

Monetary Theory with Non-degenerate Distributions*

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Abstract

At any given point of time in an actual economy, some individuals hold more money than other individuals do. This non-degenerate distribution of money holdings among individuals is a rationale for a range of policies designed for reallocating liquidity among individuals. However, monetary theory has often abstracted from this non-degenerate distribution for tractability reasons. In this paper, we construct a tractable search model of money with a non-degenerate distribution of money holdings. We model search as a directed process in the sense that buyers know the terms of trade before visiting particular sellers, as opposed to undirected search that has dominated the literature. In this model, the distribution of money holdings among individuals is non-degenerate. We show that this distribution affects individuals' decisions not directly, but rather indirectly only through a one-dimensional variable – the seller's future marginal value of money. This result drastically reduces the state space of individuals' decisions and makes the model tractable. We analytically characterize a monetary equilibrium, using lattice-theoretic techniques, and prove existence of a monetary steady state. In the equilibrium, buyers follow a stylized spending pattern over time, and the money distribution has a persistent wealth effect.

JEL classifications:

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1. Introduction

Money is unevenly distributed among individuals in an actual economy at any given point of time. This distribution of money holdings has important implications for efficiency and welfare in the economy. In addition to the financial sector, a range of monetary and banking policies are designed to reallocate liquidity. For example, open market operations by the Federal Reserve Bank and the participation of many central banks in overnight markets are intended to supply liquidity and channel liquidity from one set of individuals to another. Despite such importance of a non-degenerate distribution of money holdings, monetary theory has often abstracted from it, largely for tractability reasons. In this paper, we construct a tractable model with a microfoundation of money and with a non-degenerate distribution of money holdings. We prove existence of a monetary steady state and analyze its properties.

A model that captures the importance of the money distribution should have at least three features. First, the model should have a strong microfoundation for money; that is, it should explicitly specify the physical environment that generates the demand for money. Only with such a microfoundation can one coherently assess the welfare effect of monetary policy. Second, the money distribution should have a wealth effect in the sense that different levels of money holdings yield different marginal values of money. This wealth effect is necessary for a redistribution of money among individuals to affect aggregate welfare. Third, the model should generate the spending pattern that an individual does not rebalance money holdings every period, even if he has the ability to do so. Instead, an individual should take time to run down the money balance. With this spending pattern, the wealth effect of the money distribution is persistent.

The recent microfoundation of money (i.e., search theory of money) provides a coherent framework to explain the role of money. The theory models a market as being decentralized where each trade involves only a small group (usually two) of individuals and where the ability to keep record of individuals' trading histories is limited. In this environment, objects with no intrinsic values, such as money, can improve resource allocation. In addition to justifying a role for fiat money, search theory can in principle be a natural framework for examining the importance of the money distribution. The decentralized exchange process naturally leads to a non-degenerate money distribution, because individuals end up with different amounts of money as they experience different realizations of matches. Moreover, the dispersion in money holdings can persist because an individual with a high money balance would want to spend the balance down in a sequence of trades rather than all at once.

However, this microfoundation of money has often imposed assumptions to make the money distribution degenerate. The reason for doing so has been to keep the analysis tractable: since the

distribution is an aggregate state variable with a large dimensionality, characterizing a monetary equilibrium is difficult when the distribution directly affects individuals' decisions. To avoid this difficulty, Shi (1997) assumes that each household consists of a large number of members who share consumption and utility, and Lagos and Wright (2005) assume that individuals have quasi-linear preferences over a good which they can trade in a centralized market to immediately rebalance their money holdings. With either assumption, there is no dispersion in money holdings among the buyers who enter the decentralized market.¹

We use search theory of money as the microfoundation of money, but we do not impose the assumptions to make the money distribution degenerate. The main deviation of our model from the literature is that we model search as a directed process, as opposed to undirected search. That is, buyers in our model know the terms of trade before visiting particular sellers. In the model, there is a continuum of submarkets, each of which specifies the participants' matching probabilities and the quantities of money and goods to be traded in a match. Observing these terms across the submarkets, buyers choose which submarket to enter and sellers choose the number of trading posts to be maintained in each submarket. There is a cost of maintaining a trading post, and free entry of trading posts ensures that expected net profit is zero for every trading post. A matching function with constant returns to scale determines the total number of matches in each submarket between buyers and trading posts. In equilibrium, individuals' decisions on which submarket to enter will ensure that the ratio of buyers to trading posts in each submarket is consistent with the matching probabilities specified for that submarket.

In our model, an individual chooses in every period whether to be a buyer or a seller. Sellers are individuals who have spent their money down to sufficiently low levels and choose to produce. We assume that all sellers with the same preferences contribute to a firm's production and sales, and receives monetary payments from the firm that are proportional to his contribution. The firm chooses how many trading posts to maintain in each submarket. Because there are a large number of sellers in each firm, all sellers have the same marginal valuation of money and exit production with the same amount of money.

In contrast, buyers are heterogeneous in money holdings. This non-degenerate money distribution does not render our analysis intractable as it does in other models. Directed search implies that buyers with different money balances optimally choose to separate themselves by visiting different submarkets. With such endogenous separation of buyers, we show that the

¹Green and Zhou (1998) construct a search model of money in which the money distribution is non-degenerate. However, they assume that money and goods are indivisible, as in Shi (1995) and Trejos and Wright (1995), and they only allow individuals to accumulate money in multiples of the indivisible unit. They characterize a particular steady state in which all buyers spend the same amount of money in a trade regardless of their money holdings. This result is sensitive to the indivisibility assumptions.

money distribution affects individuals' decisions not directly, but rather indirectly only through a one-dimensional variable – the seller's marginal value of money. This result drastically reduces the state space of individuals' decisions and makes the model tractable.

An individual in our model manages his money balance as in a standard inventory model (e.g., Baumol, 1952, and Tobin, 1956).² Starting with the highest equilibrium money balance, a buyer chooses to consume frequently and consume a large quantity. He does so by visiting a submarket in which the matching probability is relatively high and the quantities of money and goods traded in each match are relatively high. As each purchase reduces the money balance, he will choose next to visit a submarket which has a lower matching probability and lower quantities of money and goods in each trade. The individual continues to follow this path until he exhausts his money. Then, he will produce and sell goods to obtain the highest level of money balance, after which another round of purchases will start anew.

The money distribution has two effects on resource allocation in our model. First, the distribution implies that there is persistent dispersion in consumption. An individual with a relatively high money balance chooses not to spend all of his money at once because it is costly to obtain money. Such an individual has a lower marginal value of money and consumes more (in probabilistic terms) than does an individual with less money. Second, by affecting the distribution of individuals across the submarkets, the money distribution affects the number of trades and, hence, affects aggregate output and consumption, even though individuals' decisions do not directly depend on the distribution.

We analytically characterize a monetary equilibrium, prove existence of a monetary steady state, and analyze the money distribution. The difficulties in analyzing a buyer's decision problem are that a buyer's objective function is not necessarily concave and that a buyer's value function is not necessarily differentiable. As a result, we cannot assume, a priori, that the first-order conditions and envelope conditions hold. To resolve these difficulties, we use lattice-theoretic techniques (see Topkis, 1998) to prove that a buyer's optimal decisions are monotonic functions of the buyer's money balance. Using this result, we prove further that *on the equilibrium path*, a buyer's value function is differentiable and a buyer's optimal choices do obey first-order conditions. The validity of these standard conditions makes the model easy to use.

Our paper is related to the growing literature on directed search originated from Peters (1984), most of which studies non-monetary economies.³ Corbae, Temzelides and Wright (2003) have

²In contrast to our model, standard inventory models do not have a strong microfoundation of money and are difficult to be aggregated into an equilibrium model.

³Other standard references on directed search are Peters (1991), Moen (1997), Acemoglu and Shimer (1999), and Burdett, Shi and Wright (2001).

employed directed search in monetary theory, but their focus was on the formulation of trading coalitions and they assume that money and goods are indivisible. Rocheteau and Wright (2005) have checked the robustness of a monetary model with respect to directed search, and Galenianos and Kircher (2008) have integrated directed search into a monetary model with auctions. Because these two papers build on the model by Lagos and Wright (2005), the money distribution either is degenerate or does not have important and persistent wealth effects.⁴

Finally, a few recent papers have explored the tractability of directed search for studying the labor market with a non-degenerate distribution (e.g., Shi, 2008). Our paper is obviously different in the issue. Moreover, a monetary economy differs from a non-monetary labor market in two important aspects. First, in a monetary economy, a seller's receipt from a trade is money whose value must be determined endogenously in an equilibrium. Thus, an individual's gain from a trade depends not only on how the match surplus is split between the two individuals in the match, but also on how all individuals in the economy value the objects traded in the match. No such counterpart exists in a labor search model, where the valuation of output produced in a match is exogenous. Second, money holdings are an individual's state variable, which the individual can accumulate by trading and carry from one match to the next. Such a state variable has not been introduced in directed search models of the labor market. Because of these differences, it is more challenging to study a monetary economy than a labor market.⁵

2. The Model Environment

Consider an economy with discrete time. There are a large number of individuals who are evenly distributed over a finite number of types. The number of types is at least three. The measure of individuals within each type is normalized to one. Goods are perishable and perfectly divisible. The individuals are specialized in production and consumption in such a way that there is no double coincidence of wants between two individuals. No record of individuals' actions can be kept between periods, and a medium of exchange is needed in every trade. This role is served by a fiat object called money. Money is perfectly divisible and can be stored without cost. The stock of money per capita is a constant $M > 0$. The utility function of consumption is $U(\cdot)$ and

⁴Galenianos and Kircher (2008) generate a non-degenerate distribution of prices and money holdings by assuming that sellers use second-price auctions to sell goods. If sellers could commit to posted prices, as they do in our model, the distribution of prices or money holdings would be degenerate in their model.

⁵Zhu (2005) has taken an important step to construct a search model with divisible money and a non-degenerate money distribution. However, his model is different. He first characterizes a monetary equilibrium with indivisible money and then, by pushing the size of indivisibility to zero, he shows that the limit of this equilibrium is an equilibrium in an economy with divisible money. For the proof to work, Zhu imposes strong assumptions on preferences and an exogenous upper bound on money holdings. Moreover, conducting comparative statics in his model is complicated, because one needs to prove that the comparative statics in a sequence of economies with indivisible money converge to the comparative statics in an economy with divisible money.

the disutility function of production is $\psi(\cdot)$, with the usual properties: $U' > 0$, $U'' < 0$, $U(0) = 0$, $U'(0) = \infty$, $U'(\infty) = 0$, $\psi' > 0$, $\psi'' \geq 0$ and $\psi(0) = 0$.

In every period, an individual can choose to be either a buyer or a seller, where a seller is also a producer.⁶ Production and sales are carried out through firms, each of which employs a large number of producers who have the same preferences. A firm chooses how much to produce and where to sell the goods, in order to maximize expected profit. The cost of maintaining a trading post for one period is $k > 0$ (as disutility). A seller chooses how much to contribute to the firm's cost of production and sales. At the end of each period, the firm pays each seller an amount of money that is proportional to the seller's contribution. Firms enter the economy competitively so that all firms make zero profit in the equilibrium.

No firm has the technology to employ producers with different preferences. This assumption is necessary for preventing barter from arising within a firm. If a firm could employ producers with different preferences, the firm would arrange multi-lateral barter trades among the producers. This role is not what a production firm performs in the market; rather, it is the role of middlemen whom we abstract from. Thus, in our model, a producer in a firm does not want to consume the good produced by the firm, despite that a firm can be large.

The goods market is decentralized and characterized by directed search. The market consists of a continuum of submarkets and each submarket can contain many trading posts whose number is determined by free entry. A submarket specifies particular terms of trade between a buyer and a seller, together with the matching probabilities. Search is directed in the sense that when buyers choose which submarket to enter and firms choose how many trading posts to maintain in each submarket, they know the terms of trade and matching probabilities across the submarkets. In spite of directed search, matching in each submarket is still random, because there is no coordination among the buyers and trading posts. The number of matches in each submarket is determined by a matching technology with constant returns to scale. In equilibrium, this matching technology and the entry decisions into the submarket together must imply matching probabilities that coincide with those specified for the submarket. Once matched in a submarket, individuals trade according to the terms of trade specified for that submarket. That is, the commitment to the terms of trade precludes bargaining.

Let (x, q) denote the terms of trade in a submarket, where x is the amount of money that a seller receives from a buyer in a trade and q the amount of goods that a buyer receives from a seller. Let b denote the matching probability for a buyer in the submarket and s the matching

⁶Note that each individual is a decision maker in this model, in contrast to the large household in the model in Shi (1997). Also, there is no general good as the one in Lagos and Wright (2005) that an individual can produce to rebalance money holdings.

probability for a trading post. To describe the matching probabilities, let N_b be the number of buyers in a submarket, N_s the number of trading posts, and N_s/N_b the tightness of the submarket. Let the number of matches in the submarket be given by a function, $\mathcal{M}(N_b, N_s)$. The matching probability for a buyer in the submarket is $b = \mathcal{M}(N_b, N_s)/N_b$, and the matching probability for a trading post is $s = \mathcal{M}(N_b, N_s)/N_s$. Assuming that the function, $\mathcal{M}(\cdot, \cdot)$, has constant returns to scale, we can write the matching probabilities as $b = \mathcal{M}(1, N_s)/N_b$ and $s = \mathcal{M}(N_b/N_s, 1)$. As buyers and firms choose which submarket to enter, the tightness in each submarket is a function of the terms of trade in that submarket. Thus, both s and b are functions of (x, q) , as we will see later in (3.2). For this reason, we refer to a submarket by (x, q) alone, although a submarket is described by (b, s) and (x, q) together.

It is convenient to express s as a function of b . To do so, solve the tightness, N_s/N_b , from $b = \mathcal{M}(1, N_s/N_b)$ and substitute into $s = \mathcal{M}(N_b/N_s, 1)$. Denote the result as

$$s = \mu(b). \quad (2.1)$$

Because all properties of the function $\mu(\cdot)$ come from those of $\mathcal{M}$, we will treat $\mu(\cdot)$ as a primitive of the model and refer to it as the *matching function*. We impose the following assumption:

Assumption 1. For all $b \in [0, 1]$, the matching function $\mu(b)$ satisfies: (i) $\mu(b) \in [0, 1]$, with $\mu(0) = 1$, (ii) $\mu'(b) < 0$, and (iii) $[1/\mu(b)]$ is strictly convex, i.e., $2(\mu')^2 - \mu\mu'' > 0$.

Part (i) is a regularity condition. Part (ii) requires that if the matching probability is high for a buyer in a submarket, it must be the case that there are a relatively large number of trading posts per buyer in the submarket; in this case, the matching probability for each trading post in the submarket must be relatively low. We will show later that part (ii) implies that for any given amount of a buyer's spending, x , the quantity of goods obtained at a trading post must decrease with b in order to induce entry of trading posts into the submarket. Thus, for any given x , the seller's cost of production is lower in submarkets with higher b . Part (iii) of the above assumption requires that this production cost be strictly concave in b for any given x . That is, producers must be compensated with increasingly larger reductions in the cost of production in order to create additional trading posts to increase buyers' matching probability. This requirement is necessary for the tradeoff between the matching probability and the terms of trade across submarkets to yield a unique optimal choice of a submarket for a buyer.⁷

The following example gives a matching function that satisfies Assumption 1:

⁷We do not impose the assumption $\mu'' \leq 0$, because it is neither necessary for the analysis nor reasonable for usual examples of the matching function.

Example 2.1. Consider the function $\mathcal{M} = N_s N_b (N_s^\rho + N_b^\rho)^{-1/\rho}$. (The special case with $\rho = 1$ is the so-called telegraph matching function.) With this function, $s = \mu(b) = (1 - b^\rho)^{1/\rho}$. For all $\rho \in (0, \infty)$, $\mu(b)$ satisfies Assumption 1.

3. Optimal Decisions and Value Functions

In every period, an individual makes two types of decisions. The individual first chooses whether to be a seller or a buyer, and then makes the trading decisions. Also, a firm chooses the number of trading posts to maintain in each submarket. In this section, we will analyze these decisions recursively. That is, we will first analyze a firm's decision, then an individual's trading decisions, and finally the choice between being a seller and a buyer.

Normalize all nominal variables by the stock of money per capita, M . Let m denote an individual's money holdings at the beginning of a period (divided by M). Let $V(m)$ be the individual's *ex ante value function*, i.e., the value function before the individual chooses to be a seller or a buyer. In the analysis of the trading decisions, we will assume that the function $V(\cdot)$ is continuous, bounded, increasing and concave. After completing this analysis, we will verify that V indeed has these properties (see Theorem 3.4). However, we do not assume that V is differentiable, because differentiability holds only in a limited form to be explained later.

3.1. A firm's decision

A firm chooses how many trading posts to operate in each submarket. Denote this choice as $dN(x, q)$ for submarket (x, q) , and $s(x, q)$ as the matching probability for a trading post in submarket (x, q) . Let Ψ denote the firm's total cost of production and K the firm's total cost of maintaining the trading posts, both being measured in terms of utility. Let D be the total amount of money that the firm receives from sales in the period. We have:⁸

$$K = k \int dN(x, q), \quad \Psi = \int \psi(q) s(x, q) dN(x, q),$$

$$D = \int x s(x, q) dN(x, q).$$

We will show in the next subsection that when exiting production, all sellers have the same marginal value of money, $W'_s(m^*)$, where $W_s(m)$ is a seller value function and m^* will be defined

⁸The formula of Ψ implicitly assumes that the cost of production must be incurred on the spot of each sale. This assumption is not necessary for our analysis, although it has been used in most search models of money. If production is centralized in a firm, instead, the marginal cost of goods in each sale is independent of the quantity of the particular sale because this quantity is negligible relative to the firm's total output. Our formulation can encompass this case as a special case by setting $\psi'' = 0$.

in (3.4). Thus, the firm's expected profit (measured in utils) is:

$$W'_s(m^*)D - \Psi - K = \int \{s(x, q) [W'_s(m^*)x - \psi(q)] - k\} dN(x, q).$$

Define the firm's cost (in utils) per revenue (in dollars) as $\omega \equiv (\Psi + K)/D$. We can rewrite the above expected profit as $[W'_s(m^*) - \omega]D$. Competitive entry of firms ensures that the firm's expected profit be zero; i.e., $\omega = W'_s(m^*)$.

The firm chooses the number of trading posts in each submarket, $dN(x, q)$, to maximize the above profit. The expected profit of operating a trading post in submarket (x, q) is the expression, $s(x, q) [\omega x - \psi(q)] - k$, where we have substituted $W'_s(m^*) = \omega$. If this profit is strictly positive, the firm will choose $dN(x, q) = \infty$, but this case will never occur in an equilibrium under competitive entry of trading posts.⁹ If this profit is strictly negative, the firm will choose $dN(x, q) = 0$. If this profit is zero, the firm is indifferent about different non-negative and finite levels of $dN(x, q)$. Thus, the optimal choice of $dN(x, q)$ satisfies the following condition:

$$k \geq s(x, q) [\omega x - \psi(q)] \quad \text{and} \quad dN(x, q) \geq 0, \quad (3.1)$$

where the two inequalities hold with complementary slackness.¹⁰

For all pairs (x, q) such that $k < \omega x - \psi(q)$, the submarket has $dN(x, q) > 0$, and (3.1) holds as equality. For all pairs (x, q) such that $k \geq \omega x - \psi(q)$, the submarket has $dN(x, q) = 0$, and so $s(x, q) = 1$ and $b(x, q) = 0$. Putting the two cases together, we obtain the following matching probability for a trading post in submarket (x, q) :

$$s(x, q) = \begin{cases} \frac{k}{\omega x - \psi(q)}, & \text{if } k \leq \omega x - \psi(q) \\ 1, & \text{otherwise.} \end{cases} \quad (3.2)$$

A buyer's matching probability in submarket (x, q) is $b(x, q) = \mu^{-1}(s(x, q))$. Equation (3.2) states intuitively that the matching probability for a selling post increases in q , the quantity of goods traded in a match, and decreases in x , the quantity of money traded. Note that for any given ω , $s(x, q)$ and $b(x, q)$ are independent of the distributions of buyers and trading posts.

3.2. A seller's decision

A seller chooses the contribution to the firm's costs of production and sales. We denote this choice as $z \in [0, 1]$ so that the seller's contribution to the firm's cost is $(\Psi + K)z$. With this

⁹More precisely, if $dN(x, q) = \infty$, the matching probability for a trading post in submarket (x, q) will be zero. In this case, the expected profit of operating a trading post in submarket (x, q) is $-k < 0$, which contradicts the optimality of the choice $dN(x, q) = \infty$.

¹⁰We impose (3.2) for all possible submarkets (x, q) , not just for the submarkets that are active in the equilibrium. It will become clear that the equilibrium has only a finite number of active submarkets. Thus, this condition imposes restrictions on beliefs out of the equilibrium. As a restriction needed to complete the markets, it is common in directed search models, e.g., Moen (1997), Acemoglu and Shimer (1999), and Shi (2008).

contribution, the seller will receive zD units of money from the firm.¹¹ The seller's value function, $W_s(m)$, obeys the following Bellman equation:

$$W_s(m) = \max_{z \in [0,1]} [\beta V(m + zD) - \omega zD], \quad (3.3)$$

where we have used the notation $\omega = (\Psi + K)/D$. Define m^* as

$$m^* = \arg \max_{y \geq 0} [\beta V(y) - \omega y]. \quad (3.4)$$

Clearly, $W_s(0) = \beta V(m^*) - \omega m^*$.

The level m^* is the money balance that a seller with $m < m^*$ wants to achieve at the end of the period through production and sales. It is so because for all $m \in [0, m^*]$, the choice $z = (m^* - m)/D$ is feasible and solves the maximization problem in (3.3). In this case, the constraint $z \geq 0$ is not binding and the seller's value function is $W_s(m) = \omega m + W_s(0)$. For all $m > m^*$, the constraint $z \geq 0$ is binding. Summarizing the two cases, we have:

$$W_s(m) = \begin{cases} \omega m + W_s(0), & \text{if } m \leq m^* \\ \beta V(m), & \text{if } m > m^*. \end{cases} \quad (3.5)$$

For a seller's optimal choice, it is intuitive to expect that the higher a seller's money holdings, the less he wants to contribute to the firm. Moreover, a seller's end-of-period money holdings are increasing in his holdings at the beginning of the period. These properties hold true even if the optimal solution for z is not unique. We summarize these properties of the optimal choice and a seller's value function in the following lemma (see Appendix A for a proof):

Lemma 3.1. *Assume that $V(m)$ is continuous, bounded, increasing and concave. Then, $W_s(m)$ is continuous, bounded, increasing, and concave. The solution for z in (3.3) is decreasing in m , and strictly so if $m < m^*$. The end-of-period money balance, $y = m + zD$, is increasing in m . Moreover, $W'_s(m^*) = \omega$.*

An individual will never hold more than m^* in an equilibrium. Even if the individual starts with a money balance higher than m^* , he will spend it down to m^* or lower levels. Once that happens, his money balance will stay in the interval $[0, m^*]$. Moreover, because a seller with $m < m^*$ always restocks his money balance to the level m^* after one period of production, all sellers have the same marginal value of money when exiting production, regardless of his current money balance. This common valuation is $W'_s(m^*) = \omega$. Because of this equality, we will refer to ω as a *seller's marginal value of money*, although ω is defined as a firm's cost/revenue ratio. Note that this equality ensures that a firm's expected profit is indeed zero.

¹¹It is straightforward to reinterpret a firm here as a standard one in which the firm pays the competitive wage to the sellers and rebates all dividends to the sellers. Our formulation simplifies the notation.

3.3. A buyer's decision

A buyer chooses which submarket to enter. This is a choice of the pair (x, q) . It is more convenient to use the pair (x, b) , instead of (x, q) , to analyze a buyer's decision. To do so, we express q as a function of (x, b) by substituting $s = \mu(b)$ into (3.2). For any submarket with $\omega x > k/\mu(b)$, the substitution yields:

$$q = Q(x, b) = \psi^{-1} \left(\omega x - \frac{k}{\mu(b)} \right). \quad (3.6)$$

For any submarket with $\omega x \leq k/\mu(b)$, we can set $Q(x, b) = 0$. The function $Q(x, b)$ gives the quantity of goods that a seller is willing to sell for the balance x with a matching probability $\mu(b)$. Denote $u(x, b) = U(Q(x, b))$.

In submarket (x, b) , if the buyer gets a match, he obtains $Q(x, b)$ units of goods which yield utility $u(x, b)$, and his money balance falls to $(m - x)$. In such a trade, the buyer faces the money constraint $x \leq m$. If the buyer does not get a match, he carries the balance, m , to the next period, which yields a discounted value $\beta V(m)$. The buyer's value function, $W_b(m)$, obeys the following Bellman equation:

$$W_b(m) = \max_{b \in [0, 1], x \in [0, m]} \{ \beta V(m) + b [u(x, b) + \beta V(m - x) - \beta V(m)] \}. \quad (3.7)$$

To analyze the buyer's optimal choices, let us use Assumption 1 on the matching function, $\mu(b)$, and the assumptions on the cost function, $\psi(q)$, to derive the following properties:

$$Q_1(x, b) > 0, Q_2(x, b) < 0, Q(x, b) \text{ is concave, and } Q_{12} \geq 0. \quad (3.8)$$

It is easy to explain the monotonicity of Q . When a buyer is willing to pay more money, a seller is willing to sell a larger quantity of goods. On the other hand, if a buyer wants to trade with a relatively high probability for any given amount of money, the buyer must accept a relatively low quantity of goods. This is because the cost of production must be relatively low in order to induce firms to set up a large number of trading posts needed to deliver the high matching probability for a buyer.

Concavity of Q in x means intuitively that the marginal benefit to a buyer from a higher money balance is diminishing. Similarly, concavity of Q in b means that the marginal cost to a buyer of having a high matching probability is increasing. Note that part (iii) of Assumption 1 is used to ensure that Q is concave in b . Moreover, Q is concave in (x, b) jointly because $\psi(q) = \omega x - [\mu(b)]^{-1} k$, which is separable in x and b , and because ψ^{-1} is a concave function. These features of ψ also explain why $Q_{12} \geq 0$, with strict inequality if ψ is strictly convex.¹²

¹²Throughout this paper, "increasing" means "non-decreasing", "concave" means "weakly concave", etc. The modifier "strictly" is added when a property is strict.

The property $Q_{12} \geq 0$ implies that Q is supermodular in (x, b) , and strictly so if ψ is strictly convex. Since this supermodularity will play an important role in our analysis, let us explain it further. Consider two pairs, (x_1, b_1) and (x_2, b_2) , with $x_2 > x_1$ and $b_2 > b_1$. In the current context, supermodularity of $Q(x, b)$ requires that $Q(x_2, b_2) - Q(x_1, b_2) \geq Q(x_2, b_1) - Q(x_1, b_1)$. That is, the additional benefit to a buyer from having a higher money balance increases with b . Our model meets this requirement for the following reason. In submarkets where a buyer has a relatively high matching probability, the amount of goods that a seller is willing to sell for any given amount of money balance is relatively low. At such a low quantity of goods, a seller's marginal cost of production is low, and so a seller is willing to increase the quantity of goods by more for any given increase in the amount of money spent by a buyer.

With the properties of the function Q and the utility function U , the composite function $u(x, b) = U(Q(x, b))$ has the following properties:

$$u_1(x, b) > 0, u_2(x, b) < 0, u(x, b) \text{ is strictly concave, and } u_{12} > 0. \quad (3.9)$$

Note that $u(x, b)$ is strictly concave in (x, b) jointly and strictly supermodular. These strict properties arise from strict concavity of the utility function, and they hold regardless of whether ψ is strictly or weakly convex.

Turn to the optimal choices. Let us denote a buyer's optimal choice of x as $x^*(m)$, and of b as $b^*(m)$. Denote the quantity of goods purchased by the buyer as $q^*(m)$ and the buyer's residual money balance after the trade as $\phi(m)$. Then,

$$q^*(m) \equiv Q(x^*(m), b^*(m)), \quad \phi(m) \equiv m - x^*(m).$$

Relative to a buyer who holds a lower quantity of money, a buyer with more money will choose to enter a submarket where he will have a higher matching probability; moreover, once he is matched in the submarket, he will spend a larger amount of money, buy a larger quantity of goods, and leave the trade with a higher money balance. As a result, a buyer with more money will obtain a higher present value. The following lemma states these intuitive results of the policy functions and the properties of W_b (see Appendix B for a proof):

Lemma 3.2. *Assume that $V(\cdot)$ is continuous, bounded, increasing and concave. Then, $W_b(\cdot)$ is continuous, bounded, and increasing. If $b^* = 0$, the choice of x is irrelevant for the buyer. If $b^* > 0$, the following results hold: (i) $x^*(m)$ and $b^*(\cdot)$ exist and are unique for each m ; (ii) $x^*(m)$, $b^*(m)$, $q^*(m)$ and $\phi(m)$ are increasing and continuous functions; (iii) $W_b(m)$ is strictly increasing if $\omega > 0$.*

Uniqueness of the optimal choices comes from quasi-concavity of the objective function in (x, b) . The main challenge in establishing the above lemma is that the value function V is not necessarily differentiable in general. In fact, the objective function in (3.7) may not even be concave jointly in (x, b, m) , despite that V is concave. One reason is that the objective function involves the product of two functions. Another reason is that the objective function involves the difference between $V(m - x)$ and $V(m)$, both of which are concave in m . The lack of joint concavity of the objective function implies that the value function, $W_b(m)$, is not necessarily concave. This feature of $W_b(m)$ renders inapplicable most of the well-known methods for establishing differentiability of the value function (e.g., Benveniste and Scheinkman, 1979). Without differentiability of V , we cannot use standard approaches such as the first-order conditions to establish monotonicity of the optimal choices stated in Lemma 3.2.

To resolve these difficulties, we employ the lattice-theoretic techniques (see Topkis, 1998) to prove that the optimal choices are monotone in the individual's money holdings. Such monotonicity enables us to establish a limited form of differentiability of the value function. Finally, we show that the optimal choices satisfy the first-order conditions and envelope conditions.

To analyze the optimal choices, we decompose the maximization problem into two steps: in the first step, we fix b and characterize the optimal choice of x ; in the second step, we characterize the optimal choice of b .¹³ For the first step, we denote the buyer's surplus from trade as

$$t(x, b, m) = u(x, b) + \beta V(m - x) - \beta V(m). \quad (3.10)$$

For any given (b, m) , the optimal choice of x maximizes t . Denote the optimal choice and the maximized function as

$$\tilde{x}(b, m) = \arg \max_{x \in [0, m]} t(x, b, m), \quad t^*(b, m) = t(\tilde{x}(b, m), b, m). \quad (3.11)$$

With the properties of u in (3.9), we prove that $t(x, b, m)$ is supermodular in (x, b, m) . Topkis's (1998) theorems then imply that the optimal choice, $\tilde{x}(b, m)$, and the maximized function, $t^*(b, m)$, are both increasing functions of (b, m) (see Appendix B). In the second step, the optimal choice of b maximizes the function, $b t^*(b, m)$. We prove that this function is supermodular in (b, m) . Thus, the optimal choice, $b^*(m)$, and the maximized function, $b^*(m) t^*(b^*(m), m)$, are increasing functions of m . By changing the choice variables from (x, b) to (x, q) and to $(m - x, b)$, in turn, we use the same procedure and techniques above to prove that the optimal quantity of goods purchased, $q^*(m)$, and the residual money balance, $\phi(m)$, are increasing functions of the money balance that the buyer brings into the trade. Such monotonicity of the policy functions

¹³We do so because it is difficult to directly verify that the objective function in (3.7) is supermodular in (x, b) .

is reported in Lemma 3.2 above. Moreover, the value function, $W_b(m)$, is a strictly increasing function when $b^*(m) > 0$ and $\omega > 0$.

For any given (x, m) , the function $t(x, b, m)$ defined in (3.10) is differentiable with respect to b . Since the optimal choices are unique for each m , then for any given (x, m) , the optimal choice of b satisfies the following first-order condition:

$$u(x, b) + b u_2(x, b) \leq \beta [V(m) - V(m - x)] \quad \text{and} \quad b \geq 0, \quad (3.12)$$

where the two inequalities hold with complementary slackness. Note that this condition does not require differentiability of V .

Let us characterize the optimal decisions and value functions in further detail. In particular, using monotonicity of the optimal choices, we establish the properties that validate the first-order conditions for (x^*, b^*) and the envelope conditions. Start with the value functions, $W_b(m)$ and $V(m)$. The following lemma (see Appendix C for a proof) states a generalized version of the envelope condition and a limited version of differentiability of the value functions.

Lemma 3.3. *Assume that $V(m)$ is continuous, bounded, increasing and concave. In addition, assume that $b^*(m) > 0$. Then, the following results hold. (i) $W_b(m)$ is differentiable at m if and only if $V(m)$ is so. (ii) For any $m > 0$ such that $W_b(m) = V(m)$, the derivatives $W'_b(m)$ and $V'(m)$ exist and are given by the envelope condition:*

$$W'_b(m) = V'(m) = \frac{b^*(m)}{1 - \beta [1 - b^*(m)]} u_1(x^*(m), b^*(m)). \quad (3.13)$$

(iii) For any m such that $\phi(m) > 0$, the derivative $V'(\phi(m))$ exists and is given as¹⁴

$$V'(\phi(m)) = \frac{1}{\beta} u_1(x^*(m), b^*(m)). \quad (3.14)$$

(iv) For any m such that $W_b(m) = V(m)$ and $\phi(m) > 0$, the functions $b^*(m)$ and $\phi(m)$ are strictly increasing at m ; the function $V(m)$ is strictly concave at m ; and $V(\phi(m)) = W_b(\phi(m))$.

(v) $V'(m^*)$ exists and

$$V'(m^*) = \omega/\beta. \quad (3.15)$$

Part (i) of the above lemma is not surprising, given that the optimal choices are unique and monotone. Part (ii) states that if the ex ante value function $V(m)$ is equal to the buyer's

¹⁴Denote $\phi^-(m) = m^- - x^*(m)$. If $\phi(m) = 0$, then (3.14) is replaced with

$$V'(\phi^-(m)) \leq \frac{1}{\beta} u_1(x^*(m), b^*(m)).$$

value function $W_b(m)$ at m , then both value functions are differentiable at m , and the common derivative is given by the envelope condition. To obtain this result, we use the fact that a concave function has both left-hand and right-hand derivatives (see Royden, 1988, pp113-114). When $W_b(m) = V(m)$ at m , we are able to use the Bellman equation, (3.7), to prove that the left-hand derivative of $W_b(m)$ is equal to the left-hand derivative of $V(m)$, that the right-hand derivative of $W_b(m)$ is equal to the right-hand derivative of $V(m)$, and that the left-hand and right-hand derivatives are equal to the right-hand side of (3.13).¹⁵

Part (iii) of the above lemma states that for any given m such that the money constraint in a trade does not bind, the optimal choice of the buyer's expenditure, $x^*(m)$, is characterized by the first-order condition. In this case, even if $V(m)$ is not differentiable at m , V is differentiable at $\phi(m)$, i.e., at the buyer's residual money balance induced by the optimal choice. Put in another way, the buyer will choose the spending optimally so that the residual money balance will not be at a level where V is not differentiable. This result can be explained as follows. Suppose that a spending level, x , results in $V(m - x)$ being non-differentiable, i.e., $V'(m - x^+) > V'(m - x^-)$. By reducing x marginally, the opportunity cost of spending decreases by a discrete amount. Since the marginal benefit, given by $u_1(x, b)$, falls continuously with the decrease in x , the net marginal gain must be positive. Thus, spending marginally less than x is strictly better than spending x .

Part (iv) provides mild conditions under which the policy functions $b^*(m)$ and $\phi(m)$ are strictly increasing at m . Under the same conditions, the ex ante value function V is strictly concave at the residual money balance, $\phi(m)$, and is equal to a buyer's value function at such a balance. Also, part (ii) of the lemma holds at this residual balance.

Finally, part (v) shows that V is differentiable at m^* , the money balance obtained by a seller from production and sales. The condition (3.15) reinforces the interpretation of ω as the seller's marginal value of money when exiting production.

Together, the results in Lemma 3.3 significantly simplify the characterization of the optimal choices. Under some regularity conditions that we will specify in section 4.3, the above lemma implies that a buyer's optimal choices in the equilibrium satisfy the first-order conditions, (3.12) and (3.14), and that the value functions satisfy envelope condition, (3.13).

¹⁵It is well known that a concave function is differentiable almost everywhere, but this result does not help our analysis. Because the support of the money distribution has only a finite number of points, as we will see later, the well-known result does not exclude the possibility that the ex ante value function might be non-differentiable at those points.

3.4. The ex ante value function and the choice between being a buyer and a seller

At the beginning of a period, an individual chooses whether to be a buyer or a seller. This decision solves:

$$\tilde{V}(m) = \max\{W_s(m), W_b(m)\}. \quad (3.16)$$

We cannot take $\tilde{V}$ as the ex ante value function V , because $\tilde{V}$ fails to be concave for some m . To see this, note that a buyer's matching probability is zero if $m < k/\omega$. Thus, at low money balances, an individual's value as a buyer cannot exceed that as a seller. On the other hand, at high money balances, an individual's value as a buyer exceeds that as a seller. Hence, $\tilde{V}$ is strictly convex for low money balances. Moreover, because $W_b(m)$ may not be concave at some m , as we explained before, $\tilde{V}(m)$ may not be concave at such levels of m even if $\tilde{V}(m) = W_b(m)$.

To make the ex ante value function concave, we allow individuals to participate in lotteries before choosing whether to be a buyer or a seller. Consider a two-point lottery, (L_1, L_2, π) , where L_1 and L_2 are the realizations of the lottery and π is the probability that L_2 occurs. For an individual with money m , the optimal lottery and the value function solve:

$$V(m) = \max_{(L_1, L_2, \pi)} \left[(1 - \pi) \tilde{V}(L_1) + \pi \tilde{V}(L_2) \right] \quad (3.17)$$

subject to:

$$\begin{aligned} (1 - \pi) L_1 + \pi L_2 &= m, \\ \pi &\in [0, 1] \text{ and } L_i \geq 0 \text{ for } i = 1, 2. \end{aligned}$$

Denote the solutions for L_1 , L_2 and π as $L_1(m)$, $L_2(m)$ and $\pi(m)$, respectively.

Equation (3.17) defines a mapping for the ex ante value function, V . To see this, note that (3.3) defines a seller's value function $W_s(m)$ as a mapping of V , (3.7) defines a buyer's value function $W_b(m)$ as a mapping of V , and (3.16) defines $\tilde{V}(m)$ as a mapping of V . As a result, the right-hand side of (3.17) is a mapping of V . Let us denote this mapping as $\mathcal{F}$ and express (3.17) as $V(m) = \mathcal{F}V(m)$. That is, V is a fixed point of $\mathcal{F}$. The following theorem states existence, uniqueness, and other properties of the fixed point; in particular, the theorem confirms that the fixed point, V , has the properties that we have assumed so far (see Appendix D for a proof).

Theorem 3.4. *$\mathcal{F}$ has a unique fixed point, V , which is continuous, bounded, increasing and concave. Moreover, if $\omega > 0$, the following results hold: (i) $V(m) > 0$ for all $m > 0$; (ii) $V(0) = W_s(0)$, and $W_s(m) \geq W_b(m)$ for all $m \in [0, k/\omega]$; (iii) There exists $m_0 > 0$ such that $V(m_0) = W_b(m_0)$, $V'(m_0) = W'_b(m_0)$, and $V(m) > W_s(m)$ for all $m \in (0, m_0]$; (iv) $b^*(m_0) > 0$ and $\phi(m_0) = 0$; (v) If $V(0) > 0$, then $V'(m_0) > V'(m^*)$ and $m_0 < m^*$; if $V(0) = 0$, then $V(m) = \omega m/\beta$ for all $m \in [0, \max\{m_0, m^*\}]$.*

Result (i) in the above theorem is straightforward. Results (ii) – (iii) together imply the intuitive property that an individual will choose to be a buyer only when his money balance is relatively high and that, when his money balance is relatively low, he will participate in a lottery whose winning prize is m_0 . Result (iv) states that after winning the lottery with prize m_0 , an individual strictly prefers buying goods to not buying, and he spends all of his money in such a trade. Result (v) compares the relative position of m_0 with m^* . Figure 1 depicts the lottery for low money holdings and for the case where $V(0) > 0$. With the lottery, the ex ante value function V is linear for $m \in [0, m_0]$, as illustrated by the tangent line connecting points A and B .

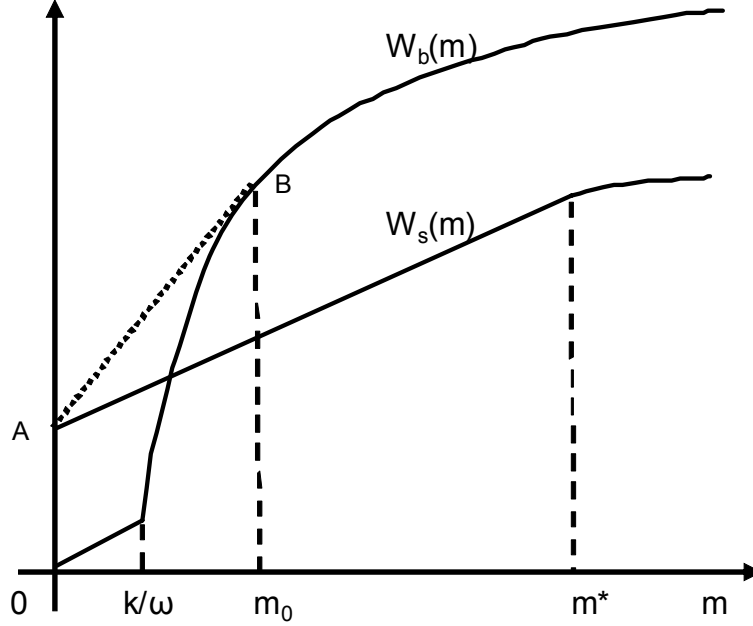


Figure 1. Lottery for low money holdings

Let us characterize m_0 more explicitly. Since the lower realization of the particular lottery is $L_1 = 0$, the winning probability for an individual who puts m into this lottery is m/m_0 . Thus, the winning size m_0 is determined as

$$m_0 = \arg \max_{L \geq m} \left[\frac{m}{L} \tilde{V}(L) + \left(1 - \frac{m}{L}\right) \tilde{V}(0) \right]. \quad (3.18)$$

It is clear that m_0 is independent of the individual's money holdings m , provided $m \leq m_0$. Moreover, $\tilde{V}$ is differentiable at m_0 , with $\tilde{V}'(m_0) = W'_b(m_0) = V'(m_0)$ (see part (iii) of Theorem 3.4). Thus, m_0 is given by the following first-order condition:

$$\frac{\tilde{V}(m_0) - \tilde{V}(0)}{m_0} = \tilde{V}'(m_0) = W'_b(m_0).$$

That is, the slope of $\tilde{V}$ (or W_b) at m_0 is equal to the slope of the tangent line connecting points A and B in Figure 1.

4. Monetary Equilibrium

In the previous section, we have analyzed the following objects: the matching probabilities in each submarket, individuals' decisions, and the value functions. For any given ω , we have shown that these objects do not depend on the distribution of individuals over money holdings. Let us recap how this independence arises in our model. Start with a firm's decision on how many trading posts to maintain in each submarket. Given ω , competitive entry of firms determines the matching probability function $s(x, q)$, independently of the distribution. In turn, the function $s(x, q)$, together with $b = \mu^{-1}(s)$, provides all the relevant information for the tradeoff between the terms of trade and the matching probabilities across the submarkets. Given this information, an individual chooses to search in the submarket that maximizes the expected gain from trading. Thus, given the functions $s(x, q)$ and $\mu^{-1}(s)$, an individual's optimal decision is independent of how many individuals are distributed in other submarkets. Clearly, the assumption of directed search is necessary for such independence.

In an equilibrium, the distribution of money holdings can possibly affect the matching probabilities and individuals' decisions only through the one-dimensional variable, ω . We characterize the distribution below.

4.1. Distributions of money holdings and trading posts

Let $G(m)$ be the measure of individuals holding a money balance less than or equal to m immediately after the outcomes of the lotteries are realized and before individuals go to the market. Denote $\text{supp}(G)$ as the support of G , and $dG(m)$ as the measure of individuals holding the particular level of money balance, m .

To characterize the distribution in an equilibrium, let us calculate the change in the distribution in an arbitrary period. Denote $G_a(m)$ as the measure of individuals whose holdings are less than or equal to m after production and trading are completed in the period. For all $m \in [0, m^*]$, the change in the measure of individuals in this group before and after trading is:

$$G_a(m) - G(m) = -dG(0) + \int_{m < m' \leq \phi^{-1}(m)} b^*(m') dG(m'). \quad (4.1)$$

The term $dG(0)$ is the measure of producers before trade. Because all producers acquire the balance m^* , they move above m after one period. The last term in (4.1) is the flow of buyers whose balances are reduced by trade from levels above m to levels less than or equal to m .

Let $G_{+1}(m)$ denote the measure of individuals whose holdings are less than or equal to m

after the lotteries are realized in the next period. Then,

$$G_{+1}(m) - G_a(m) = \int_{m < m' \leq L_1^{-1}(m)} [1 - \pi(m')] dG_a(m') - \int_{L_2^{-1}(m) < m' \leq m} \pi(m') dG_a(m'). \quad (4.2)$$

The first term on the RHS is the group of individuals who hold $m' > m$ at the end of this period and their lotteries in the next period have the realizations $L_1(m') \leq m$. The second term is the group of individuals who hold $m' \leq m$ at the end of this period and their lotteries in the next period have the realizations $L_2(m') > m$.

We can also calculate the distribution of trading posts across the submarkets. Because the buyers who hold a money balance m visit only the submarket $(x^*(m), q^*(m))$, the set of submarkets active in the equilibrium is:

$$\{(x^*(m), q^*(m)) : m \in \text{supp}(G), m \geq m_0\}.$$

In submarket $(x^*(m), q^*(m))$, the measure of buyers is $dG(m)$. As an accounting identity, the number of matched buyers in every submarket must be equal to the number of matched trading posts. That is, $b[dG(m)] = s[dN(x^*(m), q^*(m))]$. From this relationship we can compute the measure of trading posts in submarket $(x^*(m), q^*(m))$ as

$$dN(x^*(m), q^*(m)) = \frac{b^*(m)}{s(x^*(m), q^*(m))} dG(m).$$

4.2. Definition of a monetary equilibrium

A *stationary monetary equilibrium* (i.e., a monetary steady state) consists of each seller's choice: z ; each buyer's choice of the submarket to enter: (x, q) ; lotteries: $(L_1, L_2, \pi(m))$; value functions: $W_s(m), W_b(m), V(m)$; the matching probability function for each post in each submarket: $s(x, q)$; future marginal value of money: ω ; and the distribution of money holdings: G . These components satisfy $\omega > 0$ and the following requirements: (i) Given ω and s , a seller's choice solves (3.3) and induces the value function $W_s(m)$; (ii) Given ω and s , the choice of the submarket by a buyer with money m solves (3.7) and induces the value function $W_b(m)$; (iii) Given ω , each individual's choice of the lottery at the beginning of each period solves (3.17) and induces the value function $V(m)$; (iv) Free entry of trading posts: the expected profit of a trading post in each active submarket is zero, and so the function $s(x, q)$ satisfies (3.2); (v) Aggregate consistency: ω is such that the average (normalized) amount of money holdings is indeed 1, i.e., $\int m dG(m) = 1$; (vi) The distribution of money holdings, G , satisfies (4.1) and (4.2), and it is stationary, i.e., $G_{+1}(m) = G(m)$ for all m .

Parts (i) through (iv) of the definition incorporate the aforementioned property that for any given ω , the matching probabilities and individuals' decisions do not depend on the distribution of

money holdings. It should be clear that this independence holds both outside and in a steady state. Part (v) of the above definition introduces the possibility that the distribution of money holdings affects the seller's marginal value of money, ω . This is the only possible channel through which the distribution of money holdings affects the matching probabilities and individuals' decisions in the equilibrium.

4.3. Equilibrium pattern of spending

To analyze an equilibrium, we impose a regularity condition. For the difference in money holdings to have a wealth effect, it is necessary that the marginal value of money at m^* be different from that at m_0 . If $V'(m^*) = V'(m_0)$, instead, the marginal value of money would be constant in the entire domain of equilibrium money holdings. For $V'(m^*) \neq V'(m_0)$, it suffices to ensure $V(0) \neq 0$ (see part (v) of Theorem 3.4). Since the case $V(0) = 0$ is a knife edge, it is not difficult to rule it out – all we need is to assume that $\beta \neq \beta_0$, where β_0 is defined by (E.4) in Appendix E. Under this regularity condition, $m^* > m_0$ and $V'(m^*) < V'(m_0)$.

Let us define a sequence of functions $\{\phi^i\}_{i \geq 0}$ recursively by $\phi^0(m) = m$ and $\phi^{i+1}(m) = \phi^i(m)$ for $i = 0, 1, 2, \dots$. Denote T as the positive integer such that $\phi^{T-1}(m^*) \geq m_0 > \phi^T(m^*)$. That is, T is the number of purchases that a buyer with m^* can make according to the policy function ϕ before the money balance falls below m_0 . We prove the following lemma in Appendix E:

Lemma 4.1. *Assume $\omega > 0$. Then, (i) $b^*(m^*) > 0$ and $W_b(m^*) > W_s(m^*)$. If $\beta \neq \beta_0$, the following results hold: (ii) $V(0) > 0$, $m_0 < m^*$ and $V'(m_0) > V'(m^*)$, (iii) $\phi(m^*) > 0$ and $V(m^*) = W_b(m^*)$; (iv) For all $i = 1, 2, \dots, T$, if $\phi^i(m^*) > 0$, then V is strictly concave at $\phi^i(m^*)$ and $V(\phi^i(m^*)) = W_b(\phi^i(m^*))$.*

Part (i) of the above lemma states that a buyer with a balance m^* strictly prefers trading to not trading. Part (ii) was explained above. Part (iii) says that it is optimal for a buyer with a money balance m^* not to spend all of the balance in one trade, and that the ex ante value function is equal to the buyer's value function. Part (iv) describes similar properties of the ex ante value function at subsequent levels of money holdings $\phi^i(m^*)$.

An individual's money balance obeys the following dynamics, as implied by Lemma 4.1. When the individual is a seller, he acquires the balance, m^* . In the first period where the individual is a buyer and has a match, he spends $x^*(m^*)$ units of money and retains $\phi(m^*)$ units. If $\phi(m^*) \geq m_0$, the individual will continue to be a buyer in the next period, where his spending will be $x^*(\phi(m^*))$ and his residual money balance will be $\phi^2(m^*)$. This process continues until the T th time in which the individual has purchased goods. The individual will enter the T th

trade with a money balance $\phi^{T-1}(m^*)$, and will exit the trade with a balance $\phi^T(m^*) < m_0$. If $\phi^T(m^*) = 0$, the individual will become a seller after the trade. If $\phi^T(m^*) > 0$, the individual will participate in the lottery whose winning prize is m_0 . If he wins the lottery, he will make one more purchase before becoming a seller; if he does not win the lottery, he will become a seller immediately. In this round of purchases, a buyer spends less and less as each purchase reduces his money balance. The falling money balance clearly has a wealth effect, because the ex ante value function is strictly concave in the buyer's money balance along the equilibrium path.

Note that an individual does not participate in lotteries along the equilibrium path, except possibly once when his money balance reaches $\phi^T(m^*)$ and when $\phi^T(m^*) > 0$. Starting from the money balance m^* , an individual's ex ante value function is equal to the value function as a buyer (see part (iii) of Lemma 4.1), which implies that the individual with m^* does not participate in lotteries. After purchasing goods, the individual will have a residual money balance, $\phi(m^*) > 0$, at which the individual's ex ante value function will be strictly concave and equal to $W_b(\phi(m^*))$ (see part (iv) of Lemma 4.1). Again, the buyer will not use a lottery in the next period. This process continues until and including the T th time in which the buyer has a match. After the T th purchase, the individual's residual balance will be $\phi^T(m^*) < m_0$. If $\phi^T(m^*) = 0$, as in the first case listed above for the support of G , the individual runs out of money and he will produce next. If $\phi^T(m^*) > 0$, the individual will participate in the lottery whose winning prize is m_0 .¹⁶

Another implication of Lemma 4.1 is that on the equilibrium path, a buyer's optimal choices are characterized by the first-order conditions, (3.12) and (3.14), and that the value functions satisfy the envelope condition, (3.13). We have explained earlier that the optimal choice of b always satisfies the first-order condition (3.12). For the optimal choice of x and the value functions, start with a buyer whose money balance is m^* . Because $V(m^*) = W_b(m^*)$ by part (iii) of Lemma 4.1, V is differentiable at m^* , and the derivative $V'(m^*)$ satisfies the envelope condition (3.13) (and is also equal to ω/β). Because $\phi(m^*) > 0$, the buyer's optimal choice of spending, $x^*(m^*)$, satisfies the first-order condition (3.14) with $m = m^*$. If $\phi(m^*) \geq m_0$, then $V(\phi(m^*)) = W_b(\phi(m^*))$ (see part (iv) of Lemma 4.1). In this case, the derivative $V'(\phi(m^*))$ exists and satisfies the envelope condition (3.13) with $m = \phi(m^*)$. This process continues until the T th time in which the buyer has a match. When entering the T th match, the buyer has a balance $\phi^{T-1}(m^*) \geq m_0$. In this match, the derivative $V'(\phi^{T-1}(m^*))$ exists and satisfies the envelope condition (3.13) with $m = \phi^{T-1}(m^*)$. Moreover, the buyer's optimal choice of money spending either satisfies the first-order condition (3.14) if $\phi^T(m^*) > 0$, or is given by the binding

¹⁶Note that part (iv) of Lemma 4.1 implies $\phi^T(m^*) = 0$: If $\phi^T(m^*) > 0$, then V is strictly concave at $\phi^T(m^*)$, which contradicts the fact that V is linear in $[0, m_0] \ni \phi^T(m^*)$. We have not used this result in the analysis.

money constraint, $x^*(\phi^{T-1}(m^*)) = \phi^{T-1}(m^*)$.

4.4. Equilibrium distribution of money holdings

The equilibrium path of money holdings described in the previous subsection significantly simplifies the characterization of the distribution of money holdings. The equilibrium path implies that the support of the distribution of money holdings, G , is¹⁷

$$\text{supp}(G) = \begin{cases} \{\phi^i(m^*)\}_{i=0}^{T-1} \cup \{0\}, & \text{if } \phi^T(m^*) = 0 \\ \{\phi^i(m^*)\}_{i=0}^{T-1} \cup \{m_0\} \cup \{0\}, & \text{if } \phi^T(m^*) > 0. \end{cases} \quad (4.3)$$

Since the support of the distribution contains only a finite number of values, let us denote $g(m) = dG(m)$ as the measure of individuals who hold the balance m immediately before entering the goods market. Similarly, let $g_a(m) = dG_a(m)$ be the measure of individuals who hold the balance m after trading is completed in the period. The support of G_a is $\text{supp}(G) \cup \{\phi^T(m^*)\}$.

Because no lotteries are used on the equilibrium path except when an individual's money balance reaches the level, $\phi^T(m^*)$, we can write the laws of motion of the money distribution in more detail. Consider first the case where $\phi^T(m^*) > 0$. The equations (4.1) and (4.2) yield the following laws of motion:

$$g_{+1}(m^*) - g(m^*) = g(0) - b^*(m^*)g(m^*); \quad (4.4)$$

$$\begin{aligned} & g_{+1}(\phi^i(m^*)) - g(\phi^i(m^*)) \\ &= b^*(\phi^{i-1}(m^*))g(\phi^{i-1}(m^*)) - b^*(\phi^i(m^*))g(\phi^i(m^*)) \text{ for } 1 \leq i \leq T-1; \end{aligned} \quad (4.5)$$

$$g_{+1}(m_0) - g(m_0) = \frac{\phi^T(m^*)}{m_0} b^*(\phi^{T-1}(m^*))g(\phi^{T-1}(m^*)) - b^*(m_0)g(m_0); \quad (4.6)$$

$$g_{+1}(0) = b^*(m_0)g(m_0) + \left[1 - \frac{\phi^T(m^*)}{m_0}\right] b^*(\phi^{T-1}(m^*))g(\phi^{T-1}(m^*)). \quad (4.7)$$

The first two equations calculate the net change between the current and the next period in the measure of individuals who hold $\phi^i(m^*)$, where $i = 0, 1, \dots, T-1$. The outflow is the measure of individuals with $\phi^i(m^*)$ who successfully trade in the current period. For $i = 0$, the inflow is the measure of individuals who are producers in the current period; for $i \geq 1$, the inflow is the measure of individuals with $\phi^{i-1}(m^*)$ who successfully trade in the current period. Similarly, (4.6) calculates the net change in the measure of individuals with m_0 , where the inflow is the measure of individuals with $\phi^T(m^*)$ who win the lottery at the beginning of the next period. To calculate this inflow, we have used the facts that the measure of individuals with $\phi^T(m^*)$ at the

¹⁷The support of G does not include $\phi^T(m^*)$, because G is measured after the outcomes of the lotteries are realized. Because the individuals who hold a balance $\phi^T(m^*)$ participate in the lottery whose winning prize is m_0 , the realization of the lottery is either 0 or m_0 .

beginning of the next period is $g_a(\phi^T(m^*)) = b^*(\phi^{T-1}(m^*))g(\phi^{T-1}(m^*))$ and that the probability of winning the lottery is $\phi^T(m^*)/m_0$. Finally, (4.7) calculates the measure of producers in the next period, as the sum of the individuals with m_0 who successfully trade in the current period and the individuals with $\phi^T(m^*)$ who do not win the lottery.

The equations (4.4) – (4.7) also hold in the case $\phi^T(m^*) = 0$. In this case, the net flow in (4.6) is negative, which reflects the fact that there is no one who wins the lottery. Thus, in an equilibrium with $\phi^T(m^*) > 0$, no individuals hold m_0 provided that $\phi^{T-1}(m^*) > m_0$. If $\phi^{T-1}(m^*) = m_0$, the measure of individuals with m_0 is given by (4.5) with $i = T - 1$.

In a steady state, $g_{+1}(m) = g(m)$ for all m . Thus, the steady state distribution satisfies:

$$\left. \begin{aligned} g(\phi^i(m^*)) &= \frac{g(0)}{b^*(\phi^i(m^*))} \text{ for } 0 \leq i \leq T-1; \\ g(m_0) &= \frac{\phi^T(m^*)}{m_0} \frac{g(0)}{b^*(m_0)}; \\ g(0) &= \left[1 + \frac{\phi^T(m^*)}{m_0 b^*(m_0)} + \sum_{i=0}^{T-1} \frac{1}{b^*(\phi^i(m^*))} \right]^{-1}. \end{aligned} \right\} \quad (4.8)$$

The results in (4.8) are simple to explain. Because a buyer moves only to the next lower level of money holdings after a successful trade, the outflow must be the same for all equilibrium levels $\phi^i(m^*) > m_0$ in order to maintain a steady state. That is, $b^*(\phi^i(m^*))g(\phi^i(m^*))$ must be the same in a steady state for all $i = 0, 1, \dots, T-1$. Moreover, this outflow must be equal to the inflow from producers, as in the first line of (4.8). The second line of (4.8) says that the measure of individuals at m_0 must be equal to the measure of individuals at $\phi^T(m^*)$ who win the lotteries. The last line comes from substituting $g(\phi^i(m^*))$ and $g(m_0)$ into the requirement that the total measure of individuals in the economy be equal to one.

4.5. Existence of a monetary steady state

In previous subsections, we have characterized the equilibrium patterns of spending for any given money distribution and any given marginal value of money of the sellers, ω . We have also characterized the equilibrium distribution of money holdings for any given ω . For existence of a stationary monetary equilibrium, it suffices to find $\omega > 0$ that satisfies requirement (v) in the definition of an equilibrium. That is, a seller's marginal value of money must be such that all money is held by the individuals. Since the support of the money distribution is given by (4.3), we can express requirement (v) in the equilibrium definition explicitly as

$$m_0 g(m_0) + \sum_{i=0}^{T-1} \phi^i(m^*) g(\phi^i(m^*)) = 1. \quad (4.9)$$

Because ω appears in the function $Q(x, b)$ defined by (3.6) and, hence, in the function $u(x, b)$, the optimal choices, $(m^*, \phi^i(m^*), m_0)$, all depend on ω . So does the distribution of money holdings. Thus, the above equation is indeed an equation of ω . However, the optimal choices and the money distribution can, in general, depend on ω in a complicated way.

The equilibrium possesses an interesting property that enables us to simplify the task of determining ω . To describe this property, let us recognize the direct appearance of ω in a buyer's decision problem by modifying the notation for a buyer's value function as $W_b(m, \omega)$. Similarly, modify the ex ante value function as $V(m, \omega)$, a seller's value function as $W_s(m, \omega)$ and define $\tilde{V}(m, \omega) = \max\{W_s(m, \omega), W_b(m, \omega)\}$. Multiply all nominal variables by ω to obtain the "real" values of these values. Denote such real values with a caret. For example, the real value of money is $\hat{m} = \omega m$, the real value of spending is $\hat{x} = \omega x$, the real value of a buyer's residual balance after a purchase is $\hat{\phi} = \omega \phi$, and the real value of the realization of a lottery is $\hat{L} = \omega L$. The following lemma states a property of the equilibrium:

Lemma 4.2. *In a steady state, the value functions and the real values of the optimal choices depend on ω only through $\hat{m} = \omega m$ and not through ω separately. That is, (i) there exist functions $w_s, w_b, \tilde{v}$, and v such that $W_s(m, \omega) = w_s(\hat{m})$, $W_b(m, \omega) = w_b(\hat{m})$, $\tilde{V}(m, \omega) = \tilde{v}(\hat{m})$, and $V(m, \omega) = v(\hat{m})$; (ii) The real values of the optimal choices in a steady state, $(\hat{m}^*, \hat{m}_0, \hat{x}, \hat{\phi}, \hat{L}_1, \hat{L}_2)$, are independent of ω .*

Proof. We have proven that the function $V(m, \omega)$ is unique. If there exists v such that $V(m, \omega) = v(\hat{m})$ for all m , then $v(\hat{m})$ is the unique ex ante value function. In the process of verifying this result, we also show that other value functions depend on ω only through $\hat{m}$ and that the optimal choices of $(\hat{m}^*, \hat{m}_0, \hat{x}, \hat{\phi}, \hat{L}_1, \hat{L}_2)$ are independent of ω .

Suppose that $V(m, \omega) = v(\hat{m})$. First, we transform a seller's decision problem in (3.3). Note that the value function V on the right-hand side of (3.3) is the ex ante value function in the next period. According to the new notation, this value function is $V(m, \omega_{+1}) = v(\hat{m}_{+1})$, where $\hat{m}_{+1} = \hat{m}\omega_{+1}/\omega$ and ω_{+1} is the value of ω in the next period. As a result, we rewrite (3.3) as

$$w_s(\hat{m}) = \hat{m} + \max_{\hat{y} \geq \hat{m}} [\beta v(\hat{y}\omega_{+1}/\omega) - \hat{y}].$$

Clearly, the optimal choice of $\hat{y}$ depends on ω_{+1}/ω , but not on ω or ω_{+1} alone. In particular, the optimal choice with $\hat{m} = 0$ is $\hat{m}^*$, which satisfies $\beta v'(\hat{m}^*\omega_{+1}/\omega)\omega_{+1}/\omega = 1$. As a result, $\hat{m}^*$ is independent of ω in a steady state. Note that the seller's optimal choices implies that $w'_s(\hat{m}^*) = 1$, which is used in the calculation of a firm's profit.

Second, we transform a buyer's maximization problem. Since ω appears in the function s defined by (3.2) only through the term ωx , we can slightly abuse the notation to write a seller's matching

probability as $s(\hat{x}, q)$. Similarly, write the function Q defined by (3.6) as $Q(\hat{x}, b)$ and the function u as $u(\hat{x}, b)$. With the conjecture $V(m, \omega) = v(\hat{m})$, a buyer's maximization problem can be transformed as

$$w_b(\hat{m}) = \max_{b \in [0,1], \hat{x} \in [0, \hat{m}]} \{ \beta v(\hat{m}\omega_{+1}/\omega) + b[u(\hat{x}, b) + \beta v((\hat{m} - \hat{x})\omega_{+1}/\omega) - \beta v(\hat{m}\omega_{+1}/\omega)] \}.$$

Again, the ratio ω_{+1}/ω appears in the function v on the right-hand side of the equation because the function is the value function in the next period, which is defined as a function of future real money balance $\hat{m}_{+1} = \hat{m}\omega_{+1}/\omega$. The optimal choices, $(\hat{x}^*, b^*)$ depend on ω_{+1}/ω , but not on ω or ω_{+1} alone. In a steady state, they are independent of ω . So is the steady state value of the optimal choice of $\hat{\phi} = \hat{m} - \hat{x}^*$.

Third, $\tilde{v}(\hat{m}) = \max\{w_s(\hat{m}), w_b(\hat{m})\}$, and the ex ante value function is

$$v(\hat{m}) = \max_{(\hat{L}_1, \hat{L}_2, \pi)} \left[(1 - \pi) \tilde{v}(\hat{L}_1) + \pi \tilde{v}(\hat{L}_2) \right]$$

subject to:

$$\begin{aligned} (1 - \pi) \hat{L}_1 + \pi \hat{L}_2 &= \hat{m}, \\ \pi &\in [0, 1] \text{ and } \hat{L}_i \geq 0 \text{ for } i = 1, 2. \end{aligned}$$

Because the functions $w_s(\hat{m})$ and $w_b(\hat{m})$ depend on the ratio ω_{+1}/ω , so do $\tilde{v}(\hat{m})$ and $v(\hat{m})$. However, in a steady state, these functions are independent of ω or ω_{+1} . So are the steady state values of the optimal choices, $(\hat{L}_1^*, \hat{L}_2^*)$. In particular, the lottery for low money holdings yields $\hat{L}_2^* = \hat{m}_0$. As before, the above process defines a mapping of v , which is a continuous and contraction mapping. Thus, the function $v(\hat{m})$ exists and is unique. **QED**

There are two useful observations about Lemma 4.2. First, only in a steady state are the value functions and the real values of optimal choices independent of ω and ω_{+1} , as the proof clearly shows. Outside a steady state, these equilibrium objects depend on the ratio ω_{+1}/ω . Second, such independence implies that money is neutral at an individual's level in a steady state. Moreover, in a steady state, the money distribution given by (4.8) is also independent of ω and the money stock. Thus, money is neutral at the aggregate level in a steady state. We state this long-run neutrality and existence of a monetary steady state in the following theorem:

Theorem 4.3. *A stationary monetary equilibrium exists. The equilibrium is unique if $\hat{m}_0$ and $\hat{m}^*$ are unique. In a stationary equilibrium, money is neutral. The steady state distribution of money holdings satisfies: $g(m^*) > g(0)$ and $g(\phi^i(m^*)) \geq g(\phi^{i-1}(m^*))$ for all $i = 1, 2, \dots, T-1$, where the second inequality is strict if and only if $b(\phi^i(m^*)) < b(\phi^{i-1}(m^*))$. Moreover, $g(m_0) > g(\phi^{T-1}(m^*))$ if and only if $\phi^T(m^*) > m_0 b^*(m_0)/b^*(\phi^{T-1}(m^*))$.*

Proof. Substituting the steady state distribution of money holdings from (4.8) into (4.9), we obtain the following implicit equation for ω in the steady state:

$$\frac{\phi^T(m^*)}{b^*(m_0)} \left(1 - \frac{1}{m_0}\right) + \sum_{i=0}^{T-1} \frac{\phi^i(m^*) - 1}{b^*(\phi^i(m^*))} = 1. \quad (4.10)$$

Multiplying the two sides of (4.10) by ω , we get

$$\frac{\hat{\phi}^T(\hat{m}^*)}{b^*(\hat{m}_0)} \left(1 - \frac{\omega}{\hat{m}_0}\right) + \sum_{i=0}^{T-1} \frac{\hat{\phi}^i(\hat{m}^*) - \omega}{b^*(\hat{\phi}^i(\hat{m}^*))} = \omega.$$

By Lemma 4.2, $\hat{m}^*$, $\hat{m}_0$, and $\hat{\phi}^i(\hat{m}^*)$ are all independent of ω in a steady state. Thus, the steady state value of ω is uniquely solved as

$$\omega = \left[\frac{\hat{\phi}^T(\hat{m}^*)}{b^*(\hat{m}_0)} + \sum_{i=0}^{T-1} \frac{\hat{\phi}^i(\hat{m}^*)}{b^*(\hat{\phi}^i(\hat{m}^*))} \right] \bigg/ \left[1 + \frac{\hat{\phi}^T(\hat{m}^*)}{b^*(\hat{m}_0)\hat{m}_0} + \sum_{i=0}^{T-1} \frac{1}{b^*(\hat{\phi}^i(\hat{m}^*))} \right].$$

This solution satisfies $\omega \in (0, \infty)$, and so a stationary monetary equilibrium exists. If $\hat{m}_0$ and $\hat{m}^*$ are unique, then the solution for ω is unique, in which case the equilibrium is unique.

Money is neutral in a steady state because the real values of individuals' optimal choices and the money distribution are both independent of the money supply in a steady state.

It is clear that $\phi^i(m^*) = \phi^{i-1}(m^*) - x^*(\phi^{i-1}(m^*)) < \phi^{i-1}(m^*)$ for all $1 \leq i \leq T$. Because $b^*(m)$ is an increasing function, the first line of (4.8) implies that $g(m^*) > g(0)$ and $g(\phi^i(m^*)) \geq g(\phi^{i-1}(m^*))$ for all $i = 1, 2, \dots, T-1$, where the second inequality is strict if and only if $b(\phi^i(m^*)) < b(\phi^{i-1}(m^*))$. Using the first two lines of (4.8), it is easy to deduce that $g(m_0) > g(\phi^{T-1}(m^*))$ if and only if $\phi^T(m^*) > m_0 b^*(m_0)/b^*(\phi^{T-1}(m^*))$. **QED**

If the lottery for low money holdings is not used in the equilibrium (i.e., if $\phi^T(m^*) = 0$), the frequency function of money holdings among buyers is a decreasing function. That is, the higher the money balance, the fewer the number of buyers who hold that balance. This result arises because a relatively high money balance allows a buyer to trade with a relatively high probability, which reduces the measure of individuals staying at that level of money holdings in the steady state.¹⁸ Similarly, the measure of sellers is smaller than the measure of buyers at any equilibrium level of money holdings, because a seller always becomes a buyer after one period of production. If $\phi^T(m^*) > 0$, the lottery produces an additional equilibrium level of money holdings, m_0 . In this case, the frequency function of money holdings among buyers is still be a decreasing function if $\phi^T(m^*)$ is sufficiently close to m_0 . If $\phi^T(m^*)$ is not sufficiently close to m_0 , then the frequency function of money holdings has a hump at $\phi^{T-1}(m^*)$.

¹⁸This decreasing feature of the distribution of money holdings resembles that in Green and Zhou (1998), but the feature is obtained here without the restriction in Green and Zhou that goods and money are indivisible.

5. Conclusion

In this paper, we construct a tractable search model of money with a non-degenerate distribution of money holdings. We model search as a directed process in the sense that buyers know the terms of trade before visiting particular sellers, as opposed to undirected search that has dominated the literature. In this model, the distribution of money holdings among individuals is non-degenerate. We show that this distribution affects individuals' decisions not directly, but rather indirectly only through a one-dimensional variable – the seller's future marginal value of money. This result drastically reduces the state space of individuals' decisions and makes the model tractable. We analytically characterize a monetary equilibrium, using lattice-theoretic techniques, and prove existence of a monetary steady state. In the equilibrium, buyers follow a stylized spending pattern over time, and the money distribution has a persistent wealth effect.

Appendix

A. Proof of Lemma 3.1

From (3.3) it is easy to verify that $W_s(m)$ is a continuous, bounded and increasing function. Because the objective function $[\beta V(m + zD) - \omega zD]$ is concave in (z, m) jointly, its maximized value, $W_s(m)$, is concave in m . To establish the properties of the optimal choice of z , temporarily denote the objective function in (3.3) as $F(z, m)$. Let the domain of m be $[0, \bar{m}]$, where $0 < \bar{m} < \infty$. We show that $F(z, m)$ has increasing differences in $(-z, m)$. That is, for arbitrary z_1, z_2, m_1 and m_2 , where $z_2 > z_1, m_2 > m_1, z_i \in [0, 1]$ and $m_i \in [0, \bar{m}]$, the function F satisfies:

$$[F(-(-z_1), m_2) - F(-(-z_1), m_1)] - [F(-(-z_2), m_2) - F(-(-z_2), m_1)] \geq 0.$$

Computing the difference $[F(z_i, m_2) - F(z_i, m_1)]$, we get:

$$\begin{aligned} & [F(-(-z_1), m_2) - F(-(-z_1), m_1)] - [F(-(-z_2), m_2) - F(-(-z_2), m_1)] \\ = & [V(m_2 + z_1 D) - V(m_1 + z_1 D)] - [V(m_2 + z_2 D) - V(m_1 + z_2 D)] \geq 0. \end{aligned}$$

The inequality follows from concavity of V (see Royden, 1988, p113), and it is strict if V is strictly concave. Thus, $F(z, m)$ has increasing differences in $(-z, m)$ on $[-1, 0] \times [0, \bar{m}]$, and strictly so if $V(\cdot)$ is strictly concave. By Theorem 2.8.1 in Topkis (1998, p76), the optimal choice of z is decreasing in m .

To establish the properties of $y = m + zD$, we rewrite (3.3) as

$$W_s(m) = \omega m + \max_{y \geq m} [\beta V(y) - \omega y]. \quad (\text{A.1})$$

Let m and m' be two arbitrary levels of money holdings, with $m' > m$. Let the optimal solution for y under m' be $y^*(m')$, and under m be $y^*(m)$. If the constraint $y \geq m'$ does not bind, then clearly the constraint $y \geq m$ does not bind. In this case, $y^*(m') = y^*(m)$. Now consider the case where the constraint $y \geq m'$ binds. In this case, it must be the case that $y^*(m) \in [m, m']$, which implies $y^*(m') \geq y^*(m)$: if $y^*(m) > m'$, instead, then $y^*(m') = y^*(m) > m'$, which contradicts the hypothesis in the current case.

Finally, because $W_s(m)$ is a concave function, its left-hand and right-hand derivatives exist at all m (see Royden, 1988, pp113-114). At m^* , the left-hand derivative is $W'_s(m^{*-}) = \omega$, and the right-hand derivative is $W'_s(m^{*+}) = \beta V'(m^{*+})$. In Lemma 3.3, we will prove that $V'(m^*) = \omega/\beta$. Thus, $W'_s(m^{*+}) = W'_s(m^{*-}) = \omega$. **QED**

B. Proof of Lemma 3.2

The Theorem of the Maximum implies that $W_b(m)$ is continuous and bounded and that a solution exists. It is also straightforward to show that $W_b(m)$ is increasing. For the remainder of the

proof, temporarily denote $F(x, b, m) = b t(x, b, m)$, where the function t is defined in (3.10). The optimal choices (x^*, b^*) maximize $F(x, b, m)$. If $b^* = 0$, the choice of x is irrelevant for the buyer because a trade does not take place. Since the choice $b = 0$ yields $F(x, b, m) = 0$, then $b^* > 0$ is optimal only if $t(x, b, m) \geq 0$. In the remainder of the proof, we only examine the case where $b > 0$ and $t(x, b, m) > 0$.

With $b > 0$ and $t(x, b, m) > 0$, we can transform the maximization problem as

$$\max_{b, x} F(x, b, m) = \exp \left\{ \max_{b, m} [\ln b + \ln t(x, b, m)] \right\}.$$

The function $(\ln b)$ is concave in (x, b) . Recall that $u(x, b)$ is strictly concave in (x, b) jointly. Since V is concave, then $V(m - x)$ is concave in x . Thus, $t(x, b, m)$ defined in (3.10) is strictly concave in (x, b) jointly. Since the logarithmic transform is an increasing and concave transformation, the objective function on the right-hand side of the above equation is strictly concave in (x, b) jointly. Therefore, the optimal choices of (x, b) are unique.

To show that the optimal choices have monotonicity as described in the lemma, we decompose the maximization problem into two: In the first step, we fix b and find the optimal choice of x ; in the second step, we find the optimal choice of b .

Take the first step. For any given (b, m) , the optimal choice of x is denoted $\tilde{x}(b, m)$ as in (3.11). Because $u(x, b)$ is strictly concave in x and V is concave, $\tilde{x}$ exists and is unique. Note that t is separable in b and m , and so t has increasing differences in (b, m) (but not strictly so). Take arbitrary m_1, m_2, x_1, x_2, b_1 and b_2 , with $m_2 > m_1, x_2 > x_1$, and $b_2 > b_1$. Compute:

$$t(x_2, b, m) - t(x_1, b, m) = [u(x_2, b) - u(x_1, b)] + \beta [V(m - x_2) - V(m - x_1)].$$

Since $u(x, b)$ is strictly supermodular in (x, b) , we have:

$$\begin{aligned} & [t(x_2, b_2, m) - t(x_1, b_2, m)] - [t(x_2, b_1, m) - t(x_1, b_1, m)] \\ &= [u(x_2, b_2) - u(x_1, b_2)] - [u(x_2, b_1) - u(x_1, b_1)] > 0. \end{aligned}$$

That is, $t(x, b, m)$ has strictly increasing differences in (x, b) . Moreover,

$$\begin{aligned} & [t(x_2, b, m_2) - t(x_1, b, m_2)] - [t(x_2, b, m_1) - t(x_1, b, m_1)] \\ &= \beta [V(m_1 - x_1) - V(m_1 - x_2)] - \beta [V(m_2 - x_1) - V(m_2 - x_2)] \geq 0. \end{aligned}$$

The inequality follows from concavity of V (see Royden, 1988, p113) and the facts that $m_1 - x_1 < m_2 - x_1, m_1 - x_2 < m_2 - x_2$, and $(m_1 - x_1) - (m_1 - x_2) = (m_2 - x_1) - (m_2 - x_2) = x_2 - x_1 > 0$. The inequality in the above result is strict if V is strictly concave. Thus, $t(x, b, m)$ has increasing differences in (x, m) , and strictly so if V is strictly concave. Because $t(x, b, m)$ has increasing differences in (b, m) , (x, b) and (x, m) , $t(x, b, m)$ is supermodular in (x, b, m) . Since the choice

set, $[0, m]$, is also increasing in m , then $\tilde{x}(b, m)$ is increasing in (b, m) (see Topkis, 1998, p76), and the maximized value of t is supermodular in (b, m) (see Topkis, 1998, p70).

We can establish stronger properties of $\tilde{x}(b, m)$. If the constraint $x \leq m$ binds, then $\tilde{x}(b, m) = m$, in which case $\tilde{x}(b, m)$ is strictly increasing in m and independent of b . If the constraint $x \leq m$ does not bind, the derivative of $t(x, b, m)$ with respect to b is strictly increasing in x , in which case $\tilde{x}(b, m)$ is strictly increasing in b (see Edlin and Shannon, 1998).

Denote $t^*(b, m) = t(\tilde{x}(b, m), b, m)$ as in (3.11). From the above proof, $t^*(b, m)$ is supermodular in (b, m) . Because $t(x, b, m)$ strictly decreases in b for any given (x, m) , its maximized value with respect to x , i.e., $t^*(b, m)$, is strictly decreasing in b . To examine the dependence of $t^*(b, m)$ on m , take arbitrary m_1 and m_2 , with $m_2 \geq m_1$. We have:

$$t(x, b, m_2) - t(x, b, m_1) = \beta [V(m_1) - V(m_1 - x)] - \beta [V(m_2) - V(m_2 - x)] \geq 0,$$

where the inequality follows from concavity of V . Since the above result holds for all (x, b) , then

$$t^*(b, m_1) = t(\tilde{x}(b, m_1), b, m_1) \leq t(\tilde{x}(b, m_1), b, m_2) \leq t(\tilde{x}(b, m_2), b, m_2) = t^*(b, m_2).$$

Note that for the second inequality we have used the fact that $\tilde{x}(b, m_1)$ is feasible in the problem $\max_{x \leq m_2} t(x, b, m_2)$. Thus, $t^*(b, m)$ increases in m .

Now consider the optimal choice of b , denoted as $b^*(m) = \arg \max_{b \in [0, 1]} f(b, m)$, where

$$f(b, m) = F(\tilde{x}(b, m), b, m) = b t^*(b, m).$$

We show that f is supermodular in (b, m) , and so the optimal choice of b is increasing in m . Take arbitrary $b_1, b_2 \in [0, 1]$, with $b_2 > b_1$, and arbitrary $m_1, m_2 \in [0, \bar{m}]$, with $m_2 > m_1$. Compute:

$$\begin{aligned} & [f(b_2, m_2) - f(b_1, m_2)] - [f(b_2, m_1) - f(b_1, m_1)] \\ &= b_2 [t^*(b_2, m_2) - t^*(b_1, m_2)] + t^*(b_1, m_1) - t^*(b_2, m_1) \\ & \quad + (b_2 - b_1) [t^*(b_1, m_2) - t^*(b_1, m_1)]. \end{aligned}$$

Because $t^*(b, m)$ is supermodular in (b, m) , the first difference is positive; because $t^*(b, m)$ is increasing in m , the second difference is also positive. Thus, $f(b, m)$ is supermodular in (b, m) on $[0, 1] \times [0, \bar{m}]$. As a result, $b^*(m)$ is increasing in m . Since $\tilde{x}(b, m)$ is increasing in (b, m) , the optimal choice of x , given by $x^*(m) = \tilde{x}(b^*(m), m)$, is increasing in m .

Because the solutions, $b^*(m)$ and $x^*(m)$, are unique for each m and increasing in m , Lemma F.1 in Gonzalez and Shi (2007) shows that the solutions are continuous in m .

Because $t^*(b, m)$ is increasing in m for any given b , as proven above, $f(b, m)$ is increasing in m for any given b . This feature implies that $f(b^*(m), m)$ is increasing in m , and hence that the function $W_b(m)$, given by $W_b(m) = f(b^*(m), m) + \beta V(m)$, is increasing in m . We prove that $W_b(m)$ is strictly increasing (in the case $b^*(m) > 0$) in Lemma B.1 below.

To prove that $q^*(m)$ is increasing in m , temporarily denote $y = (m - x)\omega + \psi(q)$. We formulate a buyer's decision equivalently as choosing (q, y) . From the definition of y and (3.2), we can express

$$m - x = \frac{y - \psi(q)}{\omega}, \quad b = \mu^{-1} \left(\frac{k}{\omega m - y} \right).$$

Because $b \geq 0$, the relevant domain of y is $[0, \omega m - k]$. The relevant domain of q is $[0, \psi^{-1}(y)]$. A buyer chooses $(q, y) \in [0, \psi^{-1}(y)] \times [0, \omega m - k]$ to solve:

$$\max_{(q, y)} \mu^{-1} \left(\frac{k}{\omega m - y} \right) \left[U(q) + \beta V \left(\frac{y - \psi(q)}{\omega} \right) - \beta V(m) \right].$$

We can divide this problem into two steps: first solve q for any given (y, m) and then solve y .

For any given (y, m) , the optimal choice of q solves

$$J(y) = \max_{0 \leq q \leq \psi^{-1}(y)} \left[U(q) + \beta V \left(\frac{y - \psi(q)}{\omega} \right) \right].$$

Denote the solution for q as $\tilde{q}(y)$. Note that q and J do not depend on m for any given y . It is easy to see that the objective function above is supermodular in (q, y) , and strictly so if V is strictly concave. Also, the choice set, $[0, \psi^{-1}(y)]$, is increasing in y . Thus, $\tilde{q}(y)$ and $J(y)$ increase in y .

The optimal choice of y is $y^*(m) = \arg \max_{0 \leq y \leq \omega m - k} B(y, m)$, where

$$B(y, m) = \mu^{-1} \left(\frac{k}{\omega m - y} \right) [J(y) - \beta V(m)].$$

Note that if $J(y) < \beta V(m)$, the buyer can choose $y = \omega m - k$ to obtain $B = 0$. Thus, focus on the case where $J(y) \geq \beta V(m)$. Under Assumption 1, it can be verified that the function $\mu^{-1}(\frac{k}{\omega m - y})$ strictly increases in m , strictly decreases in y , and is strictly supermodular in (y, m) . Thus, for arbitrary $y_2 > y_1$ and $m_2 > m_1$, we have:

$$\begin{aligned} & B(y_2, m_2) - B(y_1, m_2) - B(y_2, m_1) + B(y_1, m_1) \\ = & \left[\mu^{-1} \left(\frac{k}{\omega m_2 - y_2} \right) - \mu^{-1} \left(\frac{k}{\omega m_1 - y_2} \right) \right] [J(y_2) - J(y_1)] \\ & + \left[\mu^{-1} \left(\frac{k}{\omega m_2 - y_1} \right) - \mu^{-1} \left(\frac{k}{\omega m_2 - y_2} \right) \right] [\beta V(m_2) - \beta V(m_1)] \\ & + \left[\mu^{-1} \left(\frac{k}{\omega m_2 - y_2} \right) - \mu^{-1} \left(\frac{k}{\omega m_2 - y_1} \right) - \mu^{-1} \left(\frac{k}{\omega m_1 - y_2} \right) + \mu^{-1} \left(\frac{k}{\omega m_1 - y_1} \right) \right] [J(y_1) - \beta V(m_1)]. \end{aligned}$$

The first term on the RHS is positive because $J(y)$ increases in y and $\mu^{-1}(\frac{k}{\omega m - y})$ increases in m . The second term is positive because $\mu^{-1}(\frac{k}{\omega m - y})$ decreases in y and $V(m)$ increases in m . The third term is strictly positive because $\mu^{-1}(\frac{k}{\omega m - y})$ is strictly supermodular in (y, m) . Therefore, $B(y, m)$ is strictly supermodular in (y, m) . Also, the choice set $[0, \omega m - k]$ is increasing in m . As a result, the solution $y^*(m)$ increases in m . Since $\tilde{q}(y)$ increases in y , then $q^*(m) = \tilde{q}(y^*(m))$ increases in m .

To show that $\phi(m)$ is increasing, we formulate a buyer's problem by letting the choices be (ϕ, y) , where y is defined as $y = \phi\omega + \psi(q)$. From the definition of y and (3.2), we can express

$$q = \psi^{-1}(y - \phi\omega), \quad b = \mu^{-1}\left(\frac{k}{\omega m - y}\right).$$

The relevant domain of ϕ is $[0, \min\{m, y/\omega\}]$, and of y is $[0, \omega m - k]$. A buyer solves:

$$\max_{(\phi, y)} \mu^{-1}\left(\frac{k}{\omega m - y}\right) \left[U\left(\psi^{-1}(y - \phi\omega)\right) + \beta V(\phi) - \beta V(m) \right]. \quad (\text{B.1})$$

As in the above formulation where the choices are (q, y) , we can divide the maximization problem into two steps. First, for any given y , the optimal choice of ϕ solves:

$$J(y) = \max_{\phi \geq 0} \left[U\left(\psi^{-1}(y - \phi\omega)\right) + \beta V(\phi) \right]. \quad (\text{B.2})$$

Note that we have written the constraint on ϕ as $\phi \geq 0$, instead of $\phi \in [0, \min\{m, y/\omega\}]$. The optimal choice satisfies $\phi < m$, because $\phi = m$ implies $x = 0$ which is not optimal (in the case with $b > 0$). Also, $\phi < y/\omega$ under the assumptions $\psi(0) = 0$ and $U'(0) = \infty$. Denote the solution for ϕ as $\tilde{\phi}(y)$. Second, the optimal choice of y solves

$$W_b(m) - \beta V(m) = \max_{0 \leq y \leq \omega m - k} \mu^{-1}\left(\frac{k}{\omega m - y}\right) [J(y) - \beta V(m)]. \quad (\text{B.3})$$

Similar to the procedure used in the above formulation of the problem where the choices are (q, y) , we can show that $\phi^*(m)$ and $y^*(m)$ increase in m . **QED**

Lemma B.1. *Consider the formulation of a buyer's problem, (B.1), where the choices are ϕ and $y = \phi\omega + \psi(q)$. Assume $b^*(m) > 0$. The following results hold: (i) $J'(y)$ exists and is given by*

$$J'(y) = \frac{U'(\tilde{q}(y))}{\psi'(\tilde{q}(y))} > 0; \quad (\text{B.4})$$

(ii) *The solution for y in (B.3) is unique and given by the following first-order condition:*

$$0 \geq J(y^*) - \beta V(m) + \frac{U'(\tilde{q}(y^*))}{\psi'(\tilde{q}(y^*))} \frac{k\mu'b^*(m)}{\mu^2} \quad \text{and} \quad y^* \leq \omega m - k, \quad (\text{B.5})$$

where the two inequalities hold with complementary slackness; (iii) $W_b(m)$ is differentiable at m if and only if $V(m)$ is so; (iv) $W_b(m)$ is strictly increasing if $\omega > 0$ (and $b^(m) > 0$).*

Proof. To prove part (i), let y and y' be arbitrary levels in $[0, \omega m - k]$. Because the constraint on the choice ϕ is $\phi \geq 0$, which does not depend on y , the choice $\tilde{\phi}(y)$ is feasible with the level y' . Thus, $J(y') \geq U(\psi^{-1}(y' - \tilde{\phi}(y)\omega)) + \beta V(\tilde{\phi}(y))$, which implies

$$J(y') - J(y) \geq U\left(\psi^{-1}(y' - \tilde{\phi}(y)\omega)\right) - U\left(\psi^{-1}(y - \tilde{\phi}(y)\omega)\right).$$

Similarly, because the choice $\tilde{\phi}(y')$ is feasible with the level y , $J(y) \geq U(\psi^{-1}(y - \tilde{\phi}(y')\omega)) + \beta V(\tilde{\phi}(y'))$, and

$$J(y') - J(y) \leq U\left(\psi^{-1}\left(y' - \tilde{\phi}(y')\omega\right)\right) - U\left(\psi^{-1}\left(y - \tilde{\phi}(y')\omega\right)\right).$$

For $y' > y$, dividing the two inequalities by $(y' - y)$ and taking the limit $y' \downarrow y$, we compute the right-hand derivative of J at y as $J'(y^+) = \frac{U'(\tilde{q}(y))}{\psi'(\tilde{q}(y))}$, where $\tilde{q}(y) \equiv \psi^{-1}(y - \tilde{\phi}(y)\omega)$. For $y' < y$, dividing the above two inequalities by $(y - y')$ and taking the limit $y' \uparrow y$, we obtain $J'(y^-) = \frac{U'(\tilde{q}(y))}{\psi'(\tilde{q}(y))}$. Therefore, $J'(y)$ exists and is given by (B.4).

For part (ii), we prove first that the objective function in (B.3) is strictly concave in y . For this result, recall that $\tilde{q}(y)$ is an increasing function, as shown in the proof of Lemma 3.2 using (q, y) as the choices. This result and (B.4) imply that $J'(y)$ is decreasing, i.e., that $J(y)$ is concave. Because $J(y)$ is increasing and concave, and $\mu^{-1}(\frac{k}{\omega m - y})$ is strictly decreasing and strictly concave in y , the objective function in (B.3) is strictly concave in y .

Strict concavity of the objective function implies that the solution for y is unique. Also, because the objective function is differentiable in y , the optimal choice of y is given by the first-order condition. Deriving the first-order condition, substituting $J'(y)$ from (B.4), and substituting $\mu^{-1}(\frac{k}{\omega m - y^*}) = b^*(m)$, we obtain (B.5).

For part (iii), consider an arbitrary m that satisfies the maintained hypothesis $b^*(m) > 0$. Note that $b^*(m) > 0$ implies $y^*(m) < \omega m - k$. Because $y^*(m) < \omega m - k$ and $y^*(m)$ is a continuous function, there exists $\varepsilon > 0$ such that $y^*(m + \varepsilon) < \omega m - k$ and $y^*(m) < \omega(m - \varepsilon) - k$. Consider the neighborhood $O(m) = (m - \varepsilon, m + \varepsilon)$. For any $m' \in O(m)$, the choice $y^*(m')$ is feasible in the problem when the money balance is m , and the choice $y^*(m)$ is feasible in the problem when the balance is m' . Using the above procedure that proved the differentiability of $J(y)$, but applying to (B.3), we can derive the formulas of the left-hand and right-hand derivatives of W_b at any m such that $b^*(m) > 0$. Substituting the first-order condition of y^* , the analysis yields the following generalized version of the envelope theorem:

$$W'_b(m^+) = b^*(m) [\omega J'(y^*) - \beta V'(m^+)] + \beta V'(m^+), \quad (\text{B.6})$$

$$W'_b(m^-) = b^*(m) [\omega J'(y^*) - \beta V'(m^+)] + \beta V'(m^-). \quad (\text{B.7})$$

Here, we have used the fact that because V is a concave function, its left-hand and right-hand derivatives exist (see Royden, 1988, pp113-114). The above equations clearly show that $W_b(m)$ is differentiable at m if and only if $V(m)$ is so.

Finally, we prove part (iv). Since V is concave and increasing, $V'(m^-) \geq V'(m^+) \geq 0$. Since $b^* \leq 1$ and $J'(y^*(m)) > 0$, (B.6) and (B.7) imply that $W'_b(m^-) \geq W'_b(m^+) \geq b^*(m)\omega J'(y^*) > 0$,

where the last inequality has used the maintained assumptions $b^*(m) > 0$ and $\omega > 0$. Therefore, $W_b(m)$ is strictly increasing under these assumptions. **QED**

C. Proof of Lemma 3.3

Maintain the assumption $b^*(m) > 0$ in this proof. We have already proven part (i) of the lemma in Lemma B.2. For future use, compute $Q_1(x, b) = \omega/\psi'(q)$ and rewrite (B.4) as

$$J'(y^*) = \frac{U'(q^*)}{\psi'(q^*)} = \frac{1}{\omega} u_1(x^*, b^*).$$

Substituting this result for $J'(y^*)$, we rewrite (B.6) and (B.7) as

$$W'_b(m^+) = b^*(m) u_1(x^*(m), b^*(m)) + \beta(1 - b^*(m)) V'(m^+) \quad (\text{C.1})$$

$$W'_b(m^-) = b^*(m) u_1(x^*(m), b^*(m)) + \beta(1 - b^*(m)) V'(m^-). \quad (\text{C.2})$$

To prove part (ii), take any $m > 0$ such that $W_b(m) = V(m)$. The lottery in (3.17) implies that $W_b(m') \leq V(m')$ for all m' . Because W_b and V are continuous functions, it is straightforward to verify that $W'_b(m^+) \leq V'(m^+)$ and $W'_b(m^-) \geq V'(m^-)$ at the particular level m . These inequalities, together with (C.1) and (C.2), imply:

$$V'(m^-) \leq \frac{b^*(m)}{1 - \beta[1 - b^*(m)]} u_1(x^*(m), b^*(m)) \leq V'(m^+).$$

On the other hand, concavity of V implies $V'(m^-) \geq V'(m^+)$. Thus, $V'(m^-) = V'(m^+)$, and the common derivative is given by the right-hand side of (3.13). By (C.1) and (C.2), $W'_b(m)$ exists and is also given by the right-hand side of (3.13).

To prove part (iii), consider any arbitrary m such that $\phi(m) > 0$ (i.e., $x^*(m) < m$) and consider the original formulation of a buyer's maximization problem, (3.7), where the choices are (x, b) . Since $x^*(m) < m$, a procedure similar to the derivation of $J'(y)$ in the previous proof but applied to (3.7) yields

$$W'_b(m^+) = \beta [b^*(m) V'(\phi^+(m)) + (1 - b^*(m)) V'(m^+)]$$

$$W'_b(m^-) = \beta [b^*(m) V'(\phi^-(m)) + (1 - b^*(m)) V'(m^-)],$$

where $\phi^+(m) = m^+ - x^*(m)$ and $\phi^-(m) = m^- - x^*(m)$. Comparing these equations with (C.1) and (C.2) yields $V'(\phi^+(m)) = V'(\phi^-(m))$. The common derivative is given by (3.14).

For part (iv), let m_1 be an arbitrary balance that satisfies the hypotheses that $W_b(m_1) = V(m_1)$, $\phi(m_1) > 0$ and $b^*(m_1) > 0$. We first prove that $b^*(m_1)$ is strictly increasing at m_1 . To this aim, let m_2 be sufficiently close to m_1 so that $\phi(m_2) > 0$ and $b^*(m_2) > 0$ (which is possible

because $\phi(m)$ and $b(m)$ are continuous functions). Shorten the notation $(x^*(m_i), b^*(m_i), \phi(m_i))$ to (x_i^*, b_i^*, ϕ_i) , where $x_i^* = m_i - \phi_i$ and $i = 1, 2$. Because $b_i^* > 0$, the first-order condition for b , i.e., the condition (3.12), yields:

$$u(x_i^*, b_i^*) + \beta[V(\phi_i) - V(m_i)] + b_i^* u_2(x_i^*, b_i^*) = 0.$$

Moreover, because m_1 satisfies the hypotheses for both (3.13) and (3.14), subtracting the two equations yields:

$$V'(\phi_1) - V'(m_1) = \frac{1 - \beta}{\beta[1 - \beta(1 - b_1^*)]} u_1(x_1^*, b_1^*) > 0.$$

This result shows that V must be strictly concave in some sections of $[\phi_1, m_1]$.

Since the proofs for strict monotonicity of $b^*(m_1)$ at m_1 are similar in the cases $m_2 > m_1$ and $m_2 < m_1$, let us consider only the case where $m_2 > m_1$. In this case, we want to prove that $b_2^* > b_1^*$. By Lemma 3.2, $x_2^* \geq x_1^*$, $b_2^* \geq b_1^*$ and $\phi_2 \geq \phi_1$. Subtract the first-order conditions for b_1^* and b_2^* (see above), and re-organize:

$$\begin{aligned} 0 = & \{u(x_2^*, b_2^*) + \beta[V(m_2 - x_2^*) - V(m_2)]\} - \{u(x_1^*, b_2^*) + \beta[V(m_2 - x_1^*) - V(m_2)]\} \\ & + \beta\{V(m_1) - V(m_1 - x_1^*) + V(m_2 - x_1^*) - V(m_2)\} \\ & + u(x_1^*, b_2^*) - u(x_1^*, b_1^*) + b_2^* u_2(x_1^*, b_2^*) - b_1^* u_2(x_1^*, b_1^*). \end{aligned}$$

The first difference on the right-hand side of the above equation is equal to $[t(x_2^*, b_2^*, m_2) - t(x_1^*, b_2^*, m_2)]$, where t is defined by (3.10). This difference is greater than or equal to zero, because x_2^* maximizes $t(x, b_2^*, m_2)$ and because x_1^* is a feasible choice of x in such a maximization problem (as $x_1^* \leq x_2^*$). The second difference is strictly positive because V is strictly concave in some sections of $[\phi_1, m_1] \subset [\phi_1, m_2]$. For the third difference on the right-hand side of the above equation, note that $u_2(x_2^*, b_2^*) \geq u_2(x_1^*, b_2^*)$, because $u_{12} > 0$ and $x_2^* \geq x_1^*$. Thus,

$$0 > u(x_1^*, b_2^*) - u(x_1^*, b_1^*) + b_2^* u_2(x_1^*, b_2^*) - b_1^* u_2(x_1^*, b_1^*).$$

The right-hand side of the above inequality is a strictly decreasing function of b_2^* , and it is equal to 0 when $b_2^* = b_1^*$. Thus, $b_2^* > b_1^*$. Therefore, $b^*(m_1)$ is strictly increasing at m_1 .

Let us continue to prove the rest of part (iv). Since (3.13) and (3.14) both hold for $m = m_1$, we can combine the two equations to obtain:

$$V'(\phi(m_1)) = V'(m_1) \left[\frac{1 - \beta}{\beta b^*(m_1)} + 1 \right].$$

Because $b^*(m_1)$ is strictly increasing at m_1 , the right-hand side above is strictly decreasing at m_1 . In this case, the above equation shows that V must be strictly concave at $\phi(m_1)$ and that $\phi(m_1)$ must be strictly increasing at m_1 . Strictly concavity of V at $\phi(m_1)$ implies that $W_b(\phi(m_1)) = V(\phi(m_1))$. To see why, suppose that the inequality does not hold. Because the lottery in (3.17)

implies $W_b(m') \leq V(m')$ for all m' , the supposition yields $W_b(\phi(m_1)) < V(\phi(m_1))$. In this case, V around $\phi(m_1)$ must be a linear segment generated by the lottery in (3.17), which contradicts strict concavity of V at $\phi(m_1)$.

Finally, we prove part (v). The definition of m^* in (3.4) implies that $V'(m^{*-}) \geq \omega/\beta \geq V'(m^{*+})$. It suffices to show that the case $V'(m^{*-}) > V'(m^{*+})$ never occurs. Suppose that $V'(m^{*-}) > V'(m^{*+})$. Since V is strictly concave at m^* under this supposition, it cannot be the linear segment generated by the lottery described later. Thus, $V(m^*) = W_b(m^*)$. By Lemma 3.3, $V'(m^*)$ exists, which contradicts the supposition that $V'(m^{*-}) > V'(m^{*+})$. **QED**

D. Proof of Theorem 3.4

It is easy to verify that the mapping $\mathcal{F}$, defined by the right-hand side of (3.17), satisfies Blackwell's sufficient conditions for contraction mapping; thus, $\mathcal{F}$ has a unique fixed point V that is continuous and bounded (see Stokey and Lucas with Prescott, 1989). With Lemmas 3.1 and 3.2, it is clear that $\mathcal{F}$ maps increasing functions into increasing functions. Thus, the fixed point of $\mathcal{F}$ is an increasing function. Moreover, since $\tilde{V}$ is a continuous function defined on a closed interval, two-point lotteries make it a concave function (see Appendix F in Menzio and Shi, 2009, for the proof). Thus, $\mathcal{F}$ maps concave functions into concave functions, and the fixed point V is concave.

To verify the other properties of V stated in the theorem, assume $\omega > 0$. For part (i), note that $V(m) \geq 0$ for all $m \geq 0$, because an individual can always choose not to trade. To show that $V(m) > 0$ for all $m > 0$, it suffices to show that there is an $m_1 \in (0, \infty)$ such that $V(m_1) > 0$. (With this result, the availability of lotteries implies strict positivity of V for all $m > 0$.) Suppose, contrary to the stated result, that $V(m) = 0$ for all $m < \infty$. Consider a buyer who holds an amount of money, $m_1 = \frac{k}{\omega} + \varepsilon_1$, where $\varepsilon_1 > 0$ is an arbitrarily small number. Let $q_1 \in (0, \psi^{-1}(\omega\varepsilon_1))$ and denote $b_1 = \mu^{-1}(\frac{k}{\omega m_1 - \psi(q_1)})$. It is clear that $b_1 > \mu^{-1}(\frac{k}{\omega m_1 - \omega\varepsilon_1}) = \mu^{-1}(1) = 0$. By trading in submarket (m_1, q_1) , the buyer obtains the following expected value:

$$b_1 [u(q_1) + \beta V(0) - \beta V(m_1)] + \beta V(m_1) = b_1 u(q_1) > 0,$$

where the equality follows from that supposition that $V(m) = 0$ for all $m < \infty$. Since the above trading strategy is feasible to the buyer, $V(m_1) \geq W_b(m_1) \geq b_1 u(q_1) > 0$. This result contradicts the supposition that $V(m) = 0$ for all $m < \infty$.

For part (ii), note that for a seller with any money balance m , the choice of not contributing to a firm yields the value $\beta V(m)$. Because this choice is always feasible, $W_s(m) \geq \beta V(m)$ for all m . For a buyer who holds $m \leq k/\omega$, all feasible choices of $x \leq m$ yield $Q(x, b) = 0$. Thus, for such a buyer, the optimal choice of b is $b^*(m) = 0$, and the value is $W_b(m) = \beta V(m) \leq W_s(m)$.

It is clear that $V(0) = \tilde{V}(0) = W_s(0)$.

For part (iii), we prove first that there is some $m' \in (0, \infty)$ such that $W_b(m') > W_s(m')$. Suppose, to the contrary, that $W_b(m) \leq W_s(m)$ for all $m \in (0, \infty)$. Then, $\tilde{V}(m) = W_s(m)$ for all m . Recall that $W_s(m) = \beta V(m^*) + \omega(m - m^*)$ for all $m \in [0, m^*]$ and $W_s(m) = \beta V(m)$ for $m > m^*$. Since $\beta V'(m^*) = \omega$, then $W_s(m)$ is concave. So is $\tilde{V}(m)$. This result and the supposition imply that $V(m) = \tilde{V}(m) = W_s(m)$ for all m . Substituting $W_s(m)$, we have $V(m) = 0$ for all $m > m^*$, which contradicts part (i).

Continue the proof of part (iii). For an individual with a money balance $m \in (0, k/\omega)$, the lottery with $L_1 = 0$ and $L_2 = m'$ yields a value higher than $\tilde{V}(m)$, where m' is described above. Thus, these individuals will participate in lotteries. However, m' may not be necessarily be the optimal winning prize of the lottery for these individuals. Let m_0 be such an optimal winning prize (defined by (3.18)). Clearly, $m_0 > k/\omega > 0$, $V(m_0) = \tilde{V}(m_0)$, and $V(m) \geq \tilde{V}(m)$ for all $m \in [0, m_0]$.

We prove the result that $V(m) > W_s(m)$ for all $m \in (0, m_0]$. Note that this result implies $V(m_0) = W_b(m_0)$ (since $V(m_0) = \tilde{V}(m_0)$) which, in turn, implies that $V'(m_0) = W'_b(m_0)$ (see Lemma 3.3). Suppose that $V(m_1) = W_s(m_1)$ at some $m_1 \in (0, m_0]$, contrary to the stated result. If $m_1 < m^*$, concavity of V implies $V'(m_1) \geq V'(m^*) = \omega/\beta$. In this case,

$$V(m_1) \geq V(0) + m_1\omega/\beta \geq W_s(0) + m_1\omega/\beta > W_s(0) + m_1\omega = W_s(m_1).$$

The last inequality has used the maintained assumption that $\omega > 0$. If $m_1 \geq m^*$, then $V(m_1) > \beta V(m_1) = W_s(m_1)$. In both cases, the result contradicts the supposition that $V(m_1) = W_s(m_1)$. Thus, $V(m) > W_s(m)$ for all $m \in (0, m_0]$.

For part (iv), suppose $b^*(m_0) = 0$, contrary to the stated result. In this case, $W_b(m_0) = \beta V(m_0)$. Since $V(m_0) = W_b(m_0)$ by part (iv) above, then $V(m_0) = 0$, which contradicts the result that $V(m_0) \geq W_s(m_0) > 0$. Thus, it must be true that $b^*(m_0) > 0$. Since $V(m_0) = W_b(m_0)$, (3.13) holds for $m = m_0$. With $b^*(m_0) > 0$, (3.13) implies $V'(m_0) < u_1(x^*(m_0), b^*(m_0))/\beta$. Since $V(m)$ is linear for $m \in [0, m_0]$, then $V'(\phi(m_0)) = V'(m_0) < u_1(x^*(m_0), b^*(m_0))/\beta$. If $\phi(m_0) > 0$, then (3.14) holds for $m = m_0$, which yields the contradiction that $V'(\phi(m_0)) = u_1(x^*(m_0), b^*(m_0))/\beta$. Thus, it must be true that $\phi(m_0) = 0$.

For part (v), we first prove that $V(0) > 0$ implies $V'(m_0) > V'(m^*)$ which, by concavity of V , implies $m_0 < m^*$. Suppose $V(0) > 0$. Recall that the definition of m_0 implies $V(m) = V(0) + V'(m_0)m$ for all $m \in [0, m_0]$. Thus, the strict inequality, $V'(m_0) < V'(m^*)$, can never hold: with the strict inequality, concavity of V would imply $m_0 > m^*$, which would imply the contradiction, $V'(m^*) = V'(m_0)$. To prove the result $V'(m_0) > V'(m^*)$, suppose that $V'(m_0) = V'(m^*)$, to the

contrary. In this case, both $V(m_0)$ and $V(m^*)$ lie on the same tangent line (or its extension) that connects $V(0)$ and $V(m_0)$. So, $V'(m^*) = [V(m^*) - V(0)]/m^*$. Substituting $V(m^*)$ from the fact that $V(0) = W_s(0) = \beta V(m^*) - \omega m^*$, we obtain

$$V'(m^*) = \omega/\beta + (1/\beta - 1)V(0)/m^* > \omega/\beta.$$

This contradicts the result that $V'(m^*) = \omega/\beta$ (see (3.15) in Lemma 3.3). Thus, if $V(0) > 0$, then $V'(m_0) > V'(m^*)$ and $m_0 < m^*$.

Conversely, $V(0) = 0$ implies that $V(m) = \omega m/\beta$ for all $m \in [0, \max\{m_0, m^*\}]$. To prove this result, suppose $V(0) = 0$. Because $0 = \beta V(0) \leq W_s(0) \leq V(0) = 0$, then $W_s(0) = 0$, i.e., $\beta V(m^*) = \omega m^*$. Since m^* maximizes the function $[\beta V(m) - \omega m]$, then $\beta V(m) - \omega m \leq \beta V(m^*) - \omega m^* = 0$ for all $m \in [0, m^*]$. On the other hand, $\beta V(m) - \omega m \geq 0$ for all $m \in [0, m^*]$, because the function $[\beta V(m) - \omega m]$ is continuous and concave for all $m \in [0, m^*]$ and because the function is equal to 0 at both $m = 0$ and $m = m^*$. Hence, the supposition $V(0) = 0$ yields $V(m) = \omega m/\beta$ for all $m \in [0, m^*]$. In this case, $V(m^*) = V(0) + V'(m^*)m^*$; that is, m^* satisfies the definition of m_0 . Thus, $V(m_0)$ lies on the tangent line (or its extension) that connects $V(0)$ and $V(m^*)$. Hence, $V'(m_0) = \omega/\beta$, and $V(m) = \omega m/\beta$ for all $m \in [0, \max\{m_0, m^*\}]$. **QED**

E. Proof of Lemma 4.1

For part (i), note that the choice $b = 0$ is always feasible to a buyer with a money balance m^* , and it yields the value $\beta V(m^*)$ for a buyer. Thus, $W_b(m^*) \geq \beta V(m^*) = W_s(m^*)$, and the inequality is strict if the optimal choice satisfies $b^*(m^*) > 0$. Suppose $b^*(m^*) = 0$, contrary to the desired result. Since $b^*(\cdot)$ is an increasing function (see Lemma 3.2), then $b^*(m) = 0$ for all $m \in [0, m^*]$, and so $W_b(m) = \beta V(m)$ for all $m \in [0, m^*]$. Because $W_s(m) \geq \beta V(m)$ for all m (as the choice $z = 0$ is always feasible to a seller), $\tilde{V}(m) = W_s(m)$ in this case for all $m \in [0, m^*]$. Since $W_s(\cdot)$ is a concave function (see Lemma 3.1), then $V(m) = W_s(m)$ for all $m \in [0, m^*]$. In particular, $V(m^*) = W_s(m^*) = \beta V(m^*)$. Since this result implies $0 = V(m^*) = \omega m^*$, which contradicts $\omega > 0$, it must be the case that $b^*(m^*) > 0$.

To prove the remainder of the lemma, let us characterize m_0 in more detail. Denote $a = \frac{\beta}{\omega} V'(m_0)$ so that $V'(m_0) = a\omega/\beta$. The lottery implies that $V(m) = V(0) + a\omega m/\beta$ for all $m \in [0, m_0]$. By Theorem 3.4, $b^*(m_0) > 0$, $\phi(m_0) = 0$, and $W_b(m_0) = V(m_0)$. With these results, the Bellman equation, (3.7), yields:

$$(1 - \beta)[V(0) + a\omega m_0/\beta] = \max_{b \in [0, 1]} b[u(m_0, b) - a\omega m_0].$$

Since $b^*(m_0) > 0$, the optimal choice of b satisfies the following first-order condition:

$$u(m_0, b) - a\omega m_0 + bu_2(m_0, b) = 0. \quad (\text{E.1})$$

Since $W_b(m_0) = V(m_0)$, Lemma 3.3 implies $W'_b(m_0) = V'(m_0)$, which can be rewritten as

$$(1 - \beta)a\omega/\beta = b[u_1(m_0, b) - a\omega]. \quad (\text{E.2})$$

Multiply (E.2) by m_0 and subtract from the Bellman equation above to obtain:

$$(1 - \beta)V(0) = b[u(m_0, b) - m_0 u_1(m_0, b)]. \quad (\text{E.3})$$

If $V(0) > 0$, then part (ii) of Lemma 4.1 is implied by part (v) of Theorem 3.4. For $V(0) > 0$, it suffices to find the condition so that if $V(0) = 0$, then (E.1) – (E.3) cannot all hold at the same time. Suppose $V(0) = 0$. Part (v) of Theorem 3.4 implies that $a = 1$ in this case. Substitute $a = 1$ and $V(0) = 0$ into (E.1) – (E.3). Using $u(m, b) = U(q)$, where $q = \psi^{-1}(\omega m - \frac{k}{\mu(b)})$, we can write (E.3) in the current case as $\omega m_0 = U(q)\psi'(q)/U'(q)$. Substituting this result for ωm_0 into the definition of q , we solve $b = b(q)$, where

$$b(q) = \mu^{-1} \left(k \left[\frac{U(q)\psi'(q)}{U'(q)} - 1 \right]^{-1} \right).$$

Substituting $a = 1$ and the results for ωm_0 and b into (E.1), we obtain:

$$U(q) \left[1 - \frac{\psi'(q)}{U'(q)} \right] + b(q) \frac{U'(q)}{\psi'(q)} \frac{k\mu'(b(q))}{[\mu(b(q))]^2} = 0.$$

The only variable in this equation is q . Let q_0 be the solution to this equation, and notice that q_0 does not depend on ω or β . So far, we have used (E.1) and (E.3). For $V(0) > 0$, it suffices to ensure that (E.2) does not hold with the quantities solved above. Substituting b , ωm_0 and q , we can deduce that (E.2) (with $a = 1$) does not hold if $\beta \neq \beta_0$, where

$$\beta_0 \equiv \left[1 + b(q_0) \left(\frac{U'(q_0)}{\psi'(q_0)} - 1 \right) \right]^{-1}. \quad (\text{E.4})$$

The proof of part (iii) of Lemma 4.1 is to be completed.

For part (iv), note that part (iii) implies that the conditions in part (iv) of Lemma 3.3 are satisfied at $m = m^*$. Thus, $V(\phi(m^*)) = W_b(\phi(m^*))$, and V is strictly concave at $\phi(m^*)$. These features imply that if $\phi^2(m^*) > 0$, then the conditions in part (iv) of Lemma 3.3 are satisfied at $m = \phi(m^*)$, in which case $V(\phi^2(m^*)) = W_b(\phi^2(m^*))$ and V is strictly concave at $\phi^2(m^*)$. Continuing this process, we conclude that if $\phi^i(m^*) > 0$, then $V(\phi^i(m^*)) = W_b(\phi^i(m^*))$ and V is strictly concave at $\phi^i(m^*)$. **QED**

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