

Risk sensitive allocations with multiple goods in international finance

Existence, survivorship, and dynamics*

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Abstract

We characterize the equilibrium of a complete markets economy with multiple agents displaying a preference for the timing of the resolution of uncertainty. Utilities are defined over an aggregate of two goods. We provide conditions under which the solution of the planner's problem exists and it features a non-degenerate invariant distribution of Pareto weights. By interpreting the model as a two country, representative consumers economy, we document that the risk-sharing scheme produces a non trivial dynamics of net exports and it is also capable of explaining the tendency of high interest rate currencies to appreciate.

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1 Introduction

In an economy with multiple agents with time additive preferences defined over a single consumption good, Pareto optimal allocations are given by a time-invariant function of aggregate endowment. Following Lucas and Stokey (1984) and Kan (1995), Anderson (2005) studies the dynamics of allocations when multiple agents have risk-sensitive preferences. These preferences are non-time additive and they allow agents to be risk-averse in future utility in addition to future consumption. As a result, optimal allocations are a function not only of aggregate endowment, but also of the distribution of wealth.¹

Anderson (2005) shows that this distribution evolves over time, provided that consumers have a different degree of risk-sensitivity. However, this result comes at the price that the wealth of the most risk-sensitive agents converges to zero over time. Alternatively, if all agents are identical, the Pareto optimal allocations would be equivalent to the case of time additive preferences.

In this paper we characterize the equilibrium of an economy with multiple agents with risk-sensitive preferences defined over two goods. We show that an interesting dynamics of Pareto optimal allocations is obtained even in the case in which all agents share the same risk-sensitivity parameter, provided that they display a different degree of preference for one of the two goods. Furthermore, we provide conditions under which a non-degenerate limiting distribution of Pareto weights exists, that is the case in which every agent in the economy survives.

Focusing on the case of a consumption aggregate of multiple goods is important in many economic applications. In a closed economy, we may think of agents displaying different elasticities of substitution between the goods produced by different firms, even if they do not differ in the degree of aversion toward intertemporal gambles. In an open economy, it is common to assume that different countries have a bias toward the consumption of different goods.

¹Lucas and Stokey (1984) characterize efficient allocations under certainty in discrete time. Epstein (1987) works with the certainty case in continuous time. Duffie, Geoffard, and Skiadas (1994) have similar results under uncertainty in continuous time. Dumas, Uppal, and Wang (2000) extend Geoffard (1996) results, by applying the variational utility concept to a stochastic environment with one good.

We follow Ma (1993) and Kan (1995) in providing a recursive method to solve the social planner's problem and we provide the conditions under which a non-degenerate distribution of Pareto weights exists. Furthermore, in a simple i.i.d. case we are able to characterize both theoretically and numerically the properties of the optimal planner's policy.

Specifically we study an economy with two agents, that we call *home* and *foreign*. A number of important results emerge. First, we show that the conditional means, variances, and covariances of consumption growth and utilities are endogenously time-varying with the wealth distribution. This is particularly interesting, because the endowments of the two goods are modeled as *i.i.d.* processes. Second, we show that the time variation of consumption and utilities' second moments introduces conditional heteroskedasticity in the marginal rates of substitutions capable of accounting for the tendency of high interest rate currencies to appreciate. Third, we document that the model produces a non trivial dynamics of net-exports, even though agents have identical risk-sensitive parameters.

The paper is organized as follows. In section 2 we provide the setup of the economy. In section 3 we define the social planner's problem and we provide a recursive method for solving it. In section 4 we provide the set of conditions under which a non-degenerate limiting distribution of Pareto weights exists. In section 5 we calibrate a simple example to show the equilibrium dynamics of consumption, Pareto weights, and continuation utilities. In this section we also document the afore mentioned international finance results. Section 6 concludes the paper.

2 Setup of the economy

In this section we lay out three assumptions about preferences, consumption, and endowments that will be retained throughout the rest of the paper.

Assumption 1 (Preferences). *Let there be two agents, indexed by 1 and 2, whose*

preferences are recursively defined as:

$$U_i(c_i, q_i) = (1 - \delta_i) \log c_i + \delta_i \theta_i \log \sum_{s'} \pi(s') \exp \left\{ \frac{q_i(s')}{\theta_i} \right\}, \quad \forall i \in \{1, 2\} \quad (1)$$

where $q_i(s')$ gives the utility remaining from next period on when next periods state is s' .

Assumption 2 (Consumption bundles). *Let consumption c_i be an aggregate of two goods, x_i and y_i . Specifically let*

$$c_i = (x_i)^{\nu_i} (y_i)^{1-\nu_i} \quad (2)$$

be the consumption bundle.

Assumption 3 (Endowments). *The endowment of the two goods follows a first-order time-homogenous Markov process $(s_0, s_1, \dots)$ which takes values in a finite set $\mathcal{N} = \{1, \dots, n\}$. The aggregate supply of the two goods at time t is such that $0 < X_t = X(s_t) < \infty$, and $0 < Y_t = Y(s_t) < \infty$.*

3 Pareto Optima

3.1 Utility

Following Ma (1993) we let A , a convex subset of R_+^2 , be the admissible current consumption set of each good in each period, $t = 0, 1, \dots$. Let (Σ, F, P) be a probability space, and let $F_0 = \{\emptyset, \Sigma\}$, $\{F_t : t \geq 0\}$ be a sub- σ -algebra of F representing the information structure. The space of consumption programs is defined as:

$$D_i \equiv \{d_i = (c_{i,0}, \dots, c_{i,t}, \dots) : c_{i,t} = (x_{i,t})^{\nu_i} (y_{i,t})^{1-\nu_i}, \quad (x_{i,t}, y_{i,t}) \in A \text{ a.e., } c_{i,t} \text{ is } F\text{-measurable, } \forall t \geq 0\}. \quad (3)$$

Now define $d_i = (c_{i,0}, d'_i)$ and consider the following recursive relation:

$$U_i(d_i) = \log\{W_i(c_{i,0}, Q_i(U(d'_i)))\} \quad (4)$$

where

$$W_i(c_{i,0}, Q_i(U(d'_i))) := c_{i,0}^{1-\delta_i} Q_i(U(d'_i))^{\delta_i} \quad (5)$$

$$Q_i(U(d'_i)) := E_0 [\exp\{U_i(d'_i)\}^{1-\gamma_i}]^{\frac{1}{1-\gamma_i}} \quad (6)$$

A real function $U_i : D \rightarrow R$ is a recursive utility function generated by (W_i, μ_i) if U_i satisfies (4). This specification of the preferences is useful since it allows a straight comparison with Ma (1993)'s results. In particular, we can immediately check that the Ma (1993)'s conditions W_1 , W'_2 , W_3 , CE_1 hold when we focus on the following monotone transformation of the preferences: $\hat{U}_i = \exp\{U_i\}$.² We assume without loss of generality that there exists a non-empty region of the parameters space where Ma (1993)'s conditions W_4 and CE_2 are satisfied.³ This is enough to guarantee the existence of a unique continuous recursive utility function U_i on the domain D_i . The function U_i is strictly increasing and it embodies risk aversion as long as $\gamma_i > 0$. Ma (1993), however, warns us that $\hat{U}_i$ is not concave when $\gamma_i > 1$. Since we are very interested in the case in which the agents have a preference for early resolution of uncertainty, i.e. $\gamma_i > 1$, we will not require concavity of the certainty equivalent. By doing so, we relax one of the assumptions in Kan (1995) and obtain more general results.

In what follows we generate F_t by a time-homogenous Markov chain $(s_0, s_1, \dots)$ whose properties are specified in assumption 3. Under this assumption, the utility of each agent is bounded above and we can also write it as:

$$\begin{aligned} U_i(s_0, d_i) &= \log\{W_i^*(s_0, c_{i,0}, \{U_i(s', d'_i)\}_{s'})\} = \log\{c_{i,0}^{1-\delta_i} Q_i(s_0, U_i(s', d'_i))^{\delta_i}\} \\ Q_i(s_0, U_i(s', d'_i)) &= [\sum_{s' \in \mathcal{N}} \exp\{U_i(s', d'_i)\}^{1-\gamma_i} \pi(s'|s_0)]^{\frac{1}{1-\gamma_i}} \end{aligned}$$

²It is well known that the risk-sensitive preferences are just a log transformation of the Epstein and Zin (1989) preferences when the intertemporal elasticity of substitution is equal to one.

³We can always adjust δ_i and γ_i to make sure that the operator $W_i(\cdot, Q(s, \cdot))$ has a modulus smaller than 1.

3.2 A Recursive method to solve the Planner's Problem

In this section we introduce a recursive method to generate the Pareto optimal allocations of our economy.

By virtue of the properties of our recursive preferences one can show that the social planner's value function, that we shall denote as $Q_p(s, \mu_1) : \mathcal{N} \times [0, 1] \rightarrow R$, satisfies the following functional equation proposed by Lucas and Stokey (1984), and Kan (1995):

$$\begin{aligned}
Q_p(s, \mu_1) &= \max_{\substack{x_1, y_1, x_2, y_2, \\ \{q_{i,s'}\}_{i \in \{1,2\}}}} \sum_{i=1}^2 \mu_i \log W_i^*(s, c_i, \{q_{i,s'}\}_{s'}) \\
&\text{s.t.} \\
&\mu_2 = 1 - \mu_1 \\
&0 \leq x_1 \leq X(s), 0 \leq x_2 \leq X(s) - x_1 \\
&0 \leq y_1 \leq Y(s), 0 \leq y_2 \leq Y(s) - y_1 \\
&c_i = (x_i)^{\nu_i} (y_i)^{1-\nu_i} \quad i = 1, 2 \\
&0 \leq \min_{\mu_1(s') \in [0,1]} Q_p(s', \mu_1(s')) - \mu_1(s') q_{1,s'} - (1 - \mu_1(s')) q_{2,s'} \quad \forall s' \in \mathcal{N} \quad (8)
\end{aligned}$$

In what follows we establish the main properties of the planner's value function and the individual utility functions with respect to the Pareto weights and the exogenous states. Most of our results are parallel to those proposed by Lucas and Stokey (1984), Kan (1995) and Anderson (2005). For this reason we omit the proofs that are just a straight generalization of those reported by these authors.

Assumption 4 (Contraction). *The parameters $\{\nu_i, \gamma_i, \delta_i\}$ are such that the composite operator W_i^* has modulus smaller than one, $i = 1, 2$.*

Remark 3.1. *Let assumption 4 hold. When $\mu_i = 0$, interpret $\mu_i \log W_i^*(s, c_i, \{q_{i,s'}\}_{s'}) = 0$. Since $0 < X(s) < \infty$ and $0 < Y(s) < \infty \quad \forall s \in \mathcal{N}$, it can be proved that there exists a unique bounded and continuous solution to (7)-(8). A feasible allocation is Pareto optimal if and only if it is generated recursively from the solution to (7)-(8).*

Remark 3.2. Let $U_i(s, \mu_i)$, $i = 1, 2$, denote the utility function of agent i evaluated at the optimum when the exogenous state is s , and μ_i is given. Because of the properties of W_i^* and Q_i , $U_i(s, \mu_i)$ is strictly increasing, and so quasi-concave, in μ_i . Furthermore, consider a consumption path that is bounded away from zero for both agents. It is possible to show that $U_i(s, \mu_i)$ is differentiable with respect to $\mu_i \in (0, 1)$, $i = 1, 2$ and $\frac{dU_i}{d\mu_i} > 0$.

Remark 3.3. It is possible to show that $Q_p(s, \mu_1)$ is differentiable at any endowment process for which each agent's consumption is bounded away from zero in the associated Pareto optimal allocation. In particular, for $\mu_1 \in (0, 1)$:

$$\frac{dQ_p}{d\mu_1}(s, \mu_1) = U_1(s, \mu_1) - U_2(s, (1 - \mu_1)),$$

while

$$\frac{d^2Q_p}{d\mu_1^2}(s, \mu_1) = \frac{dU_1}{d\mu_1}(s, \mu_1) + \frac{dU_2}{d\mu_2}(s, 1 - \mu_1) > 0.$$

As in Lucas and Stokey (1984), $Q_p(s, \mu_1)$ is strictly convex with respect to μ_1 .

The last two remarks are very important because they tell us that the unique optimal policy of the planner can be characterized using first order conditions, as pointed out by Andersen (2005) in a one good economy. Using first order conditions, we obtain:

$$\phi' := \frac{\mu'_1}{\mu'_2} = \frac{\mu_1}{\mu_2} \mathcal{M}(s', \phi') =: \phi \cdot \mathcal{M}(s', \phi') \quad (9)$$

$$\mathcal{M}(s', \phi') = \frac{\exp\{U_1(s', \phi')/\theta_1\}}{\sum_{s'} \pi(s') \exp\{U_1(s', \phi')/\theta_1\}} \frac{\sum_{s'} \pi(s') \exp\{U_2(s', \phi')/\theta_2\}}{\exp\{U_2(s', \phi')/\theta_2\}}$$

The relation in (9) implicitly generates a continuous function that we denote by f_ϕ :

$$\phi' = f_\phi(s', \phi) \quad (10)$$

Furthermore, the optimal allocation of the current available goods $\{x_i, y_i\}_{i \in \{1, 2\}}$ depends only on the couple of states $\{s, \phi\}_{\phi = \mu_1/(1 - \mu_1)}^{s \in \mathcal{N}}$ and it equals the allocation of the

following familiar one-period social planning problem:

$$\begin{aligned} \max_{\{x_i \geq 0, y_i \geq 0\}_{i \in \{1,2\}}} \quad & \sum_{i=1}^2 \mu_i \log c_i \\ x_1 + x_2 \leq X(s), \quad & y_1 + y_2 \leq Y(s). \end{aligned} \tag{11}$$

By solving the following system of equations:

$$\begin{aligned} \frac{\partial \log c_1}{\partial x_1} \phi &= \frac{\partial \log c_2}{\partial x_2}, & \frac{\partial \log c_1}{\partial y_1} \phi &= \frac{\partial \log c_2}{\partial y_2}; \\ x_1 + x_2 &= X(s), & y_1 + y_2 &= Y(s), \end{aligned}$$

we can find a pair of non-negative functions $\Theta_{i,x}(\mu_i)$ and $\Theta_{i,y}(\mu_i)$, $i = 1, 2$, such that $x_i(s, \mu_i) = \Theta_{i,x}(\mu_i)X(s)$ and $y_i(s, \mu_i) = \Theta_{i,y}(\mu_i)Y(s)$. This result is very relevant because it tells us that we can treat the optimal allocation of the current goods and the optimal allocation of the continuation utilities as two separate problems.

So far, we did not impose any specific assumption on the conditional probability of the Markov chain governing the supply of the two goods, neither we imposed any special restriction on the preference parameters of our agents. In order to have a well specified problem, all we need is that assumptions 1-4 hold. In the following sections we will impose further restrictions in order to characterize analytically the main properties of the optimal risk-sharing policy of the planner.

4 Survivorship theorems

In this section we prove that in the case of symmetric preferences defined below, a non-degenerate limiting distribution of Pareto weights exists. Equivalently, no agent dies with probability one. The following two assumptions will be retained throughout this section.

Assumption 5 (Symmetry). *Let the preference parameters δ_i and θ_i be identical $\forall i \in \{1, 2\}$. Let the consumption bundle's parameters be symmetrical across agents, that is $\nu_1 = 1 - \nu_2$.*

Assumption 6 (IID case). Assume that $\pi(s'|s) = \pi(s') \quad \forall s, s' \in \mathcal{N}$.

The following lemma characterizes the ranking of Pareto weights as function of the state of the economy.

Lemma 1 (Ranking of Pareto weights). *Let the events $\{a, b\} \subset \mathcal{N}$ be such that $X(a)/Y(a) > X(b)/Y(b)$. Then the ratio of Pareto weights is such that $\frac{\mu_1(a)}{\mu_2(a)} \leq \frac{\mu_1(b)}{\mu_2(b)}$. If $X(a)/Y(a) = X(b)/Y(b)$, then $\frac{\mu_1(a)}{\mu_2(a)} = \frac{\mu_1(b)}{\mu_2(b)}$.*

Proof. See Appendix A. □

The interpretation of lemma 1 is straightforward: whenever agent 1 receives a good shock to the endowment of the good that she likes the most, the social planner reduces her weight. In this way agent 2 can share part of the endowment risk of the economy with agent 1. Equivalently: the optimal choice of the ratio of Pareto weights is a decreasing function of the ratio of the endowments.

Next we provide a useful decomposition of the conditional expectation of the ratio of Pareto weights.

Lemma 2. *The ratio of Pareto weights can be decomposed as:*

$$E_t \left[\frac{\mu'_1}{\mu'_2} \right] = \frac{\mu_1}{\mu_2} - \frac{\text{cov}_t [\exp \{U'_2/\theta\}, \mu'_1/\mu'_2]}{E_t [U'_2/\theta]}$$

Proof. See Anderson (2005). □

Assumption 7 (Symmetric States). *Let the states $s_i, s_j \in \mathcal{N}$ be such that $s_i = \{X(i), Y(i)\}$ and $s_j = \{Y(i), X(i)\}$. Then s_i and s_j are symmetric states.*

The next lemma states that there exists a ratio of Pareto weights such that any two symmetric states will entail the same continuation utilities. This provides an helpful characterization for the above decomposition of the conditional expectation of the ratio of Pareto weights.

Lemma 3 (Crossing of continuation utilities). *For any two symmetric states s'_a and s'_b , such that $X(a) > Y(a)$, there exists a $\phi < 1$ such that $U_1(s'_a, f_\phi(s'_a, \phi)) = U_1(s'_b, f_\phi(s'_b, \phi))$.*

Proof. See Appendix A. □

Corollary 1. *For any two symmetric states s'_a and s'_b , such that $X(a) > Y(a)$, there exists a $\phi > 1$ such that $U_2(s'_a, f_\phi(s'_a, \phi)) = U_2(s'_b, f_\phi(s'_b, \phi))$.*

Proof. Follows directly from Lemma 3. □

The following lemma together with the existence of a ratio of Pareto weights which involves identical continuation utilities for any two symmetric states allows to characterize the sign of the conditional covariance term in lemma 2.

Lemma 4 (Bound on conditional covariance). *Let the sequence of endowment pairs be such that $\forall s \in \{1, \cdot, n\}$, if $s = (i, j)$, then $s + 1 = (j, i)$. Then:*

$$\text{cov}_t [\exp \{U'_2/\theta\}, \mu'_1/\mu'_2] \leq \sum_{j=1}^{n/2} \tilde{\pi}(l) \text{cov}_t [\exp \{U'_2(s)/\theta\}, \mu'_1(s)/\mu'_2(s) | s \in (2j-1, 2j)] \quad (12)$$

where the right hand side of (12) stands for the time t conditional covariance assuming that time $t + 1$ events are either $2j - 1$ or $2j$, and $\tilde{\pi}(l), \forall l \in \{1, \dots, n/2\}$ is a sequence of non-negative weights.

Proof. See Appendix A. □

We are now ready to state the main propositions of this section, according to which no agents is going to die with probability one.

Proposition 1. *The stochastic process $\phi \in [0, \bar{\phi}]$, where $\bar{\phi} < 1$, is a submartingale.*

Proof. Follows directly from Lemmas 2-4. □

Lemma 5. *A bounded submartingale cannot converge almost surely to its lower bound.*

Proof. See Sciubba (2005). □

Proposition 2. *The stochastic processes μ_1 and μ_2 cannot converge almost surely to either 0 or 1.*

Proof. Follows directly from Lemma 5 and Proposition 1. □

5 A simple example

The goal of this section is to characterize analytically the main properties of the function $f_{\mu_1} : \mu_{1,t+1} = f_{\mu_1}(s_{t+1}, \mu_{1,t})$, that is crucial for the intertemporal risk sharing.⁴ We retain assumptions 4, 5, and 6 throughout this section. For convenience, we also make the following assumption about the endowment process.

Assumption 8 (Symmetric endowments). *Let H and L be two positive scalars such that $H > L$. $\mathcal{N}$ has only 4 states: $s_1 : X(s_1) = Y(s_1) = L$, $s_2 : X(s_2) = H, Y(s_2) = L$, $s_3 : X(s_3) = L, Y(s_3) = H$, and $s_4 : X(s_4) = Y(s_4) = H$. These events have all positive probabilities: $\pi_j > 0 \forall j = 1, \dots, 4$. To ensure symmetry, we need $\pi_2 = \pi_3$ and $\pi_1 = \pi_4$. We let $\pi_1 \in (0, .5)$, and impose $\pi_2 = (1 - 2\pi_1)/2$.*

5.1 The optimal policy

When these assumptions hold, the following results— proved in Appendix B— can be obtained.

Lemma 6 (f_{μ_1} at the boundaries). *For any $s = 1, \dots, 4$, $f_{\mu_1}(s, 0) = 0$ and $f_{\mu_1}(s, 1) = 1$. Furthermore, $\lim_{\mu_1 \rightarrow 0^+} f_{\mu_1}(s, \mu_1) = 0^+$, while $\lim_{\mu_1 \rightarrow 1^-} f_{\mu_1}(s, \mu_1) = 1^-$.*

Definition 1 (The switching point). *We define $\mu_1^{sp} : U_1(2, f_{\mu_1}(2, \mu_1^{sp})) = U_1(3, f_{\mu_1}(3, \mu_1^{sp}))$.*

Lemma 7 (f_{μ_1} when $s = 2, 3$). *Assume that $\mu_1 \in (\mu_1^{sp}, 1)$, then $f_{\mu_1}(2, \mu_1) < \mu_1$. When $\mu_1 \in (0, 1 - \mu_1^{sp})$, then $f_{\mu_1}(3, \mu_1) > \mu_1$.*

Proposition 3 (f_{μ_1} when $s = 1, 4$). *It can be proved that: 1) $f_{\mu_1}(1, \mu_1) = f_{\mu_1}(4, \mu_1) \quad \forall \mu_1 \in [0, 1]$; 2) $f_{\mu_1}(1, .5) = .5$; 3) $f_{\mu_1}(1, \mu_1) > \mu_1$ if $0 < \mu_1 < .5$; $f_{\mu_1}(1, \mu_1) < \mu_1$ if $1 > \mu_1 > .5$.*

Corollary 2. *By symmetry, $f_{\mu_1}(1, \mu_1) - \mu_1 = -[f_{\mu_1}(1, 1 - \mu_1) - (1 - \mu_1)]$, $\forall \mu_1 \in [0, .5]$, and $f_{\mu_1}(2, \mu_1) - \mu_1 = -[f_{\mu_1}(3, 1 - \mu_1) - (1 - \mu_1)]$, $\forall \mu_1 \in (0, 1)$.*

⁴Note that f_{μ_1} is just a monotone transformation of f_ϕ :

$$f_{\mu_1}(s, \mu_1) = \frac{f_\phi(s, \mu_1/(1 - \mu_1))}{(1 + f_\phi(s, \mu_1/(1 - \mu_1)))}.$$

Lemma 8 (The unconditional mean of μ_1). *The ergodic distribution of μ_1 is symmetric on $[0, 1]$, implying that $E[\mu_{1,t}] = .5$.*

Lemma 7 points out that the pareto weights move countercyclically in order to smooth the utility of each agents across states of the world in which the supply of the two goods is not balanced. Consider, for example, Agent 1 who has a preference for good X, since $\nu_1 > .5$. When the state $s = 2$ is realized, $X = H > Y = L$ and agent 1 is the one that enjoys the most the high supply of X. Simultaneously, however, μ_1 decreases so that more resources are allocated to agent 2. Lemma 6 guarantees that through this policy none of the two agents obtain a Pareto weight equal to zero or one, since $\mu_{1,t} \in (0, 1) \Rightarrow \mu_{1,t+1} \in (0, 1)$.

Given an initial $\mu_{1,t}$, let us construct $f_{\mu_1}^j$ recursively as follows:

$$\begin{aligned}\mu_{1,t+1} &= f_{\mu_1}(s_{t+1}, \mu_{1,t}), \\ \mu_{1,t+2} &= f_{\mu_1}(s_{t+2}, \mu_{1,t+1}) =: f_{\mu_1}^2((s_{t+1}, s_{t+2}), \mu_{1,t}), \\ &\vdots \\ \mu_{1,t+j} &= f_{\mu_1}(s_{t+j}, \mu_{1,t+j-1}) =: f_{\mu_1}^j((s_{t+1}, \dots, s_{t+j}), \mu_{1,t}).\end{aligned}$$

Proposition 3 shows that on those histories in which the realized supply of the two goods is always equal, for any initial $\mu_{1,t} \in (0, 1)$:

$$\lim_{j \rightarrow +\infty} f_{\mu_1}^j(\vec{s}_1, \mu_{1,t}) = \lim_{j \rightarrow +\infty} f_{\mu_1}^j(\vec{s}_4, \mu_{1,t}) = .5.$$

Hence, for any initial value belonging to the interior, and $X_{t+j} = Y_{t+j} \quad j = 1, 2, \dots, +\infty$, the pareto weight tends to mean-revert. In the appendix we show that this result is driven by the convexity of the planner's utility function and by the fact that the two agents have a symmetric preference bias on the two goods.

5.2 Dynamics of Pareto weights

In this section we examine the dynamics of Pareto weights and consumption in a simple case with four equally likely pairs of endowments realizations:

$$\{(X = 103, Y = 103), (X = 103, Y = 100), (X = 100, Y = 103), (X = 100, Y = 100), \}$$

We set the coefficient of risk aversion γ to 25, the subjective discount factor δ to 0.95 and the consumption bias coefficient ν_1 to 0.98.

Figure 1 shows the difference between the time $t + 1$ and time t Pareto weights on agent 1, in each of the four endowment states. Two important outcomes ought to be noticed. First, whenever the two goods are not in equal supply (i.e. when $(X = 103, Y = 100)$ or $(X = 100, Y = 103)$), the Pareto weight on agent 1 is always increasing or decreasing. Specifically, when good X (the one toward which the preferences of agent 1 are biased) is more abundant relative to good Y , the time $t + 1$ Pareto weight on agent 1 is lower than its time t counterpart. The opposite is true when X is in scarce supply. Second, whenever the two goods are in equal supply (i.e. when $(X = 103, Y = 103)$ or $(X = 100, Y = 100)$), the time $t + 1$ Pareto weight on agent 1 is increasing if her the time t Pareto weight is lower than 0.5 and decreasing if her time t Pareto weight is larger than 0.5. This is the essence of the risk-sharing mechanism in this economy.

Figure 2 shows the expected increase in the Pareto weight as a function of the current wealth distribution. As we proved in the previous sections, the expected Pareto weight is always increasing if the current Pareto weight is smaller than 0.5 and decreasing otherwise. In a nutshell, this is the key element that allows every agent to survive in this economy.

In figure 3 we report the invariant distribution of Pareto weights for agent 1.⁵ Given the symmetric preference specification, the distribution of agent 2 Pareto weights is going to be identical. The distribution is symmetric around 0.5 and the probability mass on the tails of the distribution is very small. Hence the unconditional mean of the Pareto weights is 0.5 and the probability of one agent being allocated all the goods

⁵The graph is obtained by simulating 50,000 samples of length 2,000.

in the economy is negligible.

In figure 7 we show the differences in the invariant distributions of Pareto weights and the phase diagrams, when the value of γ is lowered from 25 to 10. A smaller risk aversion coefficient reduces the risk-sharing motives, as it is documented by the decreased degree of dynamics in the phase diagrams. As a consequence the Pareto weights have a lower volatility with a smaller γ .

5.3 Dynamics of consumption

Figure 4 depicts the risk-sharing mechanism in this economy. Assume to start from a situation in which the endowment of both goods is low. The consumption of agent 1 is going to increase if the following state is such that the endowment of good X is high and the endowment of good Y is low. However, it is going to increase less than proportionally, due to the additional adjustment of the Pareto weight (see top-right panel of figure 1). The second channel would not be present if the two agents had standard time-additive preferences or if they had identical risk-sensitive preferences in a one good economy, as in Anderson (2005).

As agent 1 is willing to consume a smaller share of current total endowment, she will benefit both in terms of future expected consumption growth and future consumption volatility. Figure 5 shows this. In the region of highest probability mass, expected consumption growth is decreasing in the current Pareto weight and the conditional volatility of consumption growth is increasing in the current Pareto weight. Hence this economy features both time-varying expected consumption growth and time-varying volatility of consumption growth. This is endogenously generated by the risk-sharing mechanism of the model. Indeed the dynamics of the endowment are characterized by constant conditional means and volatilities.

The conditional volatility of the utility of agent 1 is reported in figure 6. This shows the crossing of the utilities property that we previously documented. As a consequence of this, the conditional volatility of the utility is going to be increasing at the right of the crossing point and decreasing at its left.

5.4 An application to international finance

A straightforward application of this setup can be made, by interpreting the two agents as the representative consumers of two countries, that we shall call *home* and *foreign*. We decentralize the economy by endowing the home country with good X and the foreign country with good Y , and by using one period state contingent Arrow-Debreu securities that can be traded in the world market and that are zero in net supply. The investor of the home country faces the price $Q^{h'}(\zeta')$ for the security that delivers one unit of good X on the occurrence of state ζ' . The foreign consumer faces the price $Q^{f'}(\zeta')$ for the security that delivers one unit of good Y on the occurrence of the same state. As a consequence, the two budget constraint are

$$\begin{aligned} X^h + pY^h + \int_{\zeta'} A^{h'}(\zeta') Q^{h'}(\zeta') &= A^h + X \\ X^f + pY^f + \int_{\zeta'} A^{f'}(\zeta') Q^{f'}(\zeta') &= A^f + pY \end{aligned} \quad (13)$$

where $A^i(\zeta)$ are holdings of the ζ contingent security by country i , and p is the relative price of goods Y and X . Market clearing implies that $A^h(\zeta) + A^f(\zeta) = 0, \forall \zeta$.

By definition net exports are the value of exports minus the value of imports. In this model, this translates into a mapping with the ratio of Pareto weights:

$$\begin{aligned} \frac{NX^h}{X} &= \frac{X^f - p_y Y^h}{X} \\ &= \frac{1 - \phi}{1 - \kappa\phi} \end{aligned} \quad (14)$$

where we also normalized by the value of domestic output and denoted κ as $\frac{\nu_1}{1-\nu_1}$. Hence the dynamics of the Pareto weights that we described in the previous section induces dynamics in net exports, that would otherwise be absent under the assumption of time additive preferences.

This also suggests an intuitive interpretation of the risk sharing scheme, as depicted in figure 8. When the supply of good X is larger than the supply of good Y , the home country in part consumes and in part exports its larger endowment. Hence the ratio of Pareto weights is going to be inversely related to the share of net exports of the home

country. Equivalently the risk-sharing mechanism impelled by the model translates into pro-cyclical net-exports.

A long literature has documented the tendency of high interest rate currencies to appreciate, contrary to what predicted by the standard theory. This setup is able replicate the negative relationship between exchange rate growth and interest rate differentials that is observed in the data.

Since markets are complete, exchange rate growth can be obtained by no arbitrage as the difference of the logarithms of the marginal rates of substitution $m^{h'}$ and $m^{f'}$:

$$\Delta e' = m^{f'} - m^{h'} \quad (15)$$

where

$$\begin{aligned} m^{i'} &= \log \frac{\partial U^i / \partial C^{i'}}{\partial U^i / \partial C^i} \\ &= \log \delta - \Delta c^{i'} + \log \frac{\exp \{U^i(s', \phi') / \theta\}}{\sum_{s'} \pi(s') \exp \{U^i(s', \phi') / \theta\}}, \quad \forall i \in \{h, f\} \end{aligned}$$

If utilities are approximatively normally distributed, it is possible to write the marginal rate of substitutions as:

$$m^{i'} = \log \delta - \Delta c^{i'} + \frac{U^{h'}}{\theta} - \frac{1}{\theta} E_{s,\phi} U^{i'} - \frac{1}{2\theta^2} V_{s,\phi} U^{i'}, \quad \forall i \in \{h, f\}$$

where $E_{s,\phi} U^{i'}$ and $V_{s,\phi} U^{i'}$ are the conditional expectation and the conditional variance of utility, respectively. Hence the conditional dynamics of exchange rates is going to be driven by consumption growth expectations and conditional variances of continuation utilities:

$$E_{s,\phi} \Delta e' = E_{s,\phi} [\Delta c^{h'} - \Delta c^{f'}] + \frac{1}{2\theta^2} [V_{s,\phi} U^{h'} - V_{s,\phi} U^{f'}] \quad (16)$$

The difference of risk-free rates can be obtained as:

$$\begin{aligned}
r^h - r^f &= -\log E_{s,\phi} \exp \left\{ m^{h'} \right\} + \log E_{s,\phi} \exp \left\{ m^{f'} \right\} \\
&\approx E_{s,\phi} \left[\Delta c^{h'} - \Delta c^{f'} \right] - \frac{1}{2} \left[V_{s,\phi} \Delta c^{h'} - V_{s,\phi} \Delta c^{f'} \right] \\
&\quad + \frac{1}{2\theta} \left[Cov_{s,\phi} \left(\Delta c^{h'}, U^{h'} \right) - Cov_{s,\phi} \left(\Delta c^{f'}, U^{f'} \right) \right]
\end{aligned} \tag{17}$$

where $Cov_{s,\phi} (\Delta c^{i'}, U^{i'})$ stands for the conditional covariance between consumption growth and utility in country i .

If agents had standard time-additive preferences, the dynamics of equations (16) and (17) would be entirely driven by consumption growth differentials, resulting in the afore mentioned counterfactual perfect correlation between exchange rates appreciation and interest rate differentials. Figure 9 documents that risk-sensitive preferences introduce two opposite effects in the dynamics of interest rate differentials and expected exchange rate growth. Indeed both differences of second order terms in equation (17) are decreasing in the Pareto weight, while the difference of the variances of continuation utilities in (16) is increasing in the Pareto weight in the region of higher probability mass. The combination of the two effects results in the negative relationship displayed in figure 10.

6 Concluding remarks

We have characterized the equilibrium of a complete markets economy with multiple agents and multiple goods in which agents have risk-sensitive preferences. The assumption of multiple goods is an important one, that changes dramatically the dynamics of Pareto optimal allocations. The results shown in the application to international finance suggest that this framework should be a useful tool for modeling dynamic open economies.

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A Proof of theorems

Proof of Lemma 1. Define:

$$\begin{aligned}\varphi(s, \phi) &= \frac{1}{\theta} \log c_1(s, \phi) - \frac{1}{\theta} \log c_2(s, \phi) \\ \Phi(s, \phi) &= \frac{1}{\theta} U_1(s, \phi) - \frac{1}{\theta} U_2(s, \phi) \\ F(s, \phi) &= \delta \log \sum_{s'} \pi(s') \exp \left\{ \frac{U_1(s', \phi')}{\theta} \right\} - \delta \log \sum_{s'} \pi(s') \exp \left\{ \frac{U_2(s', \phi')}{\theta} \right\}\end{aligned}$$

Note that $\Phi(s, \phi) = \varphi(s, \phi) + F(s, \phi)$. Let $k = \nu_1 / (1 - \nu_1)$ and let $X(s)$ and $Y(s)$ be the endowments of the two goods in state s . Then, the optimal allocations imply:

$$\varphi(s, \phi) = \frac{1}{\theta} \left[(2\nu_1 - 1) \log \left(\frac{X(s)}{Y(s)} \right) + \log \phi + (1 - 2\nu_1) \log(1 + k\phi) - (1 - 2\nu_1) \log(k + \phi) \right]$$

Since $\theta < 0$, if $\nu_1 > 1/2$ it follows that $\varphi(s, \phi)$ is always: 1) decreasing in $X(s)/Y(s)$ and 2) decreasing in ϕ . Furthermore, $F(s, \phi)$ is decreasing in ϕ , since $U_1(s, \phi)$ ($U_2(s, \phi)$) is increasing (decreasing) in ϕ , by the optimality of the social planner problem. Therefore, it has to be the case that $\Phi(s, \phi)$ is decreasing in ϕ .

Take two states, a and b , such that $X'(a)/Y'(a) < X'(b)/Y'(b)$ and let $\phi'_a = f_\phi(a, \phi)$ and $\phi'_b = (b, \phi)$ be the respective ratios of Pareto weights. It is possible to characterize the ratio:

$$\frac{\phi'_b}{\phi'_a} = \exp \{ \Phi(b, \phi'_b) - \Phi(a, \phi'_a) \}$$

Since $\Phi(s, \phi')$ is decreasing in s holding ϕ' fixed, it follows that $\exp \{ \Phi(b, \phi'_a) - \Phi(a, \phi'_a) \} < 1$. Hence, if $X'(a)/Y'(a) < X'(b)/Y'(b)$, it is never optimal to set $\phi'_a = \phi'_b$.

Additionally, the fact that $\Phi(s, \phi')$ is decreasing in ϕ' implies that

$$1 > \exp \{ \Phi(b, \phi'_a) - \Phi(a, \phi'_a) \} > \{ \Phi(b, \phi'_b) - \Phi(a, \phi'_a) \}$$

It follows that $\phi'_b < \phi'_a$. □

Proof of Lemma 3. By subtracting the first order conditions of the Pareto problem:

$$U_1(s_a, \phi') - U_1(s_b, \phi') = \frac{\partial Q(s_a, \phi')}{\partial \mu'_1} - \frac{\partial Q(s_b, \phi')}{\partial \mu'_1} + U_2(s_a, \phi') - U_2(s_b, \phi')$$

Let $\mu'_1 < 1/2$. By symmetry, we can replace the last two terms of the previous equation to get:

$$\begin{aligned} U_1(s_a, \phi') - U_1(s_b, \phi') &= \frac{\partial Q(s_a, \phi')}{\partial \mu'_1} - \frac{\partial Q(s_b, \phi')}{\partial \mu'_1} + U_1(s_b, 1/\phi') - U_1(s_a, 1/\phi') \\ &< \frac{\partial Q(s_a, \phi')}{\partial \mu'_1} - \frac{\partial Q(s_b, \frac{\mu'_1 + \varepsilon}{\mu'_2 - \varepsilon})}{\partial \mu'_1} + U_1(s_b, 1/\phi') - U_1(s_a, 1/\phi') \end{aligned} \quad (\text{A.1})$$

for small enough ε and where the inequality follows from the properties of the planner's problem. Since $U_1(s_b, 1/\phi') - U_1(s_a, 1/\phi') < 0$, it amounts to showing that there exists a μ'_1 such that

$$\frac{\partial Q(s_a, \phi')}{\partial \mu'_1} - \frac{\partial Q(s_b, \frac{\mu'_1 + \varepsilon}{\mu'_2 - \varepsilon})}{\partial \mu'_1} = 0$$

for small enough ε . From Lemma 1, $f_\phi(s'_a, \phi) = \phi'_a < \phi'_b = f_\phi(s'_b, \phi)$. Since $Q(s'_a, 0) < Q(s'_b, 0)$, $\frac{\partial Q(s'_a, \phi')}{\partial \mu'_1} < 0$, and $\frac{\partial Q(s'_b, \phi')}{\partial \mu'_1} < 0$, $\forall \mu'_1 < 1/2$, it must be the case that $\frac{\partial Q(s'_a, \phi')}{\partial \mu'_1} > \frac{\partial Q(s'_b, \phi')}{\partial \mu'_1}$. Hence, there exists $\tilde{\varepsilon}$ such that:

$$Q'_a(\mu'_1 + \tilde{\varepsilon}) := \frac{\partial Q(s'_a, \frac{\mu'_1 + \tilde{\varepsilon}}{\mu'_2 - \tilde{\varepsilon}})}{\partial \mu'_1} = \frac{\partial Q(s'_b, \phi')}{\partial \mu'_1} =: Q'_b(\mu'_1)$$

By the properties of Q :

$$\lim_{\phi' \rightarrow 0} \tilde{\varepsilon} = \lim_{\mu'_1 \rightarrow 0} Q'^{-1}_a(Q'_b(\mu'_1)) - \mu'_1 = 0 \quad (\text{A.2})$$

Hence, combining (A.1) and (A.2):

$$\lim_{\mu'_1 \rightarrow 0} U_1(s_a, \phi') - U_1(s_b, \phi') < 0 \quad (\text{A.3})$$

Since $\nu_1 > 1/2$, it follows that

$$\lim_{\mu'_1 \rightarrow 1} U_1(s_a, \phi') - U_1(s_b, \phi') > 0 \quad (\text{A.4})$$

Combining (A.3) and (A.4) concludes the proof. $\square$

Proof of Lemma 4. By the definition of conditional covariance:

$$\begin{aligned}
cov_t [\exp \{U'_2/\theta\}, \mu'_1/\mu'_2] &= \sum_{j=1}^n \pi(j) \exp \{U'_2(j)/\theta\} \mu'_1(j)/\mu'_2(j) - \\
&\quad \left[\sum_{j=1}^n \pi(j) \exp \{U'_2(j)/\theta\} \right] \left[\sum_{j=1}^n \pi(j) \mu'_1(j)/\mu'_2(j) \right] \\
&= \sum_{j=1}^{n/2} \sum_{l=2j-1}^{2j} \pi(l) \exp \{U'_2(l)/\theta\} \mu'_1(l)/\mu'_2(l) \\
&\quad - \left[\sum_{j=1}^{n/2} \sum_{l=2j-1}^{2j} \pi(l) \exp \{U'_2(l)/\theta\} \right] \left[\sum_{j=1}^{n/2} \sum_{l=2j-1}^{2j} \pi(l) \mu'_1(l)/\mu'_2(l) \right] \\
&= \sum_{j=1}^{n/2} \tilde{\pi}(l) cov_t [\exp \{U'_2(s)/\theta\}, \mu'_1(s)/\mu'_2(s) | s \in (2j-1, 2j)] \\
&\quad - \left[\sum_j \tilde{\pi}(j) \left(\sum_l \pi(l) \exp \{U'_2(l)/\theta\} \right) \right] \left[\sum_j \tilde{\pi}(j) \left(\sum_l \pi(l) \mu'_1(l)/\mu'_2(l) \right) \right]
\end{aligned} \tag{A.5}$$

where $\tilde{\pi}(j) = \sum_{l=2j-1}^{2j} \pi(l)$ is a rescaling factor, $\tilde{\pi}(j), \forall j \in [1, n/2]$ is a sequence of non-negative weights and $cov_t(\cdot, \cdot | s \in (2j-1, 2j))$ stands for the time t conditional covariance assuming that time $t+1$ events are either $2j-1$ or $2j$. Since the last two terms in the square brackets of (A.5) are always non-negative by definition, it follows that:

$$cov_t [\exp \{U'_2/\theta\}, \mu'_1/\mu'_2] \leq \sum_{j=1}^{n/2} \tilde{\pi}(l) cov_t [\exp \{U'_2(s)/\theta\}, \mu'_1(s)/\mu'_2(s) | s \in (2j-1, 2j)] \tag{A.6}$$

which concludes the proof. $\square$

B Appendix for section 4

Remark B.1. From remark 3.3 it immediately follows that:

$$\begin{aligned}
\frac{dU_1}{d\mu_1}(s, \mu_1) &= (1 - \mu_1) \frac{d^2 Q_p}{d\mu_1^2}(s, \mu_1) \\
\frac{dU_2}{d\mu_2}(s, 1 - \mu_1) &= \phi \frac{dU_1}{d\mu_1}(s, \mu_1)
\end{aligned} \tag{B.7}$$

Remark B.2. Given the properties of $U_i(s, \mu_i)$, $\lim_{\mu_i \rightarrow 0} g_i(\mu_i) = -\infty$. This implies that $\forall s \in \mathcal{N}$, $\lim_{\mu_1 \rightarrow 1} \frac{dQ_p}{d\mu_1} = +\infty$, and $\lim_{\mu_1 \rightarrow 0} \frac{dQ_p}{d\mu_1} = -\infty$. From the continuity of $\frac{dQ_p}{d\mu_1}$, it follows that $\forall s \in \mathcal{N}$, $\exists \mu_s^0 \in (0, 1) : \frac{dQ}{d\mu_1}(s, \mu_s^0) = 0$. Furthermore, it is possible to prove that $\forall s, s' \in \mathcal{N}$ such that $\Upsilon_1(s) - \Upsilon_2(s) >$

$$\Upsilon_1(s') - \Upsilon_2(s'), \mu_s^0 < \mu_{s'}^0.$$

Lemma 9 (Separability). *Let assumption 6 hold. Let $U_i(s, \mu_i)$ and $Q_p(s, \mu_1)$ be the value function, respectively, of agent i , and the planner evaluated at the optimum. There exists a function $g_i(\mu_i) : [0, 1] \rightarrow R$, such that $U_i(s, \mu_i) = g_i(\mu_i) + \Upsilon_i(s)$, where $\Upsilon_i(s) = (1 - \delta_i) \log(X(s)^{\nu_i} Y(s)^{1-\nu_i})$. It follows that $\frac{dQ_p}{d\mu_1}(s, \mu_1)$ and $\frac{dQ_p}{d\mu_1}(s', \mu_1)$ differ only by a constant that depends only on s' and s , $\forall s', s \in \mathcal{N}$, $\forall \mu_1 \in (0, 1)$.*

Proof. Just define $g_i(\mu_i) := (1 - \delta_i) \log(\Theta_{i,x}(\mu_i)^{\nu_i} \Theta_{i,y}(\mu_i)^{1-\nu_i}) + \frac{\delta_i}{1-\gamma_i} E[\exp\{U_i(s', \mu_i')\}^{1-\gamma_i} | \mu_i]$. From remark 3.3 we have: $\frac{dQ_p}{d\mu_1}(s, \mu_1) = g_1(\mu_1) - g_2(1 - \mu_1) + \Upsilon_1(s) - \Upsilon_2(s)$. $\square$

Definition 2. *Let assumption 8 hold. A function $F : \mathcal{N} \times [0, 1] \rightarrow R$ is symmetric if the following is true:*

$$\begin{aligned} F(s, \mu_1) &= F(s, 1 - \mu_1), \quad s \in \{1, 4\}, \forall \mu_1 \in [0, 1] \\ F(2, \mu_1) &= F(3, 1 - \mu_1), \forall \mu_1 \in [0, 1] \end{aligned} \quad (\text{B.8})$$

Remark B.3. *Let assumption 1-6 hold. The operator described in (7)-(8) maps symmetric convex functions into symmetric convex functions. This implies that at the optimum Q_p is symmetric, and $g_1(\mu) = g_2(\mu) = g(\mu)$, $\forall \mu \in [0, 1]$.*

Before showing our main results, we rewrite the problem of the planner for the allocation of the future utilities.

Lemma 10 (Future Utilities). *Let $Q_p(s, \mu_1)$ be the convex value function of the planner evaluated at the optimum for $\mu_1 \in (0, 1)$. The optimal allocation of the future utilities can be obtained by solving the following problem:*

$$\begin{aligned} \text{choose} \quad & \{D_s := q_{1,s} - q_{2,s}\}_{s=1,\dots,4} \\ \text{to minimize} \quad & \Lambda(\mu_1) := \mu_1 \log \left[\sum_{s=1}^4 \exp\{(1 - \gamma)(q_{2,s} + D_s)\} \pi_s \right] + \\ & (1 - \mu_1) \log \left[\sum_{s=1}^4 \exp\{(1 - \gamma)(q_{2,s})\} \pi_s \right] \end{aligned} \quad (\text{B.9})$$

$$\begin{aligned} \text{s.t.} \quad & q_{2,s} = Q_p \left(s, \frac{dQ_p}{d\mu_1}^{-1}(s, D_s) \right) - D_s \frac{dQ_p}{d\mu_1}^{-1}(s, D_s) \quad \forall s = 1, \dots, 4 \\ & -\infty < D_s < +\infty \end{aligned} \quad (\text{B.10})$$

Proof. From remark 3.3 we know that $\frac{dQ_p}{d\mu_1}(s, \cdot)$ is strictly increasing and so invertible. From optimality we have $\mu_1(s) = \frac{dQ_p}{d\mu_1}^{-1}(s, q_{1,s} - q_{2,s}) \quad \forall s = 1, \dots, 4$. $\square$

Differentiating equation (B.10) and using optimality, we obtain:

$$\frac{dq_{2,s}}{dD_s} |_{Q(s, D_s)} := - \frac{dQ_p}{d\mu_1}^{-1}(s, q_{1,s} - q_{2,s}) = -\mu_1(s) \quad (\text{B.11})$$

Given remark B.2-B.3 and lemma 9 we can say that:

$$\lim_{D_s \rightarrow -\infty} \frac{dq_{2,s}}{dD_s} |_Q(s, D_s) = 0^- \quad s = 1, \dots, 4 \quad (\text{B.12})$$

$$\lim_{D_s \rightarrow +\infty} \frac{dq_{2,s}}{dD_s} |_Q(s, D_s) = -1^+$$

$$\frac{dq_{2,s}}{dD_s} |_Q(s, 0) = -.5 \quad s \in \{1, 4\}$$

$$\frac{dq_{2,s}}{dD_s} |_Q(3, D_s) < \frac{dq_{2,s}}{dD_s} |_Q(1, D_s) = \frac{dq_{2,s}}{dD_s} |_Q(4, D_s) < \frac{dq_{2,s}}{dD_s} |_Q(2, D_s), D_s \in (-\infty, +\infty) \quad (\text{B.13})$$

Differentiating equation (B.9), we obtain:

$$\frac{dq_{2,s}}{dD_s} |_\Lambda(s, \mu_1, D_s, \{q_{1,s}, q_{2,s}\}_{s=1}^4) := \frac{\frac{\partial \Lambda}{\partial D_s}}{\frac{\partial \Lambda}{\partial q_{2,s}}} = - \frac{\exp\{(1-\gamma)D_s\}}{\exp\{(1-\gamma)D_s\} + \frac{1-\mu_1}{\mu_1} \cdot \frac{E[\exp\{(1-\gamma)q_{1,s}\}|\mu_1]}{E[\exp\{(1-\gamma)q_{2,s}\}|\mu_1]}} \quad (\text{B.14})$$

It is important to notice what follows:

$$\lim_{D_s \rightarrow -\infty} \frac{dq_{2,s}}{dD_s} |_\Lambda = -1^+ \quad s = 1, \dots, 4 \quad (\text{B.15})$$

$$\lim_{D_s \rightarrow +\infty} \frac{dq_{2,s}}{dD_s} |_\Lambda = 0^-$$

$$\lim_{\mu_1 \rightarrow 0^+} \frac{dq_{2,s}}{dD_s} |_\Lambda = 0^-$$

$$\lim_{\mu_1 \rightarrow 1^-} \frac{dq_{2,s}}{dD_s} |_\Lambda = -1^+$$

The solution of the problem described in lemma 10 has to satisfy the following tangency condition:

$$\frac{dq_{2,s}}{dD_s} |_Q = \frac{dq_{2,s}}{dD_s} |_\Lambda \quad s = 1, \dots, 4, \forall \mu_1 \in (0, 1) \quad (\text{B.16})$$

Equation (B.10) has to hold too, so that the future utilities are feasible.

Proof of lemma 6. When $\mu_1 = 0$, agent 1 plays no role in the planner's problem. The planner continues to give all the resources to agent 2. In a symmetric way, when $\mu_1 = 1$, then $\mu_1(s) = 1, s = 1, \dots, 4$. See Anderson (2005) for more details. Equation (B.12)-(B.16) imply that when $\mu_1 \rightarrow 0^+$, at the optimum $D_s \rightarrow -\infty$ and $\mu_1(s) \rightarrow 0^+$, so $\lim_{\mu_1 \rightarrow 0^+} f_{\mu_1}(s, \mu_1) = 0^+$. When $\mu_1 \rightarrow 1^-$, equation (B.12)-(B.16) imply that at the optimum $D_s \rightarrow -\infty$, and $\mu_1(s) \rightarrow 1^-$, i.e. $\lim_{\mu_1 \rightarrow 1^-} f_{\mu_1}(s, \mu_1) = 1^-$. $\square$

Claim B.4 (f_{μ_1} evaluated in $\mu_1 = .5$). When $s \in \{1, 4\}$, $f_{\mu_1}(1, .5) = .5$. Furthermore, $f_{\mu_1}(2, .5) < f_{\mu_1}(1, .5) < f_{\mu_1}(1, 3)$.

Proof. By symmetry, at the optimum $E[\exp\{(1-\gamma)U_1\}|\mu_1 = .5] = E[\exp\{(1-\gamma)U_2\}|\mu_1 = .5]$. In the states $s \in \{1, 4\}$, the supply of the two goods is perfectly equal, symmetry implies that $D_1 = D_4 = 0$.

Using these conditions in equation (B.14), we obtain $\frac{dq_{2,1}}{dD_1}|_\Lambda = \frac{dq_{2,4}}{dD_4}|_\Lambda = .5$. From remark B.2 and B.3, and lemma 9 we know that $\mu_1^0 = \mu_4^0 = .5$, implying that $\frac{dq_{2,s}}{dD_s}|_Q(1,0) = \frac{dq_{2,s}}{dD_s}|_Q(4,0) = .5$. Hence, when $s \in \{1,4\}$ and $\mu_1 = .5$, the tangency condition for the states $s \in \{1,4\}$ is satisfied if $\mu_1(1) = \mu_1(4) = .5$. According to (B.12)-(B.13) the tangency condition in other states can be satisfied only if $f_{\mu_1}(2, .5) < .5 < f_{\mu_1}(3, .5)$. $\square$

Claim B.5 ($f_{\mu_1}(2, \cdot)$ and $f_{\mu_1}(3, \cdot)$). *There exists no $\bar{\mu}_1 \in (\mu_1^{sp}, 1 - \mu_1^{sp}) : f_{\mu_1}(s, \bar{\mu}_1) = \bar{\mu}_1$, when $s \in \{2, 3\}$. $\forall \mu_1 \in (0, \mu_1^{sp}]$, $f_{\mu_1}(3, \mu_1) > \mu_1$; $\forall \mu_1 \in [1 - \mu_1^{sp}, 1)$, $f_{\mu_1}(2, \mu_1) < \mu_1$.*

Proof. We prove this claim by contradiction. Focus on state $s = 2$, and assume that $\exists \bar{\mu}_1 \in (\mu_1^{sp}, 1 - \mu_1^{sp}) : f_{\mu_1}(2, \bar{\mu}_1) = \bar{\mu}_1$. The tangency condition would imply:

$$\frac{dq_{2,2}}{dD_2}|_\Lambda = \bar{\mu}_1 = \frac{\exp\{(1-\gamma)D_2\}}{\exp\{(1-\gamma)D_2\} + \frac{1-\bar{\mu}_1}{\bar{\mu}_1} \cdot \frac{E[\exp\{(1-\gamma)q_{1,s}\}|\bar{\mu}_1]}{E[\exp\{(1-\gamma)q_{2,s}\}|\bar{\mu}_1]}} = \frac{dq_{2,2}}{dD_2}|_Q.$$

Simplifying the previous expression:

$$\exp\{(1-\gamma)(q_{1,2} - q_{2,2})\} = \frac{E[\exp\{(1-\gamma)q_{1,s}\}|\bar{\mu}_1]}{E[\exp\{(1-\gamma)q_{2,s}\}|\bar{\mu}_1]} \quad (\text{B.17})$$

Remember that $X(2) = H$ and that agent 1 has a bias for good X , while $Y(2) = L$. The symmetric bias over the two goods implies the following contradiction:

$$\exp\{(1-\gamma)(q_{1,2} - q_{2,2})\} < \frac{\exp\{(1-\gamma)q_{1,2}\}}{E[\exp\{(1-\gamma)q_{2,s}\}|\bar{\mu}_1]} < \frac{E[\exp\{(1-\gamma)q_{1,s}\}|\bar{\mu}_1]}{E[\exp\{(1-\gamma)q_{2,s}\}|\bar{\mu}_1]} \quad \forall \bar{\mu}_1 \in (\mu_1^{sp}, 1 - \mu_1^{sp}). \quad (\text{B.18})$$

The same type of contradiction can be obtained for state $s = 3$. The fact that $\forall \mu_1 \in (0, \mu_1^{sp}]$, $f_{\mu_1}(3, \mu_1) > \mu_1$, and $\forall \mu_1 \in [1 - \mu_1^{sp}, 1)$, $f_{\mu_1}(2, \mu_1) < \mu_1$ follows directly from proposition 1 and corollary 1. $\square$

Proof of lemma 7. It follows directly from claim B.4 and B.5, the continuity of $f_{\mu_1}(s, \cdot)$. $\square$

Proof of proposition 3. We prove the proposition point by point.

Point 1) Lemma 9 and remark B.3 imply:

$$\frac{dQ_p}{d\mu_1}(4, \mu_1) = \frac{dQ_p}{d\mu_1}(1, \mu_1) \quad \forall \mu_1 \in (0, 1)$$

Combine the tangency condition and equation (B.11) to obtain that $f_{\mu_1}(1, \mu_1) = f_{\mu_1}(4, \mu_1)$.

Point 2) See claim B.4.

Point 3) Focus first on state $s = 1$ for simplicity. Denote the current pareto weight for agent 1 as μ_1 ,

and the future state contingent pareto weight as $\mu_1(s)$, $s = 1, \dots, 4$. At the optimum, we can write:

$$\begin{aligned}
\frac{\exp\{(1-\gamma)U_1(1, \mu_1(1))\}}{E[\exp\{(1-\gamma)U_1(s, \mu_1(s))\}|\mu_1]} &= \frac{1}{\sum_{s=1}^4 \pi_s \exp\{(1-\gamma)[\Upsilon_1(s) - \Upsilon_1(1) + g(\mu_1(s)) - g(\mu_1(1))]\}} \\
&= \frac{1}{\pi_1 + \pi_4 e^{(1-\gamma)(\Upsilon_1(4) - \Upsilon_1(1))} + (\pi_2 + \pi_3) \tilde{E}_{1, \mu_1}} \\
&= \frac{1}{\pi_1 + \pi_4 \left(\frac{H}{L}\right)^{(1-\gamma)(1-\delta)} + (\pi_2 + \pi_3) \tilde{E}_{1, \mu_1}} \tag{B.19}
\end{aligned}$$

where

$$\begin{aligned}
\tilde{E}_{1, \mu_1} &:= \sum_{s=2}^3 .5 \exp\{(1-\gamma)[\Upsilon_1(s) - \Upsilon_1(1) + g(\mu_1(s)) - g(\mu_1(1))]\} \\
&= \sum_{s=2}^3 .5 \exp\{(1-\gamma)[\Upsilon_1(s) - \Upsilon_1(1) + \int_{\mu_1(1)}^{\mu_1(s)} \frac{dg}{d\mu_1}(r) dr]\} \tag{B.20}
\end{aligned}$$

For the second agent we have:

$$\begin{aligned}
\frac{\exp\{(1-\gamma)U_2(1, 1 - \mu_1(1))\}}{E[\exp\{(1-\gamma)U_2(s, 1 - \mu_1(s))\}|\mu_1]} &= \frac{1}{\sum_{s=1}^4 \pi_s \exp\{(1-\gamma)[\Upsilon_2(s) - \Upsilon_2(1) + g(1 - \mu_1(s)) - g(1 - \mu_1(1))]\}} \\
&= \frac{1}{\pi_1 + \pi_4 \left(\frac{H}{L}\right)^{(1-\gamma)(1-\delta)} + (\pi_2 + \pi_3) \tilde{E}_{2, \mu_1}} \tag{B.21}
\end{aligned}$$

where

$$\tilde{E}_{2, \mu_1} = \sum_{s=2}^3 .5 \exp\{(1-\gamma)[\Upsilon_2(s) - \Upsilon_2(1) + \int_{1-\mu_1(1)}^{1-\mu_1(s)} \frac{dg}{d\mu_1}(r) dr]\} \tag{B.22}$$

Using equation (B.7) we obtain also:

$$\tilde{E}_{2, \mu_1} = \sum_{s=2}^3 .5 \exp\{(1-\gamma)[\Upsilon_2(s) - \Upsilon_2(1) + \int_{\mu_1(1)}^{\mu_1(s)} \frac{r}{1-r} \cdot \frac{dg}{d\mu_1}(r) dr]\}$$

At the optimum, we have:

$$\frac{\mu_1(1)}{1 - \mu_1(1)} = \frac{\mu_1}{1 - \mu_1} \cdot \frac{\pi_1 + \pi_4 \left(\frac{H}{L}\right)^{(1-\gamma)(1-\delta)} + (\pi_2 + \pi_3) \tilde{E}_{2, \mu_1}}{\pi_1 + \pi_4 \left(\frac{H}{L}\right)^{(1-\gamma)(1-\delta)} + (\pi_2 + \pi_3) \tilde{E}_{1, \mu_1}}.$$

When $\mu_1 = .5$, by symmentry we have $\tilde{E}_{1, .5} = \tilde{E}_{2, .5}$ and $\mu_1(1) = \mu_1(4) = .5$. When $0 < \mu_1 < .5$, instead,

the following is true:

$$\Upsilon_2(2) - \Upsilon_2(1) + \int_{\mu_1(1)}^{\mu_1(2)} \frac{r}{1-r} \frac{dg}{d\mu_1}(r) dr < \Upsilon_1(2) - \Upsilon_1(1) + \int_{\mu_1(1)}^{\mu_1(2)} \frac{dg}{d\mu_1}(r) dr \quad (\text{B.23})$$

$$\Upsilon_2(3) - \Upsilon_2(1) + \int_{\mu_1(1)}^{\mu_1(3)} \frac{r}{1-r} \frac{dg}{d\mu_1}(r) dr > \Upsilon_1(3) - \Upsilon_1(1) + \int_{\mu_1(1)}^{\mu_1(3)} \frac{dg}{d\mu_1}(r) dr \quad (\text{B.24})$$

This implies that $\tilde{E}_{1,\mu_1} < \tilde{E}_{2,\mu_1}$, and $f_{\mu_1}(1, \mu_1) > \mu_1$. The opposite is true when $.5 < \mu_1 < 1$. $\square$

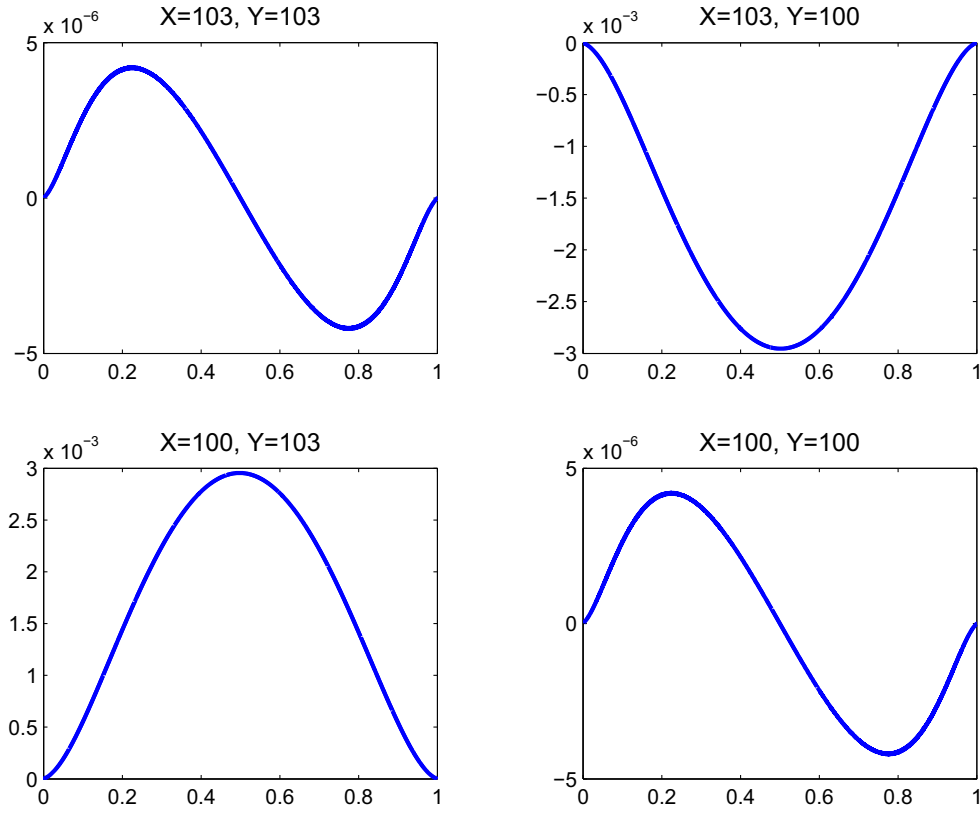


FIG. 1 - Phase diagrams of Pareto weights. The four panels report the difference between time $t + 1$ and time t Pareto weights on agent 1, $f_{\mu_1}(s_{t+1}, \mu_t) - \mu_t$ for every realization of the two endowments, s_{t+1} . The preference parameters are: $\gamma = 25$, $\delta = 0.95$, and $\nu_1 = 0.98$.

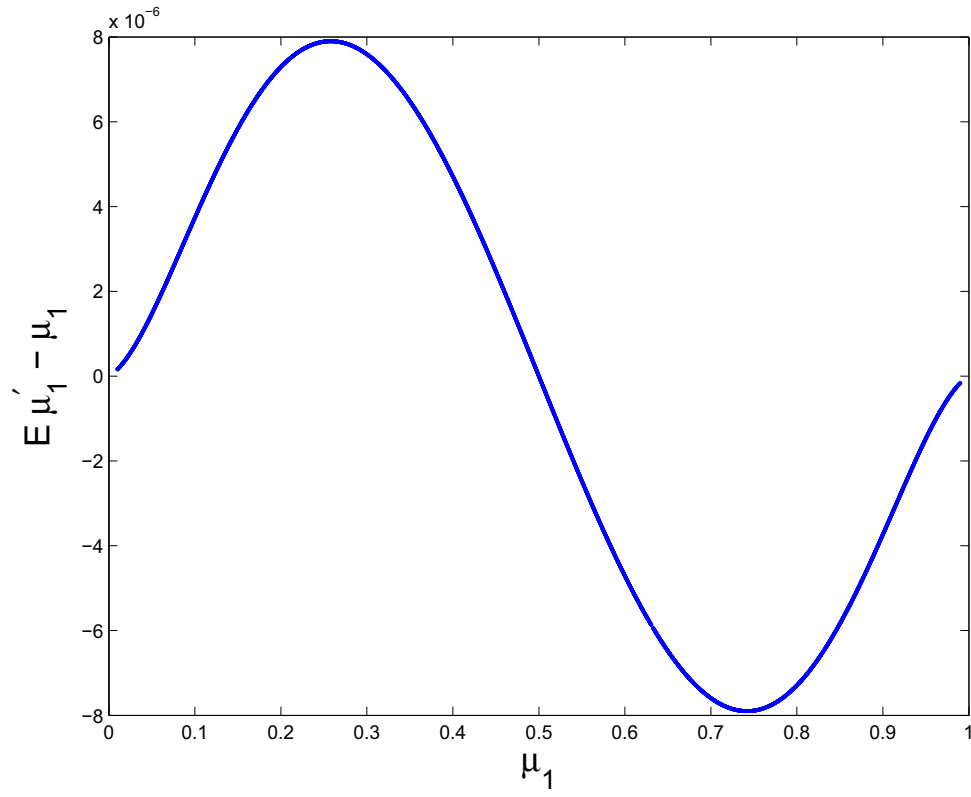


FIG. 2 - Conditional expectation of Pareto weight for agent 1 as a function of the current Pareto weight. The preference parameters are: $\gamma = 25$, $\delta = 0.95$, and $\nu_1 = 0.98$.

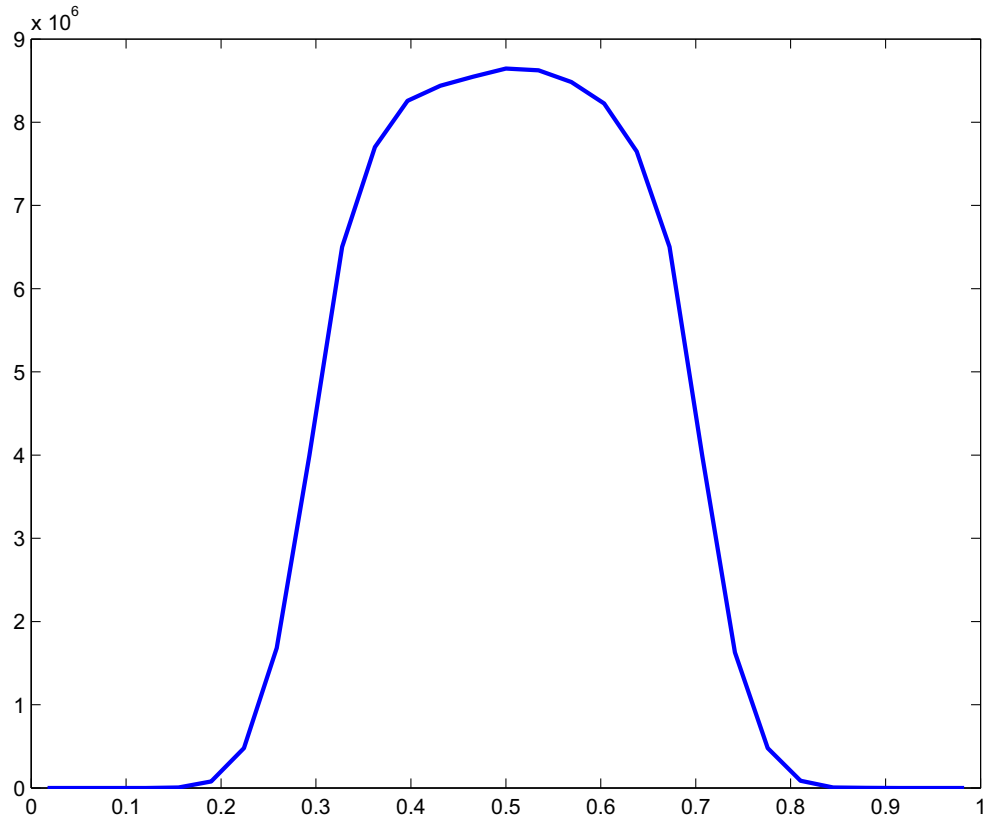


FIG. 3 - Invariant distribution of Pareto weights for agent 1 as a function of the current Pareto weight. The preference parameters are: $\gamma = 25$, $\delta = 0.95$, and $\nu_1 = 0.98$.

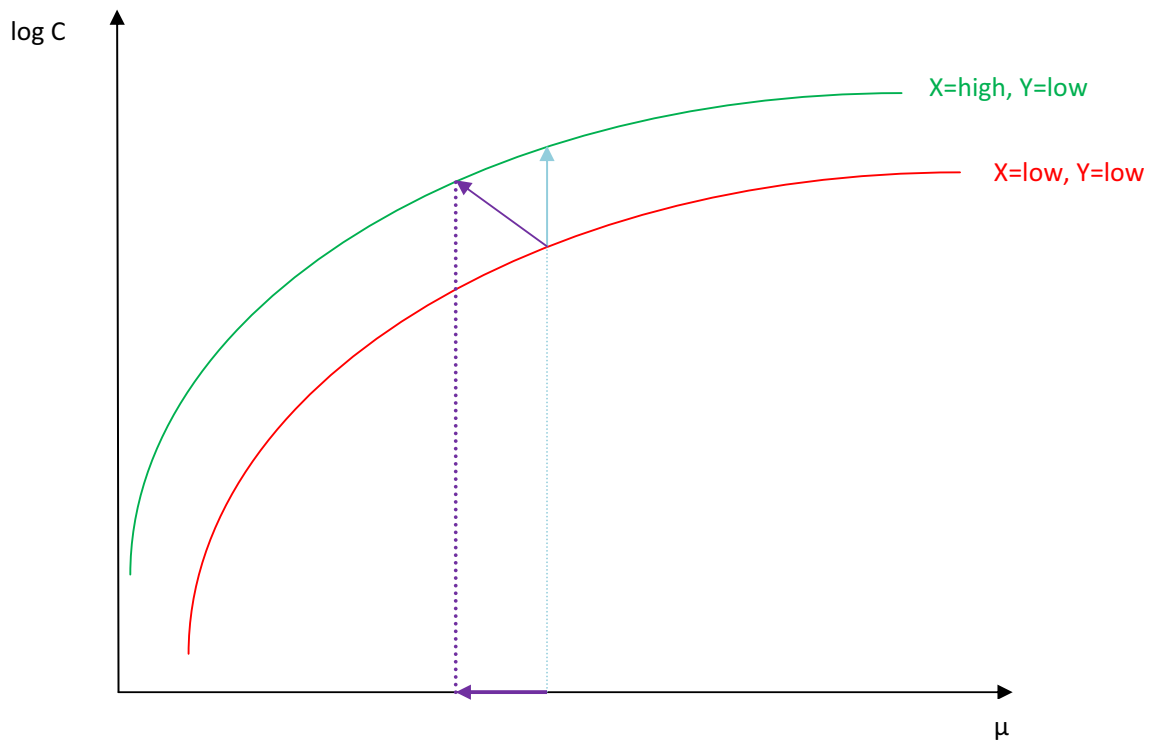


FIG. 4 - Profiles of the logarithm of consumption for two realizations of the endowments.

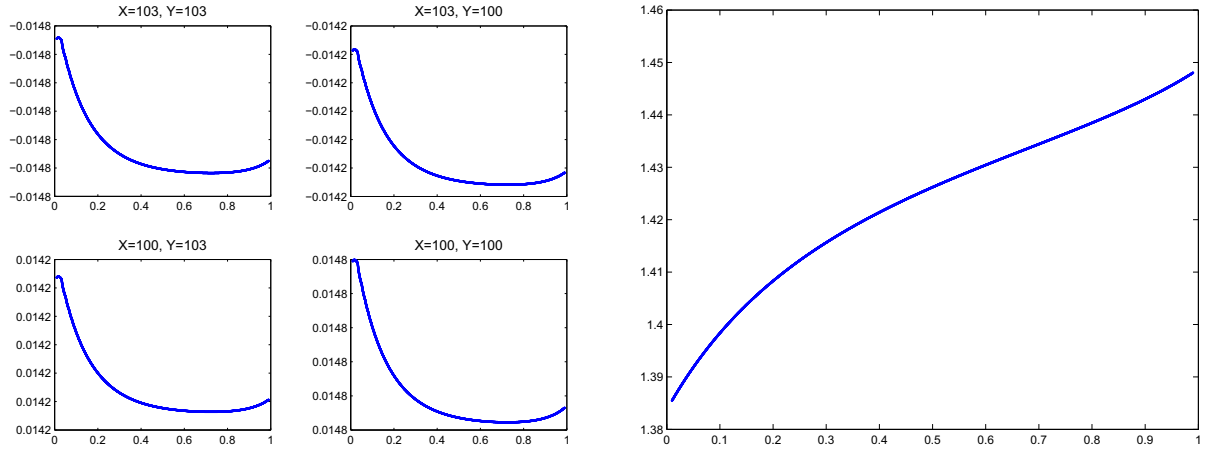


FIG. 5 - Conditional means (left panels) and conditional volatility (right panel) of consumption growth of agent 1 as a function of her current Pareto weight. The conditional volatility of consumption growth does not depend on the current endowment state, since endowments are i.i.d..

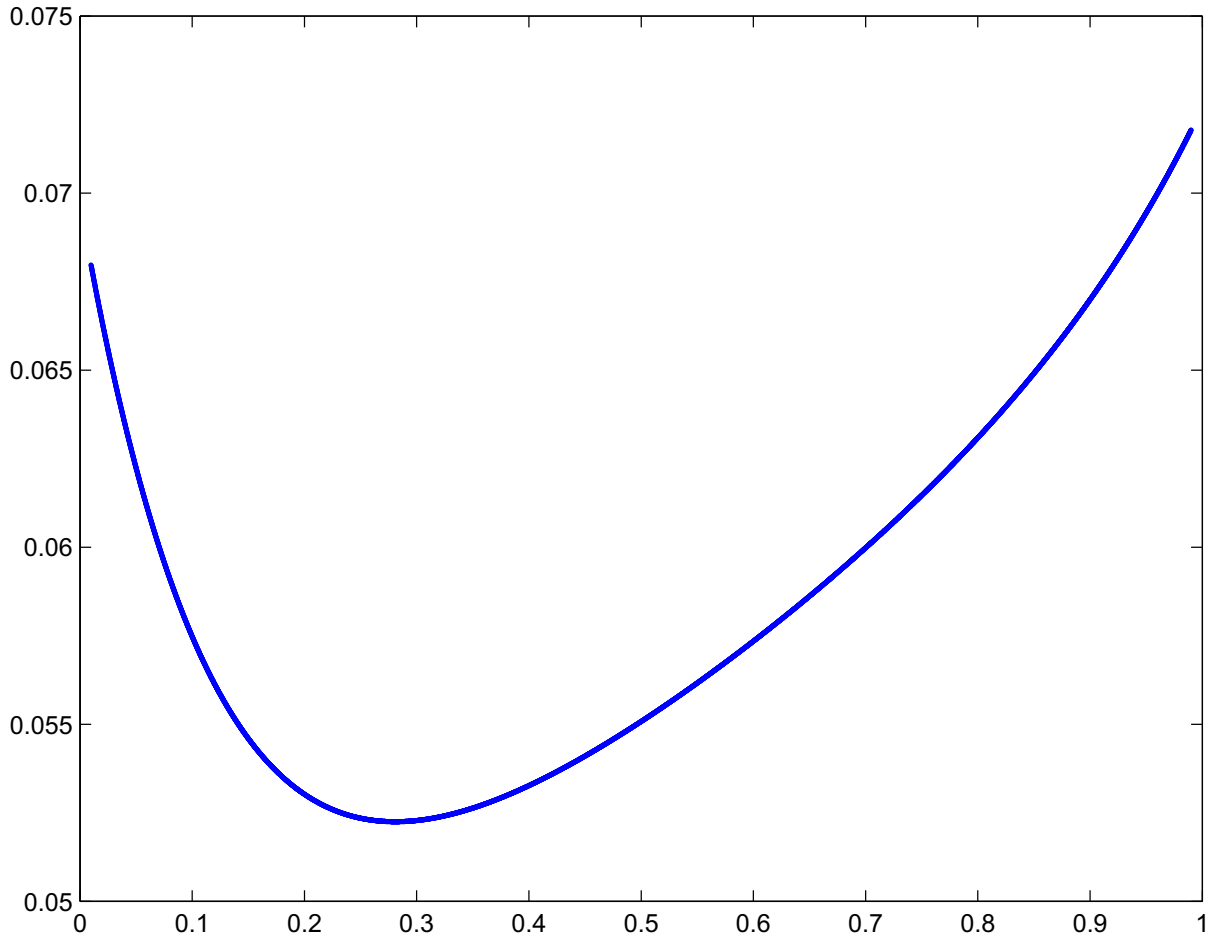


FIG. 6 - Conditional volatility of the utility of agent 1 as a function of the current Pareto weight. The preference parameters are: $\gamma = 25$, $\delta = 0.95$, and $\nu_1 = 0.98$.

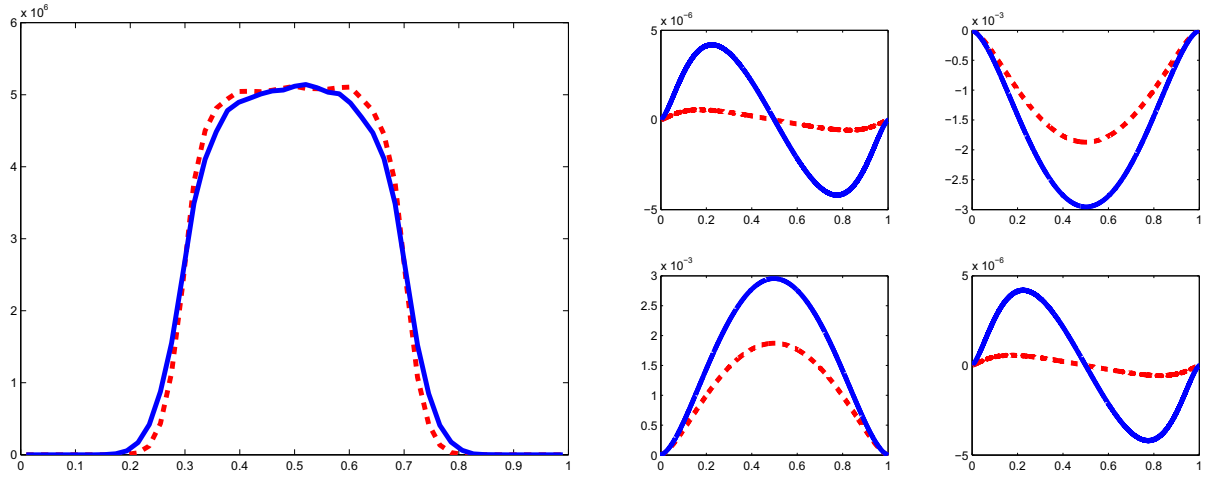


FIG. 7 - Comparison of invariant distributions of Pareto weights (left) and phase diagrams (right). The dashed lines refer to the case of $\gamma = 10$, while the thick lines refer to the baseline case of $\gamma = 25$.

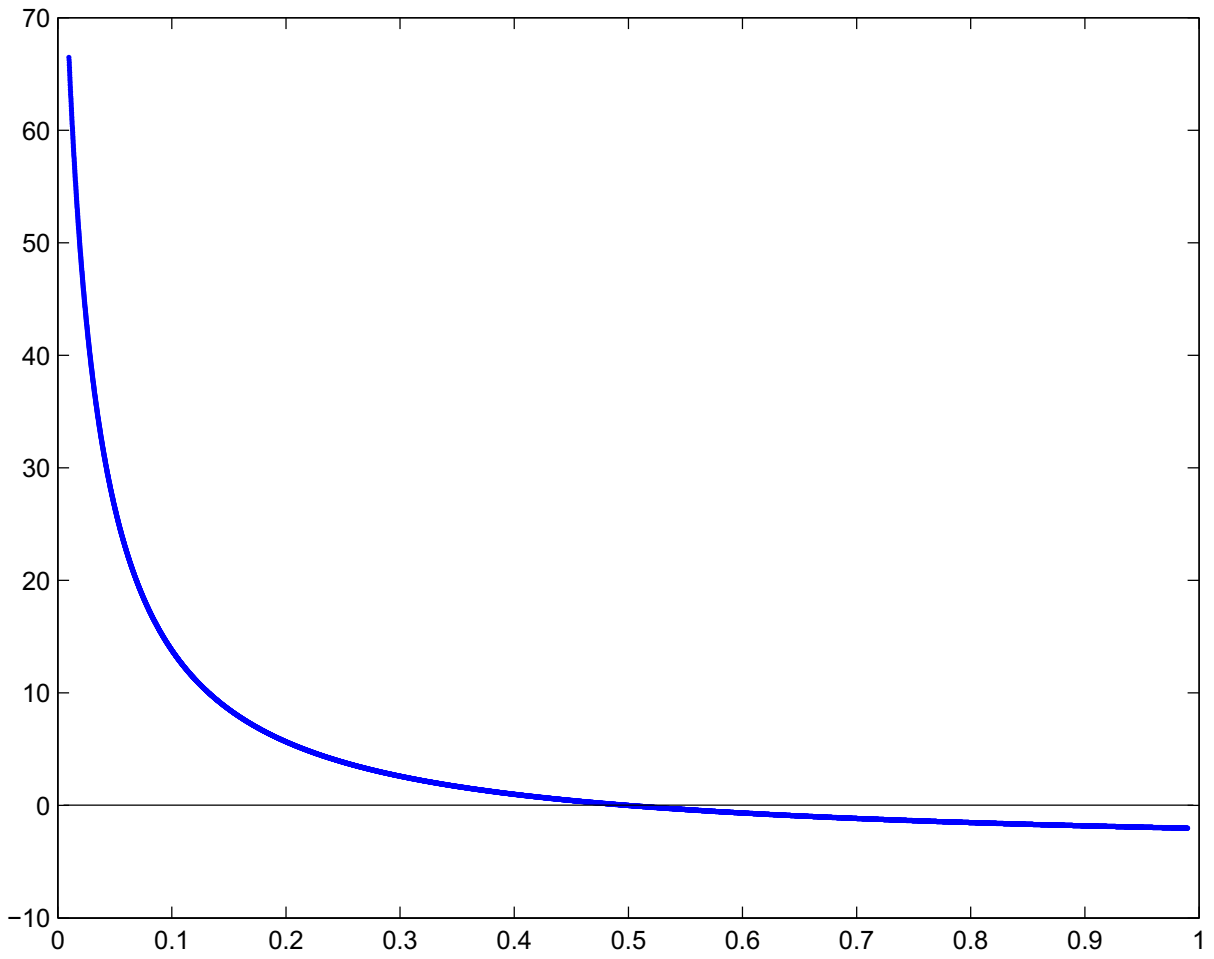


FIG. 8 - Net Export-Output ratio for the home country. The preference parameters are: $\gamma = 25$, $\delta = 0.95$, and $\nu_1 = 0.98$.

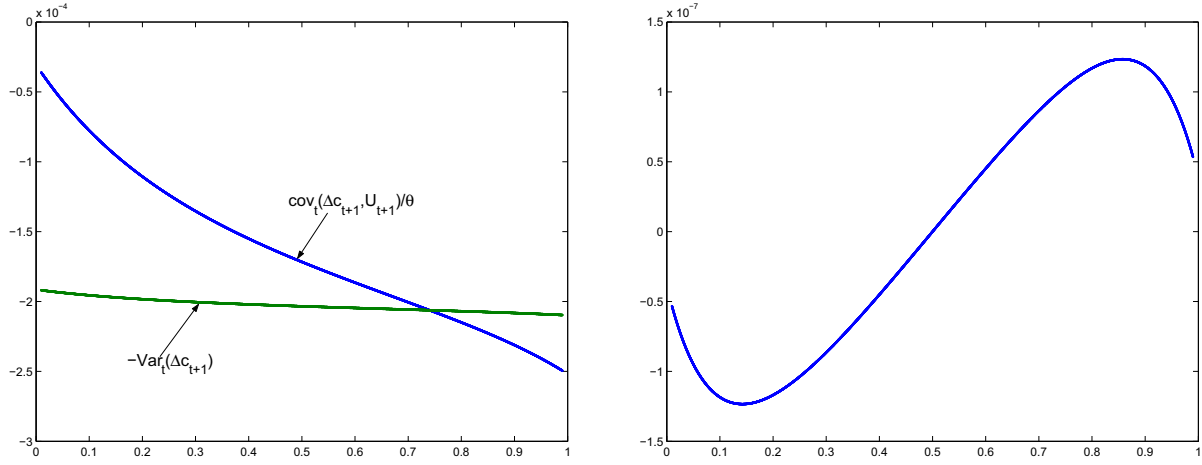


FIG. 9 - The left panel shows the profiles of the two major determinants of risk-free rates differentials as a function of the Pareto weight: $-\text{Var}_t(\Delta c_{t+1}^h) + \text{Var}_t(\Delta c_{t+1}^f)$ and $\text{Cov}_t(\Delta c_{t+1}^h, U_{t+1}^h)/\theta - \text{Cov}_t(\Delta c_{t+1}^f, U_{t+1}^f)$. The right panel shows the profile of the major determinant of expected exchange rate growth as a function of the Pareto weight: $\text{Var}_t(U_{t+1}^h) + \text{Var}_t(U_{t+1}^f)$. The preference parameters are: $\gamma = 25$, $\delta = 0.95$, and $\nu_1 = 0.98$.

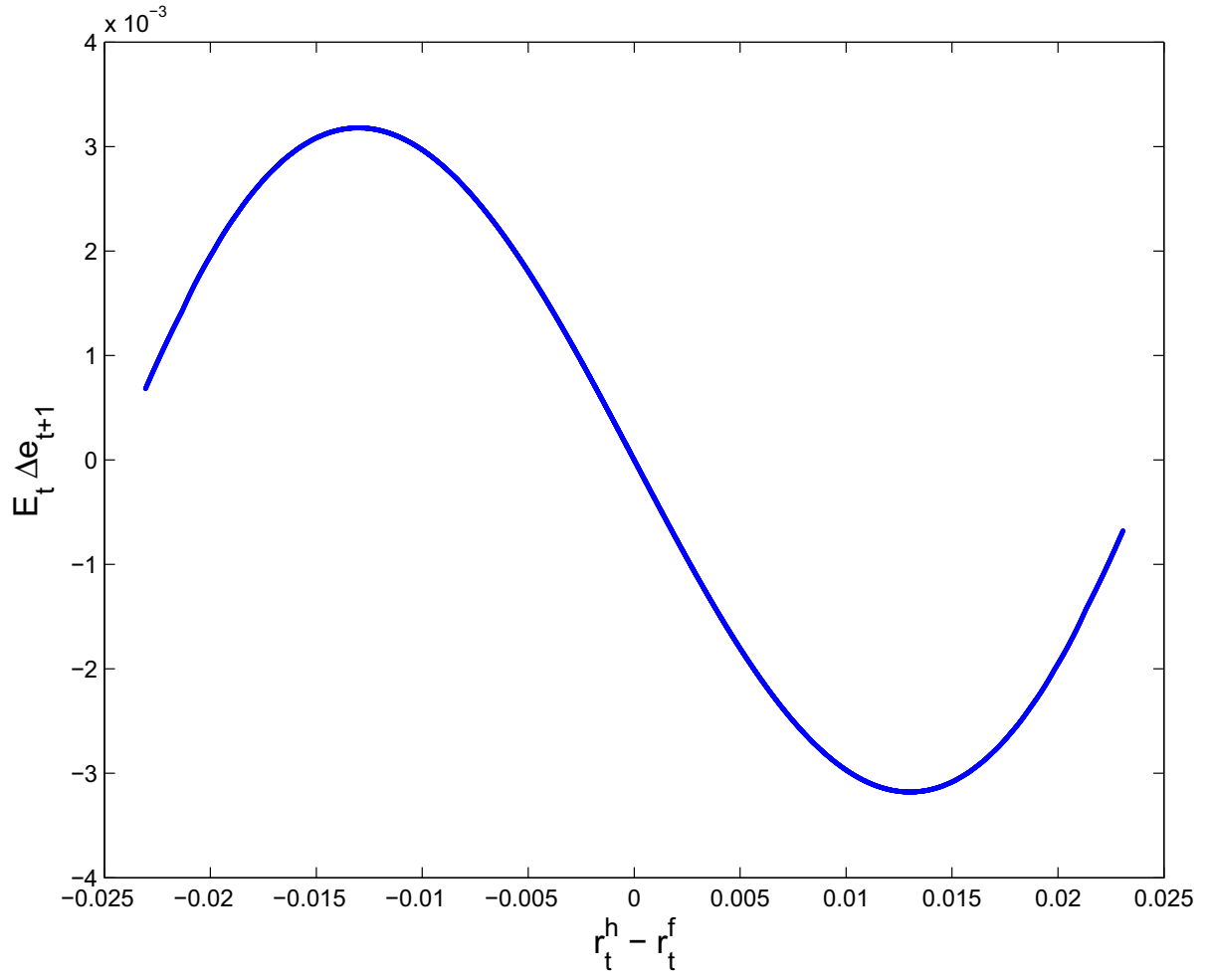


FIG. 10 - Relationship between expected exchange rate growth and interest rate differentials as a function of the current Pareto weight. The preference parameters are: $\gamma = 25$, $\delta = 0.95$, and $\nu_1 = 0.98$.