=0}^\infty$.

To assess model's predictions, analysts usually calculate moments of the simulated paths $((x_t(z^t), y_t(z^t)))_{t=0}^\infty$ from a numerical approximation. The idea is that the simulated moments should approach steady-state moments of the true model. Under continuity of the policy function, Santos and Peralta-Alva (2005) establish various convergence properties of the simulated moments. They also provide examples of non-existence of stochastic steady-state solutions, and lack of convergence of empirical distributions to some invariant distribution of the model. Hence, it is not clear how economies with distortions should be simulated, since for these economies the continuity of the policy function does not usually follow from standard economic assumptions.

3.2 Convergence of the Simulated Moments

We now outline a reliable simulation procedure that circumvents the lack of continuity of the equilibrium law of motion. We append two further steps to the standard model simulation. First, we discretize the image space of the approximate equilibrium selection so that this function can take on a finite number of points. Then, the simulated moments are generated by a finite Markov chain that has an invariant distribution, and every empirical distribution converges almost surely to some ergodic invariant distribution of the Markov chain. Second, following Blume (1982) and Duffie *et al.* (1994) we randomize over continuation values of operator B . We construct a new operator B^{cv}

that is a convex-valued correspondence in the space of probability measures. This operator has an invariant distribution μ^* . Finally, as we refine the approximations the simulated moments from our numerical approximations are shown to converge to the moments of some invariant distribution μ^* .

(i) *Discretization of the equilibrium law of motion*: In order to make the analysis more transparent, let $S = X \times M$. Let $\chi_n^{h,N} : S \times Z \rightarrow S \times Z$ be a selection from $\text{graph}[\mathcal{C}^N([B(V_n^{h,N})]^h)]$. Note that function $\chi_n^{h,N}$ is simply defined from the above functions $y = g_n^{y,h,N}(x, z, m)$, and $m_+(z_+) = g_n^{m,h,N}(x, z, m; z_+)$ and the law of motion for state variable x as given by equation (1). Then, $\chi_n^{h,N}$ gives rise to a time-homogeneous Markov process $(s, z) \rightarrow s_+(z_+)$ for $s = (x, m)$ and all $z_+ \in Z$. Now, let A_γ be a set with a finite number of points in S such that $d(A_\gamma, S) < \gamma$ so that each point in S is within a γ -ball of some point in A . Let $\chi_n^{h,N,A_\gamma}(s, z) = \arg \min_{s_+ \in A_\gamma} d(s_+, \chi_n^{h,N}(s, z))$. If there are various solution points s_+ we just pick arbitrarily one solution s_+ . Hence, the new discretized function χ_n^{h,N,A_γ} takes values in the finite set $A_\gamma \times Z$, and gives rise to a Markov chain that has an invariant distribution ν_n^{*,h,N,A_γ} . Further, the moments of a simulated path $(s_t, z_t)_{t=0}^\infty$ converge almost surely to those of some ergodic invariant distribution ν_n^{*,h,N,A_γ} [e.g., see Stokey, Lucas and Prescott (1989)].

(ii) *Randomization over continuation equilibrium sequences*: We can view operator $B : V^* \rightarrow V^*$ as a correspondence in the space of probability measures μ on $S \times Z$. That is, $\nu \in B(\mu)$ if there is a selection χ of B such that $\nu = \chi \cdot \mu$, where $\chi \cdot \mu$ denotes the action of function χ on probability measure μ [e.g., see Stokey, Lucas and Prescott (1989)]. Following Blume (1982) and Duffie *et al.* (1994) we convexify the image of B . Thus, if ν and ν' are two probability measures that belong to the range of B we assume that every convex combination $\lambda\nu + (1 - \lambda)\nu'$ also belongs to the range of B . We let B^{cv} denote this convexification of operator B over the space of probability measures μ on $S \times Z$. [Duffie *et al.* (1994) argue that such convexification amounts to a weak form of sunspot equilibria since the randomization proceeds over equilibrium distributions rather than over an external parameter or extraneous sunspot variable.] The new operator B^{cv} is a convex-valued, upper semicontinuous correspondence. Since $S \times Z$ is assumed to be compact, the set of probability measures μ on $S \times Z$ is also compact in the weak topology of measures. Therefore, operator B^{cv} has a fixed point solution; i.e., there exists an invariant probability, $\mu^* \in B^{cv}(\mu^*)$.

(iii) *Convergence of the simulated moments to population moments of the model*: For given function χ_n^{h,N,A_γ} and a randomly selected sequence $(z^t)_{t=0}^\infty$, we generate an approximate equilibrium

path $(s_t)_{t=0}^\infty$. Let $f : S \times Z \rightarrow R_+$ be a function of interest. Then, $\frac{1}{T} \sum_{t=0}^T f(s_t, z_t)$ represents a simulated moment or some other statistic. Since χ_n^{h,N,A_γ} defines a Markov chain, it follows that $(s_t, z_t)_{t=0}^\infty$ must enter an ergodic set in finite time. Therefore, $\frac{1}{T} \sum_{t=0}^T f(s_t, z_t)$ must converge almost surely to $\int f(s, z) d\nu_n^{*,h,N,A_\gamma}$ as $T \rightarrow \infty$ for some ergodic invariant distribution ν_n^{*,h,N,A_γ} . We now link convergence of invariant distributions ν_n^{*,h,N,A_γ} of numerical approximations to some invariant distribution of the original model μ^* so that the simulated statistics converge almost surely to those of some invariant distribution μ^* .

Theorem 3.2 *Let (ν_n^{*,h,N,A_γ}) be a sequence of invariant distributions corresponding to functions (χ_n^{h,N,A_γ}) . Then, every limit point of (ν_n^{*,h,N,A_γ}) converges weakly to some invariant distribution $\mu^* = B^{cv}(\mu^*)$.*

To summarize our work in this section, convergence of the simulated moments involves discrete approximations over the following margins:

1. *Discretization of the domain:* h mesh size of the family of simplices X^j .
2. *Discretization of the image:* N number of elements in every outer approximation.
3. *Finite iterations:* n number of iterations over operator $B^{h,N}$.
4. *Finite Markov chain:* γ maximum distance of every point in S to some point in set A_γ .
5. *Finite simulations:* T length of a simulated path $(s_t, z_t)_{t \geq 0}$.
6. *Convexification of equilibrium distributions:* B^{cv} regularized operator in the space of distributions with a convex image.

Thus, for a given a path $(s_t, z_t)_{t=0}^\infty$ generated under function χ_n^{h,N,A_γ} , for every $\epsilon > 0$ there are invariant distributions μ^*, μ'^* of B^{cv} such that $\int f(s, z) d\mu^* + \epsilon > \frac{1}{T} \sum_{t=0}^T f(s_t, z_t) > \int f(s, z) d\mu'^* - \epsilon$ almost surely whenever the aforementioned parameters (1) – (5) are all sufficiently close to their limits. Therefore, for a sufficiently fine approximation the moments from simulated paths are close to the set of moments of the invariant distributions of the model. Of course, if B^{cv} has a unique invariant distribution μ^* then $\mu'^* = \mu^*$ and the above expression reads as $\int f(s, z) d\mu^* + \epsilon > \frac{1}{T} \sum_{t=0}^T f(s_t, z_t) > \int f(s, z) d\mu^* - \epsilon$.

4 Non-Optimal Growth Models

In this section we study a standard stochastic growth model with taxes, heterogeneous agents, and incomplete markets. This framework comprises several macroeconomic models that are often simulated by numerical methods. Because of the presence of distortions, equilibrium allocations are not Pareto optimal, and hence they cannot be computed by solving a social planner problem. Reliable computational methods for distorted growth models [cf. Bizer and Judd (1989), Coleman (1991), and Greenwood and Huffman (1995)] are only valid for monotone equilibrium dynamics. However, monotone dynamics are hard to obtain in models with heterogeneous agents, multiple goods, externalities, non-linear taxation, and government expenditures.

We illustrate the applicability of our algorithm within two alternative specifications of the model, and contrast its performance against standard numerical methods. The first version of the model we study is a representative-agent deterministic economy with capital income taxes. Under the chosen parameterization, continuous Markov equilibrium on the standard state-space fails to exist. We show that, due to the lack of existence of a continuous policy function, standard methods may easily fail to converge. More important, a standard algorithm yields decision rules with dynamic properties that do not correspond to those of the true equilibrium solution. Our algorithm, nevertheless, is guaranteed to converge and it can accurately capture the set of recursive equilibria. The second specification of the model we consider is a stochastic economy with heterogeneous agents and no taxation. For the parameter values we employ, our method yields a single valued equilibrium correspondence. Our theory implies that a policy function on the enlarged state space exists. We compare this solution against one obtained with a method based on the approximate aggregation strategy. We find that our algorithm yields a more accurate policy function (as measured by its lower Euler equation residuals). Furthermore, we show that the low of accuracy of the policy obtained with the approximate aggregation strategy delivers substantially biased simulated statistics. This example illustrates the approximate aggregation principle is indeed a fragile one. The algorithm we propose does not require approximate aggregation to deliver accurate solutions.

4.1 Economic Environment

Consider an economy populated by a finite number of agents, I , who live forever. There is an aggregate shock, A , that affects the overall productivity level, as well as individual income shocks. The only asset in this economy is capital and each agent is constrained to hold positive amounts of it. Hence, markets are incomplete.

Each agent, i maximizes an intertemporal objective as given by:

$$E \left[\sum_{t=0}^{\infty} (\beta^i)^t U^i(c_t^i) \right], \quad (4.1)$$

where $E[\cdot]$ is the expectation operator conditional on information available at period 0, $\beta \in (0, 1)$ is the discount factor, and c_t is consumption for a given state history up to period t . The function U is strictly increasing, strictly concave, and twice continuously differentiable with $U'(0) = \infty$.

Stochastic consumption plans $(c_t^i)_{t \geq 0}$ are financed from after-tax capital returns, wages, and non-labor endowments. These values are expressed in terms of the single good, which is taken as the numeraire commodity of the system at each date-event. For a given rental rate r_t and wage w_t household i offers $k_t^i \geq 0$ units of capital to the production sector, and supplies inelastically $l_t^i(A_t) \geq 0$ units of labor. For simplicity, we abstract from leisure considerations.

The endowments of labor $l_t^i(A_t)$ and of the aggregate good $e_t^i(A_t)$ follow a Markov chain that depends on the current realization of the aggregate shock. However, the total labor supply is assumed to be constant $\sum_i l_t^i(A_t) = 1$ for all A_t . Hence, the vector of exogenous shocks in this model is

$$z_t = (A_t, e_t^1, \dots, e_t^I, l_t^1, \dots, l_t^I).$$

Each household i is subject to the following sequence of budget constraints

$$\begin{aligned} k_{t+1}^i(z^t) + c_t^i(z^t) &= (1 - \delta) k_t^i(z^{t-1}) + (1 - \tau_k(K)) r_t(z^t) k_t^i(z^{t-1}) + \\ &\quad + w_t(z^t) l_t^i + e_t^i + T_t^i(z^t) \\ k_{t+1}^i(z^t) &\geq 0, \text{ for all state histories } z^t = (z_0, \dots, z_t), \text{ and } k_0^i \text{ given.} \end{aligned} \quad (4.2)$$

Capital income is taxed according to function τ_k , which depends on the aggregate capital stock, K_t . This tax function is assumed to be positive, continuous, and bounded away from 1. All tax revenues

collected from agent i are rebated back in a lump-sum fashion to the same agent. Government rebates are denoted by T_t^i , and are taken as given by all agents.

The production sector is made up of a continuum of identical units that have access to a constant returns to scale technology in individual factors. Hence, without loss of generality we shall focus on the problem of a representative firm. After observing the current shock vector z the firm hires K units of capital and L units of labor. The total quantity produced of the single aggregate good is given by a production function $A_t F(K_t, L_t)$, where A_t is the firm's total factor productivity. $F(K_t, L_t)$ is the direct contribution of the firm's inputs to the production of the aggregate good. Hence, at each date event z^t one-period profits of the representative firm are defined as

$$A_t F(K_t, L_t) - r_t K_t - w_t L_t. \quad (4.3)$$

We shall maintain the following standard conditions on function F :

Assumption 4.1 $F : \mathbf{R}_+ \times \mathbf{R}_+ \rightarrow \mathbf{R}_+$ is increasing, concave, continuous and linearly homogeneous.

This function is continuously differentiable at each interior point (K, L) ; moreover, $\lim_{K \rightarrow \infty} D_1 F(K, L) = 0$ for each given $L > 0$.

4.2 Sequential and Recursive Competitive Equilibrium

The present framework contemplates several deviations from a frictionless world in which a competitive equilibrium may be recast as the solution to an optimal planning program. Our model includes individual uninsurable shocks to labor, capital income taxes, and an aggregate shock to production. Households may hold capital to transfer wealth, but they may be unable to smooth out consumption since there is only one single asset and capital holdings must be non-negative.

Definition 4.1 A sequential competitive equilibrium (SCE) is a tax function $\tau_k(K)$, and a collection of vectors $(\{c_t^i(z^t), k_{t+1}^i(z^t)\}_i, K_t(z^t), L_t(z^t), w_t(z^t), r_t(z^t))_{t \geq 0}$ that satisfy

- (i) *Constrained utility maximization:* For each household i , the sequence $(c_t^i, k_{t+1}^i)_{t \geq 0}$ maximizes the objective (4.1) subject to the sequence of budget constraints (4.2).
- (ii) *Profit maximization:* For each z^t , vector $(K_t(z^t), L_t(z^t))$ maximizes profits (4.3).

(iii) *Market clearing: For each z^t and its predecessor node z^{t-1} ,*

$$K_t(z^t) + \sum_{i=1}^I c_t^i(z^t) = A_t F(K_t(z^t), L_t(z^t)) + (1 - \delta)K_t(z^t) + \sum_{i=1}^I e_t^i(z^t),$$

$$\sum_{i=1}^I k_t^i(z^{t-1}) = K_t(z^t) \text{ and } \sum_{i=1}^I l_t^i(z^t) = L_t(z^t).$$

Note that the equilibrium quantities $(K_t(z^t), L_t(z^t))_{t \geq 0}$ may be inferred from households' aggregate supply of these factors. Hence, we may refer to a SCE as simply a sequence of vectors $(\{c_t^i(z^t), k_{t+1}^i(z^t)\}_i, r_t(z^t), w_t(z^t))_{t \geq 0}$. There does not seem to be a general proof of existence of competitive equilibria for infinite-horizon economies with distortions. We are aware of a related contribution by Jones and Manuelli (1999), but their analysis is not directly applicable to sequential markets economies. Hence, in the Appendix we outline a proof of the following result.

Proposition 4.2 *A SCE exists.*

For computational purposes we need to bound the equilibrium values of the key variables of the model. For it, we show first (in the Appendix) there are positive constants K^{max} and K^{min} such that for every equilibrium sequence of physical capital vectors $(k_{t+1}^i(z^t))_{t \geq 0}$ if $K^{max} \geq \sum_{i=1}^I k_0^i(z^0) \geq K^{min}$ then $K^{max} \geq \sum_{i=1}^I k_{t+1}^i(z^t) \geq K^{min}$ for all z^t . Moreover, $K^{min} > 0$ if $\lim_{K \rightarrow 0} D_1 F(K, L) = \infty$ for some positive L . Hence, in what follows the domain of aggregate capital will be restricted to the interval $[K^{min}, K^{max}]$, and it should be understood that $K^{min} = 0$ only if $\lim_{K \rightarrow 0} D_1 F(K, L)$ is bounded for all given $L > 0$. This implies that every equilibrium sequence of factor prices $(r_t(z^t), w_t(z^t))_{t \geq 0}$ is bounded.

We also need to bound the equilibrium shadow values of investment. To accomplish this task, we define an auxiliary value function of an individual sequential optimization problem (SOP). For a given sequence of factor prices and aggregate capital $(\mathbf{r}_0(\mathbf{z}_0), \mathbf{w}_0(\mathbf{z}_0), \mathbf{K}(\mathbf{z}_0)) = (r_t(z^t), w_t(z^t), K_t(z^t))_{t \geq 0}$, let

$$\begin{aligned}
J^i(k_0^i, z_0, \mathbf{r}_0(\mathbf{z}_0), \mathbf{w}_0(\mathbf{z}_0), \mathbf{K}(\mathbf{z}_0)) &= \max E \sum_{t=0}^{\infty} \beta^t u^i(c_t(z^t), z_t) \\
&\text{s.t.} \\
k_{t+1}^i(z^t) + c_t^i(z^t) &= (1 - \delta) k_t^i(z^{t-1}) + \\
&\quad (1 - \tau_k(K_t(z^t))) r_t(z^t) k_t^i(z^{t-1}) + \\
&\quad + w_t(z^t) l_t^i + e_t^i + T_t^i(z^t), \\
k_{t+1}^i(z^t) &\geq 0, k_0^i \text{ given.}
\end{aligned}$$

For every bounded sequence $(\mathbf{r}_0(\mathbf{z}_0), \mathbf{w}_0(\mathbf{z}_0), \mathbf{K}(\mathbf{z}_0)) = (r_t(z^t), w_t(z^t), K_t(z^t))_{t \geq 0}$, the value function $J^i(k_0^i, z_0, \mathbf{r}_0(\mathbf{z}_0), \mathbf{w}_0(\mathbf{z}_0), \mathbf{K}(\mathbf{z}_0))$ is well defined, and continuous. Moreover, mapping $J^i(\cdot, z_0, \mathbf{r}_0(\mathbf{z}_0), \mathbf{w}_0(\mathbf{z}_0), \mathbf{K}(\mathbf{z}_0))$ is increasing, concave, and differentiable with respect to the initial condition k_0^i . Further, the partial derivative $D_1 J^i(k_0^i, z_0, \mathbf{r}_0(\mathbf{z}_0), \mathbf{w}_0(\mathbf{z}_0), \mathbf{K}(\mathbf{z}_0))$ varies continuously with $(k_0^i, \mathbf{r}_0(\mathbf{z}_0), \mathbf{w}_0(\mathbf{z}_0), \mathbf{K}(\mathbf{z}_0))$ [cf. Rincon-Zapatero and Santos (2005)]. The next result readily follows from these regularity properties of the value function.

Proposition 4.3 *For all SCE $(\{c_t^i(z^t), k_{t+1}^i(z^t)\}_i, r_t(z^t), w_t(z^t))_{t \geq 0}$ with $K^{max} \geq \sum_{i=1}^I k_0^i(z^0) \geq K^{min}$, there is a constant γ such that $0 \leq D_1 J^i(k_0^i, z_0, \mathbf{r}_0(\mathbf{z}_0), \mathbf{w}_0(\mathbf{z}_0), \mathbf{K}(\mathbf{z}_0)) \leq \gamma$ for all z^t .*

Observe that these bounds apply to the following types of utility functions: (i) Both function $u(\cdot, z)$ and its derivative are bounded, (ii) function $u(\cdot, z)$ is bounded, and its derivative function is unbounded, and (iii) both function $u(\cdot, z)$ and its derivative are unbounded. Phelan and Stacchetti (2001) deal with case (i) and Krebs (2004) and Kubler and Schmedders (2003) consider utility functions of type (iii). We provide a uniform method of proof that covers all the three cases, as well as production functions with bounded and unbounded derivatives. As a matter of fact, Proposition 4.3 fills an important gap in the literature, since no general results are available on upper and lower bounds for factor prices and marginal utilities for production economies with heterogeneous consumers and market frictions.

RECURSIVE SPECIFICATION

For any initial distribution of capital k_0 and a given shock z_0 , we define the Markov equilibrium correspondence $V^* : \mathbf{K} \times \mathbf{Z} \rightarrow \mathbf{R}_+^I$ as

$$V^*(k_0, z_0) = \left\{ \begin{array}{l} (\cdots, D_1 J^i(k_0^i, z_0, \mathbf{r}_0(\mathbf{z}_0), \mathbf{w}_0(\mathbf{z}_0), \mathbf{K}(\mathbf{z}_0)), \cdots) : \\ (\{c_t^i(z^t), k_{t+1}^i(z^t)\}_i, r_t, w_t)_{t \geq 0} \text{ is a SCE} \end{array} \right\}, \quad (4.4)$$

where $\mathbf{K} = \{k : K^{max} \geq \sum_{i=1}^I k^i \geq K^{min}\}$. Hence, the set $V^*(k_0, z_0)$ contains all current equilibrium shadow values of investment $m_0 = (\cdots, m_0^i, \cdots)$, for every household i .

The following result can be used to establish the existence of recursive equilibria for the class of models under consideration. We also derive the compactness and uppersemicontinuity of the equilibrium correspondence, which are key requirements of our computational algorithm.

Corollary 4.4 *Correspondence V^* is nonempty, compact-valued, and upper semicontinuous.*

Note that by the envelope theorem we must have $D_1 J^i(k_0^i, z_0, \mathbf{r}_0(\mathbf{z}_0), \mathbf{w}_0(\mathbf{z}_0), \mathbf{K}(\mathbf{z}_0)) \geq (1 - \delta + (1 - \tau_k)r_0(z_0))D_1 u^i(c_0^i, z_0)$, with equality when $c_0^i > 0$. Moreover, Proposition 4.3 implies $0 \leq D_1 J^i(k_0^i, z_0, \mathbf{r}_0(\mathbf{z}_0), \mathbf{w}_0(\mathbf{z}_0), \mathbf{K}(\mathbf{z}_0)) \leq \gamma$, and so $c_0^i = 0$ is only possible if the derivative of the utility function u^i is bounded at $c_0^i = 0$. Hence, the above corollary is a straightforward consequence of Propositions 4.2 and 4.3.

The second key element of our analysis is operator B . For the class of models under consideration this operator can be defined as follows. Take any given correspondence $V : \mathbf{K} \times \mathbf{Z} \rightarrow \mathbf{R}_+^I$. Then, for every $(k, z) \in \mathbf{K} \times \mathbf{Z}$, the set $B(V)(k, z)$ is constituted by the values $m = (\cdots, (1 - \delta + (1 - \tau_k)r)[D_1 u^i(c^i, z) + \zeta^i], \cdots)$, with $m^i \leq \gamma$ for all i , such that there is some vector $(c, k_+, r, w, m_+, \lambda, \zeta) \in \mathbf{R}_+^I \times \mathbf{R}_+^I \times \mathbf{R}_+ \times \mathbf{R}_+ \times (\mathbf{R}_+^I)^N \times \mathbf{R}_+^I \times \mathbf{R}_+^I$, with $m_+(z_+) \in V(k_+, z_+)$ for all $z_+ \in \mathbf{Z}$, that satisfies all individual and aggregate *temporary equilibrium conditions*.²

4.3 Numerical example 1: A model with capital income taxes

We study first a deterministic version of the above model with a representative agent and capital income taxes. To further simplify our analysis we assume capital is the only input required in production, a logarithmic utility function, and full capital depreciation, $\delta = 1$. Greenwood and Huffman

²As already pointed out, the bound $\gamma \geq m^i \geq 0$ for all i entails that a solution $c^i = 0$ is only attained if function u^i has bounded derivatives. Bounding the vector of values m is useful to select a correspondence W such that $B(W) \subset W$.

(1995) study existence of recursive equilibria for representative agent economies with distortions like the present one. These authors then show that if the equilibrium dynamics of the model satisfy certain monotonicity requirements then recursive equilibrium $K' = g(K)$ exists. The version of the model we evaluate in this section satisfies the aforementioned monotonicity conditions when the capital income tax rate function is increasing. However, when taxes are given by a decreasing function monotonicity conditions may be easily broken. More importantly, Santos (2002, Proposition 3.4) shows that for the following specification of the model continuous Markov equilibrium in the natural state space fails to exist:

$$f(k) = k^{1/3}, \beta = 0.95, \quad (4.5)$$

with a continuous, piecewise linear, tax schedule given by

$$\tau(K) = \begin{cases} 0.10 & \text{if } K \leq 0.160002 \\ 0.05 - 10(K - 0.165002) & \text{if } 0.160002 \leq K \leq 0.170002 \\ 0 & \text{if } K \geq 0.170002. \end{cases} \quad (4.6)$$

It is easy to verify, nevertheless, that the conditions of Proposition 4.3 hold so that recursive equilibrium in an adequately expanded state space does exist.

Implementation of our algorithm

Following the notation of our general theoretical framework, the state vector of endogenous variables of this model, x , is constituted by one element, capital, which according to the proof of Proposition 4.3, takes values on the interval $[K_{\min}, K_{\max}]$. The vector of current endogenous variables, y , also has one element, consumption (profits, and the interest rate in equilibrium can be easily written as functions of the current capital stock). Further, the representative agent's capital holdings equal those of the aggregate economy. Hence, $k_t = K_t$ for all t . Employing all of the previous observations, we can write:

$$\varphi(k_{t+1}, c_t) = f(k_t) - c_t - k_{t+1}, \text{ and} \quad (4.7)$$

$$m_t = h(k_t, c_t) = \frac{r_t(1 - \tau_k(K_t))}{c_t} = \frac{\frac{1}{3}k_t^{-2/3}(1 - \tau(k_t))}{c_t}. \quad (4.8)$$

Similarly, aggregate feasibility and the intertemporal optimality conditions for the household can be

summarized by the Euler equation

$$\Phi(k_t, c_t, m_{t+1}) = \frac{1}{c_t} - \beta m_{t+1}. \quad (4.9)$$

Finally, for a given correspondence V , $B(V)(k_t)$ is given by the set of values $m_t \in V(k_t)$ such that there is c_t, k_{t+1} and $m_{t+1} \in V(k_{t+1})$ satisfying the temporary equilibrium conditions (4.7- 4.9).

The strategy we followed to implement our algorithm in the computer exploits the low dimensionality and compactness of the equilibrium correspondence. Specifically, notice that for each given k_t the shadow values associated to investment, $m(k_t)$, lie in a compact interval $[\underline{m}(k_t), \overline{m}(k_t)]$. Hence, our numerical algorithm starts by approximating the upper and lower bound functions $\underline{m}(k_t)$ and $\overline{m}(k_t)$ using step functions. Notice however, that these functions may be discontinuous. Hence, the standard strategy of approximating these functions only at the vertex points of the triangulation and employing interpolation to obtain approximate values in the overall state space may not work. In this case it is necessary to obtain bounds for all values within each of the simplices. The interested reader can find a detailed algorithm in our computational appendix. In what follows we illustrate the properties of our numerical approximation.

Figure 4.1 presents our initial guess (left panel), $V_0^{h,N}$, and the correspondence defined by the area (right panel) between the upper and lower approximated functions $\underline{m}(k_t)$ and $\overline{m}(k_t)$. A useful feature of this example is that the backwards shooting algorithm can be used to obtain highly accurate solutions. The dots in the Figure below represent an approximate solution derived via the backwards shooting algorithm.

Since the limiting correspondence is not single valued near the middle steady state, our method is signaling the possibility of a multiple valued equilibrium correspondence. To approximate equilibrium time series from the model we used the method described in Section 3.1. The resulting policy correspondence is illustrated in Figure 4.2 below together with the solution obtained via the shooting method. In this specific example our method and the shooting algorithm result in highly accurate solutions. However, the shooting algorithm is of limited applicability. For instance, it cannot be used once it introduces uncertainty into the analysis while our method can easily handle it.

Comparison with standard computational algorithms

A standard practice in quantitative analysis is to assume that a continuous policy function

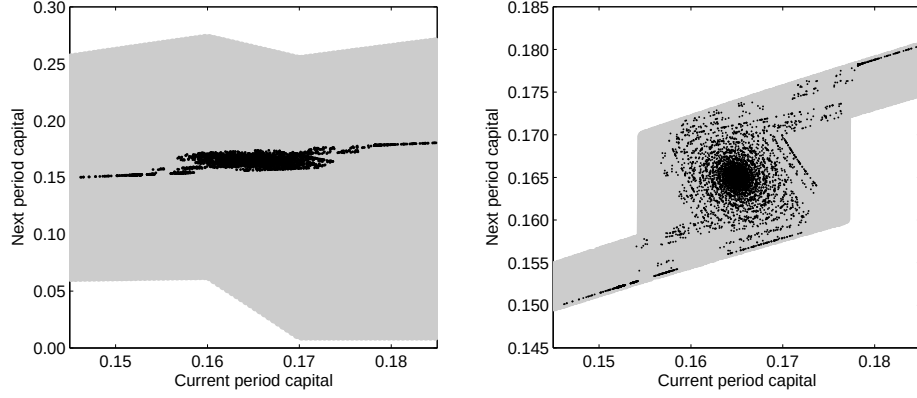


Figure 4.1: Initial (left grey area) and limiting correspondence (right grey area) vs solution obtained via the backwards shooting method (black dots).

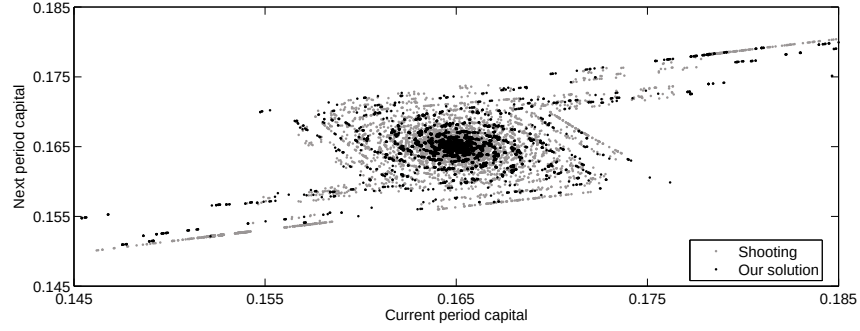


Figure 4.2: Equilibrium correspondence computed with our method vs the backwards shooting method.

exists. This entails assuming that equilibrium is Markovian on the minimal state space, and that a continuous policy function exists. To solve this model using such standard methods, we impose

$$k_1 = g(k, \xi),$$

where g belongs to the class of functions that can approximate any continuous function, and that are completely determined by a finite vector of parameters ξ . We obtain an estimate for ξ by solving the Euler equation at as many points in the domain as the dimension of ξ

$$u'(k^i, g(k^i, \xi)) = \beta u'(g(k^i, \xi), g(g(k^i, \xi), \xi)) \cdot [f'(g(k^i, \xi))(1 - \tau(g(k^i, \xi))) + (1 - \delta)]$$

The choice of the grid points, k^i , where the Euler equation is solved may be correlated with the functional form chosen to approximate the policy function (e.g. in the case of Chebyshev polynomials). We aimed at computing function $g(k, \xi)$ from the class of piecewise linear functions, or finite elements. For this model, the method failed to converge. In particular, we found that vector ξ oscillated with no discernible pattern across different iterations. As expected, the area of the domain where the lack of convergence occurred was close to the middle steady state. Figure 4.3 below illustrates representative functions from different iterations of the algorithm.

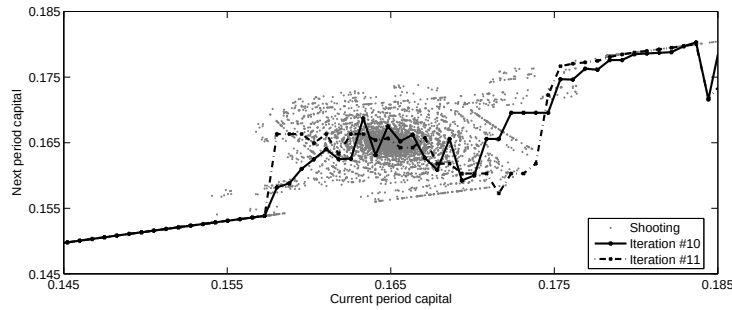


Figure 4.3: Different iterations of a standard solution method vs Shooting solution

Observe that the distance between candidate policy functions may become relatively small, and a researcher may be tempted to employ one of those policies to perform quantitative analysis. The jumps displayed by the above policy functions near the middle steady state imply that these policies have a multiplicity of steady states. Some of these steady states may even be stable.

In summary, we presented a calibrated dynamic model for which equilibria cannot be represented by a continuous policy function over the standard state space. We have also illustrated that under such circumstances standard computational methods based on iterations of continuous functions may not converge. More important, such methods may result in a policy function with very different dynamics from those of the true model solution. In contrast, our solution method is guaranteed to converge and approximate the true model's solution.

4.4 Numerical example 2: A model with two agents and no taxes

We now consider a specification of the model with two agents who face idiosyncratic and aggregate uncertainty. There are no taxes. Both agents have the same utility function, $U^i(c^i) = \frac{(c^i)^{1-\sigma}}{1-\sigma}$,

and preference parameter, $\beta^1 = \beta^2 = 0.95$. The capital share is $\alpha = 0.34$ and depreciation rate $\delta = 0.06$. Total factor productivity (the aggregate shock) is a random variable with two possible values: $A_g = 1.0807$ and $A_b = 1.0593$. Each agent has a random endowment of labor, l^i , which can take two possible values, $l_b = 0$ and $l_g = 1$. Further, we assume that idiosyncratic shocks do not affect the aggregate labor supply, namely, $l_t^1 + l_t^2 = 1$ at all date-events. Finally, productivity and labor endowments are assumed to be jointly driven by a Markov process with transition matrix

$$\Pi = \begin{pmatrix} 0.5 & 0.4 & 0.06 & 0.04 \\ 0.6 & 0.3 & 0.06 & 0.04 \\ 0.45 & 0.35 & 0.15 & 0.05 \\ 0.5 & 0.3 & 0.15 & 0.05 \end{pmatrix}.$$

We follow the usual convention where element $\Pi_{p,q}$ is the probability of moving to state p conditional on the current state q .³

Implementation of our algorithm

Mapping this model into the notation of our general theoretical framework is simple. The vector of endogenous predetermined variables is given by the capital holdings of each agent, $x_t = (k_t^1, k_t^2)$, while the vector of current endogenous variables contains the consumption and investment choices of each agent, $y_t = (c_t^1, c_t^2, i_t^1, i_t^2)$. Interest rates and wages can be explicitly written, in equilibrium, in terms of the aggregate capital and labor supply $(K_t, 1)$. Hence, the functions defining the laws of motion for capital, and the shadow multipliers for investment are given by

$$\varphi(x_{t+1}, x_t, y_t, z_t) = (i_t^1 + (1 - \delta)k_t^1 - k_{t+1}^1, i_t^2 + (1 - \delta)k_t^2 - k_{t+1}^2), \quad (4.10)$$

$$(m_t^1, m_t^2) = h(x_t, y_t, z_t) = ((r_t + 1 - \delta)(c_t^1)^{-\sigma}, (r_t + 1 - \delta)(c_t^2)^{-\sigma}). \quad (4.11)$$

Here $r_t = \theta A_t K_t^{\theta-1}$, $w_t = (1 - \theta) A_t K_t^\theta$. Finally, the vector-valued function capturing intertemporal

³Notice there are four possible states (A_g, l_g, l_b) , (A_g, l_b, l_g) , $(A_b, l_g, L\varepsilon_b)$, and (A_b, l_b, l_g) .

optimality and all individual and aggregate constraints, $\Phi(x_t, y_t, z_t, E_t[m_{t+1}]) =$

$$\begin{aligned} & ((r_t + 1 - \delta)k_t^i + w_t l_t^i - c_t^i - i_t^i, \\ & (c_t^j)^{-\sigma} - \beta E_t m_{t+1}^j(z_{t+1}) + \lambda_t^j, \text{ for } j = 1, 2, \\ & \sum_i (c_t^i + i_t^i) = A_t K_t^\theta). \end{aligned} \tag{4.12}$$

The multipliers associated to the constraints $k_{t+1}^i \geq 0$ are denoted by λ_t^i .

Our algorithm operates as follows. Let V be any given correspondence, then $BV(x, z)$ is the set of all values (m_t^1, m_t^2) for which one can find values $c_t^1, c_t^2, i_t^1, i_t^2, k_{t+1}^1, k_{t+1}^2$, and $(m_{t+1}^1, m_{t+1}^2) \in V(k_{t+1}^1, k_{t+1}^2, z_{t+1})$ at all possible successors z_{t+1} of z_t , that satisfy (4.10-4.12). Notice that the set of values for the shadow multipliers of investment lie in a higher dimensional space (at least R^2 for the model to be interesting). Our computational appendix provides one possible way of implementing our algorithm in the computer in this type of environment with multiple agents.

Comparing Alternative Solution Methods

A commonly employed method to solve this type of models is based on the method of "approximate aggregation" pioneered by Krusell and Smith (1998). The key insight of this algorithm is that in equilibrium aggregate variables may be well approximated as functions of simple statistics. In particular, the stochastic process driving aggregate capital is assumed to be characterized by a finite vector of moments. Individual decisions are computed on the basis of such expectations for aggregate variables. Finally, a fixed point problem is solved where the moments of the aggregate time series from the simulated individual are required to be close to the moments assumed by each individual for the law of motion for aggregate capital.

In our baseline model, the algorithm is applied in the following way. Start with a guess on a parameterized functional form for the first moment of aggregate capital [cf. 4.14 below]. Then dynamic programming techniques and value function iteration can be used for solving the problem

of the representative household

$$\nu(k^i; K, z) = \max\{U(c) + \beta E[\nu(k'^i; K', z')|z, \varepsilon]\} \quad (4.13)$$

$$\begin{aligned} s.t. \quad c + k^{i'} &= r(K, z)k^i + w(K, z)\varepsilon^i + (1 - \delta)k^i \\ k^{i'} &\geq B \\ \log K_{t+1} &= a(z) \log K_t + b(z) \end{aligned} \quad (4.14)$$

The algorithm iterates estimates of coefficients $(a(z_g), b(z_g), a(z_b), b(z_b))$ and individual policy functions in the following fashion: i) Depart from a given value for coefficients; (ii) Solve the dynamic programming problem of each agent (4.13); (iii) Construct aggregate capital time series by aggregating the resulting individual time series simulations; (iv) Perform a regression over the stationary region to obtain new estimates for such coefficients. An approximate equilibrium is found when two subsequent estimated coefficients are close to each other and if the R^2 and standard error of the aforementioned regression are accurate enough.

The problem in (4.13) can be considered as an approximation to the dynamic programming problem that characterizes Markov Equilibria over the natural state space. In particular, it assumes that aggregate capital tomorrow depends only on the first moment of aggregate capital today, and on the aggregate shock. A Markovian solution over the natural state space of this economy, when it exists, requires replacing the expectations operator on the right hand of (4.13) by an expectation on the joint distribution of capital, individual shocks, and the aggregate shock.

An obvious advantage of this type class of algorithms is that the model can be easily extended to include an arbitrary number of agents that are ex-ante equal but that receive different idiosyncratic shocks. However, the fixed point problem required for obtaining an approximate equilibrium may not have a solution. Furthermore, little is known about the accuracy of the simulated moments and individual decisions that derives from this method.

In Table 4.4 below, we compare the quantitative properties of the ‘‘Approximate Aggregation’’ method described above to those of our algorithm.

Method	$Mean(EE_1)$	$Mean(EE_2)$	$Mean(k_1)$	$Mean(k_2)$
Approx. Aggregation	1.57×10^{-2}	2.71×10^{-2}	2.8196	4.5210
Our Algorithm	5.14×10^{-4}	7.58×10^{-4}	3.0898	3.8623

Table 4.4: Euler equation residuals and simulated moments of alternative solution methods.

In spite of the fact that our algorithm again results in a single valued equilibrium correspondence, we can see here that our method yields higher accuracy of approximation as measured by Euler equation residuals. More important, our non-linear equilibrium approximation results in substantially different simulated statistics for individual wealth than those obtained from the approximate aggregation method.

5 A Stochastic OLG Economy

Overlapping generation have become central in the analysis of several macro issues such as the funding of social security, the optimal profile of savings and investment over the life cycle, the effects of various fiscal and monetary policies, and the evolution of future interest rates and asset prices under current demographic trends.⁴

As already stressed, there are no known convergent procedures for the computation of sequential competitive equilibria in OLG models even for frictionless economies with complete financial markets. Our main goal here is to illustrate that our approach delivers a reliable algorithm for the solution of competitive equilibria in a general class of OLG models.

The economy we consider is conformed by a sequence of overlapping generations that live for two periods. As in our previous analysis we will focuss on stationary Markov equilibria. Hence, the primitive characteristics of the economy are defined by a stationary Markov chain, and generations will only differ by their birth date.

At every time period $t = 0, 1, 2, \dots$ a new generation is born. Each generation is made up of I agents, who are present in the economy for two periods. More specifically, for a household of type i born at time t preferences are defined over consumption bundles of the goods available at times t and $t + 1$, and the agent can only trade goods and assets in these two periods. The economy starts with an initial generation who is only present in the initial period $t = 0$. This generation is endowed with the aggregate supply of assets.

⁴For instance, see Champ and Freeman (2002), Conesa and Krueger (1999), Geanakoplos, Magill and Quinzii (2003), Gourinchas and Parker (2002), Imrohoroglu, Imrohoroglu, and Joines (1995), Storelesletten, Telmer and Yaron (2004), and Ventura (1999).

For simplicity, we assume that every utility function U^i is separable over consumption of different dates. For an agent i born in period t , let $c_{y,t}^i(z^t)$ denote the consumption of the aggregate good in period t over the history of shocks z^t , and let $c_{o,t+1}^i(z^{t+1}|z^t)$ denote the consumption in period $t+1$ for every successor node $z^{t+1}|z^t$. Then the intertemporal objective U^i is defined as

$$U^i(c_{y,t}^i, c_{o,t+1}^i; z^t, z^{t+1}) = u^i(c_{y,t}^i(z^t), z_t) + \beta \sum_{z^{t+1} \in \mathbf{Z}} v^i(c_{o,t+1}^i(z^{t+1}), z_{t+1}) \pi(z^{t+1}|z^t) \quad (5.1)$$

where the one-period utilities u^i and v^i satisfy the following conditions:

Assumption 5.1 *For each $z \in \mathbf{Z}$ the one-period utility functions $v^i(\cdot, z), u^i(\cdot, z) : \mathbf{R}_+ \rightarrow \mathbf{R} \cup \{-\infty\}$ are increasing, strictly concave, and continuous. These functions are also continuously differentiable at every interior point $c > 0$.*

At each node z^t , there exist spot markets for the consumption good and J securities. These securities are specified by the current vector of prices, $q_t(z^t) = (\dots, q_t^j(z^t), \dots)$, and the vectors of future dividends $d_r(z^r) = (\dots, d_r^j(z^r), \dots)$ promised to deliver at future information sets $z^r|z^t$ for $r > t$. We assume that the vector of security prices $q_t(z^t)$ is non-negative – a condition implied by free disposal of securities – and the vector of dividends $d_t(z^t)$ is positive in all components and depends only on the current realization of the vector of shocks z_t ; hence, $(d_t(z_t))_{t \geq 0}$ is a time invariant Markov chain.

Each agent i born at $t = 1, 2, \dots$ is endowed with a vector of goods $e_t^i = (e_{y,t}^i, e_{o,t+1}^i)$ and trades an asset portfolio θ^i to attain desirable amounts of consumption. The endowment process $(e_t^i(z^t)) = (e_{y,t}^i(z^t), e_{o,t+1}^i(z^{t+1}|z^t))$ follows a time invariant Markov chain; hence $e_{y,t}^i(z^t) = e_y^i(z_t)$, and $e_{o,t+1}^i(z^{t+1}|z^t) = e_o^i(z_{t+1})$ for every agent i and every t . Likewise, the dividend process $(d_t(z_t))_{t \geq 0}$ only depends on the current realization z_t . Given prices $(q_t(z^t))_{t \geq 0}$, a consumption-savings plan $(c_{y,t}^i(z^t), c_{o,t+1}^i(z^{t+1}), \theta_t^i(z^t))$ must obey the following two-period budget constraints:

$$\theta_{t+1}^i(z^t) \cdot q_t(z^t) + c_{y,t}^i(z^t) \leq e_{y,t}^i(z_t), \text{ for } \theta_{t+1}^i(z^t) \geq 0, \quad (5.2)$$

$$c_{o,t+1}^i(z^{t+1}) \leq \theta_{t+1}^i(z^t) \cdot (q_{t+1}(z^{t+1}) + d_t(z_{t+1})) + e_{o,t+1}^i(z_{t+1}), \text{ all } z^{t+1}|z^t. \quad (5.3)$$

As already pointed out, the initial stock of securities θ_0 is owned by the members of the starting

generation at $t = 0$. Each agent i in this generation seeks to maximize the total quantity of consumption $c_{o,0}^i(z_0)$ for given endowments of the aggregate good e_o^i and the vector of securities θ_0^i . More precisely,

$$c_{o,0}^i(z_0) = \theta_0^i \cdot (q_0(z_0) + d_0(z_0)) + e_o^i(z_0). \quad (5.4)$$

In this economy the aggregate commodity endowment is bounded by a portfolio-trading plan [Santos and Woodford (1997)], and hence asset pricing bubbles cannot exist in a SCE. As discussed above, to circumvent technical issues concerning existence of a SCE, we still maintain the short-sale constraint $\theta_t \geq 0$ for all t .

Definition 5.1 *A SCE is a collection of vectors $\{(c_{y,t}^i(z^t), c_{o,t+1}^i(z^{t+1}|z^t), \theta_{t+1}^i(z^t))_{i=1}^I, q_t(z^t)\}_{t \geq 0}$ such that*

(i) *Utility maximization: For every household i and all t vector $(c_{y,t}^i(z^t), c_{o,t+1}^i(z^{t+1}|z^t), \theta_{t+1}^i(z^t))$ maximizes the objective (5.1) subject to (5.2)-(5.3). For every household i of the starting generation $c_{o,0}^i(z_0)$ satisfies (5.4).*

(ii) *Market clearing: For each z^t ,*

$$\begin{aligned} \sum_{i=1}^I (c_{y,t}^i(z^t) + c_{o,t}^i(z^t)) &= \sum_{j=1}^J d_t^j(z_t) + \sum_{i=1}^I (e_{yt}^i(z_t) + e_{ot}^i(z_t)) \\ \sum_{i=1}^I \theta_{t+1}^{ji}(z^t) &= 1, j = 1, \dots, J. \end{aligned}$$

Under the present assumptions, the existence of a SCE can be established by standard methods [e.g., Balasko and Shell (1980), and Schmachtenberg (1988)]. Moreover, by similar arguments used by these authors it is easy to show that every sequence of equilibrium asset prices $(q_t(z^t))_{t \geq 0}$ is bounded.

Then, we define the Markov equilibrium correspondence $V^* : \Theta \times \mathbf{Z} \rightarrow \mathbf{R}_{++}^{JI}$ as

$$V^*(\theta_0, z_0) = \left\{ \left(\dots \left(q_0^j(z_0) + d_0^j(z_0) \right) D_1 v^i(c_0^i(z_0), z_0) \dots \right) : (c_y, c_o, \theta, q) \text{ is a SCE} \right\}.$$

Then, from the above results on existence of SCE for OLG economies we obtain

Proposition 5.2 *Correspondence V^* is nonempty, compact-valued, and upper semicontinuous.*

5.1 Numerical Example: A monetary model

We consider a simplified version of a deterministic OLG model with money of Grandmont (1985). This example is useful because the deterministic version of the model can be solved with arbitrary accuracy using standard methods. Hence, it is possible to compare the true solution of the model to a numerical approximation derived with our algorithm, and to other approximations derived with standard solution methods where continuity of Markov equilibrium is assumed. Extensions to a stochastic environment are easy to handle by our algorithm, but do not allow for closed form solutions.

Each individual receives an endowment of e_1 when young and e_2 when old. The endowment is perishable, and the only way to save is to hold money. Relative to the general OLG framework discussed before, the present example corresponds to an economy with one asset, money, that pays zero dividends at each given period. The initial old agent is endowed with the initial money supply M . Let P_t be the price level at time t . An agent born in period t chooses consumption c_{1t} when young, c_{2t+1} when old, and money holdings M_t to solve

$$\max u(c_{1t}) + \beta v(c_{2t+1})$$

subject to

$$\begin{aligned} c_{1t} + \frac{M_t}{P_t} &= e_1, \\ c_{2t+1} &= e_2 + \frac{M_t}{P_{t+1}}. \end{aligned}$$

A sequential competitive equilibrium for this economy is a sequence of prices $(P_t)_{t \geq 0}$, consumption for the period-zero consumer c_{20} , and sequences of consumption and money holdings $\{c_{1t}, c_{2t+1}, M_t\}_{t \geq 0}$ such that individuals optimize and markets clear:

$$c_{1t} + c_{2t} = e_1 + e_2, \text{ and } M_t = M \text{ for all } t.$$

A sequential competitive equilibrium can be characterized by the following condition:

$$\frac{1}{P_t} u' \left(e_1 - \frac{M}{P_t} \right) = \frac{1}{P_{t+1}} \beta v' \left(e_2 + \frac{M}{P_{t+1}} \right).$$

We define real money balances as $b_t = M/P_t$. We then have

$$b_t u' (e_1 - b_t) = b_{t+1} \beta v' (e_2 + b_{t+1}).$$

We can express an equilibrium by an offer curve in the (b_t, b_{t+1}) space.⁵ A recursive equilibrium can be characterized by a function g such that $b_{t+1} = g(b_t)$. A recursive equilibrium does not exist when the offer curve is backward bending.

We can cast this model in our general framework under certainty described in Section 2.1 by setting $x_t = m_t$, $y_t = b_t$, and

$$\begin{aligned} h(x_t, y_t) &= x_t v' (x_t + e_2), \\ \Phi(x_t, y_t, m_{t+1}) &= x_t u' (e_1 - x_t) - \beta m_{t+1}. \end{aligned}$$

In addition, the function φ is endogenous.

In the remainder of this section, we restrict our attention to the following functional specification and parameterization:

$$u(c) = c^{0.45}, \quad v(c) = -\frac{1}{7} c^{-7}, \quad \beta = 0.8,$$

$M = 1$, $e_1 = 2$, and $e_2 = 2^{6/7} - 2^{1/7}$. For this simple example, the offer curve is backward bending. We may write it as $b_t = G(b_{t+1})$ for some function G . We can compute the “exact” offer curve using the following functional equation:

$$G(b) u' (e_1 - G(b)) = b \beta v' (e_2 + b).$$

The solution is illustrated in Figure 5.1.

As we can clearly see here, the equilibrium correspondence is multiple valued and standard methods, based on the assumption that a continuous function describes equilibrium, may display undesirable features. The implementation of the numerical algorithm of Section 3, however, is

⁵ Alternative, we can use the (c_{1t}, c_{2t+1}) space as in Cass, Okuno, and Zilcha (1979).

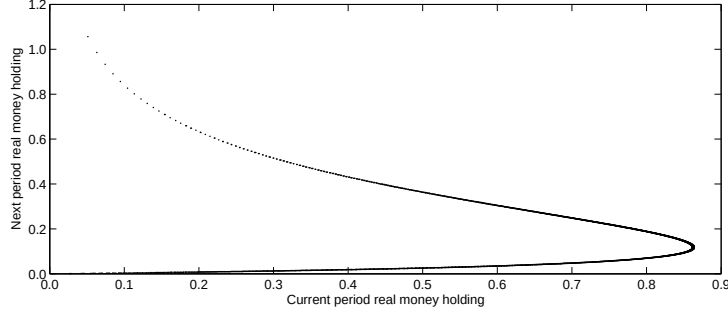


Figure 5.1: “Exact” offer curve for the OLG model.

straightforward. In fact, since the shadow multipliers of investment lie on a compact subset of R , we can compute equilibrium using algorithm similar to the one we used to solve the growth model of the previous section. Hence upper and lower bound functions are computed, and once the fixed point is found a selection algorithm may be used to generate equilibrium time series. The results from this algorithm are reported in Figure 5.2 where the dark grey area represents the initial correspondence, the light grey area represents the fixed point of algorithm $B^{h,N}$, and the dotted line is the equilibrium correspondence constructed using the equilibrium selection algorithm of Section 3.

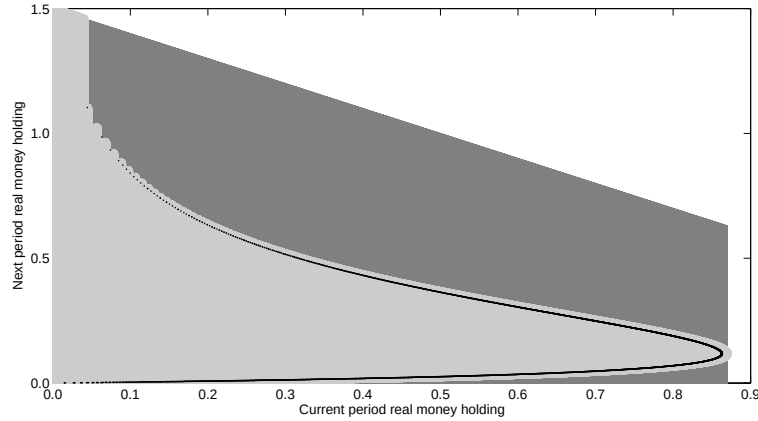


Figure 5.2: Initial guess, limiting correspondence, and approximated equilibrium policy correspondence from operator $B^{h,N}$.

For this example, we find that the policy correspondence and time series from our method generate an Euler equation residual of order 10^{-6} , so that the solution obtained with our algorithm

is indistinguishable from the “exact” solution.

Comparison with existing computational algorithms.

A common practice to approximate equilibria for these models consists of searching for a recursive equilibrium of the form $b' = g(b)$. Following such a procedure, we find that one can either obtain the upper part or the lower part of the offer curve. Which one one will obtain depends on the initial guess employed. This strong dependence on initial conditions is a rather undesirable feature of this method. Notice, in particular, that for initial conditions where the method yields the lower part of the actual equilibrium correspondence one will end up with time series simulations where monetary equilibrium does not exist. Indeed, zero real monetary holdings are the unique absorbing steady state associated with the lower part of the equilibrium correspondence.

In general, the time series properties from simulations from approximations to continuous Markov equilibrium may be quite different from an approximation to the equilibrium correspondence, even in the deterministic version of the model. In the approximate equilibrium correspondence a cyclical equilibrium exists whereby real money holdings oscillate between 0.85296237892 and 0.09517670718. We illustrate this dynamic behavior by the solid line in Figure 5.3 below. With the same initial price and money holdings, but following the solution based on the existence of continuous Markov equilibrium, money holdings converge monotonically to $\frac{\bar{M}}{p} = 0.418142579084$, as illustrated by the dashed line in this same figure.

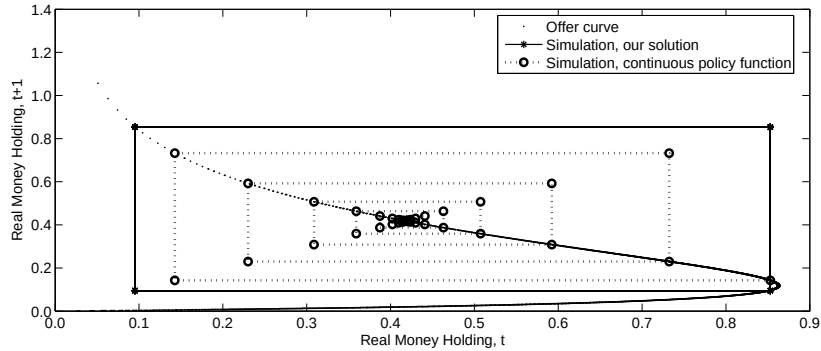


Figure 5.3: Time-series behavior of different numerical solutions.

Hence, standard methods find an equilibrium for this model. What is impossible to do using continuous function approximations, is to obtain the full set of equilibria.

6 Asset Pricing Models with Incomplete Markets

There is an important family of macroeconomic models that incorporate financial frictions in the form of sequentially incomplete markets, borrowing constraints, transactions costs, cash-in-advance constraints, and margin and collateral requirements. These models are commonly used to assess the effects of monetary policies, and the variability of asset prices, consumption, interest rates, inflation, and other macro aggregates.⁶ Fairly general conditions rule out the existence of financial bubbles in these economies, and hence equilibrium asset prices are determined by the expected value of future dividends [Santos and Woodford (1997)]. However, financial frictions invalidate the welfare theorems, and may also break monotonicity or continuity conditions that are commonly used to establish existence of Markovian equilibrium. Hence, the current literature offers no reliable method for the computation of this important class of models. The purpose of this section is to illustrate the applicability of our proposed algorithm in this class of models.

We study an economy with a finite number of agents who, at each node of an uncertainty tree, can trade quantities of the unique aggregate good as well as a fixed set of assets that span the horizon of the economy. For simplicity, we only incorporate two financial frictions: (i) Asset markets are sequentially incomplete; that is, there are fewer securities than possible realizations of the vector of shocks z , and (ii) Short-sales are not allowed except for one type of asset. Further, short-sales are subject to collateral constraints.

Each agent i maximizes an intertemporal objective as given by:

$$E \left[\sum_{t=0}^{\infty} (\beta^i)^t U^i(c_t^i) \right], \quad (6.1)$$

where $\beta^i \in (0, 1)$, and U^i is strictly increasing, strictly concave and continuously differentiable with $U_1^i(0) = \infty$.

At each node z^t , there exist spot markets for the consumption good and J securities. These

⁶For instance, see Campbell (1999), Heaton and Lucas (1996), Huggett (1993), Krebs and Wilson (2004), Mankiw (1986), and Telmer (1993). For some monetary models see Bewley (1980), Lucas (1982), and Santos (2004).

securities are specified by the current vector of prices, $q_t(z^t) = (\dots, q_t^j(z^t), \dots)$, and the vectors of dividends $d(z^r) = (\dots, d^j(z^r), \dots)$ promised to deliver at future information sets $z^r|z^t$ for $r > t$. We assume that the vector of security prices $q_t(z^t)$ is non-negative – a condition implied by free disposal of securities – and that the vector of dividends $d_t(z^t)$ is positive in all components and depends only on the current realization of the vector of shocks z_t ; hence, $(d(z_t))_{t \geq 0}$ follows a stationary Markov process. For convenience of the presentation, we assume that each security j is in unit supply.

In addition to the above securities, there is a market for one period bonds available at each node of the tree. A bond is a promise of one unit of the consumption good at all z_{t+1} successors of z^t . Bonds are traded at price $p_t(z^t)$, they are in zero net supply, and they are the only assets for which short-sales are allowed. Collateral requirements imply agents must hold $k^j \geq 0$ consumption-worth units of each security j to short-sale one unit of the bond. Agents who short-sale bonds can default on their promised payments. In case of default, the buyer of the bond gets the collateral associated to the short-sale.

Every household i enters the financial markets at state z_0 with an initial endowment of securities θ_0^i . In addition, at each node z^t the agent receives $e^i(z_t) > 0$ units of the consumption good. As before, the endowment vector $(e(z_t))_{t \geq 0}$ varies only with the current realization of the shock z_t . Given a price process $(q_t(z^t), p_t(z^t))_{t \geq 0}$, agent i chooses desired quantities of consumption and portfolio holdings $(c_t^i(z^t), \theta_{t+1}^i(z^t), \phi_{t+1}^i(z^t))_{t \geq 0}$ subject to the following sequence of budget constraints

$$c_t^i(z^t) - \phi_t^i(z^{t-1}) \min \left\{ 1, \sum_j k^j \frac{q_t^j(z^t)}{q_{t-1}^j(z^{t-1})} \right\} + \theta_{t+1}^i(z^t) \cdot q_t(z^t) = \quad (6.2)$$

$$e^i(z_t) + \theta_t^i(z^{t-1}) \cdot (q_t(z^t) + d(z_t)) - \phi_{t+1}^i(z^t) p_t(z^t),$$

$$-k^j \phi_{t+1}^i(z^t) \leq q_t^j(z^t) \theta_{t+1}^{ij}(z^t), \text{ for } j = 1..J, \quad (6.3)$$

$$0 \leq \theta_{t+1}^i(z^t), \text{ all } z^t, \theta_0^i \text{ given.} \quad (6.4)$$

Note that agents cannot freely transfer wealth across states, as portfolio holdings in shares must be non-negative, and there are collateral constraints on bond short-sales. The minimum in expression (6.2) above reflects it is optimal to default on previous short-sales whenever the promised payment is larger than the cost of losing the collateral.

Definition 6.1 A sequential competitive equilibrium (SCE) for this economy is a collection of sequences $(c_t(z^t), \theta_{t+1}(z^t), \phi_{t+1}(z^t), p_t(z^t), q_t(z^t))_{t \geq 0}$ such that (i) for each i the sequence $(c_t^i(z^t), \theta_{t+1}^i(z^t), \phi_{t+1}^i(z^t))_{t \geq 0}$ maximizes the objective (6.1) subject to (6.2)-(6.4), and (ii) markets clear:

$$\sum_i^I c_t^i(z^t) = \sum_j^J d^j(z_t) + \sum_i^I e_t^i(z^t), \quad (6.5)$$

$$\sum_i^I \theta_{t+1}^{ji}(z^t) = 1, \text{ for } j = 1, \dots, J, \quad (6.6)$$

$$\sum_i^I \phi_{t+1}^i(z^t) = 0, \text{ at all } z^t. \quad (6.7)$$

The methods of Hernandez and Santos (1996) and Levine and Zame (1996) can easily be extended to show the existence of a sequential competitive equilibrium for versions of this model with no bonds nor collateral constraints.

It is straightforward to cast this type of economies into our general framework. To economize on space, we will skip some details in what follows. One important difference between this model and our previous examples is that we include a vector of one-period-behind security prices into the state space. In particular, notice that, due to collateral constraints, the optimal default strategy on short-sales requires comparing the cost of paying the associated promise against that of acquiring shares today, selling short, and then defaulting on the short-sale. Hence, the state space for the recursive specification of the model includes the exogenous shocks, the space of possible values for share prices, Q , and the distribution of share and bond holdings, $\Theta = \left\{ \theta \in \mathbf{R}_+^{JI} : \sum_{i=1}^I \theta^{ji} = 1 \text{ for all } j \right\}$ and $\Delta = \left\{ \phi \in \mathbf{R}_+^I : \sum_{i=1}^I \phi^i = 0 \right\}$, respectively. The equilibrium shadow value correspondence $V^* : Q \times \Theta \times \Delta \times \mathbf{Z} \rightarrow \mathbf{R}_+^{JI}$ is therefore defined as

$$V^*(q_-, \theta_0, \phi_0, z_0) = \left\{ \left(\dots, \left(q_0^j(z_0) + d^j(z_0) \right) U_1^i(c_0^i(z_0)), \dots \right) : (c_t, \theta_{t+1}, \phi_{t+1}, q_t, p_t, \lambda_t, \gamma_t)_{t \geq 0} \text{ is a SCE} \right\}.$$

Observe that, for every $(q_-, \theta_0, \phi_0, z_0)$, the set $V^*(q_-, \theta_0, \phi_0, z_0)$ contains all equilibrium JI -vectors $m_0 = (\dots, m_0^{ji}, \dots)$ of shadow values of investing in each asset j for every agent i . For a correspondence V , we apply operator $B : V \mapsto B(V)$ defined as follows: For each $(q_-, \theta, \phi, z) \in Q \times \Theta \times \Delta \times \mathbf{Z}$, the set $B(V)(q_-, \theta, \phi, z)$ contains all values $m = (\dots, m^{ji}, \dots)$ for all j and i such that, for each $m_+ = (\dots, m_+^{ji}(z_+), \dots) \in V(q, \theta_+, \phi_+, z_+)$, and for each $z_+ \in \mathbf{Z}$, there is some vector

$(c, \theta_+, \phi_+, q, q_+, p, \lambda, \gamma)$ satisfying the optimality conditions and budget constraints of all agents, as well as all aggregate feasibility and market clearing conditions.

For this model to be solved with our algorithm, all equilibrium variables must lie in a compact set. The main issue in this class of economies consists of finding bounds for asset prices, and for the shadow values of investment. As shown by Rincon-Zapatero and Santos (2005), in a version of the model without collateral constraints, the shadow values of investment can be computed as derivatives of auxiliary individual value functions that take prices as given. Bounds on those derivatives are then easy to establish. Similarly, additional regularity conditions allow Kubler and Schmedders (2003) to establish compactness, and existence of equilibrium in an economy similar to ours. Building on the previous literature we derive the following result.

Proposition 6.2 *Correspondence V^* is nonempty, compact-valued, and upper semicontinuous.*

We now illustrate an application of our algorithm for a model with two agents and two assets.

6.1 Numerical Example

There are two infinitely lived agents, and a single perishable consumption good. At each period, each agent i receives a random endowment of the consumption good. This endowment can take three possible values $e_g^i > e_m^i > e_b^i > 0$. Every period there are spot markets for the consumption good, for the shares of ownership of a Lucas tree, and for one period bonds. The Lucas tree generates a random sequence of dividends in terms of the consumption good, which can take two possible values $d_h > d_l > 0$. A Markov process drives the sequence of shocks $\{(e_t^1, e_t^2, d_t)\}_{t=0}^\infty$. The total number of shares of the Lucas tree is normalized to one, and the fraction owned by each agent at node z^t is denoted by $\theta^i(z^t)$. Bond holdings of agent i are denoted by $\phi^i(z^t)$.

The model specification we have chosen, as well as the functional forms and parameterization we will employ later in this section, are taken from Kubler and Schmedders (2003). These authors prove the existence of recursive equilibria when the state space is enlarged to include all exogenous and endogenous variables. However, in their paper they make the important that employing wealth,

$$\omega = \frac{\theta q + \phi \min \left\{ 1, k \frac{q_+}{q} \right\}}{q},$$

as auxiliary variable reduces the set of predetermined variables to $y = \left(\omega, d_t, \{e_t^h\}_{h=1}^2 \right)$. Further, the budget constraints also simplify to

$$c_t^1 = e_t^1 + \omega_t q_t + \theta_t (d_t - q_t) - \phi_t p_t \quad (6.8)$$

$$c_t^2 = e_t^2 + (1 - \omega_t) q_t + (1 - \theta_t) (d_t - q_t) + \phi_t p_t \quad (6.9)$$

$$0 \leq \theta_t \leq 1.$$

With this simplification it is no longer necessary to keep track of all previous period, or one period ahead, prices. To make the comparison between this alternative algorithm and ours straightforward we also adopt the above strategy although it is not required by our method.

Implementation of our algorithm

The shadow value of investment here are defined as

$$m_t^i \equiv q_t U_1^i(c_t^i). \quad (6.10)$$

From the above definition, and the individual constraints (6.8-6.9) we can solve for θ_t and q_t as functions of $m_t^1, m_t^2, y_t, p_t, \phi_t$. Hence, given a correspondence V , and any given pair of shadow values $(m_t^1, m_t^2) \in V(y)$, we have that (m_t^1, m_t^2) will also belong to BV if we can find bond holdings ϕ_t and prices (p_t, q_{t+1}) , a wealth level, ω_{t+1} , and continuation values for the shadow investment values, $(m_{y_{t+1}}^1, m_{y_{t+1}}^2) \in V(y_{t+1}, \omega_{t+1})$, for all successors of y_t, y_{t+1} , that satisfy all individual's constraints as well as the intertemporal optimality conditions

$$(d_t - q_t) U_1^i(c_t^i) + \beta E_t m_{y_{t+1}}^i + q_t \lambda_{c,t}^i + \lambda_{ss}^i = 0 \quad (6.11)$$

$$-p_t U_1^i(c_t^i) + \beta E_t \left[\frac{k}{q_t} m_{y_{t+1}}^i | \Omega_A \right] + \beta E_t \left[\frac{m_{y_{t+1}}^i}{q_{t+1}} | \Omega_B \right] + k \lambda_{c,t}^i = 0 \quad (6.12)$$

where $\Omega_A = \left\{ (b_t, q_t, q_{t+1}) : \min \left\{ b_t, k \frac{q_{t+1}}{q_t} \right\} = k \frac{q_{t+1}}{q_t} \right\}$ and $\Omega_B = \left\{ (b_t, q_t, q_{t+1}) : \min \left\{ b_t, k \frac{q_{t+1}}{q_t} \right\} = b_t \right\}$.

Finally, the above variables have to be consistent with the definition of wealth so that the following condition must also be met

$$\omega_{t+1} = \frac{\theta_t q_{t+1} + \phi_t \min \left\{ b_t, k \frac{q_{t+1}}{q_t} \right\}}{q_{t+1}}. \quad (6.13)$$

Comparison with Alternative Solution Methods

Kubler and Schmedders (2003) prove the existence of recursive equilibrium over an enlarged state space that includes all exogenous and endogenous variables, and wealth. Recursive equilibrium is constructed from a correspondence that maps the enlarged state space into the set of all endogenous variables. As we have seen in our previous examples, the computational cost of approximating a set operator grows exponentially in the dimension of the domain and range of the operator. Hence, even in this simple example, computing equilibrium based on such method seems to be a formidable task. Indeed, Kubler and Schmedders (2003) offer as alternative a computational algorithm that iterates over functions mapping from the enlarged state space into the set of all endogenous variables. Unfortunately, and as expected in light of the numerical examples of other sections, there is no guarantee of convergence for such algorithm. Further, a method based on iteration of functions cannot (by construction) find more than one equilibrium at a time. In contrast with this, our proposed algorithm constructs recursive equilibria from an operator that maps from an enlarged state space into the space of shadow multipliers of investment, which is a lower dimensional object. This reduction in the dimensionality of the domain of our operator makes it computable.

To illustrate the performance of our algorithm, assume both agents have identical Bernoulli utilities $u = \frac{c^{1-\sigma}}{1-\sigma}$, with a common coefficient of risk aversion of $\sigma = 2$. Agents also have identical impatience parameters $\beta_1 = \beta_2 = 0.95$. There are four possible values for the aggregate endowment, $\bar{e} \in (9.9, 10.5, 9.9, 10.5)$, dividends are $d = 0.3 \cdot \bar{e}$, while the supports for individual endowments are

$$\begin{aligned} e^1 &\in (1.386, 2.205, 5.544, 5.145), \\ e^2 &= 0.7\bar{e} - e^1. \end{aligned}$$

Finally, the transition matrix driving individual shocks is given by

$$\Pi(z'|z) = \begin{bmatrix} 0.4 & 0.4 & 0.1 & 0.1 \\ 0.4 & 0.4 & 0.1 & 0.1 \\ 0.1 & 0.1 & 0.4 & 0.4 \\ 0.1 & 0.1 & 0.4 & 0.4 \end{bmatrix}.$$

We simulate the economy using the decision rules obtained from our method, as well as those from

an algorithm where a continuous Markov equilibrium is assumed to exist. The resulting simulated statistics are summarized in Table 6.1 below. There, EE_i denotes the Euler equation residual for agent i associated to each numerical approximation, and c^i denotes consumption of agent i .⁷

	$mean\ c_1$ ($\sigma(c_1)$)	$mean\ c_2$ ($\sigma(c_2)$)	$mean(EE_1)$	$mean(EE_2)$
Continuous Markov equilibrium	4.96 (0.78)	5.26 (0.78)	5.05×10^{-6}	3.27×10^{-8}
Our algorithm	4.96 (0.78)	5.26 (0.78)	2.41×10^{-5}	9.01×10^{-6}

Table 6.1: Simulated moments from alternative solution methods.

Hence, for the parameterization we have chosen, both methods deliver identical simulated moments for asset holdings. More important, for this example our algorithm converged to a function that coincides with the one obtained from a method that *assumes* such a function exists. It is important to stress, however, that it is because of our theory that we can use the above numerical result to derive a key equilibrium property of this model. In particular, a straightforward Corollary of Theorem 3.3 is that convergence to a function can only occur when the true equilibrium correspondence is indeed a function. As a result, equilibrium in this model is unique. In contrast, algorithms where the existence of a continuous function is assumed do not offer any guidance on whether other equilibria may exist.

⁷Euler equation residuals are commonly employed to estimate the accuracy of a given numerical approximation. The basic idea is the fact that along the true solution of the model the Euler equation must equal zero. Hence, smaller Euler equations should reflect more accurate solutions. Santos (2000) studies approximation errors based on Euler equations for optimal models.

4 Concluding Remarks

This paper provides a theoretical framework for the computation and simulation of dynamic competitive-markets economies in which the welfare theorems may fail to hold because of market frictions or the existence of an infinite number of generations. We have applied these methods to various macro and finance models with heterogeneous agents, taxes, sequentially incomplete markets, borrowing limits, and short-sales and collateral requirements.

For optimal economies, sequential competitive equilibria are generated by a continuous policy function which is the fixed-point solution of a contractive operator. Continuity of the policy function allows for various methods of approximation and functional interpolation, and it is essential to validate laws of large numbers for the simulated paths. Differentiability and contractive properties are useful for the derivation of error bounds that can guide the computation process. But for OLG models and economies with distortions several papers [e.g. Krebs (2004), Kubler and Polemarchakis (2004), Kubler and Schmedders (2002), and Santos (2002)] have shown that a continuous Markov equilibrium may not exist. We establish a general result on the existence of a Markovian equilibrium solution in a suitably expanded space of state variables. We construct a numerical algorithm that has desirable approximation properties and guarantees convergence of the moments computed from simulated paths.

There are three main features of our algorithm that should be of interest for quantitative work in this area. First, the existence of a Markovian competitive equilibrium is obtained in an enlarged space of state variables. Our choice of the shadow utility values of assets returns is dictated by computational considerations. This is a minimal addition to the state space to restore existence of a Markovian equilibrium and with the property that the extra added variables enter linearly into the Euler equation. Second, Markovian equilibrium solutions may not be continuous. Hence, the algorithm iterates in a space of candidate equilibrium sets – rather than in a space of functions. We still establish some desirable approximation properties of our operator. And third, we provide a reliable method for model simulation. We resort to a further discretization of the equilibrium law of motion so that it becomes a Markov chain. The usual simulation over a continuum of values cannot be justified on theoretical grounds: The simulated moments may fail to converge to the set of moments of the invariant distributions of the model. Other ways to restore laws of large numbers

for the simulated paths of these economies would be by imposing monotonicity assumptions on the equilibrium dynamics [Santos (2006)] or by expanding artificially the noise process [e.g., Blume (1979)]. These latter approaches seem to be of more limited economic interest.

Of course, our methods have to face some computational challenges. Iteration over sets is computationally much more costly than iteration over functions. Therefore, the expansion of the state space along with iteration over sets should certainly be manifested into an additional computational burden. Besides, our general convergence results lack error bounds. Note that our models cannot be restated as optimization programs, and miss some common concavity, differentiability and contractive properties. In terms of numerical implementation the innovative techniques for error estimation proposed by Judd, Yeltkin, and Conklin (2003) seem to be of limited application for our economies. These authors use outer and inner approximations over convex sets. It is not clear to us that an outer approximation over convex sets will converge to the convex hull of the equilibrium correspondence. Moreover, inner convex approximations may be hard to find. Still, these techniques may work well in some applications.

There are several directions in which our analysis can be extended. For example, in the preceding sections we considered exogenous short-sales constraints and exogenous borrowing limits. We could incorporate borrowing constraints that depend on future income [e.g., Miao and Santos (2005)]. These general borrowing schemes arise in financial models and in the modelling of the public sector so as to allow for various types of fiscal policy rules. In most quantitative studies of recursive equilibrium with fiscal policy, the government must balance the budget in each state of the world. This is a rather strong assumption. Another extension is to the area of policy games. As our algorithm includes all the shadow values of investment, it can deal with heterogeneity and market frictions. For example, we can generalize the model of Phelan and Stacchetti (2001) to include heterogeneous agents and various types of financial frictions.

A Appendix: Proofs

In this Appendix we prove all results formally stated in Section 3 and Proposition 4. All remaining results follow from similar arguments.

Proof of Proposition 1 (Sketch): As in the original work of Bewley (1972), the existence of a SCE can be established by approximating the infinite-horizon economy by a sequence of finite economies. This is the strategy followed by Jones and Manuelli (1999), but their proof is incomplete and does not apply to sequential competitive economies. As is usual in this approximation argument the hardest part of the proof is to provide upper bounds for equilibrium allocations and prices over all the finite-horizon economies. We nevertheless skip this part since these bounds follow from the proof of Proposition 2 below.

Hence, following Jones and Manuelli (1999), we consider the following steps for the proof of a SCE: (i) Existence of an equilibrium for a finite horizon economy. This result is covered by the general proofs of existence of competitive equilibria for economies with taxes and externalities of Arrow and Hahn (1971), Mantel (1975), and Shafer and Sonnenschein (1976). (ii) Uniform bounds for equilibrium allocations and prices of finite-horizon economies. As already pointed out, these bounds can be established by the method of proof of Proposition 2. (iii) Existence of SEC as a limit point of finite equilibria. The preceding steps (i) and (ii) guarantee that there is a collection of vectors $(c_t(z^t), k_{t+1}(z^t), K_t(z^t), L_t(z^t), \bar{K}_t(z^t), w_t(z^t), r_t(z^t))$ that can be obtained as a limit of equilibria of finite economies. It is obvious that for such limiting solution the market clearing conditions must be satisfied at each z^t , and that one period-profits must be maximized. Moreover, for each agent i the limiting allocation $(c_t^i(z^t), k_{t+1}^i(z^t))$ must satisfy the sequence of budget constraints (2), and it is optimal since the discounted objective (1) is continuous in the product topology over the set of feasible consumption plans $(c_t^i(z^t))_{t \geq 0}$ which are preferred to the endowment allocation $(e_t^i(z_t))_{t \geq 0}$. This is because feasible consumption plans $(c_t^i(z^t))_{t \geq 0}$ are bounded above (see below) and the endowment process $(e_t^i(z_t))_{t \geq 0}$ is bounded below by a positive quantity.

Proof of Proposition 2: We first show that there are positive constants K^{max} and K^{min} such that for every equilibrium sequence of physical capital vectors $(k_{t+1}^i(z^t))_{t \geq 0}$ if $K^{max} \geq \sum_{i=1}^I k_0^i(z^0) \geq K^{min}$ then $K^{max} \geq \sum_{i=1}^I k_t^i(z^{t+1}) \geq K^{min}$ for all z^t . The existence of K^{max} follows directly

from Assumptions 2 and 3. In particular, A is bounded by Assumption 2, and by Assumption 3 the marginal productivity of capital converges to zero as K goes to ∞ for every fixed $L > 0$. Also, it is obvious that $K^{min} \geq 0$. We now claim that if $\lim_{K \rightarrow 0} D_1 F(K, L) = \infty$ for some given positive L , then $K^{min} > 0$. For if not, there is a sequence of equilibrium capitals $(k_{t+1}^i(z^t))_{t \geq 0}$ such that $\sum_{i=1}^I k_t^i(z^{t+1})$ is arbitrarily close to 0 for some z^{t+1} . Under the system of budget constraints (2), it follows that there is an arbitrarily small number $\varepsilon \geq 0$ such that $c_t^i(z^t) \geq e_t^i(z^t) - \varepsilon$ for every i . Therefore, modulo an arbitrarily small number the derivative $D_1 u(c_t^i(z^t), z_t)$ is bounded by $D_1 u(e_t^i(z^t), z_t)$, and $D_1 F(K_t, L_t)$ is arbitrarily large. These latter two conditions together are not compatible with utility maximization, since the existence of K^{max} implies that future consumption $c_t^i(z^r|z^t)$ for $r > t$ is uniformly bounded. Consequently, if $\lim_{K \rightarrow 0} D_1 F(K, L) = \infty$ for some positive L , then $K^{min} > 0$.

Since L takes on a finite number of positive values, our bounds K^{max} and K^{min} imply that there are constants r^{max} and w^{max} such that for every equilibrium sequence of factor prices $(r_t^i(z^t), w_t^i(z^t))_{t \geq 0}$ we have $0 \leq r_t(z^t) \leq r^{max}$ and $0 \leq w_t(z^t) \leq w^{max}$ for all z^t . Hence, the value function $J^i(k_0^i, z_0, \mathbf{r}_0(\mathbf{z}_0), \mathbf{w}_0(\mathbf{z}_0))$ is well defined, and as already pointed out the derivative $D_1 J^i(\cdot, z_0, \mathbf{r}_0(\mathbf{z}_0), \mathbf{w}_0(\mathbf{z}_0))$ is continuous in $(k_0^i, \mathbf{r}_0(\mathbf{z}_0), \mathbf{w}_0(\mathbf{z}_0))$. Moreover, by a simple notational change it follows from (2) that function J^i can be rewritten as $J^i(a_0^i(z_0), z_0, \mathbf{r}_0(\mathbf{z}_0), \mathbf{w}_0(\mathbf{z}_0))$, where $a_0^i = e_0^i(z_0) + r_0 k_0^i$. Then we can conclude that $0 \leq D_1 J^i(k_0^i, z_0, \mathbf{r}_0(\mathbf{z}_0), \mathbf{w}_0(\mathbf{z}_0)) \leq \gamma$, since $e_0^i(z_0)$ is bounded below by a positive number, and as shown above all feasible vectors $(k_0^i, \mathbf{r}_0(\mathbf{z}_0), \mathbf{w}_0(\mathbf{z}_0))$ lie in a compact set.

Proof of Theorem 1: For the proof of Theorem 1, we shall invoke the following version of Bellman's equation.

$$J^i(k_0^i, z_0, \mathbf{r}_0(\mathbf{z}_0), \mathbf{w}_0(\mathbf{z}_0)) = \max u^i(c_0^i(z_0), z_0) + \beta E[J^i(k_1^i(z^1), z^1, \mathbf{r}_1(\mathbf{z}^1), \mathbf{w}_1(\mathbf{z}^1))] \quad (\text{BE})$$

$$\begin{aligned} \text{s. t.} \quad & k_1^i(z_0) + c_0^i(z_0) = r_0(z_0) k_0^i(z_0) + w_0(z_0) l_0^i(z_0) + e_0^i(z_0), \\ & k_1^i(z^t) \geq 0, k_0^i \text{ given.} \end{aligned}$$

We now divide the proof into three parts:

(i) $V^* \subset B(V^*)$: This part essentially follows from (BE). Since

$m_0^i = D_1 J^i(k_0^i, z_0, \mathbf{r}_0(\mathbf{z}_0), \mathbf{w}_0(\mathbf{z}_0))$ is the derivative of the value function, and (5)-(6) in the definition of operator B provide necessary conditions for utility maximization. Conditions (7)-(10) are also

satisfied in every SCE.

(ii) $B(V^*) \subset V^*$: This is the sufficiency part; again, the most difficult step of the proof follows from (BE). More specifically, since value function $J^i(k_0^i, z_0, \mathbf{r}_0(\mathbf{z}_0), \mathbf{w}_0(\mathbf{z}_0))$ is concave in k_0^i and $m_0^i = D_1 J^i(k_0^i, z_0, \mathbf{r}_0(\mathbf{z}_0), \mathbf{w}_0(\mathbf{z}_0))$, conditions (5)-(7) imply that (BE) holds. But by the well-known arguments of dynamic programming the Bellman equation (BE) implies that the original sequential optimization problem (SOP) attains a global solution. Hence, constrained utility maximization in the definition of SCE is satisfied. Conditions (8)-(9) imply profit maximization, and condition (10) implies market clearing.

(iii) V^* is the largest fixed point of B . First note that by the same arguments as in the proof of Proposition 2 we can show that every fixed-point solution $\hat{W} = B(\hat{W})$ is a compact correspondence. Now, as in Section 3 we may consider a measurable selection $f(k, z, m)$ for every (k, z, m) in the graph of $\hat{W}$. Let $(k_{t+1}(z^t), z^{t+1}, m_{t+1}(z^{t+1}), c_t(z^t), r_t(z^t), w_t(z^t), \lambda_t(z^t), \delta_t(z^t)) = f(k_t(z^{t-1}), z_t, m_t(z^t))$ for each $z^{t+1}|z^t$. Then we claim that $(c_t(z^t), k_{t+1}(z^t), r_t(z^t), w_t(z^t))_{t \geq 0}$ is a SCE. Indeed, operator B is compact, and hence the sequence of factor prices $(r_t(z^t), w_t(z^t))_{t \geq 0}$ is bounded. Moreover, individual consumptions and capital holdings $(c_t^i(z^t), k_{t+1}^i(z^t))_{t \geq 0}$ are bounded, and the sequence of shadow values of investment $(m_{t+1}(z^{t+1}))_{t \geq 0}$ is bounded. Along these sequences, the Euler equations and the budget constraints are satisfied for every agent i , and so the individual (SOP) attains a global maximum [e.g., see Rincon-Zapatero and Santos (2005)]. Also, Conditions (8)-(9) imply profit maximization, and condition (10) implies market clearing. It follows that every selection f generates a SCE. Therefore, $\hat{W} = V^*$.

Proof of Theorem 2: Let $\hat{W} = \bigcap_n W_n$. Hence, $\hat{W} = \{(k, z, m) : m \in B^n(W)(k, z) \text{ for every } n\}$. Consequently, $B(\hat{W}) = \{(k, z, m) : \text{There are some continuation values } (k_+, z_+, m_+) \text{ such that } m_+ \in B^n(W)(k_+, z_+) \text{ for every } n \geq 1 \text{ and all } z_+ \in \mathbf{Z}\}$. It follows that $B(\hat{W}) \subset \{(k, z, m) : m \in B^{n+1}(W)(k, z) \text{ for every } 1\}$. Therefore, $B(\hat{W}) \subset \hat{W}$ as the sequence $(W_n)_{n \geq 0}$ is decreasingly monotone. We next show that $\hat{W} \subset B(\hat{W})$.

Consider any (k, z, m) that belongs to the graph of $\hat{W}$. Hence, $m \in B^n(W)(k, z)$ for every n . Let Ψ_n be the set of continuation values (k_+, z_+, m_+) of (k, z, m) such that $m_+ \in B^{n-1}(W)(k_+, z_+)$. This set is non-empty and compact, and so $\Psi = \bigcap_n \Psi_n$ is not empty. Consequently, for every (k, z, m) that belongs to the graph of $\hat{W}$, there exists a non-empty set of continuation values (k_+, z_+, m_+)

that belong to the the graph of $\hat{W}$. This proves that (k, z, m) belongs to the graph of $B(\hat{W})$, and so $\hat{W} \subset B(\hat{W})$.

We thus obtain that $\hat{W} = B(\hat{W})$. Finally, by the monotonicity of B the assumed condition $V^* \subset W$ implies $V^* \subset B^n(W)$ for every n . Hence, $V^* \subset \hat{W}$. Moreover, $\hat{W} \subset V^*$ since V^* is the largest fixed point of B . We thus obtain $V^* = \hat{W}$.

Proof of Theorem 3: This follows trivially from part (iii) of the proof of Theorem 1.

Proof of Proposition 4: This result is proved along the lines of Levine and Zame (1996) and Magill and Quinzii (1994). For every agent i the objective in (1) satisfies Assumption (A.2) of Santos and Woodford (1997). Hence, for every optimal consumption-portfolio plan $\{c_t^i(z^t), \theta_{t+1}^i(z^t)\}_{t \geq 0}$ the sequence of portfolio values $\{\theta_{t+1}^i(z^t)q_t(z^t)\}_{t \geq 0}$ is bounded above, and a uniform bound can be found that applies to all equilibrium sequences of asset prices $\{q_t(z^t)\}_{t \geq 0}$. Since there is a finite number I of agents and one unit of the asset, equilibrium condition (18) implies the existence of the lower bound $-M$.

The second part of the proposition is proved by contradiction. Note that the dividend process $(d_t(z_t))_{t \geq 0}$ is bounded below by a positive number. Also, by the argument above all equilibrium sequences of asset prices $\{q_t(z^t)\}_{t \geq 0}$ are bounded above. By a similar argument, it is easy to see that $\{q_t(z^t)\}_{t \geq 0}$ is bounded below by a positive number. Hence, it follows from (19) that any initial small debt $\theta_0 < 0$ that is rolled over at every period, it will grow to $-\infty$. Indeed, at every z^t the debt θ_{t+1} must be incremented to pay for the dividend $d(z^t)$, and these negative increments can be bounded uniformly. Since every sequence of asset prices $\{q_t(z^t)\}_{t \geq 0}$ is bounded below, the value $\{q_t(z^t)\theta_{t+1}(z^t)\}_{t \geq 0}$ must also converge to $-\infty$. Now, by the definition of $\pi_{e^i}(z^t)$, if (22) is violated at some date-event z^t , it means that the debt cannot be repaid in finite time, and hence it must grow without bound along some history of date-events $\{z^r | z^t\}_{t \geq 0}$. The proposition is thus established.

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