Preliminary and Incomplete

Accounting for Low-Frequency Variation in Tobin's q

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Abstract

The distinction between firm value and the value of a firm's tangible assets — Tobin's q — exhibits high variability and persistence. Why? We ask to what extent *growth options* are important. We append a model of industry equilibrium to the growth option model of what a firm is. Growth options are positive net-present-value investment opportunities which certain firms have access to upon the arrival of a new technology. The model of industry equilibrium puts restrictions on how large the value of these growth options can be. The restrictions arise from firm entry and exit and from a growth-option exercise game played by surviving firms. These restrictions make it possible to ask questions about the aggregate scale of the growth-option component of firm value and, therefore, to examine the relationship between the value of the U.S. corporate sector and the value of its physical capital.

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1 Introduction

The difference between the value of U.S. firms and the value of their physical capital — Tobin's q — has grown increasingly large over time.

Figure 1[1]
Tobin's q

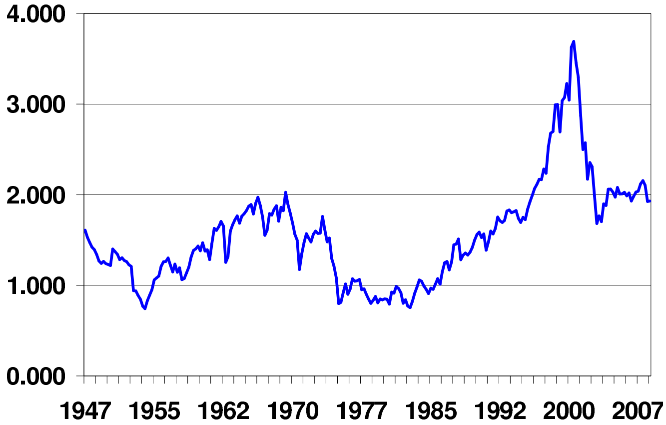


Figure 1 represents but one among many alternative measures of what Tobin meant by q. The good news is that they all yield roughly the same picture. They all leave us with the same question. Why does the value of a firm vary so much relative to the value of its tangible assets?

One answer, of course, focuses on intangible assets. According to this view — *e.g.*, Hall (2001), McGrattan and Prescott (2005) and many others — the variation in Figure 1 is largely an artifact of poor measurement of the capital stock in the denominator of q. Or, equivalently, the variation in Figure 1 is mostly a reflection of the accumulation and decumulation (or depreciation) of intangible capital. Other answers have ranged from secular changes in taxes, to adjustment costs of capital, to bubbles. We don't deny that each of these explanations have likely played some role. Nevertheless we focus on an alternative explanation, one that is quite different in nature.

According to the growth option view of what a firm is, one should look at Figure 1 and think about *rents*. This view attributes an important part of firm

[1]Quarterly data, 1952Q1-2008Q2. Replacement cost of capital is computed by taking fixed investment of non-farm, non-financial businesses — Table F102 from the Flow of Funds — and accumulating-up the capital stock assuming an annual depreciation rate of 10%. Initial conditions and some scaling assumptions taken from Robert Hall. Market value for non-farm, non-financial U.S. corporations is computed by adding the market value of equity to the market value of debt. Data are from the Flow of Funds and incorporate an adjustment for difference between the book value of debt and the market value of debt developed in Hall (2001). Thanks to Robert Hall, Hanno Lustig and Stijn Van Nieuwerburgh for providing their data and code for the latter.

value to having made positive net-present-value (NPV) investments in past, and to the ability to undertake them in the future. Indeed, this is the conventional view of how corporations create value for their owners. Froot, Scharfstein, and Stein (1994) for example, make the case for financial risk management by stating that the "overarching goal" should be to "ensure that a company has the cash available to make value-enhancing investments." Or, stronger yet, they state that

> *The key insight of Franco Modigliani and Merton Miller, each of whom won a Nobel Prize for his work in this area, is that value is created on the left-hand side of the balance sheet when companies make good investments — in, say, plant and equipment, R&D, or market share — that ultimately increase operating cash flows.*

This view of the firm's investment process — what we'll call the *growth option model* — stands in contrast to many macroeconomic models of the firm, in which investment is a zero-value transaction. It suggests that when one looks at Figure 1, one should consider the 60s and the 90s as periods of relatively abundant, unrealized investment opportunities and the 70s and 00s as periods of scarcity and/or periods in which the opportunities had been realized (in some cases, obviously, with disappointing results). Our goal is to ask how plausible this view is.

The growth option model of the firm has seen much recent success in accounting for equity returns, especially in the cross-sectional distribution. Berk, Green, and Naik (1999), Carlson, Fisher, and Giammarino (2004) Gomes, Kogan, and Zhang (2003), Obreja (2007) and Panageas and Yu (2006) have shown that firm-specific characteristics that empirical studies have identified as predictors of expected equity returns arise naturally as a consequence of the growth-option investment process. These papers serve as a useful reference point for our analysis. We ask whether the growth option model can account for the time-series behavior of firm *value* as well as it can the cross-sectional behavior of firm equity returns.

A second link between our paper and the above papers, a somewhat more challenging link, is as follows. These papers have many differences — partial versus general equilibrium for instance — but they share one important commonality. Their models are silent on where the positive NPV projects come from and why firms are lucky enough to receive them. They cannot address questions such as "why don't the firms invest in two projects instead of just one?" Firms would, if they could. For our question — what explains the difference between the *size* of the corporate capital stock and its value? —- these issues cannot be avoided. Therefore, our point of departure is to append a model of industry equilibrium onto the growth options model. Doing so imparts discipline onto how positive the positive NPV projects can be and allows us to ask how much of Figure 1 can reasonably be thought of as being related to rents. Our model is a simplified version of Panageas and Yu (2006) appended with a model of industry equilibrium similar to Jovanovic and MacDonald (1994).

2 Overview

A useful reference point for our model is what has become a benchmark model of the firm as a portfolio of growth options: the Berk, Green, and Naik (1999) (BGN) model. A firm in the BGN model is a portfolio of existing and future *projects*. At each date j the firm gets one take-it-or-leave-it, irreversible opportunity to acquire an additional project. The project costs I_j and at each subsequent date either pays $C_j(t)$, $t > j$, or, with probability π, it dies. At some date t the value of a project undertaken at date $j \leq t$ is

$$V_j(t) = E_t \sum_{s=t+1}^{\infty} m_{t,s}\, C_j(s)\, \chi_j(s) \ ,$$

where $m_{t,s}$ is the date t pricing kernel for valuing cash flows at date $s > t$ and $\chi_j(s)$ is a 0–1 variable indicating whether or not the project has survived to date $s \geq j$. Note that $\chi_j(s) = 0$ if either the project has died prior to (or at) date s or if the project was never initiated to begin with at date j. The value of "assets in place" for the firm is

$$V_a(t) = \sum_{j=0}^{t} V_j(t)\, \chi_j(t) \ .$$

The investment rule is to invest if the NPV is positive, where NPV is

$$npv_j = V_j(j) - I_j \ .$$

The value of the firm's growth options is

$$V_g(t) = E_t \sum_{s=t+1}^{\infty} m_{t,s} \max\left(npv_s\,,\, 0\right) \ .$$

The value of the firm is

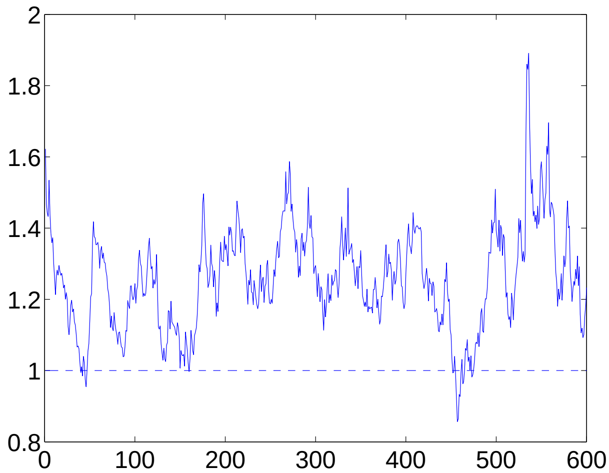
$$V(t) = V_a(t) + V_g(t) \ .$$

Tobin's q, with book-value in the denominator, is

$$q(t) = V(t) / \sum_{j=0}^{t} I_j\, \chi_j(t) \ .$$

Figure 2 plots Tobin's q for a simulation of the BGN model that corresponds to their benchmark calibration. This calibration primarily targets interest-rate behavior and the mean risk premium and standard deviation of the return on a portfolio of 50 firms. Their model is quite successful, given this calibration, in accounting for a variety of cross-sectional facts. Therefore, the point of Figure 2 boils down to whether or not BGN's quantitatively-successful model of return behavior has realistic implications for firm value (relative to physical asset value).

Figure 2[1]
Tobin's q, BGN Model



The mean, standard deviation and autocorrelation of q are 1.2563, 0.1426 and 0.9329, respectively. Each is somewhat lower than what we see in the data (Figure 1), where the corresponding estimates are 1.55, 0.55 and 0.97, respectively. Why?

Figure 3 plots the components of q associated with Figure 2. What's clear is the following.

1. The growth option component of firm value contributes relatively little to total value. It averages about 1/7th the magnitude of the value of assets-in-place. This contributes to a relatively small average *level* of q in Figure 2.

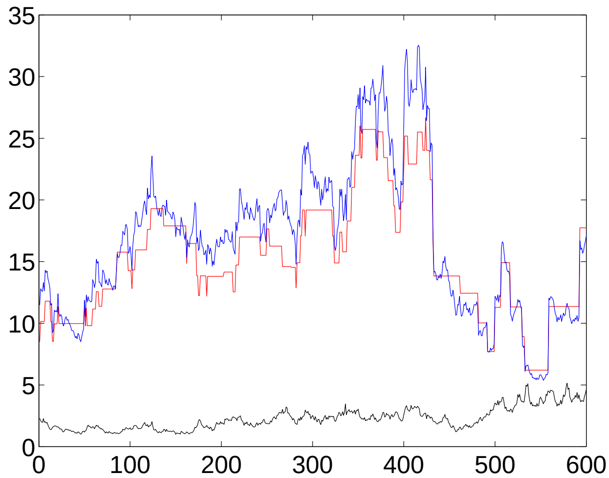
2. The birth and death of projects drives much of the dynamics in the BGN model. Consider time periods over which the number of existing projects is fixed. These are periods over which book value is constant. Most of the *variation* in q arises because of variation in the value of assets-in-place relative to their book value. Variation in the growth option piece, $V_g(t)$, is small. Small variation in $V_t(t)$ is due, in large part, to *i.i.d.* cash flows, $C_j(t)$. That

[1]The calibration corresponds to monthly frequency data, so the simulation is of length 50 years. The first 200 observations are omitted, so the data correspond to a firm with a number of projects varying around the long-term mean.

is, to draw an analogy, stock options derive value from volatility in stock *prices*, not (necessarily) volatility in *cash flow*. The lack of variability in the conditional distribution of the BGN cash flows leads to low variability in $V_g(t)$ (and a low mean). In a way, the BGN model of firm *value* is driven more by rents than it is options. This will also be a feature of our model.

3. Consider the entire time period. Most of the variation in q comes from the process of birth and death of projects, not from variation in the value of the firm's growth options. When a project dies (is born), q tends to go up (go down) as the part of firm value deriving from assets-in-place diminishes (increases). This will also be a feature of our model.

Figure 3[2]
Components of Tobin's q: BGN Model



An alternative calibration and/or specification of the BGN model can surely do a better job accounting for q. Likely candidates are dynamics in the distribution of $C_j(t)$ or a looser relationship between book value, I_j, and future cash flow, $C_j(t)$ (BGN make them proportional). But this is not our objective. We find the above exercise useful in part because it serves as a useful reference point for our model, and in part because it highlights several qualitative properties that any dynamic model of rents accruing to investment is likely to share.

Our objective is to put some structure on the rents, the npv_j variable. The above calculations make it clear that this is important for understanding how rents affect q. With sufficient freedom to choose arbitrary processes for $C_j(t)$ and I_j, it is likely that we can generate any pattern in q that we like. We need to go beyond exogenous rents.

[2]The lower line is the value of the growth options. The jagged upper line is the value of assets-in-place. The box-like remaining line is the book value of assets.

What we'll do is as follows. The $npv_j > 0$ rents will arise because of *luck*. Some firms will be better positioned than others to take advantage of a large-scale technological change. These incumbent firms will be fortunate but not so fortunate as to earn the rents forever. Over time, as new firms enter and other incumbent firms improve their capabilities and invest, output will increase, the product-market price will fall, and the rents will be dissipated. In relation to the above discussion, we develop a model of product-market equilibrium to endogenize $C_j(t)$ so that it depends on the price of output, p_t. The evolution of p_t will govern the evolution of $V_j(t)$ and, we hope, allow us to say something about the quantitative importance of the growth options.

3 Model

In the BGN model, the firm gets one, take-it-or-leave-it real option each period. There is no scale decision. The cash flows from the real option are exogenous. For our question it is important to depart from each of these assumptions. We do so as follows.

First, the firm's investment time horizon will be richer. Firms in our model will not receive one take-it-or-leave-it investment option per period, but instead one option per 'technological epoch' (described below). The firm will then decide if and when to exercise the option within the epoch. Second, cash flows — and therefore rents — will be endogenized through competition to sell output to an aggregate demand curve. Third, competition will primarily take the form of an option-exercise game similar to that developed in Grenadier (2002). We begin by discussing the time horizon.

3.1 Technological Change and the Investment Time Horizon

The time horizon over which a firm makes investment decisions is of particular importance for our question. Our basic idea is that the value of the investment opportunities, $V_g(t)$ above, is limited by the dissipation of rents through product-market competition, and that this process takes *time*. We need a model to tell us *how much* time is reasonable and what are the associated implications for the magnitude of $V_g(t)$ relative to book value and $V_a(t)$. A natural time-frame in this context is that over which major technological innovations are discovered, commercialized and refined. Panageas and Yu (2006) incorporate such a time frame — a 'technological epoch' — in an BGN-flavored model of investment options. We use their setting as starting point. To it, we add the Jovanovic and MacDonald (1994) model of industry equilibrium. Their model is also set in an environment of low-frequency technological change. They describe how incumbent firms at the onset of a technological epoch earn rents during a period of innovation and refinement, how these rents are dissipated by entry and technological improvement from competitors, and how the entire structure has sharp implications for how product-market price and the distribution of firms evolves over time.

Time is discrete and indexed with t. Following Panageas and Yu (2006), time is divided into an increasing sequence of technological epochs indexed with $n \in (-\infty..., -1, 0, 1, ...\infty)$. At each date t, an aggregate shock z_t is realized and, in addition, a major technological improvement — a change from epoch n to epoch $n+1$ — occurs with probability λ. To this we graft Jovanovic and MacDonald's (1994) Schumpetarian model of innovation and refinement that induce firm entry and exit. The arrival of a new epoch represents *invention*: the discovery of a fundamentally new technology in its raw form. Invention is followed by a period of *innovation* — the primitive commercialization of the invention — followed by a period of further innovation labeled *refinement*. For example, the invention of the microprocessor was followed by many important innovations, one being the development of the PC operating system. Microsoft's DOS and Windows represent a refinement of the operating system. As Jovanovic and MacDonald (1994) effectively argue, this cycle of invention, innovation and refinement typically generates major changes in the industrial organization of the production process. Our model attempts to capture how such changes interact with a firm's value and its investment policy. Jovanovic and MacDonald (1994) consider the former but not the latter (there is no investment decision in their model).

There is also technological change *within* an epoch. The onset of a new epoch coincides with the arrival of a new invention. For ease of explanation, we shift the timeline so that time $t = 0$ marks the beginning of the epoch. For all firms in the economy the invention is associated with potentially new growth opportunities. However, to be able to exploit these opportunities firms have to devise new technologies able to put the invention to work. In the language of Jovanovic and MacDonald (1994), firms have to *innovate* to be able to exploit the new invention. Innovation is equivalent to devising a new technology which exploits the new invention. However, innovation is only the first step towards understanding how the invention can be put to practice. For this reason, we refer to the firms who innovate as low know-how (LKH) firms. The new technologies devised by the LKH firms are not public yet, so they are not subject to imitation from the firms who have not had the chance to innovate yet. Nevertheless, the production output of the LKH firms is publicly observable. We assume that firms innovate at a constant rate r_0.

We assume that the invention is rather general at first and that the new technologies of the LKH firms are ways to tap into the potential of the invention. However, after a while, the true potential of the invention will be revealed and some LKH firms will be better positioned for growth than others. We follow Jovanovic and MacDonald (1994) and refer to this time as the *refinement* of the invention. This event, denoted T, occurs with probability ρ. We refer to the LKH firms which fit better with the potential of the invention as high know-how (HKH) firms. These firms are the first to capitalize on the potential of the invention and their productivity essentially jumps at the refinement. The rest of the LKH firms adjust their technologies in an effort to catch up with the HKH firms. At any date $t \geq T$ an existing LKH firm acquires HKH ability with probability r.

After the refinement HKH firms have a first-mover advantage over LKH firms. HKH firms can make the most of their advantage by scaling up their production before LKH firms get a chance to do the same. We assume that scaling up entails both fixed and variable costs and so the option to scale up is essentially a real option. This is where the option exercise game comes into play.

The nature of the investment costs is a critical issue for us. Following Grenadier (2002), we characterize a Nash equilibrium in exercise strategies. Back (2008), however, has criticized the 'open loop' nature of Grenadier's equilibrium concept. The issue is that any single firm has the incentive to deviate from the equilibrium by preempting its competitors investment expansion. We circumvent this issue with the following 'information hold-up' assumption. We assume that at any date following the refinement, $t \geq T$, there is a certain amount of information, ν_t, regarding the HKH technology that is public and a certain amount that is known only to HKH firms. Over time, word gets out and the fraction that is public increases. 'Early exercise' of the investment option accelerates the process of giving up one's informational advantage. It also gives one an increased share of the demand curve if one's competitors do not follow suit. It is these two offsetting forces that allow us to characterize a Nash equilibrium in investment behavior that is 'closed loop:' not subject to preemption. This is formalized in Section #.

3.2 The Firm's Problem

At date t the firm produces quantity q_t using only capital k_t. The production function is

$$q_t = A_t z_t k_t \quad , \tag{1}$$

where A_t and z_t are, respectively, high and low-frequency aggregate productivity shocks,

$$\begin{aligned} A_t &= A \text{ if } t < T \text{ or LKH} && (2)\\ &= A(1+\mu) \text{ if } t \geq T \text{ and HKH} && (3)\\ \log z_{t+1} &= \bar{\theta} + \log z_t + \sigma \epsilon_{t+1} \quad , && (4) \end{aligned}$$

with $\mu > 0$ and $\epsilon_t \sim \mathrm{N}(0,1)$. Variable costs of producing are zero, but fixed costs of installing capital are not. The firm's profit maximization problem is

$$\max_{\{k_s\}} E_t \sum_{s=t+1}^{\infty} \gamma^{s-t} \pi(q_s; p_s) \ , \tag{5}$$

subject to equation (1–4) and

$$\begin{aligned} \pi(q_t; p_t) &= q_t \, p_t & (6)\\ \Phi(k; z_t, \nu_t) &= \text{capital installation costs} & (7)\\ p_t &= D(Q_t, z_t) & (8) \end{aligned}$$

Equation (6) states that per-period operating profit is simply quantity times sales price, p_t. Equation (7) states that the firms costs arise entirely from installing its capital and that these costs depend on the level of capital, the aggregate shock, and an aggregate information variable, ν_t, described below along with the function Φ. Equation (8) states that the product price derives from an inverse demand function, D, that depends on z_t and *aggregate* quantity, Q_t. Equilibrium will be imperfectly competitive, so that the firm's decision will depend on how q_t affects Q_t, and how its investment policy affects that of its competitors.

In terms of the technological process described in Section 3.1, the recursive formulation of (5) is as follows. The state variables are z_t, ν_t, $\bar{n}_t \equiv (n_t^l, n_t^h)$ and $\bar{k}_t \equiv (k_t, k_t^{-i})$, where n_t^l and n_t^h are the number of active LKH and HKH firms in the industry, respectively, k_t is an *individual* firm's level of capital and k_t^{-i} is that of each of their competitors. Define the state vector as $x_t = (z_t, \nu_t, \bar{n}_t, \bar{k}_t)$. These variables are sufficient because $\bar{k}_t$ describes the distribution of total capital and $\bar{n}_t$ describes the distribution of how productive it is (*i.e.*, firm-specific values for A_t). Given z_t, aggregate output, Q_t is then known and, therefore, so is the product-market price, $p_t = D(Q_t, z_t)$. We use the notation $p(x_t)$. The variable ν_t is required to characterize the outcome of the option exercise game, which then determines the long-run value for $\bar{k}_t$.

Denote the date t value of an LKH firm with capital k_t, prior to the refinement date T, as $U^l(k_t; x_t)$. Similarly, $V^l(k_t; x_t)$ and $V^h(k_t; x_t)$ denote the value of LKH and HKH firms after the refinement, $t \geq T$. We now show the general structure that connects these value functions, omitting terminal conditions and so on. Subsequent sections provide more explicit derivations. U^l must satisfy,

$$U^l(k_t; x_t) = A z_t k_t p(x_t) + \gamma E_t\Big(\rho V^l(k_{t+1}; x_{t+1}) + (1-\rho) U^l(k_{t+1}; x_{t+1}\Big) \tag{9}$$

where, recall, ρ is the probability that the refinement date arrives. Once this happens a fraction r of firms find that their innovations are 'aligned' with the refinement. They become HKH firms. A fraction $(1-r)$ find otherwise and remain as LKH firms. The latter's value is described by,

$$V^l(k_t;x_t) \quad = Az_tk_tp(x_t) + \gamma E_t\Big(rV^h(k_{t+1};x_{t+1}) + (1-r)V^l(k_{t+1};x_{t+1}\Big) \tag{10}$$

Equation (10) says that for $t \geq T$ an LKH firm is able to 'upgrade' by date $t+1$ with probability r, thus attaining HKH status.

Note that decision making about k_t seems absent from equations (10) and (11), although it is not absent in the primitive problem (5). This is slightly misleading. LKH firms *do* make decisions, but only *entry* decisions. A property of the equilibrium that we implement is that, once in the industry, LKH firms find it unprofitable to scale-up their capital beyond the level that they (endogenously) chose at the entry point. All post-entry investment will turn out to be undertaken by HKH firms that participate in the option exercise game.

An HKH firm's problem is where the growth options appear. As is described in detail below, an HKH firm can increase its scale by I (a choice variable) by paying a cost $\Phi(I,z,\nu)$. In the equilibrium that we implement, a firm's investment strategy depends on the information variable, ν_t, and the industry composition, $\bar{n}_t$. We represent this strategy with the pair (s^i_t, k^i_t), where $s^i_t = s^i(\bar{n}_t, \nu_t)$, for some function s^i that takes the value 1 if firm i exercises or 0 otherwise, while $k^i_t = k^i(\bar{n}_t, \nu_t)$ represents the firm's new (higher) level of capital. Suppose that all firms except i follow the strategy s^{-i}. An HKH firm's problem is

$$\begin{aligned} V^h(k_t;x_t,s^{-i},k^{-i}) = \max_{s^i,k^i} \Big\{ & A(1+\mu)z_tk_tp(x_t) + s^i_tG(k^i_{t+1};x_t\bar{s},\bar{k}) \\ & + (1-s^i_t)\gamma E_tV^h(k_t;x_{t+1},\bar{s},\bar{k})\Big\} \end{aligned} \tag{11}$$

where $\bar{s} \equiv (s^i, s^{-i})$ and $\bar{k} \equiv (k^i, k^{-i})$, and

$$G(k^i_{t+1};x_t,\bar{s},\bar{k}) \equiv -\Phi(k^i_{t+1}-k_t,z_t,\nu_t) + E_t\sum_{u\geq t+1}\gamma^{u-t}A(1+\mu)z_uk^i_up(x_u) \tag{12}$$

A symmetric Nash equilibrium in the exercise strategies for the HKH firms consists of a strategy pair $(s^*,k^*) = (s^*,k^*)(\bar{n},\nu)$ such that if all the HKH firms, with the exception of firm i, play the strategy $s^{-i} = s^*$ at any point in time, then $s^i = s^*$ solves the problem (11–12).

An industry equilibrium for this economy consists of a time path for the product price, p_t, the distribution of firms, $\bar{n}_t$, and their capital stocks, k_t, such that each firm's capital stock solves the above problem and $p_t = D(Q_t, z_t)$, where aggregate output $Q_t = n^l_tq^l_t + n^h_tq^h_t$, with q^l_t and q^h_t being the outputs of LKH and HKH firms, respectively.

We now turn to a more detailed analysis of the model.

3.3 Participation in the New Industry

The economy consists of n_0 firms in total. At the onset of an epoch, denoted $t = 0$, a fraction r_0 of these firms obtain the ability to begin LKH innovation. The capital that they require can be purchased and installed at cost $\Phi(k, z, \nu)$ where, recall, k is their choice of capital, z is the aggregate productivity shock and ν is the aggregate information variable. The exact functional form of Φ is discussed later. For now, it's important to assume that Φ is increasing convex in k and decreasing convex in ν.

We assume that at this point the level of information ν is very low making the adjustment costs $\Phi(k, z, \nu)$ very large for relatively large productions scales k. To encourage firms to invest in the new technology, however, we assume that $\Phi(k, z, \nu) = 0$ as long as $k \leq k_0$ for some small k_0 (to be calibrated). As we will see later, the value of an LKH firm is increasing in the initial productions scale and therefore, under the above assumptions, an LKH firm invests k_0 in the new technology.

As time goes on, $t > 0$, a new fraction r_0 of the n_0 firms enter each period. For dates prior to the refinement, $t < T$, the number of LKH firms obeys

$$n^l_{t+1} \quad = \quad n^l_t + r_0 n_0 \ . \tag{13}$$

Recall that U^l and V^l denote the value of an LKH firm prior to and after the refinement, respectively. Conditional on acquiring the LKH prior to the refinement, the value of an LKH firm is given by:

$$U^l(z_t, n^l_t) = \pi^l(q^l_t, p_t) + \gamma E_t\left[\rho V^l(z_{t+1}, n^l_{t+1})\right] + \gamma(1-\rho)E_t\left[U^l(z_{t+1}, n^l_{t+1})\right] \tag{14}$$

with the terminal condition $U^l(z_T, n^l_T) = 0$. For reasons that will become apparent, we now use the notation $\pi(q, p)$ to denote profits, from equation (6) above.

3.4 The Refinement

At any date $t > 0$ the refinement event occurs with probability ρ. Denote this stopping time T. At date T a fraction r of existing LKH firms find that their technology is 'aligned' with the refinement and they upgrade to the technology level HKH, effective at date $T + 1$. At this point we allow for *entry* by firms that have not yet acquired LKH. Entry is critical for the overall point of our paper — a model of *competition* for growth options. But there are many ways to model it. We constrain the model so as to not have another free parameter by allowing the new HKH firms to *choose* a fee at which to sell the LKH technology to firms that have not yet acquired it. We label this fee F. An entrant is willing to pay F at date T for LKH as long as:

$$V^l(z_T, n_T^h, n_T^l) = V^l(z_T, n_T^l) \geq F \tag{15}$$

where T denotes the refinement time and $n_T^h = 0$.

The fee is determined by the HKH firms so as to maximize their value at the refinement date T:

$$V^h(z_T, n_T^h, n_T^l) = \max_{F, n^\phi} \left\{ F \frac{n^\phi}{n_{T+1}^h} + E_T \left[\gamma V^h(z_{T+1}, n_{T+1}^h, n_{T+1}^l) \right] \right\} \tag{16}$$

where n^ϕ is the mass of new entrants at date T that are willing to pay F and n_{T+1}^h is the mass of initial HKH firms, at the refinement date. Note the trade-off faced by the HKH firms. Increasing F leads to fewer firms entering the industry and therefore larger output prices. Nevertheless, fewer entrants also means less proceeds from selling the LKH technology.

For the new entrants that acquire LKH, the condition (15) must bind:

$$F(n^\phi) = V^l(z_T, n_T^h, n_T^l + n^\phi) \tag{17}$$

The problem of the HKH firms at time T can then be rewritten as:

$$V^h(z_T, n_T^h, n_T^l) = \max_{n^\phi} \left\{ F(n^\phi) \frac{n^\phi}{n_{T+1}^h} + E_T \left[\gamma V^h(z_{T+1}, n_{T+1}^h, n_{T+1}^l) \right] \right\} \tag{18}$$

Thus, as long as the HKH firms are allowed to sell their information on the LKH, the HKH firms effectively can control the mass of new entrants at the refinement.

The next section describes the dynamics of the LKH and HKH firms after the refinement as well as the type of industry equilibrium that we are after.

3.5 The Investment Problem

After the refinement, the true potential of the invention is revealed and a fraction r of the existing LKH firms emerge as HKH firms. For dates $t > T + 1$ 'entry' continues as the remaining LKH firms (including those who just purchased the technology) acquire the HKH with probability r. The main difference between LKH and HKH firms is the productivity. One unit of input capital is more productive for the later type than it is for the former type.

The increase in productivity gives HKH firms an incentives to scale up production and capture market share. For simplicity, we assume that HKH firms can only scale up once. Thus each HKH firm holds exactly one real option. In equilibrium, each HKH firm chooses the optimal exercise strategy by internalizing the impact of its decision on the existing *and future* HKH firms.

The investment cost associated with each real option plays a crucial role in pinning down an equilibrium in exercise strategies. We assume that these investment costs depend on the scale, the productivity shock and, most importantly, the level of available information about the HKH technology. The level of publicly-available information about the HKH technology depends on the number of HKH firms as well as the production scale of these firms.[3]

Let $\Phi(I, z, \nu)$ denote the adjustment cost to investment for production scale I when the technology shock is z and the level of information is ν. Denote Y as (deterministic) scaled aggregate output,

$$Y_t \equiv Q_t z_t^{-1} \ . \tag{19}$$

We assume that Φ has the following properties:

i. Φ is homogeneous in z, so that $\Phi(I, z, \nu) = z\Phi(I, \nu)$

ii. $\Phi(I, \nu) = \phi_0(\nu) + \phi_1(\nu)I^2$

iii. ϕ_0 and ϕ_1 are decreasing convex in ν

iv. There is a threshold ν^* such that for $\nu > \nu^*$ both $\phi_0(\nu)$ and $\phi_1(\nu)$ are constant

v. $\nu = \nu(Y)$, i.e. ν is a function of the non-random component of the aggregate output

The threshold ν^* should be thought of as the level of information beyond which the HKH technology becomes completely public and any firm from or outside the industry can invest in this technology.

We implement a symmetric feedback Nash equilibrium in exercise strategies. The strategies followed by each firm are contingent on the industry composition $\bar{n} = (n^l, n^h)$ as well as the level of information ν. Thus the exercise strategy of

[3]Intuitively, one can imagine that the portion of the investment costs associated with the information about the HKH technology is essentially a substitute for the number of specialized workers needed to operate the HKH technology. Prior to exercising their real option HKH firms hire and train specialized workers. These workers are in high demand across HKH firms. As such, one can imagine that these workers have sufficient incentives to go work for a different HKH firm and train other workers. As the number of specialized workers increases the information about the HKH technology becomes slowly public. Eventually, if enough specialized workers are around the HKH technology becomes public and in particular it could be exploited by other firms which could not participate in the industry up to that point.

firm i can be defined by the pair (s_t^i, k_t^i), where $s_t^i = s^i(\bar{n}_t, \nu_t)$, for some function s^i which takes the value 1 if firm i exercises or 0, otherwise, while $k_t^i = k^i(\bar{n}_t, \nu_t) \geq k_0$.

Suppose that all firms except firm i follow the strategy s^{-i}. The decision problem for firm i can now be stated as:

$$
\begin{aligned}
V^{H,i}(z_t, \bar{n}_t, \nu_t; s^{-i}, k^{-i}) = \max_{s^i, k^i_{t+1}} \Big\{ & \pi^h(q_t^h(k_0), p_t) + s_t^i G(k_{t+1}^i, z_t, \bar{n}_t, \nu_t; \bar{s}, \bar{k}) \\
& + [1 - s_t^i]\gamma E_t V^h(z_{t+1}, \bar{n}_{t+1}, \nu_{t+1}; \bar{s}, \bar{k}) \Big\}
\end{aligned}
\tag{20}
$$

where $\bar{s} = (s^i, s^{-i})$, $\bar{k} = (k^i, k^{-i})$ and

$$
G(k_{t+1}^i, z_t, \bar{n}_t, \nu_t; \bar{s}) = -\Phi(k_{t+1}^i - k_t^i, z_t, \nu_t) + E_t \sum_{u \geq t+1} \gamma^{u-t} \pi^h(q_u^h(k_u^i), p_u) \tag{21}
$$

Notice that the exercise decision of the firm i internalizes the price externality associated with this decision.

3.6 Industry Equilibrium

The output goods price p_u is given by (8) and it is a direct function of aggregate output Q_u. Aggregate output $Q_s = Q_s^l + Q_s^h$ combines the output of the LKH firms, Q_u^l and the output of the HKH firms, Q_u^h. The dynamics of each of the components of aggregate output are given for all $u > t$ by:

$$
\begin{aligned}
Q_u^l &= n_u^l q_u^l(k_0) \\
Q_u^h &= Q_{u-1}^h \frac{z_u}{z_{u-1}} + [n_{u-1}^h - 1] q_u^h(k_u^{-i}) s_{u-1}^{-i} + q_u^h(k_u^i) s_{u-1}^i + r n_{u-1}^l q_u^h(k_0)
\end{aligned}
\tag{22}
$$

while the dynamics of the number of LKH and HKH firms are given by:

$$
\begin{aligned}
n_u^l &= (1 - r) n_{u-1}^l \\
n_u^h &= n_{u-1}^h + r n_{u-1}^l
\end{aligned}
\tag{23}
$$

Definition (Industry Equilibrium): A symmetric Nash equilibrium in the exercise strategies for the HKH firms consists of a strategy pair $(s^*, k^*) = (s^*, k^*)(\bar{n}, \nu)$ such that if all the HKH firms, with the exception of firm i, play the strategy $s^{-i} = s^*$ at any point in time, then firm i will find it optimal to also play the strategy $s^i = s^*$, in the sense of the optimization program in (20) - (23).

3.7 Characterizing the Industry Equilibrium

We begin with several important features of the long-run industry equilibrium.

Before beginning we need to be specific about the shape of the inverse demand function used to back out the output prices in (8). Let Y^* be such that $\nu^* = \nu(Y^*)$. We assume the following functional form for D:

$$\begin{aligned} D(Q, z) &= Q^{-\eta} z^{\eta} = Y^{-\eta}, \text{ if } Y < Y^* \\ &= C_0 - C_1 Q z^{-1} = C_0 - C_1 Y, \text{ if } Y \geq Y^* \end{aligned} \tag{24}$$

where $0 < \eta < 1$ and the coefficients C_0 and C_1 are chosen such that D is differentiable everywhere:

$$\begin{aligned} C_0 &= (1+\eta)[Y^*]^{-\eta} \\ C_1 &= \eta[Y^*]^{-1-\eta} \end{aligned} \tag{25}$$

An important feature of the functional form in the above definition is that the elasticity of demand for the output price is constant. This feature is quite standard in the real options and industry equilibria literature.[4] Also standard is to have the 'demand shock,' z_t, enter multiplicatively. However, we depart slightly from the previous literature by assuming that the demand shock affects directly the level of supply rather than the output price *per se*. This is straightforward to see as the aggregate output and the shock enter with similar powers but different signs.

3.7.1 Parameter Restrictions

First, notice that when the aggregate output reaches Y^*, the information about the HKH technology becomes public and therefore any non LKH firm can invest in the same technology as any HKH firm. Since investment costs at this point are no longer affected by the information level all firms invest immediately and rents go to zero. We now derive restrictions on the optimal informational threshold $\nu^* = \nu(Y^*)$ (or equivalently Y^*) so that this happens.

In equilibrium outside firms (non-HKH) compete in investment strategies to the point where the economic rents are reduced to zero. Let $\bar{Y}$ denote the long run aggregate output. Then the industry is in a competitive equilibrium, in the long run, if there are no economic rents left:

$$0 \geq \bar{G} = \max_{I \geq 0} -\Phi(I, z, \nu^*) + E \sum_{u \geq 1} \gamma^u \pi^h(q_u^h(I), p(\bar{Y}, I)) \tag{26}$$

Notice that the output price is given by $p(\bar{Y}, I) = C_0 - C_1 \left[\bar{Y} + (1+\mu)AI\right]$. Solving for the optimal production scale and substituting back in gives:

$$\frac{1}{z}\bar{G} = -\phi_0(\nu^*) + \frac{1}{1-\gamma f} \frac{[\gamma f A(1+\mu)]^2 \bar{p}}{(1-\gamma f)\phi_1(\nu^*) + 2\gamma f[A(1+\mu)]^2 C_1} \left[-\frac{1}{2}\bar{p} + C_0\right] \tag{27}$$

[4]See, for instance, Dixit and Pindyck (1993), Leahy (1993), Grenadier (2002), Novy-Marx (2007), Jovanovic and MacDonald (1994).

where $\bar{p} = C_0 - C_1\bar{Y}$ and $\log f = \bar{\mu} + \frac{1}{2}\sigma^2$. Substituting in $\bar{p}$, the restriction $\bar{G} \leq 0$ becomes:

$$\left[\frac{C_0}{C_1}\right]^2 - 2\frac{(1-\gamma f)\phi_1(\nu^*) + 2\gamma f[A(1+\mu)]^2 C_1}{[\gamma f A(1+\mu)]^2 C_1^2}\phi_0(\nu^*) \leq \bar{Y}^2 \leq \left[\frac{C_0}{C_1}\right]^2 \tag{28}$$

where the second inequality follows from the restriction that $\bar{p} \geq 0$.

After the HKH firms have exercised, the information on the HKH technology becomes public and immediately all the LKH firms and the rest of the non-participating firms in the economy are now given access to the HKH technology and allowed to invest. These firms will invest in the HKH technology to the point where rents are reduced to zero.

Let $Y_{\underline{T}+1}$ denote the aggregate output generated by the HKH firms immediately after they exercise their real options. Let n^0 denote the number of new entrants and I^0 the amount they invest. Then in equilibrium rents are zero and aggregate output equals $\bar{Y}$:

$$\begin{aligned} 0 &= -\Phi(I^0, \nu^*) + \frac{\gamma f}{1-\gamma f}A(1+\mu)[C_0 - C_1[Y_{\underline{T}+1} + n^0 I^0 A(1+\mu)]] \\ \bar{Y} &= Y_{\underline{T}+1} + n^0 I^0 A(1+\mu) \end{aligned} \tag{29}$$

Solving for n^0 and I^0 yields:

$$\begin{aligned} n^0 &= \frac{\gamma f}{1-\gamma f}\frac{1}{\phi_0(\nu^*)}[C_0 - C_1 Y_{\underline{T}+1}][\bar{Y} - Y_{\underline{T}+1}]\left[1 - \frac{1}{2}\beta\right] \\ I^0 &= \beta\frac{\gamma f A(1+\mu)}{(1-\gamma f)\phi_1(\nu^*) + 2\gamma f[A(1+\mu)]^2 n^0 C_1}[C_0 - C_1 Y_{\underline{T}+1}] \end{aligned} \tag{30}$$

for some $0 < \beta < 1$ which is uniquely determined as long as:

$$\phi_0(\nu^*)\phi_1(\nu^*)(1-\gamma f)^2 < [\gamma f A(1+\mu)]^2[C_0 - C_1 Y^*]\left\{\frac{1}{2}C_0 - \left[\bar{Y} - \frac{1}{2}Y^*\right]C_1\right\} \tag{31}$$

Further details are provided in Appendix 1.

This section has demonstrated that the long-run equilibrium is fully determined as long as the parameters are such that the conditions in (28) and (31) are satisfied.

3.7.2 The Nash equilibrium in exercise strategies

We first characterize the benchmark case when all HKH firms pre-commit to the same exercise strategy. In this case, the Nash equilibrium is easy to derive because no firm deviates from the equilibrium path.

Consider the following pre-commitment strategy $(\underline{s}, \underline{k})$ given by $\underline{s}(\bar{n}, \nu) = 1$ for $t = \underline{\tau}(\bar{n}, \nu)$ and $\underline{s}(\bar{n}, \nu) = 0$ otherwise where $\underline{\tau}(\bar{n}, \nu)$ is such that:

$$\begin{aligned} Y_{\underline{\tau}} &< Y^* \\ Y_{\underline{\tau}+1} &\geq Y^* \end{aligned} \tag{32}$$

while $\underline{k}(\bar{n}, \nu)$ solves:

$$\underline{V}^h(z, \bar{n}, \nu) = \max_{\underline{k}} \Big\{ E_0 \sum_{u \geq 0} \gamma^u \pi^h(q_u^h(k_0), p_u) + E_0[\gamma^{\underline{\tau}} G(\underline{k}, z_{\underline{\tau}}, \bar{n}_{\underline{\tau}}, \nu_{\underline{\tau}})] \Big\} \tag{33}$$

where $E_0[\,\cdot\,] = E[\,\cdot\,|\bar{n}, \nu]$ while G is defined as in (20) - (23) with $s^{-i} = s^i = \underline{s}$ and $k^{-i} = k^i = \underline{k}$. The price process p is characterized by the following aggregate output process $Y_u = Y_u^l + Y_u^h$, where:

$$\begin{aligned} Y_u^l &= n_u^l q_u^l(k_0) z_u^{-1}, \text{ for } u \leq \underline{\tau}(\bar{n}, \nu) \\ Y_u^l &= 0, \text{ for } u > \underline{\tau}(\bar{n}, \nu) \\ Y_u^h &= n_u^h q_u^h(k_0) z_u^{-1}, \text{ for } u \leq \underline{\tau}(\bar{n}, \nu) \\ Y_u^h &= n_\tau^h q_{\underline{\tau}}^h z_{\underline{\tau}}^{-1}(k_0 + \underline{k}(\bar{n}, \nu)) > Y^*, \text{ for } u > \underline{\tau}(\bar{n}, \nu) \end{aligned} \tag{34}$$

Notice that since the information about the HKH technology becomes public after $u \geq \underline{\tau}$, the output price falls down to $p_u = \bar{p}$, for $u \geq \underline{\tau}$. In particular, $\underline{V}^h$ becomes:

$$\begin{aligned} \underline{V}^h(z_{\underline{\tau}}, \bar{n}_{\underline{\tau}}, \nu_{\underline{\tau}}) &= z_{\underline{\tau}} \max_{\underline{k}} \left\{ -\Phi(\underline{k}, \nu) + \gamma f A(1+\mu) \frac{\gamma f}{1 - \gamma f} \bar{p}[k_0 + \underline{k}] \right\} \\ &= z_{\underline{\tau}} \left\{ -\phi_0(\nu_{\underline{\tau}}) + \frac{1}{2} \frac{1}{\phi_1(\nu_{\underline{\tau}})} \left[A(1+\mu) \frac{\gamma f}{1 - \gamma f} \right]^2 \bar{p}^2 \right\} \end{aligned} \tag{35}$$

We next argue that under certain parameter restrictions this strategy supports a feedback Nash equilibrium in exercise strategies in the sense of the definition in the previous section.

Suppose that all HKH firms in the industry with the exception of firm i follow this strategy. That is $(s^{-i}, k^{-i}) = (\underline{s}, \underline{k})$. Then firm i will decide on its optimal exercise strategy (s^i, k^i) as in (20) - (23).

We now look for conditions under which firm i has no incentive to deviate from $(\underline{s}, \underline{k})$ by either preempting or free-riding on the other firms.

We notice first that firm i has no incentive to free-ride on the other firms in the hope of investing at a lower cost after the other firms have exercised. Recall that according to the strategy ($\underline{\text{s}}$, $\underline{\text{k}}$), all firms but firm i plan to exercise at time $t = \underline{\tau}$, right before Y_t reaches the informational threshold Y^*. Suppose firm i decides to deviate by investing at time $\underline{\tau}+1$ so it can economize on the investment costs after all the other firms exercised. But at time $\underline{\tau}+1$ the information about the HKH technology becomes public while $Y_{\underline{\tau}+1} \geq Y^*$. Furthermore, according to the definition of Y^* the value of the real option for firm i at $t = \underline{\tau}+1$ is at most 0. Thus, firm i will be strictly worse than if it were to exercise at time $t = \underline{\tau}$ with the rest of the firms.

Let's analyze now the possibility that firm i deviates from the strategy $(\underline{s}, \underline{k})$ by preempting the rest of the firms. Let $\tau^i < \underline{\tau}$ denote the exercise time of firm i. Let k^i solve:

$$V^{H,i}(z, \bar{n}, \nu) = \max_{k^i} \left\{ E_0 \sum_{u \geq 0} \gamma^u \pi^h(q_u^h(k_0), p_u) + E_0[\gamma^{\tau^i} G(k^i, z_{\tau^i}, \bar{n}_{\tau^i}, \nu_{\tau^i})] \right\} \quad (36)$$

where $E_0[\,\cdot\,] = E[\,\cdot\,|\bar{n}, \nu]$ while G is defined as in (20) - (23). The price process p is characterized by the following aggregate output process $Y_u = Y_u^l + Y_u^h$, where:

$$\begin{aligned}
Y_u^l &= n_u^l q_u^l(k_0) z_u^{-1}, \text{ for } u \leq \underline{\tau}(\bar{n}_{\tau^i}, \nu_{\tau^i}) \\
Y_u^l &= 0, \text{ for } u > \underline{\tau}(\bar{n}_{\tau^i}, \nu_{\tau^i}) \\
Y_u^h &= n_u^h q_u^h(k_0) z_u^{-1}, \text{ for } u \leq \tau^i \\
Y_u^h &= [n_u^h - 1] q_u^h(k_0) z_u^{-1} + q_{\tau^i}^h z_{\tau^i}^{-1}(k_0 + k^i), \text{ for } \tau^i < u \leq \underline{\tau}(\bar{n}_{\tau^i}, \nu_{\tau^i}) \\
Y_u^h &= [n_{\underline{\tau}}^h - 1] q_{\underline{\tau}}^h z_{\underline{\tau}}^{-1}(k_0 + \underline{k}) + q_{\tau^i}^h z_{\tau^i}^{-1}(k_0 + k^i), \text{ for } u > \underline{\tau}(\bar{n}_{\tau^i}, \nu_{\tau^i})
\end{aligned} \quad (37)$$

We notice that as a result of firm i preempting the industry, the aggregate output increases at time $t = \tau^i$ and so does the amount of information ν. Then according to the strategy $(\underline{s}, \underline{k})$, the rest of the firms respond by exercising sooner at time $t = \underline{\tau}(\bar{n}_{\tau^i}, \nu_{\tau^i}) \leq \underline{\tau}(\bar{n}, \nu)$. In this sense the strategies considered here are feedback strategies.

To see whether it is feasible for firm i to preempt we look first at the case when $\tau^i = \underline{\tau}(\bar{n}, \nu) - 1$. Suppose firm i invests k^i large enough so that $Y_{\tau^i} + A(1+\mu)k^i = Y_{\tau^i+1} = Y_{\underline{\tau}} > Y^*$. In this case the output price falls down to $p_u = \bar{p}$ starting with $u = \tau^i + 1 = \underline{\tau}$ and the value of the firm at time $t = \tau^i$ becomes:

$$\begin{aligned}
V^{H,i}(z_{\tau^i}, \bar{n}_{\tau^i}, \nu_{\tau^i}) &= z_{\tau^i} \max_{k^i} \left\{ -\Phi(k^i, \nu_{\tau^i}) + A(1+\mu) \frac{\gamma f}{1 - \gamma f} \bar{p}[k_0 + k^i] \right\} \\
&= z_{\tau^i} \left\{ -\phi_0(\nu_{\tau^i}) + \frac{1}{2} \frac{1}{\phi_1(\nu_{\tau^i})} \left[A(1+\mu) \frac{\gamma f}{1 - \gamma f} \right]^2 \bar{p}^2 \right\}
\end{aligned} \quad (38)$$

where $k^i = \frac{1}{\phi_1(\nu_{\tau^i})} \left[A(1+\mu) \frac{\gamma f}{1-\gamma f} \right] \bar{p} \geq \frac{Y^* - Y_{\tau^i}}{A(1+\mu)}$.

If firm i were to wait instead and exercise with everyone else the value of firm

i at $t = \tau^i$ is:

$$
\begin{aligned}
V^{H,i}(z_{\tau^i}, \bar{n}_{\tau^i}, \nu_{\tau^i}) &= z_{\tau^i}\gamma f \left[-\Phi(\underline{k}, \nu_{\underline{\tau}}) + A(1+\mu)\frac{\gamma f}{1-\gamma f}\bar{p}[k_0 + \underline{k}] \right] \\
&= z_{\tau^i}\gamma f \left\{ -\phi_0(\nu_{\underline{\tau}}) + \frac{1}{2}\frac{1}{\phi_1(\nu_{\underline{\tau}})}\left[A(1+\mu)\frac{\gamma f}{1-\gamma f} \right]^2 \bar{p}^2 \right\}
\end{aligned} \tag{39}
$$

Thus, firm i has no incentive to preempt the rest of the firms when:

$$
\begin{aligned}
-\phi_0(\nu_{\tau^i}) + \gamma f \phi_0(\nu_{\underline{\tau}}) &< \frac{1}{2}\left[-\frac{1}{\phi_1(\nu_{\tau^i})} + \frac{\gamma f}{\phi_1(\nu_{\underline{\tau}})} \right] \left[A(1+\mu)\frac{\gamma f}{1-\gamma f} \right]^2 \bar{p}^2 \\
\frac{Y^* - Y_{\tau^i}}{A(1+\mu)} &\leq \frac{1}{\phi_1(\nu_{\tau^i})}\left[A(1+\mu)\frac{\gamma f}{1-\gamma f} \right] \bar{p}
\end{aligned} \tag{40}
$$

Since Y^* and $\bar{p}$ are constructed from the long-run equilibrium restrictions determined in the previous section and Y_{τ^i} depends on the initial number of LKH firms at the refinement as well as the rate of acquiring the HKH technology, the two restrictions above are essentially restrictions on the parameter space.

The more general case when $\tau^i < \underline{\tau} - 1$ is similar to this one because preemption triggers the rest of the firm to exercise as early as $\tau^i + 1$ since the costs of investments are now significantly lower.[5] The details are in the Appendix.

Thus the strategy $(\underline{s}, \underline{k})$ supports a symmetric feedback Nash equilibrium in exercise strategies.

We finish this section with the dynamics of an LKH firm after the refinement. Given that now we know how the HKH firms plan to exercise their real options and what happens in the long-run equilibrium we can characterize the dynamics of V^l:

$$
\begin{aligned}
V^l(z_t, n_t^h, n_t^l) = \pi^l(q^l(k_0), p_t) + \\
\gamma E_t \left[r V^h(z_{t+1}, n_{t+1}^h, n_{t+1}^l) + (1-r)V^l(z_{t+1}, n_{t+1}^h, n_{t+1}^l) \right]
\end{aligned} \tag{41}
$$

for $t < \underline{\tau}$ with the terminal condition:

$$
\begin{aligned}
V^l(z_{\underline{\tau}}, n_{\underline{\tau}}^h, n_{\underline{\tau}}^l) = \pi^l(q^l(k_0), p_{\underline{\tau}}) + \\
r V^h(z_{\underline{\tau}}, n_{\underline{\tau}}^h, n_{\underline{\tau}}^l) + (1-r)\mathbf{1}_{\{n^0 \leq n_{\underline{\tau}}^l\}} \frac{\gamma f}{1-\gamma f}\pi^h(q^h(k_0), \bar{p})
\end{aligned} \tag{42}
$$

Consistent with the long-run equilibrium determined in the previous sections, the last equation accounts for the fact that only n^0 firms (both LKH and non-participating) will be able to invest in the HKH technology immediately after $\underline{\tau}$. The rest of $n_{\underline{\tau}}^l - n^0$ will exit the industry and earn 0. This is reminiscent of the industry *shake-out* from Jovanovic and MacDonald (1994).

[5]This follows from the shape of the adjustment cost functions $\phi_0(\nu)$ and $\phi_1(\nu)$. Both these functions are convex decreasing at a large pace when ν is relatively small

4 Numerical Implementation

The model presented in the previous section has to be solved numerically. The choice of the type of equilibria that we implement reduces substantially the computational burden associated with numerical implementation. Bellow we present our choice of parameters and the qualitative as well as quantitative properties of the model.

4.1 Constraints

Here we analyze the parameter constraints derived in the previous section. Notice first that the LHS of the constraint in (40) is negative as by construction $\phi_0(\nu_{\tau^i}) > \phi_0(\underline{\nu})$. Also recall that $\tau^i = \underline{\tau} - 1$, meaning that:

$$Y^* - Y_{\tau^i} \leq k_0 A[(n^h_{\underline{\tau}} - n^h_{\underline{\tau}-2})(1+\mu) + (n^l_{\underline{\tau}} - n^l_{\underline{\tau}-2})] \tag{43}$$

or furthermore:

$$Y^* - Y_{\tau^i} \leq k_0 A[r(2-r)(1+\mu) + ((1-r)^2 - 1)]n^l_{\underline{\tau}-2} = k_0 A\mu[1-(1-r)^2]n^l_{\underline{\tau}-2} \tag{44}$$

Thus a sufficient condition for (40) to hold is:

$$\begin{gathered} \gamma f \geq \frac{\phi_1(\nu_{\underline{\tau}})}{\phi_1(\nu_{\tau^i})} \\ k_0 \frac{\mu}{1+\mu}[1-(1-r)^2]n^l_{\underline{\tau}-2} \leq \frac{1}{\phi_1(\nu_{\tau^i})}\left[A(1+\mu)\frac{\gamma f}{1-\gamma f}\right]\bar{p} \end{gathered} \tag{45}$$

Notice now that the optimal scale chosen by the firms following the $(\underline{k}, \underline{s})$ policy is $\underline{k} = \frac{1}{\phi_1(\underline{\tau})}\frac{\gamma f}{1-\gamma f}A(1+\mu)\bar{p}$. The aggregate output generated by all the HKH firms at $t = \underline{\tau}$ is given by $Y_{\underline{\tau}+1} = [k_0 + \underline{k}]A(1+\mu)n^h_{\underline{\tau}}$. If rents are still positive it has to be the case that $Y_{\underline{\tau}+1} \leq \bar{Y}$. This happens if:

$$\bar{Y} \geq \frac{C_0}{C_1} + \frac{k_0 A(1+\mu)n^h_{\underline{\tau}} - \frac{C_0}{C_1}}{1 + \frac{1}{\phi_1(\underline{\tau})}\frac{\gamma f}{1-\gamma f}[A(1+\mu)]^2 n^h_{\underline{\tau}} C_1} \tag{46}$$

If the inequality is strict then the constraint in (31) needs to be satisfied as well. That is:

$$\bar{Y} \leq \frac{1}{2}\left[\frac{C_0}{C_1} + Y^*\right] - \frac{1}{C_1(C_0 - C_1 Y^*)}\frac{1}{[A(1+\mu)]^2}\left[\frac{1-\gamma f}{\gamma f}\right]^2 \phi_0(\nu^*)\phi_1(\nu^*) \tag{47}$$

Notice that the RHS of the above equation is strictly dominated by $\frac{C_0}{C_1}$, meaning that the upper bound on $\bar{Y}$ is strictly lower than the one in the constraint (28).

If the constraint in (46) is binding then, according to the lower bound constraint in (28), it has to be the case that:

$$\left[\frac{C_0}{C_1} + \frac{k_0 A(1+\mu)n^h_{\underline{\tau}} - \frac{C_0}{C_1}}{1+\frac{\gamma f}{1-\gamma f}[A(1+\mu)]^2 n^h_{\underline{\tau}} C_1}\right]^2 \geq \left[\frac{C_0}{C_1}\right]^2 - 2\frac{1-\gamma f}{\gamma f}\frac{(1-\gamma f)\phi_1(\nu^*) + 2\gamma f[A(1+\mu)]^2 C_1}{\gamma f[A(1+\mu)]^2 C_1^2}\phi_0(\nu^*) \tag{48}$$

The equilibrium that we implement in this paper assumes that Y^* is chosen such that the constraint in (48) is satisfied. Furthermore, we define $\bar{Y}$ such that the constraint in (46) is binding.

To determine the thresholds Y^* and $\bar{Y}$, we have to pin down first the exercise strategies of the HKH firms. Here we make use of a moment condition, namely the exercise time for the HKH firms in the industry matches a certain target, say τ_0, which comes from the data. Following the definition in (32), we choose Y^* as the average of Y_{τ_0} and Y_{τ_0+1} such that the optimal exercise time for the HKH firms matches the target: $\underline{\tau} = \tau_0$. Recall that according to the Nash equilibrium in exercise strategies all HKH firms exercise precisely at τ_0 periods following the refinement. This means that both Y_{τ_0} and Y_{τ_0+1} are easily determined as a function of the number of LKH firms at the refinement, the rate of acquiring the HKH and the exercise time τ_0. The other threshold $\bar{Y}$ is defined by the binding constraint (46). Notice that both Y^* and $\bar{Y}$ will depend on the number of firms that have acquired the LKH prior or at the refinement.

Finally, the rest of the parameters are chosen such that we match several important macroeconomic target moments and such that the constraints (45) and (48) are satisfied. The next section presents numerical values for the parameters of the model.

4.2 Choice of parameters

The goal is to calibrate the model to capture several important macroeconomic properties of the US economy. We start with the choice of the time step which we set equal to a quarter. Since our focus is on the low-frequency properties of Tobin's Q, a quarter seems like an appropriate choice. The annual average returns of the S&P 500 stocks for the past 75 years is around 12%, while the average inflation is around 3%. This results in a quarterly average real return of about 2.2%. In our model, the equivalent of this real rate is γ. We therefore set $\gamma = 0.978$. The drift and the standard deviation for the aggregate shock are set to match the first two moments of the time series of the US real GDP quarterly rates over the past 60 years. We set $\bar{\theta} = 0.84\%$ and $\sigma = 1\%$.

In our simulations we only consider one epoch (one invention). As such, the vintage capital parameter A is not identifiable from mu and we normalize it to $A = 1$. This parameter will play an important role if we consider an economy with more than one epoch.

In the data, Tobin's q is increasing continuously from the mid 80's to the late 90's, a period spanning almost 16 years. We notice further that Tobin's q grows at a lower pace in the first half of this period than in the second half. This sudden change in the growth rate of Tobin's q is consistent with the predictions of our model prior and after the refinement, where the refinement happens precisely after 8 years (around 1992). Therefore, we set both the target length of the period prior to the refinement as well as the target period after the refinement to equal eight years. This choice for the average arrival time for the refinement is also consistent with the choice in Jovanovic and MacDonald (1994).

We calibrate the rate r_0 of acquiring the LKH prior to the refinement and the likelihood of refinement arrival ρ so that the average refinement arrival time is about 32 quarters. We obtain $r_0 = 0.2\%$ and $\rho = 0.2$.

Next, we calibrate the increase in output per unit capital μ for the HKH firms in a new industry, the rate r of acquiring the HKH, and the parameters associated with the adjustment cost function Φ so that the constraints (45) and (48) are satisfied and the average exercise time for the HKH firms is about 32 quarters.

We use the following specification for the fixed and variable components of the adjustment cost function:

$$\begin{aligned} \phi_0(Y) &= ae^{-Y^{0.5}+(Y^*)^{0.5}}, \text{ for } Y \le Y^* \\ \phi_1(Y) &= be^{-Y^{0.5}+(Y^*)^{0.5}}, \text{ for } Y \le Y^* \end{aligned} \tag{49}$$

for some positive a and b.

This specification captures two important features which are important for deriving the Nash equilibrium in exercise strategies. First, both the fixed and the variable components decrease as the more information about the HKH technology becomes public. And second, both the fixed and the variable costs decrease in a convex fashion so that the costs become more comparable as the threshold Y^* is approached.

Defining Y^* and $\bar{Y}$ as indicated in the previous section, we find $\mu = 10 \log \bar{\theta}$, $r = 0.2\%$, $a = 1$ and $b = 0.1$.

The elasticity η of the demand function is set equal to 0.8, consistent with the choice in other studies (see for instance Jovanovic and MacDonald (1994)).

Finally, for the quantitative results that we present bellow, we consider an economy with $n_0 = 5000$ firms. The firms that participate in the industry pledge

$k_0 = 1$ units of capital which they can acquire costlessly from the outside capital markets (see Section 3.3). The rest of the firms in the economy which cannot participate in the industry produce at a rate which matches the GDP growth rate $\bar{\theta}$. We assume that all firms that are outside the industry have the same amount of capital k_0^a which we calibrate so that the Tobin's q in the model (for the entire economy) at the refinement equals 1.4, which is approximately the value for Tobin's q in the data around year 1992 (the year when Tobin's q switches from a lower growth rate to a higher growth rate, as described above).

4.3 Qualitative results

As described so far, a new invention comes along and firms try to innovate in order to take advantage of the new growth opportunities. Some firms succeed, others don't. Over time, a new industry takes shape. The epoch marks the interval between the arrival of the invention and the time when the industry reaches a competitive equilibrium. Using the parameters calibrated in the previous section we simulate an equilibrium with one epoch (or one invention).

We now illustrate several qualitative features of our model. We focus first on the price dynamics and the composition of participating firms within an epoch. Figure 4 presents the price dynamics within the epoch.

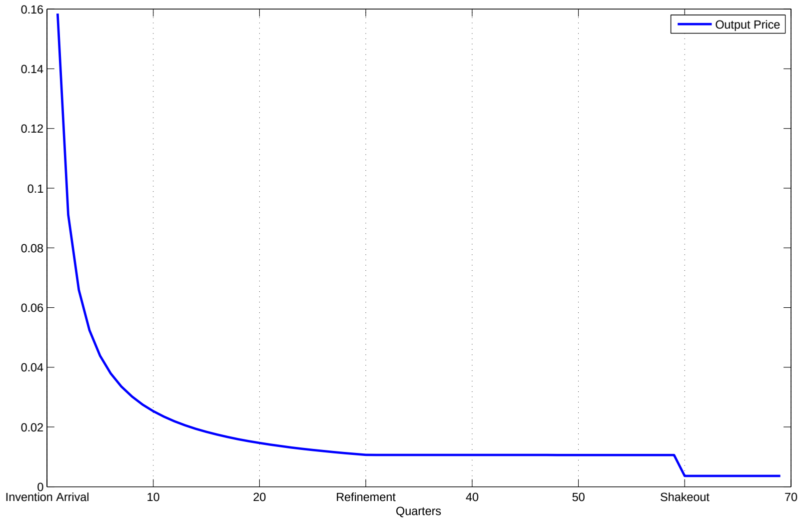


Figure 4: Output price dynamics

In figure 4 we notice that, consistent with our equilibrium concept, the price declines slowly up to the point where the HKH firms exercise their real options. In the period following the exercise time, the price drops sharply inducing an industry shake-out, in which some LKH firms manage to stay in the industry while the rest leave the industry. After the shake-out the price stabilizes around the long-run industry equilibrium price $\bar{p}$.

Figure 5 presents the evolution of the industry composition during the epoch. We notice a steady increase in the mass of LKH firms prior and at the refinement, consistent with the fact that firms which were not successful in acquiring the LKH up to refinement can still do so by purchasing it from the HKH firms. As time passes, more and more LKH firms are acquiring the HKH reducing the mass of LKH firms. At the shake-out, all LKH firms exit the industry.

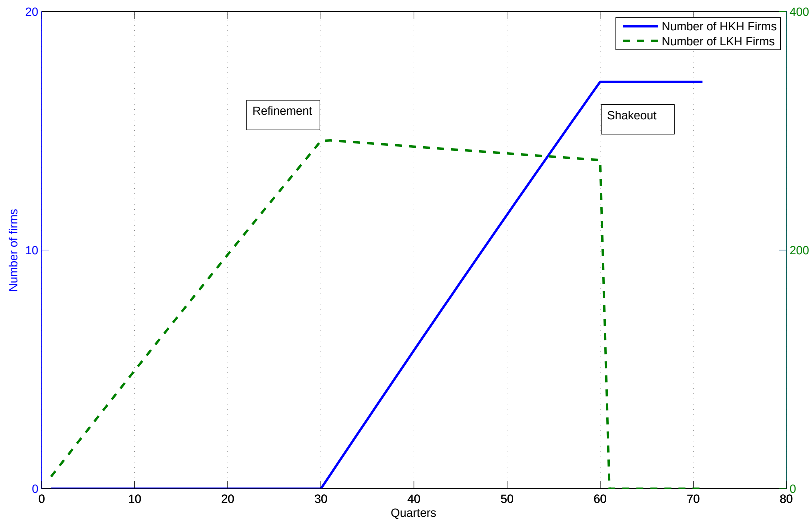


Figure 5: Industry composition

Finally, the aggregate output in this economy is plotted in Figure 6. We notice that the output increases steadily up to the exercise point as the LKH firms acquire gradually HKH. At the exercise point we notice a jump in the output. This output is the combined result of the HKH firms increasing substantially their market share but also the LKH firm leaving the industry and not producing anymore.

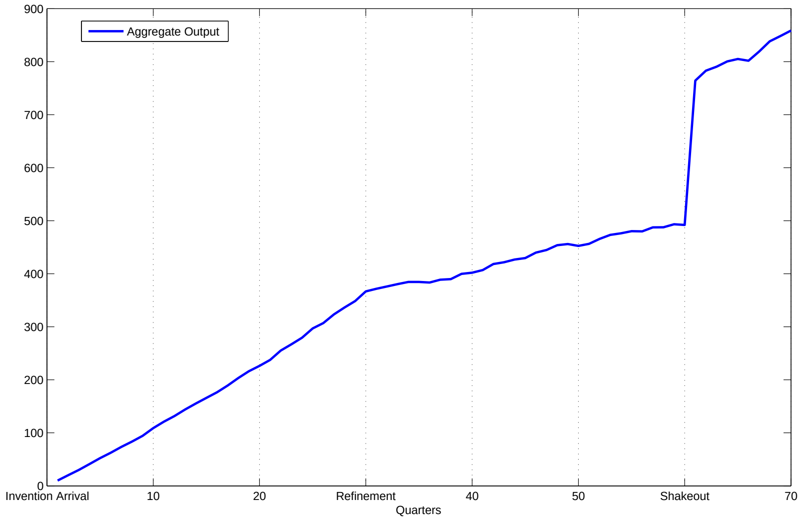


Figure 6: Aggregate output

4.4 Quantitative results

We now investigate the ability of the model to generate the low-frequency time-series variation in the value of the US aggregate stock of capital relative to the replacement value. We measure the value of capital by aggregating the value of the firms' equity at every point in time. We measure the replacement value of the stock of capital by aggregating the value of the installed capital inside the firms.

Figure 7 plots the Tobin's q generated by the model. We notice that prior to the refinement the Tobin's q increases steadily for about eight years from almost 1 to a little bit over 1.25. The refinement is perceived as good news by the market participants and Tobin's q jumps slightly to about 1.4. This is the value of Tobin's q generated by the model which matches the value of Tobin's in the data for year 1992. After the refinement, Tobin's q trend up at a much faster pace than before for about 8 more years and reaches a magnitude as high as 4.3 prior to the industry shake-out. Thus, the model is able to generated both qualitatively and quantitatively the behavior of Tobin's q observed in the data, especially during the period between mid 80's and late 90's. More importantly, the model suggests that valuation effects induced by delaying investment are more than enough to explain the large magnitude in the Tobin's q observed in the data.

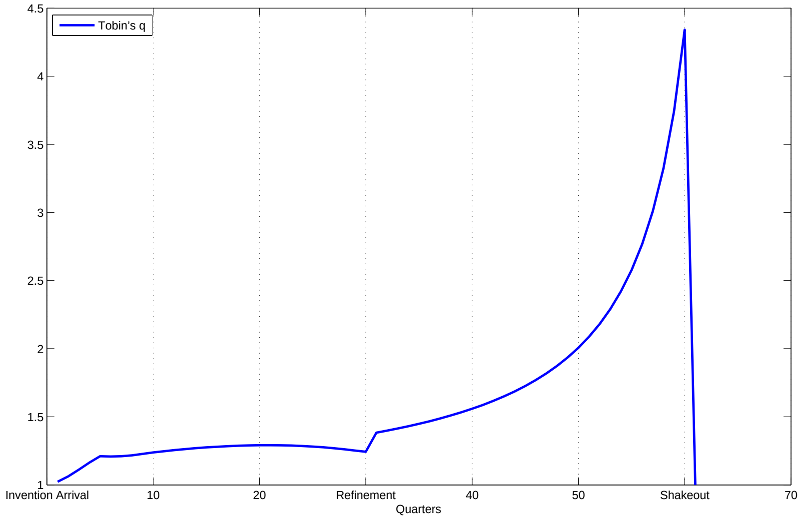


Figure 7: Tobin's q

5 Conclusion

Can growth options explain the observed variation in a firm's value relative to it's stock of capital? What is the nature of these positive net-present-value growth options? We construct a model of industry equilibrium in which new inventions present firms with new growth opportunities. To unlock these growth opportunities firms must devise new technologies. Some firms are more successful than others. The firms with the successful technologies extract economic rents that they try to magnify through investment. The adjustment costs associated with investment depend a variable that we interpret as information. The more of this information is public the less expensive it is to exercise the investment option. The level of information is tied to the aggregate output of these firms. In particular if one firm exercises its real option prior to the other ones, the rest of the firms benefit from the lower adjustment costs. Thus, firms face the following trade-off. On the one hand they want to exercise early so as to capture market share while output prices remain high. On the other hand, they want to wait until a competitor exercises so that the adjustment costs to investment become lower.

As the number of competitor firms increases, these firms have to play a strategic game in exercise strategies. We find that under certain parameter restrictions there exists a symmetric, "closed loop" Nash 1equilibrium in exercise strategies in which firms find it valuable to wait before they exercise their real option.

We calibrate the model relative to several U.S. aggregate data targets. Qualitatively, the model can deliver the hump shape in the number of participating firms in an industry as well as a continuous decline in the size of economic rents available to the firms in the industry. Quantitatively, the model can generate substantial variation and persistence in Tobin's q, reaching values in the ballpark of what we see in U.S. data.

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