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Theoretical Foundations of Buffer Stock Saving

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Abstract

"Buffer-stock" versions of the dynamic stochastic optimizing model of saving are now standard in the consumption literature. This paper builds theoretical foundations for rigorous understanding of the main features of buffer stock models, including the existence of a target level of wealth and the proposition that aggregate consumption growth equals aggregate income growth in a small open economy populated by buffer stock consumers.

Keywords: Precautionary saving, buffer-stock saving, consumption, marginal propensity to consume, permanent income

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1 Introduction

Stimulated by the success of Modigliani and Brumbergx(1954)'s Life Cycle model and Friedman'sx(1957) Permanent Income Hypothesis, a vast literature in the 1960s and 1970s formalized the idea that consumption spending reflects optimal intertemporal choice. This literature culminated in famous papers by Schectman and Escuderox(1977) and Bewleyx(1977), which provided the foundation for the flourishing modern theory of dynamic stochastic optimization in economics.

Given this intellectual pedigree, it is surprising that the now-standard method for analyzing such problems, contraction mapping theory, has not yet established some basic properties of the benchmark consumption problem with unbounded (e.g.xconstant relative risk aversion) utility, uncertain labor income, and no liquidity constraints. The gap exists because standard theorems from the contraction mapping literature (those in Stokey et.xal.x(1989)) cannot be used for this problem (for reasons explained below).

This paper fills that gap, deriving the key 'impatience' conditions that must be satisfied for the problem to have a unique well-defined solution.

The reader could be forgiven for not having noticed the existence of a gap. A large literature solving precisely these kinds of problems has emerged over the past decade, made possible by advances in numerical solution methods (following Zeldesx(1989)). But numerical solutions are a 'black box:' They make it possible to use a model without really understanding its properties. Indeed, without foundational theory, it can even be difficult to be confident that a specific numerical solution is correct. And without theoretical underpinnings, an analyst often has little intuition for how any given simulation result might change with the structure or calibration of the model.

A good example is the finding that when consumers are both impatient and 'prudent' (Kimballx(1990)) there will be a target level of nonhuman wealth ('cash' for short) such that if actual cash exceeds the target, the consumer will spend freely and cash will fall (in expectation), while if actual cash is below the target the consumer will save and cash will rise. Carrollx(1992;x1997) showed that target saving behavior arises under plausible parameter values for both finite and infinite horizon models. Gourinchas and Parkerx(2002) estimate the model on household data and conclude that for the mean household the buffer-stock phase of life lasts from age 25 until around age 40-45; using the same model with different data Cagettix(2003) finds target saving behavior into the 50s for the median household. But none of these papers provides a rigorous explanation for why target saving behavior arises.

This paper provides the analytical foundations for target saving and many other model characteristics that have become familiar from the simulation literature. The key step is a proof that the problem defines a contraction mapping, which is shown by application of a theorem ofxBoydx(1990). The paper pairs its theoretical results with illustrative simulation examples, providing an integrated framework for understanding buffer-stock saving behavior.[1]

[1] The software that generates these simulation results is available on the author's website.

The paper proceeds in three parts.

The first part demonstrates that the maximization problem problem can be rewritten in terms of ratios of key variables to permanent noncapital income, and proves that the rewritten problem defines a contraction mapping, with a unique limiting consumption function, under certain parametric restrictions that can be interpreted as requiring the consumer to be sufficiently 'impatient.' It then shows that a related class of models (exemplified by Deatonx(1991)) constitute a particular limit of the model.

The next section articulates and examines five key properties the model. First, as cash approaches infinity the expected growth rate of consumption and the marginal propensity to consume (MPC) converge to their values in the perfect foresight case. Second, as cash approaches zero the expected growth rate of consumption approaches infinity, and the MPC approaches a specific simple analytical limit. Third, there exists a unique 'target' cash-to-permanent-income ratio. Fourth, at the target cash ratio, the expected growth rate of consumption is slightly less than the expected growth rate of permanent labor income. Finally, the expected growth rate of consumption is declining in the level of cash. The first four propositions are proven generally; the last is shown to hold if there are no transitory shocks, but may fail in extreme cases if there are both transitory and permanent shocks.

The final section briefly examines aggregate behavior in an economy populated by buffer-stock consumers. Szeidlx(2006) has recently proven that such an economy will be characterized by stable invariant distributions for the consumption ratio and other key

2Szeidl's proof supplants the analysis in an earlier draft of this paper, which conjectured that such a result held and provided simulation evidence but no proof.

variables.2

This section shows that even with a fixed aggregate interest rate that differs from the time preference rate, the economy converges to a balanced growth equilibrium in which the growth rate of aggregate consumption tends toward the (exogenous) growth rate of aggregate permanent income. A similar proposition holds at the level of individual households.

2 The Problem

2.1 Setup

The consumer solves an optimization problem from the current period t until the end of life at T defined by the objective

.

$$\max \; E_t \left[\sum_{n=0}^{T-t} \beta^n \mathrm{u}(\boldsymbol{c}_{t+n}) \right]$$

where $\mathrm{u}(\bullet)$ is a constant relative risk aversion utility function $\mathrm{u}(\bullet) = \bullet^{1-\rho}/(1-\rho)$

3The main results also hold for logarithmic utility which is the limit as $\rho \to 1$ but incorporating the logarithmic special case in the proofs is cumbersome and therefore omitted.

for $\rho > 1$.3
(We will ultimately be interested in the limit as the 'horizon' $T-t$ approaches infinity, but it is useful start with a finite horizon). The consumer's initial condition is defined by market resources $\boldsymbol{m}_t$ (what Deatonx(1991) calls 'cash-on-hand') and permanent noncapital income $\mathbf{p}_t$.

In the usual analysis, a dynamic budget constraint (DBC) incorporates all of the factors that determine next period's $\boldsymbol{m}$ given this period's choices; for purposes of this paper, it will be useful to disarticulate the various steps so that they can be examined individually. Thus, we capture the dynamic budget constraint using the following equations:

.

$$
\begin{aligned}
\boldsymbol{a}_t &= \boldsymbol{m}_t - \boldsymbol{c}_t, \\
\boldsymbol{b}_{t+1} &= \boldsymbol{a}_t \mathsf{R} \\
\mathbf{p}_{t+1} &= \mathbf{p}_t \underbrace{\Gamma \psi_{t+1}}_{\equiv \Gamma_{t+1}}, \\
\boldsymbol{m}_{t+1} &= \boldsymbol{b}_{t+1} + \mathbf{p}_{t+1} \xi_{t+1},
\end{aligned}
$$

where $\boldsymbol{a}_t$ indicates the consumer's assets at the end of period t, which grow by a fixed interest factor $\mathsf{R} = (1 + \mathsf{r})$ between periods, and $\boldsymbol{b}_{t+1}$ is the consumer's financial balances before next period's consumption

4Allowing a stochastic interest factor is straightforward but adds little insight to the analysis. The effects are more interesting for analysis of the invariant distribution (Szeidlx(2006)).

choice;4

$\boldsymbol{m}_{t+1}$ is the sum of financial ('bank') balances $\boldsymbol{b}_{t+1}$ and next-period noncapital income, equal to permanent noncapital income $\mathbf{p}_{t+1}$ multiplied by a mean-one iid transitory income shock factor ξ_{t+1} (from the perspective of period t, all future transitory shocks are assumed to satisfy $E_t[\xi_{t+n}] = 1x\forall xn \geq 1$); and permanent noncapital income in period $t + 1$ is equal to its previous value, multiplied by a growth factor Γ, and modified by a mean-one iid shock ψ_{t+1}, $E_t[\psi_{t+n}] = 1x\forall xn \geq 1$ satisfying $\psi \in [\psi, \bar{\psi}]$ for $0 < \psi \leq 1 \leq \bar{\psi} < \infty$ where $\psi = \bar{\psi} = 1$ is the degenerate case with no permanent

5It is useful to emphasize that permanent noncapital income as defined here differs from what Deatonx(1992) calls permanent income (which is often adopted in the macro literature). Deaton defines permanent income as the amount that a perfect foresight consumer could spend while leaving total (human and nonhuman) wealth constant. Relatedly, we refer to $\boldsymbol{m}_t$ as 'cash-on-hand' or 'market resources' rather than as wealth to avoid any confusion for readers accustomed to thinking of the discounted value of future labor income as a part of wealth. The 'market resources' terminology is motivated by the model's assumption that human wealth cannot be capitalized, an implication of anti-slavery laws.

shocks.5

(Hereafter for brevity we occasionally drop time subscripts, e.g.x$E[\psi^{-\rho}]$ will signify $E_t[{\psi_{t+1}}^{-\rho}]$).

Following Carrollx(1992), assume that in future periods $t + nx\forall xn \geq 1$ there is a small probability, represented by $\wp$, that income will be zero (a 'zero-income event'),

.

$$\xi_{t+n} = \begin{cases} 0 & \text{with probability } \wp > 0 \\ \theta_{t+n}/\not\wp & \text{with probability } \not\wp \equiv (1-\wp) \end{cases}$$

(1)

where θ_{t+n} is an iid mean-one random variable ($E_t[\theta_{t+n}] = 1 x \forall x n > 0$) that has a distribution satisfying $\theta \in [\theta, \bar{\theta}]$ where $0 < \theta \leq 1 \leq \bar{\theta} < \infty$ (degenerately $\theta = \bar{\theta} = 1$). Call the cumulative distribution functions F_{ψ} and F_{θ} (and F_{ξ} is derived trivially from (1) and F_{θ}). Permanent income and cash start out strictly positive, $\mathbf{p}_t \in (0, \infty)$ and $\boldsymbol{m}_t \in (0, \infty)$, and the consumer cannot die in debt,

.

$$\boldsymbol{c}_T \leq \boldsymbol{m}_T. \tag{2}$$

The model looks more special than it is. In particular, the assumption of a positive probability of zero-income events may seem questionable. However, it is easy to show that a model with a nonzero minimum value of ξ (motivated, for example, by the existence of unemployment insurance) can be redefined by capitalizing the PDV of perfectly certain income into current market

[6]So long as this PDV is a finite number and unemployment benefits are proportional to $\mathbf{p}_t$; see the discussion in sectionx2.12.

assets,[6] transforming that model back into the model analyzed here. Also, the assumption of a positive point mass (as opposed to positive density) for the worst realization of the transitory shock is inessential, but simplifies and clarifies the proofs.

This model differs from Bewley'sx(1977) classic formulation in several ways. The CRRA utility function does not satisfy Bewley's assumption that u(0) is well defined, or that u′(0) is well defined and finite, so neither the value function nor the marginal value function will be bounded. It differs from Schectman and Escuderox(1977) in that they impose liquidity constraints and positive minimum income. It differs from both of these formulations in that it permits permanent growth, and permanent shocks to income, which a large empirical literature finds are quite substantial in micro data (MaCurdyx(1982); Abowd and Cardx(1989); Carroll and Samwickx(1997); Jappelli and Pistaferrix(2000); Storesletten, Telmer, and Yaronx(2004); Blundell, Low, and Prestonx(2008)) and which the model suggests are far more consequential for household welfare than are transitory fluctuations. It differs from Deatonx(1991) because liquidity constraints are absent; there are separate transitory and permanent shocks (*a la* Muthx(1960)); and the transitory shocks here can occasionally cause income to reach

7Below it will become clear that the Deaton model can be thought of as a particular limit of this paper's model.

zero.7

Finally, it differs from models found in Stokey et.xal.x(1989) because neither liquidity constraints nor bounds on utility or marginal utility are

8Similar restrictions to those in the cited literature are made in the well known papers by Scheinkman and Weissx(1986) and Claridax(1987). See Tochex(2005) for an elegant analysis of a related but simpler continuous-time model.

imposed.8

2.2 Perfect Foresight Benchmarks

The analytical solution to the perfect foresight specialization of the model, obtained by setting $\wp = 0$ and $\theta = \bar{\theta} = \psi = \bar{\psi} = 1$, provides a useful benchmark.

2.2.1 Human Wealth

The dynamic budget constraint plus the can't-die-in-debt condition (2) imply an exactly-holding intertemporal budget constraint (IBC)

.

$$\mathrm{PDV}_t(\boldsymbol{c}) \;=\; \overbrace{\boldsymbol{m}_t - \mathbf{p}_t}^{\boldsymbol{b}_t} + \overbrace{\mathrm{PDV}_t(\mathbf{p})}^{\boldsymbol{h}_t}, \tag{3}$$

where $\boldsymbol{h}_t$ is 'human wealth,' the discounted value of noncapital earnings. With constant growth and interest factors

.

$$\begin{aligned} \boldsymbol{h}_t &= \mathbf{p}_t + (\Gamma/\mathsf{R})\mathbf{p}_t + (\Gamma/\mathsf{R})^2\mathbf{p}_t + ... + (\Gamma/\mathsf{R})^{T-t}\mathbf{p}_t \\ &= \underbrace{\left(\frac{1-(\Gamma/\mathsf{R})^{T-t+1}}{1-(\Gamma/\mathsf{R})}\right)}_{\equiv h_t}\mathbf{p}_t. \end{aligned} \tag{4}$$

The infinite horizon solution is defined as the limit of the finite horizon solution as the horizon $T-t$ approaches

9This is not necessarily the same as the solution to a truly infinite horizon problem; to the extent that they differ, the limit of the finite horizon solution is more economically sensible.

infinity,[9]
and (4) makes plain that in order for $\lim_{n\rightarrow\infty}\boldsymbol{h}_{T-n}$ to be finite, we must impose the Finite Human Wealth ('FHW') condition

.

$$\left(\frac{\Gamma}{\mathsf{R}}\right) < 1. \tag{5}$$

Intuitively, for human wealth to be finite, labor income must grow more slowly than the rate at which it is being discounted.

2.2.2 Unconstrained Solution

In the absence of a liquidity constraint, the consumption Euler equation must hold in every period; with $\mathrm{u}'(\boldsymbol{c}) = \boldsymbol{c}^{-\rho}$, this says

.

$$\begin{aligned} \boldsymbol{c}_t^{-\rho} &= \mathsf{R}\beta\boldsymbol{c}_{t+1}^{-\rho} \\ (\boldsymbol{c}_{t+1}/\boldsymbol{c}_t) &= (\mathsf{R}\beta)^{1/\rho} \end{aligned} \tag{6}$$

and defining the 'return patience factor' (the terminology will be justified

10The character is an Old English letter pronounced 'thorn.'

shortly)10
as

.

$$\text{Þ}_{\mathsf{R}} \equiv \left(\frac{(\mathsf{R}\beta)^{1/\rho}}{\mathsf{R}}\right) \tag{7}$$

we have

.

$$\begin{aligned}
\text{PDV}_t(\boldsymbol{c}) &= \left(1 + \text{Þ}_{\mathsf{R}} + \text{Þ}_{\mathsf{R}}^2 + ...\right) \boldsymbol{c}_t \\
&= \left(\frac{1 - \text{Þ}_{\mathsf{R}}^{T-t+1}}{1 - \text{Þ}_{\mathsf{R}}}\right) \boldsymbol{c}_t
\end{aligned}$$

so that the IBC (3) implies

.

$$\boldsymbol{c}_t = \underbrace{\left(\frac{1-\text{Þ}_{\mathsf{R}}}{1-\text{Þ}_{\mathsf{R}}^{T-t+1}}\right)}_{\equiv \underline{\kappa}_t}(\boldsymbol{b}_t + \boldsymbol{h}_t) \tag{8}$$

where κ_t is the marginal propensity to consume.

Since $(\mathsf{R}\beta)^{1/\rho}$ is the growth factor for consumption (eq (6)), if we start with any positive value of consumption, (8) implies that we must impose

.

$$\text{Þ}_{\mathsf{R}} < 1 \tag{9}$$

for the limit of the PDV of consumption to be finite as the horizon extends to infinity.

The formula for κ_t in (8) illuminates another interpretation of this restriction: It guarantees a positive finite limit to the marginal propensity to consume,

.

$$0 < \underline{\kappa} \;\equiv\; 1 - \text{Þ}_{\mathsf{R}} = \lim_{n\to\infty} \underline{\kappa}_{T-n}. \tag{10}$$

Equation (9) thus imposes a maximum degree of 'patience,' in the sense that the perfect foresight consumer cannot be so pathologically patient as to wish to spend nothing out of a marginal increase in

11Note that the strict inequality rules out the knife-edge case in which consumption is equal to wealth divided by the remaining horizon; as the horizon approaches infinity, the limiting consumption function for the $\text{Þ}_{\mathsf{R}} = 1$ case would be $\mathrm{c}(m) = 0$.

wealth.[11]
Motivated by this interpretation, we will henceforth refer to (9) as the 'return impatience condition' or RIC for short.

2.2.3 Constrained Solution

Defining a 'perfect foresight growth patience factor' as

.

$$\text{Þ}_\Gamma \;=\; \left(\frac{(\beta \mathsf{R})^{1/\rho}}{\Gamma}\right), \tag{11}$$

we can now specify what we call the 'perfect foresight growth impatience condition' (PF-GIC)

.

$$\text{Þ}_\Gamma \;<\; 1, \tag{12}$$

and notice that if the PF-GIC and the FHW condition $\mathsf{R} > \Gamma$ both hold, then the RIC must hold (substitue R for Γ in (11) and use (12)).

If the PF-GIC and the FHW conditions hold but the consumer is subject to a liquidity constraint $\boldsymbol{c}_t \leq \boldsymbol{m}_t x \forall xt$, the optimal policy for an infinite horizon perfect foresight consumer is to run wealth down to zero over some finite horizon; an increase in current wealth pushes the zero-wealth date farther into the future but cannot postpone it forever. Value between the present date and the zero-wealth date can be shown to be

12See Parkx(2006) for a continuous-time demonstration of these points, and references therein for discrete time treatments.

finite.12

Once wealth has reached zero (that is, for a consumer who enters period t with $\boldsymbol{b}_t = 0$), the PF-GIC ensures that the consumer will desire to spend more than all of his current income. But the constraint limits spending to a maximum of $\boldsymbol{m}_t = \mathbf{p}_t$, so the consumer spends all his income and arrives again in the next period with $\boldsymbol{b}_{t+1} = 0$, and so on *ad infinitum* (Deatonx(1991)). Thus, once assets have been run down, the consumer will spend his permanent labor income $\mathbf{p}_{t+n}$ in every subsequent period.

Value for such a perpetually constraint-bound consumer can be calculated from

.

$$\begin{aligned}
\boldsymbol{v}_t &= \mathrm{u}(\mathbf{p}_t) + \beta \mathrm{u}(\mathbf{p}_t \Gamma) + \beta^2 \mathrm{u}(\mathbf{p}_t \Gamma^2) + ... \\
&= \mathrm{u}(\mathbf{p}_t) \left(1 + \beta \Gamma^{1-\rho} + (\beta \Gamma^{1-\rho})^2 + ...\right) \\
&= \left(\frac{\mathrm{u}(\mathbf{p}_t)}{1 - \beta \Gamma^{1-\rho}} \right),
\end{aligned}$$

a derivation that is valid if we impose the Perfect Foresight Finite Value Condition (PF-FVC):

.

$$\begin{aligned} \beta\Gamma^{1-\rho} &< 1. \\ \beta \mathsf{R}\Gamma^{-\rho} &< \mathsf{R}/\Gamma. \end{aligned} \tag{13}$$

Thus, the combination of PF-GIC and PF-FVC guarantees that value is finite for the constrained consumer with a bank balance of zero. Finiteness of the constrained consumer's value, combined with finiteness of value between the present moment and the date at which the constraint will bind, yield the conclusion that value is finite even for perfect foresight consumers for whom the constraint is not (presently) binding.

2.3 The Growth Impatience Condition

The PF-GIC (12) is not sufficient to guarantee existence of a well-behaved solution in the more general model with permanent income shocks, because the precautionary saving motive increases the consumer's effective degree of patience (desire to save). The most transparent way to generalize the PF-GIC is to define an uncertainty-adjusted growth factor

.

$$\acute{\Gamma} \;=\; \left(E[\psi^{-\rho}]\right)^{-1/\rho}\Gamma \tag{14}$$

(which reduces to Γ if there is no uncertainty about permanent income, and is strictly less than Γ in the presence of such

13Note further that under the usual assumption of lognormality of ψ, $E[\psi^{-\rho}] = \exp(\rho({\sigma_\psi}^2/2) + \rho^2 {\sigma_\psi}^2/2)$ so that $(E[\psi^{-\rho}])^{-1/\rho} = \exp(-(1+\rho){\sigma_\psi}^2/2)$; in this connection it is worth pointing out that Kimballx(1990) shows that the term $(1+\rho){\sigma_\psi}^2/2$ is approximately proportional to the 'prudence' (precautionary saving) induced by a risk with variance ${\sigma_\psi}^2/2$.

uncertainty).[13]
Using this, we specify a 'growth patience factor'

.

$$\text{Þ}_{\acute{\Gamma}} \;=\; \left(\frac{(\mathsf{R}\beta)^{1/\rho}}{\acute{\Gamma}}\right) \tag{15}$$

using which we can define a generalized version of (12) which we call the 'growth impatience condition' (GIC)

.

$$\text{Þ}_{\acute{\Gamma}} \ < \ 1, \tag{16}$$

that sectionx2.8 shows is necessary for the problem to define a contraction

14Equation (16) can be raised to the ρ power yielding the alternative form $(\mathsf{R}\beta)\acute{\Gamma}^{-\rho} < 1$ which Deatonx(1991) imposed to guarantee that his problem defined a contraction mapping.

mapping.[14]

2.4 Demonstration That The Problem Can Be Rewritten In Ratio Form

As written, the problem has two state variables, $\mathbf{p}_t$ and $\boldsymbol{m}_t$. But for relative risk aversion $\rho > 1$, the number of relevant state variables can be reduced to one as follows. Defining nonbold variables as the boldface counterpart normalized by $\mathbf{p}_t$ (e.g.x$m_t = \boldsymbol{m}_t/\mathbf{p}_t$), assume that $\mathbf{v}_{T+1} = 0$, and consider the problem in the second-to-last period of life,

.

$$
\begin{aligned}
\mathbf{v}_{T-1}(\boldsymbol{m}_{T-1}, \mathbf{p}_{T-1}) &= \max_{\boldsymbol{c}_{T-1}} \ \mathrm{u}(\boldsymbol{c}_{T-1}) + \beta E_{T-1}[\mathrm{u}(\boldsymbol{m}_T)] \\
&= \max_{c_{T-1}} \ \mathrm{u}(\mathbf{p}_{T-1} c_{T-1}) + \beta E_{T-1}[\mathrm{u}(\mathbf{p}_T m_T)] \\
&= \mathbf{p}_{T-1}^{1-\rho} \left\{ \max_{c_{T-1}} \ \mathrm{u}(c_{T-1}) + \beta E_{T-1}[\mathrm{u}(\Gamma_T m_T)] \right\}
\end{aligned}
$$

where we have used the fact that for CRRA utility, $\mathrm{u}(xy) = x^{1-\rho}\mathrm{u}(y)$.

Now consider the problem

.

$$
\begin{aligned}
\mathrm{v}_t(m_t) &= \max_{c_t} \ \left\{ \mathrm{u}(c_t) + \beta E_t[\Gamma_{t+1}^{1-\rho} \mathrm{v}_{t+1}(m_{t+1})] \right\} \qquad (17) \\
&\text{s.t.} \\
a_t &= m_t - c_t, \\
m_{t+1} &= \underbrace{\mathcal{R}_{t+1}}_{\equiv \mathsf{R}/\Gamma_{t+1}} a_t + \xi_{t+1}
\end{aligned}
$$

with first order condition

.

$$c_t^{-\rho} = \mathsf{R}\beta E_t[\Gamma_{t+1}^{-\rho} c_{t+1}^{-\rho}].$$

If we specify $\mathrm{v}_T(m_T) = \mathrm{u}(m_T)$ and define $\mathrm{v}_{T-1}(m_{T-1})$ from (17) for $t = T-1$ we have

.

$$\mathbf{v}_{T-1}(\boldsymbol{m}_{T-1}, \mathbf{p}_{T-1}) = \mathbf{p}_{T-1}^{1-\rho} \mathrm{v}_{T-1}(\boldsymbol{m}_{T-1}/\mathbf{p}_{T-1}).$$

Similar logic can be applied inductively to all earlier periods, which means that if we solve the normalized one-state-variable problem specified in (17) we will have solutions to the original problem for any $t < T$ from:

.

$$\begin{aligned}\mathbf{v}_t(\boldsymbol{m}_t, \mathbf{p}_t) &= \mathbf{p}_t^{1-\rho}\mathrm{v}_t(\boldsymbol{m}_t/\mathbf{p}_t), \\ \boldsymbol{c}_t(\boldsymbol{m}_t, \mathbf{p}_t) &= \mathbf{p}_t\mathrm{c}_t(\boldsymbol{m}_t/\mathbf{p}_t),\end{aligned}$$

and so on.

2.5 The Baseline Solution

Figurex1 depicts the successive consumption rules that apply in the last period of life ($\mathrm{c}_T(m)$), the second-to-last period, and various earlier periods under the set of baseline parameter values listed in Tablex1, which are representative of common calibrations in the micro literature. (The 45 degree line is labelled as $\mathrm{c}_T(m) = m$ because in the last period of life it is optimal to spend all remaining resources.)

Tablex1: Microeconomic Model Calibration

Calibrated Parameters		
Description	Parameter	Value
Permanent Income Growth Factor	Γ	1.03
Interest Factor	R	1.04
Time Preference Factor	β	0.96
Coeff of Relative Risk Aversion	ρ	2
Probability of Zero Income	$\wp$	0.005
Std Dev of Log Permanent Shock	σ_{ψ}	0.1
Std Dev of Log Transitory Shock	σ_{θ}	0.1
Model Characteristics Calculated From Parameters		
		Approximate
	Formula	Calculated
Description	or Symbol	Value
Finite Human Wealth Measure	Γ/R	0.990
Level Patience Factor	$(\mathsf{R}\beta)^{1/\rho}$	0.999
Uncertainty Adjustment To Growth	$(E[\psi^{-\rho}])^{-1/\rho}$	0.985
Uncertainty-Adjusted Growth	$\grave{\Gamma}$	1.015
Return Patience Factor	$Þ_{\mathsf{R}} \equiv \mathsf{R}^{-1}(\mathsf{R}\beta)^{1/\rho}$	0.961
Perfect Foresight Growth Patience Factor	$Þ_{\Gamma} \equiv \Gamma^{-1}(\mathsf{R}\beta)^{1/\rho}$	0.970
Growth Patience Factor	$Þ_{\acute{\Gamma}} \equiv \acute{\Gamma}^{-1}(\mathsf{R}\beta)^{1/\rho}$	0.985

PIC

Figurex1: Convergence of the Consumption Rules

In the figure, the consumption rules appear to converge as the horizon recedes; we call the limiting infinite-horizon consumption rule

.

$$\mathrm{c}(m) \;=\; \lim_{n\to\infty} \mathrm{c}_{T-n}(m). \tag{18}$$

2.6 Concave Consumption Function Characteristics

We now need to show that the maximization problem (17) defines a sequence of continuously differentiable strictly increasing strictly concave functions

15Carroll and Kimballx(1996) proved concavity but not the other desired properties.

$\{\mathrm{c}_T, \mathrm{c}_{T-1}, ..., \mathrm{c}_{T-k}\}$.15

The proof is straightforward but tedious, and so is relegated to appendix A. For present purposes, the key point is the following intuition: $\mathrm{c}_t(m) < m$ for all periods $t < T$ because if the consumer spent all available resources, he would arrive in period $t + 1$ with balances b_{t+1} of zero, then might earn zero noncapital income for the rest of his life (an unbroken series of zero-income events is unlikely but possible). In such a case, the budget constraint and the can't-die-in-debt condition mean that the consumer would ultimately be forced to spend zero, incurring negative infinite utility. To avoid this disaster, the consumer never spends everything. (This is an example of the 'natural borrowing constraint' induced by a precautionary motive

16It would perhaps be better to call it the 'utility-induced borrowing constraint' as it follows from the assumptions on the utility function (in particular, $u'(0) = \infty$); for example, no such constraint arises if utility is of the Constant Absolute Risk Aversion form.

(Zeldesx(1989))).16

2.7 Bounds For The Consumption Functions

The consumption functions depicted in figurex1 appear to have limiting slopes as $m \downarrow 0$ and as $m \uparrow \infty$. This section confirms that impression and derives those slopes, which turn out to be useful in the contraction mapping proofs.

Assume that a continuously differentiable concave consumption function exists in period $t + 1$, with an origin at $c_{t+1}(0) = 0$, a minimal MPC $\kappa_{t+1} > 0$, and maximal MPC $\bar{\kappa}_{t+1} \leq 1$. (If $t + 1 = T$ these will be $\kappa_T = \bar{\kappa}_T = 1$; for earlier periods they will exist by recursion from the arguments below).

For $m_t > 0$ we can define $e_t(m_t) = c_t(m_t)/m_t$ and using this the Euler equation can be rewritten

.

$$e_t(m_t)^{-\rho} = \beta \mathsf{R} E_t \left[\left(e_{t+1}(m_{t+1}) \left(\frac{\overbrace{\mathsf{R}a_t(m_t) + \Gamma_{t+1}\xi_{t+1}}^{=m_{t+1}\Gamma_{t+1}}}{m_t} \right) \right)^{-\rho} \right] \tag{19}$$

$$\begin{aligned} &= \cancel{\wp}\beta \mathsf{R} m_t^{\rho} E_t \left[(e_{t+1}(m_{t+1}) m_{t+1}\Gamma_{t+1})^{-\rho} \mid \xi_{t+1} > 0 \right] \\ &\quad + \wp\beta \mathsf{R}^{1-\rho} E_t \left[\left(e_{t+1}(\mathcal{R}_{t+1} a_t(m_t)) \frac{m_t - c_t(m_t)}{m_t} \right)^{-\rho} \mid \xi_{t+1} = 0 \right] \end{aligned} \tag{20}$$

Consider the first conditional expectation in (20), recalling that if $\xi_{t+1} > 0$ then $\xi_{t+1} \equiv \theta_{t+1}/\cancel{\wp}$. Since $\lim_{m \downarrow 0} a_t(m) = 0$, $E_t[(e(m_{t+1})m_{t+1}\Gamma_{t+1})^{-\rho} x | x\xi_{t+1} > 0]$ is contained within bounds defined by $(e_{t+1}(\underline{\theta}/\cancel{\wp})\Gamma\underline{\psi}\underline{\theta}/\cancel{\wp})^{-\rho}$ and $(e_{t+1}(\bar{\theta}/\cancel{\wp})\Gamma\bar{\psi}\bar{\theta}/\cancel{\wp})^{-\rho}$ both of which are finite numbers, implying that the whole term multiplied by $\cancel{\wp}$ goes to zero as ${m_t}^{\rho}$ goes to zero. As $m_t \downarrow 0$ the expectation in the other term goes to ${\bar{\kappa}_{t+1}}^{-\rho}(1 - \bar{\kappa}_t)^{-\rho}$. (This follows from the strict concavity and differentiability of the consumption function.) It follows that the limiting $\bar{\kappa}_t$ satisfies $(\bar{\kappa}_t)^{-\rho} = \beta\wp\mathsf{R}^{1-\rho}(\bar{\kappa}_{t+1})^{-\rho}(1 - \bar{\kappa}_t)^{-\rho}$. We can conclude that

.

$$
\begin{aligned}
\bar{\kappa}_t &= \wp^{-1/\rho}(\beta \mathsf{R})^{-1/\rho}\mathsf{R}(1-\bar{\kappa}_t)\bar{\kappa}_{t+1} \\
\underbrace{\wp^{1/\rho}\overbrace{\mathsf{R}^{-1}(\beta \mathsf{R})^{1/\rho}}^{Þ_{\mathsf{R}}}}_{\equiv \underline{\lambda}}\bar{\kappa}_t &= (1-\bar{\kappa}_t)\bar{\kappa}_{t+1} \qquad (21)
\end{aligned}
$$

which implies

.

$$
\begin{aligned}
(\underline{\lambda}\bar{\kappa}_t)^{-1} &= (1-\bar{\kappa}_t)^{-1}\bar{\kappa}_{t+1}^{-1} \\
\bar{\kappa}_t^{-1}(1-\bar{\kappa}_t) &= \underline{\lambda}\bar{\kappa}_{t+1}^{-1} \\
\bar{\kappa}_t^{-1} &= 1+\underline{\lambda}\bar{\kappa}_{t+1}^{-1}. \qquad (22)
\end{aligned}
$$

Then $\left\{\bar{\kappa}_{T-n}^{-1}\right\}_{n=0}{}^{\infty}$ is an increasing convergent sequence if

.

$$
\begin{array}{rcl}
0 \leq & \underline{\lambda} & < 1 \\
0 \leq & \wp^{1/\rho}\text{\textbf{Þ}}_{\mathsf{R}} & < 1,
\end{array}
$$

a condition that we dub the 'Weak Return Impatience Condition' (WRIC) because with $\wp < 1$ it will hold more easily (for a larger set of parameter values) than the RIC. This condition is unlikely to be binding unless $\wp$ is close to one (that is, unless labor income is almost always zero).

Since $\bar{\kappa}_T = 1$, iterating (22) backward to infinity (since we are interested in the consumption function of an infinitely lived agent) we obtain:

.

$$
\lim_{n \to \infty} \bar{\kappa}_{T-n} = \underbrace{1 - \wp^{1/\rho}\text{\textbf{Þ}}_{\mathsf{R}}}_{\equiv \bar{\kappa}} \tag{23}
$$

so we will call $\bar{\kappa}$ the limiting maximal MPC.

The minimal MPC's are obtained by considering the case where $m_t \uparrow \infty$.

If the FHW condition holds, then as $m_t \uparrow \infty$ the proportion of consumption that will be financed out of capital income approaches 1, so that the terms involving Ξ_{t+1} in (19) can be neglected, leading to a revised Euler equation

.

$$(m_t \mathrm{e}_t(m_t))^{-\rho} = \beta \mathsf{R} E_t \left[(\mathrm{e}_{t+1}(\mathrm{a}_t(m_t)\mathcal{R}_{t+1}) \left(\mathsf{R}\mathrm{a}_t(m_t)\right))^{-\rho} \right]$$

but concavity of the consumption function implies that $\lim_{m\to\infty} \mathrm{e}_t(m) = \kappa_t$ so the limiting Euler equation is

.

$$\begin{aligned} (m_t \underline{\kappa}_t)^{-\rho} &= \beta \mathsf{R} E_t [(\underline{\kappa}_{t+1} \mathsf{R}(1-\underline{\kappa}_t)m_t)^{-\rho}] \\ \underline{\kappa}_t &= \left(\beta \mathsf{R}^{1-\rho}((1-\underline{\kappa}_t)\underline{\kappa}_{t+1})^{-\rho}\right)^{-1/\rho} \end{aligned}$$

and the same sequence of derivations used above yields the conclusion that

.

$$\underline{\kappa}_t^{-1} \;=\; 1 + \bar{\lambda}\underline{\kappa}_{t+1}^{-1} \tag{24}$$

where $\bar{\lambda} = 1 - \kappa$ so that $\left\{\underline{\kappa}_{T-n}^{-1}\right\}_{n=0}{}^{\infty}$ is an increasing convergent sequence (so κ_{T-n} is a decreasing convergent sequence) if $0 \leq \text{Þ}_{\mathsf{R}} < 1$ which is the return impatience condition.

Under some circumstances described below, the problem will have a solution even if the RIC does not hold. That solution will have the feature that the marginal propensity to consume is always strictly positive, even if κ as defined in (10) is negative. To encompass this possibility, we define

.

$$\hat{\underline{\kappa}}_t \;=\; \max[0, \underline{\kappa}_t] \tag{25}$$

and call $\hat{\underline{\kappa}} = \lim_{n\to\infty}\hat{\underline{\kappa}}_{T-n}$ the limiting minimal MPC.

We are now in position to observe that the optimal level of consumption must satisfy

.

$$\hat{\underline{\kappa}}_t m_t \ \leq \ c_t \ \leq \ \bar{\kappa}_t m_t \tag{26}$$

because consumption starts at zero and is continuously differentiable (as argued above), is strictly concave (Carroll and Kimballx(1996)), and always exhibits a slope between $\hat{\underline{\kappa}}_t$ and $\bar{\kappa}_t$.

2.8 Conditions Under Which the Problem Defines a Contraction Mapping

To prove that the consumption rules converge, we need to show that the problem defines a contraction mapping. This cannot be proven using the standard theorems in, say, Stokey et.xal.x(1989), which require marginal utility to be bounded over the space of possible values of m, because the possibility (however unlikely) of an unbroken string of zero-income events for the remainder of life means that as m approaches zero c must approach zero (see the discussion in 2.6); thus, marginal utility is unbounded. Fortunately, Boydx(1990) provides a weighted contraction mapping theorem that can be used. To use Boyd's theorem we need

Theorem 1. *Consider any function* $\bullet \in \mathcal{C}(\mathcal{A}, \mathcal{B})$ *where* $\mathcal{C}(\mathcal{A}, \mathcal{B})$ *is the space of continuous functions from* $\mathcal{A}$ *to* $\mathcal{B}$*. Suppose* $F \in \mathcal{C}(\mathcal{A}, \mathcal{B})$

with $\mathcal{B} \subseteq \mathbb{R}$ *and* $\digamma > 0$*. Then* $\bullet$ *is* $\digamma$*-bounded if the* $\digamma$*-norm of* $\bullet$*,*

.

$$\| \bullet \|_{\digamma} = \sup_{m} \left[\frac{| \bullet (m)|}{\digamma(m)} \right],$$

(27)

is finite.

For $\mathcal{C}_{\digamma}(\mathcal{A}, \mathcal{B})$ defined as the set of functions in $\mathcal{C}(\mathcal{A}, \mathcal{B})$ that are $\digamma$-bounded; w, x, y, and z as examples of $\digamma$-bounded functions; and using $\mathbf{0}(m) = 0$ to indicate the function that returns zero for any argument, Boydx(1990) proves the following.

Boyd's Weighted Contraction Mapping Theorem. *Let* $\mathsf{T} : \mathcal{C}_{\digamma}(\mathcal{A}, \mathcal{B}) \to \mathcal{C}(\mathcal{A}, \mathcal{B})$ *such*

17We will usually denote the function that results from the mapping as, e.g., $\{\mathsf{T}\mathrm{w}\}$.

18To non-theorist
and 3) are taken to mea
so that, e.g., x $\leq$ y is sh
which can be applied to

*that*17 ’18

.

1) T *is non-decreasing, i.e.* $\mathrm{x} \le \mathrm{y} \Rightarrow \{\mathsf{T}\mathrm{x}\} \le \{\mathsf{T}\mathrm{y}\}$

2) $\{\mathsf{T}\mathbf{0}\} \in \mathcal{C}_{\digamma}(\mathcal{A}, \mathcal{B})$

3) $\{\mathsf{T}(\mathrm{w} + \zeta \digamma)\} \le \{\mathsf{T}\mathrm{w}\} + \zeta \xi \digamma$ *for some real* $0 < \xi < 1$ *and all real* $\zeta > 0$.

Then T *defines a contraction with a unique fixed point.*

For our problem, take $\mathcal{A}$ as $\mathbb{R}_{++}$ and $\mathcal{B}$ as $\mathbb{R}$, and define

.

$$\{\mathsf{E}\mathrm{z}\}(a_t) \;=\; E_t\left[\Gamma_{t+1}^{1-\rho}\mathrm{z}(a_t\mathcal{R}_{t+1} + \xi_{t+1})\right].$$

Using this, we introduce the mapping

19Note that the existence of the maximum is assured by the continuity of $\{\mathsf{E}\mathrm{z}\}($
continuous because it is the sum of continuous $\digamma$-bounded functions z).

$\mathcal{T} : \mathcal{C}_{\digamma}(\mathcal{A}, \mathcal{B}) \to \mathcal{C}(\mathcal{A}, \mathcal{B})$,19

.

$$\{\mathcal{T}\mathrm{z}\}(m_t) \;=\; \max_{c_t \in [\underline{\hat{\kappa}} m_t, \bar{\kappa} m_t]} \{\mathrm{u}(c_t) + \beta \left(\{\mathsf{E}\mathrm{z}\}(m_t - c_t)\right)\} \quad (28)$$

Our first goal is to show that our operator $\mathcal{T}$ satisfies the conditions that Boyd requires of his operator T, if we impose certain restrictions on parameter values. The first required condition is the WRIC necessary for convergence of the maximal MPCs, equation (23) above.

A more serious restriction is

.

$$\underbrace{\beta E[\Gamma^{1-\rho}]}_{\equiv \beth} \;<\; 1 \quad (29)$$

which is the uncertainty-adjusted version of (13) and has a similar interpretation as the condition necessary to guarantee that discounted value is finite. We refer to this henceforth as the Finite Value Condition (FVC).

The final restriction is the Growth Impatience Condition (GIC) (16) presented above. Of the three restrictions, this is the only one that is likely to be binding for many potential parameterizations of the model.

We discuss the interpretation of these restrictions in further detail below. Having articulated the restrictions, we are now in position to state the main theorem of the paper.

Theorem 1. *The mapping $\mathfrak{T}$ is a contraction mapping if the restrictions on parameter values* (16), (23), *and* (29) *are true.*

2.8.1 Proof That $\mathfrak{T}$ Is A Contraction Mapping

We must show that our operator $\mathfrak{T}$ satisfies all of Boyd's conditions.

Boyd's operator T maps from $\mathcal{C}_{F}(\mathcal{A}, \mathcal{B})$ to $\mathcal{C}(\mathcal{A}, \mathcal{B})$. A preliminary requirement is therefore that $\{\mathfrak{T}\mathrm{z}\}$ be continuous for any $F-$bounded z, $\{\mathfrak{T}\mathrm{z}\} \in x\mathcal{C}(\mathbb{R}_{++}, \mathbb{R})$. This is not difficult to show; see Hiraguchix(2003).

Consider next condition 1). For this problem,

.

$$\{\mathcal{T}\mathrm{x}\}(m_t) \text{ is } \max_{c_t \in [\underline{\hat{\kappa}} m_t, \bar{\kappa} m_t]} \left\{ \mathrm{u}(c_t) + \beta E_t \left[\Gamma_{t+1}^{1-\rho} \mathrm{x}\left(m_{t+1}\right) \right] \right\}$$
$$\{\mathcal{T}\mathrm{y}\}(m_t) \text{ is } \max_{c_t \in [\underline{\hat{\kappa}} m_t, \bar{\kappa} m_t]} \left\{ \mathrm{u}(c_t) + \beta E_t \left[\Gamma_{t+1}^{1-\rho} \mathrm{y}\left(m_{t+1}\right) \right] \right\},$$

so $\mathrm{x}(\bullet) \leq \mathrm{y}(\bullet)$ implies $\{\mathcal{T}\mathrm{x}\}(m_t) \leq \{\mathcal{T}\mathrm{y}\}(m_t)$ by

20For a fixed m_t, recall that m_{t+1} is just a function of c_t and the stochastic shocks.

inspection.[20]

Condition 2) requires that $\{\mathcal{T}\mathbf{0}\} \in \mathcal{C}_{F}\left(\mathcal{A}, \mathcal{B}\right)$. By definition,

.

$$\{\mathcal{T}\mathbf{0}\}(m_t) = \max_{c_t \in [\underline{\hat{\kappa}} m_t, \bar{\kappa} m_t]} \left\{ \left(\frac{c_t^{1-\rho}}{1-\rho} \right) + \beta 0 \right\}$$

the solution to which is patently $\mathrm{u}(\bar{\kappa} m_t)$. Thus, condition 2) will hold if $(\bar{\kappa} m_t)^{1-\rho}$ is F-bounded. We use the bounding function

.

$$\digamma(m) \;=\; \eta + m + m^{1-\rho}, \tag{30}$$

for some real scalar $\eta > 0$ whose value will be determined in the course of the proof. Under this definition of $\digamma$, $\{\mathfrak{T}\mathbf{0}\}(m_t) = \mathrm{u}(\bar{\kappa} m_t)$ is clearly $\digamma$-bounded.

Finally, we turn to condition 3), $\{\mathfrak{T}(\mathrm{z}+\zeta\digamma)\}(m_t) \leq \{\mathfrak{T}\mathrm{z}\}(m_t)+\zeta\xi\digamma(m_t)$. The proof will be more compact if we define č and ă as the consumption and assets

21Section 2.6 proves existence of a continuously differentiable consumption function, which implies the existence of a corresponding continuously differentiable assets function.

functions21
associated with $\mathfrak{T}$z and ĉ and â as the functions associated with $\mathfrak{T}(\mathrm{z}+\zeta\xi\digamma)$; using this notation, condition 3) can be rewritten

.

$$\mathrm{u}(\hat{\mathrm{c}}) + \beta\{\mathsf{E}(\mathrm{z}+\zeta\digamma)\}(\hat{\mathrm{a}}) \;\leq\; \mathrm{u}(\breve{\mathrm{c}}) + \beta\{\mathsf{E}\mathrm{z}\}(\breve{\mathrm{a}}) + \zeta\xi\digamma.$$

Now note that if we force the $\smile$ consumer to consume the amount that is optimal for the $\wedge$ consumer, value for the $\smile$

consumer must decline (at least weakly). That is,

.

$$\mathrm{u}(\hat{\mathrm{c}}) + \beta\{\mathsf{E}\mathrm{z}\}(\hat{\mathrm{a}}) \;\leq\; \mathrm{u}(\check{\mathrm{c}}) + \beta\{\mathsf{E}\mathrm{z}\}(\breve{\mathrm{a}}).$$

Thus, condition 3) will certainly hold under the stronger condition

.

$$\begin{aligned}
\mathrm{u}(\hat{\mathrm{c}}) + \beta\{\mathsf{E}(\mathrm{z} + \zeta \mathcal{F})\}(\hat{\mathrm{a}}) &\leq \mathrm{u}(\hat{\mathrm{c}}) + \beta\{\mathsf{E}\mathrm{z}\}(\hat{\mathrm{a}}) + \zeta\xi\mathcal{F} \\
\beta\{\mathsf{E}(\mathrm{z} + \zeta \mathcal{F})\}(\hat{\mathrm{a}}) &\leq \beta\{\mathsf{E}\mathrm{z}\}(\hat{\mathrm{a}}) + \zeta\xi\mathcal{F} \\
\beta\zeta\{\mathsf{E}\mathcal{F}\}(\hat{\mathrm{a}}) &\leq \zeta\xi\mathcal{F} \\
\beta\{\mathsf{E}\mathcal{F}\}(\hat{\mathrm{a}}) &\leq \xi\mathcal{F} \\
\beta\{\mathsf{E}\mathcal{F}\}(\hat{\mathrm{a}}) &< \mathcal{F}.
\end{aligned}$$

Using $\mathcal{F}(m) = \eta + m + m^{1-\rho}$ and defining $\hat{a}_t = \hat{\mathrm{a}}(m_t)$, this condition is

.

$$\beta E_t \Gamma_{t+1}^{1-\rho} \left(\hat{a}_t \mathcal{R}_{t+1} + \xi_{t+1} + (\hat{a}_t \mathcal{R}_{t+1} + \xi_{t+1})^{1-\rho}\right) - m_t - m_t^{1-\rho} \;\; < \;\; \eta(1 - \underbrace{\beta E_t \Gamma_{t+1}^{1-\rho}}_{=\daleth})$$

which by imposing the FVC condition $\daleth < 1$ can be rewritten as:

.

$$\eta > \frac{\beta E_t \left[\Gamma_{t+1}^{1-\rho}(\hat{a}_t \mathcal{R}_{t+1} + \xi_{t+1})\right] - m_t + \beta E_t \left[\Gamma_{t+1}^{1-\rho}(\hat{a}_t \mathcal{R}_{t+1} + \xi_{t+1})^{1-\rho}\right] - m_t^{1-\rho}}{1 - \daleth} \tag{31}$$

Since η is an arbitrary constant that we can pick, the proof thus reduces to showing that the numerator of (31) is bounded from above. Consider first the part:

.

$$\begin{aligned}
\beta E_t \left[\Gamma_{t+1}^{1-\rho}\left(\hat{a}_t \mathcal{R}_{t+1} + \xi_{t+1}\right)\right] - m_t &\leq \beta E_t \left[\Gamma_{t+1}^{1-\rho}\left((1-\underline{\hat{\kappa}}) m_t \mathcal{R}_{t+1} + \xi_{t+1}\right)\right] - m_t \\
&= m_t((1-\underline{\hat{\kappa}})\beta E_t \Gamma_{t+1}^{1-\rho}\mathcal{R}_{t+1} - 1) + \beta E_t \Gamma_{t+1}^{1-\rho}\xi_{t+1} \\
&= m_t((1-\underline{\hat{\kappa}})\underbrace{\beta \mathsf{R} E_t \Gamma_{t+1}^{-\rho}}_{<1 \text{ by GIC}} - 1) + \beta E_t \Gamma_{t+1}^{1-\rho} \\
&< \beta E_t \Gamma_{t+1}^{1-\rho} = \beth, \qquad (32)
\end{aligned}$$

so the first part is bounded from above by $\beth$.

Turning to the remaining part of the numerator:

.

$$
\begin{aligned}
& \cancel{\wp}\beta E_t\left[\Gamma_{t+1}^{1-\rho}(\hat{a}_t\mathcal{R}_{t+1}+\theta_{t+1}/\cancel{\wp})^{1-\rho}\right]+\wp\beta E_t\left[\Gamma_{t+1}^{1-\rho}(\hat{a}_t\mathcal{R}_{t+1})^{1-\rho}\right]-m_t^{1-\rho} \\
\leq\ & \cancel{\wp}\beta E_t\left[\Gamma_{t+1}^{1-\rho}((1-\bar{\kappa})m_t\mathcal{R}_{t+1}+\theta_{t+1}/\cancel{\wp})^{1-\rho}\right]+\wp\beta\mathsf{R}^{1-\rho}((1-\bar{\kappa})m_t)^{1-\rho}-m_t^{1-\rho} \\
=\ & \cancel{\wp}\beta E_t\left[\Gamma_{t+1}^{1-\rho}((1-\bar{\kappa})m_t\mathcal{R}_{t+1}+\theta_{t+1}/\cancel{\wp})^{1-\rho}\right]+m_t^{1-\rho}\left(\wp\beta\mathsf{R}^{1-\rho}\left(\wp^{1/\rho}\frac{(\mathsf{R}\beta)^{1/\rho}}{\mathsf{R}}\right)^{1-\rho}-1\right) \\
=\ & \cancel{\wp}\beta E_t\left[\Gamma_{t+1}^{1-\rho}((1-\bar{\kappa})m_t\mathcal{R}_{t+1}+\theta_{t+1}/\cancel{\wp})^{1-\rho}\right]+m_t^{1-\rho}\left(\underbrace{\wp^{1/\rho}\frac{(\mathsf{R}\beta)^{1/\rho}}{\mathsf{R}}}_{<1\text{ by WRIC}}-1\right) \qquad (33) \\
<\ & \cancel{\wp}\beta E_t\left[(\underline{\theta}/\cancel{\wp})^{1-\rho}\right]=\beth\cancel{\wp}^{\rho}\underline{\theta}^{1-\rho}.
\end{aligned}
$$

We can thus conclude that equation (31) will certainly hold for any:

.

$$
\eta>\underline{\eta}=\frac{\beth(1+\cancel{\wp}^{\rho}\underline{\theta}^{1-\rho})}{1-\beth} \qquad (34)
$$

which is a positive finite number under our assumptions.

The proof that $\mathfrak{T}$ defines a contraction mapping under the conditions (16), (23), and (29) is now complete.

2.9 $\mathfrak{T}$ and v

In defining our operator $\mathfrak{T}$ we made the restriction $\underline{\hat{\kappa}} m_t \leq c_t \leq \bar{\kappa} m_t$. However, in the discussion of the consumption function bounds, we showed only (in (26)) that $\underline{\hat{\kappa}}_t m_t \leq c_t \leq \bar{\kappa}_t m_t$. (The difference is in the presence or absence of time subscripts on the MPC's). We have therefore not proven (yet) that the sequence of value functions (17) defines a contraction mapping.

Fortunately, the proof of that proposition is identical to the proof in 2.8, except that we must replace $\bar{\kappa}$ with $\bar{\kappa}_{T-1}$ and the WRIC must be replaced by a stronger condition. The place where these conditions have force is in the step at (33). Consideration of the prior two equations reveals that a sufficient stronger condition is

.

$$\begin{aligned}
\wp\beta(\mathsf{R}(1-\bar{\kappa}_{T-1}))^{1-\rho} &< 1 \\
(\wp\beta)^{1/(1-\rho)}(1-\bar{\kappa}_{T-1}) &< 1 \\
(\wp\beta)^{1/(1-\rho)}(1-(1+\underline{\lambda})^{-1}) &< 1
\end{aligned}$$

where we have used (22) for $\bar{\kappa}_{T-1}$. For small values of $\wp$ this expression can be further simplified using $(1+\lambda)^{-1} \approx 1-\lambda$ (where λ is defined in (21)) so that it becomes

.

$$
\begin{aligned}
(\wp\beta)^{1/(1-\rho)}\underline{\lambda} &< 1 \\
(\wp\beta)^{1/(1-\rho)}\wp^{1/\rho}\text{Þ}_{\mathsf{R}} &< 1 \\
(\wp\beta)\wp^{(1-\rho)/\rho}\text{Þ}_{\mathsf{R}}^{1-\rho} &< 1 \\
\beta\wp^{1/\rho}\text{Þ}_{\mathsf{R}}^{1-\rho} &< 1
\end{aligned}
$$

which for small values of $\wp$ is plainly easy to satisfy.

The upshot is that under these slightly stronger conditions the value functions for the original problem define a contraction mapping with a unique v(m). But since $\lim_{n\rightarrow\infty}\underline{\hat{\kappa}}_{T-n} = \underline{\hat{\kappa}}$ and $\lim_{n\rightarrow\infty}\bar{\kappa}_{T-n} = \bar{\kappa}$, it must be the case that the v(m) toward which these v$_{T-n}$'s are converging is the *same* v(m) that was the endpoint of the contraction defined by our operator $\mathfrak{T}$. Thus, under our slightly stronger (but still quite weak) condition not only do the value functions defined by (17) converge, they converge to the same unique v defined by

22It seems likely that convergence of the value functions for the original problem could be proven even if only the WRIC were imposed; but that proof is not an essential part of the enterprise of this paper and is therefore left for future work.

$\mathfrak{T}$.22

2.10 Convergence of Consumption Functions

We have shown that the v$_{T-n}$ functions converge in a F−bounded space. What we are really interested in, however, is convergence of the consumption policy functions in Euclidian space. The proof that the former implies the latter is uninteresting and is relegated to appendices C and D.

2.11 Liquidity Constraints as a Limit

This section shows that a related problem commonly considered in the literature (e.g.xby Deatonx(1991)), with a liquidity constraint and a positive minimum value of income, is the limit of the model considered here as the probability $\wp$ of the zero-income event approaches zero.

Formally, suppose we change the description of the problem by making the following two assumptions:

.

$$\begin{aligned} \wp &= 0 \\ c_t &\leq m_t, \end{aligned}$$

and suppose we designate the solution to this consumer's problem $\grave{\mathrm{c}}_t(m)$. We will henceforth refer to this as the problem of the 'restrained' consumer (and, to avoid a common confusion, we will refer to the consumer as 'constrained' only in circumstances when the constraint is actually binding).

Redesignate the consumption function that emerges from our original problem for a given fixed $\wp$ as $\mathrm{c}_t(m;\wp)$ where we separate the arguments by a semicolon to distinguish between m, which is a state variable, and $\wp$, which is not. The proposition we wish to demonstrate is

.

$$\lim_{\wp \downarrow 0} \mathrm{c}_t(m;\wp) \;=\; \grave{\mathrm{c}}_t(m).$$

We will first examine the problem in period $T-1$, then argue that the key result propagates to earlier periods. For simplicity, suppose that the interest, growth, and time-preference factors are $\beta = \mathsf{R} = \Gamma = 1$, and there are no permanent shocks, $\psi = 1$; the results below are easily generalized to the full-fledged version of the problem.

The solution to the restrained consumer's optimization problem can be obtained as follows. Assuming that the consumer's behavior in period T is given by $\mathrm{c}_T(m)$ (in practice, this will be $\mathrm{c}_T(m) = m$), consider the unrestrained optimization problem

.

$$\grave{\mathrm{a}}^*_{T-1}(m) \;=\; \arg\max_{a} \left\{ \mathrm{u}(m-a) + \int_{\underline{\theta}}^{\bar{\theta}} \mathrm{v}_T(a+\theta)dF_\theta \right\}. (35)$$

As usual, the envelope theorem tells us that $\mathrm{v}_T{}'(m) = \mathrm{u}'(\mathrm{c}_T(m))$ so the expected marginal value of ending period $T-1$ with assets a can be defined as

.

$$\grave{\mathfrak{v}}'_{T-1}(a) \;\equiv\; \int_{\underline{\theta}}^{\bar{\theta}} \mathrm{u}'(\mathrm{c}_T(a+\theta))dF_\theta,$$

and the solution to (35) will satisfy

.

$$\mathrm{u}'(m-a) \;=\; \grave{\mathfrak{v}}'_{T-1}(a). \tag{36}$$

$\grave{\mathrm{a}}_{T-1}{}^{*}(m)$ therefore answers the question "With what level of assets would the restrained consumer like to end period $T-1$ if the constraint $c_{T-1} \leq m_{T-1}$ did not exist?" (Note that the restrained consumer's income process remains different from the process for the unrestrained consumer so long as $\wp > 0$). The restrained consumer's actual asset position will be

.

$$\grave{\mathrm{a}}_{T-1}(m) \;=\; \max[0, \grave{\mathrm{a}}^{*}_{T-1}(m)],$$

reflecting the inability of the restrained consumer to spend more than current resources, and note (as pointed out by Deatonx(1991)) that

.

$$m^{\#}_{T-1} \;=\; \left(\grave{\mathfrak{v}}'_{T-1}(0)\right)^{-1/\rho}$$

is the cusp value of m at which the constraint makes the transition from binding to not binding.

Analogously to (36), defining

.

$$\mathfrak{v}'_{T-1}(a;\wp) \;\equiv\; \left[\wp a^{-\rho} + \cancel{\wp}\int_{\underline{\theta}}^{\bar{\theta}} \left(\mathrm{c}_T(a+\theta/\cancel{\wp})\right)^{-\rho} dF_\theta\right], \quad (37)$$

the Euler equation for the original consumer's problem implies

.

$$(m - a)^{-\rho} \;=\; \mathfrak{v}'_{T-1}(a; \wp) \tag{38}$$

with solution $\mathrm{a}_{T-1}(m; \wp)$. Now note that for a fixed $a > 0$, $\lim_{\wp \downarrow 0} \mathfrak{v}_{T-1}{}'(a; \wp) = \grave{\mathfrak{v}}_{T-1}{}'(a)$. Since the LHS of (36) and (38) are identical, this means that for such an m $\lim_{\wp \downarrow 0} \mathrm{a}_{T-1}(m; \wp) = \grave{\mathrm{a}}_{T-1}{}^{*}(m)$. That is, for any fixed value of $m > m_{T-1}{}^{\#}$ such that the consumer subject to the restraint would voluntarily choose to end the period with positive assets, the level of end-of-period assets for the unrestrained consumer approaches the level for the restrained consumer as $\wp \downarrow 0$. With the same a and the same m, the consumers must have the same c, so the consumption functions are identical in the limit.

Now consider values $m \leq m_{T-1}{}^{\#}$ for which the restrained consumer is constrained. It is obvious that the baseline consumer will never choose $a \leq 0$ because the first term in in (37) is $\lim_{a \downarrow 0} \wp a^{-\rho} = \infty$, while $\lim_{a \downarrow 0}(m - a)^{-\rho}$ is finite (the marginal value of end-of-period assets approaches infinity as assets approach zero, but the marginal utility of consumption has a finite limit for $m > 0$). The subtler question is whether it is possible to rule out strictly positive a for the unrestrained consumer.

The answer is yes. Suppose, for some $m < m^{\#}$, that the unrestrained consumer is considering ending the period with any positive amount of assets $a = \delta > 0$. For any such δ we have that $\lim_{\wp \downarrow 0} \mathfrak{v}_{T-1}{}'(a; \wp) = \grave{\mathfrak{v}}_{T-1}{}'(a)$. But by assumption we are considering a set of circumstances in which $\grave{\mathrm{a}}_{T-1}{}^{*}(m) < 0$, and we showed earlier that $\lim_{\wp \downarrow 0} \mathrm{a}_{T-1}{}^{*}(m; \wp) = \grave{\mathrm{a}}_{T-1}{}^{*}(m)$. So, having assumed $a = \delta > 0$, we have proven that the consumer would optimally choose $a < 0$, which is a contradiction. A similar argument holds for $m = m_{T-1}{}^{\#}$.

These arguments demonstrate that for any $m > 0$, $\lim_{\wp \downarrow 0} \mathrm{c}_{T-1}(m;\wp) = \grave{\mathrm{c}}_{T-1}(m)$ which is the period $T-1$ version of (35). But given equality of the period $T-1$ consumption functions, backwards recursion of the same arguments demonstrates that the limiting consumption functions in previous periods are also identical in the limit.

Note finally that another intuitive confirmation of the equivalence between the two problems is that our formulax(23) for the maximal marginal propensity to consume satisfies

.

$$\lim_{\wp \downarrow 0} \bar{\kappa} = 1,$$

which makes sense because the marginal propensity to consume for a constrained restrained consumer is 1 by the definition of 'constrained.'

2.12 Discussion of Conditions Required for Convergence

This section interprets the conditions (16) and (29) that were imposed in proving that the problem defines a contraction mapping.

To put these conditions on a comparable footing, rewrite the GIC as

.

$$\overbrace{\underbrace{(\mathsf{R}\beta)\Gamma^{-\rho}}_{=\text{Þ}_{\Gamma}} E[\psi^{-\rho}]}^{\mathcal{P}_{\acute{\Gamma}}} \; < \; 1 \tag{39}$$

which is stronger than the perfect foresight PF-GIC (12) because for nondegenerate ψ, if $E[\psi] = 1$ then $E[\psi^{-\rho}] > 1$ by Jensen's inequality. The GIC is stronger under uncertainty because the extra precautionary saving motive induced by uncertainty has the potential to convert a consumer who would be growth-impatient in a perfect foresight world (running down m as far as possible) into a consumer who is growth-patient (will accumulate m without bound) in a world containing uninsurable permanent

23This insight is useful because it contradicts a commonly held intuition that since consumption cannot be insured against permanent shocks, it will always be optimal for consumption to respond one-for-one to a permanent shock (regardless of parameter values and for any size of the permanent shock). Deatonx(1991) proves such a proposition in his model for a liquidity constrained consumer with zero initial assets (though Carrollx(2001) shows that if assets are nonzero then the MPC will be strictly less than one). One way to express the caveat is that for zero-asset liquidity constrainted consumers the MPC out of permanent shocks is one *so long as the GIC holds*, unless uncertainty increases to a sufficient degree to convert the consumer from growth-impatient to growth-patient. As a sidenote, if we imagine that consumers harbor even the smallest degree of Bayesian uncertainty about the magnitude of the shocks that they face, a large enough negative shock might have a substantial effect on the consumer's perception of the variance of future risks, perhaps inducing patience endogenously. This train of thought leads to an interesting set of questions that could fruitfully be pursued in future research.

risks.23

The Finite Value Condition (29) can be expressed as

.

$$\begin{aligned} \mathsf{R}\beta\Gamma^{1-\rho}E[\psi^{1-\rho}] &< \mathsf{R} \\ \text{Þ}_{\Gamma}\underbrace{E[\psi^{1-\rho}]}_{<E[\psi^{-\rho}]} &< \mathsf{R}/\Gamma \end{aligned} \qquad (40)$$

where $E[\psi^{1-\rho}] < E[\psi^{-\rho}]$ follows from $\rho > 1$ and Jensen's inequality. Thus, a sufficient condition for the problem to define a contraction mapping is

.

$$\text{Þ}_{\Gamma} E[\psi^{-\rho}] \;<\; \min[1, \mathsf{R}/\Gamma]. \tag{41}$$

Two cases are worth considering. The first is where $\mathsf{R}/\Gamma > 1$, which corresponds to the Finite Human Wealth condition imposed for solving the perfect foresight model. In this case, obviously (39) is stronger than (40) and so the finite value condition will be guaranteed by the GIC and therefore is not binding.

More interesting (though less likely) is the case where $\mathsf{R}/\Gamma < 1$; here, the rewritten FVC (40) will be tighter than the rewritten GIC (39). This case is interesting because it represents a circumstance where the model with uncertainty produces a contraction mapping with a well defined solution even though the corresponding unconstrained perfect foresight model does not have a well defined solution (because perfect foresight human wealth is infinite).

The intuition for why the model has a solution despite infinite human wealth comes from the realization that the precautionary motive acts like a self-imposed liquidity constraint. To see this most clearly, notice that a consumer who spent his permanent income in every period would experience a value given by

.

$$
\begin{aligned}
\boldsymbol{v}_t &= \mathrm{u}(\mathbf{p}_t) + \beta E_t[\mathrm{u}(\mathbf{p}_t\Gamma_{t+1})] + ... \\
&= \mathrm{u}(\mathbf{p}_t)\left(1 + \beta E_t[\Gamma_{t+1}^{1-\rho}] + \beta^2 E_t[\Gamma_{t+1}^{1-\rho}\Gamma_{t+2}^{1-\rho}] + \ldots\right) \\
&= \mathrm{u}(\mathbf{p}_t)\left(1 + \beta E_t[\Gamma_{t+1}^{1-\rho}] + \beta^2 E_t[\Gamma_{t+1}^{1-\rho}]E_t[\Gamma_{t+2}^{1-\rho}] + \ldots\right)
\end{aligned}
$$

which will be finite so long as the FVC $\beth < 1$ holds. This says that the FVC corresponds to a requirement that value for a *constrained* consumer is finite, which is obviously possible even if the expectation of the consumer's future labor income is infinity, because for a constrained consumer infinite but uncertain human wealth cannot be translated into infinite current consumption. The FVC therefore can be understood as defining the circumstances under which the precautionary motive with respect to permanent shocks guarantees finite consumption even if the expectation (in a present discounted value sense) of human wealth is

24Recall that we are defining the infinite horizon solution as the limit of the finite horizon solution as the horizon approaches infinity.

unbounded.[24]

A particularly transparent case is $\mathsf{R} = 1$ and $\Gamma \geq 1$. Human wealth is infinite so the perfect foresight infinite horizon model has no solution, but if $\beta E[\Gamma^{1-\rho}] < 1$ our model does have a limiting solution.

3 Analysis of the Converged Consumption Function

Figuresx2 and 3a,b capture the main properties of the converged consumption

25These figures reflect the converged rule corresponding to the parameter values indicated in tablex1.

rule.[25]

Figurex2 shows the expected consumption growth factor $E_t[\boldsymbol{c}_{t+1}/\boldsymbol{c}_t]$ for a consumer using the converged consumption rule, while Figuresx3a,b illustrate theoretical bounds for the consumption function and the marginal propensity to consume.

Five features of behavior are captured, or suggested, by the figures. First, as $m_t \rightarrow \infty$ the expected consumption growth factor goes to $(\mathsf{R}\beta)^{1/\rho}$, indicated by the lower bound in figure 2, and the marginal propensity to consume approaches $\kappa = (1 - \text{Þ}_{\mathsf{R}})$ (figurex3), the same as the perfect foresight MPC. Second, as $m_t \rightarrow 0$ the consumption growth factor approaches ∞ (figurex2) and the MPC approaches $\bar{\kappa} = (1 - \mathsf{R}^{-1}(\mathsf{R}\beta\wp)^{1/\rho})$ (figurex3). Third (figure 2), there is a target cash-on-hand-to-income ratio $\check{m}$ such that if $m_t = \check{m}$ then $E_t[m_{t+1}] = m_t$, and (as indicated by the arrows of motion on the $E_t[\boldsymbol{c}_{t+1}/\boldsymbol{c}_t]$ curve), the model's dynamics are 'stable' around the target in the sense that if $m_t < \check{m}$ then cash-on-hand will rise (in expectation), while if $m_t > \check{m}$, it will fall (in expectation). Fourth (figure 2), at the target m, the expected rate of growth of consumption is slightly less than the expected growth rate of permanent labor income. The final proposition suggested by figurex2 is that the expected consumption growth factor is declining in the level of the cash-on-hand ratio m_t. This turns out to be true in the absence of permanent shocks, but in extreme cases it can be false if permanent shocks are

26Throughout the remaining analysis I make a final assumption that is not strictly justified by the foregoing. We have shown that the finite-horizon consumption functions $\mathrm{c}_{T-n}(m)$ are twice continuously differentiable and strictly concave, and that they converge to a continuous function $\mathrm{c}(m)$. It does not strictly follow that $\mathrm{c}(m)$ is twice continuously differentiable, but I will assume that it is.

present.26

PIC

Figurex2: Target Saving, Expected Consumption Growth, and Permanent Income Growth

3.1 Limits as $m_t \to \infty$

Define

.

$$\underline{c}(m) = \underline{\kappa} m$$

which is the solution to an infinite-horizon problem with no labor income ($\xi_t = 0 x \forall x t$); clearly $\underline{c}(m) < c(m)$, since allowing the possibility of future labor income cannot reduce current

27We the RIC here and subsequently so that $\kappa > 0$; the situation is a bit more complex when the RIC does not hold. The key point is that in that case the bound on consumption is given by the spending that would be undertaken by a consumer who faced binding liquidity constraints. Detailed analysis of this special case is not sufficiently interesting to warrant inclusion in the paper.

consumption.27

Assume that the FHW condition holds so that $\Gamma < \mathsf{R}$ and the infinite horizon perfect foresight solution,

.

$$\bar{\mathrm{c}}(m) \;=\; (m-1+h)\underline{\kappa},$$

exists (to obtain this, divide (8) by $\mathbf{p}_t$; the $\Gamma \geq \mathsf{R}$ case is discussed below). This constitutes an upper bound on consumption in the presence of uncertainty, since Carroll and Kimballx(1996) show that the introduction of uncertainty strictly decreases the level of consumption at any m.

Thus, we can write

.

$$\begin{aligned} \underline{\mathrm{c}}(m) < &\quad \mathrm{c}(m) &< \bar{\mathrm{c}}(m) \\ 1 < &\quad \mathrm{c}(m)/\underline{\mathrm{c}}(m) &< \bar{\mathrm{c}}(m)/\underline{\mathrm{c}}(m). \end{aligned}$$

But

.

$$
\begin{aligned}
\lim_{m\to\infty} \bar{\mathrm{c}}(m)/\underline{\mathrm{c}}(m) &= \lim_{m\to\infty} (m-1+h)/m \\
&= 1.
\end{aligned}
$$

Hence, as $m \to \infty, \mathrm{c}(m)/\mathrm{c}(m) \to 1$, and the continuous differentiability and strict concavity of $\mathrm{c}'(m)$ therefore imply

.

$$
\lim_{m\to\infty} \mathrm{c}'(m) = \underline{\mathrm{c}}'(m) = \bar{\mathrm{c}}'(m) = \underline{\kappa}
$$

because any other fixed limit would eventually lead to a level of consumption either exceeding $\bar{\mathrm{c}}(m)$ or lower than $\mathrm{c}(m)$.

Figurex3 confirms these limits visually. The top plot shows the converged consumption function along with its upper and lower bounds, while the lower plot shows the marginal propensity to consume.

PIC

PIC

Figurex3: Bounds (in Dark) of c(m) and c$'$(m) as $m_t \to \infty$ and $m_t \to 0$

Next we establish the limit of the expected consumption growth factor as $m_t \to \infty$:

.

$$\lim_{m_t \to \infty} E_t[\boldsymbol{c}_{t+1}/\boldsymbol{c}_t] \;=\; \lim_{m_t \to \infty} E_t[\Gamma_{t+1} c_{t+1}/c_t].$$

But

.

$$\lim_{m_t \to \infty} \Gamma_{t+1} \underline{\mathrm{c}}(m_{t+1})/\bar{\mathrm{c}}(m_t) = \lim_{m_t \to \infty} \Gamma_{t+1} \bar{\mathrm{c}}(m_{t+1})/\underline{\mathrm{c}}(m_t) = \lim_{m_t \to \infty} \Gamma_{t+1} m_{t+1}/m_t,$$

and

.

$$
\begin{aligned}
\lim_{m_t \to \infty} \Gamma_{t+1} m_{t+1}/m_t &= \lim_{m_t \to \infty} \left(\frac{\mathsf{R}\mathrm{a}(m_t) + \Gamma_{t+1}\xi_{t+1}}{m_t} \right) \\
&= (\mathsf{R}\beta)^{1/\rho}
\end{aligned}
$$

because $\lim_{m_t \to \infty} \mathrm{a}'(m) = \text{Þ}_{\mathsf{R}}$ and $\Gamma_{t+1}\xi_{t+1}/m_t \leq (\Gamma \bar{\psi}\bar{\theta}/\wp\!\!\!/)/m_t$ which goes to zero as m_t goes to infinity.

Hence we have

.

$$
(\mathsf{R}\beta)^{1/\rho} \leq \lim_{m_t \to \infty} E_t[\boldsymbol{c}_{t+1}/\boldsymbol{c}_t] \leq (\mathsf{R}\beta)^{1/\rho}
$$

so as cash goes to infinity, consumption growth approaches its value in the perfect foresight model.

This argument applies equally well to the problem of the restrained consumer, because as m approaches infinity the constraint becomes irrelevant.

Of course, the constraint never becomes irrelevant if human wealth is infinite. We ruled out infinite human wealth at the beginning of this section by assuming $\mathsf{R} > \Gamma$. If this finite human wealth condition does not hold, it is possible to show that for any finite horizon consumer the marginal propensity to consume approaches the finite-horizon perfect foresight MPC as wealth approaches infinity. However, as the horizon gets longer, the perfect foresight MPC approaches zero. It can be shown therefore that the limiting MPC for the converged consumption function approaches (but never reaches) zero. (This is why we chose $\underline{\hat{\kappa}} = 0$ if the FHW condition fails in the proofs above.)

3.2 Limits as $m_t \rightarrow 0$

Now consider the limits of behavior as m_t gets arbitrarily small.

Equation (22) implies that the limiting value of $\bar{\kappa}$ is

.

$$\bar{\kappa} \;=\; 1 - \mathsf{R}^{-1}(\wp\mathsf{R}\beta)^{1/\rho}.$$

Defining $\mathrm{e}(m) = \mathrm{c}(m)/m$ as before we have

.

$$\lim_{m \downarrow 0} \mathrm{e}(m) \;=\; (1 - \wp^{1/\rho}\text{Þ}_{\mathsf{R}}) = \bar{\kappa}.$$

Now differentiate

.

$$\begin{aligned} \mathrm{e}'(m) &= m^{-1}\mathrm{c}'(m) - m^{-2}\mathrm{c}(m) \\ m\mathrm{e}'(m) &= \mathrm{c}'(m) - \mathrm{c}(m)/m \\ \mathrm{c}'(m) &= \mathrm{e}(m) + m\mathrm{e}'(m) \end{aligned}$$

and using the continuous differentiability of the consumption function along with $0 < \mathrm{e}(m) < 1$ we have

.

$$\lim_{m \downarrow 0} \mathrm{c}'(m) \quad = \quad \lim_{m \downarrow 0} \mathrm{e}(m) = \bar{\kappa}.$$

Figurex3 confirms that standard numerical solution methods obtain this limit for the MPC as m approaches zero.

For consumption growth, we have

.

$$\begin{aligned}
\lim_{m_t \downarrow 0} E_t \left[\left(\frac{\mathrm{c}(m_{t+1})}{\mathrm{c}(m_t)} \right) \Gamma_{t+1} \right] \quad &> \quad \lim_{m_t \downarrow 0} E_t \left[\left(\frac{\underline{\mathrm{c}}(\mathcal{R}_{t+1}\mathrm{a}(m_t) + \xi_{t+1})}{\bar{\kappa} m_t} \right) \Gamma_{t+1} \right] \\
&> \quad \lim_{m_t \downarrow 0} E_t \left[\left(\frac{\underline{\mathrm{c}}(\xi_{t+1})}{\bar{\kappa} m_t} \right) \Gamma_{t+1} \right] \\
&= \quad \infty
\end{aligned}$$

where the last line follows because the minimum possible realization of ξ_{t+1} is $\theta > 0$ so the minimum possible value of expected next-period consumption is

28The same arguments establish $\lim_{m \downarrow 0} E_t[\boldsymbol{c}_{t+1}/\boldsymbol{c}_t] = \infty$ for the problem of the restrained consumer.

positive.28

3.3 There exists exactly one target cash-on-hand ratio, which is stable

Define the target cash-on-hand-to-income ratio $\check{m}$ as the value of m such that

.

$$E_t[m_{t+1}/m_t] = 1 \text{ if } m_t = \check{m}. \tag{42}$$

where the $\vee$ accent is meant to invoke the fact that this is the value that other m's 'point to.'

We prove existence by arguing that $E_t[m_{t+1}/m_t]$ is continuous on $m_t > 0$, and takes on values both above and below 1, so that it must equal 1 somewhere by the intermediate value theorem.

Specifically, the same logic used in sectionx3.2 shows that $\lim_{m_t \downarrow 0} E_t[m_{t+1}/m_t] = \infty$.

The limit as m_t goes to infinity is

.

$$
\begin{aligned}
\lim_{m_t \to \infty} E_t[m_{t+1}/m_t] &= \lim_{m_t \to \infty} E_t\left[\frac{\mathcal{R}_{t+1}\mathrm{a}(m_t) + \xi_{t+1}}{m_t}\right] \\
&= E_t[(\mathsf{R}/\Gamma_{t+1})\text{\thorn}_{\mathsf{R}}] \\
&= E_t[(\mathsf{R}\beta)^{1/\rho}/\Gamma_{t+1}]
\end{aligned}
$$

and Jensen's inequality implies that

.

$$
E_t[(\mathsf{R}\beta)^{1/\rho}/\Gamma_{t+1}] < 1 \tag{43}
$$

under our assumptions.

Stability means that in a local neighborhood of $\check{m}$, values of m_t above $\check{m}$ will result in a smaller ratio of $E_t[m_{t+1}/m_t]$ than at $\check{m}$. That is, if $m_t > \check{m}$ then $E_t[m_{t+1}/m_t] < 1$. This will be true if

.

$$\left(\frac{d}{dm_t}\right) E_t[m_{t+1}/m_t] \;<\; 0$$

at $m_t = \check{m}$. But

.

$$\begin{aligned}
\left(\frac{d}{dm_t}\right) E_t[m_{t+1}/m_t] &= E_t\left[\left(\frac{d}{dm_t}\right)[\mathcal{R}_{t+1}(1 - \mathrm{c}(m_t)/m_t) + \xi_{t+1}/m_t]\right] \\
&= E_t\left[\frac{\mathcal{R}_{t+1}(\mathrm{c}(m_t) - \mathrm{c}'(m_t)m_t) - \xi_{t+1}}{m_t^2}\right]
\end{aligned}$$

which will be negative if its numerator is negative. Define $\boldsymbol{\zeta}(m_t)$ as the expectation of the numerator,

.

$$\boldsymbol{\zeta}(m_t) = \underbrace{E_t[\mathcal{R}_{t+1}]}_{\equiv\bar{\mathcal{R}}}(\mathrm{c}(m_t) - \mathrm{c}'(m_t)m_t) - 1.$$

(44)

The target level of market resources $\check{m}$ satisfies

.

$$\begin{aligned} E_t[m_{t+1}] &= E_t[\mathcal{R}_{t+1}(m_t - c_t) + \xi_{t+1}] \\ \check{m} &= \bar{\mathcal{R}}(\check{m} - \mathrm{c}(\check{m})) + 1 \\ \bar{\mathcal{R}}\mathrm{c}(\check{m}) &= 1 - (1 - \bar{\mathcal{R}})\check{m}. \end{aligned} \tag{45}$$

At the target, equation (44) is

.

$$\boldsymbol{\zeta}(\check{m}) = \bar{\mathcal{R}}\mathrm{c}(\check{m}) - \bar{\mathcal{R}}\mathrm{c}'(\check{m})\check{m} - 1.$$

Substituting for the first term in this expression using (45) gives

.

$$
\begin{aligned}
\boldsymbol{\zeta}(\check{m}) &= 1-(1-\bar{\mathcal{R}})\check{m}-\bar{\mathcal{R}}\mathrm{c}'(\check{m})\check{m}-1 \\
&= \check{m}\left(\bar{\mathcal{R}}-1-\bar{\mathcal{R}}\mathrm{c}'(\check{m})\right) \\
&= \check{m}\left(\bar{\mathcal{R}}(1-\mathrm{c}'(\check{m}))-1\right) \\
&< \check{m}\left(\bar{\mathcal{R}}(1-(1-\mathsf{R}^{-1}(\mathsf{R}\beta)^{1/\rho}))-1\right) \\
&= \check{m}\left(\bar{\mathcal{R}}(\text{\TH}_{\mathsf{R}})-1\right) \\
&= \check{m}\left(\underbrace{E_t[(\mathsf{R}\beta)^{1/\rho}/\Gamma_{t+1}]}_{<1 \text{ from } (43)}-1\right) \\
&< 0.
\end{aligned}
$$

We have now proven that some target $\check{m}$ must exist, and that at any such $\check{m}$ the solution is stable. Nothing we have said so far, however, rules out the possibility that there will be multiple values of m that satisfy the definition (42) of a target.

Multiple targets can be ruled out as follows. Suppose there exist multiple targets; these can be arranged in ascending order and indexed by an i superscript, so that the target with the smallest value is, e.g., $\check{m}^1$. The argument just completed implies that since $E_t[m_{t+1}/m_t]$ is continuously differentiable there must exist some small ϵ such that $E_t[m_{t+1}/m_t] < 1$ for $m_t = \check{m}^1 + \epsilon$. (Continuous differentiability of $E_t[m_{t+1}/m_t]$ follows from the continous differentiability of $\mathrm{c}(m_t)$).

Now assume there exists a second value of m satisfying the definition of a target, $\check{m}^2$. Since $E_t[m_{t+1}/m_t]$ is continuous, it must be approaching 1 from below as $m_t \rightarrow \check{m}^2$, since by the intermediate value theorem it could not have gone above 1 between $\check{m}^1 + \epsilon$ and $\check{m}^2$ without passing through 1, and by the definition of $\check{m}^2$ it cannot have passed through 1 before reaching $\check{m}^2$. But saying that $E_t[m_{t+1}/m_t]$ is approaching 1 from below as $m_t \rightarrow \check{m}^2$ implies that

.

$$\left(\frac{d}{dm_t}\right) E_t[m_{t+1}/m_t] \ > \ 0 \tag{46}$$

at $m_t = \check{m}^2$. However, we just showed above that precisely the opposite of equation (46) holds for any m that satisfies the definition of a target. Thus, assuming the existence of more than one target implies a contradiction, and so there must be only one target $\check{m}$.

The foregoing arguments rely on the continuous differentiability of $\mathrm{c}(m)$, so the arguments do not directly go through for the restrained consumer's problem in which the existence of liquidity constraints can lead to discrete changes in the slope $\mathrm{c}'(m)$ at particular values of m. But we can use the fact that the restrained model is the limit of the baseline model as $\wp \downarrow 0$ to conclude that there is likely a unique target cash level even in the restrained model.

If consumers are sufficiently impatient, however, the target level in the restrained model will be $\check{m} = E_t[\xi_{t+1}] = 1$. That is, if a consumer starting with $m = 1$ will save nothing, $\mathrm{a}(1) = 0$, then the target level of m in the restrained model will be 1; if a consumer with $m = 1$ would choose to save something, then the target level of cash-on-hand will be greater than the expected level of income.

3.4 Expected consumption growth at target m is less than expected permanent income growth

In figurex2 the intersection of the target cash-on-hand ratio locus at $\check{m}$ with the expected consumption growth curve lies below the intersection with the horizontal line representing the expected growth rate of permanent income. This can be proven as follows.

Strict concavity of the consumption function implies that if $E_t[m_{t+1}] = \check{m} = m_t$ then

.

$$
\begin{aligned}
E_t\left[\frac{\Gamma_{t+1}\mathrm{c}(m_{t+1})}{\mathrm{c}(m_t)}\right] &< E_t\left[\left(\frac{\Gamma_{t+1}(\mathrm{c}(\check{m})+\mathrm{c}'(\check{m})(m_{t+1}-\check{m}))}{\mathrm{c}(\check{m})}\right)\right] \\
&= E_t\left[\Gamma_{t+1}\left(1+\left(\frac{\mathrm{c}'(\check{m})}{\mathrm{c}(\check{m})}\right)(m_{t+1}-\check{m})\right)\right] \\
&= \Gamma+\left(\frac{\mathrm{c}'(\check{m})}{\mathrm{c}(\check{m})}\right)E_t\left[\Gamma_{t+1}\left(m_{t+1}-\check{m}\right)\right] \\
&= \Gamma+\left(\frac{\mathrm{c}'(\check{m})}{\mathrm{c}(\check{m})}\right)\left[E_t[\Gamma_{t+1}]\underbrace{E_t[m_{t+1}-\check{m}]}_{=0}+\mathrm{cov}_t(\Gamma_{t+1},m_{t+1})\right]
\end{aligned} \tag{47}
$$

and since $m_{t+1}=(\mathsf{R}/\Gamma_{t+1})\mathrm{a}(\check{m})+\xi_{t+1}$ and $\mathrm{a}(\check{m})>0$ it is clear that $\mathrm{cov}_t(\Gamma_{t+1},m_{t+1})<0$ which implies that the entire term added to Γ in (47) is negative, as required.

3.5 Expected consumption growth is a declining function of m_t (or is it?)

Figurex2 depicts the expected consumption growth factor as a strictly declining function of the cash-on-hand ratio. To investigate this, define

.

$$\boldsymbol{\Upsilon}(m_t) \equiv \Gamma_{t+1}\mathrm{c}(\mathcal{R}_{t+1}\mathrm{a}(m_t)+\xi_{t+1})/\mathrm{c}(m_t) = \boldsymbol{c}_{t+1}/\boldsymbol{c}_t$$

and the proposition in which we are interested is

.

$$(d/dm_t)E_t[\underbrace{\boldsymbol{\Upsilon}(m_{t+1})}_{\boldsymbol{\Upsilon}_{t+1}}] < 0$$

or differentiating through the expectations operator, what we want is

.

$$E\left[\Gamma_{t+1}\left(\frac{\mathrm{c}'(m_{t+1})\mathcal{R}_{t+1}\mathrm{a}'(m_t)\mathrm{c}(m_t)-\mathrm{c}(m_{t+1})\mathrm{c}'(m_t)}{\mathrm{c}(m_t)^2}\right)\right] < 0 \quad (48)$$

Henceforth indicating appropriate arguments by the corresponding subscript (e.g.x$\mathrm{c}_{t+1}{}' \equiv \mathrm{c}'(m_{t+1})$), since $\Gamma_{t+1}\mathcal{R}_{t+1} = \mathsf{R}$, the portion of the LHS of equation (48) in brackets can be manipulated to yield

.

$$\begin{aligned}
\mathrm{c}_t \boldsymbol{\Upsilon}'_{t+1} &= \mathrm{c}'_{t+1}\mathrm{a}'_t\mathsf{R} - \mathrm{c}'_t \Gamma_{t+1}\mathrm{c}_{t+1}/\mathrm{c}_t \\
&= \mathrm{c}'_{t+1}\mathrm{a}'_t\mathsf{R} - \mathrm{c}'_t \boldsymbol{\Upsilon}_{t+1}. \qquad (49)
\end{aligned}$$

Now differentiate the Euler equation with respect to m_t:

.

$$\begin{aligned}
1 &= \mathsf{R}\beta E_t[\boldsymbol{\Upsilon}_{t+1}^{-\rho}] \\
0 &= E_t[\boldsymbol{\Upsilon}_{t+1}^{-\rho-1}\boldsymbol{\Upsilon}'_{t+1}] \\
&= E_t[\boldsymbol{\Upsilon}_{t+1}^{-\rho-1}]E_t[\boldsymbol{\Upsilon}'_{t+1}] + \mathrm{cov}(\boldsymbol{\Upsilon}_{t+1}^{-\rho-1}, \boldsymbol{\Upsilon}'_{t+1}) \\
E_t[\boldsymbol{\Upsilon}'_{t+1}] &= -\mathrm{cov}(\boldsymbol{\Upsilon}_{t+1}^{-\rho-1}, \boldsymbol{\Upsilon}'_{t+1})/E_t[\boldsymbol{\Upsilon}_{t+1}^{-\rho-1}] \qquad (50)
\end{aligned}$$

but since $\boldsymbol{\Upsilon}_{t+1} > 0$ we can see from (50) that (48) is equivalent to

.

$$\mathrm{cov}(\boldsymbol{\Upsilon}_{t+1}^{-\rho-1}, \boldsymbol{\Upsilon}'_{t+1}) \;>\; 0$$

which, using (49), will be true if

.

$$\mathrm{cov}(\boldsymbol{\Upsilon}_{t+1}^{-\rho-1}, \mathrm{c}'_{t+1}\mathrm{a}'_t\mathsf{R} - \mathrm{c}'_t\boldsymbol{\Upsilon}_{t+1}) \;>\; 0$$

which in turn will be true if both

.

$$\mathrm{cov}(\mathbf{\Upsilon}_{t+1}^{-\rho-1}, \mathrm{c}'_{t+1}) \;\; > \;\; 0$$

and

.

$$\mathrm{cov}(\mathbf{\Upsilon}_{t+1}^{-\rho-1}, \mathbf{\Upsilon}_{t+1}) \;\; < \;\; 0.$$

The latter proposition is obviously true. The former will be true if

.

$$\text{cov}\left((\Gamma\psi_{t+1}\text{c}(m_{t+1}))^{-\rho-1}, \text{c}'(m_{t+1})\right) \;>\; 0$$

where recall that $m_{t+1} = (\mathsf{R}/\Gamma\psi_{t+1})a_t + \xi_{t+1}$.

The two shocks cause two kinds of variation in m_{t+1}. Variations due to ξ_{t+1} satisfy the proposition, since a higher draw of ξ both reduces ${c_{t+1}}^{-\rho-1}$ and reduces the marginal propensity to consume. However, permanent shocks have conflicting effects. On the one hand, a higher draw of ψ_{t+1} will reduce m_{t+1}, thus increasing both ${c_{t+1}}^{-\rho-1}$ and ${c_{t+1}}'$. On the other hand, the ${c_{t+1}}^{-\rho-1}$ term is multiplied by $\Gamma\psi_{t+1}$, so the effect of a higher ψ_{t+1} could be to decrease the first term in the covariance, leading to a negative covariance with the second term.

The software archive associated with this paper presents an example in which this perverse effect dominates. However, extreme assumptions were required ($\wp < 0.000000001$, very small transitory shocks) and the region in which ${\boldsymbol{\Upsilon}_{t+1}}' > 0$ was tiny. In practice, for plausible parametric choices, $E_t[{\boldsymbol{\Upsilon}_{t+1}}'] < 0$ should generally hold.

4 The Aggregate and Idiosyncratic Relationship Between Consumption Growth and Income Growth

This section examines the behavior of large collections of buffer-stock consumers with identical parameter values. Such a collection can be thought of as either a subset of the population within a single country (say, members of a given education or occupation group), or as the whole population in a small open economy (we will continue to take the aggregate interest rate as exogenous and constant). It is also possible, though more difficult, to solve a closed-economy version of the model where the interest rate is endogenous.

Formally, we assume a continuum of *ex ante* identical households on the unit interval, with constant total mass normalized to one and indexed by $i \in [0, 1]$, all behaving according to the model specified

[29]One inconvenient aspect of the model as specified is that it does not exhibit a stationary distribution of idiosyncratic permanent labor income; the longer the economy lasts, the wider is the distribution. This problem can be remedied by assuming a constant probability of death, and replacing deceased households with newborns whose initial idiosyncratic permanent income matches the mean idiosyncratic permanent income of the population. For a fully worked-out general equilibrium version of such a model, see Carroll and Tokuokax(2009).

above.[29]

4.1 Convergence of the Cross-Section Distribution

A recent paper by Szeidlx(2006) proves that such a population will be characterized by an invariant distribution of m which induces invariant distributions for c and a; designate these $\mathcal{F}^m$, $\mathcal{F}^a$, and $\mathcal{F}^c$. (Szeidl's proof supplants simulation evidence of ergodicity that appeared in an earlier version of this paper).

The proof of convergence does not yield any sense of how quickly convergence occurs, which in principle depends on all of the parameters of the model as well as the initial conditions. To build intuition, Figurex4 supplies an example in which a population begins with a particularly simple distribution that is far from the invariant one:

.

$$m_{1,i} = \xi_{1,i},$$

which would characterize a population in which all assets had been wiped out immediately before the receipt of period 1's labor income. The figure plots the distributions of a (for technical reasons, this is slightly better than plotting m) at the ends of 1, 4, 10, and 40

30These figures reflect results for the calibration detailed in Table 1, which are representative of the micro literature which has mainly focused on matching behavior of median or "typical" households, who hold little liquid wealth. A higher time preference factor would be necessary to match the behavior of richer households who hold much of the aggregate capital stock. However, for these richer households, the precautionary motives highlighted in this model may be less relevant than other motives which are not well understood.

periods.[30]

PIC

Figurex4: Convergence of $\mathcal{F}^a$ to Invariant Distribution

The figure illustrates the fact that, under these parameter values, convergence to the invariant distribution has largely been accomplished within 10 periods. By 40 periods, the distribution is indistinguishable from the invariant distribution.

4.2 Consumption and Income Growth at the Household Level

It is useful to define the operator $\mathcal{M}[\bullet]$ which yields the mean value of its argument in the population, as distinct from the expectations operator $E[\bullet]$ which represents beliefs about the future.

An economist with a microeconomic dataset could calculate the average growth rate of idiosyncratic consumption, and would find

.

$$\begin{aligned}
\mathcal{M}[\Delta \log \boldsymbol{c}_{t+1}] &= \mathcal{M}[\log c_{t+1}\mathbf{p}_{t+1} - \log c_t\mathbf{p}_t] \\
&= \mathcal{M}[\log \mathbf{p}_{t+1} - \log \mathbf{p}_t + \log c_{t+1} - \log c_t] \\
&= \mathcal{M}[\log \mathbf{p}_{t+1} - \log \mathbf{p}_t] + \mathcal{M}[\log c_{t+1} - \log c_t] \\
&= \gamma - \sigma_\psi^2/2,
\end{aligned}$$

where the last equality follows because the invariance of $\mathcal{F}^c$ means that $\mathcal{M}[\log c_{t+1}] = \mathcal{M}[\log c_t]$.

Papers in the simulation literature have observed an approximate equivalence between the average growth rates of idiosyncratic consumption and permanent income, but formal proof was not possible until Szeidl's proof of ergodicity.

4.3 Growth Rates of Aggregate Income and Consumption

Attanasio and Weberx(1995) point out that concavity of the consumption function (or other nonlinearities) can imply that it is quantitatively important to distinguish between the growth rate of average consumption and the average growth rate of consumption. We have just examined the average growth rate; we now examine the growth rate of the average.

Using capital letters for aggregate variables, the growth factor for aggregate income is given by:

.

$$\begin{aligned} \boldsymbol{Y}_{t+1}/\boldsymbol{Y}_t &= \mathcal{M}[\xi_{t+1}\Gamma\psi_{t+1}\mathbf{p}_t]/\mathcal{M}[\mathbf{p}_t\xi_t] \\ &= \Gamma \end{aligned}$$

because of the independence assumptions we have made about ξ and ψ.

Aggregate assets are:

.

$$\begin{aligned} \boldsymbol{A}_t &= \mathcal{M}\left[a_t \mathbf{p}_t\right] \\ &= A\mathbf{P}_t + \text{cov}_t(a_{t,i}, \mathbf{p}_{t,i}) \end{aligned} \tag{51}$$

where $\mathbf{P}_t$ designates the mean level of permanent income across all individuals, and we are assuming that $a_{t,i}$ was distributed according to the invariant distribution with a mean value of A. Since permanent income grows at mean rate Γ while the distribution of a is invariant, if we normalize $\mathbf{P}_t$ to one we will similarly have for any period $n \geq 1$

.

$$\boldsymbol{A}_{t+n} = A\Gamma^n + \text{cov}(a_{t+n,i}, \mathbf{p}_{t+n,i}).$$

Unfortunately, Szeidlx(2006)'s proof of the invariance of $\mathcal{F}^a$ does not yield the information about how the cross-sectional covariance between a and $\mathbf{p}$ evolves required to show that the covariance term grows by a factor smaller than Γ.

The desired result can be proven if there are no permanent shocks; see appendix E for that proof, along with a discussion of the characteristics of a covariance term that prevents proof in the general case with both transitory and permanent shocks.

A wide range of simulation experiments confirms that the role of that covariance term is more an irritating theoretical curiosum than an important practical consideration. An example is given in Figurex5, which plots $\boldsymbol{C}_{t+1}/\boldsymbol{C}_t$ for the economy whose converging CDFs were depicted inx4.

After the 40 periods of simulation that generated CDFs plotted inx4, we conduct an experiment designed to flush out the role of annoying covariance term: We reset the level of permanent income to be identical for all consumers ('the revolution'):

.

$$\mathbf{p}_{41,i} \;=\; \Gamma \mathbf{P}_{40} \;\forall\; i \tag{52}$$

and we redistribute cash among consumers in such a way as to leave each consumer with the same value of $m_{41,i}$ that they would have had in the absence of the revolution. This leaves us with the same distribution of m as before the revolution, but no covariance between m and $\mathbf{p}$.

PIC

Figurex5: Consumption Growth in Simulated Economy with $\Gamma = 1.03$

The effect on aggregate consumption growth of even such an extreme revolution in convariance is small, and dissipates immediately (no effect is visible after the period of revolution itself). This experiment is representative of many that suggest that the practical effects of time-variaton in the covariance between m and $\mathbf{p}$ are negligible in practice.

5 Conclusions

This paper provides theoretical foundations for many characteristics of buffer stock saving models that have heretofore been observed in simulations but not proven. The main results apply either to a model without liquidity constraints, or to the model with constraints (e.g.xDeatonx(1991)) considered as a limiting case. Perhaps the most important such proposition is the existence of a target cash-to-permanent-income ratio toward which actual cash will tend.

Another contribution of the paper is that it provides a set of tools for simulation analysis (available on the author's web page) that confirm and illustrate the theoretical propositions. These programs demonstrate how the incorporation of the theoretical results can make numerical solution algorithms more efficient and simpler. A goal of the paper has been to make these tools accessible and easy to use while incorporating the full rigor of the theoretical results in the structure. The programs could therefore be a useful starting point for many kinds of future research.

Appendices

A Existence of a Concave Consumption Function

To show that (17) defines a sequence of continuously differentiable strictly increasing concave functions $\{c_T, c_{T-1}, ..., c_{T-k}\}$, we start with a definition. We will say that a function $n(z)$ is 'nice' if it satisfies

$n(z)$ is well-defined iff $z > 0$

$n(z)$ is strictly increasing

$n(z)$ is strictly concave

$n(z)$ is $\mathbf{C}^3$ (its first three derivatives exist)

$n(z) < 0$

$\lim_{z \downarrow 0} x n(z) = -\infty$

(Notice that an implication of niceness is that $\lim_{z \downarrow 0} n'(z) = \infty$.)

Assume that some v_{t+1} is nice. Our objective is to show that this implies v_t is also nice; this is sufficient to establish that v_{t-n} is nice by induction for all $n > 0$ because $\mathrm{v}_T(m) = \mathrm{u}(m)$ and $\mathrm{u}(m) = m^{1-\rho}/(1-\rho)$ is nice by inspection.

Now define an end-of-period value function $\mathfrak{v}_t(a)$ as

.

$$\mathfrak{v}_t(a) = \beta E_t \left[\Gamma_{t+1}^{1-\rho} \mathrm{v}_{t+1}(\mathcal{R}_{t+1} a + \xi_{t+1}) \right]. \tag{53}$$

Since there is a positive probability that ξ_{t+1} will attain its minimum of zero and since $\mathcal{R}_{t+1} > 0$, it is clear that $\lim_{a \downarrow 0} \mathfrak{v}_t(a) = -\infty$ and $\lim_{a \downarrow 0} \mathfrak{v}_t{}'(a) = \infty$. So $\mathfrak{v}_t(a)$ is well-defined iff $a > 0$; it is similarly straightforward to show the other properties required for $\mathfrak{v}_t(a)$ to be nice. (See Hiraguchix(2003)).

Next define $\mathrm{v}_t(m, c)$ as

.

$$\underline{\mathrm{v}}_t(m,c) = \mathrm{u}(c) + \mathfrak{v}_t(m-c)$$

(54)

which is $\mathbf{C}^3$ since $\mathfrak{v}_t$ and u are both $\mathbf{C}^3$, and note that our problem's value function defined in (17) can be written as

.

$$\mathrm{v}_t(m) \;=\; \max_c \; \underline{\mathrm{v}}_t(m,c). \tag{55}$$

v_t is well-defined only if $0 < c < m$. Furthermore, $\lim_{c\downarrow 0}\mathrm{v}_t(m,c) = \lim_{c\uparrow m}\mathrm{v}_t(m,c) = -\infty$, $\frac{\partial^2 \underline{\mathrm{v}}_t(m,c)}{\partial c^2} < 0$, $\lim_{c\downarrow 0}\frac{\partial \underline{\mathrm{v}}_t(m,c)}{\partial c} = +\infty$, and $\lim_{c\uparrow m}\frac{\partial \underline{\mathrm{v}}_t(m,c)}{\partial c} = -\infty$. It follows that the $\mathrm{c}_t(m)$ defined by

.

$$\mathrm{c}_t(m) \;=\; \arg\max_{0<c<m} \underline{\mathrm{v}}_t(m,c) \tag{56}$$

exists and is unique, and (17) has an internal solution that satisfies

.

$$\mathrm{u}'(\mathrm{c}_t(m)) = \mathfrak{v}_t'(m - \mathrm{c}_t(m)).$$

(57)

Since both u and $\mathfrak{v}_t$ are strictly concave, both $\mathrm{c}_t(m)$ and $\mathrm{a}_t(m) = m - \mathrm{c}_t(m)$ are strictly increasing. Since both u and $\mathfrak{v}_t(m)$ are three times continuously differentiable, using (57) we can conclude that $\mathrm{c}_t(m)$ is continuously differentiable and

.

$$\mathrm{c}_t'(m) \;=\; \frac{\mathfrak{v}_t''(\mathrm{a}_t(m))}{\mathrm{u}''(\mathrm{c}_t(m)) + \mathfrak{v}_t''(\mathrm{a}_t(m))}. \tag{58}$$

Similarly we can easily show that $\mathrm{c}_t(m)$ is twice continuously differentiable

31See AppendixxB.

(as is $\mathrm{a}_t(m)$).31
This implies that $\mathrm{v}_t(m)$ is nice, since $\mathrm{v}_t(m) = \mathrm{u}(\mathrm{c}_t(m)) + \mathfrak{v}_t(\mathrm{a}_t(m))$.

B $\mathrm{c}_t(m)$ Is Twice Continuously Differentiable

First we show that $\mathrm{c}_t(m)$ is continuous $\mathbf{C}^1$. Define y as $y \equiv x + dx$
. Since $\mathrm{u}'\left(\mathrm{c}_t(y)\right) - \mathrm{u}'\left(\mathrm{c}_t(m)\right) = \mathfrak{v}_t{}'(\mathrm{a}_t(y)) - \mathfrak{v}_t{}'(\mathrm{a}_t(m))$ and
$\frac{\mathrm{a}_t(y)-\mathrm{a}_t(m)}{dm} = 1 - \frac{\mathrm{c}_t(y)-\mathrm{c}_t(m)}{dm}$,

.

$$\frac{\mathfrak{v}_t'(\mathrm{a}_t(y)) - \mathfrak{v}_t'(\mathrm{a}_t(m))}{\mathrm{a}_t(y) - \mathrm{a}_t(m)} = \left(\frac{\mathrm{u}'\left(\mathrm{c}_t(y)\right) - \mathrm{u}'\left(\mathrm{c}_t(m)\right)}{\mathrm{c}_t(y) - \mathrm{c}_t(m)} + \frac{\mathfrak{v}_t'(\mathrm{a}_t(y)) - \mathfrak{v}_t'(\mathrm{a}_t(m))}{\mathrm{a}_t(y) - \mathrm{a}_t(m)}\right)\frac{\mathrm{c}_t(y) - \mathrm{c}_t(m)}{dm}$$

Since c_t and a_t are continuous and increasing, $\lim_{dm\to+0}\frac{\mathrm{u}'(\mathrm{c}_t(y))-\mathrm{u}'(\mathrm{c}_t(m))}{\mathrm{c}_t(y)-\mathrm{c}_t(m)} < 0$ and $\lim_{dm\to 0}\frac{\mathfrak{v}_t'(\mathrm{a}_t(y))-\mathfrak{v}_t'(\mathrm{a}_t(m))}{\mathrm{a}_t(y)-\mathrm{a}_t(m)} \leq 0$ are satisfied. Then $\frac{\mathrm{u}'(\mathrm{c}_t(y))-\mathrm{u}'(\mathrm{c}_t(m))}{\mathrm{c}_t(y)-\mathrm{c}_t(m)} + \frac{\mathfrak{v}_t'(\mathrm{a}_t(y))-\mathfrak{v}_t'(\mathrm{a}_t(m))}{\mathrm{a}_t(y)-\mathrm{a}_t(m)} < 0$ for sufficiently small dx. Hence we obtain a well-defined equation:

.

$$\frac{\mathrm{c}_t(y)-\mathrm{c}_t(m)}{dm} = \frac{\frac{\mathfrak{v}_t'(\mathrm{a}_t(y))-\mathfrak{v}_t'(\mathrm{a}_t(m))}{\mathrm{a}_t(y)-\mathrm{a}_t(m)}}{\frac{\mathrm{u}'(\mathrm{c}_t(y))-\mathrm{u}'(\mathrm{c}_t(m))}{\mathrm{c}_t(y)-\mathrm{c}_t(m)} + \frac{\mathfrak{v}_t'(\mathrm{a}_t(y))-\mathfrak{v}_t'(\mathrm{a}_t(m))}{\mathrm{a}_t(y)-\mathrm{a}_t(m)}}$$

This implies that the right-derivative, ${\mathrm{c}_t}'^{+}(m)$ is well-defined and

.

$$c_t'^{+}(m) = \frac{\mathfrak{v}_t''(\mathrm{a}_t(m))}{\mathrm{u}''(\mathrm{c}_t(m)) + \mathfrak{v}_t''(\mathrm{a}_t(m))}$$

Similarly we can show that ${\mathrm{c}_t}'^{+}(m) = {\mathrm{c}_t}'^{-}(m)$, which means ${\mathrm{c}_t}'(m)$ exists. Since $\mathfrak{v}_t$ is $\mathbf{C}^3$, ${\mathrm{c}_t}'(m)$ exists and is continuous. ${\mathrm{c}_t}'(m)$ is differentiable because ${\mathfrak{v}_t}''$ is $\mathbf{C}^1$, $\mathrm{c}_t(m)$ is $\mathbf{C}^1$ and $\mathrm{u}''(\mathrm{c}_t(m)) + {\mathfrak{v}_t}''(\mathrm{a}_t(m)) < 0$. ${\mathrm{c}_t}''(m)$ is given by

.

$$\mathrm{c}_t''(m) = \frac{a_t'(m)\mathfrak{v}_t'''(a_t)\left[\mathrm{u}''(c_t) + \mathfrak{v}_t''(a_t)\right] - \mathfrak{v}_t''(a_t)\left[c_t'\mathrm{u}'''(c_t) + a_t'\mathfrak{v}_t'''(a_t)\right]}{\left[\mathrm{u}''(c_t) + \mathfrak{v}_t''(a_t)\right]^2}$$

(59)

Since ${\mathfrak{v}_t}''(\mathrm{a}_t(m))$ is continuous, ${\mathrm{c}_t}''(m)$ is also continuous.

C Convergence of v_t in Euclidian Space

Boyd's theorem shows that $\mathfrak{T}$ defines a contraction mapping in a $\digamma$-bounded space. We now show that $\mathfrak{T}$ also defines a contraction mapping in Euclidian space.

Since $\mathrm{v}^*(m) = \mathcal{T}\mathrm{v}^*(m)$,

.

$$\left\|\mathrm{v}_{T-n+1}(m) - \mathrm{v}^*(m)\right\|_{F} \leq \xi^{n-1} \left\|\mathrm{v}_T(m) - \mathrm{v}^*(m)\right\|_{F}$$

(60)

On the other hand, $\mathrm{v}_T - \mathrm{v}^* \in \mathcal{C}_{F}(\mathcal{A}, \mathcal{B})$ and $\kappa = \left\|\mathrm{v}_T(m) - \mathrm{v}^*(m)\right\|_{F} < \infty$ because v_T and v^* are in $\mathcal{C}_{F}(\mathcal{A}, \mathcal{B})$. It follows that

.

$$\left|\mathrm{v}_{T-n+1}(m) - \mathrm{v}^*(m)\right| \leq \kappa \xi^{n-1} \left|F(m)\right|$$

(61)

Then we obtain

.

$$\lim_{n\to\infty} \mathrm{v}_{T-n+1}(m) = \mathrm{v}^*(m).$$

(62)

Since $\mathrm{v}_T(m) = \frac{m^{1-\rho}}{1-\rho}$, $\mathrm{v}_{T-1}(m) \leq \frac{(\bar{\kappa}m)^{1-\rho}}{1-\rho} < \mathrm{v}_T(m)$. On the other hand, $\mathrm{v}_{T-1} \leq \mathrm{v}_T$ means $\mathfrak{T}\mathrm{v}_{T-1} \leq \mathfrak{T}\mathrm{v}_T$, in other words, $\mathrm{v}_{T-2}(m) \leq \mathrm{v}_{T-1}(m)$. Inductively one gets $\mathrm{v}_{T-n}(m) \geq \mathrm{v}_{T-n-1}(m)$. This means that $\{\mathrm{v}_{T-n+1}(m)\}_{n=1}{}^{\infty}$ is a decreasing sequence, bounded below by v^*.

D Convergence of c_t

Given the proof that the value functions converge, we now show the pointwise convergence of consumption functions $\{\mathrm{c}_{T-n+1}(m)\}_{n=1}{}^{\infty}$.

We start by showing that

.

$$\mathrm{c}(m) = \underset{c_t \in [\underline{\hat{\kappa}} m, \bar{\kappa} m]}{\arg\max} \left\{ \mathrm{u}(c_t) + \beta E_t \left[\Gamma_{t+1}^{1-\rho} \mathrm{v}(m_{t+1}) \right] \right\}$$

(63)

is uniquely determined. We show this by contradiction. Suppose there exist c_1 and c_2 that both attain the supremum for some m, with mean $\tilde{c} = (c_1 + c_2)/2$. c_i satisfies

.

$$\mathfrak{T}\mathrm{v}(m) = \mathrm{u}(c_i) + \beta \underbrace{E_t \left[\Gamma_{t+1}^{1-\rho} \mathrm{v}(\mathrm{m}_{t+1}(m, c_i)) \right]}_{\equiv \mathfrak{v}}$$

(64)

where $\mathrm{m}_{t+1}(m, c) = (m - c)\mathcal{R}_{t+1} + \xi_{t+1}$ and $i = 1, 2$. $\mathfrak{T}\mathrm{v}$ is concave for concave $\mathfrak{v}$. Since the space of continuous and concave functions is closed, $\mathfrak{v}$ is also concave and satisfies

.

$$\frac{1}{2}\sum_{i=1,2} E_t\left[\Gamma_{t+1}^{1-\rho}\mathrm{v}(\mathrm{m}_{t+1}(m,c_i))\right] \leq E_t\left[\Gamma_{t+1}^{1-\rho}\mathrm{v}(\mathrm{m}_{t+1}(m,\tilde{c}))\right].$$

(65)

On the other hand, $\frac{1}{2}\left\{\mathrm{u}(c_1)+\mathrm{u}(c_2)\right\} < \mathrm{u}(\tilde{c})$. Then one gets

.

$$\mathfrak{T}\mathrm{v}(m) < \mathrm{u}(\tilde{c}) + \beta E_t\left[\Gamma_{t+1}^{1-\rho}\mathrm{v}(\mathrm{m}_{t+1}(m,\tilde{c}))\right]$$

(66)

Since $\tilde{c}$ is a feasible choice for c_j, the RHS of this equation is not a maximum, which contradicts the definition.

Using uniqueness of $\mathrm{c}(m)$ we can now show

.

$$\lim_{n\to\infty} \mathrm{c}_{T-n+1}(m) = \mathrm{c}(m).$$

(67)

Suppose this does not hold for some $m = m^*$. In this case, $\{\mathrm{c}_{T-n+1}(m^*)\}_{n=1}{}^{\infty}$ has a subsequence $\left\{c_{T-n(i)}(m^*)\right\}_{i=1}{}^{\infty}$ that satisfies $\lim_{i\to\infty}\mathrm{c}_{T-n(i)}(m^*) = c^*$ and $c^* x \mathrm{c}(m^*)$. Now define $c_{T-n+1}{}^* = \mathrm{c}_{T-n+1}(m^*)$. $c^* > 0$ because $\lim_{i\to\infty}\mathrm{v}_{T-n(i)+1}(m^*) \leq \lim_{i\to\infty}\mathrm{u}(c_{T-n(i)}{}^*)$. Because $\mathrm{a}(m^*) > 0$ and $\psi \in [\psi, \bar{\psi}]$ there exist $\{m_+{}^*, \bar{m}_+{}^*\}$ satisfying $0 < m_+{}^* < \bar{m}_+{}^*$ and $m_{T-n+1}(m^*, c_{T-n+1}{}^*) \in [\underline{m}_+^*, \bar{m}_+^*]$. It follows that $\lim_{n\to\infty}\mathrm{v}_{T-n+1}(m) = \mathrm{v}(m)$xand the convergence is uniform onx$m \in [\underline{m}_+^*, \bar{m}_+^*]$. (Uniform convergence is obtained from Dini's

32 [**Dini's theorem**] For a monotone sequence of continuous functions $\{\mathrm{v}_n(m)\}_{n=1}{}^{\infty}$ which is defined on a compact space and satisfies $\lim_{n\to\infty}\mathrm{v}_n(m) = \mathrm{v}(m)$ where $\mathrm{v}(m)$ is continuous, convergence is uniform.

theorem.32)
Hence for any $\delta > 0$, there exists an n_1 such that

.

$$\beta E_{T-n}\left[\Gamma_{T-n+1}^{1-\rho}\left|\mathrm{v}_{T-n+1}(m_{T-n+1}(m^*, c^*_{T-n+1})) - \mathrm{v}(m_{T-n+1}(m^*, c^*_{T-n+1}))\right|\right] \; < \; \delta$$

for all $n \geq n_1$. It follows that if we define

.

$$\mathrm{w}(m^*, z) = \mathrm{u}(z) + \beta E_{T-n}\left[\Gamma_{T-n+1}^{1-\rho}\mathrm{v}(m_{T-n+1}(m^*, z))\right]$$

(68)

then $\mathrm{v}_{T-n}(m^*)$ satisfies

.

$$\lim_{n\to\infty} |\mathrm{v}_{T-n}(m^*) - \mathrm{w}(m^*, c^*_{T-n+1})| = 0.$$

(69)

On the other hand, there exists an $i_1 \in \mathbb{N}$ such that

.

$$\left|\mathrm{v}(\mathrm{m}_{T-n(i)}(m^*, c^*_{T-n(i)})) - \mathrm{v}(\mathrm{m}_{T-n(i)}(m^*, c^*))\right| \leq \delta \text{ for all } i \geq i_1$$

(70)

because v is uniformly continuous on $[m_+{}^*, \bar{m}_+{}^*]$.
$\lim_{i\to\infty} \left|\mathrm{c}_{T-n(i)}(m^*) - c^*\right| = 0$ and

.

$$\left|\mathrm{m}_{T-n(i)}(m^*, c^*_{T-n(i)}) - \mathrm{m}_{T-n(i)}(m^*, c^*)\right| \leq \frac{\mathsf{R}}{\Gamma\underline{\psi}} \left|c^*_{T-n(i)} - c^*\right|$$

(71)

This implies

.

$$\lim_{i\to\infty}\left|\mathrm{w}(m^*, c^*_{T-n(i)+1}) - \mathrm{w}(m^*, c^*)\right| = 0.$$

(72)

From (69) and (72), we obtain $\lim_{i\to\infty}\mathrm{v}_{T-n(i)}(m^*) = \mathrm{w}(m^*, c^*)$ and this implies $\mathrm{w}(m^*, c^*) = \mathrm{v}(m^*)$. This implies that $\mathrm{c}(m)$ is not uniquely determined, which is a contradiction.

Thus, the consumption functions must converge.

E Equality of Aggregate Consumption And Income Growth With Transitory Shocks

The text asserted that in the absence of permanent shocks it is possible to prove that the growth factor for aggregate consumption approaches that for aggregate permanent income. This section establishes that results.

Suppose the population starts in period t with an arbitrary value for $\mathrm{cov}(a_{t+1,i}, \mathbf{p}_{t+1,i})$. Then if $\breve{m}$ is the invariant mean level of m we can define a 'mean MPS away from $\breve{m}$' function

.

$$\acute{\mathrm{a}}(\Delta) \;\; = \;\; \Delta^{-1} \int_{\breve{m}}^{\breve{m}+\Delta} \mathrm{a}'(z) dz$$

and since $\psi_{t+1,i} = 1$, $\mathcal{R}_{t+1,i}$ is a constant at $\mathcal{R}$ we can write

.

$$a_{t+1,i} \;\; = \;\; \mathrm{a}(\breve{m}) + (m_{t+1,i} - \breve{m})\acute{\mathrm{a}}(\overbrace{\mathcal{R}a_{t,i} + \xi_{t+1,i}}^{m_{t+1,i}} - \breve{m})$$

so

.

$$\mathrm{cov}(a_{t+1,i}, \mathbf{p}_{t+1,i}) \;\; = \;\; \mathrm{cov}\left(\acute{\mathrm{a}}(\mathcal{R}a_{t,i} + \xi_{t+1,i} - \breve{m}), \Gamma \mathbf{p}_{t,i}\right)$$

But since $\mathsf{R}^{-1}(\wp \mathsf{R}\beta)^{1/\rho} < \acute{a}(m) < \text{Þ}_{\mathsf{R}}$,

.

$$|\mathrm{cov}((\wp \mathsf{R}\beta)^{1/\rho} a_{t+1,i}, \mathbf{p}_{t+1,i})| < |\mathrm{cov}(a_{t+1,i}, \mathbf{p}_{t+1,i})| < |\mathrm{cov}((\mathsf{R}\beta)^{1/\rho} a_{t+1,i}, \mathbf{p}_{t+1,i})|$$

and for the version of the model with no permanent shocks the GIC says that $(\mathsf{R}\beta)^{1/\rho} < \Gamma$, which implies

.

$$|\mathrm{cov}(a_{t+1,i}, \mathbf{p}_{t+1,i})| < \Gamma |\mathrm{cov}(a_{t,i}, \mathbf{p}_{t,i})|.$$

This means that from any arbitrary starting value, the relative size of the covariance term shrinks to zero over time (compared to the $A\Gamma^n$ term which is growing steadily by the factor Γ). Thus,

$\lim_{n \to \infty} \boldsymbol{A}_{t+n+1}/\boldsymbol{A}_{t+n} = \Gamma$.

This logic unfortunately does not go through when there are permanent shocks, because the $\mathcal{R}_{t+1,i}$ terms are not independent of the permanent income shocks.

To see the problem clearly, define $\breve{\mathcal{R}} = \mathcal{M}\left[\mathcal{R}_{t+1,i}\right]$ and consider a second order Taylor expansion of $\acute{\mathrm{a}}(m_{t+1,i})$ around $\breve{m}_{t+1,i} = \breve{\mathcal{R}} a_{t,i} + 1$,

.

$$\acute{\mathrm{a}}_{t+1,i} \;\approx\; \acute{\mathrm{a}}(\breve{m}_{t+1,i}) + \acute{\mathrm{a}}'(\breve{m}_{t+1,i})\left(m_{t+1,i} - \breve{m}_{t+1,i}\right).$$

The problem comes from the $\acute{\mathrm{a}}'$ term. The concavity of the consumption function implies convexity of the a function, so this term is strictly positive but we have no theory to place bounds on its size as we do for its level $\acute{\mathrm{a}}$. We cannot rule out by theory that a positive shock to permanent income (which has a negative effect on $m_{t+1,i}$) could have an unboundedly positive effect on $\acute{\mathrm{a}}'$ (as for instance if it pushes the consumer arbitrarily close to the self-imposed liquidity constraint).

F Endogenous Gridpoints Solution Method

The final two appendices describe briefly some details of the solution methods.

The model is solved using the method of endogenous gridpoints (Carrollx(2006)). A grid of values of end-of-period assets $\vec{a}$ is defined, using which marginal end-of-period value and consumption are computed using the Euler equation:

.

$$
\begin{aligned}
\mathrm{u}'(\mathfrak{c}_t(\vec{a})) &= \mathsf{R}\beta E_t[\mathrm{u}'(\Gamma_{t+1}\mathrm{c}_{t+1}(\mathcal{R}_{t+1}\vec{a}+\xi))] \qquad (73)\\
\vec{c}_t &= \left(\mathsf{R}\beta E_t\left[(\Gamma_{t+1}\mathrm{c}_{t+1}(\mathcal{R}_{t+1}\vec{a}+\xi))^{-\rho}\right]\right)^{-1/\rho}
\end{aligned}
$$

and we can construct the 'consumed' function $\mathfrak{c}_t$ (for any end-of-period a, what is the amount of consumption that must have taken place such that it would have been optimal to end the period with that a) by interpolation.

The consumption function can be obtained using the dynamic budget constraint to generate the corresponding m's:

.

$$\vec{m}_t = \vec{a} + \vec{c}_t$$

using which an interpolation can be constructed.

The same logic can also be used to obtain a recursive formula for the marginal propensity to *have consumed*, by differentiating (73) with respect to a:

.

$$\begin{aligned} \mathrm{u}''(\mathfrak{c}_t(\vec{a}))\mathfrak{c}_t^a(\vec{a}) &= \mathsf{R}\beta E[\mathrm{u}''(\Gamma \mathrm{c}_{t+1})\Gamma \mathrm{c}_{t+1}^m \mathcal{R}] \\ &= \mathsf{R}\beta E[\mathrm{u}''(\Gamma \mathrm{c}_{t+1})\mathsf{R} \mathrm{c}_{t+1}^m] \\ \mathfrak{c}_t^a(\vec{a}) &= \mathsf{R}\beta E[\mathrm{u}''(\Gamma \mathrm{c}_{t+1})\mathsf{R} \mathrm{c}_{t+1}^m]/\mathrm{u}''(\mathfrak{c}_t(\vec{a})) \end{aligned}$$

and the marginal propensity to consume is obtained from the marginal propensity to have consumed by noting that

.

$$
\begin{aligned}
c &= m - a \\
\mathfrak{c}^a + 1 &= m^a \\
\mathrm{c}^m m^a &= \mathfrak{c}^a \\
\mathrm{c}^m(1 + \mathfrak{c}^a) &= \mathfrak{c}^a \\
\mathrm{c}^m &= \mathfrak{c}^a/(1 + \mathfrak{c}^a)
\end{aligned}
$$

so that we can generate $\vec{\kappa}_t$ and $\vec{\lambda}_t$ directly from $\mathfrak{c}^a(\vec{a})$.

G Approximating Precautionary Saving

Define the precautionary saving function as

.

$$
\mathrm{s}_t(m) = \bar{\mathrm{c}}_t(m) - \mathrm{c}_t(m)
$$

which yields the difference between the consumption of a perfect foresight consumer and the consumption that is optimal in the presence of uncertainty. It is straightforward to show that if the RIC holds,

.

$$\lim_{m \to \infty} \mathrm{s}_t(m) = 0$$

because as assets approach infinity the proportion of spending that will be financed out of the uncertain labor income approaches zero.

This fact motivates us to use $\log s_t$ as the object to be approximated. That is, having computed $\vec{c}_t$ and $\vec{m}_t$ as described above, we calculate

.

$$\vec{s}_t = \bar{\mathrm{c}}_t(\vec{m}_t) - \vec{c}_t$$

and, noting that $\mathrm{s}_t{}'(\vec{m}_t) = \kappa_t - \vec{\kappa}_t$ while $(d/dm) \log \mathrm{s}_t(m) = \mathrm{s}_t{}'(m)/\mathrm{s}_t(m)$ we can construct the points needed for a polynomial interpolating approximation of $\varsigma_t(m) \equiv \log \mathrm{s}_t(m)$ from

.

$$\{\vec{m}_t, \{\log \vec{s}_t, (\underline{\kappa}_t - \vec{\kappa}_t)/\vec{s}_t\}^\top\}^\top.$$

With the interpolating approximation to $\varsigma_t(m)$ in hand, we can generate the implied consumption and marginal propsensity to consume functions from

.

$$\begin{aligned}
\mathrm{c}_t(m) &= \bar{\mathrm{c}}_t(m) - e^{\varsigma_t(m)} \\
\mathrm{c}_t^m(m) &= \underline{\kappa}_t - \varsigma_t^m(m)\varsigma_t(m).
\end{aligned}$$

Among the many advantages of approximating the log of s, the most useful is that when the $\varsigma_t(m)$ function is extended linearly beyond the boundaries over which the approximation is constructed, the resulting representation of the consumption function automatically asymptotes to the perfect foresight consumption function as required by theory.

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