

Creditor Rights, Inequality and Development in a Neoclassical Growth Model*

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Abstract

I study the implications of the limited enforceability of credit contracts for inequality and economic growth. I introduce limited enforcement into a deterministic neoclassical growth model. Two types of agents differ in their initial wealth, ability and patience and each operate a private firm. The agents can borrow and lend to each other but face enforcement constraints. This results in capital being misallocated across agents. Three main conclusions are obtained from this model. First, capital misallocation disappears in the long run if agents are equally patient. In contrast, if agents' discount rates differ, capital misallocation persists asymptotically. Second, poor creditor rights magnify the effect of heterogeneity in ability on long run wealth inequality, because wealth accumulation functions as a substitute for poor creditor rights. Third, the interest rate is generally *lower* than in an economy without enforcement constraints.

Introduction

I study how the limited enforceability of credit contracts affects the allocation of capital in a neoclassical growth model with heterogeneous agents. Poor creditor rights lead to capital misallocation and hence lower aggregate total factor productivity. I ask under which conditions

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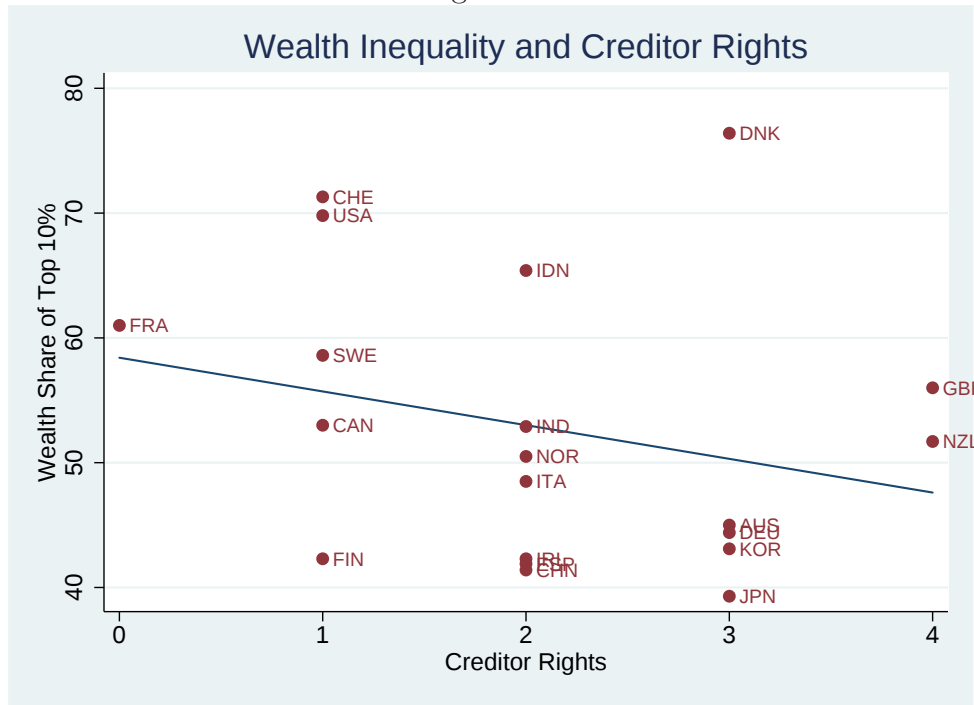
this misallocation persists or disappears over time, and show that differential impatience of agents is a force towards persistent misallocation. I further show that poor creditor rights typically increase wealth inequality and lower interest rates. In contrast to the leading papers on financial market imperfections in growth models (Banerjee and Newman, 1993; Aghion and Bolton, 1997; Piketty, 1997; Banerjee and Duflo, 2005; Buera and Shin, 2008; Buera et al., 2008), I emphasize the endogenous and forward-looking nature of credit constraints while maintaining a high degree tractability. To my knowledge, the conclusions about the role of discounting for persistent misallocation, and the effect of poor creditor rights on wealth inequality are also new.

The main purpose of this paper is to understand the implications of the limited enforceability of credit contracts for inequality and economic growth. Limited enforceability of contracts in general and of credit contracts in particular seems to be an important component of modern economies. Legal creditor rights, that is how easy it is to enforce a credit contract in court, differ substantially across countries and are important determinants of private credit (e.g. La Porta et al., 1997, 1998; Djankov et al., 2007). Poor creditor rights in turn imply that capital markets don't function smoothly which potentially results in the misallocation of capital. See Banerjee and Duflo (2005) and Banerjee and Moll (2009) and the references cited therein for evidence on misallocation. A growing body of work has been interested in the effects of misallocation on aggregate output and Total Factor Productivity (e.g. Banerjee and Duflo, 2005; Hsieh and Klenow, 2007). However, as Banerjee and Moll (2009) argue one important question, that I will also try to address here, is why misallocation is in fact persistent.

Most work in this area concentrates on the effect that credit constraints and misallocation have on aggregate variables such as TFP and GDP, that is on inequality across countries. In my view, another interesting question is: What are the implications of credit constraints for inequality within a given country? Within-country inequality differs immensely across countries and there is (at least anecdotal) evidence that inequality is particularly high in developing countries. I want to argue in this paper that poor creditor rights are a potential reason for high within-country wealth inequality. As I will explain below, poor creditor rights can magnify the effect of heterogeneity in ability on long run wealth inequality, because wealth accumulation functions as a substitute for poor creditor rights. Figure 1 presents some suggestive evidence that such a negative correlation between the quality of creditor rights and wealth inequality is

in fact present in the data.¹ To summarize, I ask the following question: How do poor creditor

Figure 1:



rights affect inequality and development?

To tackle this question, I introduce limited enforcement into a deterministic neoclassical growth model. Two types of agents differ only in their initial wealth and each operate a private firm. The agents can borrow and lend to each other but face enforcement constraints. As such the model is a reinterpretation of Kehoe and Perri (2002).² Limited enforcement is an endogenous credit market imperfection which results in capital misallocation. Because credit markets are imperfect, the representative agent framework is invalid. As such this paper presents an example of what Banerjee and Duflo (2005) term “non-aggregate growth theory”. At the same time it incorporates elements from contract theory. Both contract theory and non-aggregate growth theory usually rely heavily on numerical simulations. The main methodological contribution of this paper is then to provide some analytic results in a model that combines elements from both fields. These analytic results aid in understanding the economic mechanisms at work. I will briefly comment on the setup of the model and draw some connections to related theoret-

¹The data sources used for this regression are creditor rights data from Djankov et al. (2007) and wealth inequality data from James B. Davies (2008)

²Kehoe and Perri study the implications of limited enforcement for international business cycles in a two-country model. Their two countries correspond to my two types of agents.

ical literature. As already mentioned, the setup is very similar to Kehoe and Perri (2002) albeit with a different interpretation. In contrast, to their paper however, my model is deterministic. This delivers a majority of the analytic tractability. Two other papers with deterministic models the mathematical structure of which are very similar are Ray (Ray) and Acemoglu, Golosov, and Tsyvinski (Acemoglu et al.). They also stress the importance of discount rates. In contrast, to each of them my model is set up in continuous time. This delivers additional tractability.

Solving the model in continuous time discloses a simple connection between two branches of the literature on optimal contracts: The literature using the Lagrangian method developed Marcet and Marimon (1999) and the promised value literature started by Spear and Srivastava (1987). Time-varying Pareto weights of Marcet and Marimon (1999) are simply the costates corresponding to promised value as a state.³ The continuous time setup also implies that the equilibrium of the economy is characterized by a system of ordinary differential equations. While the dimension of this system of ODEs is greater than two and cannot be analyzed with a standard phase diagram, the system is very similar in structure to the standard neoclassical growth model. In particular, these ODEs have a steady state around which they can be linearized. I prove that the linearized system of differential equations is saddle-path stable, using a result from a field of linear algebra known as "Inertia Theory". The results I use are from Datta (1999). For another example of their application to economics see Moll (2008).

Other related literature. In addition to the empirical papers already cited above, my paper is related to a broad theoretical literature on the macro-implications of credit market imperfections. Early examples are Banerjee and Newman (1993), Aghion and Bolton (1997), and Piketty (1997). Often these papers predict interesting phenomena such as persistent inequality and poverty traps. However, one main shortcoming of these analyses is that they are based on strong assumptions. Generations are assumed to live for a single time period and the evolution of wealth is determined by a warm-glow bequest motive that is not forward-looking. In contrast to such Solow-type overlapping-generations models of economic growth, modern macroeconomics typically assumes that agents are forward-looking and accumulate assets optimally as in the neoclassical growth model. Also, in the above papers usually either

³Marcet and Marimon (1999) already observed that a time-varying Pareto weight "acts as a costate variables". This statement is made more precise here.

the interest rate and/or the identity of borrowers and lenders is exogenous. More recent contributions (Townsend and Ueda, 2006; Buera et al., 2008; Banerjee and Moll, 2009) address these problems. As already mentioned, one main contribution of this paper is to emphasize the forward-looking nature of both savings decisions and credit constraints while keeping the model relatively tractable.

The paper is organized in four sections. Section 1 presents the model. Section 2 analyzes the dynamics. Section 3 discusses the extension. Section 4 concludes.

1 Setup

1.1 Preferences and Technology

Time is continuous. There is a continuum of two types of households indexed by $i = 1, 2$. Preferences for both types of households are given by

$$\int_0^\infty e^{-\rho_i t} u(c_i) dt, \quad i = 1, 2. \quad (1)$$

Note that agents potentially differ in their discount rates ρ_i . Without loss of generality, I assume that type 2 agents are the more patient of the two so that $\rho_1 \geq \rho_2$. I assume that the function u is strictly increasing and strictly concave and satisfies standard Inada conditions. Each household owns a private firm which uses k_i units of capital to produce $z_i f(k_i)$ units of output. z_i is an agents entrepreneurial ability. I assume that the function f is strictly increasing and strictly concave and satisfies standard Inada conditions. Capital depreciates at the rate δ .

1.2 Market Structure and Equilibrium

Denote by $a_i(t)$ the wealth of agent i and by $r(t)$ the (endogenous) interest rate. Then agent i 's wealth evolves according to⁴

$$\dot{a}_i = z_i f(k_i) - (r + \delta)k_i + r a_i - c_i, \quad a_i(0) = a_{i0} \quad (2)$$

⁴I am using here that by arbitrage the rental rate of capital is $R = r + \delta$. More formally, consider an agent financing a given amount of capital k . He can split his wealth a between own capital usage k^o and bank deposits $a - k^o$ which earn interest r ; to supplement k^o , he can rent capital k^R in the rental market; he pays depreciation costs only on own capital k^o . His problem is to minimize $\delta k^o + R k^R - r(a - k^o)$ subject to $k^R + k^o \geq k$. An interior solution requires that $R = r + \delta$.

It is important to draw a distinction between wealth and capital here. Agent i 's wealth a_i is the assets in his possession *before* engaging in any trade. In contrast, agent i 's capital k_i is the amount of assets (machines) *used in production*, that is *after trading*. Only in the absence of capital markets, we would have $a_i = k_i, i = 1, 2$.

In addition, at each point in time, agents face borrowing or enforcement constraints. In particular, I assume that agents who borrow capital can default and run away with a fraction $\phi \in [0, 1]$ of the capital they are using. There is some outside value $v_i(\phi k_i)$ associated with defaulting. I will consider different alternative default values below. For now, I only impose $u(0)/\rho_i \geq v_i(0), i = 1, 2$. The enforcement constraints are therefore:

$$\int_t^\infty e^{-\rho_i(\tau-t)} u(c_i(\tau)) d\tau \geq v_i(\phi k_i(t)), \quad i = 1, 2, t \geq 0 \quad (3)$$

Note that the standard growth model can be nested by letting $\phi = 0$. It is also worth noting that enforcement constraints act as *borrowing constraints*. This is because (3) can always be satisfied by choosing k_i on the right hand side small enough.

The structure of the model here is almost precisely the same as outlined in section 6.1 of Banerjee and Duflo (2005). The only difference is that borrowing constraints are endogenous rather than exogenous and that I restrict the types of agents to two.

Definition 1 *An equilibrium with enforcement constraints consists of a time path for the interest rate $r(t), t \geq 0$ and allocations $(a_i(t), k_i(t), c_i(t)), i = 1, 2, t \geq 0$ such that*

- *Given the interest rate $r(t)$ and outside options $v_i(\phi k_i(t))$, for all a_{i0} , the allocations $(a_i(t), k_i(t), c_i(t))$ maximize (1) subject to (2) and (3).*
- *markets clear for all t :*

$$a_1(t) + a_2(t) = k_1(t) + k_2(t) \quad (4)$$

[Three sources of potential heterogeneity: The agents' initial wealth a_{i0} , their ability z_i and their discount rates ρ_i .]

This completes the entirely standard description of the economy. My ultimate goal is to analyze the dynamics of the capital stocks k_1 and k_2 under limited enforcement. It is, however, instructive to have a closer look at a competitive equilibrium without any such imperfections, that is the case where $\phi = 0$. As is standard, such an equilibrium can be solved as a planning problem. This prelude to the actual problem of interest is contained in the next section,

allowing me to both establish some useful benchmarks and introduce some notation in an entirely standard setting.

2 Unconstrained Planning Problem

Consider now the case $\phi = 0$ so that (3) never binds. The easiest way to characterize the competitive equilibrium is to solve a *planning problem*:

$$\begin{aligned} \max_{c_1, c_2, k_1, k_2} \quad & \alpha_0 \int_0^\infty e^{-\rho_1 t} u(c_1) dt + \int_0^\infty e^{-\rho_2 t} u(c_2) dt \quad \text{s.t.} \\ & \dot{k} = z_1 f(k_1) + z_2 f(k_2) - \delta k - c_1 - c_2, \quad k_1 + k_2 \leq k, \quad k(0) = k_0 \end{aligned}$$

The state variable in this dynamic program is k , the aggregate capital stock or aggregate wealth. In contrast, the capital usage of the two types of agents, k_1 and k_2 are controls with the restriction that

$$k_1 + k_2 \leq k \tag{5}$$

I will refer to this constraint as the "market clearing constraint". α_0 is the Pareto weight that the planner places on agent 1. This Pareto weight is determined by the three sources of potential heterogeneity: The agents' initial wealth a_{i0} , their ability z_i and their discount rates ρ_i . It is convenient to define a time-varying Pareto weight

$$\alpha(t) \equiv \alpha_0 e^{-(\rho_1 - \rho_2)t}.$$

The planner's objective can then be written as

$$\int_0^\infty e^{-\rho_2 t} \{ \alpha(t) u(c_1(t)) + u(c_2(t)) \} dt.$$

If type 2 agents are more patient $\rho_1 > \rho_2$, then the Pareto weight goes to zero asymptotically, so that the more patient type 2 agents get all the surplus in the long run.

The Hamiltonian for the problem is

$$\mathcal{H}(k, \lambda) = \alpha u(c_1) + u(c_2) + \lambda [z_1 f(k_1) + z_2 f(k_2) - \delta k - c_1 - c_2] + q[k - k_1 - k_2]$$

where λ is the co-state corresponding to k and q is the Lagrange multiplier on the market clearing constraint (5). It is well known that the solution to the planner's problem can be split up into two stages. First, solve the following two sharing rules:⁵

$$U(c, \alpha) = \max_{c_1, c_2} \alpha u(c_1) + u(c_2) \quad \text{s.t.} \quad c_1 + c_2 \leq c \tag{6}$$

⁵For example with CRRA utility with parameter σ , we have $U(c, \alpha) = [\alpha^{1/\sigma} + 1]^\sigma u(c)$.

$$F(k) = \max_{k_1, k_2} z_1 f(k_1) + z_2 f(k_2) \quad \text{s.t.} \quad k_1 + k_2 \leq k \quad (7)$$

Second, solve the system of three ODEs

$$\begin{aligned} \dot{\lambda} &= \lambda[\rho_2 + \delta - F'(k)] \\ \dot{k} &= F(k) - \delta k - U_c^{-1}(\lambda, \alpha) \\ \dot{\alpha} &= -[\rho_1 - \rho_2]\alpha \end{aligned} \quad (8)$$

where $U_c^{-1}(\cdot, \lambda)$ is the inverse of the marginal utility of consumption with respect to its first argument. Also, any solution to (8) must satisfy the boundary conditions

$$k(0) = k_0, \quad \alpha(0) = \alpha_0, \quad \lim_{t \rightarrow \infty} e^{-\rho_2 t} \lambda(t) k(t) = 0$$

Equation (8) completely characterizes the evolution of the aggregate capital stock k and the co-state λ .

Note that the marginal products of capital are equalized at all points in time

$$z_1 f'(k_1) = z_2 f'(k_2). \quad (9)$$

That is, capital is allocated optimally across agents. This will not necessarily be true anymore when agents can run away with a positive fraction of their operating capital, $\phi > 0$.

I'm particularly interested in the evolution of *individual* welfare over time. To this end, define type 1's value in the unconstrained planning problem

$$w^u(k(t), \alpha) \equiv \int_t^\infty e^{-\rho_1(t-s)} u(c_1(s)) ds \quad (10)$$

where $c_1(t)$ solves (6) for all t , and $\lambda(t)$ and $k(t)$ satisfy the ODE (8) with initial condition $k(0) = k_0$.

3 Constrained Planning Problem

Now, consider the case $\phi > 0$ so that defaulting agents get to keep a positive fraction of their operating capital stock. We can still solve a planning problem to describe the competitive equilibrium of this economy. In particular, one simply adds the constraints (3) to the constraint set of the planning problem. Denote by $\alpha \in (0, 1]$, the relative Pareto weight that the planner places on type 1 agents. As discussed above, only the enforcement constraints of type 1 agents might ever bind so that I can ignore the constraints of type 2 agents. To make the problem interesting, I impose the following restrictions on the default values $v_i(\phi k_i)$.

Assumption 1

$$w^u(k, 1) \geq v_i(\phi k/2), \quad \text{all } k$$

where $w^u(k, \alpha)$ is type 1's unconstrained value function defined in the previous section. Recall also, that $k/2$ is the unconstrained capital allocation. This assumption says that if we remove the asymmetry of initial wealth from the economy and set $\alpha = 1$, the enforcement constraints will be satisfied. For example, it is easy to see that Assumption 1 is satisfied if the consequence of default is being cast into autarky forever.

The *constrained planning problem* is:

$$\begin{aligned} \max_{c_1, c_2, k_1, k_2} \int_0^\infty e^{-\rho_2 t} u(c_2) dt \quad \text{s.t.} \quad & \int_0^\infty e^{-\rho_1 t} u(c_1) dt \geq w_{10} \\ \dot{k} = f(k_1, z_1) + f(k_2, z_2) - \delta k - c_1 - c_2, \quad & k_1 + k_2 \leq k, \quad k(0) = k_0 \\ \int_t^\infty e^{-\rho_i(\tau-t)} u(c_i(\tau)) d\tau \geq v_i(\phi k_i(t)), \quad & i = 1, 2, \quad t \geq 0. \end{aligned} \quad (11)$$

3.1 The Optimal Contract

To solve the problem, it will be convenient to define an agent's continuation value or equivalently his value from sticking to the contract

$$w_i(t) \equiv \int_t^\infty e^{-\rho_i(\tau-t)} u(c_i(\tau)) d\tau \quad (12)$$

Clearly, the enforcement constraints reduces to

$$w_i(t) \geq v_i(\phi k_i(t)), \quad i = 1, 2 \quad t \geq 0. \quad (13)$$

Differentiating (12) with respect to time results in the continuous time Bellman equation

$$\dot{w}_i = \rho_i w - u(c_i) \quad (14)$$

I follow the branch of the literature that uses continuation values as state variables (Spear and Srivastava, 1987) and include w_i in the state space. The law of motion (14) is the continuous time version of the “promise-keeping constraint”. Since the problem is stated in continuous time, the next logical step is to define corresponding costates and to write down a Hamiltonian. Denote these co-states by $\psi_i, i = 1, 2$. There are now three states (k, w_1, w_2) and three costates

$(\lambda, \psi_1, \psi_2)$. The corresponding Hamiltonian is

$$\begin{aligned} \mathcal{H}(k, w_1, w_2, \lambda, \psi_1, \psi_2) = & u(c_2) + \lambda[z_1 f(k_1) + z_2 f(k_2) - \delta k - c_1 - c_2] \\ & + q[k - k_1 - k_2] + \sum_{i=1,2} \psi_i[\rho_i w_i - u(c_i)] + \mu_i[w_i - v_i(\phi k_i)] \end{aligned} \quad (15)$$

where $\mu_i(t)$ are the Lagrange multiplier on the enforcement constraints (13).

3.2 Euler Equations

The Euler equations are derived in the usual way. It is useful to define

$$R \equiv \frac{q}{\lambda}$$

As will shortly become clear, R is the rental rate of capital. The following system of differential equations holds

$$\begin{aligned} \dot{\lambda} &= \lambda[\rho_2 + \delta - R] \\ \dot{\psi}_1 &= -[\rho_1 - \rho_2]\psi_1 - \mu_1 \\ \dot{\psi}_2 &= -\mu_2 \\ \dot{k} &= z_1 f(k_1) + z_2 f(k_2) - \delta k - c_1 - c_2 \\ \dot{w}_1 &= \rho_1 w_1 - u(c_1) \\ \dot{w}_2 &= \rho_2 w_2 - u(c_2) \end{aligned} \quad (16)$$

They are complemented by the first order conditions for c_1, c_2, k_1 and k_2 and the complementary slackness condition for the enforcement constraint

$$-\psi_1 u'(c_1) = (1 - \psi_2) u'(c_2) = \lambda \quad (17)$$

$$f'(k_i) - \mu_i \frac{\phi v'_i(\phi k_i)}{\lambda} = R, \quad i = 1, 2 \quad (18)$$

$$\mu_i[w_i - v_i(\phi k_i)] = 0, \quad i = 1, 2 \quad (19)$$

Now consider equations (18) and (19). These two equations determine how the aggregate capital stock k is split between the two types. If type 1's enforcement constraint is slack, then $\mu = 0$ and (18) collapses to the familiar condition of equalized marginal products. This condition is crucial and will be discussed in more detail below.

3.3 Steady State

The steady state of the economy $(\lambda^*, \psi_1^*, \psi_2^*, k^*, w_1^*, w_2^*)$ is defined by setting all time derivatives equal to zero in (24).

Proposition 1 *There is a continuum of steady states $(\lambda^*, \psi_1^*, \psi_2^*, k^*, w_1^*, w_2^*)$. In any steady state*

(1) *When $\rho_1 = \rho_2$, there is no capital misallocation in steady state and the steady state capital stock k^* is equal to the first-best steady state capital stock of the standard neoclassical growth model.*

(2) *When $\rho_1 \neq \rho_2$, there is capital misallocation in steady state.*

(All proofs are in the Appendix.)

3.4 Dynamics

For simplicity, consider the case $z_1 > z_2$, and $a_{10} = a_{20}$. Then at time zero, the more able type 1 agents will be borrowing from type 2 agents. Of course, only borrowing agents can default, which is equivalent to saying that only their enforcement constraints might bind.

As above, I continue to assume that $\rho_1 \geq \rho_2$. We can then also say something about the binding pattern of the enforcement constraints over time. Consider first the case $\rho_1 = \rho_2$. Because there is no uncertainty and everything except initial wealth is symmetric, it is also easy to see that only type 1 agents, will *ever* borrow. Therefore, only their enforcement constraints will *ever* bind. Next, consider the case $\rho_1 > \rho_2$. I have argued above that in the unconstrained allocation consumption of the more impatient type 1 agents will go to zero in the long-run. This implies that also type 1's wealth goes to zero. Also in this case, type 1 agents will always be borrowing. I can therefore ignore the enforcement constraints of type 2 agents and drop the ODEs for ψ_2 and w_2 . Because there is then no ambiguity about an agent's identity, I write (w, ψ, μ) instead of (w_1, ψ_1, μ_1) so that the system of ODEs becomes

$$\begin{aligned}\dot{\lambda} &= \lambda[\rho_2 + \delta - R] \\ \dot{\psi} &= -\psi[\rho_1 - \rho_2] - \mu \\ \dot{k} &= z_1 f(k_1) + z_2 f(k_2) - \delta k - c_1 - c_2 \\ \dot{w} &= \rho_1 w - u(c_1)\end{aligned}\tag{20}$$

Consider the first of these equations (17), together with the law of motion for ψ . An easy connection can be drawn to the time-varying Pareto-weights of Marcet and Marimon (1999) and Kehoe and Perri (2002). In particular, define

$$\alpha(t) \equiv -\psi(t)$$

If we choose w_0 such that $-\psi(0) = \alpha_0$, $\alpha(t)$ can be interpreted as a time-varying Pareto weight

$$\frac{u'(c_2(t))}{u'(c_1(t))} = \alpha(t), \quad \dot{\alpha}(t) = -[\rho_1 - \rho_2]\alpha(t) + \mu(t), \quad \alpha(0) = \alpha_0 \quad (21)$$

The interpretation of (21) is clear: $\alpha(t)$ is agent 1's Pareto weight. Suppose for the moment that agents are equally patient, $\rho_1 = \rho_2$. Then, this Pareto weight is the initial Pareto weight plus Lagrange multipliers accumulated over the past.⁶ Marcet and Marimon (1999) already observed that this time-varying Pareto weight "acts as a costate variable." Stating the problem in continuous time makes this observation very precise.

Embedded in equation (21) is the effect that limited enforcement has on the allocation of consumption across agents. Suppose for the moment that the utility function $u(\cdot)$ is of the CRRA type with parameter σ . Differentiating (21) with respect to time yields

$$\sigma \left[\frac{\dot{c}_1}{c_1} - \frac{\dot{c}_2}{c_2} \right] = -[\rho_1 - \rho_2] + \frac{\mu}{\alpha}$$

In the benchmark case with equal discounting, $\rho_1 = \rho_2$, the right-hand side is positive whenever type 1's enforcement constraint binds. In that case

$$\frac{\dot{c}_1}{c_1} > \frac{\dot{c}_2}{c_2}.$$

That is agents with binding constraints experience higher consumption growth. This result is typical in the literature on optimal contracts: In effect, the planner "bribes" the agents not to default by assigning them a higher continuation value.

Proposition 2 *In an equilibrium with enforcement constraints:*

⁶Equation (21) is the continuous time analogue of equation (7) in Kehoe and Perri (2002). The same differential equation for α could have been derived directly from the Lagrangian for (11) by use of an integration by parts argument. This is not surprising for two reasons: (i) deriving the differential equations for a Hamiltonian system from a Lagrangian fomulation makes use of an integration by parts argument; (ii) Marcet and Marimon (1999) use Abel's summation formula which is the discrete time version of integration by parts.

(1) When $\rho_1 = \rho_2$, capital misallocation disappears asymptotically. That is (9) holds as $t \rightarrow \infty$.

(2) When $\rho_1 > \rho_2$, capital misallocation persists even asymptotically.

When $\rho_1 = \rho_2$ capital misallocation disappears in the long-run. The reason is similar to that discussed in Buera and Shin (2008) and Banerjee and Moll (2009): Being credit constraint creates big incentives to save in the form of a high marginal product of capital which implies a high marginal rate of substitution between consumption today and consumption tomorrow. Therefore, individuals use wealth accumulation as a substitute for poorly functioning credit markets. This logic breaks down, however, if agents discount at differential rates. In that case, the marginal rate of substitution also depends on the discount rate so that an constrained agents might actually have lower incentives to save than a constrained one.

3.5 Local Stability

More can be said about the behavior of the dynamic system (20) for the case where individuals are equally patient $\rho_1 = \rho_2 (= \rho)$. In particular, I here show that the dynamics of the economy are very similar to that of the standard neoclassical growth model and that steady states are locally stable. I next define two useful constructs which I term "pseudo-aggregate production function" and "pseudo-aggregate utility function". While the representative agent framework is invalid, these constructs will play exactly the same role as the aggregate production function and the aggregate utility function in the unconstrained planning problem.⁷

The pseudo-aggregate production function is

$$F(k, w) = \max_{k_1, k_2} z_1 f(k_1) + z_2 f(k_2) \quad \text{s.t.} \quad k_1 + k_2 \leq k, \quad w \geq v_1(\phi k_1) \quad (22)$$

If we define the Lagrange multipliers on the first and second constraint to be R and μ/λ respectively, then the first order and complementary slackness conditions coincide with (18) and (19).

Similarly, the pseudo-aggregate utility function is

$$U(c, x) = \max_{c_1, c_2} \alpha u(c_1) + u(c_2) \quad \text{s.t.} \quad c_1 + c_2 \leq c, \quad u(c_1) \geq x \quad (23)$$

⁷I add the qualifier "pseudo" because these are clearly not an aggregate production/utility functions in the usual sense. In particular, they depend on agent 1's utility and continuation value.

Now, denote by λ and $-\psi$ the Lagrange multipliers on the first and second constraint respectively. Then the first order condition is (17).⁸

Lemma 1 *The pseudo-aggregate utility function $U(c, x)$ is strictly concave.*

Just as in the unconstrained planning problem, the (pseudo-) aggregate production and utility functions can be used to simplify (16). In particular, using the envelope theorem in (22), we see that

$$F_k(k, w) = R, \quad F_w(k, w) = \mu/\lambda$$

Similarly, applying the envelope theorem in (23), we have that

$$U_c(c, x) = \lambda, \quad U_x(c, x) = \psi \quad \text{or} \quad \partial U(c, x) = (\lambda, \psi)$$

where $\partial U : R^2 \rightarrow R^2$ is the gradient of U . Because U is concave, ∂U has an inverse. Then

$$(c, x) = (\partial U)^{-1}(\lambda, \psi) \quad \text{or} \quad (c, x) = [c(\lambda, \psi), x(\lambda, \psi)]$$

Using these properties of the (pseudo-) aggregate production and utility functions, (16) can be rewritten as a system of autonomous differential equations in (λ, ψ, k, w) only.

$$\begin{aligned} \dot{\lambda} &= \lambda[\rho + \delta - F_k(k, w)] \\ \dot{\psi} &= -\lambda F_w(k, w) \\ \dot{k} &= F(k, w) - \delta k - c(\lambda, \psi) \\ \dot{w} &= \rho w - x(\lambda, \psi) \end{aligned} \tag{24}$$

The boundary conditions are

$$k(0) = k_0, \quad w(0) = w_0 \quad \lim_{t \rightarrow \infty} e^{-\rho t} \lambda(t) k(t) = 0, \quad \lim_{t \rightarrow \infty} e^{-\rho t} \psi(t) w(t) = 0$$

These four ODEs and boundary conditions together with the solutions to the pseudo-aggregate production and utility functions (22) and (23) completely characterize the competitive equilibrium with enforcement constraints.

Having established the existence of a steady state, I analyze the local stability of the system of ODEs (24). I'm first interested in the binding pattern of the enforcement constraint (3). Figure 5 depicts the state space $K \times W$. The state space is partitioned into two regions, the

⁸Substituting in for the second constraint, and differentiating one finds that $U_x(c, x) = \alpha - u'(c_2)/u'(c_1) = \psi$ which can be positive or negative depending on α .

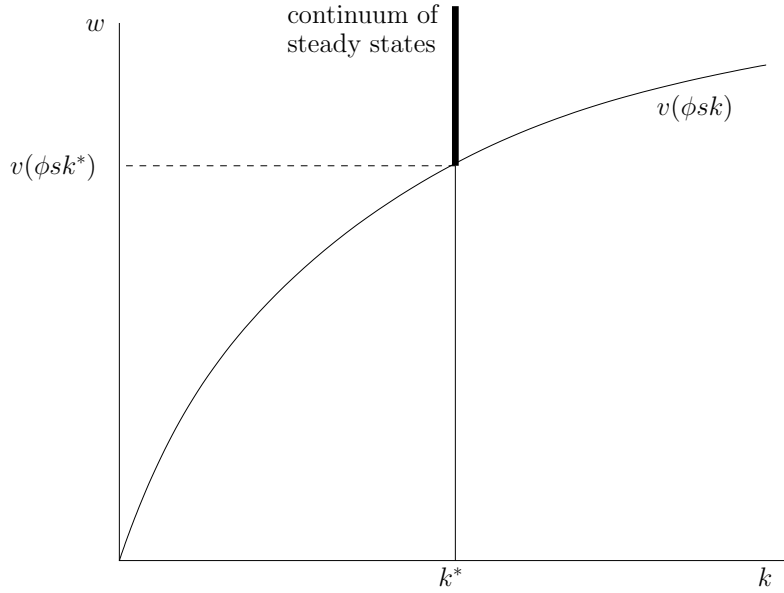


Figure 2: State Space $K \times W$

“constrained and unconstrained regions”. For all combinations of (k, w) below the line $v(\phi sk)$, agent 1’s enforcement constraint binds. For all combinations above this line, it does not. The system of ODEs is relatively easy to analyze, if the economy is always in the constraint region of the state space for all $t \geq 0$.⁹ I first provide conditions that guarantee that this is the case. Since I only analyze the local stability of the system of ODEs (20), these conditions only need to be valid in a neighborhood of the steady states.

Define the following critical value of type 1’s Pareto weight $\alpha(\phi; k)$ by

$$w^u(k, \alpha(\phi; k)) = v_1(\phi k/2)$$

where $w^u(k, \alpha)$ is type 1’s value in the unconstrained planning problem defined in (10). It is useful to work with the time-varying Pareto weights $\pi(t) = \alpha - \psi(t)$ defined above.

Lemma 2 *Agent 1’s enforcement constraint is slack if and only if $\pi(t) \geq \alpha(\phi, k(t))$.*

The following two assumptions imply that the economy will always be in the constraint region of the state space.

Assumption 2 *The outside option $v_1(\phi k_1)$ is such that $\alpha(\phi; k^*)$ is independent of k^* .*

⁹It is also easy to analyze if the economy is always in the unconstrained region. The dynamics are then identical with the ones of the standard neoclassical growth model. I therefore concentrate on the constraint case.

Assumption 3 *Type 1's enforcement constraint binds at date zero, $w_0 < v_1(\phi sk_0)$.*

Lemma 3 *Let Assumptions 2 and 3 be satisfied and let $\rho_1 = \rho_2$. Then the enforcement constraint (3) binds for all $t \geq 0$, in a neighborhood of the steady state. That is, for (k_0, w_0) sufficiently close to (k^*, w^*)*

$$w(t) \leq v_1(\phi sk(t)) \quad t \geq 0.$$

This Lemma establishes conditions under which the economy is always in the constraint region of the state space (Figure 5). It can then be seen that the steady state must lie at the boundary of the half-line $w^* \geq v_1(\phi k^*/2)$. That is, it satisfies $w^* = v_1(\phi k^*/2)$ so that constraint (3) is "barely binding". For all practical purposes, then the steady state is unique.

Linearizing (24) around this unique steady state yields

$$\dot{z} = Az, \quad A = \begin{bmatrix} 0 & 0 & \partial \dot{\lambda} / \partial k & \partial \dot{\lambda} / \partial w \\ 0 & 0 & \partial \dot{\psi} / \partial k & \partial \dot{\psi} / \partial w \\ \partial \dot{k} / \partial \lambda & \partial \dot{k} / \partial \psi & \rho & 0 \\ \partial \dot{w} / \partial \lambda & \partial \dot{w} / \partial \psi & 0 & \rho \end{bmatrix}, \quad z \equiv \begin{bmatrix} \lambda - \lambda^* \\ \psi - \psi^* \\ k - k^* \\ w - w^* \end{bmatrix}. \quad (25)$$

This can be written more compactly as

$$A = \begin{bmatrix} 0 & X \\ Y & \rho I \end{bmatrix}$$

where 0 is a 2×2 matrix of zeros and I is the 2×2 identity matrix. Denote the 2×2 Hessian matrices of the aggregate production function and aggregate utility function by $\partial^2 F(k, w)$ and $\partial^2 U(c, x)$ respectively. It is easy to show that X and Y are simply given by

$$X = -\lambda^* \partial^2 F(k^*, w^*), \quad Y = -[\partial^2 U(c^*, x^*)]^{-1}$$

As show in Lemma 1, $U(c, x)$ is strictly concave. It can also be show that $F(k^*, w^*)$ is negative definite. This immediately implies that both X and Y are positive definite matrices.

The stability proof makes use of some results from a field of linear algebra called "Inertia Theory". The following results are taken from Datta (1999).¹⁰ I'm interested in applying the theory to the 4×4 matrix A . However, the theory applies to matrices of any dimension. I choose to present it in its most general form. It will then also be applicable in the extension of the model to n types of agents.

¹⁰Benhabib and Nishimura (1981) use a similar result by Wielandt (1973) in a stability proof for n -sector growth.

Definition 2 *The inertia of a matrix A , denoted by $In(A)$, is the triplet $(\pi(A), \nu(A), \delta(A))$ where $\pi(A)$, $\nu(A)$ and $\delta(A)$ are, respectively, the number of eigenvalues of A with positive, negative, and zero real parts, counting multiplicities.*

A linear system of $2n$ differential equations such as (24), is saddle path stable if exactly half of the eigenvalues of A have negative real parts. Saddle path stability is therefore equivalent to the statement

$$In(A) = (n, n, 0)$$

The following theorem is very useful in establishing sufficient conditions under which indeed $In(A) = (n, n, 0)$.¹¹ In the following, $M \geq 0$ means that the matrix M is positive semidefinite.

Proposition 3 *Let $\delta(A) = 0$, and let W be a nonsingular symmetric matrix such that*

$$WA + A^T W = M \geq 0 \tag{26}$$

Then $In(A) = In(W)$.

This Theorem is a generalization of the Lyapunov Stability Theorem for matrices (see Theorem 3.2. in Datta (1999)). It is used in the same way, and its advantages and disadvantages are similar. In particular, the exercise boils down to finding a matrix W whose inertia we can easily determine. If we are not able to find such a matrix, the theorem is worthless.

Proposition 4 *$In(A) = (2, 2, 0)$ and hence the steady state $(\lambda^*, \psi^*, k^*, w^*)$ is locally stable.*

The linearized system (25) can be used to compute solutions to the constrained planning problem (11). In the Appendix I derive in more details the entries of the matrix A .

4 Discussion

I wrote in the introduction that the main purpose of this paper is to understand the interaction between limited enforceability of credit contracts, inequality and economic growth. I here discuss some of the predictions of the model. To do so I provide simulations with the following

¹¹It is natural to ask whether corresponding techniques are also available for systems of *difference* equations. Datta (1999) also defines a unit circle inertia as the triplet of the number of eigenvalues outside, inside and on the unit circle. Theorem 4.5. in this paper is the analogue of Theorem 3 for the unit circle inertia.

standard functional forms and parameter values: CRRA utility with parameter $\sigma = 2$; Cobb-Douglas production with capital share $\gamma = 0.3$; Discount and depreciation rates $\rho = \delta = 0.05$; The value of default is given by being cast into autarky forever. I vary the parameter capturing the quality of creditor rights ϕ , and the initial condition (k_0, w_0) .

4.1 Creditor Rights and Growth

I have already shown that capital misallocation disappears asymptotically if agents have the same discount rates, $\rho_1 = \rho_2$ so that limited enforcement does not matter for aggregate outcomes in the long run. However, one might ask how the transition path to the steady state is affected. To answer this question, consider the following experiment. Two countries are identical in every respect except their quality of creditor rights ϕ . In particular, they are characterized by the same initial conditions for aggregate capital and inequality (k_0, w_0) , but one has a higher ϕ (worse creditor rights) than the other. I have in mind here, say, Argentina and Chile in the 1970s. The value of default is assumed to be autarky. I assume that one country has perfect credit markets, $\phi = 0$ whereas the other is characterized by an intermediate level of creditor rights, $\phi = 0.5$. Initial conditions are $(k_0, w_0) = (0.8k^*, 0.7w^*)$. Figure 3 plots the time paths for the aggregate capital stocks, for individual marginal products of capital, for aggregate production and total factor productivity.¹² Consider first the paths of the aggregate capital stock for the two economies. Limited enforcement has an effect on aggregate savings. Perhaps somewhat surprisingly, capital accumulation is higher in the country with worse creditor rights, that is the *savings effect* is positive. This savings effect can go either way however, and depends crucially on the specification of the outside option. For example, with a linear outside option $v_i(\phi k) = \phi k$, it goes the other way. Consider, next the individual marginal products of capital. The presence of limited enforcement implies that marginal products are not equalized so that there is capital misallocation. More formally, from (18) a binding enforcement constraint implies that

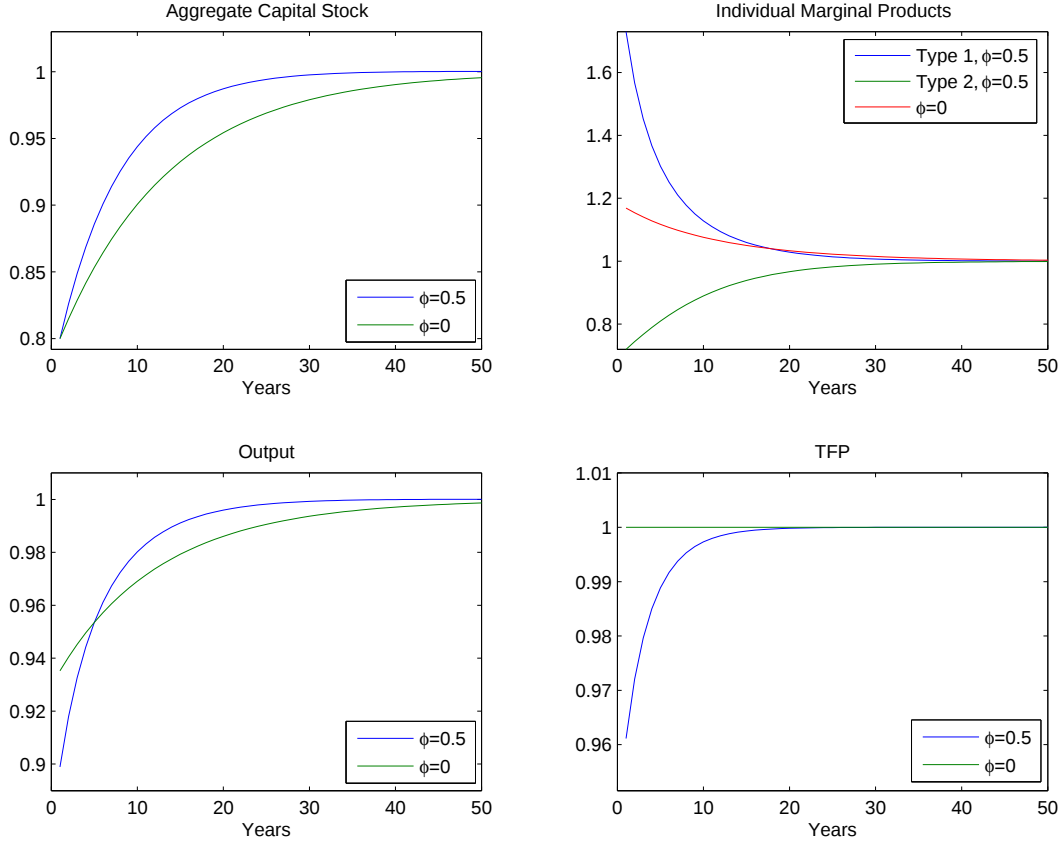
$$z_1 f'(k_1) > z_2 f'(k_2)$$

which can be observed in the second panel of Figure . This misallocation shows up as low output or equivalently as low TFP (panel 4). I term this the *TFP effect* of limited enforcement.¹³ I

¹²Total factor productivity is simply defined as the residual Y^{LE}/Y^{FB} .

¹³In the present example, the TFP loss is always below four percent. The size of the TFP loss depends on the number of agents in the economy. Consider the upper bound on the TFP loss which is given by the loss

Figure 3: Different Creditor Rights



have proved in proposition 4 that this misallocation disappears over time so that marginal products are equalized eventually. This result is very much in the spirit of Banerjee and Moll (2009) and the references therein. Finally, consider GDP in the two economies. The effect on GDP is the combination of the *savings* and *TFP* effects. Because the former is positive while the latter is negative, the overall effect is ambiguous. A similar conclusion holds when considering GDP *growth* as opposed to GDP *levels*: because of the two counteracting effects, the overall impact of bad creditor rights on growth is again ambiguous.¹⁴

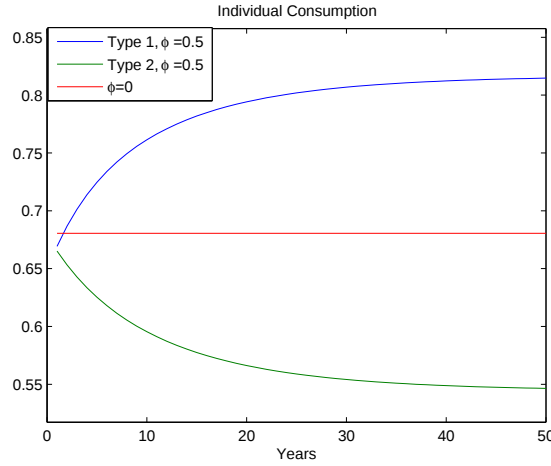
incurred from giving the all capital to one agent and none to others. It is $1 - k^\gamma / [n(k/n)^\gamma] = 1 - (1/n)^{1-\gamma}$, which is increasing in n .

¹⁴One might also be interested in the relation between initial inequality and the time path of aggregate output. One measure of initial inequality is type 1's promised value w_0 . Consider then two countries that have the same ϕ and k_0 but different k_0 . The time paths will be qualitatively identical to the ones in Figure 3. In particular, the effect of initial inequality on output levels and growth is ambiguous. This complex relationship is in line with the argument made by Banerjee and Duflo (2005) that "estimating the relationship between inequality and growth in a cross-country dataset [...] has, at best, very limited use."

[ADD GRAPH WITH STEADY STATE EFFECTS FOR THE CASE $\rho_1 \neq \rho_2$]

4.2 Creditor Rights and Inequality

Figure 1 provides some suggestive evidence that wealth inequality is higher in countries with worse creditor rights. I want to argue in this section that poor creditor rights magnify the effect of heterogeneity in ability on long run wealth inequality in the present model, providing a potential rationale for the relation observed in Figure 1. Compare again two countries: one with perfect credit markets, $\phi = 0$, and one with an intermediate quality of credit rights, $\phi = 0.5$. In both countries, agents are equally wealthy at time zero but type 1 has higher ability $z_1 > z_2$. The aggregate capital stock is in steady state $k_0 = k^*$. Figure 4.2 depicts the evolution of (consumption) inequality. In the country with perfect credit markets, $\phi = 0$



there is a complete separation between ownership and production. Individual consumptions are equal, $c_1(t) = c_2(t)$ for all t . In contrast, with poor creditor rights there is a fanning out of inequality. The consumption of the more able type 1s increases and that of the less able type 2's decreases. As discussed above, capital misallocation disappears asymptotically because agents use wealth accumulation as a substitute for poor creditor rights. The same effect is also responsible for the fanning out of inequality observed in Figure 3. With perfect credit markets, the more able type 1 agents, simply borrow some capital from the less able type 2 agents. This is not possible with poor creditor rights. Instead, the more able agents accumulate wealth in order to be able to exploit their superior production technologies.

4.3 Interest Rate

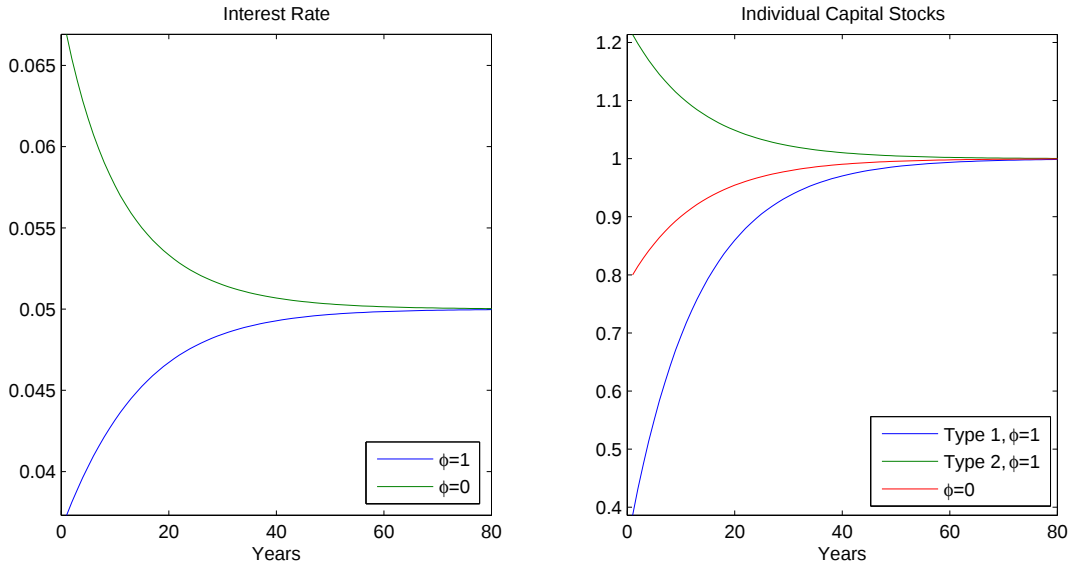
In their paper entitled "Growth Diagnostics", Hausmann et al. (2005) suggest that it should be possible to identify binding credit constraints by looking at interest rate data. I examine whether this is true in the present model. The interest rate in the economy is given by

$$r = \rho - \frac{\dot{\lambda}}{\lambda},$$

where $\lambda = u'(c_2)$ is the marginal utility of the unconstrained type 2 agents. That is, the interest rate is given by the continuous time analogue of the marginal rate of substitution across time for the unconstrained type 2 agents. In the model above, we also have that $r = R - \delta$. This is the usual arbitrage argument that says that the interest rate must equal the rental rate minus the cost of depreciation. The rental rate of capital R is given by the marginal product of the pseudo-aggregate production function which is in turn equal to the marginal product of the unconstrained type 2 agents: $R = F_k(k, w) = f'(k_2)$, so that

$$r = R - \delta = f'(k_2) - \delta.$$

Figure 4: Interest Rate



The time path of the interest rate is shaped by the same TFP and savings effects as in Figure 3 above. Consider first the case where the savings effect is absent so that aggregate capital accumulation is the same as in the unconstrained neoclassical growth model. In that case, the

interest rate is *always lower* compared to an economy without enforcement constraints. This follows immediately from the fact that the interest rate is determined by the marginal product of the *unconstrained* agent who gets a bigger share of the aggregate capital stock in the presence of enforcement constraints. In general equilibrium, the flip-side of the constrained marginal product being relatively high, is that the unconstrained marginal product is relatively low. Similar results about the interest rate are obtained in Kehoe and Levine (1993) and Alvarez and Jermann (2000). However, this logic ignores the presence of a savings effect: the aggregate capital stock might be higher which might overturn the TFP effect. This might result in the interest rate being higher in an economy with enforcement constraints. Overall, the effect of poor creditor rights on the interest rate is ambiguous. In that sense, the level of the interest rate is uninformative about the presence of credit constraints in an economy.

Low interest rates are also relevant with respect to an argument put forth by King and Rebelo (1993): If one wants to explain sustained growth by transitional dynamics of the standard neoclassical growth model, one generates extremely counterfactual implications for the time path of the interest rate. For example, King and Rebelo (1993) argue that if the neoclassical growth model were to explain the postwar growth experience of Japan, the interest rate in 1950 should have been around 500 percent. As can be seen in Figure 4, it is theoretically possible that both the capital stock and the interest rate approach the steady state *from below*, invalidating that part of the argument by King and Rebelo (1993).

5 Conclusion

TO BE DONE

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Appendix

5.1 Proof of Proposition 1

Part 1: Setting $\dot{\psi}_i = 0$, since $\rho_1 = \rho_2 = \rho$ it must be that in steady state $\mu_i^* = 0, i = 1, 2$. This immediately implies that

$$z_1 f'(k_1^*) = z_2 f'(k_2^*) = \rho + \delta$$

so that $k^* = k_1^* + k_2^*$ is the steady state capital stock of the standard neoclassical growth model. Any $w_i^* \geq v_i(\phi k_i^*), i = 1, 2$ satisfies $\mu_i^* = 0$. Now pick any such w_1^* . c_1^* is pinned down by $w_1^* = u(c_1^*)/\rho_1$. $c_2^* = c^* - c_1^*$ where $c^* = z_1 f(k_1^*) + z_2 f(k_2^*) - \delta k^*$. Any combination of $(\lambda^*, \psi_1^*, \psi_2^*)$ satisfying

$$-\psi_1^* u'(c_1^*) = (1 - \psi_2^*) u'(c_2^*) = \lambda^*$$

is consistent with it. Then all of $(\lambda^*, \psi_1^*, \psi_2^*, k^*, w_1^*, w_2^*)$ are pinned down.

Part 2: Without loss of generality, consider the case $\rho_1 > \rho_2$. Then $\mu_1^* = -[\rho_1 - \rho_2]\psi_1^* > 0$. Since only one type's enforcement constraint can bind, $\mu_2^* = 0$ so that

$$z_1 f'(k_1^*) > z_2 f'(k_2^*) = \rho_2 + \delta.$$

That is, marginal products aren't equalized so that capital is misallocated. $\square$

5.2 Proof of Lemma 1

First consider $U(c, x)$. Substituting in for the constraints

$$U(c, x) = \alpha x + u[c - u^{-1}(x)]$$

Pick any $\theta \in (0, 1)$ and any c, c', x, x' . Define $c^\theta = \theta c + (1 - \theta)c'$, and similarly for x^θ . I want to show that $U(c^\theta, x^\theta) > \theta U(c, x) + (1 - \theta)U(c', x')$. This follows because

$$u[c^\theta - u^{-1}(x^\theta)] > u[c^\theta - (\theta u^{-1}(x) + (1 - \theta)u^{-1}(x'))] > \theta u[c - u^{-1}(x)] + (1 - \theta)u[c' - u^{-1}(x')],$$

where the first inequality follows from the fact that the inverse of a concave function is convex.

5.3 Proof of Proposition 2

TO BE DONE.

5.4 Proof of Lemma 2

TO BE DONE.

5.5 Proof of Lemma 3

It is useful to work with the Marcet and Marimon (1999) state space. That is, recall from above the time-varying Pareto weight $\pi = \alpha - \psi$ and let the state space be $S \equiv \Pi \times K$. Figure 5 is a phase diagram in this state space. The plane is divided into two parts depending on whether

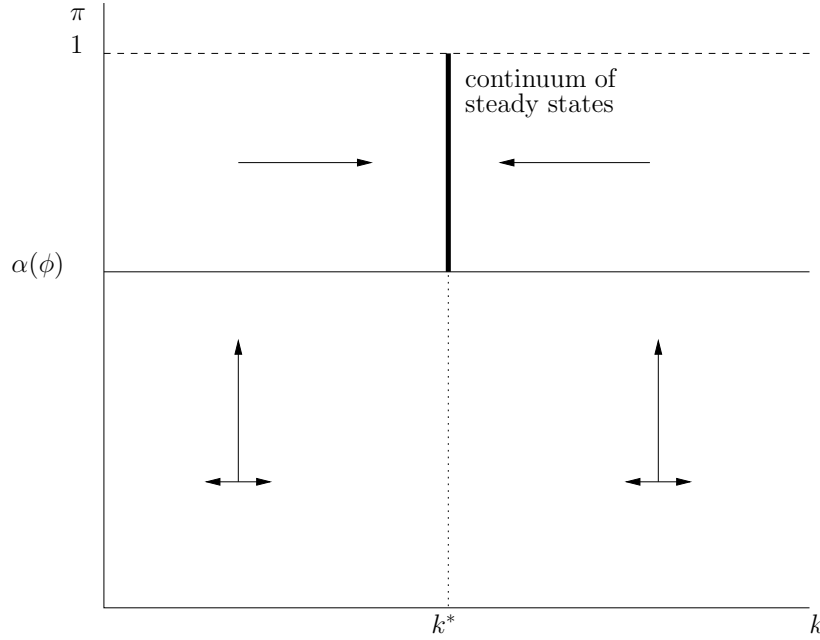


Figure 5: State Space $\Pi \times K$

$\pi \geq \alpha(\phi)$. Define $C \equiv \{(\pi, k) \in S : \pi < \alpha(\phi)\}$. By Assumption 2, $\alpha(\phi)$ does not depend on k . The enforcement constraint (3) binds if and only if $(\pi, k) \in C$. The theorem is proven if I can show that an economy starting out in C , it will never leave that region. The arrows indicate what we know about the dynamics of (π, k) . For $\pi < \alpha(\phi)$, we know that $\dot{\pi} = \mu > 0$, but the dynamics for k are unclear. For the part of the state space where $\pi \geq \alpha(\phi)$, the dynamics of the system are those of the unconstrained benchmark in section 1.3, and we know that the aggregate capital stock converges to its steady state value k^* while Pareto weight is constant. It can easily be seen from the figure 5 that an economy starting out in C will never leave that region. $\square$

5.6 Proof of Proposition 3

See Theorem 4.4. in Datta (1999). The theorem is originally due to Carlson and Schneider (1963).

5.7 Proof of Proposition 4

First, recall that X and Y are both positive definite matrices. Define

$$W = \begin{bmatrix} -Y & 0 \\ 0 & X \end{bmatrix}, \quad M = \begin{bmatrix} 0 & 0 \\ 0 & 2\rho X \end{bmatrix}$$

We have that

$$WA + A^T W = M \geq 0$$

Applying Theorem 3 we see that

$$In(A) = In(W) = (n, n, 0)$$

where the second equality follows because W is block diagonal and its eigenvalues are those of $-Y$ and X .

5.8 Computation

TO BE DONE.