

Too big to fail, but a lot to bail: Optimal financing of large bailouts

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Abstract

The termination of a representative financial firm due to excess leverage may lead to substantial bankruptcy costs. A benevolent government in the tradition of Ramsey (1927) may be inclined to provide transfers to the firm so as to prevent its liquidation and the associated deadweight costs. The paper studies the optimal way to finance such a “bailout” with distortionary taxes and obtains two results. First, some degree of “fiscal stimulus” through procyclical taxation (low taxes in bad times, high taxes in good times) is always optimal. This is true even when markets are complete and government expenditure is set to zero. Second, taxes exhibit history dependence, even in a complete market. These results are in contrast with pre-existing results in the literature on optimal fiscal policy, and are driven by the endogeneity of the transfer payments that are required to salvage the financial firm. The paper also considers extensions whereby bailouts are financed partly by diluting existing shareholders, or by obtaining a fraction of the capital of the underlying company, and discusses the relative merits of these alternatives.

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1 Introduction

Had a forecaster in 2005 predicted the magnitude of the financial crisis of 2007-2008 with perfect foresight, most economists at the time would probably have reacted with scepticism. The decline of housing values, the demise of major financial institutions, and the necessity for large capital injections by the government to preserve the stability of the financial system would probably have sounded unlikely to many economic analysts in 2005.

Predictably, the magnitude and the financial origins of the current crisis have focused the attention of economists on a new set of issues. For instance, should governments undertake bailouts? If so, how should such bailouts be financed? Should fiscal policy be used to stimulate the economy in the midst of a financial crisis?

The present paper presents a theoretical framework that can help address these questions. The framework is neoclassical, and follows the tradition of Ramsey (1927). A benevolent government adjusts tax rates, anticipating the effects of tax changes on equilibrium allocations, and ultimately consumer welfare. There are two types of (competitive) firms. Financial firms supply financial services using capital. Final goods firms produce consumption goods using labor and the services of financial firms as inputs. Financial firms are partly financed by short term debt. If the short term debt cannot be rolled over, then the financial firm must be liquidated. Liquidation leads to deadweight costs, which makes the government willing to provide the financial firm with capital injections in order to prevent its liquidation and the associated deadweight losses.

In the baseline model, the bailout payments are financed by distortionary labor taxes similar to Lucas and Stokey (1983), and markets are complete. Because taxation is distortionary, Ricardian equivalence fails and the optimal welfare-maximizing way to raise taxes becomes a non-trivial problem.

A new feature of the analysis is that taxes are not raised to finance exogenous government expenditures, but rather to finance transfer payments from the government to the financial firm. Hence, government outlays do not drive a wedge between consumption and output; they merely reallocate existing consumption goods through taxes and transfers. Importantly,

the net present value of the transfer payments is not exogenous to the model, but rather is determined in general equilibrium, since the value of the firm and the necessary transfer payments to prevent liquidation are endogenous to the model.

Somewhat surprisingly, the endogeneity of the transfer payments has profound implications for optimal taxation.

On the one hand, a standard labor tax smoothing argument (see e.g. Barro (1979)) favors a constant tax rate, irrespective of current economic conditions. Indeed, a first result of the model states that if the government were to ignore the endogeneity of transfer payments, then the optimal tax rate should be constant. This result is reminiscent of Lucas and Stokey (1983), who find that in the presence of complete markets and constant government expenditure (government expenditure is set to zero in the present model), tax rates should be constant.

On the other hand, countercyclical fiscal policy (raising tax rates in good times and lowering them in bad times) can help boost output, the demand for financial services and accordingly the value of the representative firm in bad times. In turn, increasing the firm's value in bad times can lower the amount of transfers that are required to salvage the firm, and can lead to a lower overall net present value of required (distortionary) taxes.

The main result of the paper is that the tradeoff between smoothing tax distortions and reducing the net present value of transfers makes some degree of “fiscal stimulus” always optimal during a crisis. Specifically, optimal taxes are a non-decreasing function of total factor productivity. An additional result of theoretical interest is that optimal tax rates are history dependent *despite complete markets*; tax rates at some time t do not only depend on the productivity level at time t , but are also a non-decreasing function of past transfer payments. This result contrasts with Lucas and Stokey (1983), where tax rates are history independent, when government expenditure is Markovian.

Extensions of the model consider cases where bailouts are financed partly by having the government obtain either shares of the financial company or shares of its underlying assets. Assuming that markets are complete, it is only the total value of these claims that matters,

while the composition of the claims (the fraction of total value due to equity shares or asset shares) is irrelevant. However, were one to introduce additional frictions (such as a ceiling for debt), then asset shares are always associated with a lower required debt ceiling.

The analysis is related to two strands of the literature. The first strand uses continuous time finance techniques to price government guarantees as contingent claims¹. In this literature, the stochastic discount factor is taken as given, and the issue of how the government should raise the revenue to pay for the guarantees is not considered. By contrast, in the present framework the endogeneity of the stochastic discount factor and the optimal way to finance these guarantees are explicitly taken into account. The second strand of the literature studies optimal (labor-distortionary) taxation². This literature is mostly considered with the optimal timing of taxes in the presence of exogenous expenditures. As mentioned above, the distinguishing feature of the present paper is that taxes are raised in order to finance endogenous transfer payments rather than *exogenous government expenditures*. Karantounias, Hansen, and Sargent (2008) also obtain history dependence of the tax rate in a framework involving complete markets. However, their results are driven by fears of model mis-specification rather than endogenous transfers.

The paper is structured as follows. Section 2 describes the model and the problem of the government. Section 3 derives the optimal taxation policy. Section 4 considers additional forms of funding government transfers and derives the process of optimal debt holdings. Section 5 concludes. All proofs are in the appendix.

¹A representative sample of papers in this literature includes Merton (1978), Ronn and Verma (1986), Lucas and McDonald (2005), Panageas (2008), Pennacchi and Lewis (1994), Ramaswamy, Brix, and Schick (2002).

²A representative sample of papers in this literature includes Barro (1979), Lucas and Stokey (1983), Aiyagari, Sargent, Marcet, and Seppala (2002), Angeletos (2002).

2 Model

2.1 Consumers, firms and assets

The representative consumer has preferences given by

$$E_0 \left\{ \int_0^\infty e^{-\rho t} \left[\log(c_t) + \frac{(1-h_t)^{1-\phi}}{1-\phi} \right] dt \right\}, \quad (1)$$

where c_t is an adapted consumption process, and h_t denotes hours worked, $\rho > 0$ is the subjective discount rate, and $\phi > 0$, $\phi \neq 1$ controls the elasticity of labor supply. The representative agent's endowment of hours is normalized to one. Specification (1) is attractive, since it allows hours to be stationary in the long run, while keeping the utility of consumption and leisure separable.

Firms in the economy are competitive and fall into two groups: Non-financial (NF) and financial (F). Within each group all firms are identical, so that one can speak of a single representative financial and non-financial firm. The services produced by the representative financial firm are used by the non-financial firm as intermediate goods. Specifically, the (representative) non-financial firm is the sole producer of consumption goods and it utilizes the production function

$$Y_t = Z_t (F_t)^{1-\alpha} (h_t)^\alpha, \quad (2)$$

where F_t denotes the amount of financial services, h_t the hours worked, $\alpha \in (0, 1)$ controls the shares of production factors and Z_t captures total factor productivity, which follows a geometric brownian motion

$$\frac{dZ_t}{Z_t} = \mu dt + \sigma dB_t. \quad (3)$$

The parameters $\mu > 0$ and $\sigma > 0$ control the drift and the volatility of the geometric brownian motion and satisfy $\mu - \frac{\sigma^2}{2} > 0$. The price of a unit of financial services is given by p_t . Accordingly, the optimization problem of the (representative) non-financial firm is given by

$$\max_{h_t, F_t} Y_t - w_t h_t - p_t F_t. \quad (4)$$

The financial firm employs a simple production technology. It owns K_t units of capital goods, and uses one unit of capital goods to produce one unit of financial services. To simplify matters, I follow Lucas (1978) and assume that K_t cannot be accumulated, nor does it depreciate, so that it remains constant.

The financial firm receives an income stream from its operations equal to $p_t F_t$. Financial markets are complete and there exists a (unique) stochastic discount factor ξ_t .

Accordingly, the total value of the financial firm's capital stock, which I will denote as W_t , is given by the present value of the revenue that it produces

$$W_t = \left(E_t \int_t^\infty \frac{\xi_s}{\xi_t} p_s F_s ds \right). \quad (5)$$

2.2 Default and Bailouts

The financial firm is levered; it owes debtholders an amount L_t . The coupon of this debt is variable and equals the rate of return on an instantaneously maturing risk-free bond³ The presence of debt introduces the possibility of default. Default is modeled in a standard way. If a firm defaults, ownership of the entire capital stock is transferred to debtholders. To simplify the presentation, I assume that in the event of bankruptcy, the capital stock of the firm is sold by the debtholders to a newly formed set of financial firms that raise their funds by issuing equity to the representative households. However, this redeployment process is costly. Specifically, a fraction δ of the capital stock is lost in the process of redeployment, so that K_t jumps downward to $(1 - \delta) K_t$.

³As Duffie (2001), and Karatzas and Shreve (1998) show, the stochastic discount factor in a brownian filtration model can be written as

$$\frac{d\xi_t}{\xi_t} = -r_t dt - dA_t - \kappa_t dB_t, \quad (6)$$

where (r_t) is the equilibrium interest rate, κ_t is the Sharpe ratio or price of risk and A_t is a continuous process of bounded variation. The rate of return of an instantaneously maturing risk-free bond is given by $r_t dt + dA_t$. (In many cases $dA_t = 0$ and the rate of return of an instantaneously maturing risk-free bond is simply $r_t dt$.)

To simplify the analysis, I will assume that debt includes a protective covenant that allows debtholders to liquidate the firm once

$$W_t \leq L_t \tag{7}$$

Equation (7) can be viewed as a constraint, that arises naturally in the presence of collateralized borrowing⁴.

Since bankruptcy leads to deadweight costs (because a fraction δ of the capital stock gets destroyed), a benevolent government may have an incentive to intervene and “bail out” the financial firm⁵. Bailouts are modeled in a way similar to Panageas (2008). Specifically, if the firm is threatened by imminent default, the government makes a transfer that allows the firm to pay down an amount $dL_t < 0$ of debt, so that⁶

$$W_t \geq L_{t+} \equiv L_t + dL_t. \tag{8}$$

An implication of Karatzas and Shreve (1991) p. 210 is that there exists a unique minimal process dL_t that safeguards that (8) holds and it is given by

$$\int_0^t dL_t = \min \left(L_0, \min_{0 \leq s \leq t} W_s \right) - L_0. \tag{9}$$

Later, I derive a sufficient condition on δ so that the government always finds it optimal to bail out the firm and provide the transfers implied by equation (9). I refer to such a situation as a “perpetual bailout”. For what follows, I simply assume that the bailout is perpetual and proceed as if all agents anticipate that the government will provide the transfers implied by equation (9).

⁴A similar constraint would arise in the presence of short term debt that needs to be rolled over.

⁵Here I make the same implicit assumption as Leland (1998), namely that contracting frictions or hold-up problems between various classes of debt prevent renegotiation between shareholders and debtholders to avoid default.

⁶This specification of the transfer and its use to reduce debt is stylized, but could be easily relaxed without changing the main insights of the analysis. For instance, replacing a fraction of the fair-market-rate loans to the private sector with below-market-rate loans to the government is equivalent to a government transfer whose value dL_t is equal to the difference in the economic value of the two loans.

2.3 Taxes to finance the bailout

The government needs to raise taxes in order to finance the transfers to the firm. Taxation is distortionary and the only source of funding for the government is labor income taxation. This simple assumption allows a comparison with the results in Lucas and Stokey (1983). Later I enrich the model and consider what happens when the government can obtain a fraction of company stock, or a fraction of the firm's assets (or both) in exchange for providing the associated transfers.

Specifically, the government levies a (proportional) labor tax on workers. This tax raises a revenue given by $\tau_t w_t h_t$ at time t . For simplicity, there are no government expenditures and no initial debt. Hence taxes are raised only in order to finance the transfers to the firm. Given the above assumptions, and recalling that markets are complete and $dL_s < 0$, the government's intertemporal budget constraint is

$$E_0 \int_0^\infty \xi_t \tau_t w_t h_t dt = -E_0 \int_0^\infty \xi_t dL_t. \quad (10)$$

2.4 Formulation of the government's problem

Given a path of τ_t , a market equilibrium is defined as a tuple of adapted process for c_t , h_t , F_t , ξ_t , w_t and p_t , so that

1. Consumers maximize (1) over c_t, h_t subject to their intertemporal budget constraint

$$E_0 \int_0^\infty \xi_t c_t dt = E_0 \int_0^\infty \xi_t (1 - \tau_t) w_t h_t dt + E_0 \int_0^\infty \xi_t p_t F_t ds - E_0 \int_0^\infty \xi_t dL_t. \quad (11)$$

2. Firms maximize (4) over h_t, F_t
3. Goods markets, financial services markets, and labor markets clear, i.e. $c_t = Y_t$, $F_t = K_t$ and hours supplied by workers are equal to hours demanded by firms.
4. All asset markets clear⁷.

⁷Alternatively put, the markets for all Arrow Debreu securities clear.

Equation (11) has a natural interpretation. It states that the consumer's present value of consumption (left hand side of equation [11]) should equal the present value of after tax labor income (first term on the right hand side of [11]) plus the total value (sum of debt and equity) of the financial firm which is given by the last two terms on the right hand side of (11). Observe that the last term in(11) captures the increase in total firm value due to the government transfers.

Constructing an equilibrium for a given path τ_t is straightforward. Attaching a Lagrange multiplier ν to the intertemporal budget constraint (11) and maximizing over c_t, h_t leads to the pair of first order conditions

$$e^{-\rho t} \frac{1}{c_t} = \nu \xi_t, \quad (12)$$

$$e^{-\rho t} (1 - h_t)^{-\phi} = \nu \xi_t (1 - \tau_t) w_t \quad (13)$$

Combining (12), and (13) leads to

$$(1 - h_t)^{-\phi} = (1 - \tau_t) \frac{w_t}{c_t} \quad (14)$$

Equation (14) is the familiar first order condition stating that the marginal disutility of an extra hour of work should equal the after tax wage times the marginal utility of an extra unit of consumption.

Turning to firms, the first order conditions with respect to h_t and F_t yield

$$\alpha Y_t = w_t h_t \quad (15)$$

$$(1 - \alpha) Y_t = p_t F_t \quad (16)$$

These are familiar relationships for factor payments when the production function is of the Cobb-Douglas form. Multiplying both sides of (14) by h_t , using (15) and recognizing that in equilibrium $c_t = Y_t$ leads to

$$(1 - h_t)^{-\phi} h_t = (1 - \tau_t) \alpha. \quad (17)$$

Equation (17) expresses the hours worked at time t exclusively as a function of the tax rate τ_t . Equation (17) implies a one to one relationship between a given tax rate and the

hours worked in equilibrium. To capture this relationship, let

$$\tau(h_t) \equiv 1 - \frac{(1 - h_t)^{-\phi} h_t}{\alpha} \quad (18)$$

denote the tax rate that is required to induce an equilibrium outcome of h_t hours worked. Straightforward calculations yield $\tau'(h_t) < 0, \tau''(h_t) < 0$.

Before proceeding with the formulation of the government's problem it is useful to use (12)-(17) in order to obtain the present value of taxes and bailout payments in a market equilibrium. Using (12) and (15) inside (10) and recognizing that in equilibrium $c_t = Y_t$ implies that the budget constraint of the government can be written as

$$\alpha E_0 \int_0^\infty e^{-\rho t} \tau(h_t) dt = -E_0 \int_0^\infty e^{-\rho t} \frac{1}{c_t} dL_t. \quad (19)$$

Furthermore, equations (5) together with (16), (12) and $c_t = Y_t$ yield

$$W_t = \left(\frac{1 - \alpha}{\rho} \right) Y_t. \quad (20)$$

Equation (20) is a well known property of economies where the representative agent has logarithmic utility over consumption; in such economies the price to earnings ratio for a claim that pays a constant fraction $(1 - \alpha)$ of aggregate consumption ($c_t = Y_t$) is simply $\frac{1}{\rho}$. Defining

$$m_t \equiv \min_{0 \leq s \leq t} Y_s \quad \text{and} \quad \chi \equiv \frac{(1 - \alpha)}{\rho}, \quad (21)$$

and using equations (20) and (9) leads to

$$dL_t = \begin{cases} 0 & \text{if } \chi m_t \geq L_0 \\ \chi dm_t & \text{otherwise.} \end{cases} \quad (22)$$

Equation (22) implies that dL_t changes only when $c_t = m_t$ and hence

$$\begin{aligned} -E_0 \int_0^\infty e^{-\rho t} \frac{1}{c_t} dL_t &= -\chi E_0 \int_0^\infty e^{-\rho t} 1_{\{\chi m_t \leq L_0\}} \frac{1}{m_t} dm_t \\ &= -\chi E_0 \int_0^\infty e^{-\rho t} 1_{\{\log m_t \leq \log(\frac{L_0}{\chi})\}} d \log m_t, \end{aligned} \quad (23)$$

where the last equality obtains because m_t is a bounded variation process. Combining (19) and (23) leads to the following problem for the government

Problem 1 Let $J(Z_0, K_0; h_t)$ denote the representative consumer's welfare

$$U(Z_0, K_0; h_t) \equiv E_0 \left\{ \int_0^\infty e^{-\rho t} \left[\log(Z_t) + (1 - \alpha) \log(K_0) + \alpha \log h_t + \frac{(1 - h_t)^{1-\phi}}{1 - \phi} \right] dt \right\}. \quad (24)$$

Then the optimal taxation problem for a government that extends a perpetual bailout is

$$\max_{h_t \in [0,1]} U(Z_0, K_0; h_t)$$

subject to

$$\alpha E_0 \int_0^\infty e^{-\rho t} \tau(h_t) dt = -\chi E_0 \int_0^\infty e^{-\rho t} 1_{\{\log m_t \leq \log(\frac{L_0}{\chi})\}} d \log m_t, \quad (25)$$

where $\tau(h_t)$ is given by (18) and m_t is given by (21).

Equation (24) is simply equation (1), where c_t has been replaced with Y_t in equation (2). Given the supposition of a perpetual bailout, $K_t = K_0$. Equation (25) is the government's intertemporal budget constraint, which follows from (23) and (19).

Before proceeding it is useful to make three remarks.

First, the one-to-one correspondence between hours worked and tax rates implies an equivalence between choosing τ_t and h_t . For convenience, it is easiest to have the government choose h_t rather than τ_t .

Second, as is well known (see e.g. Ljungqvist and Sargent (2004), Chapter 15) in a market equilibrium the government's budget constraint (10) implies the consumer's budget constraint⁸. As a result, to check feasibility of an allocation it suffices to ensure that the government's intertemporal budget constraint holds (equation [25]). Alternatively put, after solving for the optimal h_t^* of problem 1, it is always possible to find a market equilibrium that supports the resulting allocation. (In that equilibrium output and consumption are given by $c_t = Y_t = Z_t K_0^{1-\alpha} (h_t^*)^\alpha$, the taxes that yield the optimal allocation as a market

⁸To see this add $E_0 \int_0^\infty \xi_t (1 - \tau_t) w_t h_t dt + E_0 \int_0^\infty \xi_t p_t F_t dt$ to both sides of (10) and use the fact that $w_t h_t + p_t F_t = Y_t = c_t$ on the left hand side of the resulting expression to obtain (11).

equilibrium are given by $\tau(h_t^*)$ and the price processes ξ_t, w_t, p_t are given by equations (12), (14) and (16) (evaluated at $F_t = K_0$).)

Third, problem 1 assumes that the government can commit to a sequence of taxes. However, as Lucas and Stokey (1983) show, this assumption can be relaxed if the government can issue contingent debt at all maturities.

The next section solves problem 1.

3 Optimal taxation

The key difficulty in solving problem 1 is the endogeneity of the cost of the bailout, which is reflected in the fact that output Y_t and as a result its running minimum m_t are affected by the choice of h_t .

To obtain intuition, it is useful to examine what would happen if the government behaved “naively” and optimized over h_t as if m_t were an exogenous process beyond its influence. In that case, the solution to problem 1 is straightforward. Letting λ denote the Lagrange multiplier on the government’s budget constraint and maximizing

$$U(Z_0, K_0; h_t) + \lambda \left[\alpha E_0 \int_0^\infty e^{-\rho t} \tau(h_t) dt + \chi E_0 \int_0^\infty e^{-\rho t} 1_{\{\log m_t \leq \log(\frac{L_0}{\chi})\}} d \log m_t \right]$$

amounts to simple “point by point” maximization problem with solution

$$h^* = \arg \max_{h_t} \left[\alpha \log h_t + \frac{(1 - h_t)^{1-\phi}}{1 - \phi} + \lambda \alpha \tau(h_t) \right]. \quad (26)$$

By concavity of the objective in (26), there is a unique value h^* that maximizes (26). Furthermore, the one-to-one correspondence between h_t and τ_t (equation [17]) implies that the government would end up choosing a *constant* tax rate. This result is reminiscent of the well known labor tax smoothing results in the literature on optimal taxation (see e.g. Ljungqvist and Sargent (2004) Chapter 15), and serves as an illustration of the labor tax smoothing forces that are present in the model.

However, matters are more complex, because a fully rational government takes into account that m_t (and hence the cost of the bailout) are both endogenous. This introduces two

opposing forces: On the one, smoothing the distortions associated with taxation tends to favor a stable tax rate. On the other hand, lowering taxes in states where the financial firm is threatened with bankruptcy can boost demand for labor, and hence increase the marginal product of the financial firm and the value of its assets W_t . In turn this boost in value implies that the total cost of the bailout may be reduced by “stimulating” the economy through tax cuts.

The remainder of this section derives the optimal h_t^* (and hence the optimal τ_t^*) that solves problem 1. It is easiest to start by assuming that τ_0 and hence h_0 is given, so that $\log m_0 = \log Y_0 = \log Z_0 + (1 - \alpha) \log K_0 + \alpha \log h_0$ is also given. I will assume that bailout payments haven’t yet started at time 0

$$\log Y_0 = \log m_0 \geq \log \left(\frac{L_0}{\chi} \right) \quad (27)$$

Remark 1 in the appendix allows the government to freely choose τ_0 (and hence Y_0 and m_0) and shows that condition (27) is always satisfied for sufficiently high values of Z_0 . Next, attach a Lagrange multiplier $\lambda > 0$ to (25) and define

$$f(h_t) \equiv \alpha \log h_t + \frac{(1 - h_t)^{1-\phi}}{1 - \phi} + \lambda \alpha \tau(h_t). \quad (28)$$

Observe that the first two terms inside the square brackets of (24) are exogenous and hence maximizing U is equivalent to maximizing

$$\Phi(h_t) = E_0 \int_0^\infty e^{-\rho t} f(h_t) dt + \lambda \chi E_0 \int_0^\infty e^{-\rho t} 1_{\{\log m_t \leq \log(\frac{L_0}{\chi})\}} d \log m_t. \quad (29)$$

To derive the optimal h_t , I first derive an upper bound to Φ over all processes h_t , and then show how to attain it with an appropriate choice of h_t .

To this end, consider an arbitrary path of $\tilde{h}_t$ and fix the associated process $\tilde{m}_t$. By definition of $\tilde{m}_t$,

$$\log Y_t = \log Z_t + (1 - \alpha) \log K_0 + \alpha \log \tilde{h}_t \geq \log \tilde{m}_t. \quad (30)$$

In a next step, define

$$g(Z_t, \tilde{m}_t) \equiv \max_{h_t \text{ s.t. } \log Z_t + (1-\alpha) \log K_0 + \alpha \log h_t \geq \log \tilde{m}_t} f(h_t). \quad (31)$$

and let

$$J(\tilde{m}_t) = E_0 \int_0^\infty e^{-\rho t} g(Z_t, \tilde{m}_t) dt + \lambda \chi E_0 \int_0^\infty e^{-\rho t} 1_{\{\log \tilde{m}_t \leq \log(\frac{L_0}{\chi})\}} d \log \tilde{m}_t. \quad (32)$$

By the definition of $g(Z_t, \tilde{m}_t)$ it follows that⁹ $g(Z_t, \tilde{m}_t) \geq f(\tilde{h}_t)$. Letting $D^{(m_0)}$ denote the set of decreasing adapted processes starting at m_0 , and comparing (29) and (32) gives

$$\Phi(\tilde{h}_t) \leq J(\tilde{m}_t) \leq \max_{m_t \in D^{(m_0)}} J(m_t) \quad (33)$$

Hence, $\max_{m_t \in D^{(m_0)}} J(m_t)$ provides an upper bound to the feasible payoffs $\Phi(\tilde{h}_t)$ for any admissible process $\tilde{h}_t$. Furthermore the optimal process h_t^* that attains this upper bound is given by

$$h_t^* = \arg \max_{h_t \text{ s.t. } \log Z_t + (1-\alpha) \log K_0 + \alpha \log h_t \geq \log m_t^*} f(h_t), \quad (34)$$

where $m_t^* = \arg \max_{m_t \in D^{(m_0)}} J(m_t)$. Letting

$$\bar{h} \equiv \arg \max_{h_t} f(h_t), \quad (35)$$

denote the unconstrained maximum of $f(h_t)$, and noting that $f(\cdot)$ is concave, the h_t^* that solves (34) is simply given by

$$\log h_t^* = \max \left[\log \bar{h}, \frac{1}{\alpha} (\log m_t^* - \log Z_t - (1-\alpha) \log K_0) \right] \quad (36)$$

Equation (36) suggests a simple two-step strategy for solving the government's problem. First, determine the optimal m_t that solves $\max_{m_t \in D^{(m_0)}} J(m_t)$. In a second step, use equations (36) and (18) to determine the optimal process for hours and taxes given the optimal m_t^* from the first step. Intuitively, in the first step, the government determines a minimum level of output that it wants to achieve, while in the second step it determines the hours and the taxes that will implement that minimum level.

⁹Equation (30) implies that $\tilde{h}_t$ is one of the feasible choices in the maximization problem defined in equation (31).

3.1 Determining m_t^*

To determine m_t^* one needs to solve the optimization problem

$$V(Z_0, m_0) = \max_{m_t \in D(m_0)} J(m_t). \quad (37)$$

This maximization problem shares several similarities with problems of irreversible investment. The only difference is that irreversible investment problems feature maximization over increasing rather than decreasing processes.

The next proposition derives the solution to problem (37).

Proposition 1 *Define*

$$\varphi_1 \equiv \frac{-\left(\mu - \frac{\sigma^2}{2}\right) + \sqrt{\left(\mu - \frac{\sigma^2}{2}\right)^2 + 2\rho\sigma^2}}{\sigma^2}$$

Using the notation $x^- \equiv \min[x, 0]$, define

$$y(u) \equiv \frac{1}{\alpha} u^{-\frac{1}{\alpha}} K_0^{-\frac{1-\alpha}{\alpha}} \left[f' \left(u^{-\frac{1}{\alpha}} K_0^{-\frac{1-\alpha}{\alpha}} \right) \right]^- + \lambda \chi \rho \quad (38)$$

and let β be given by the solution to

$$\inf_{\beta} \text{ s.t. } \int_{\beta}^{\infty} \frac{y(u)}{(u)^{\varphi_1+1}} du = 0. \quad (39)$$

Equation (39) has a unique solution and the optimal m_t^* that maximizes (37) is given by

$$m_t^* = \min \left[\frac{L_0}{\chi}, \frac{1}{\beta} \left(\min_{0 \leq s \leq t} Z_s \right) \right]. \quad (40)$$

Figure 1 illustrates the solution (40) for values of $m_t \leq \frac{L_0}{\chi}$. The diagram is split into two regions that are referred to as the “inaction” region and the “forbidden” region. In the inaction region $Z_t > \beta m_t$ and hence $m_t = \frac{1}{\beta} \min_{0 \leq s \leq t} Z_s < \frac{1}{\beta} Z_t$ or $Z_t > \min_{0 \leq s \leq t} Z_s$. Accordingly, it is optimal to set $dm_t^* = 0$, i.e. take no action. By contrast if Z_t becomes (instantaneously) smaller than βm_t , then it is optimal to immediately reduce m_t^* until the inequality $Z_t \geq \beta m_t^* = \min_{0 \leq s \leq t} Z_s$ is restored.

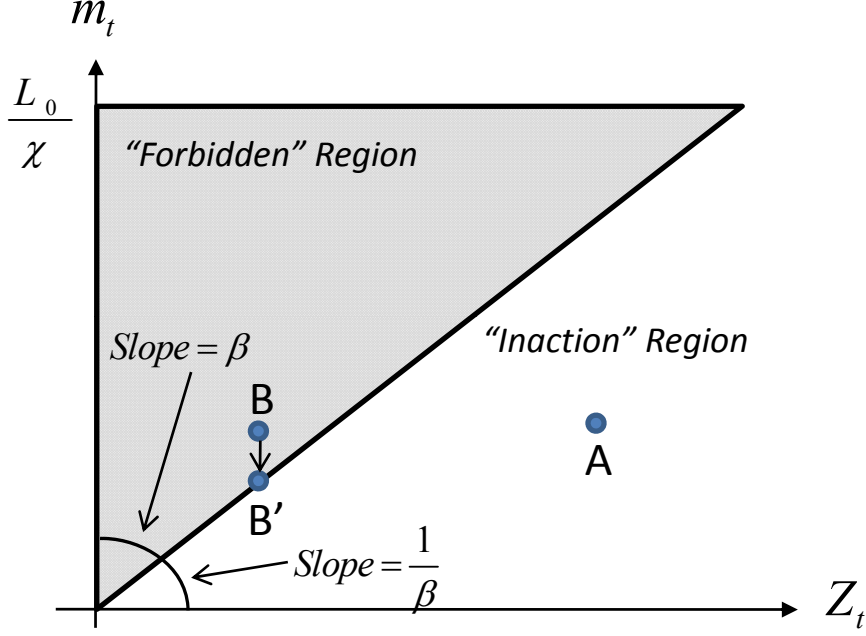


FIGURE 1: An illustration of the optimal policy m_t^* . At point A , $Z_t > \beta m_t^*$ and it is optimal to keep m_t^* unchanged. By contrast at point B it is optimal to instantaneously decrease m_t^* so as to move to point B' and restore the inequality $Z_t \geq \beta m_t^*$.

3.2 Procyclical tax rates and history dependence

Using the definition of $\tau(h_t)$ (equation[18]), noting that $\tau'(h_t) < 0$ and using (36) leads to the following expression for the optimal tax rate

$$\tau\left(Z_t, \min_{0 \leq s \leq t} Z_s\right) = \min \left\{ \tau(\bar{h}), \tau \left[\beta^{-\frac{1}{\alpha}} K_0^{-\frac{1-\alpha}{\alpha}} \left(\frac{Z_t}{\min_{0 \leq s \leq t} Z_s} \right)^{-\frac{1}{\alpha}} \right] \right\}, \quad (41)$$

where m_t^* is given by (40) and $\bar{h}$ is given by (35). Equation (41) shows that the optimal tax rate can be expressed as the ratio of two state variables, namely Z_t and $\min_{0 \leq s \leq t} Z_s$.

Inspecting (41), and recalling that τ is a decreasing function, reveals that the tax rate is non-decreasing in Z_t , i.e. the tax rate is higher in times of higher productivity. Figure 2 illustrates the dependence of τ on Z_t . When $Z_t = \min_{0 \leq s \leq t} Z_s$, (i.e. at times when the

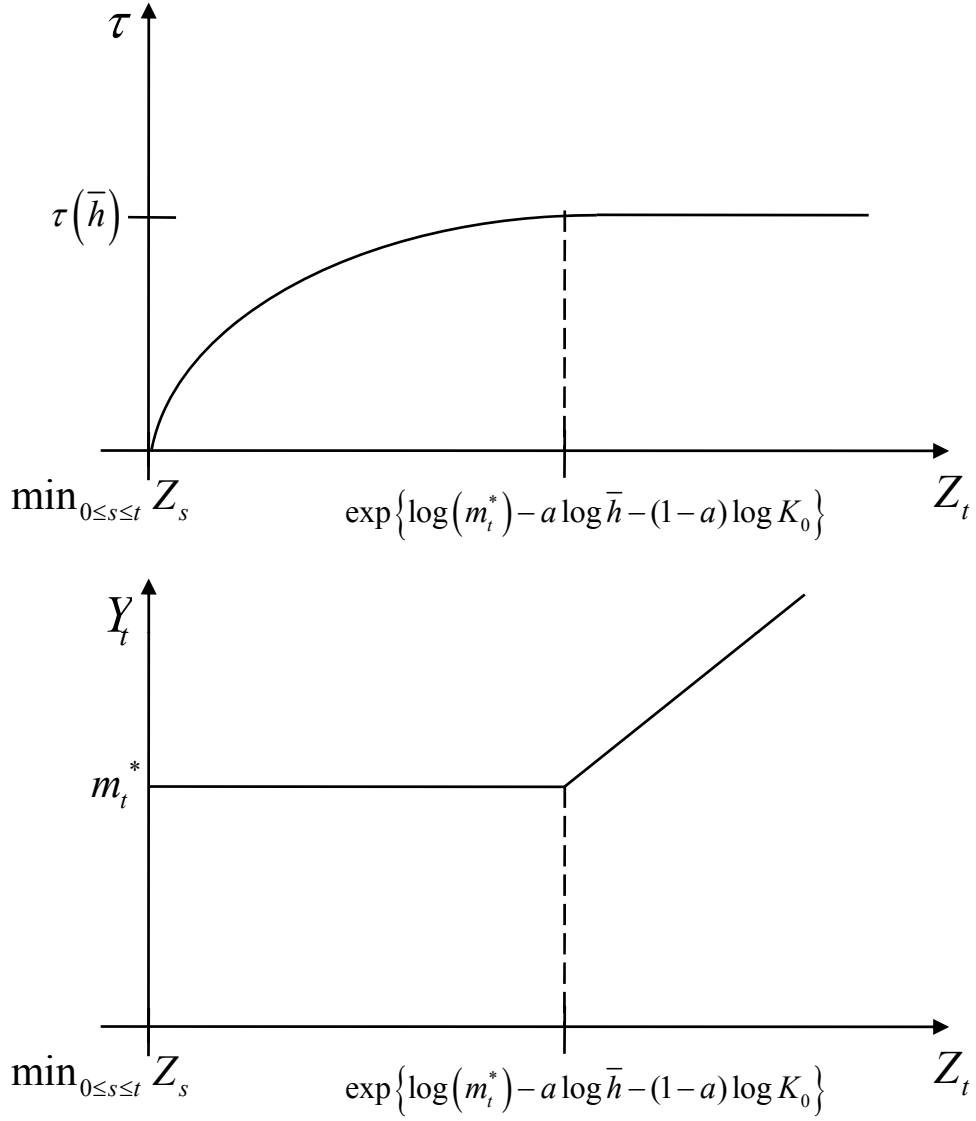


FIGURE 2: Procyclical taxes and output stabilization

government makes transfers to the private sector), the tax rate attains its lowest value. For values of Z_t that are to the right of $\min_{0 \leq s \leq t} Z_s$, the tax rate increases monotonically until the level $\tau(\bar{h})$.

The lower panel of the figure depicts the associated behavior of Y_t . For values of Z_t such

that τ is increasing in Z_t , output remains constant. Output starts increasing with Z_t only once τ becomes constant. This behavior of Y_t is a manifestation of the two forces present in the model; on the one hand, the government would like to set a constant tax rate for standard labor tax smoothing reasons. On the other hand however, by making the tax rate procyclical, the government can stabilize output Y_t and accordingly ensure that the value of capital W_t does not fall further. This helps the government save transfers to the firm and reduce the overall cost of the bailout.

A further and somewhat surprising property of the optimal tax rate is its history dependence *despite complete markets*. Equations (40) and (41) imply that the tax rate is a non-increasing function of the running minimum of output and hence an increasing function of the total payments to the financial firm by time t (by equation [22]). Alternatively put, *fixing* a given level of productivity (Z_t) at time t , the model implies a positive relationship between total bailout payments *prior* to time t and tax rates at time t .

3.3 Quantitative Evaluation

Equation (41) gives the optimal tax rate up to two constants $\bar{h}$ and β that depend on the Lagrange multiplier λ . The next Lemma shows how to compute λ .

Proposition 2 *The value of λ that enforces equation (25) is given as the solution to the equation*

$$V_\lambda \left(Z_0, \frac{L_0}{\chi}; \lambda \right) = 0. \quad (42)$$

An explicit expression for $V_\lambda \left(Z_0, \frac{L_0}{\chi}; \lambda \right)$ is contained in the proof of Proposition 2. Table 1 solves equation (42) for various parameter combinations and reports the difference between the maximal and the minimal tax rate $\tau(\bar{h}) - \tau \left(\beta^{-\frac{1}{\alpha}} K_0^{-\frac{1-\alpha}{\alpha}} \right)$ along with the value of the government's guarantee (i.e. the right hand side of [25]) as a fraction of Y_t .

The baseline scenario sets $\phi = 2$. This is a popular choice in the literature since it implies that the hours worked in the absence of taxes are approximately 1/3. K_0 is normalized to 1

Parameters	Value of guarantee	Maximum minus minimum tax rate
Baseline	0.074	0.055
$\sigma = 0.04$	0.151	0.082
$\mu = 0.015$	0.112	0.061
$\rho = 0.03$	0.093	0.060
$\alpha = 0.8$	0.062	0.036
$\phi = 4$	0.077	0.060

TABLE 1: Value of the guarantee to the financial sector as a fraction of GDP and difference between maximum and minimum tax rates. The baseline scenario assumes $\phi = 2$, $\rho = 0.04$, $\mu = 0.02$, $\sigma = 0.03$, $\alpha = 0.7$, $K_0 = 1$, $\frac{L_0}{x} = 1$ and Z_0 is chosen so that $Y_0 = 1$ in the absence of taxation. The results for the baseline case are reported in the first row, while the rest of the rows change one parameter and keep the other parameters at their baseline levels.

throughout, while Z_0 is set so as to ensure that $Y_0 = Z_0 K_0^{1-\alpha} h_{\tau=0}^\alpha = Z_0 h_{\tau=0}^\alpha = 1$. This choice facilitates the interpretation of the cost of the guarantee in the last column of the table as a fraction of GDP. The choice of α is motivated by the magnitude of the labor share in GDP, while the choices of μ and σ reflect (approximately) the mean growth rate and the volatility of the logarithm of per-capita consumption. Finally, ρ is chosen to match an earnings to price ratio of 0.04.

Table 1 shows that guarantees costing between 6.2% and 15.1% of GDP (depending on parameters) the distance between the highest and the lowest taxes is between 3.6% and 8.2%.

3.4 The optimality of a “perpetual” bailout

Sofar the analysis has assumed that a “perpetual” bailout is optimal. This subsection shows that this is always the case as long as δ is large enough.

Equations (36) and (40) imply that $h_t^* \in [\beta^{-\frac{1}{\alpha}} K_0^{-\frac{1-\alpha}{\alpha}}, \bar{h}]$. Importantly, neither β nor $\bar{h}$

depend on δ . Accordingly, at any point in time, the representative agent's welfare -assuming a perpetual bailout- is at least as large as

$$E_t \left\{ \int_t^\infty e^{-\rho(s-t)} [\log(Z_s) + (1-\alpha) \log(K_0) + \psi_1] ds \right\}, \quad (43)$$

where $\psi_1 = \min_{h_t \in [\beta^{-\frac{1}{\alpha}} K_0^{-\frac{1-\alpha}{\alpha}}, \bar{h}]} \left(\alpha \log h_t + \frac{(1-h_t)^{1-\phi}}{1-\phi} \right)$. Similarly, assuming that the government lets the financial firm fail at time t , the representative agent's welfare is bounded above by

$$E_t \left\{ \int_t^\infty e^{-\rho(s-t)} [\log(Z_s) + (1-\alpha) \log((1-\delta)K_0) + \psi_2] ds \right\}, \quad (44)$$

where $\psi_2 = \max \left(\alpha \log h_t + \frac{(1-h_t)^{1-\phi}}{1-\phi} \right)$. Comparing (43) and (44) reveals that a *sufficient* condition for a perpetual bailout is

$$(1-\alpha) \log(1-\delta) + \psi_2 < \psi_1. \quad (45)$$

Since neither ψ_1 nor ψ_2 depend on δ , and $\log(1-\delta)$ can be made arbitrarily small as $\delta \rightarrow 1$, there always exist values of δ that make condition (45) hold.

It is useful here to make two remarks. First, even though condition (45) is sufficient to ensure the optimality of a perpetual bailout, it is not necessary. Second, the values of δ that are required to ensure the optimality of a perpetual bailout are quantitatively very small. For the parametric examples of table (1) values as low as $\delta = 0.02$ are sufficient to ensure the optimality of a perpetual bailout. The reason is that if the firm fails, there is a permanent loss of capital that has a first order impact on the productive capacity of the economy. By contrast, if the firm is bailed out, the capital stock remains intact. The only source of inefficiency comes from the distorted incentives to work, which are of second order, especially when labor supply is relatively inelastic.

It is important to stress that re-distribution concerns (which have been ignored so far) may have a substantial impact on the welfare implications of a perpetual bailout. In a representative agent economy, the taxes raised through labor taxation get indirectly rebated back to the consumer in the form of an increased value of his total (financial and non-financial) wealth. In reality, a large fraction of the population has little or no financial

wealth and has to rely on labor income alone to finance consumption. If the government cares mostly about these parts of the population, then the welfare calculations need to be modified. In these modified calculations, the benefit of a bailout would stem from the fact that the deadweight costs of bankruptcy reduce the capital stock and hence the wages of workers.

4 Additional forms of financing bailouts and the evolution of debt

Sofar, bailouts could only be financed with distortionary labor taxes. In reality, bailouts are at least partly financed by the beneficiaries of bailout payments. For instance, the government may provide cash injections in exchange for equity holdings in the underlying company.

Within the context of the model this amounts to effectively taxing existing capital holders. As is well understood in the literature, taxation of the existing capital stock amounts to a lump sum, non-distortionary, and hence efficient tax.

Despite the apparent appeal of such forms of raising funds, it is likely that in reality only a fraction of the funds necessary for a “perpetual” bailout can be raised in this way. For instance, it is likely that at times prior to time 0 the government might have entered commitments through institutions that protect capital in order to promote capital accumulation. It may also be the case that shareholders and management need to always have a minimum stake in the company in order to provide the right amount of monitoring and work effort.

In light of the above considerations, in this section I allow for the possibility that part of the funds needed for the bailout may be raised by having the government obtain either a) a fraction $\pi_1 \geq 0$ of the equities of the financial firm and/or b) a share $\pi_2 \geq 0$ of the revenues of the representative financial firm. However, I continue to assume that no tax can be raised on debtholders; one potential motivation for this assumption is that debtholders may have outside options of investing (such as municipal bonds) that the government cannot

tax for institutional reasons. The rights and the behavior of debtholders - and in particular the constraint implied by equation (7) - are as in the baseline model, irrespective of the allocation of the cash flow rights between shareholders and the government.

The motivation for considering equities and revenue fractions as two alternative forms for raising funds is based on current proposals to fund part of the cost of bailouts by either diluting current shareholders or by placing some of the troubled assets in the hands of the government (or in a government-sponsored “bad bank”). Since the goal of this section is mostly illustrative, I assume that π_1 and π_2 remain constant throughout time. Letting

$$P_t \equiv \left(\frac{1}{\xi_t} E_t \int_t^\infty \xi_s (1 - \pi_2) p_s F_s ds - L_t - \frac{1}{\xi_t} E_t \int_t^\infty \xi_s dL_s \right) \quad (46)$$

denote the total value of equity¹⁰ at time t , I assume that

$$\pi_1 \xi_0 P_0 + \pi_2 E_0 \int_0^\infty \xi_t p_t F_t dt \leq -\zeta E_0 \int_0^\infty \xi_t dL_t, \quad \zeta \in (0, 1). \quad (47)$$

Specification (47) is attractive, because it ensures that the value of equity is at least as large as $-(1 - \zeta) E_0 \int_0^\infty \frac{\xi_t}{\xi_0} dL_t \geq 0$, so that it implies allocations that are compatible with limited liability of equity.

The government’s modified budget constraint is given by

$$E_0 \int_0^\infty \xi_t \tau_t w_t h_t dt + \pi_1 \xi_0 P_0 + \pi_2 E_0 \int_0^\infty \xi_t p_t F_t dt = -E_0 \int_0^\infty \xi_t dL_t. \quad (48)$$

Combining (47) and (48) leads to

$$E_0 \int_0^\infty \xi_t \tau_t w_t h_t dt \geq -(1 - \zeta) E_0 \int_0^\infty \xi_t dL_t. \quad (49)$$

¹⁰To see that this is the total value of equity, note that equity value is given as the difference between post-tax revenue and interest payments. By footnote 3 the payments to debtholders when debt is equal to L_t are given by $L_t (r_t dt + dA_t)$. Letting

$$P_t \equiv \frac{1}{\xi_t} \left(E_t \int_t^\infty \xi_s (1 - \pi_2) p_s F_s ds - E_t \int_t^\infty \xi_s L_s (r_s ds + dA_s) \right)$$

denote the total value of equity, using the fact that $L_t (r_t dt + dA_t) = -L_t (d\xi_t/\xi_t - \kappa_t \xi_t dB_t)$ (by footnote 3), integrating by parts and ignoring terms having expectation equal to zero leads to (46).

Since labor taxes are distortionary, (49) will hold with equality at the optimum. There are two obvious, yet important, implications of equation (49). First, obtaining a share of the dividends or the revenues of the financial firm is equivalent to limiting the cost of the guarantee and hence the distortions associated with labor taxation. And second, since π_1 and π_2 do not appear in (49), the choice between taxation of the profits of the financial firm or the underlying capital has no welfare consequences (since neither the maximization objective [1], nor the modified budget constraint [49] depend on π_1 or π_2 .)

The second implication of (49) depends crucially on the assumption of complete markets. If one were to assume restrictions on the ability of the government to trade in contingent securities, then one form of financing may become more preferable compared to the other. To illustrate this point, the next subsection considers the stochastic process of government debt under the two alternatives (obtaining a fraction of dividends or obtaining a fraction of revenues) and shows that in the presence of a debt ceiling, the latter form of financing may be preferable.

4.1 Evolution of debt and debt ceilings

Letting B_t denote the government's debt, one obtains

$$\begin{aligned} \xi_t B_t = & E_t \int_t^\infty \xi_s dL_s + E_t \int_t^\infty \xi_s \tau(h_s) w_s h_s ds \\ & + \pi_1 \left(E_t \int_t^\infty \xi_s (1 - \pi_2) p_s F_s ds - \xi_t L_t - E_t \int_t^\infty \xi_s dL_s \right) + \pi_2 E_t \int_t^\infty \xi_s p_s F_s ds \end{aligned} \quad (50)$$

The next proposition shows that the maximal value of the debt to gdp ratio $b_t \equiv \frac{B_t}{Y_t}$ (across all $t > 0$) is smallest when $\pi_1 = 0$.

Proposition 3 *Assume that constraint (47) is satisfied with equality. Then the value of π_1 that minimizes $\sup_{0 \leq s \leq t} b_s$ is given by $\pi_1 = 0$.*

5 Conclusion

The present paper considered optimal fiscal policy, when distortionary taxes are used to finance large bailouts. The key departure from pre-existing literature is that taxes are not used to finance exogenous *government expenditure*, but rather endogenous *transfer payments*.

The baseline model supports two broad conclusions. First, some degree of procyclical taxation is always optimal. Taxes are not only used to finance the bailout, but also to support real activity, raise the value of financial firms and hence reduce the net present value of the taxes required to finance the transfers. This result is in contrast to a large literature (e.g. Barro (1979), Lucas and Stokey (1983)), that prescribes a constant, acyclical tax rate, when government expenditures do not vary. Second, tax rates are dependent on both current productivity and past transfer payments. This history dependence of the tax rate in a complete market is in contrast to Lucas and Stokey (1983), where the tax rate depends only on current government expenditure.

An extended version of the model considers additional forms of funding a bailout such as obtaining equity shares of the company and diluting shareholders or obtaining shares of the underlying capital stock. In a complete market it is only the total value of these claims that affects welfare, and not the choice between the two. However, the stochastic process for debt implied by the two alternatives is different. For instance, obtaining equity shares may lead to higher levels of the debt to gdp ratio, which may be unattractive if the government is subject to a debt ceiling.

A Appendix

Proof of Proposition 1. The first step towards proving proposition 1 is to show that (without loss of generality) one can focus attention to policies that set $m_0 = \frac{L_0}{\chi}$.

Lemma 1 $m_0^* \leq \frac{L_0}{\chi}$.

Proof of Lemma 1. Suppose otherwise and consider an optimal m_t^* such that $m_0^* > \frac{L_0}{\chi}$. Let τ^{L_0} denote the first time that $m_{\tau^{L_0}}^* = \frac{L_0}{\chi}$. Then, by Bellman's principle of optimality,

$$\begin{aligned} V(Z_0, m_0) &= E_0 \int_0^{\tau^{L_0}} e^{-\rho t} g(Z_t, m_t^*) dt + E e^{-\rho \tau^{L_0}} V\left(Z_{\tau^{L_0}}, \frac{L_0}{\chi}\right) \\ &\leq E_0 \int_0^{\tau^{L_0}} e^{-\rho t} g\left(Z_t, \frac{L_0}{\chi}\right) dt + E e^{-\rho \tau^{L_0}} V\left(Z_{\tau^{L_0}}, \frac{L_0}{\chi}\right) \\ &\leq V\left(Z_0, \frac{L_0}{\chi}\right), \end{aligned}$$

where the first inequality follows from $g_m < 0$. Since $V(Z_0, m_0) \leq V\left(Z_0, \frac{L_0}{\chi}\right)$, and it is always possible to decrease m_0 instantaneously, one may assume without loss of generality that $m_0^* = \frac{L_0}{\chi}$. ■

In light of Lemma 1, one can set $m_0 = \frac{L_0}{\chi}$ without loss of generality and accordingly $1_{\{\log m_t \leq \log(\frac{L_0}{\chi})\}} = 1$. Letting

$$\eta(Z_t, m_t) \equiv g(Z_t, m_t) + \lambda \chi \rho \log m_t,$$

and applying integration by parts to (32) gives¹¹

$$J\left(Z_0, \frac{L_0}{\chi}; m_{t>0}\right) = \hat{J}\left(Z_0, \frac{L_0}{\chi}; m_{t>0}\right) - \lambda \chi \log\left(\frac{L_0}{\chi}\right), \quad (51)$$

¹¹In applying integration by parts, note that

$$\lim_{T \rightarrow \infty} e^{-\rho T} E \log m_T = 0$$

which follows from the fact that h_t is bounded and that $\lim_{T \rightarrow \infty} e^{-\rho T} E \min_{0 \leq s \leq T} \log Z_T = 0$. (The fact that $\lim_{T \rightarrow \infty} e^{-\rho T} E \min_{0 \leq s \leq T} \log Z_T = 0$ follows from the closed form expression for $\Pr(\min_{0 \leq s \leq T} \log Z_T \geq x)$ in Corollary B.3.4. of Musiela and Rutkowski (1998) (p.470) after using integration by parts to compute $E e^{-\rho T} \min_{0 \leq s \leq T} \log Z_T$ and sending T to infinity.)

where

$$\hat{J}\left(Z_0, \frac{L_0}{\chi}; m_{t>0}\right) = E_0 \int_0^\infty e^{-\rho t} \eta(Z_t, m_t) dt.$$

Clearly, maximizing J is equivalent to maximizing $\hat{J}$ (since $\lambda\chi \log\left(\frac{L_0}{\chi}\right)$ is a constant that does not depend on the choice of $m_{t>0}$). Before proceeding, it is useful to prove the following Lemma.

Lemma 2 *Let $\bar{Z}$ denote a value of Z that solves the equation*

$$\inf_{\bar{Z}} \text{ s.t. } \int_{\bar{Z}}^\infty \frac{\eta_m(x, m_t)}{x^{\varphi_1+1}} dx = 0. \quad (52)$$

Such a value exists, is unique, and is increasing in m_t .

Proof of Lemma 2. Differentiating $\eta(Z_t, m_t)$ with respect to m_t gives

$$\eta_m(Z_t, m_t) = \frac{1}{\alpha} Z_t^{-\frac{1}{\alpha}} K_0^{-\frac{1-\alpha}{\alpha}} m_t^{\frac{1}{\alpha}-1} \left[f'(m_t^{\frac{1}{\alpha}} Z_t^{-\frac{1}{\alpha}} K_0^{-\frac{1-\alpha}{\alpha}}) \right]^- + \lambda\chi\rho \frac{1}{m_t}$$

Since $f'(m_t^{\frac{1}{\alpha}} x^{-\frac{1}{\alpha}} K_0^{-\frac{1-\alpha}{\alpha}})$ is positive once $m_t^{\frac{1}{\alpha}} x^{-\frac{1}{\alpha}} K_0^{-\frac{1-\alpha}{\alpha}} < \bar{h}$, it follows that $\eta_m = \lambda\chi\rho \frac{1}{m_t} > 0$ whenever $x > \bar{h}^{-\alpha} K_0^{-(1-\alpha)} m_t$. Furthermore, since¹² $\lim_{h \rightarrow 1} f'(h) = -\infty$, it follows that $\lim_{x \rightarrow m K_0^{-(1-\alpha)}} f'(m_t^{\frac{1}{\alpha}} x^{-\frac{1}{\alpha}} K_0^{-\frac{1-\alpha}{\alpha}}) = -\infty$. Additionally, since f is concave, differentiating $\frac{1}{\alpha} x^{-\frac{1}{\alpha}} K_0^{-\frac{1-\alpha}{\alpha}} m_t^{\frac{1}{\alpha}-1} f'(m_t^{\frac{1}{\alpha}} x^{-\frac{1}{\alpha}} K_0^{-\frac{1-\alpha}{\alpha}})$ with respect to x implies that $\eta_m(x, m_t)$ is a non-decreasing function of x , and by the intermediate value theorem there exists a unique $\bar{x} \in (m K_0^{-(1-\alpha)}, +\infty)$ such that $\frac{\eta_m(\bar{x}, m_t)}{\bar{x}^{\varphi_1+1}} = 0$. Finally, $\frac{\eta_m(x, m_t)}{x^{\varphi_1+1}} > 0$ for all $x > \bar{x}$ and $\frac{\eta_m(x, m_t)}{x^{\varphi_1+1}} < 0$ for all $x < \bar{x}$. Accordingly, there exists a unique $\bar{Z}$ that solves equation (52). Inspection reveals that for $m_1 > m_2$, $\eta_m(x, m_1) < \eta_m(x, m_2)$ and hence $\bar{Z}(m_1) > \bar{Z}(m_2)$. (To see this, note that since η_m is decreasing¹³, $\eta_{mm} < 0$). An application of the implicit

¹²A straightforward computation leads to

$$f'(h_t) = \frac{\alpha}{h_t} + (1-h_t)^{-\phi} \left[1 - \lambda - \phi \frac{h_t}{(1-h_t)} \right]$$

and since $\lim_{h_t \rightarrow 1} (1-h_t)^{-\phi} = +\infty$ and $\lim_{h_t \rightarrow 1} \left(-\phi \frac{h_t}{(1-h_t)} \right) = -\infty$, it follows that $\lim_{h_t \rightarrow 1} f'(h_t) = -\infty$.

¹³ η_{mm} is defined everywhere except at the point where $m_t^{\frac{1}{\alpha}} x^{-\frac{1}{\alpha}} K_0^{-\frac{1-\alpha}{\alpha}} = \bar{h}$. At that point one can arbitrarily set η_{mm} equal to either its left- or right-derivative (or any other value) without affecting the remainder of the proof.

function theorem to (52) implies that

$$\frac{d\bar{Z}}{dm_t} = \frac{\int_{\bar{Z}}^{\infty} \frac{\eta_{mm}(x, m_t)}{x^{\varphi+1}} dx}{\bar{Z}^{-(\varphi_1+1)} \eta_m(\bar{Z}, m_t)} > 0$$

since¹⁴ $\eta_{mm} < 0$ and $\eta_m(\bar{Z}, m_t) < 0$ (note that $\bar{Z} < \bar{x}$ and hence $\eta_m(\bar{Z}, m_t) < 0$.) ■

The remainder of the proof proceeds via a verification argument; first I postulate a solution for the value function

$$\hat{V}\left(Z_t, \frac{L_0}{\chi}\right) = \sup_{m_t > 0} \hat{J}\left(Z_0, \frac{L_0}{\chi}; m_t > 0\right) \quad (53)$$

and then verify directly that $\hat{V}\left(Z_t, \frac{L_0}{\chi}\right)$ is indeed the value function.

Specifically, let $\bar{Z}(m)$ denote the solution of equation (52), and let

$$\omega(x) \equiv \bar{Z}^{-1}(\cdot)$$

denote the inverse function of $\bar{Z}(\cdot)$. Observe that since $\bar{Z}(m)$ is increasing, $\omega(x)$ is also increasing. Next consider the following expression for the value function

$$\hat{V}(Z_t, m_t) = \int_0^{\bar{Z}(m_t)} \mathcal{G}(Z_t, x) \eta(x, \omega(x)) dx + \int_{\bar{Z}(m_t)}^{\infty} \mathcal{G}(Z_t, x) \eta(x, m_t) dx, \quad (54)$$

where $\mathcal{G}(Z_t, x)$ is defined as follows¹⁵

$$\mathcal{G}(Z_t, x) \equiv \begin{cases} \frac{2}{(\varphi_1 - \varphi_2)\sigma^2} Z_t^{\varphi_2} x^{-\varphi_2 - 1} & \text{if } x \leq Z_t \\ \frac{2}{(\varphi_1 - \varphi_2)\sigma^2} Z_t^{\varphi_1} x^{-\varphi_1 - 1} & \text{if } x > Z_t \end{cases} \quad (55)$$

and the constant φ_2 is defined as

$$\varphi_2 \equiv \frac{-\left(\mu - \frac{\sigma^2}{2}\right) - \sqrt{\left(\mu - \frac{\sigma^2}{2}\right)^2 + 2\rho\sigma^2}}{\sigma^2}$$

The first step towards verifying that $\hat{V}(Z_t, m_t)$ is the value function of (53) is contained in the following Lemma

¹⁴Note that the non-differentiability of η_m at a single point is irrelevant for the computation of the integral $\int_{\bar{Z}}^{\infty} \frac{\eta_{mm}(m_t, x)}{x^{\varphi_1+1}} dx$.

¹⁵In the literature $\mathcal{G}(Z_t, x)$ is known as the Green function, see e.g. Kabilia (1993), or Øksendal (2003) Ch. 9.

Lemma 3 $\widehat{V}(Z_t, m_t)$ satisfies the following differential inequality

$$\sigma^2 Z_t^2 \widehat{V}_{ZZ} + \mu Z_t \widehat{V}_Z - \rho \widehat{V} + \eta(Z_t, m_t) \leq 0 \quad (56)$$

Proof of Lemma 3. By applying Ito's Lemma to (54) and using (55), it is simple to verify that¹⁶

$$\sigma^2 Z_t^2 \widehat{V}_{ZZ} + \mu Z_t \widehat{V}_Z - \rho \widehat{V} = - \begin{cases} \eta(Z_t, \omega(Z_t)) & \text{if } Z_t \leq \overline{Z}(m_t) \\ \eta(Z_t, m_t) & \text{if } Z_t > \overline{Z}(m_t) \end{cases} \quad (57)$$

When $Z_t \leq \overline{Z}(m_t)$ it follows that $\omega(Z_t) \leq m_t$ and hence

$$\eta(Z_t, m_t) - \eta(Z_t, \omega(Z_t)) = \int_{\omega(Z_t)}^{m_t} \eta_m(Z_t, u) du < 0 \quad (58)$$

since¹⁷ $\eta_m(Z_t, \omega(Z_t)) < 0$ and η_m is declining in m . Combining (57) with (58) gives (56). ■

The second step of the verification argument is contained in the following statement

Lemma 4 The derivative of $\widehat{V}$ with respect to m satisfies the following set of (in)equalities

$$\widehat{V}_m(Z_t, m_t) = \begin{cases} = 0 & \text{if } Z_t < \overline{Z}(m_t) \\ \leq 0 & \text{if } Z_t \geq \overline{Z}(m_t) \end{cases}$$

Proof of Lemma 4. Differentiating (54) with respect to m_t , noting that $\eta(\overline{Z}(m_t), \omega(\overline{Z}(m_t))) = \eta(\overline{Z}(m_t), m_t)$ and using the definition of $\mathcal{G}(Z_t, x)$ in equation (55) implies that

$$V_m(Z_t, m_t) = \begin{cases} \frac{2}{(\varphi_1 - \varphi_2)\sigma^2} Z_t^{\varphi_1} \int_{\overline{Z}(m_t)}^{\infty} \frac{\eta_m(x, m_t)}{x^{\varphi_1+1}} dx & \text{if } Z_t < \overline{Z}(m_t) \\ \frac{2}{(\varphi_1 - \varphi_2)\sigma^2} \left[Z_t^{\varphi_2} \int_{\overline{Z}(m_t)}^{Z_t} \frac{\eta_m(x, m_t)}{x^{\varphi_2+1}} dx + Z_t^{\varphi_1} \int_{Z_t}^{\infty} \frac{\eta_m(x, m_t)}{x^{\varphi_1+1}} dx \right] & \text{if } Z_t \geq \overline{Z}(m_t) \end{cases}$$

In light of (52) it follows that $V_m(Z_t, m_t) = 0$ whenever $Z_t < \overline{Z}(m_t)$. Similarly, when $Z_t \geq \overline{Z}(m_t)$

$$\begin{aligned} V_m(Z_t, m_t) &= \frac{2Z_t^{\varphi_1}}{(\varphi_1 - \varphi_2)\sigma^2} \left[Z_t^{\varphi_2 - \varphi_1} \int_{\overline{Z}(m_t)}^{Z_t} \frac{\eta_m(x, m_t)}{x^{\varphi_1+1}} \frac{x^{\varphi_1+1}}{x^{\varphi_2+1}} dx + \int_{Z_t}^{\infty} \frac{\eta_m(x, m_t)}{x^{\varphi_1+1}} dx \right] \\ &= \frac{2Z_t^{\varphi_1}}{(\varphi_1 - \varphi_2)\sigma^2} \left[\int_{\overline{Z}(m_t)}^{Z_t} \frac{\eta_m(x, m_t)}{x^{\varphi_1+1}} \left(\frac{x}{Z_t} \right)^{\varphi_1 - \varphi_2} dx + \int_{Z_t}^{\infty} \frac{\eta_m(x, m_t)}{x^{\varphi_1+1}} dx \right] \end{aligned}$$

¹⁶See Kobila (1993) for some technical details.

¹⁷As was shown above $\eta_m(\overline{Z}, m_t) < 0$, which implies that $\eta_m(Z_t, \omega(Z_t)) < 0$, since $Z_t = \overline{Z}(m_t)$ if and only if $m_t = \omega(Z_t)$.

Since $\int_{\bar{Z}(m_t)}^{Z_t} \frac{\eta_m(x, m_t)}{x^{\varphi_1+1}} dx + \int_{Z_t}^{\infty} \frac{\eta_m(x, m_t)}{x^{\varphi_1+1}} dx = 0$ by (52), and $\eta_m(x, m_t)$ is an increasing function of x it follows that $\int_{\bar{Z}(m_t)}^{Z_t} \frac{\eta_m(x, m_t)}{x^{\varphi_1+1}} dx < 0$. Additionally, since $\left(\frac{x}{\bar{Z}_t}\right)^{\varphi_1-\varphi_2} < 1$ for all $x \in [\bar{Z}(m_t), Z_t]$, it follows that $V_m(Z_t, m_t) \geq 0$ for all $Z_t \geq \bar{Z}(m_t)$. ■

The rest of the verification argument follows similar steps to Kobila (1993), Proposition 6.1. To save space, I simply outline the argument and refer the reader to Kobila (1993) for additional technical details.

Take any arbitrary process $m_t \in D^{(L_0/\chi)}$. Applying Ito's Lemma to $\hat{V}$ gives

$$\begin{aligned} E_0 \left(e^{-\rho T} \hat{V}(Z_T, m_T) \right) - \hat{V}(Z_0, m_0) &= E_0 \int_0^T e^{-\rho t} \left(\sigma^2 Z_t^2 \hat{V}_{ZZ} + \mu Z_t \hat{V}_Z - \rho \hat{V} \right) dt \\ &\quad + E_0 \int_0^T e^{-\rho t} \sigma Z_t \hat{V}_Z dB_s \\ &\quad + E_0 \int_0^T e^{-\rho t} \hat{V}_m dm_t. \end{aligned}$$

Letting $T \rightarrow \infty$, and using arguments similar to Kobila (1993), one obtains the limit

$$\begin{aligned} \hat{V}(Z_0, m_0) &= -E_0 \int_0^\infty e^{-\rho t} \left(\sigma^2 Z_t^2 \hat{V}_{ZZ} + \mu Z_t \hat{V}_Z - \rho \hat{V} \right) dt - E_0 \int_0^\infty e^{-\rho t} \hat{V}_m dm_t \\ &\geq E_0 \int_0^\infty e^{-\rho t} \eta(Z_t, m_t) dt, \end{aligned} \tag{59}$$

where the second line follows from equation (56) and the fact that $E_0 \int_0^\infty e^{-\rho t} V_m dm_t < 0$ (since $V_m \geq 0$ and m_t is decreasing). Since $m_t \in D^{(L_0/\chi)}$ was arbitrary, equation (59) implies that $\hat{V}(Z_0, m_0)$ provides an upper bound to any attainable payoff. Additionally, by the Skorohod equation (Karatzas and Shreve (1991) p. 210) equality in (59) holds only for the process

$$m_t^* = \min \left[\frac{L_0}{\chi}, \omega \left(\min_{0 \leq s \leq t} Z_s \right) \right] \tag{60}$$

of equation (40). The next Lemma allows one to obtain an explicit expression for $\omega(\cdot) = \bar{Z}^{-1}(\cdot)$.

Lemma 5 *Let $y(u)$ be given by (38) and β by (39). Then the value $\bar{Z}(m_t)$ that solves (52) is given by $\bar{Z}(m_t) = \beta m_t$.*

Proof of Lemma 5. By Lemma 2, equation (52) has a unique solution $\overline{Z}(m_t)$. Accordingly, in order to prove the statement of Lemma 5, it suffices to show that

$$\int_{\beta m_t}^{\infty} \frac{\eta_m(x, m_t)}{x^{\varphi_1+1}} dx = 0 \text{ for all } m_t. \quad (61)$$

To this end, observe first that $\eta_m(x, m_t) = \frac{1}{m_t} y\left(\frac{x}{m_t}\right)$. Next, let $x = um_t$ and apply a change of variables to (39) to obtain

$$0 = \int_{\beta m_t}^{\infty} \frac{y\left(\frac{x}{m_t}\right)}{\left(\frac{x}{m_t}\right)^{\varphi_1+1}} \frac{1}{m_t} dx = m_t^{\varphi_1+1} \left(\int_{\beta}^{\infty} \frac{\eta_m(x, m_t)}{x^{\varphi_1+1}} dx \right).$$

This proves (61). ■

Combining (60), Lemma 5 and the definition of $\omega(\cdot)$ yields (40). This concludes the proof of Proposition 1. ■

Remark 1 *The purpose of this remark is to show that $\sup_{\lambda \in [0, \infty)} \beta(\lambda) < \infty$. To see this, note that when $\lambda = 0$, inspection of (38) and (39) reveals that $\beta = \overline{h}^{-\alpha} K_0^{1-\alpha}$. Additionally, for any β*

$$\lim_{\lambda \rightarrow \infty} \frac{1}{\lambda} \int_{\beta}^{\infty} \frac{y(u)}{(u)^{\varphi_1+1}} du = \int_{\beta}^{\infty} \frac{u^{-\frac{1}{\alpha}} K_0^{-\frac{1-\alpha}{\alpha}} \tau' \left(u^{-\frac{1}{\alpha}} K_0^{-\frac{1-\alpha}{\alpha}} \right) + \chi \rho}{(u)^{\varphi_1+1}} du.$$

Since $\lim_{u \rightarrow \infty} u^{-\frac{1}{\alpha}} K_0^{-\frac{1-\alpha}{\alpha}} \tau' \left(u^{-\frac{1}{\alpha}} K_0^{-\frac{1-\alpha}{\alpha}} \right) = 0$ and $\lim_{u^{-\frac{1}{\alpha}} K_0^{-\frac{1-\alpha}{\alpha}} \rightarrow 1} u^{-\frac{1}{\alpha}} K_0^{-\frac{1-\alpha}{\alpha}} \tau' \left(u^{-\frac{1}{\alpha}} K_0^{-\frac{1-\alpha}{\alpha}} \right) = -\infty$ there exists a finite root of (39) at the limit $\lambda \rightarrow \infty$. Since β is a continuous function of λ , $\sup_{\lambda \in [0, \infty)} \beta(\lambda)$ is finite. This implies that if $Z_0 \geq \frac{L_0}{\chi} [\sup_{\lambda \in [0, \infty)} \beta(\lambda)]$, then one can guarantee that $m_0 < \frac{L_0}{\chi}$ is not optimal. It is useful to note, that even though $Z_0 \geq \frac{L_0}{\chi} [\sup_{\lambda \in [0, \infty)} \beta(\lambda)]$ is a sufficient condition for $m_0 \geq \frac{L_0}{\chi}$, it is not necessary; lower values of Z_0 could (and in the numerical example of Table 1 do) imply that $m_0 = \frac{L_0}{\chi}$, since $Z_0 > \beta(\lambda^*) \frac{L_0}{\chi}$, where λ^* is the Lagrange multiplier that solves (25).

Proof of Proposition 2. By equation (51) one obtains $J_{\lambda} \left(Z_0, \frac{L_0}{\chi}; m_{t>0}, \lambda \right) = \hat{J}_{\lambda} \left(Z_0, \frac{L_0}{\chi}; m_{t>0}, \lambda \right) - \chi \log \left(\frac{L_0}{\chi} \right)$ for any process $m_{t>0}$. In particular for $m_{t>0} = m_{t>0}^*$,

$$V_{\lambda} \left(Z_0, \frac{L_0}{\chi}; \lambda \right) = \hat{V}_{\lambda} \left(Z_0, \frac{L_0}{\chi}; \lambda \right) - \chi \log \left(\frac{L_0}{\chi} \right). \quad (62)$$

The next Lemma establishes two properties of $\widehat{V}_\lambda(Z_0, m_t; \lambda)$.

Lemma 6 *Let $B(Z_t, m_t^*; \lambda) \equiv \widehat{V}_\lambda(Z_t, m_t^*; \lambda)$ and $h_t^* = h^*(Z_t, m_t^*; \lambda)$ denote the hours that maximize (34), assuming that $m_t = m_t^*$. Then, the function $B(Z_t, m_t^*; \lambda)$ satisfies the following properties*

$$\sigma^2 Z_t^2 B_{ZZ} + \mu Z_t B_Z - \rho B = -\alpha \tau(h(Z_t, m_t^*)) - \chi \rho \log m_t^* \quad (63)$$

$$B_m(\beta m_t^*, m_t^*) = 0. \quad (64)$$

Proof of Lemma 6. By (54) and Proposition 1, $\widehat{V}(Z_t, m_t^*; \lambda) = \int_0^{\beta m_t^*} \mathcal{G}(Z_t, x) \eta\left(x, \frac{x}{\beta}\right) dx + \int_{\beta m_t^*}^\infty \mathcal{G}(Z_t, x) \eta(x, m_t^*) dx$. Differentiating this expression with respect to λ and using the definition of $\widehat{V}_\lambda(Z_t, m_t^*; \lambda) = B(Z_t, m_t^*; \lambda)$ gives

$$\begin{aligned} B(Z_t, m_t^*; \lambda) &= \int_0^{\beta m_t^*} \mathcal{G}(Z_t, x) \eta_\lambda\left(x, \frac{x}{\beta}\right) dx + \int_{\beta m_t^*}^\infty \mathcal{G}(Z_t, x) \eta_\lambda(x, m_t^*) dx \\ &\quad - \frac{d\beta}{d\lambda} \int_0^{\beta m_t^*} \mathcal{G}(Z_t, x) \eta_m\left(x, \frac{x}{\beta}\right) \frac{x}{\beta} dx. \end{aligned} \quad (65)$$

For $Z_t \geq \beta m_t^*$, the definition of $\mathcal{G}(Z_t, x)$, along with an application of Ito's Lemma to (65) leads to $\sigma^2 Z_t^2 B_{ZZ} + \mu Z_t B_Z - \rho B = -\eta_\lambda(Z_t, m_t^*)$. Using the definition of $\eta(x, m_t^*)$ and applying the envelope theorem to compute g_λ yields $\eta_\lambda(Z_t, m_t^*) = \alpha \tau(h(Z_t, m_t^*; \lambda)) + \chi \rho \log m_t^*$. This proves (63). To prove (64) note that when $Z_t = \beta m_t^*$,

$$B_m(\beta m_t^*, m_t^*) = \int_{\beta m_t^*}^\infty \mathcal{G}(\beta m_t^*, x) \eta_{\lambda m}(x, m_t^*) dx - \frac{d\beta}{d\lambda} \mathcal{G}(\beta m_t^*, \beta m_t^*) \eta_m(\beta m_t^*, m_t^*) m_t^*. \quad (66)$$

Substituting in for $\mathcal{G}(\beta m_t^*, x)$ gives

$$\int_{\beta m_t^*}^\infty \mathcal{G}(\beta m_t^*, x) \eta_{\lambda m}(x, m_t^*) dx = \frac{2(\beta m_t^*)^{\varphi_1}}{(\varphi_1 - \varphi_2) \sigma^2} \int_{\beta m_t^*}^\infty \frac{\eta_{\lambda m}(x, m_t^*)}{x^{\varphi_1 + 1}} dx$$

Since $\eta_m(x, m_t^*) = \frac{1}{m_t^*} y\left(\frac{x}{m_t^*}\right)$ it follows that $\eta_{\lambda m}(x, m_t^*) = \frac{1}{m_t^*} y_\lambda\left(\frac{x}{m_t^*}\right)$. Hence,

$$\begin{aligned}
\int_{\beta m_t^*}^{\infty} \mathcal{G}(\beta m_t^*, x) \eta_{\lambda m}(x, m_t^*) dx &= \frac{2(\beta m_t^*)^{\varphi_1}}{(\varphi_1 - \varphi_2) \sigma^2} \int_{\beta m_t^*}^{\infty} \frac{y_{\lambda}\left(\frac{x}{m_t^*}\right)}{x^{\varphi_1+1}} \frac{1}{m_t^*} dx \\
&= \frac{2}{(\varphi_1 - \varphi_2) \sigma^2} \frac{\beta^{\varphi_1}}{m_t^*} \int_{\beta}^{\infty} \frac{y_{\lambda}(u)}{u^{\varphi_1+1}} du.
\end{aligned} \tag{67}$$

An application of the implicit function theorem to (39) yields $\frac{d\beta}{d\lambda} = \left(\frac{\beta^{\varphi_1+1}}{y(\beta)}\right) \left(\int_{\beta}^{\infty} \frac{y_{\lambda}(u)}{(u)^{\varphi_1+1}} du\right)$. Substituting this expression for $\frac{d\beta}{d\lambda}$, along with (67) into (66), and recalling the definition of $\mathcal{G}(\beta m_t^*, x)$ and that $\eta_m(x, m_t^*) m_t^* = y\left(\frac{x}{m_t^*}\right)$ gives (64). ■

Applying Ito's Lemma to $B(Z_t, m_t^*; \lambda)$, and ignoring terms with expectation equal to zero¹⁸ leads to

$$\begin{aligned}
B(Z_t, m_t^*; \lambda) &= -E_t \int_t^{\infty} e^{-\rho(t-s)} [\sigma^2 Z_t^2 B_{ZZ} + \mu Z_t B_Z - \rho B] ds + E_t \int_t^{\infty} e^{-\rho(s-t)} V_m dm_s^* \\
&= E_t \int_t^{\infty} e^{-\rho(t-s)} [\alpha \tau(h(Z_s, m_s^*)) + \chi \rho \log m_s^*] ds.
\end{aligned} \tag{68}$$

Furthermore, integration by parts implies that

$$E_t \int_t^{\infty} e^{-\rho(t-s)} \chi \rho \log m_s^* ds - \chi \log m_t^* = \chi E_t \int_t^{\infty} e^{-\rho(t-s)} d \log m_s^*. \tag{69}$$

By Lemma 1 $m_0^* = \frac{L_0}{\chi}$. Using (62), and evaluating (68), (69) at $t = 0$ leads to $V_{\lambda}\left(Z_0, \frac{L_0}{\chi}; \lambda\right) = E_0 \int_0^{\infty} e^{-\rho s} \alpha \tau(h(Z_s, m_s^*)) ds + \chi E_0 \int_0^{\infty} e^{-\rho(t-s)} d \log m_s^*$. This implies that equation (25) is satisfied if and only if $V_{\lambda}\left(Z_0, \frac{L_0}{\chi}; \lambda\right) = 0$.

This concludes the proof of Proposition 2 ■

Remark 2 Equation (42) always has a solution as long as Z_0 is large enough. To see this, notice first that $V_{\lambda} < 0$ when $\lambda = 0$ since in that case $\tau(h(Z_t, m_t^*; \lambda = 0)) = 0$, while $-\chi E_t \int_t^{\infty} e^{-\rho(t-s)} d \log m_s^* > 0$. Also, fixing any $\lambda > 0$, inspection of (65) and (55) reveals that

$$\lim_{Z_0 \rightarrow \infty} \widehat{V}_{\lambda}\left(Z_0, \frac{L_0}{\chi}; \lambda\right) = \frac{1}{\rho} \left[\alpha \tau(\bar{h}; \lambda) + \chi \rho \log\left(\frac{L_0}{\chi}\right) \right]$$

By equation (62) it follows that $\lim_{Z_0 \rightarrow \infty} V_{\lambda}\left(Z_0, \frac{L_0}{\chi}; \lambda\right) = \frac{1}{\rho} \alpha \tau(\bar{h}; \lambda) > 0$. Since $B(Z_t, m_t^*; \lambda)$ is continuous in λ , equation (42) always has a positive root for large enough Z_0 .

¹⁸Panageas (2008) contains a more elaborate proof of the next expression. The reader is referred to that paper for technical details.

Proof of Proposition 3. Reading (47) with equality and using (12), (16) and $c_t = Y_t$ leads to

$$[\pi_1 (1 - \pi_2) + \pi_2] \frac{1 - \alpha}{\rho} = -(\zeta - \pi_1) E_0 \int_0^\infty e^{-\rho s} \frac{1}{c_s} dL_s + \pi_1 \frac{L_0}{c_0}. \quad (70)$$

Re-arranging (50) gives

$$\begin{aligned} \xi_t B_t &= (1 - \pi_1) E_t \int_t^\infty \xi_s dL_s - \pi_1 \xi_t L_t + E_t \int_t^\infty \xi_s \tau(h_s) w_s h_s ds \\ &\quad + [\pi_1 (1 - \pi_2) + \pi_2] E_t \int_t^\infty \xi_s p_s F_s ds \end{aligned} \quad (71)$$

Using (12), (15), $c_t = Y_t$ and substituting into (50) implies that the debt to gdp ratio $b_t = \frac{B_t}{Y_t} = \frac{B_t}{c_t}$ can be expressed as

$$\begin{aligned} b_t &= (1 - \pi_1) E_t \int_t^\infty e^{-\rho(s-t)} \frac{1}{c_s} dL_s - \pi_1 \left(\frac{L_t}{c_t} \right) + \alpha E_t \int_t^\infty e^{-\rho(s-t)} \tau(h_s) ds \\ &\quad + [\pi_1 (1 - \pi_2) + \pi_2] \frac{1 - \alpha}{\rho} \\ &= (1 - \pi_1) E_t \int_t^\infty e^{-\rho(s-t)} \frac{1}{c_s} dL_s - \pi_1 \left(\frac{L_t}{c_t} \right) + \alpha E_t \int_t^\infty e^{-\rho(s-t)} \tau(h_s) ds \\ &\quad - (\zeta - \pi_1) E_0 \int_0^\infty e^{-\rho s} \frac{1}{c_s} dL_s + \pi_1 \frac{L_0}{c_0}, \end{aligned} \quad (72)$$

where the second equation follows from (70). Re-arranging, one obtains

$$\begin{aligned} b_t &= (1 - \pi_1) E_t \int_t^\infty e^{-\rho(s-t)} \frac{1}{c_s} dL_s - \zeta E_0 \int_0^\infty \frac{1}{c_s} dL_s + \alpha E_t \int_t^\infty e^{-\rho(s-t)} \tau(h_s) ds \\ &\quad + \pi_1 \left[\frac{L_0}{c_0} - \frac{L_t}{c_t} \right] + \pi_1 E_0 \int_0^\infty e^{-\rho s} \frac{1}{c_s} dL_s \\ &\leq -\zeta E_0 \int_0^\infty \frac{1}{c_s} dL_s + \alpha \frac{1}{\rho} \tau(\bar{h}) + \pi_1 \left(\frac{L_0}{c_0} + E_0 \int_0^\infty e^{-\rho s} \frac{1}{c_s} dL_s \right), \end{aligned} \quad (73)$$

where the last inequality follows from i) $(1 - \pi_1) E_t \int_t^\infty e^{-\rho(s-t)} \frac{1}{c_s} dL_s < 0$, ii) $\tau(h_s) \leq \tau(\bar{h})$ and iii) $\frac{L_t}{c_t} > 0$. (Moreover, the upper bound (73) is approached arbitrarily closely as $Z_t \rightarrow \infty$). To conclude the proof, note that $\frac{L_0}{c_0} + E_0 \int_0^\infty e^{-\rho s} \frac{1}{c_s} dL_s > 0$. To see this, note that methods similar to Panageas (2008), can be used to explicitly compute the value of the

guarantee $E_0 \int_0^\infty e^{-\rho s} \frac{1}{c_s} dL_s$ as $\frac{\chi}{\varphi_2} \left(\frac{Z_0}{\beta m_0^*} \right)^{\varphi_2}$. Using this fact, one obtains

$$\begin{aligned} \frac{L_0}{c_0} + E_0 \int_0^\infty e^{-\rho s} \frac{1}{c_s} dL_s &= \frac{L_0}{m_0^*} \left(\frac{m_0^*}{c_0} \right) + \frac{\chi}{\varphi_2} \left(\frac{Z_0}{\beta m_0^*} \right)^{\varphi_2} \\ &= \chi \left[\frac{\beta m_0^* h_{\min}^\alpha}{Z_0 h_0^\alpha} + \frac{1}{\varphi_2} \left(\frac{Z_0}{\beta m_0^*} \right)^{\varphi_2} \right], \end{aligned}$$

where $h_{\min}$ denotes the lower bound on hours worked. By (36), $\frac{\beta m_0^* h_{\min}^\alpha}{Z_0 h_0^\alpha}$ is equal to 1, as long as Z_0 is below some cutoff $\overline{Z}_0$ and then becomes equal to $\frac{\beta m_0^* h_{\min}^\alpha}{Z_0 h_{\max}^\alpha}$ for value larger than $\overline{Z}_0$. Since $|\varphi_2| > 1$, it follows that $\frac{L_0}{c_0} + E_0 \int_0^\infty e^{-\rho s} \frac{1}{c_s} dL_s > 0$, irrespective of Z_0 .

Since $\frac{L_0}{c_0} + E_0 \int_0^\infty e^{-\rho s} \frac{1}{c_s} dL_s > 0$, equation (73) implies that $\sup_{0 \leq s \leq t} b_s$ is minimal when $\pi_1 = 0$. ■

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