Non-Stationary Search Equilibrium*

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Abstract

We study aggregate equilibrium dynamics of a frictional labor market where firms post employment contracts and workers search randomly on and off the job for such contracts, while the economy is hit by aggregate productivity shocks. Our exercise provides the first analysis of aggregate dynamics of a popular class of search wage-posting models.

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*This paper fully develops the theory that we first sketched in our companion paper "The Timing of Labor Market Expansions: New Facts and a New Hypothesis". We acknowledge useful comments to that paper from seminar and conference audiences at numerous venues. The usual disclaimer applies.

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1 Introduction

We study the aggregate equilibrium dynamics of a frictional labor market where workers search randomly on and off the job for employment contracts posted by firms. Our exercise provides the first analysis of aggregate dynamics of a popular class of search wage-posting models, originating with Burdett and Mortensen (1998, henceforth BM). They provide the first successful formalization of the hypothesis that cross-sectional wage dispersion is largely a consequence of labor market frictions. In so doing the BM model has started what has now established itself as the most promising line of research in the analysis of wage inequality, as the vibrant and empirically very successful literature organized around that hypothesis continues to show.

That literature, however, is invariably cast in deterministic steady state. Ever since the inception of the BM model, job search scholars have regarded the characterization of its out-of-steady-state behavior as a daunting problem, essentially because one of the model's state variables, which is also the main object of interest, is the endogenous *distribution* of wage (or job value) offers. This is an infinite-dimensional object, endogenously determined in equilibrium as the distribution across a continuum of firms of strategies that are all best responses to one another.

We find a way around this problem by considering a class of equilibria satisfying what we call the *Rank-Preserving property*, i.e. equilibria in which the workers' ranking of firms is time-invariant. We show that this class of equilibria is generic if all firms are equally productive and the environment is not subject to aggregate stochastic shocks. We further show that the same property holds in equilibrium when firms have heterogeneous productivity, where more productive firms offer a larger value and employ more workers at all points in time, if (but not only if) they have more employees to begin with. We view the fact that the workers' ranking of firms also reflects the hierarchy of productivity in a Rank-Preserving Equilibrium in the presence of productive heterogeneity across firms as a very appealing property of the model. It parallels a similar property of BM's static equilibrium, and in ensures constrained-efficient labor reallocation at all dates. Finally, we investigate existence of Rank-Preserving Equilibria in a stochastic environment.

Besides being of intrinsic theoretical interest, our characterization of the dynamics of the BM model opens the analysis of aggregate labor market dynamics as a whole potential new field of application of search/wage-posting models. Unlike the typical representative-agent model, the BM model makes predictions about the business cycle behavior of wage distributions, firm size distributions, or patterns of labor reallocation across firms. In a companion paper (Moscarini and Postel-Vinay (2008), [MPV08]), we quantitatively gauge those predictions against facts documented using various (often new) data sets. Moreover, by advocating the BM model as a potentially useful tool for the study of aggregate labor market dynamics, we hope to contribute to a synthesis between the BM approach and the "other", equally successful side of the search literature, organized around the matching framework (Pissarides, 1990; Mortensen and Pissarides, 1994), initially designed for the understanding of labor market flows and equilibrium unemployment.[1]

The paper has five sections after this Introduction and before the Conclusions. In those Sections we (resp.) present the model economy, analyze the contract-posting problem in discrete time, then in continuous time, we characterize a class of tractable equilibria and a computation algorithm, and we show that this class is non restrictive on perfect foresight paths udner weak assumptions.

2 The economy

We study a stochastic economy where firms commit to employment contracts and workers search randomly for those contracts. The special case of a stationary and deterministic economy where contracts are restricted to a constant wage is the BM wage posting model with heterogeneous firm types. We present our model in discrete time, as it affords more clarity in the presentation of the contract posting problem under one-sided commitment as a recursive problem, following the seminal insights of Thomas and Worrall (1988). Then, we analyze equilibrium in a continuous-time equivalent economy, which is more tractable and easier to relate to the search literature.

The labor market is populated by a unit-mass of workers, who can be either employed or unemployed, and by a unit measure of firms,[2] all risk neutral, infinitely lived, maximizers of payoffs discounted with factor $\beta \in (0,1)$. Firms operate constant-return technologies with heterogeneous productivity levels $\omega\theta$, where ω is an aggregate component, evolving within some set of values $\Omega \subset \mathbb{R}_+$ according to a discrete time Markov process $H(d\omega' \mid \omega)$, and θ is a fixed, idiosyncratic heterogeneity component, distributed across firms $\theta \sim \Gamma(\cdot)$ with density $\gamma = \Gamma'$ over $\left[\underline{\theta}, \overline{\theta}\right]$.

The labor market is affected by search frictions in that unemployed workers enjoy a flow value of leisure b and can only sample job offers sequentially with some probability $\lambda_0^\omega \in (0,1)$ each period. Employed workers earn a wage, are allowed to search on the job, and face a per-period sampling chance of job offers of $\lambda_1^\omega \in (0,1)$. For notational simplicity we will assume uniform sampling of firms by workers, in that any worker receiving a job offer draws the type of the firm

[1] Rudanko (2008) and Menzio and Shi (2008) formulate and solve wage contract-posting models with aggregate productivity shocks, where job search is directed, rather than random in the spirit of BM. This assumption greatly simplifies the analysis, by severing the link between the individual firm's contract-posting problem and the distribution of contract offers. This is the main hurdle that we face, and that we resolve by exploiting the idea and emergence of Rank-Preserving Equilibrium, while maintaining BM's assumption of random search common to the vast majority of the search literature. From a theoretical viewpoint, we see both programs as fruitful directions of exploration. From a quantitative viewpoint, the directed search approach is focused on the response of the job-finding rate to aggregate shocks. This approach does not generate a well-defined notion of employer size. Hence, it is silent on the wealth of new evidence on cyclical patterns of the size distribution and employer size/growth relationship that we offer in MPV08, and that we envision as central to our understanding of the propagation of aggregate shocks in labor markets.

[2] We thus implicitly fix the measure of active firms, thus remaining mostly silent on the question of entry and exit. A simple extension of the model to make it capture entry and exit of firms over the business cycle is illustrated in our companion paper MPV08.

from which the offer emanates from the distribution $\Gamma(\cdot)$.[3] Firm-worker matches are dissolved with chance $\delta^{\omega} \in (0,1)$. Upon match dissolution, the worker becomes unemployed. Note that all these transition probabilities, although exogenous, are allowed to depend on the aggregate state ω. All workers are ex-ante identical: they are infinitely lived, risk-neutral, equally capable at any job, and they attach a common lifetime value of U_t to being unemployed at date t.

In each period, the timing is as follows. Given a current state ω of aggregate labor productivity and size (measure of workers employed) L:

1. a firm of type θ produces and pays wages in state ω, earns a flow profit of $(\omega\theta - w)\,L$;
2. the new state ω' is realized;
3. firms post employment contracts;
4. jobs are destroyed exogenously with chance $\delta^{\omega'}$;
5. the remaining employed workers receive an outside offer with chance $\lambda_1^{\omega'}$ and decide whether to accept it or to stay with the current employer;
6. new workers are hired from unemployment, each with probability $\lambda_0^{\omega'}$.

A firm of idiosyncratic productivity θ chooses and commits to an employment contract, which specifies a wage as a function of a vector $\mathbf{s}$ of state variables, to maximize the present discounted value of profits, given other firms' contract offers. The firm is further subjected to an *equal treatment constraint*, whereby it must pay the same wage to all their workers. This is the sense in which we generalize the BM restrictions placed on the set of feasible wage contracts to a non-steady-state environment.[4] Under commitment, such a wage function implies a value V for any worker to work for that firm, which is also a function of the state $\mathbf{s}$. Equivalently, the firm chooses a promised value for the future period, the present discounted value of wages, after observing the new realization of the state ω'.

[3] Later, we extend the model to allow for non-uniform sampling, in that different types-θ firms have different chances of being sampled by job searchers. This extension is theoretically straightforward and useful in quantitative applications

[4] We thus rule out, among other things, wage-tenure contracts (Stevens, 2004; Burdett and Coles, 2003), offer-matching or individual bargaining (Postel-Vinay and Robin, 2002; Dey and Flinn, 2005; Cahuc, Postel-Vinay and Robin, 2006), contracts conditioned on employment status (Carrillo-Tudela, 2007). Note, however, that the model can be generalized to allow for time-varying individual heterogeneity under the assumption that firms offer the type of piece-rate contracts described in Barlevy (2008). In that sense experience and/or tenure effects can be introduced into the model. Shimer (2008) proposes an alternative formulation, which maintains BM's restriction of a constant posted wage, even out of steady state, and delivers a few of the same results.

3 The contract-posting problem

We look for a symmetric equilibrium, a promised value function $V(\mathbf{s})$, where all firms of equal type θ offer their workers a contract that guarantees the same value and have the same size at all points in time and in all states of nature, and where $V(\mathbf{s})$ is the best response to the same value generated by the offer policy chosen by other firms.

3.1 State variables

Our first, non-trivial task is to identify the minimal set of state variables $\mathbf{s}$ on which offered values (contracts) can be conditioned under commitment. To do so, we have to show that, for the chosen state vector $\mathbf{s}$ and the candidate equilibrium strategy $V(\mathbf{s})$, the best-response problem of a firm only depends on $\mathbf{s}$.

We will show that the state vector $\mathbf{s}$, beyond firm productivity θ and the new state of aggregate productivity ω',[5] comprises current firm size L and the current non-normalized distribution of employment across firms, Λ. Here $\Lambda(\theta)$ is total (cumulated) employment at all firms with productivity less than θ — so that $\Lambda\left(\overline{\theta}\right)$ is total employment in the economy. Note that Λ is an infinite-dimensional object, and that in a symmetric equilibrium (ruling out atoms), knowledge of Λ implies knowledge of the size $L^*(\theta) := \Lambda'(\theta)/\gamma(\theta)$ of a type-θ firm which is following the putative equilibrium strategy.

We first establish that, for any value W offered by a type-θ firm of size L in state ω', all components of the state vector $\mathbf{s} = \{\theta, L, \omega', \Lambda\}$ evolve in the new state according to a given first-order difference equation. This is trivially true of θ, which is fixed, and of ω', which evolves exogenously. Let $\mathbb{I}$ denote indicator function, and

$$F\left(x \mid \omega', \Lambda\right) = \int_{\underline{\theta}}^{\overline{\theta}} \mathbb{I}\left\{V\left(\theta, L^{\star}\left(\theta'\right), \omega', \Lambda\right) \leq x\right\} d\Gamma(\theta) \tag{1}$$

$$G\left(x \mid \omega', \Lambda\right) = \frac{1}{\Lambda\left(\overline{\theta}\right)} \int_{\underline{\theta}}^{\overline{\theta}} \mathbb{I}\left\{V\left(\theta, L^{\star}\left(\theta'\right), \omega', \Lambda\right) \leq x\right\} L^{\star}(\theta)\, d\Gamma(\theta) \tag{2}$$

denote, respectively, the (normalized) cross-sectional distribution of values offered by other firms and of values earned by employed workers. These are probabilities, respectively, that another randomly chosen firm offers less than value x and that a randomly chosen employed worker earns less than value x. Let $\overline{F} = 1 - F$. Firm size L then evolves according to

$$L_{+} = L\left(1 - \delta^{\omega'}\right)\left(1 - \lambda_1^{\omega'}\overline{F}\left(W \mid \omega', \Lambda\right)\right) + \lambda_0^{\omega'}\left(1 - \Lambda\left(\overline{\theta}\right)\right) + \lambda_1^{\omega'}\Lambda\left(\overline{\theta}\right) G\left(W \mid \omega', \Lambda\right). \tag{3}$$

Next-period employment at a firm offering W results from the sum of three terms. First, a fraction of the L existing workers are not exogenously separated and do not receive an outside offer better

[5] The reason why the relevant state on which the contract choice is conditioned is ω' and not ω is that firms post values after observing ω' and before knowing how many workers will arrive and accept.

than W. Second, some of the $1-\Lambda\left(\overline{\theta}\right)$ unemployed workers are hired by the firm. Finally, workers employed elsewhere receive an outside offer from the firm under consideration with probability $\lambda_1^{\omega'}$ and this offer yields a value W that is better than what they are currently earning with probability $G\left(W\mid\omega',\Lambda\right)$. One can verify by inspection of (3) that the evolution of firm size indeed only depends on the state $\mathbf{s}=\{\theta,L,\omega',\Lambda\}$ given the current offer W: $L_+:=\mathcal{L}\left(\mathbf{s},W\right)$ with "chance" $H\left(d\omega'\mid\omega\right)$.

Finally, in equilibrium, the size $L^\star\left(\theta'\right)=\Lambda'\left(\theta'\right)/\gamma\left(\theta'\right)$ of any other firm θ' offering $V\left(\theta',L^\star\left(\theta'\right),\omega',\Lambda\right)$ evolves according to that same law of motion $\mathcal{L}$, so that , integrating over θ', we obtain a law of motion for the distribution of employment:

$$\Lambda_+\left(\theta\right)=\int_{\underline{\theta}}^{\theta}\mathcal{L}\left(\theta',L^\star\left(\theta'\right),\omega',\Lambda,V\left(\theta',L^\star\left(\theta'\right),\omega',\Lambda\right)\right)d\Gamma\left(\theta'\right) \tag{4}$$

with chance $H\left(d\omega'\mid\omega\right)$ before the state ω' is drawn by nature. This establishes that $\Lambda_+=\mathcal{T}\left(\omega',\Lambda\right)$, which satisfies our requirement as $\left(\omega',\Lambda\right)$ is a subset of $\mathbf{s}$. This concludes the proof that $\mathbf{s}$ is Markovian given a value offered by the firm in each ω'.

3.2 The firm's contract-posting problem

To show that the optimal value offer made by firm θ in response to the strategy $V\left(\mathbf{s}\right)$ is indeed a function of $\mathbf{s}$ only, we write the optimal contract-posting problem as one of choosing the current wage and continuation values $W\left(\omega'\right)$ which could, in principle, depend also on states other than aggregate productivity ω':

$$\begin{aligned}\Pi\left(\theta,L,\omega,\Lambda,\overline{W}\right)=\max_{w,W(\cdot)}\Big\langle&\left(\omega\theta-w\right)L\\&+\beta\cdot\int_{\Omega}\Pi\left(\theta,\omega',\mathcal{L}\left(\theta,L,\omega',\Lambda,W\left(\omega'\right)\right),\mathcal{T}\left(\omega',\Lambda\right),W\left(\omega'\right)\right)H\left(d\omega'\mid\omega\right)\Big\rangle\end{aligned}$$

subject to a worker participation constraint and a promise-keeping constraint to deliver at least the promised $\overline{W}$:[6, 7]

$$W\left(\omega'\right)\geq U\left(\omega',\mathcal{T}\left(\omega',\Lambda\right)\right) \tag{5}$$

$$\begin{aligned}&\max\left\{U\left(\omega,\Lambda\right),\overline{W}\right\}\leq w+\beta\cdot\int_{\Omega}\Bigg\{\delta^{\omega'}U\left(\omega',\Lambda\right)+\left(1-\delta^{\omega'}\right)\cdot\\&\left[\left(1-\lambda_1^{\omega'}\overline{F}\left(W\left(\omega'\right)\mid\omega',\Lambda\right)\right)W\left(\omega'\right)+\lambda_1^{\omega'}\int_{W(\omega')}^{\infty}vdF\left(v\mid\omega',\Lambda\right)\right]\Bigg\}H\left(d\omega'\mid\omega\right)\end{aligned} \tag{6}$$

These constraints must be appended to the firm's Bellman equation with their associated Lagrange multipliers, $m^{\omega'}$ and π^{ω}, respectively. The value of unemployment U only depends on aggregate

[6] Plus, possibly, a minimum wage constraint, ignored here.

[7] Note that we are ignoring the possibility for a firm optimally to want to shut down in certain states of nature.

state variables, because it solves:

$$U(\omega,\Lambda)=b+\beta\int\left[\lambda_0^{\omega'}\int_{\underline{\theta}}^{\overline{\theta}}V\left(\theta,\omega',L^\star(\theta),\Lambda\right)d\Gamma(\theta)+\left(1-\lambda_0^{\omega'}\right)U\left(\omega',\mathcal{T}\left(\omega',\Lambda\right)\right)\right]H\left(d\omega'\mid\omega\right) \tag{7}$$

Writing down the Lagrangian for the above DP problem and taking the first-order condition w.r.t. the wage w, one immediately obtains:

$$\pi^\omega=L,$$

which implies that the promise-keeping constraint binds at all dates.[8] This fact has two implications. First, by the envelope theorem, $\pi^\omega=-\Pi_W\left(\theta,L,\omega,\Lambda,\overline{W}\right)$, so that $\Pi_W=-L$ and we can define the joint value of the firm-worker collective as a function independent of the value promised to the worker:

$$S(\theta,L,\omega,\Lambda):=\Pi\left(\theta,L,\omega,\Lambda,\overline{W}\right)+\overline{W}L.$$

Second, we can approach the firm's problem with the promise-keeping constraint (6) holding as an equality and substitute it into the Bellman equation. Substituting the definitions of S above and $\mathcal{L}(\cdot)$ in (3), the firm's problem becomes

$$\begin{aligned}S(\theta,L,\omega,\Lambda)=\omega\theta L+\beta\cdot\int_\Omega\Bigg\{&\delta^{\omega'}U\left(\omega',\mathcal{T}\left(\omega',\Lambda\right)\right)L\\&+\max_{W(\omega')}\Bigg\langle S\left(\theta,\omega',\mathcal{L}\left(\theta,L,\omega',\Lambda,W\left(\omega'\right)\right),\mathcal{T}\left(\omega',\Lambda\right)\right)+L\left(1-\delta^{\omega'}\right)\lambda_1^{\omega'}\int_{W(\omega')}^{\infty}vdF\left(v\mid\omega',\Lambda\right)\\&-W\left(\omega'\right)\left(\lambda_0^{\omega'}\left(1-\Lambda\left(\overline{\theta}\right)\right)+\lambda_1^{\omega'}\Lambda\left(\overline{\theta}\right)G\left(W\left(\omega'\right)\mid\omega',\Lambda\right)\right)\Bigg\rangle\Bigg\}H\left(d\omega'\mid\omega\right).\end{aligned} \tag{8}$$

subject to the constraint (5) for each state ω', with Lagrange multiplier $m^{\omega'}$.

3.3 Optimality conditions and equilibrium definition

The first-order conditions are, for each state ω' (using the definition of $\mathcal{L}(\cdot)$ again and using subscripts to denote partial derivatives):

$$\begin{aligned}&\lambda_0^{\omega'}\left(1-\Lambda\left(\overline{\theta}\right)\right)+\lambda_1^{\omega'}\Lambda\left(\overline{\theta}\right)G\left(W\left(\omega'\right)\mid\omega',\Lambda\right)\\&\quad=\left[S_L\left(\theta,\omega',\mathcal{L}\left(\theta,L,\omega',\Lambda,W\left(\omega'\right)\right),\mathcal{T}\left(\omega',\Lambda\right)\right)-W\left(\omega'\right)\right]\\&\quad\times\left[\left(1-\delta^{\omega'}\right)\lambda_1^{\omega'}Lf\left(W\left(\omega'\right)\mid\omega',\Lambda\right)+\lambda_1^{\omega'}\Lambda\left(\overline{\theta}\right)g\left(W\left(\omega'\right)\mid\omega',\Lambda\right)\right]-m^{\omega'}\end{aligned} \tag{9}$$

[8] Except possibly for start-up firms that have $L=0$ initially, but then for those the concept of a promised value inherited from the past is moot and the choice of w is irrelevant.

with complementary slackness $m^{\omega'}\left[W\left(\omega'\right)-U\left(\omega',\mathcal{T}\left(\omega',\Lambda\right)\right)\right]$. The Envelope condition w.r. to firm size writes down as:

$$\beta^{-1}S_L\left(\theta,L,\omega,\Lambda\right)=\beta^{-1}\omega\theta+\int_{\Omega}\left\{\delta^{\omega'}U\left(\omega',\mathcal{T}\left(\omega',\Lambda\right)\right)+\left(1-\delta^{\omega'}\right)\lambda_1^{\omega'}\int_{W(\omega')}^{\infty}vdF\left(v\mid\omega',\Lambda\right)\right.$$
$$\left.+S_L\left(\theta,\omega',\mathcal{L}\left(\theta,L,\omega',\Lambda,W\left(\omega'\right)\right),\mathcal{T}\left(\omega',\Lambda\right)\right)\left(1-\delta^{\omega'}\right)\left(1-\lambda_1^{\omega'}\overline{F}\left(W\left(\omega'\right)\mid\omega',\Lambda\right)\right)\right\}H\left(d\omega'\mid\omega\right). \tag{10}$$

Finally, a Transversality Condition (TVC) requires that the discounted value of the collective vanishes in expectation w.r. to the stochastic path of ω

$$\lim_{t\to\infty}E\left[\beta^t S_L\left(\mathbf{s}_t\right)L_t|\mathbf{s}_0\right]=0. \tag{11}$$

This set of optimality conditions depends only on $\mathbf{s}=\left(\theta,L,\omega',\Lambda\right)$. Therefore, the value optimally offered by the firm after each realization of ω' is indeed a function of the state $\mathbf{s}=\left(\theta,L,\omega',\Lambda\right)$, as claimed, and we can proceed to construct an equilibrium where contracts depend only on $\mathbf{s}$.

An *equilibrium* of this economy is a pair of functions (S,V) of the state $\mathbf{s}=\left(\theta,L,\omega',\Lambda\right)$ such that $S\left(\mathbf{s}\right)$ and $W\left(\omega'\right)=V\left(\mathbf{s}\right)$ solve the system (7), (9), (10) subject to (1), (2), the TVC (11), a given initial condition for Λ, that we assume atomless, thus for the corresponding density Λ', with firm size $L=L^{\star}\left(\theta\right)=\Lambda'\left(\theta\right)/\gamma\left(\theta\right)$ solving (3). Notice that this definition implies that the value promised to these initial workers is the same that would be promised in the future to an equal number of workers employed at the same firm in the same aggregate state ω,Λ.

Once equilibrium has been found, the wages that implement it are backed out from the binding promise-keeping constraint (6) evaluated at $V\left(\mathbf{s}\right)$.

4 An equivalent continuous-time economy

We now consider the limit of the optimality conditions (9), (10) derived in the previous section in an approximately equivalent continuous time economy. We then show how those continuous-time-limit conditions can be derived as the optimality conditions in a continuous-time version of the firm's dynamic programming problem (8). While it is more common to cast optimal contracting problem with limited commitment in discrete time, the continuous-time approximation affords more transparent economics of the optimality conditions, and makes our contribution more directly comparable with the literature, almost invariably cast in continuous time.

4.1 The continuous-time limit of optimality conditions

Specifically, let Δ be a (small) period length and further define the discount factor $\beta=\left(1+r\Delta\right)^{-1}$ for some $r>0$, flow productivity $\omega\theta\Delta$, value of leisure $b\Delta$, offer probabilities $\lambda_i^{\omega}\Delta$ and $\delta^{\omega}\Delta$.

To save on notation, we denote the continuous time equivalent objects λ_i^ω, δ^ω, $\omega\theta$, b, all positive scalars, by the same symbols as their discrete time counterparts, although they have to be correctly reinterpreted. We further assume that the process governing the aggregate state is a continuous time stationary Markov process such that $\Pr(\omega_{t+\Delta} = \omega' \mid \omega_t = \omega, t)$ is independent of t for all (ω, ω'), and that the process has an infinitesimal generator A^ω, such that the limit

$$A^\omega[\phi] = \lim_{\Delta \downarrow 0} E\left[\frac{\phi(\omega_{t+\Delta}) - \phi(\omega_t)}{\Delta} \mid \omega_t = \omega\right]$$

exists for all ω and appropriately well-behaved functions ϕ (the specific class depends on the specific process.). Classic examples are a simple (conditional) Poisson process: $\Pr(\omega_{t+\Delta} = \omega' \mid \omega_t = \omega) = \sigma(\omega' \mid \omega) \cdot \Delta + o(\Delta)$, where $A^\omega[\phi] = \sigma(\omega' \mid \omega)[\phi(\omega') - \phi(\omega)]$ for all ϕ, and a diffusion $d\omega = \mu(\omega)\,dt + \sigma(\omega)\,dz_t$, where z_t is Wiener and $A^\omega[\phi] = \mu(\omega)\phi'(\omega) + \sigma^2(\omega)\phi''(\omega)/2$ for all $\phi \in \mathbb{C}^2$.

Under these assumptions, the process for employment solves the counterpart of (3), which converges to the following ODE as $\Delta \to 0$:

$$dL = \left[-\left(\delta^\omega + \lambda_1^\omega \overline{F}(W|\omega,\Lambda)\right)L + \lambda_0^\omega u + \lambda_1^\omega(1-u)G(W|\omega,\Lambda)\right]dt \tag{12}$$

where unemployment is $u = 1 - \Lambda(\overline{\theta})$, employment is $1 - u = \Lambda(\overline{\theta})$, and the value distributions F and G depend on ω and Λ. Thus, employment L and the size distribution Λ are continuous in time as $\Delta \to 0$. Notice that in continuous time the state is $(\theta, \omega, L, \Lambda)$ rather than $(\theta, \omega', L, \Lambda)$ as in discrete time, because the difference between ω and ω' becomes immaterial in continuous time, any jump in ω cannot be anticipated by the players.

We let

$$\mu(\theta, L, \omega', \Lambda) = S_L(\theta, L, \omega', \Lambda)$$

denote the shadow joint marginal value of an additional unit of labor to the firm-worker collective. Then, as $\Delta \to 0$, the FOC for an optimal value offer W, adapted from (9), reads

$$\lambda_0^\omega u + \lambda_1^\omega(1-u)G(W|\omega,\Lambda) = \lambda_1^\omega[\mu(\theta,\omega,L,\Lambda) - W][Lf(W|\omega,\Lambda) + (1-u)g(W|\omega,\Lambda)] - m^\omega. \tag{13}$$

The Euler equation — which is the continuous-time counterpart of the Envelope condition (10) — becomes:

$$\begin{aligned}&\left(r + \delta^\omega + \lambda_1^\omega\overline{F}(W|\omega,\Lambda)\right)\mu(\theta,\omega,L,\Lambda)\,dt \\ &\quad = A^\omega[\mu(\theta,\omega,L,\Lambda)]\,dt + \mu_L dL + \partial_\Lambda\mu + \left[\omega\theta + \delta^\omega U(\omega,\Lambda) + \lambda_1^\omega\int_W^\infty v dF(v|\omega,\Lambda)\right]dt\end{aligned} \tag{14}$$

where $\partial_\Lambda\mu$ is a shorthand for the effect of the (continuously) changing size distribution on the shadow marginal value of employment. Specifically, in equilibrium, at each θ the population cdf of employment solves

$$\frac{d\Lambda(\theta)}{dt} = \frac{d}{dt}\int_{\underline{\theta}}^{\theta} \Lambda'(\theta')\, d\theta' = \int_{\underline{\theta}}^{\theta} \frac{d\Lambda'(\theta')}{dt} d\theta'$$
$$= \int_{\underline{\theta}}^{\theta} \big[-\left(\delta^{\omega} + \lambda_1^{\omega}\overline{F}\left(V\left(\theta',\omega,L^{\star}(\theta'),\Lambda\right)|\omega,\Lambda\right)\right)L^{\star}(\theta')$$
$$+ \lambda_0^{\omega}u + \lambda_1^{\omega}(1-u)\,G\left(V\left(\theta',\omega,L^{\star}(\theta'),\Lambda\right)|\omega,\Lambda\right)\big]d\Gamma(\theta'),$$

where $V(\theta,\omega,L,\Lambda)$ is the equilibrium value-offer function, such that $W = V(\theta,\omega,L^{\star}(\theta),\Lambda)$ must solve the FOC and Euler equation for all θ.

Finally, the TVC reads

$$\lim_{t\to\infty} E\left[e^{-rt}\mu(\mathbf{s}_t)\,L_t|\mathbf{s}_0\right] = 0. \tag{15}$$

4.2 The firm's optimization problem in continuous time

Consider now the following continuous-time Dynamic Programming problem of choosing an optimal flow wage to maximize the present discounted value of profits Π, that is the value of the firm, taking into account the effect of wages on hiring and retention given wage offers by other firms:

$$r\Pi(\theta,\omega,L,\Lambda,W)\,dt = \max_{w}(\omega\theta - w)\,Ldt$$
$$+ A^{\omega}\left[\Pi(\theta,\omega,L,\Lambda,W)\right]dt + \Pi_L(\theta,\omega,L,\Lambda,W)\,dL + \partial_{\Lambda}\Pi(\theta,\omega,L,\Lambda,W)$$

subject to

$$rWdt = \left[w + A^{\omega}[W] - \delta^{\omega}(W - U(\omega,\Lambda)) + \lambda_1^{\omega}\int_{W}^{+\infty}(x-W)\,dF(x|\omega,\Lambda)\right]dt + W_L dL + \partial_{\Lambda}W$$
$$dL = \left[-\left(\delta^{\omega} + \lambda_1^{\omega}\overline{F}(W|\omega,\Lambda)\right)L + \lambda_0^{\omega}u + \lambda_1^{\omega}(1-u)\,G(W|\omega,\Lambda)\right]dt$$
$$W \geq U, \text{ and } L_0, W_0 \text{ given}$$

The discounting cost on total profits Π equals the maximized cash flow plus continuation value, which includes the effects of changing aggregate productivity state, firm size, and firm size distribution. The wage translates into a value W for the worker, which has to be at least as large as the value of unemployment U for the worker to participate.[9] In turn, this value translates into a change in the firm size through the usual turnover, exogenous to and from unemployment and endogenous to and from other firms. Note that the shorthand ∂_{Λ} was used here again to denote the effect on

[9]Note that the value of unemployment solves in continuous time:

$$rU(\omega,\Lambda) = b + \lambda_0^{\omega}\left(\int_{\underline{\theta}}^{\overline{\theta}} V(\theta,\omega,L^{\star}(\theta),\Lambda)\,d\Gamma(\theta) - U(\omega,\Lambda)\right) + AU(\omega,\Lambda) + \partial_{\Lambda}U(\omega,\Lambda).$$

values of the continuously evolving state variable Λ. Because Λ is an uncontrolled state variable, which is exogenous to any individual firm, it would pose no particular problem to subsume it in a pure effect of time (which has the advantage of being a one-dimensional state variable). We believe, however, that making the dependence of the firm's problem on Λ explicit, while costly in notation terms, helps making the exposition clearer.

Denoting by $w\left(\theta', \omega, L, \Lambda\right)$ the equilibrium wage policy and plugging it into the worker value equation, together with the firm size dynamic equation we can solve for the equilibrium value $V\left(\theta', \omega, L\left(\theta'\right), \Lambda\right)$ and firm size $L\left(\theta'\right)$ for all other firms θ'. Then $\dot{\Lambda}\left(\theta\right) = \int^{\theta} \dot{L}\left(\theta'\right) d\Gamma\left(\theta'\right)$ describes the equilibrium evolution of the size distribution Λ, thus of the value distributions F and G, that firm θ takes as given. This is the equilibrium consistency (fixed point) condition.

To solve this continuous time DP problem for the best response contract, we ignore the evolution of the exogenous state variable Λ (and U), as unaffected by the firm's choice, append the two constraints to the RHS of the HJB equation, with Lagrange multipliers π^{ω} on the promise-keeping constraint, namely, the equation for W, with a sign convention so that $\pi^{\omega} = -\Pi_W$. Taking a FOC for the wage, we obtain again $L = \pi^{\omega}$. Therefore, this continuous time problem also delivers the decomposition of profits into a collective surplus minus a value promised to workers $\Pi\left(\theta, \omega, L, \Lambda, W\right) = S\left(\theta, \omega, L, \Lambda\right) - WL$. Substituting these equalities in the DP, we obtain an equivalent problem:

$$\begin{aligned} rS\left(\theta, \omega, L, \Lambda\right) dt = \max_{W \geq U(\omega,\Lambda)} \left[\omega\theta + \delta^{\omega} U\left(\omega, \Lambda\right) + \lambda_1^{\omega} \int_W^{\infty} v dF\left(v\right) + A^{\omega}\left[S\left(\theta, \omega, L, \Lambda\right)\right]\right] L dt \\ + S_L\left(\theta, \omega, L, \Lambda\right) dL + \partial_{\Lambda} S\left(\theta, \omega, L, \Lambda\right), \quad (16) \end{aligned}$$

subject to (12). The Lagrange multiplier on this constraint is $\mu^{\omega} = S_L$ and must be equal to $\pi^{\omega} + W$. With this notation, the FOC for W, the Euler equation (derived from the Envelope condition for L in problem (16) and the TVC are none other than (13), (14) and (15). Therefore, the discrete time contract-posting problem is approximately equivalent, in the $\Delta-$parameterization for Δ small, to a continuous time DP problem of wage posting.

Finally, this continuous-time version of the firm's problem encompasses the deterministic economy studied in our companion paper MPV08, which is obtained from (16) by fixing ω forever, i.e. setting $A^{\omega}\left[\cdot\right] = 0$. We shall return to the deterministic case below.

5 Rank-Preserving Equilibrium (RPE)

Solving for equilibrium directly is an intractable problem, because the size distribution of firms Λ is an infinitely-dimensional state variable. In this section we present a strategy to circumvent that problem.

5.1 Definition and basic properties of a RPE

We define a tractable and natural class of equilibria, which have the following property:

Definition 1 (Rank-Preserving Property) *An equilibrium is Rank-Preserving if equilibrium worker values are such that* $\theta \mapsto V(\theta, \omega, L^{\star}(\theta), \Lambda)$ *is increasing in* θ*. That is a more productive firm always pays its workers more.*

We use this restriction on the equilibrium set to simplify the calculations involved in solving for equilibrium in the stochastic model. We then prove that, in a perfect foresight "deterministic" model in which aggregate productivity ω is not expected to change, but firm sizes are not necessarily stationary, under some weak sufficient conditions on the initial size distribution or discounting all equilibria must have this property. Since the aggregate stochastic dynamics must closely resemble the deterministic ones for small aggregate shocks, this finding provides support to the relevance of RPE in general. Finally, we relate the asymptotic steady state RPE of the deterministic model to the steady state equilibrium of the Burdett and Mortensen (1998) model.

A direct consequence of the above definition is that in a RPE workers rank firms according to productivity at all dates. The following two properties hold true at all dates under the RP assumption: the proportion of firms that offer less than θ is simply that proportion of firms that are less productive than θ

$$F(V(\theta, \omega, L^{\star}(\theta), \Lambda) \mid \omega, \Lambda) \equiv \Gamma(\theta),$$

and the number of employed workers who earn a value that is lower than that offered by θ equals employment at firms less productive than θ:

$$(1-u)\, G(V(\theta, \omega, L^{\star}(\theta), \Lambda) \mid \omega, \Lambda) = \Lambda(\theta).$$

Therefore:

$$f(V(\theta, \omega, L^{\star}(\theta), \Lambda) \mid \omega, \Lambda) = \frac{dF}{dV} = \frac{dF/d\theta}{dV/d\theta} = \frac{\gamma(\theta)}{dV/d\theta}$$

and

$$(1-u)\, g(V(\theta, \omega, L^{\star}(\theta), \Lambda) \mid \omega, \Lambda) = \frac{dG}{dV} = \frac{dG/d\theta}{dV/d\theta} = \frac{\Lambda'(\theta)}{dV/d\theta} = \frac{L^{\star}(\theta)\, \gamma(\theta)}{dV/d\theta}.$$

Notice that the same conditions hold in discrete time if we replace, given our timing assumptions, ω with the new state of aggregate productivity ω'. We derive their implications in continuous time due to its greater notational simplicity, but the same derivations would hold in discrete time after appropriate changes.

5.2 Evolution of the firm size distribution in RPE

In a given aggregate state ω, firm sizes evolve following (12). In a RPE, in which firm size equals $L^{\star}(\theta) = \Lambda'(\theta)/\gamma(\theta)$, (12) reads as:

$$d\Lambda'(\theta) = \gamma(\theta)\left\{-\left[\delta^{\omega} + \lambda_1^{\omega}\overline{\Gamma}(\theta)\right]\frac{\Lambda'(\theta)}{\gamma(\theta)} + \lambda_0^{\omega}\left[1 - \Lambda\left(\overline{\theta}\right)\right] + \lambda_1^{\omega}\Lambda(\theta)\right\}dt$$

which is a Partial Differential Equation in Λ, a function of time and θ. We can solve it forward from any initial condition for any realization of the history of ω, independently of wages and offered values. Indeed integration with respect to θ yields:

$$d\Lambda(\theta) = \left\{\lambda_0^{\omega}u\Gamma(\theta) - \left[\delta^{\omega} + \lambda_1^{\omega}\overline{\Gamma}(\theta)\right]\Lambda(\theta)\right\}dt. \tag{17}$$

Now suppose that the state ω has an almost everywhere constant sample path. For any initial condition $\Lambda_0(\theta)$ at some (renormalized) initial date $t = 0$ such that the aggregate state remained at ω between 0 and t, the last law of motion is an ODE which solves as:

$$\Lambda_t(\theta) = e^{-\left[\delta^{\omega}+\lambda_1^{\omega}\overline{\Gamma}(\theta)\right]t}\left(\Lambda_0(\theta) + \lambda_0^{\omega}\Gamma(\theta)\int_0^t u_s e^{\left[\delta^{\omega}+\lambda_1^{\omega}\overline{\Gamma}(\theta)\right]s}ds\right), \tag{18}$$

where $u_s = 1 - \Lambda_s\left(\overline{\theta}\right)$ is the unemployment rate at date s, which solves the simple first-order ODE $\dot{u}_s = \delta^{\omega}(1 - u_s) - \lambda_0^{\omega}u_s$ over $[0, t]$ with initial condition $u_0 = 1 - \Lambda_0\left(\overline{\theta}\right)$. Next differentiating with respect to θ, one obtains a closed-form expression for the workforce of any type-θ firm:

$$\begin{aligned} L_t^{\star}(\theta) &= \frac{\Lambda_t'(\theta)}{\gamma(\theta)} \\ &= e^{-\left[\delta^{\omega}+\lambda_1^{\omega}\overline{\Gamma}(\theta)\right]t}\left[\frac{\Lambda_0'(\theta)}{\gamma(\theta)} + \lambda_1^{\omega}t\Lambda_0(\theta) + \lambda_0^{\omega}\int_0^t\left[1 + \lambda_1^{\omega}(t-s)\Gamma(\theta)\right]u_s e^{\left[\delta^{\omega}+\lambda_1^{\omega}\overline{\Gamma}(\theta)\right]s}ds\right] \end{aligned} \tag{19}$$

where $L_0^{\star}(\theta)$ was the value of this solution under state ω' at the time of the last state switch from ω' to ω.

If instead ω evolves continuously, as for example a diffusion process, then (17) describes a stochastic law of motion, as λ_i^{ω} and δ^{ω} potentially change continuously and stochastically in time.

In either case, the steady-state versions of (18) and (19) (assuming the aggregate state forever stays at ω) are:

$$L_{\infty}^{\star}(\theta) = \frac{\delta^{\omega}\lambda_0^{\omega}\left(\delta^{\omega} + \lambda_1^{\omega}\right)}{\left(\delta^{\omega} + \lambda_0^{\omega}\right)\left[\delta^{\omega} + \lambda_1^{\omega}\overline{\Gamma}(\theta)\right]^2} \quad \text{and} \quad \Lambda_{\infty}(\theta) = \frac{\delta^{\omega}\lambda_0^{\omega}\Gamma(\theta)}{\left(\delta^{\omega} + \lambda_0^{\omega}\right)\left[\delta^{\omega} + \lambda_1^{\omega}\overline{\Gamma}(\theta)\right]}, \tag{20}$$

which are the familiar steady-state expressions found in the BM model.[10]

[10] Incidentally, this is the point at which the necessity for sampling weights appears. Note from equation (20) that the steady-state size ratio of the largest to the smallest firm in the market in units of (non-normalized) employment is $L_{\infty}^{\star}\left(\overline{\theta}\right)/L_{\infty}^{\star}\left(\underline{\theta}\right) = \left(1 + \lambda_1/\delta\right)^2$, a ratio which is in the order of 25-30 given standard estimates of λ_1 and δ. Now of course the data counterpart of that size ratio is virtually infinite. It appears that the BM model requires a sampling distribution that is very heavily skewed toward high-productivity firms in order to replicate the observed distribution of firm sizes.

Before going any further into characterizing Rank-Preserving Equilibria, we should notice that the analysis of firm size and employment dynamics carried out in this paragraph would apply to any job ladder model in which a similar concept of RPE can be defined. Indeed nothing in the dynamics of $L_t^\star$ or Λ_t depends on the particulars of the wage setting mechanism, so long as this is such that employed jobseekers move from lower-ranking into higher-ranking jobs in the sense of a time-invariant ranking. Therefore, this model's predictions about everything relating to firm sizes are in fact much more general than the wage- (or value-) posting assumption retained in the BM model.

5.3 Further characterization of RPE

The FOC (13) is solved by $W = V\left(\theta, \omega, L^\star\left(\theta\right), \Lambda\right)$:

$$\begin{aligned}\lambda_0^\omega u + \lambda_1^\omega \Lambda\left(\theta\right) &= \lambda_1^\omega \left(\mu - W\right)\left[L^\star\left(\theta\right) f\left(W\right) + \left(1-u\right) g\left(W\right)\right] - m^\omega \\ &= 2\lambda_1^\omega \frac{L^\star\left(\theta\right)\gamma\left(\theta\right)}{dW/d\theta}\left(\mu - W\right) - m^\omega\end{aligned}$$

with complementary slackness $m^\omega\left(W-U\right)=0$.

The Euler equation (14) reads:

$$\left(r + \delta^\omega + \lambda_1^\omega \overline{\Gamma}\left(\theta\right)\right)\mu = A^\omega\left[\mu\right] + \mu_L \frac{\dot{\Lambda}'\left(\theta\right)}{\gamma\left(\theta\right)} + \mu_\Lambda \dot{\Lambda}\left(\theta\right) + \omega\theta + \delta^\omega U\left(\omega, \Lambda\right) + \lambda_1^\omega \int_W^\infty v dF\left(v\right) \tag{21}$$

where dots indicate time derivatives, and now the shadow marginal value $\mu\left(\theta, \omega, L^\star\left(\theta\right), \Lambda\right)$ depends on the distribution of employment Λ only through the distribution of employment across firms up to θ, $\Lambda\left(\theta\right)$ and the corresponding density $L^\star\left(\theta\right)$. Both are scalars, and the notation μ_Λ (or μ_L) makes sense as a partial derivative. In fact, the state reduces from $\left(\theta, \omega', L, \Lambda\right)$, which is infinite-dimensional due to the relevance of the entire firm size distribution Λ, to the four-dimensional vector $\mathbf{s} = \left(\theta, \omega', L, \Lambda\left(\theta\right)\right)$: in order to make its decisions, the firm only needs to know the mass of employment at less productive firms and not the entire size distribution Λ.

In addition to considerably simplifying equilibrium determination, the RP assumption is theoretically appealing for at least two more reasons. First, it parallels a well-known property of the static equilibrium characterized by BM, which is to have a unique equilibrium where workers rank firms according to productivity. Second, RPE feature constrained-efficient labor reallocation at all dates: if workers consistently rank more productive firms higher than less productive ones, then job-to-job moves will always be up the productivity ladder.

5.4 A strategy to solve for for stochastic RPE

For any given specification of the stochastic process for ω, we can tackle the solution of the RPE. For the sake of illustration, we focus on the Poisson example.

As long as the state ω does not change, Λ evolves deterministically. Therefore, in RPE we can subsume the effects of the state Λ on μ into that of time. With a slight abuse of notation:

$$\dot{\mu}_t(\theta,\omega) = \mu_L(\theta,\omega,L^\star(\theta),\Lambda(\theta))\,\dot{L}^\star(\theta) + \mu_\Lambda(\theta,\omega,L^\star(\theta),\Lambda(\theta))\,\dot{\Lambda}(\theta)$$

and the Euler equation reads

$$\begin{aligned}\dot{\mu}_t(\theta,\omega) = {} & \left(r+\delta^\omega+\lambda_1^\omega\overline{\Gamma}(\theta)\right)\mu_t(\theta,\omega) \\ & - \sigma^\omega\left(\mu_t(\theta,\omega') - \mu_t(\theta,\omega)\right) - \omega\theta - \delta^\omega U_t(\omega) - \lambda_1^\omega \int_{V_t(\theta,\omega,L^\star(\theta),\Lambda(\theta))}^{\infty} v dF(v)\end{aligned}$$

Further let

$$\pi_t(\theta,\omega) = \mu_t(\theta,\omega) - V_t(\theta,\omega)$$

and take a derivative w.r. to θ on both sides of the Euler equation

$$\frac{\partial\dot{\mu}_t(\theta,\omega)}{\partial\theta} = \left(r+\delta^\omega+\lambda_1^\omega\overline{\Gamma}(\theta)\right)\frac{\partial\mu_t(\theta,\omega)}{\partial\theta} - \sigma^\omega\left(\frac{\partial\mu_t(\theta,\omega')}{\partial\theta} - \frac{\partial\mu_t(\theta,\omega)}{\partial\theta}\right) - \omega - \lambda_1^\omega\gamma(\theta)\,\pi_t(\theta,\omega). \tag{22}$$

Together with the FOC for an interior solution of the promised value

$$\frac{\partial\pi_t(\theta,\omega)}{\partial\theta} = \frac{\partial\mu_t(\theta,\omega)}{\partial\theta} - \frac{2\lambda_1^\omega\Lambda'(\theta)}{\lambda_0^\omega\left[1-\Lambda(\overline{\theta})\right]+\lambda_1^\omega\Lambda(\theta)}\pi_t(\theta,\omega) \tag{23}$$

this gives a system of four PDEs in $\pi_t(\theta,\omega), \partial\mu(\theta,\omega)/\partial\theta$ in θ and t, a pair for each value of ω.

The main difficulty in solving this system lies in the interdependence of $\partial\mu_t(\theta,\omega)/\partial\theta$ and $\partial\mu_t(\theta,\omega')/\partial\theta$, that is, on the jump in the shadow marginal value of one worker, differentiated across firms, when the aggregate state switches. To get around this problem, we can approximate that "jump term" by a known function J (e.g. polynomials) of the state variables $\Lambda(\theta)$ and $L^\star(\theta)$, depending on a finite vector of unknown coefficients, $\mathbf{a}$ in the following fashion:

$$\sigma^\omega\left(\frac{\partial\mu_t(\theta,\omega')}{\partial\theta} - \frac{\partial\mu_t(\theta,\omega)}{\partial\theta}\right) \simeq J\left(\Lambda(\theta), L^\star(\theta) \mid \mathbf{a}\right).$$

Given a specific vector of coefficients $\mathbf{a}$, system (22,23), together with the transversality condition (15), becomes a pair of independent systems of PDEs, one for each aggregate state, which can be separately numerically solved over the infinite future for any initial value of $(\Lambda(\cdot), L^\star(\cdot))$ using the algorithm described in MPV08.

We thus proceed in the following steps:

0. Pick an initial state of the economy $(\omega_0, \Lambda_0(\cdot), L_0^\star(\cdot))$ and simulate a path of ω. Denote switching dates as $(s_1, s_2, \dots)$.

1. Fix a parameter $\mathbf{a}$.

2. Given the choice of **a** made at step 1 and the implied J-function, solve (22,23,15) using the appropriate initial conditions. More specifically:

   (a) Solve (22,23,15) with initial condition $(\Lambda_0(\cdot), L_0^\star(\cdot))$ as if state ω_0 prevailed forever. This implies certain values for $\partial\mu_0(\theta,\omega_0)/\partial\theta$, $\partial\mu_{s_1}(\theta,\omega_0)/\partial\theta$ and $\left(\Lambda_{s_1}(\cdot), L_{s_1}^\star(\cdot)\right)$ at date s_1 when the next aggregate shock occurs.

   (b) Solve (22,23,15) with initial condition $\left(\Lambda_{s_1}(\cdot), L_{s_1}^\star(\cdot)\right)$ as if state ω_1 prevailed over $t \in [s_1, +\infty)$. This implies certain values for $\partial\mu_{s_1}(\theta,\omega_1)/\partial\theta$, $\partial\mu_{s_2}(\theta,\omega_1)/\partial\theta$ and $\left(\Lambda_{s_2}(\cdot), L_{s_2}^\star(\cdot)\right)$.

   (c) Solve (22,23,15) with initial condition $\left(\Lambda_{s_2}(\cdot), L_{s_2}^\star(\cdot)\right)$ as if state ω_2 prevailed over $t \in [s_2, +\infty)$, etc. That is, repeat step 2, mutatis mutandis, for the first K jumps in ω (how many jumps depends on the dimension of the vector **a**).

3. The simulations performed at stage 3 provide a vector of jumps in $\partial\mu/\partial\theta$:

$$\left[\frac{\partial\mu_{s_k}(\theta,\omega')}{\partial\theta} - \frac{\partial\mu_{s_k}(\theta,\omega)}{\partial\theta}\right], \quad k = 1, \cdots, K.$$

   Compare those with the jumps predicted from the initially chosen function $J(\cdot \mid \mathbf{a})$ and the simulated path of $(\Lambda(\cdot), L^\star(\cdot))$. If different, update **a** and start over at step 1.[11]

The proposed algorithm is akin to a projection method to solve the system of PDEs that characterize equilibrium. Its specific feature is that projection is only used to approximate the jumps in $\partial\mu/\partial\theta$ caused by aggregate shocks, the rest of the system being solved "exactly".

Once the algorithm has converged, it produces a solution for $\{\Lambda(\cdot), L^\star(\cdot), \partial\mu_t(\cdot)/\partial\theta, \pi_t(\cdot)\}$ over $t \in [0, +\infty)$ given the initially simulated sequence of aggregate states. Wages are finally retrieved from:

$$w_t(\theta,\omega) = \omega\theta - \left(r + \delta + \lambda_1\overline{\Gamma}(\theta)\right)\pi_t(\theta,\omega) + \dot{\pi}_t(\theta,\omega).$$

5.5 Preliminary results [in progress]

We calibrate parameters as in our MPV08 companion paper, and the aggregate productivity process to reflect the average duration of booms and recessions in the US. Figures 1-5 illustrate the results of some representative simulations. The results do not differ qualitatively from those of the transitional dynamics illustrated in MPV08.

The model predicts procyclical wages and EE rate, although the wage jumps in a direction opposite to labor productivity when aggregate shocks hit. As documented in MPV08, the growth

[11] Exactly how **a** is updated depends on the chosen functional form for the approximate jump function J. In practice we use a projection on a base of polynomials, and the updated vector of coefficients **a** is obtained by regression of the "simulated jumps" $\partial\mu_{s_k}(\theta,\omega')/\partial\theta - \partial\mu_{s_k}(\theta,\omega)/\partial\theta$ on the monomial elements of J.

rate differential of employment at (initially) large minus small firms collapses upon a recession and rises slowly through an expansion, reflecting the slow upgrading of workers to better jobs through on the job search. This upgrading also strongly smoothes and propagates the effects of the aggregate labor productivity shock on measured average labor productivity. Reallocation across firms is quantitatively important.

6 RPE transitional dynamics with no aggregate shocks

6.1 Main results

In light of the tractability of Rank-Preserving Equilibria, it is natural to ask how general they are. Specifically: under which conditions is every equilibrium Rank-Preserving? So far, we have obtained results in this sense for the case where the aggregate state ω is fixed for ever, but the initial distribution of employment Λ_0 is not stationary (as it is in BM), thus Λ_t evolves as the economy makes it transitional dynamics to the eventual steady state. Although special, this case does feature aggregate dynamics and is significantly more complex than the steady state on which the literature has, by and large, focused so far. Also, transitional dynamics, and its RPE properties, provide a good approximation to stochastic dynamics when the aggregate state ω is slow-moving.

As ω is fixed, we omit it from the state vector and normalize it to unity. Clearly, in the perfect foresight equilibrium, the firm size distribution Λ_t, thus the value of unemployment U_t and value distributions F_t, G_t, evolve smoothly over time. Therefore, we can subsume their effects on the firm's problem into time, and recast wage posting as an optimal control problem, subject to an initial promised value and firm size, which are the only two relevant "state variables".

Specifically, with $A^\omega[\cdot]$, the DP programming problem is equivalent to the following control problem. Firms post wage time profiles over an infinite horizon to solve

$$\Pi_0(L_0;\theta) = \max_{\{w_t\}} \int_0^{+\infty} (\theta - w_t) L_t e^{-rt} dt \tag{24}$$

$$\text{subject to:} \quad rW_t = \dot{W}_t + w_t - \delta(W_t - U_t) + \lambda_1 \int_{W_t}^{+\infty} (x - W_t)\, dF_t(x) \tag{25}$$

$$\dot{L}_t = -\left[\delta + \lambda_1 \overline{F}_t(W_t)\right] L_t + \lambda_0 u_t + \lambda_1 (1-u_t) G_t(W_t) \tag{26}$$

$$W_t \geq U_t, \;\; L_0, u_t, F_t, G_t \text{ given} \tag{27}$$

It is easy to show that the optimality conditions are the same as in the DP continuous time problem with $A^\omega[\cdot]$: (13), (14), and the TVC (15) without expectation operator. The solution is now a wage and resulting value functions that depend only on θ and time.

Given the definition of a RPE spelled out above, from the comparative dynamics (w.r. to θ) of the solution of this control problem we establish two propositions.

Proposition 1 (Ranked Initial Firm Size Implies Rank-Preserving Equilibrium) *If the aggregate productivity state ω is constant and the initial state of the economy is such that L_0 is non decreasing in θ (i.e. higher-θ firms are no smaller), then any equilibrium of the dynamic value-posting game is necessarily Rank-Preserving.*

The proof of this first proposition is in Appendix (A). It builds on Caputo's (2003) comparative dynamic characterization of optimal controls in infinite horizon problems, which itself is based on the second-order condition of the primal-dual problem corresponding to (24).

This Proposition has a simple economic intuition, thus it appears to be a robust conclusion. In BM's steady state model, more productive firms offer higher wages due to a single-crossing property of their steady state profits, which in turn reflects two very basic economic forces. First, a higher wage implies a larger firm size, as a more generous offer makes it easier to poach workers and to fend off competition. Second, a larger firm size is more valuable to a more productive firm, because each worker produces more. Therefore, by a simple monotone comparative statics argument, it must be the case that more productive firms offer more, employ more workers, and earn higher profits. Simply put, a productive firm can afford paying more, and is willing to do so to attract workers, because its opportunity cost of not producing is higher. Key to this argument is the fact that firm size is an endogenous object, and BM look for an appropriate firm size distribution which guarantees a stationary allocation.

In our dynamic model, firm size is a state variable, and its *initial* value is a parameter of the model, arbitrarily fixed , not an endogenous object. Therefore, in order to get a start on monotone comparative statics, it is sufficient (but not necessary) that the initial size distribution shares the key property of BM's steady state distribution; namely, it is increasing in productivity. In the proof, we begin by invoking Theorem 2 of Caputo (2003), which in this case is equivalent to a single-crossing property of the Hamiltonian of the value-posting problem. Given a ranked initial size, a more productive firm still wants and can afford to pay more, now in terms of values accruing to workers. The initial ranking of sizes by productivities is preserved throughout, so values offered to workers remain ranked by firm productivity at all points in the future, even if the firm were to stop and re-optimize. This condition is only sufficient. We conjecture that it is not necessary, and we are exploring this issue.

Proposition 2 (Rank-Preserving Stationary Allocations) *For r in a neighborhood of zero, the set of necessary optimality conditions for problem* (24) *has a unique steady-state symmetric solution which is "rank-preserving" in the sense that:*

- *the steady-state worker value $V_\infty(\theta)$ is non decreasing in θ;*
- *steady-state firm size $L_\infty(\theta)$ is non decreasing in θ.*

The proof is in Appendix (B). This latter result should not come as a surprise to those familiar with the BM model: if $r \to 0$, then firms only care about steady-state profits and our initially dynamic optimization problem becomes confounded with the static BM problem, which has a unique solution that is RP in the sense indicated in Proposition (2).

Taken together, Propositions (1) and (2) are statements about the generality of RPE. Specifically these propositions establish that RPE are generic within the set of dynamic equilibria such that there is no dispersion in firm size among firms of a common type θ. Note that steady-state symmetric equilibria are necessarily in that class—as can be seen from the steady-state version of (26) which shows that steady-state firm size only depends on the value offered in steady-state, $V_\infty(\theta)$. The flip side of those arguments is that the RP property can transitionally break down because of the entry of new firms. For example an entrant firm with a productivity level somewhere in the interior of Γ's support and an initial size of zero might be tempted to break ranks in one direction or the other, depending on the shape of $f_t(\cdot)$ and $g_t(\cdot)$. With this caveat in mind, we now proceed to a characterization of RPE.

6.2 Wage contracts

We now go back to the dynamical system characterizing the behavior of the typical individual firm, and analyze it in a RPE. Ignoring the worker participation constraint, the system becomes

$$\left(\lambda_0 u_t + \lambda_1 \int_{\underline{p}}^{\theta} L_t(x)\, d\Gamma(x)\right) V_t'(\theta) = 2\lambda_1 \gamma(\theta) L_t(\theta) \left(\mu_t(\theta) - V_t(\theta)\right) \tag{28}$$

$$\dot{\mu}_t(\theta) = \left(r + \delta + \lambda_1 \overline{\Gamma}(\theta)\right) \mu_t(\theta) - \lambda_1 \int_{\theta}^{+\infty} V_t(x)\, d\Gamma(x) - \delta U_t - \theta \tag{29}$$

$$\lim_{t \to +\infty} e^{-rt} \mu_t(\theta) = 0. \tag{30}$$

Differentiation of (29) w.r.t. θ yields:

$$\dot{\mu}_t'(\theta) = \left(r + \delta + \lambda_1 \overline{\Gamma}(\theta)\right) \mu_t'(\theta) - \lambda_1 \gamma(\theta) \left(\mu_t(\theta) - V_t(\theta)\right) - 1. \tag{31}$$

This, together with (28) and the definition of the firm's shadow value of the marginal worker $\pi_t(\theta) := \mu_t(\theta) - V_t(\theta)$, gives the following system of two PDEs in $(\mu_t'(\theta), \pi_t(\theta))$:

$$\begin{aligned} \dot{\mu}_t'(\theta) &= R(\theta)\, \mu_t'(\theta) + R'(\theta)\, \pi_t(\theta) - 1 \\ \mu_t'(\theta) &= \pi_t'(\theta) + B_t(\theta)\, \pi_t(\theta). \end{aligned} \tag{32}$$

where $R(\theta) := r + \delta + \lambda_1 \overline{\Gamma}(\theta)$ is the effective discount factor of the firm, and

$$B_t(\theta) := \frac{2\lambda_1 \gamma(\theta) L_t(\theta)}{\lambda_0 u_t + \lambda_1 \int_{\underline{p}}^{\theta} L_t(x)\, d\Gamma(x)}.$$

The system (32) can be solved numerically subject to some initial and boundary conditions. 'Initial' conditions are given by the steady-state solution to (32), which is characterized as:

$$\mu'_{\infty}(\theta) = \frac{1+\lambda_1\gamma(\theta)\pi_{\infty}(\theta)}{r+\delta+\lambda_1\overline{\Gamma}(\theta)}$$
$$\pi_{\infty}(\theta) = \frac{\left[\delta+\lambda_1\overline{\Gamma}(\theta)\right]^2}{r+\delta+\lambda_1\overline{\Gamma}(\theta)}\left(\int_{\underline{p}}^{\theta}\frac{dx}{\left[\delta+\lambda_1\overline{\Gamma}(x)\right]^2}+\frac{\pi_{\infty}\left(\underline{p}\right)(r+\delta+\lambda_1)}{(\delta+\lambda_1)^2}\right). \tag{33}$$

Now turning to boundary conditions, standard arguments prove that the lowest-type firms have no reason to pay more than the minimum wage: type $\underline{p}$ firms can only hire from unemployment and lose workers to poachers anyway, so trying to prevent poaching by raising wages is pointless for those firms in a RPE. While this implies that the minimum wage constraint (27) will bind at all dates for the lowest-type firm, it also implies that the following (time-invariant) boundary conditions are satisfied:

$$\pi_t\left(\underline{p}\right) \equiv \frac{\underline{p}-\underline{w}}{r+\delta+\lambda_1}$$
$$\mu'_t\left(\underline{p}\right) \equiv \frac{1+\lambda_1\gamma\left(\underline{p}\right)\pi_t\left(\underline{p}\right)}{r+\delta+\lambda_1}, \tag{34}$$

where the second condition is obtained by combining the first one with the $\dot{\mu}'_t(\theta)$ equation in (32). These boundary conditions can be further simplified by imposing $\underline{p}=\underline{w}$, a kind of free-entry condition holding throughout the adjustment toward the new steady state, which implies $\pi_t\left(\underline{p}\right)\equiv 0$. The minimum productivity $\underline{p}$ that can survive in the market is $\underline{w}$, as any firm with $\theta > \underline{w}$ can make positive profits by offering $\underline{w}$, and possibly even more by offering a higher wage while no firm with $\theta < \underline{w}$ can ever make any profits.

We note that (32) can also be written more compactly as one PDE in the firm's shadow value of the marginal worker:

$$\frac{\partial^2\pi_t(\theta)}{\partial t\partial\theta}+\frac{\partial}{\partial t}\left[B_t(\theta)\pi_t(\theta)\right] = \frac{\partial}{\partial\theta}\left[R(\theta)\pi_t(\theta)\right]+R(\theta)B_t(\theta)\pi_t(\theta)-1. \tag{35}$$

Once either the PDE in (35) is solved for $\pi_t(\theta)$ or (32) is solved for $(\mu'_t(\theta),\pi_t(\theta))$, wages can be retrieved from:

$$w_t(\theta) = \theta-\left(r+\delta+\lambda_1\overline{\Gamma}(\theta)\right)\pi_t(\theta)+\dot{\pi}_t(\theta),$$

which has the following familiar steady-state solution:

$$w_{\infty}(\theta) = \theta-\left(\delta+\lambda_1\overline{\Gamma}(\theta)\right)^2\left(\int_{\underline{p}}^{\theta}\frac{dx}{\left(\delta+\lambda_1\overline{\Gamma}(x)\right)^2}+\frac{\underline{p}-\underline{w}}{(\delta+\lambda_1)^2}\right). \tag{36}$$

This is exactly the BM solution for the heterogeneous firm case (see equation (47) in Burdett and Mortensen, 1998). This confirms that our contracts are consistent with the BM steady-state wage-posting equilibrium *if the labor market is at a steady state*. It is no longer the case off steady-state, however: posting a time-invariant wage is not, in general,[12] a firm's best response to all other firms posting time-invariant wages.[13]

We now look back to the worker participation constraint. The only firm for which this can be binding at the steady state characterized above is the lowest-type firm, $\underline{p}$. It may be the case, however, that the constraint temporarily binds for some higher-type firms over the transition to that steady state, in which case the economy no longer behaves according to (32). In our companion paper, MPV08, we describe an algorithm that constructs an equilibrium in which we impose a minimum wage constraint that makes the participation constraint always slack, and that is allowed to temporarily bind for some firms (at the lower end of the θ-distribution) with the restriction that it only bind over some initial period.

The aforementioned numerical algorithm can be used to numerically solve (32) and simulate our model's dynamic equilibrium to study its quantitative properties. This is the objective pursued in MPV08. In the appendix, we show that if workers and firms discount payoffs at different rates, and workers are myopic (care only about flow wages), then the only equilibrium entails constant wages independently of the evolution of unemployment and firm sizes. This suggests that a major force behind aggregate dynamics is the wage backloading that firms always strive to attain, leveraging the forward-looking preferences of their employees.

As for the present theoretical analysis, we thus conclude our equilibrium characterization with the following two remarks. First, as we already mentioned, our constructive characterization of a RPE implies uniqueness within that class of equilibria. Second, our analysis still leaves two open questions: (conditions for) existence of an equilibrium, which in the RP case reduces to the relatively tractable problem of existence of a solution to a system of PDEs; and the much harder issue of equilibrium play when our sufficient conditions for RP fails.

6.3 Homogeneous firms

The original motivation of the search literature exemplified by BM is to explain the large cross-sectional variation in worker wages that remains even after controlling for observable and unob-

[12] A pedagogically interesting exception is the case of myopic workers ($\rho = +\infty$), fully characterized and discussed in the Appendix.

[13] To see this, notice that the worker's value equation, the FOC and the Euler equation yield two distinct growth rates for $\pi_t(\theta) = \mu_t(\theta) - V_t(\theta)$ if all wages are constant and the economy is off its steady state (so that firm sizes change over time). Under constant wages, the same equations give a $\pi_t(\theta)$ which evolves as an exponential of time. But then with a constant wage and constant wages offered elsewhere, $V_t(\theta)$ is constant over time, so dividing (28) by $L_t(\theta)$ tells us that $\pi_t(\theta)$ is proportional to the gross hiring rate, and so it cannot be exponential in time (because the hiring rate is not an exponential function of time in a RPE). All this implies that posting a constant wage in the face of competitors themselves posting constant wages violates the firm's set of necessary optimality conditions.

servable worker and firm characteristics. The hallmark of the BM research program is to produce a robust failure of the law of one price, whereas identical agents make identical trades a different prices. In the light, the most striking and meaningful version of the BM model is the simplest setting where all firms and workers are identical. BM prove that the unique equilibrium must be in asymmetric strategies and entail wage dispersion among identical workers.

The proof of Proposition (1) can be adapted to show that, in this case, whenever we start with a continuous size distribution $L_0(\theta)$ with no atoms, equilibrium is always RP. The logic is simple. Now θ plays the role of the rank in the initial $L_0(\theta)$. An initially larger firm wants to offer more to its workers, although they produce just as much as anywhere else, because the retention effect of a more generous offer is more powerful the larger firm size. Since there exists always a ranking of firms when $L_0(\theta)$ has no atoms, then the RP always holds in equilibrium.

6.4 Comparison with BM's steady state analysis

The exact relationship between our and BM's analyses is subtle and an explicit comparison is now in order. In our approach, a choice of parameters $L_0(\theta)$, λ_0, δ implies a unique initial unemployment rate u_0 and stationary unemployment rate $\overline{u}$:

$$u_0 = 1 - \int L_0(x)\, d\Gamma(x) \lesseqgtr \frac{\delta}{\delta + \lambda_0} = \overline{u}$$

Notice the initial distribution of employment $L_0(\theta)$, thus the initial unemployment rate, are parameters that can be fixed arbitrarily. That is, they are given initial conditions, inherited somehow from past history. In contrast, BM assume steady state: given δ and λ_0, they *choose* (solve for) the only possible employment distribution $L^{BM}(\theta) = L_0(\theta)$ which replicates itself and thus makes the initial unemployment rate stationary: $u_0 = \overline{u}$. This equilibrium distribution $L^{BM}(\theta)$ turns out to be RP.

If we also impose steady state, after solving for the optimal time-dependent contracts, we get the same employment distribution as BM: we need to choose $L_0(\theta) = L^{BM}(\theta)$ to obtain stationarity. That is, a best-response time-dependent wage is constant if the economy is in steady state. However, if we fix $L_0(\theta)$ arbitrarily, so generically the economy is not in steady state $u_0 \neq \overline{u}$, but moves, then no equilibrium can have constant wages: the best-response to constant wages paid by all firms is not a constant wage when unemployment changes over time.

Finally, if r is small and thus close (but not necessarily identical) to the $r = 0$ assumed by BM, in any equilibrium of our model the size distribution converges necessarily to a RP stationary limit $L_\infty(\theta)$, which is the same as BM's steady state $L_\infty(\theta) = L^{BM}(\theta)$.

6.5 Vacancy creation [in progress]

So far, we have taken the arrival rates of offers λ_0^ω and λ_1^ω as given constants, given the current state of aggregate labor productivity ω. This shuts down potentially an important part of labor demand, the endogenous change in labor market tightness when unemployment and the size distribution of firms evolve over time conditional on a state. Also, we return to the issue of sampling weights. We remarked in a note that all the previous analysis goes through if one reinterprets γ as the sampling-weighted density of firms, the product of a population density (how many firms of type θ) and a sampling density (the chance of drawing an offer from a firm θ). This reformulation, though, works directly for time-invariant sampling weights.

A matching technology with vacancy posting can take care of both restrictions: it endogenizes and makes time-invariant the arrival rates of offers and the sampling weights. We now pursue this extension in the transitional dynamics model, where ω is fixed. We discover that, due to the slow-moving size distribution of firms, the job market tightness (vacancy/unemployment ratio) that determines matching rates does not jump immediately to its steady state, as it does in standard search-and-matching bargaining models. Therefore, the question of how offer arrival rates change over time is meaningful even in the absence of aggregate shocks. We address it in the contest of RPE, keeping in mind that we have not extended the previous two propositions to this case, thus the RP restriction binds. We leave the extension to vacancy posting with aggregate shocks for future research.

Suppose that a firm θ needs to post vacancies in order to hire. Suppose that posting v_t vacancies has a flow cost $c(v_t)$, where c is convex and smooth. Let $v_t(\theta)$ be the measure of vacancies posted by $\theta-$ firms at time t. Let $\overline{v}_t = \int_{\underline{p}}^{\overline{\theta}} v_t(\theta)\, d\Gamma(\theta)$ denote the aggregate stock of vacancies.[14] Suppose that a random CRS matching function m mediates meetings between unemployed and employed job searchers with open vacancies, where the latter type of job seeker have a relative search intensity of η. Then contact rates are endogenous and time-dependent:

$$\lambda_{0t} = \frac{m\left(u_t + \eta\left(1-u_t\right), \overline{v}_t\right)}{u_t + \eta\left(1-u_t\right)} \qquad \text{and} \qquad \lambda_{1t} = \eta\lambda_{0t},$$

and so are sampling weights: $q_t(\theta) = v_t(\theta)/\overline{v}_t$.[15] Note that $\Phi_t(\theta) = \int_{\underline{p}}^{\theta} q_t(x)\, d\Gamma(x)$ remains a proper cdf at all points in time.

The choice of vacancies must be added to the control problem. While it cannot affect the arrival rate of offers to workers, as each firm is too small to make itself easily visible, it does affect the chance that the job application will land on a particular firm type, namely, the sampling distribution. Indeed the law of motion of employment at a type-θ firm, given an implemented path

[14] To avoid cluttering the notation, we normalize the total mass of firms, N, at 1 throughout this section. Amending this normalization is straightforward.

[15] Note that we are treating both types of job seekers as perfectly substitutable inputs into the matching process (up to the constant relative search intensity σ). A possible alternative would have been to model vacancies as completely directed toward a given type of job seeker, with separate search markets and different matching functions for employed and unemployed workers. While the available evidence seems to buttress the latter option (Van Ours, 1995), we have opted for a theoretically slightly simpler route.

of worker values and posted vacancies $\{W_t, v_t\}_{t\geq 0}$ writes down as:

$$\dot{L}_t(\theta) = -\left[\delta + \lambda_{1t}\overline{F}_t(W_t)\right] L_t(\theta) + \frac{v_t}{\overline{v}_t} \cdot \left[\lambda_{0t} u_t + \lambda_{1t}(1-u_t) G_t(W_t)\right]. \tag{37}$$

The current-value Hamiltonian of the modified firm's problem, which the firm now maximizes with respect to the pair of control variables (W_t, v_t), now writes as:[16]

$$\begin{aligned}\mathcal{H}_t(\theta) = &\left[\theta + \lambda_{1t}\int_{W_t}^{+\infty} x dF_t(x) + \delta U_t\right] L_t(\theta) - W_t \cdot \frac{v_t}{\overline{v}_t} \cdot \left[\lambda_{0t} u_t + \lambda_{1t}(1-u_t) G_t(W_t)\right] \\ &- c(v_t) + \mu_t(\theta)\left\{-\left[\delta + \lambda_{1t}\overline{F}_t(W_t)\right] L_t(\theta) + \frac{v_t}{\overline{v}_t} \cdot \left[\lambda_{0t} u_t + \lambda_{1t}(1-u_t) G_t(W_t)\right]\right\}.\end{aligned}$$

Optimality conditions are:

$$c'(v_t(\theta)) = \left[\mu_t(\theta) - W_t(\theta)\right] \cdot \frac{1}{\overline{v}_t} \cdot \left[\lambda_{0t} u_t + \lambda_{1t}(1-u_t) G_t(W_t)\right] \tag{38}$$

$$\begin{aligned}&\frac{v_t(\theta)}{\overline{v}_t} \cdot \left[\lambda_{0t} u_t + \lambda_{1t}(1-u_t) G_t(W_t(\theta))\right] \\ &\qquad = \lambda_{1t}\left[\mu_t(\theta) - W_t(\theta)\right]\left[f_t(W_t(\theta)) L_t(\theta) + (1-u_t) g_t(W_t(\theta)) \frac{v_t(\theta)}{\overline{v}_t}\right]\end{aligned} \tag{39}$$

$$\dot{\mu}_t(\theta) = \left[r + \delta + \lambda_{1t}\overline{F}_t(W_t(\theta))\right]\mu_t(\theta) - \left[\theta + \lambda_{1t}\int_{W_t(\theta)}^{+\infty} x dF_t(x) + \delta U_t\right] \tag{40}$$

$$\lim_{t\to+\infty} e^{-rt}\mu_t(\theta) L_t(\theta) = 0. \tag{41}$$

We now focus on RPE with endogenous vacancies. Such equilibria are characterized by the set of conditions derived in the previous subsection, together with the condition $W_t'(\theta) > 0$ for all (t, θ), which now implies:

$$F_t(W_t(\theta)) = \int_{\underline{p}}^{\theta} \frac{v_t(x)}{\overline{v}_t} d\Gamma(x) = \Phi_t(\theta) \qquad \text{and} \qquad (1-u_t) G_t(W_t(\theta)) = \int_{\underline{p}}^{\theta} L_t(x) d\Gamma(x).$$

First substituting into the law of motion of $L_t(\theta)$, (37), and proceeding as we did in Subsection (5.2), it is easy to establish the following:

$$\int_{\underline{p}}^{\theta} L_t(x) d\Gamma(x) = e^{-\int_0^t [\delta + \lambda_{1s}\overline{\Gamma}_s(\theta)] ds}\left(\int_{\underline{p}}^{\theta} L_0(x) d\Gamma(x) + \int_0^t \lambda_{0s} u_s \Phi_s(\theta) e^{\int_0^s [\delta + \lambda_{1\tau}\overline{\Gamma}_\tau(\theta)] d\tau} ds\right), \tag{42}$$

which can be differentiated w.r.t. θ to obtain an expression for $L_t(\theta)$.

Now turning to what the RP assumption implies for the necessary conditions (38)-(40) and re-introducing the firm's shadow value of the marginal worker $\pi_t(\theta) := \mu_t(\theta) - W_t(\theta)$, some

[16] Again to streamline the following derivations we focus on the case of equal discount rates, $r = \rho$. This is inconsequential.

straightforward algebra leads to:

$$c'\left(v_t\left(\theta\right)\right)=\frac{\pi_t\left(\theta\right)}{\overline{v}_t}\left(\lambda_{0t}u_t+\lambda_{1t}\int_{\underline{p}}^{\theta}L_t\left(x\right)d\Gamma\left(x\right)\right) \tag{43}$$

$$\mu_t'\left(\theta\right)=\pi_t'\left(\theta\right)+\frac{2\lambda_{1t}L_t\left(\theta\right)\gamma\left(\theta\right)\pi_t\left(\theta\right)}{\lambda_{0t}u_t+\lambda_{1t}\int_{\underline{p}}^{\theta}L_t\left(x\right)d\Gamma\left(x\right)} \tag{44}$$

$$\dot{\mu}_t'\left(\theta\right)=\left(r+\delta+\lambda_{1t}\overline{\Gamma}_t\left(\theta\right)\right)\mu_t'\left(\theta\right)-\lambda_{1t}\gamma\left(\theta\right)\frac{v_t\left(\theta\right)}{\overline{v}_t}\pi_t\left(\theta\right)-1. \tag{45}$$

Note that (45) is a differentiated version of (40) (following the same steps as in Subsection (6.2), and that (44) can be slightly simplified by appealing to the relationship between contact rates, $\lambda_{1t}=\eta\lambda_{0t}$.

Dynamic RPE are characterized by the solution to the above system of PDEs, ((43)-(45), combined with (42) and the law of motion of unemployment, $\dot{u}_t=-\lambda_{0t}u_t+\delta\left(1-u_t\right)$. While there is little hope to derive a closed-form solution to that system, a simple fixed-point algorithm can be used to obtain numerical solutions. Details of that algorithm are given in Appendix (C).

7 Conclusions [TBA]

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Appendix

A Proof of Proposition (1)

The wage-posting control problem (24) subject to (25), (26) and (27) can be simplified into a value-posting problem subject only to (26)and (27). Solve for the wage from (25), replace it into (24) and integrate by parts w.r. to time to obtain the following equivalent value-posting control problem starting from any time:

$$\begin{aligned}\Pi_t(\theta, L_t) = \max_{\{W_s\}} \int_t^{+\infty} & \left\{ \left[\omega\theta + \lambda_1 \int_{W_s}^{+\infty} x dF_s(x) + \delta U_s \right] L_s \right. \\ & \left. - W_s \left[\lambda_0 u_s + \lambda_1 (1 - u_s) G_s(W_s) \right] \right\} e^{-r(s-t)} ds \qquad (46)\end{aligned}$$

$$\text{subject to: } \dot{L}_s = -\left(\delta + \lambda_1 \overline{F}_s(W_s)\right) L_s + \lambda_0 u_s + \lambda_1 (1 - u_s) G_s(W_s) \qquad (47)$$

$$L_t \text{ given.} \qquad (48)$$

The proof of Proposition (1) involves an application of Caputo's (2003) comparative dynamic results.[17] In order to apply those results we let $W_t^\star(L_t; \theta)$ denote the closed-loop optimal control to problem (46), and we also take up Caputo's (2003) notation $D_{xy}[W_t^\star(L_t; \theta), (L_t, t; \theta)]$ for the (x, y) element of the Hessian matrix of the primal-dual problem associated with (46) (see Caputo, 2003, equations 14-23 and Theorem 2).

We begin by establishing two intermediate results, from which the proposition will follow.

Lemma 1 *Along the optimal path, for all $t \geq 0$:*

$$\frac{\partial W_t^\star}{\partial \theta}(L_t; \theta) \geq 0 \qquad \text{and} \qquad \frac{\partial^2 \Pi_t}{\partial \theta \partial L_t}(L_t; \theta) > 0.$$

Proof. We apply Theorem 2 in Caputo (2003), which is a statement of the second-order necessary condition for the primal-dual problem corresponding to (θ_a). That second-order condition implies, inter alia, the following:

$$-D_{\theta\theta}[W_t^\star(L_t; \theta), (L_t, t; \theta)] \equiv \frac{\partial W_t^\star}{\partial \theta} \cdot \frac{\partial^2 \Pi_t}{\partial \theta \partial L_t} \cdot [\lambda_1 f_t(W_t^\star) L_t + \lambda_1 (1 - u_t) g_t(W_t^\star)] \geq 0. \qquad (49)$$

Application of the Dynamic Envelope Theorem (e.g. Caputo, 1990) next establishes that:

$$\frac{\partial \Pi_t}{\partial \theta}(L_t; \theta) = \int_t^{+\infty} L_s e^{-r(s-t)} ds,$$

implying:

$$\frac{\partial^2 \Pi_t}{\partial \theta \partial L_t}(L_t; \theta) = \int_t^{+\infty} \frac{\partial L_s}{\partial L_t} e^{-r(s-t)} ds. \qquad (50)$$

The proof of the proposition is then completed by establishing the following:

[17] Although the control problem under consideration is nonautonomous, it can be reexpressed as an autonomous problem by treating time as an additional predetermined state variable τ_s such that $\dot{\tau}_s = 1$ and $\tau_t = t$. This is the generic type of problem analyzed by Caputo (2003).

Claim: $\partial L_s / \partial L_t > 0$ *for* $s \geq t$ *along the optimal path.*

To prove this claim, we go back to the law of motion of L_s along the optimal path which is given by:

$$L_s = -\left(\delta + \lambda_1 \overline{F}_s \left(W_s^\star \left(L_s; \theta\right)\right)\right) L_s + \lambda_0 u_s + \lambda_1 \left(1 - u_s\right) G_s \left(W_s^\star \left(L_s; \theta\right)\right).$$

Differentiation w.r.t. L_t yields:

$$\begin{aligned}\frac{\partial L_s}{\partial L_t} &= \left\{ -\left(\delta + \lambda_1 \overline{F}_s \left(W_s^\star\right)\right) + \left[\lambda_1 f_s \left(W_s^\star\right) L_s + \lambda_1 \left(1 - u_s\right) g_s \left(W_s^\star\right)\right] \cdot \frac{\partial W_s^\star}{\partial L_s} \right\} \cdot \frac{\partial L_s}{\partial L_t} \\ &\equiv \Psi_s \left(W_s^\star, L_s\right) \cdot \frac{\partial L_s}{\partial L_t}.\end{aligned}$$

Given the initial condition $\partial L_s / \partial L_t = 1$ at $s = t$, the differential equation above can be rewritten as:

$$\frac{\partial L_s}{\partial L_t} = \exp \int_t^s \Psi_x \left(W_x^\star \left(L_x; \theta\right), L_x\right) dx > 0. \tag{51}$$

□

Thus $\partial W_t^\star / \partial \theta$ has the same sign as $\partial^2 \Pi_t / \partial \theta \partial L_t$.

An initially larger firm will stay larger for ever, for given productivity. The reason is that, given the same productivity, future hiring and retention would differ between the two firms only because the size differential itself could affect the values they offer to their workers, so an initial differential can never be "undone" by employment's smooth dynamics. The two paths of employment can never cross. Next, by CRS in production, size is the marginal return to the firm-worker collective from a higher productivity θ. Therefore, higher productivity raises the marginal value to the collective of having one more worker today, which translates into a larger size for ever. Hence, a higher productivity leads the firm to optimally raise the value it currently offers to its workers, in order to increase retention and hiring and gain that additional worker now, and a larger size for ever.

Lemma (1) is not sufficient to establish that the RP property must hold in equilibrium: for that we need to determine how the optimal $W_t^\star$ responds to differences in the state variable, i.e. firm size (L_t). This is what the next proposition is about.

Lemma 2 *Along the optimal path, for all* $t \geq 0$*:*

$$\frac{\partial W_t^\star}{\partial L_t} \left(L_t; \theta\right) \geq 0 \qquad \textit{and} \qquad \frac{\partial^2 \Pi_t}{\partial L_t^2} \left(L_t; \theta\right) > 0.$$

Proof. We apply Theorem 2 in Caputo (2003) again, which also implies:

$$\begin{aligned} - D_{LL} \left[W_t^\star \left(L_t; \theta\right), \left(L_t, t; \theta\right)\right] \equiv \frac{\partial W_t^\star}{\partial L_t} \cdot \Bigg\{ & \lambda_1 f_t \left(W_t^\star\right) \cdot \left[\frac{\partial \Pi_t}{\partial L_t} - W_t^\star\right] \\ & + \frac{\partial^2 \Pi_t}{\partial L_t^2} \cdot \left[\lambda_1 f_t \left(W_t^\star\right) L_t + \lambda_1 \left(1 - u_t\right) g_t \left(W_t^\star\right)\right] + \left(r - \rho\right) \Bigg\} \geq 0. \end{aligned} \tag{52}$$

The term in {} measures the effect of a marginal change in the offered value on the costate variable, which is the shadow value $\partial \Pi_t / \partial L_t = \mu_t$ of the marginal worker to the firm-worker collective (think of the marginal worker as an asset worth μ_t to that match.) This effect on the asset price can be decomposed into a direct impact on the dividend from that marginal worker, which is the reduction in the likelihood of a quit, saving the match an amount $\partial \Pi_t / \partial L_t - W_t^*$ and not the whole $\partial \Pi_t / \partial L_t$ as the worker will no longer accept an outside offer W_t^*, plus the impact on hiring and retention, thus on total firm size, which in turn affects the shadow value of the future marginal worker. This inequality states that the firm, given a larger size, should optimally raise its offer when doing so raises the asset value of the marginal worker, and conversely.

Next, the Euler equation for problem (46) (which can also be viewed as one of the envelope conditions for the HJB equation associated with (46); see e.g. Caputo, 2003) writes down as:[18]

$$\frac{\partial^2 \Pi_t}{\partial L_t \partial t} + \frac{\partial^2 \Pi_t}{\partial L_t^2} \cdot L_t = \left(r + \delta + \lambda_1 \overline{F}_t \left(W_t^\star\right)\right) \frac{\partial \Pi_t}{\partial L_t} - \theta - \lambda_1 \int_{W_t^\star}^{+\infty} x dF_t \left(x\right) - \delta U_t - \left(r - \rho\right) W_t^\star. \quad (53)$$

Differentiating w.r.t. L_t along the optimal path yields:

$$\begin{aligned}
&\frac{\partial}{\partial t} \frac{\partial^2 \Pi_t}{\partial L_t^2} + \frac{\partial}{\partial L_t} \frac{\partial^2 \Pi_t}{\partial L_t^2} \cdot L_t - \left(\delta + \lambda_1 \overline{F}_t \left(W_t^\star\right)\right) \frac{\partial^2 \Pi_t}{\partial L_t^2} \\
&\qquad + \left[\lambda_1 f_t \left(W_t^\star\right) L_t + \lambda_1 \left(1 - u_t\right) g_t \left(W_t^\star\right)\right] \frac{\partial W_t^\star}{\partial L_t} \cdot \frac{\partial^2 \Pi_t}{\partial L_t^2} \\
&\qquad = \left(r + \delta + \lambda_1 \overline{F}_t \left(W_t^\star\right)\right) \frac{\partial^2 \Pi_t}{\partial L_t^2} - \lambda_1 f_t \left(W_t^\star\right) \frac{\partial W_t^\star}{\partial L_t} \cdot \left[\frac{\partial \Pi_t}{\partial L_t} - W_t^\star\right] - \left(r - \rho\right) \frac{\partial W_t^\star}{\partial L_t}.
\end{aligned}$$

Rearranging:

$$\begin{aligned}
\frac{d}{dt} \frac{\partial^2 \Pi_t}{\partial L_t^2} = &\left(r + 2\delta + 2\lambda_1 \overline{F}_t \left(W_t^\star\right)\right) \frac{\partial^2 \Pi_t}{\partial L_t^2} - \frac{\partial W_t^\star}{\partial L_t} \cdot \left\{\lambda_1 f_t \left(W_t^\star\right) \cdot \left[\frac{\partial \Pi_t}{\partial L_t} - W_t^\star\right]\right. \\
&\left. + \frac{\partial^2 \Pi_t}{\partial L_t^2} \cdot \left[\lambda_1 f_t \left(W_t^\star\right) L_t + \lambda_1 \left(1 - u_t\right) g_t \left(W_t^\star\right)\right] + \left(r - \rho\right)\right\}, \quad (54)
\end{aligned}$$

which can be integrated as:

$$\begin{aligned}
\frac{\partial^2 \Pi_t}{\partial L_t^2} = &\int_t^{+\infty} \frac{\partial W_s^\star}{\partial L_s} \cdot \left\{\lambda_1 f_s \left(W_s^\star\right) \cdot \left[\frac{\partial \Pi_s}{\partial L_s} - W_s^\star\right]\right. \\
&\left. + \frac{\partial^2 \Pi_s}{\partial L_s^2} \cdot \left[\lambda_1 f_s \left(W_s^\star\right) L_s + \lambda_1 \left(1 - u_s\right) g_s \left(W_s^\star\right)\right] + \left(r - \rho\right)\right\} \cdot e^{-\int_t^s \left(r + 2\delta + 2\lambda_1 \overline{F}_x \left(W_x^\star\right)\right) dx} ds \geq 0,
\end{aligned}$$

where the positive sign follows from (52). Lemma (2) then follows from (52) again, the positive sign of $\partial^2 \Pi_t / \partial L_t^2$ just established and the fact that $\partial \Pi_t / \partial L_t - W_t^\star > 0$ (which is directly implied by the first order condition for problem (46) and otherwise reflects the fact that any given firm makes a positive profit on its marginal worker). □

We are now in a position to complete the proof of Proposition (1). Going back to the law of motion of L_t along the optimal path for a given firm type θ:

$$L_t = -\left(\delta + \lambda_1 \overline{F}_t \left(W_t^\star \left(L_t; \theta\right)\right)\right) L_t + \lambda_0 u_t + \lambda_1 \left(1 - u_t\right) G_t \left(W_t^\star \left(L_t; \theta\right)\right)$$

[18] Note that $\partial \Pi_t / \partial \ell_t$ is the shadow joint value to the firm and the worker of the marginal match in the specific firm considered, which coincides with μ_t in the main text.

and differentiating w.r.t. θ, we obtain:

$$\begin{aligned}\frac{\partial}{\partial t}\frac{dL_t}{d\theta} &= \left\{-\left(\delta+\lambda_1\overline{F}_t\left(W_t^\star\right)\right)+\left[\lambda_1 f_t\left(W_t^\star\right)L_t+\lambda_1\left(1-u_t\right)g_t\left(W_t^\star\right)\right]\cdot\frac{\partial W_t^\star}{\partial L_t}\right\}\cdot\frac{dL_t}{d\theta}\\ &+\left[\lambda_1 f_t\left(W_t^\star\right)L_t+\lambda_1\left(1-u_t\right)g_t\left(W_t^\star\right)\right]\cdot\frac{\partial W_t^\star}{\partial\theta}\\ &\equiv \Psi_t\left(W_t^\star,L_t\right)\cdot\frac{dL_t}{d\theta}+\left[\lambda_1 f_t\left(W_t^\star\right)L_t+\lambda_1\left(1-u_t\right)g_t\left(W_t^\star\right)\right]\cdot\frac{\partial W_t^\star}{\partial\theta},\end{aligned}\tag{55}$$

which can be integrated as:

$$\frac{dL_t}{d\theta}=\left.\frac{dL_t}{d\theta}\right|_{t=0}\cdot e^{\int_0^t\Psi_x\left(W_x^\star,L_x\right)dx}+\int_0^t\left[\lambda_1 f_s\left(W_s^\star\right)L_s+\lambda_1\left(1-u_s\right)g_s\left(W_s^\star\right)\right]\cdot\frac{\partial W_s^\star}{\partial\theta}\cdot e^{\int_s^t\Psi_x\left(W_x^\star,L_x\right)dx}ds,$$

which is positive from Lemma (1) and the assumption about the initial condition. Firm size $L_t(\theta)$ is thus increasing in θ throughout, which proves the proposition as:

$$\frac{dW_t^\star}{d\theta}=\frac{\partial W_t^\star}{\partial\theta}+\frac{\partial W_t^\star}{\partial L_t}\cdot\frac{dL_t}{d\theta},\tag{56}$$

where all terms are positive from the result above and Lemmas (1) and (2). □

B Proof of Proposition (2)

In this appendix we only prove that a steady-state solution for r not too large is necessarily RP. Uniqueness is established by construction later in the main text. Also in this proof we take up some of the notation introduced in Appendix (A) and drop all time subscripts when alluding to steady-state quantities.

The steady-state version of (55) writes as:

$$\frac{dL}{d\theta}=\frac{\lambda_1 f\left(W^\star\right)L+\lambda_1\left(1-u\right)g\left(W^\star\right)}{\Psi\left(W^\star,L\right)}\cdot\frac{\partial W^\star}{\partial\theta}.$$

Substitution into (56) yields (using the definition of $\Psi(\cdot)$):

$$\frac{dW^\star}{d\theta}=-\frac{\delta+\lambda_1\overline{F}\left(W^\star\right)}{\Psi\left(W^\star,L\right)}\cdot\frac{\partial W^\star}{\partial\theta}.$$

Showing that $\Psi\left(W^\star,L\right)\leq 0$ is therefore necessary and sufficient to prove the proposition (as $\partial W^\star/\partial\theta\geq 0$ from Lemma (1) in Appendix (A). This is what we now do.

Writing (54) in steady-state, we obtain:

$$\frac{\partial^2\Pi}{\partial L^2}\cdot\frac{\partial W^\star}{\partial\theta}=\frac{\partial^2\Pi}{\partial\theta\partial L}\cdot\left(\frac{\partial W^\star}{\partial L}\right)^2\cdot\frac{\lambda_1 f\left(W^\star\right)L+\lambda_1\left(1-u\right)g\left(W^\star\right)}{r+2\delta+2\lambda_1\overline{F}\left(W^\star\right)}.$$

Substitution into (52) (written in steady-state) yields:

$$\begin{aligned}\frac{\partial W^\star}{\partial\theta}\cdot\lambda_1 f\left(W^\star\right)\cdot\left[\frac{\partial\Pi}{\partial L}-W^\star\right]+\frac{\partial^2\Pi}{\partial\theta\partial L}\cdot\left(\frac{\partial W^\star}{\partial L}\right)^2\cdot\frac{\left[\lambda_1 f\left(W^\star\right)L+\lambda_1\left(1-u\right)g\left(W^\star\right)\right]^2}{r+2\delta+2\lambda_1\overline{F}\left(W^\star\right)}+\left(r-\rho\right)\frac{\partial W^\star}{\partial\theta}\\ =\frac{\partial^2\Pi}{\partial L\partial\theta}\cdot\frac{\partial W^\star}{\partial L}\cdot\left[\lambda_1 f\left(W^\star\right)L+\lambda_1\left(1-u\right)g\left(W^\star\right)\right].\end{aligned}\tag{57}$$

Now substituting (51) into (50) and time-differentiating leads to:

$$\frac{\partial}{\partial t}\frac{\partial^2\Pi_t}{\partial\theta\partial L_t}=-1+\left(r-\Psi_t\right)\frac{\partial^2\Pi_t}{\partial\theta\partial L_t},$$

so that in steady state:

$$\frac{\partial^2 \Pi}{\partial\theta\partial L} = \frac{1}{r-\Psi}. \tag{58}$$

First note that from Lemma (1), which establishes that $\partial^2\Pi_t/\partial\theta\partial L_t > 0$ throughout in a dynamic equilibrium, the only consistent steady-state solution thus has $\Psi < r$. Then once again substituting into (57) and rearranging yields the following quadratic equation in Ψ:

$$\Psi^2 - [r + K(r+2\Delta)]\Psi - \Delta(r+\Delta) + rK(r+2\Delta) = 0, \tag{59}$$

where $\Delta := \delta + \lambda_1 \overline{F}(W^\star)$ and $K := \frac{\partial W^\star}{\partial\theta} \cdot \left(r - \rho + \lambda_1 f(W^\star) \cdot \left[\frac{\partial \Pi}{\partial L} - W^\star\right]\right)$. As $r \to 0$, (59) has one strictly positive and one strictly negative root (the product of which equals $-\Delta^2$), the consistent one being the latter from the remark after Equation (58) above. Because all the coefficients of (59) are continuous functions of r, so are its roots, which proves the proposition. □

C Computation of RPE with Endogenous Vacancies

To compute a dynamic equilibrium with endogenous vacancies, we proceed as follows:

Step 1. Guess an initial path of vacancies in each firm, $\left\{v_t^{(i)}(p)\right\}_{t\geq 0, p\in[\underline{p},\overline{p}]}$. Given that initial guess and the (given) state of the economy at time 0, construct $\overline{v}_t^{(i)} = \int_{\underline{p}}^{\overline{p}} v_t^{(i)}(p)\, d\Gamma(p)$ and compute the time path of unemployment from:

$$\dot{u}_t^{(i)} = \delta\left(1 - u_t^{(i)}\right) - m\left(u_t^{(i)} + \sigma\left(1 - u_t^{(i)}\right), \overline{v}_t^{(i)}\right) \cdot \frac{u_t^{(i)}}{u_t^{(i)} + \sigma\left(1 - u_t^{(i)}\right)},$$

and deduce the time paths for contact rates, $\lambda_{1t}^{(i)} = \sigma\lambda_{0t}^{(i)} = m\left(u_t^{(i)} + \sigma\left(1 - u_t^{(i)}\right), \overline{v}_t^{(i)}\right) / \left(u_t^{(i)} + \sigma\left(1 - u_t^{(i)}\right)\right)$ for $t \geq 0$.

Next construct the time path for the sampling distribution: $\Phi_t^{(i)}(p) = \int_{\underline{p}}^{p} \frac{v_t^{(i)}(x)}{\overline{v}_t^{(i)}} d\Gamma(x)$ and use all those initial guesses in the RHS of (42) to solve for the implied time path of firm sizes, $\left\{L_t^{(i)}(p)\right\}_{t\geq 0, p\in[\underline{p},\overline{p}]}$.

Step 2. Use the time paths obtained at Step 1 to substitute in the RHS of (44) and (45), and solve the resulting system of PDEs for $\left\{\pi_t^{(i)}(p), \mu_t'^{(i)}(p)\right\}_{t\geq 0, p\in[\underline{p},\overline{p}]}$. This is achieved using a solution algorithm similar to the one described in MPV08.

Step 3. Finally solve (43) for vacancies:

$$v_t^{(i+1)}(p) = c'^{-1}\left(\frac{\pi_t^{(i)}(p)}{\overline{v}_t^{(i)}}\left(\lambda_{0t}^{(i)} u_t^{(i)} + \lambda_{1t}^{(i)} \int_{\underline{p}}^{p} L_t^{(i)}(x)\, d\Gamma(x)\right)\right)$$

and check consistency with the initial guess (i.e. $v_t^{(i+1)}(p)$ “close to” $v_t^{(i)}(p)$ for all (p,t). If inconsistent, start over at step one using $\left\{v_t^{(i+1)}(p)\right\}_{t\geq 0, p\in[\underline{p},\overline{p}]}$ as an initial guess.

D The Case of Myopic Workers

Suppose workers discount payoffs at rate ρ and firms at rate r. Suppose workers are (infinitely) impatient, they only care about current wages and $\rho = \infty$. The firm's original problem simplifies to:

$$\Pi_0^\star(\theta) = \max_{\{w_t\}} \int_0^{+\infty} (\theta - w_t) L_t(\theta) e^{-rt} dt \tag{60}$$

$$\text{subject to:} \quad \dot{L}_t(\theta) = -\left(\delta + \lambda_1 \overline{F}_t(w_t)\right) L_t(\theta) + \frac{q(\theta)}{N}\left(\lambda_0 u_t + \lambda_1 (1-u_t) G_t(w_t)\right), \tag{61}$$

which has one less state variable ($W_t(\theta)$) than the original problem (24). (Readers will pardon the notational abuse whereby $F(\cdot)$ and $G(\cdot)$ now take w_t, rather than W_t, as an argument.)

Denoting the costate associated with $L_t(\theta)$ (i.e. the firm's shadow value of the marginal worker) as $\pi_t(\theta)$, the optimality conditions for (60) write down as:

$$1 = \pi_t(\theta) \times \left[\lambda_1 f_t(w_t(\theta)) + \lambda_1 \frac{q(\theta)}{NL_t(\theta)}(1-u_t) g_t(w_t(\theta))\right] \tag{62}$$

$$\dot{\pi}_t(\theta) = \left(r + \delta + \lambda_1 \overline{F}_t(w_t(\theta))\right)\pi_t(\theta) + w_t(\theta) - \theta, \tag{63}$$

$$\lim_{t\to+\infty} e^{-rt}\pi_t(\theta) = 0. \tag{64}$$

Now focusing on RPE's, where $(1-u_t) g_t(w_t(\theta)) = NL_t(\theta) f_t(w_t(\theta))/q(\theta) = NL_t(\theta)\gamma(\theta)/w_t'(\theta)$, the first order condition (62) becomes:

$$w_t'(\theta) = 2\lambda_1 \gamma(\theta)\pi_t(\theta). \tag{65}$$

Substitution into (63) delivers the following PDE in $w_t(\theta)$:

$$\dot{w}_t'(\theta) = \left(r + \delta + \lambda_1 \overline{\Gamma}(\theta)\right) w_t'(\theta) + 2\lambda_1 \gamma(\theta)(w_t(\theta) - \theta). \tag{66}$$

This has a simple time-invariant solution, which is rank preserving (it is the customary steady-state wage equation in the BM model with heterogeneous firms):

$$w_\infty(\theta) = \theta - \left(r + \delta + \lambda_1 \overline{\Gamma}(\theta)\right)^2 \int_{\underline{p}}^{\theta} \frac{dx}{\left(r + \delta + \lambda_1 \overline{\Gamma}(x)\right)^2} - \left(\underline{p} - w_\infty(\underline{p})\right). \tag{67}$$

This time-invariant solution satisfies the RP property and the optimality conditions (62) - (64). It is therefore an RPE, in which all firms jump right on to the new steady-state wage policy after a shock. Firm sizes then evolve according to (19) and the cross-section distribution of wages also gradually shifts toward its new steady-state shape as labor gets reallocated between firms.

The model can be closed by assuming the free-entry condition $\underline{p} = \underline{w}$. Under this assumption, $w_t(\underline{p}) = \underline{p} = \underline{w}$ for all t. To show that the invariant solution is unique, integrate equation (66) between $\underline{p}$ and θ:

$$\int_{\underline{p}}^{\theta} \dot{w}_t'(x)\,dx = (r+\delta)\left[w_t(\theta) - w_t(\underline{p})\right] + \lambda_1 \int_{\underline{p}}^{\theta} \overline{\Gamma}(x) w_t'(x)\,dx + 2\lambda_1 \int_{\underline{p}}^{\theta} \gamma(x)(w_t(x) - x)\,dx.$$

Integrating by parts the middle term on the RHS yields (using $d\underline{w}/dt = 0$):

$$\dot{w}_t(\theta) = \left(r + \delta + \lambda_1 \overline{\Gamma}(\theta)\right) w_t(\theta) - \left(r + \delta + \lambda_1 \overline{\Gamma}(\underline{p})\right)\underline{p} + 3\lambda_1 \int_{\underline{p}}^{\theta} w_t(x)\,d\Gamma(x) - \int_{\underline{p}}^{\theta} x\,d\Gamma(x),$$

and

$$\ddot{w}_t(\theta) = \left(r + \delta + \lambda_1 \overline{\Gamma}(\theta)\right)\dot{w}_t(\theta) + 3\lambda_1 \int_{\underline{p}}^{\theta} \dot{w}_t(x)\,d\Gamma(x).$$

We now establish that the invariant distribution is the unique solution, so the equilibrium jumps right away to the new steady state. Since $\dot{w}_t(\theta)$ is differentiable in θ, there exists $\theta > \underline{p}$ such that $\dot{w}_t(\theta)$ preserves the sign for $\theta \in \left[\underline{p}, \theta\right]$. If this sign is zero, $\dot{w}_t(\theta) = 0$ for all θ: we have the stationary solution. If it is weakly positive with strict inequality on a set of positive measure, then from the above equation $\ddot{w}_t(\theta) > 0$. But then $\dot{w}_t(\theta)$ rises and becomes even more positive on some set of θ's. By induction, $\dot{w}_t(\theta)$ and thus $w_t(\theta)$ grow unbounded, ultimately make profits negative, and cannot converge to the new steady state. By the same reasoning, if $\dot{w}_t(\theta) \leq 0$ for all $\theta \in \left[\underline{p}, \theta\right]$ with strict inequality on a set of positive measure, then $w_t(\theta)$ grows unbounded below on some set of productivities, violating the minimum wage requirement and any reservation wage. Using the entry condition, wages are

$$w_t(\theta) = \theta - \left(r + \delta + \lambda_1 \overline{\Gamma}(\theta)\right)^2 \int_{\underline{p}}^{\theta} \frac{dx}{\left(r + \delta + \lambda_1 \overline{\Gamma}(x)\right)^2}.$$

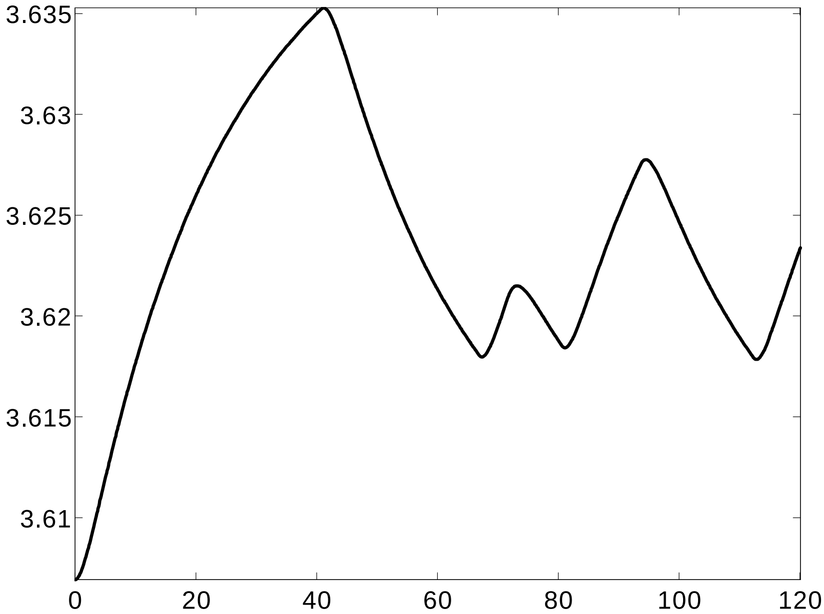

Fig. 1: Mean output per worker

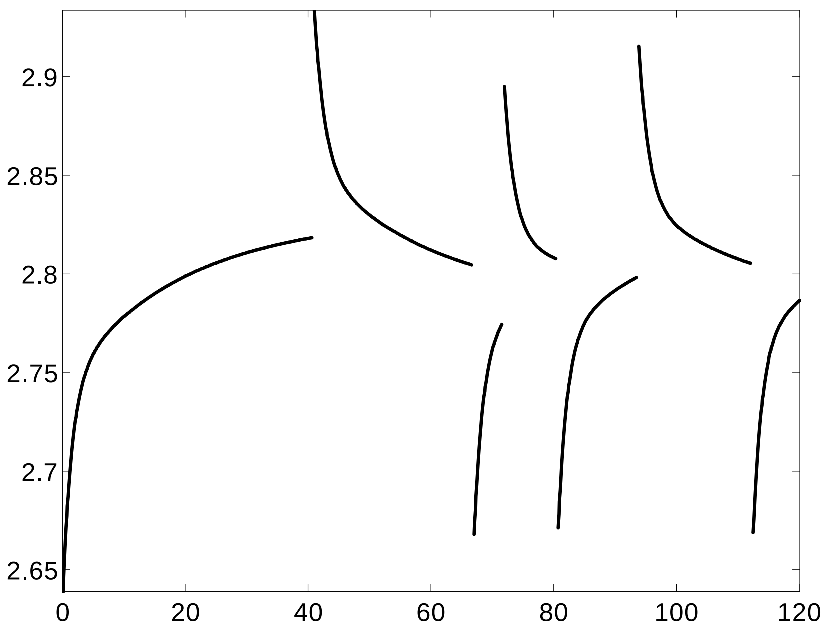

Fig. 2: Mean wage

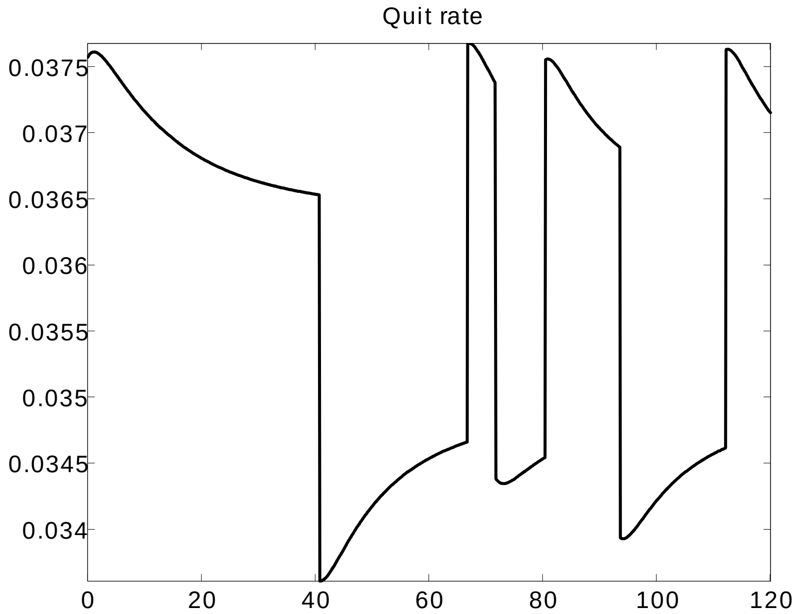

Fig. 3: Mean EE rate

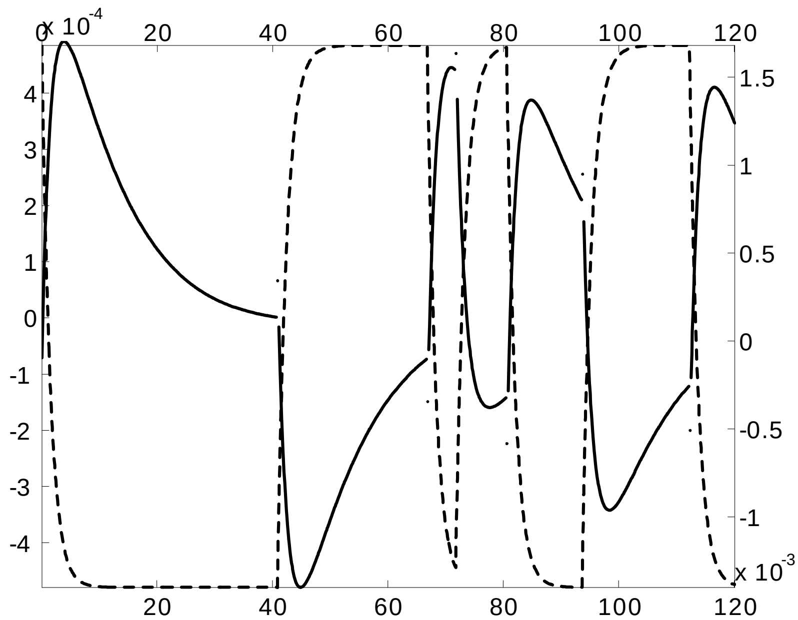

Fig. 4: Firm size growth differential (solid) and unemployment (dashed)

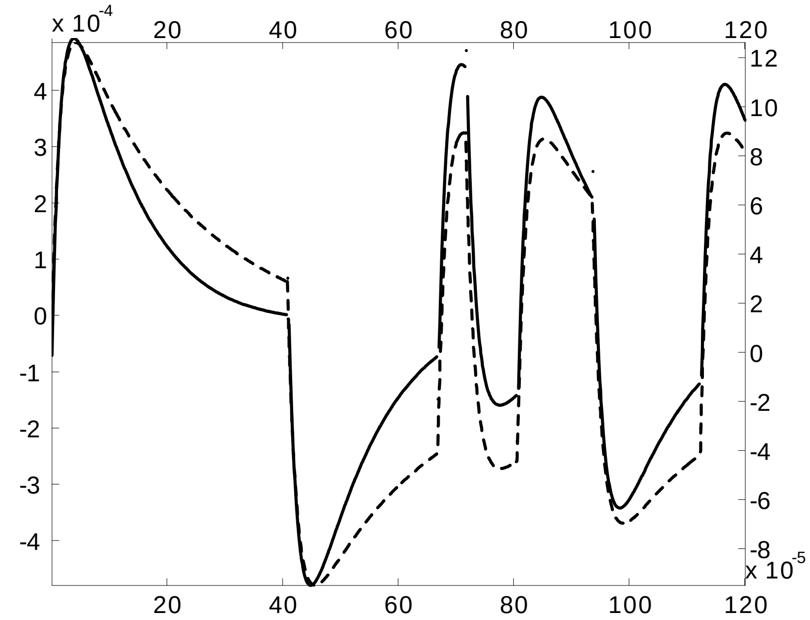

Fig. 5: Firm size growth differential (solid) and productivity (dashed)