The dynamics of partially-revealing rational expectations equilibria*

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February 14, 2009

Abstract

This paper investigates the qualitative properties of a dynamic rational expectations equilibrium model where incomplete private information revelation is possible. The information revelation properties of prices are endogenous to the model. Periods of complete and incomplete information revelation are examined and it is shown that the private information of ambiguity averse investors tends to be revealed following periods of poor market performance. It is further demonstrated that the differences in long-run market prices that arise from partially- and fully-revealing equilibria are significant. These results shed some light on the market selection hypothesis under asymmetric information.

1 Introduction

The *rational expectations* paradigm posits that market participants use the true distribution governing the resolution of future uncertainty when making deci-

*The paper is preliminary and incomplete, we welcome comments and suggestions.

sions. This paper studies the dynamic properties of rational expectations equilibria in asymmetric information economies where market participants learn through privately-held signals and observations of market prices. It does so in the framework of a standard dynamic exchange economy of the type used in many asset-pricing applications. In equilibrium it is shown that there are periods when all privately-held information is revealed throught market prices, and periods of incomplete information revelation when market prices do not reveal all privately-held information. It is demonstrated that full-revelation tends to occur following periods of relatively poor asset market performance. Furthermore, it is shown that eventually the private information that is unique to some individuals becomes lost to the market as a whole, that is, eventually it will never be revealed through market prices.

The model used in this paper is a dynamic extension of Radner (1979), similar to Tirole (1982). For the purposes of this paper we will view the rational expectations assumption as being composed of two smaller assumptions, one a strengthening of the other. The first is the assumption that all market participants are able to correctly interpret the information that allows them to make predictions about the future. The second, stronger assumption is that all information that could possibly be revealed to aid market participants in predicting the future is actually revealed and correctly used by them.

As an example, consider a person that is to bet on the outcome of a tossed coin. The first assumption of the rational expectations paradigm is that if a person observes that the coin to be tossed has exactly two sides and further observes or otherwise knows that the coin is a fair coin, then she correctly formulates the belief that the probability of heads occuring is 1/2. The second assumption is that the individual actually can observe that the coin has exactly two sides and is fair.

Condie and Ganguli (2008) demonstrate that in models with ambiguity averse investors, prices need not reveal all privately held information. As such prices may not reveal all information that could be learned by market participants. Under the weaker of the rational expectations assumptions, equilibrium would then require that agents optimize relative to the information that they can interpret. In this equilibrium prices will differ from what they would be if all information were revealed. Sections 3 and 4 deal with this weaker assumption of rational expectations equilibrium. In these it is shown that prices can endogenously go through periods of full and partial revelation. The intuition behind this relies on the nature of the preferences of ambiguity averse investors.

Investors who display ambiguity aversion tend to hold riskless portfolios for a range of asset prices. Under certain conditions they will hold riskless conditions even under several different signals. This leads prices to be partially-revealing under these signals. If these riskless portfolios have a high payoff in the market (i.e. risky portfolios fair poorly) then the relative wealth of the ambiguity averse investor will increase. This increase in wealth leads to an increased demand for assets by the ambiguity averse investor. As the ambiguity averse investor's demand increases, it becomes increasingly expensive to purchase a risk free portfolio (especially in states of the world where aggregate consumption is low). This increase in price eventually causes the ambiguity averse investor to abandon a risk-free portfolio. When this happens prices again become fully-revealing. However, because of the nature of ambiguity averse investors, when they hold portfolios that bear market risk, they tend to earn lower average returns than an expected utility maximizing investor with correct beliefs. As such, their wealth will tend to decrease over time to the point that it becomes possible to hold a risk-free portfolio under several signals and prices will again become partially-revealing. Thus prices can cycle between between periods of full- and partial-revelation.

In section 5, we discuss the difference between the weak and strong rational expectations assumptions in the context of the survival of investors in the market. It is shown by example that two distinct sets of investors may survive in this market and that the revelation properties of prices differ across these two, which helps shed light on the market selection hypothesis under asymmetric information. Our preliminary results suggest that this framework is a very tractable one in which to examine the market selection hypothesis in greater generality.

2 Model

Time in this model is discrete and indexed by $t \in \{0, 1, 2, \ldots, T\}$, $T \leq \infty$. There is one consumption good and $\mathcal{N} = \{1, \ldots, N\}$ is the set indexing the (types of) investors in this economy. At each time period t, one of a finite set of payoff states labelled $\mathcal{Z} = \{1, \ldots, Z\}$ occurs and is denoted z_t. Let $\Omega = \mathcal{Z}^T$ be the space of all sequences of payoff states with representative element, called a *payoff-state path* ω.

At each time period t, after the realization of a payoff state z in that period, each investor $n \in \mathcal{N}$ receives a signal $s^n \in \mathcal{S}^n = \{s^n_1, \ldots, s^n_S\}$ that conveys possibly ambiguous information about the probability distribution governing which payoff state will occur at time $t+1$. The set $\mathcal{S} = \times_{n=1}^N \mathcal{S}^n$, with representative element s_t is the space of all joint signals that can be received by participants in this market at any time period t. These joint signals contain all of the private information that can possibly be known about the distribution of payoffs next period. That is, if an investor knows the joint signal that has been received currently then she knows all information that could possibly affect her beliefs about the true probability distribution governing which state $z \in \mathcal{Z}$ will occur next period. Let $\Sigma = \mathcal{S}^T$ be the space of sequences of joint signals with representative element, called a *signal path*, σ.

Define $\Theta = (\mathcal{Z} \times \mathcal{S})^T$, the set of all sequences of payoff states and signals, as the state space with representative element, called a *path*, θ. We denote by $\theta^t = (\theta_0, \ldots, \theta_t)$ the t-period partial history on the path θ, where $\theta_\tau = (z_\tau, s_\tau)$ for $\tau \geq 0$. The set $\Theta^t = (\mathcal{Z} \times \mathcal{S})^t$ is the collection of all t-period histories, and abusing notation, we denote a representative element from Θ^t by θ^t. Similarly, $\omega^t \in \times_{\tau=0}^t \mathcal{Z}$ (respectively, $\sigma^t \in \times_{\tau=0}^t \mathcal{S}$) denotes the t-period partial history on the payoff-state path ω (respectively, signal path σ).

The filtration generated by Θ is denoted $\mathcal{F} = \{\mathcal{F}_t\}$, where $\mathcal{F}_t$ is the σ-algebra generated by Θ^t for $t \geq 1$ and $\mathcal{F}_0$ is the trivial σ-algebra. When $T = \infty$, $\mathcal{F}_\infty$ is the σ-algebra generated by $\cup_{t=1}^\infty \mathcal{F}_t$. The "true" probability measure on Θ is denoted ρ.

At any time t, consumption for each investor takes values in $\mathbb{R}_+$. A *consumption plan* $x : \Theta \to \times_{t=1}^T \mathbb{R}_+$ is a sequence of $\mathbb{R}_+$-valued functions $\{x_t\}$ in which x_{t+1} is $\mathcal{F}_{t+1}$-measurable. We further make the assumption that all consumption plans are bounded, i.e. for some $K > 0$, $x_t(\theta) \leq K$ for all θ and $t \geq 0$. The space of these measurable and bounded consumption plans for each investor is denoted X.

Each investor n is endowed with a particular consumption plan $e^n \in X$ called the *endowment stream*, where e_t^n is $\mathcal{F}_t^{\mathcal{Z}}$-measurable with $\{\mathcal{F}_t^{\mathcal{Z}}\}_{t=0}^T$ denoting the filtration generated by $\mathcal{Z}^T$.[1] This restriction means that the signals only provide information about the distribution over states of the endowments, not on their size. The aggregate endowment stream for the economy is denoted by $e = \{e_1, e_2, \ldots, e_T\} \in X$.

Trade takes place sequentially in the economy and in each period, the investors trade Arrow securities, which are in zero net supply. Formally, in each

[1] As before, when $T = \infty$, $\mathcal{F}_\infty^{\mathcal{Z}}$ is the σ-algebra generated by $\cup_{t=1}^\infty \mathcal{F}_t^{\mathcal{Z}}$.

period $t \geq 1$ after the joint signal s_t is realized, markets open for Arrow securities (contingent claims) which pay off when the payoff state z in period $t+1$ is realized. Abusing notation, we denote by z the Arrow security which pays off if state z realizes.

The prices for securities may reveal information to the investors in this market when they make trading decisions (Radner (1979)). We discuss prices and their informational role before defining equilibrium. Let $\{q_{t+1}(\cdot|s_t)\}$ denote the Arrow security prices (state prices) when the market opens for trading securities that realize payoffs at time $t+1$ after the joint signal s_t is realized in terms of period-t consumption. We assume that investors may use these Arrow security prices to infer information about the private signals that others have received at time t.

In particular, the Arrow security prices at time $t \geq 0$ are described by a map $q_{t+1} : \mathcal{S} \to P$, where $P \subseteq \mathbb{R}^Z$ denotes the space of prices. Suppose that s_t and s'_t are two distinct joint signals that realize at time t. If the signals s_t and s'_t differ only in the signal received by investor n, then investor n will know the time t joint private signal. On the other hand, if s_t and s'_t differ in the private signal received by some other investor, then n can only differentiate the joint signals if state prices are different given these two signals, i.e., if $q_{t+1}(\cdot|s_t) \neq q_{t+1}(\cdot|s'_t)$. If $q_{t+1}(\cdot|s_t) = q_{t+1}(\cdot|s'_t)$, then n can not distinguish between the signals and formulates beliefs accordingly.

Formally, the random variable q_t is $\mathcal{F}_{t-1}$-measurable and generates a partition of the space Θ. The information conveyed by q_t is denoted by the σ-algebra $\mathcal{F}^q_{t-1}$ and the filtration corresponding to $q = \{q_t\}$ is denoted $\mathcal{F}^q$. In equilibrium, investor n must make decisions that are consistent with the information garnered from previous realizations of the state, from her own private signal and from the information that can be obtained from market prices. For any investor n, let $\hat{\mathcal{F}}^n = \{\hat{\mathcal{F}}_t^n\}_{t=0}^T$ be the filtration generated by the private signals $\mathcal{S}^n$ received in each period t. We denote by $\mathcal{F}^n = \{\mathcal{F}_t^n\}_{t=0}^T$ the filtration which is generated from these two sources of information—publicly observed security prices and state realizations and privately observed signals. Formally, $\mathcal{F}^n$ is the filtration generated by the *join* of the filtrations $\mathcal{F}^q$ and $\hat{\mathcal{F}}^n$ at each time t. In equilibrium investor n's consumption decision for time t must be $\mathcal{F}_t^n$-measurable. If prices are fully-revealing at each partial history θ_t then the filtrations $\mathcal{F}^n$ will be the same for each investor.

For any sequence of prices $\{q_t\}$, we will often speak of the price sequence along a path θ or after some $\theta^t \in \Theta^t$ In this case, we will use $q_t(\cdot|\cdot,\theta)$ or $q_t(\cdot|\cdot,\theta^t)$ respectively to explicitly denote this dependence. We will follow the same convention for all variables such as consumption plans, utilities, etc.

The preferences of investors in this economy depend on the information that is revealed to them through private signals and the state prices. Since signals are to be interpreted as containing information about the probabilities of future events, private signals and prices change the beliefs, and hence the preferences of investors in the economy.

We assume that there are at least some investors in the market whose preferences can be modelled by an EU (expected utility) functional of the form, for $t \geq 0$

$$U^n(x, \mathcal{F}^n) = \mathbb{E}_{\pi^n}\left[\sum_{t=0}^{T} \beta^t u(x_t)|\mathcal{F}_t^n\right], \tag{1}$$

where π^n denotes the beliefs of the EU investor given his information as described by $\mathcal{F}^n$. For any $f_t^n \in \mathcal{F}_t^n$ we let $\pi_{t+1}^{\mathcal{Z},n}(f_t^n)$ and $\pi_{t+1}^{\mathcal{S},n}{}_t(f_t^n)$ represent an EU investor's beliefs about payoff states and signals respectively to be realized at time $t+1$ given the information in f_t^n. Since signal that has been realized at time t may be unknown to an investor at time t, the investor's beliefs about the signal that has been realized at time t are given by $\pi_t^{\mathcal{S},n}(f_t^n)$. The set of expected-utility maximizing investors in the economy is denoted $\mathcal{N}^E$. Given π^n, the belief of $n \in \mathcal{N}^E$ at time t over Θ^{t+1} is denoted π^n_t. From these beliefs one may calculate marginal beliefs over the states in $\mathcal{Z}$ at time $t+1$, and over the signals in $\mathcal{S}$ at time $t+1$.

Furthermore, there are investors in the market whose preferences display ambiguity aversion. We will use the formulation of recursive ambiguity aversion axiomatized in Epstein and Schneider (2003) and used in Epstein and Schneider (2007). These preferences are formed by taking as primitive a mapping from each possible history of states and private signals into a set of possible distributions over states next period. Like an expected-utility maximizing investor observes information from previous states and private signals (which may have been deduced from prices) and then formulates a forecast about the probability over states tomorrow, an ambiguity averse investor observes all possible information and derives a set of possible forecast distributions over states tomorrow. The investor then maximizes the minimum expected utility over those distributions.

With $f^n_t \in \mathcal{F}^n_t$, denoting the information available to investor n at time t, let $\Gamma^n_t(f^n_t) \subseteq \Delta(\Theta)$ denote the set of forecast distributions over paths given investor n's information at time t, where $\Delta(\Theta)$ is the set of all probability distributions over Θ. Define

$$\Gamma^n = \{\pi \in \Delta(\Theta) : \pi_t(\theta) = \prod_{\tau=1}^{t} \pi_\tau(\theta), \text{ with } \pi_\tau(\theta) \in \Gamma^n_\tau(f^n_\tau) \text{ for each } f^n_\tau\}. \quad (2)$$

When we need to explicitly speak of paths or partial histries, we will also use the notation $\Gamma^n_t(\theta^t) \in \Delta(\Theta)$ to denote beliefs of investor n after history θ^t, where $\Gamma^n_{t+1}(\theta^t) \equiv \Gamma^n_{t+1}(f^n_t(\theta^t))$ when $f^n_t(\theta^t)$ is the information set for n after history θ^t. Also, $\Gamma^{\mathcal{Z},n}_{t+1}(f^n_t) \subseteq \Delta(\mathcal{Z})$ (respectively, $\Gamma^{\mathcal{S},n}_{t+1}(f^n_t) \subseteq \Delta(\mathcal{S})$) denote beliefs over payoff states to realize in $t+1$ (respectively, over signals realized at

$t+1$). Finally, we define the set

$$\Gamma_{t+1}^{\mathcal{Z},n}(x_{t+1},\theta^t) = \left\{ \pi \in \Gamma_{t+1}^n(\theta^t) : \pi \in \underset{\pi \in \Gamma_{t+1}^n(\theta^t)}{\arg\min} \left\{ u(x_t) + \beta E_\pi \left[Eu(x_{t+1}(\theta_{t+1})) \right] \right\} \right\}. \tag{3}$$

That is, $\Gamma_{t+1}^{\mathcal{Z},n}(x,\theta^t)$ is the set of distribution that minimize the expected present value of time t and time $t+1$ consumption. When, for a given history θ^t the expected value of consumption across *signals* at $t+1$ ($Eu(x_{t+1}(\theta_{t+1}))$) is independent of the state that is realized at time $t+1$ then all distributions in $\Gamma_{t+1}^{\mathcal{Z},n}(\theta^t)$ will be in $\Gamma_{t+1}^{\mathcal{Z},n}(x_{t+1},\theta^t)$. This observation is important

To maintain parsimony, we will assume that the probability distribution over signals holds no ambiguity for any investor. That is, each investor knows the true distribution representing what signals will be received, although they may not know the current signal that has been observed. The ramifications of this assumption will be shown when equilibrium is examined in the following sections.

Then these ambiguity averse (AA) investors have preferences represented by

$$U^n(x,\mathcal{F}^n) = \min_{\pi^n \in \Gamma^n} \mathbb{E}_\pi^n \left[\sum_{t=0}^T \beta^t u(x_t) | \mathcal{F}_t^n \right] \tag{4}$$

for $t \geq 0$ and the set of such investors is denoted $\mathcal{N}^A$.

Since there are no markets for trading contingent on signals, markets are inherently incomplete in this model. Let y_{t+1}^n be the vector representing a portfolio of securities (claims) purchased by investor n at time t for delivery at time $t+1$ and $y^n = \{y_{t+1}^n\}_{t=0}^{T-1}$ denote the corresponding *portfolio plan* for investor n.

After the partial history θ^t, with s_t denoting the joint signal at time t in θ_t, investor n faces the constraint

$$e_t^n(\theta) + y_t^n(\theta) - x_t^n(\theta) + \sum_{z \in \mathcal{Z}} (e_{t+1}^n(\theta_t, z) - y_{t+1}^n(\theta_t, z)) q_{t+1}(z|s_t) \geq 0. \quad (5)$$

If two paths θ^t and θ'^t differ only in the private signal s_t and s'_t respectively received at time t and the security prices are identical (non-revealing) across these private signals, prices under these two signals will be the same and hence the constraint just given will be the same under these two signals. Let $B(q, e^n)$ be the set of all feasible consumption and portfolio plans under the constraints given in (5). A rational expectations equilibrium for this dynamic economy can now be defined similarly to Tirole (1982).

Definition 1. A *rational expectations equilibrium* is an N-tuple of consumption and portfolio plans, $x = \{x^n\}$ and $y = \{y^n\}$, and a process of security prices $q = \{q_t\}$ that satisfy for all $t \geq 0$

1. x_t^n, y_{t+1}^n and q_{t+1}^n are each $\mathcal{F}_t^n$-measurable for each n,
2. $x^n \in \arg\max U_0^n(x, \mathcal{F}^n)$ subject to $(x^n, y^n) \in B(q, e^n)$, and
3. $\sum_n x_t^n = e_t$ and $\sum_n y_{t+1}^n = 0$.

Now we can define a partially-revealing rational expectations equilibrium as follows.

Definition 2. A rational expectations equilibrium (x, y, q) is said to be *partially-revealing at t* if for some (distinct) joint signals s^t, s'^t, $q_{t+1}(\cdot|s_t) = q_{t+1}(\cdot|s'_t)$.

Given the notation for dependence on paths or histories introduced above, the following definition pertains to partial revelation on a path.

Definition 3. A rational expectations equilibrium (x, y, q) is said to be *partially-revealing at θ^t* $= \big(\theta^{t-1}, (z_t, s)\big)$ if for some $s' \neq s$, $q_t(\cdot|\cdot, \big(\theta^{t-1}, (z_t, s)\big)) = q_t(\cdot|\cdot, \big(\theta^{t-1}, (z_t, s')\big))$.

3 Partially-revealing equilibria

Before beginning the explication of partially-revealing REE, some notation must be clarified. For any two vectors $x, y \in \mathbb{R}^n$, define $x \circ y = [x_1 y_1, \ldots, x_n y_n]$. Similarly, for a set of vectors $A \subseteq \mathbb{R}^n$ and the vector $x \in \mathbb{R}^n$ define $x \circ A = \{z \in \mathbb{R}^n : z = x \circ y \text{ for } y \in A\}$. Finally, for any investor n, given the partial history θ^t let

$$\mathbf{u}_t^n(\theta^t) = \begin{pmatrix} \frac{u^{n\prime}(x_{t+1}(1,\theta^t))}{u^{n\prime}(x_t(\theta^t))} \\ \vdots \\ \frac{u^{n\prime}(x_{t+1}(Z,\theta^t))}{u^{n\prime}(x_t(\theta^t))} \end{pmatrix} \tag{6}$$

Notice that $\mathbf{u}_t^n$ is a random vector that depends on the allocation of investor n. To simplify the explication, this dependency will be implicitly understood.

Consider first an expected utility maximizing investor. After any history of signals and states, this investor must select a quantity of Arrow securities to be received and/or delivered in each state next period. The utility to the investor of each of these securities depends on the investor's beliefs about consumption next period. These beliefs in turn depend on the signal that the investor believes has been realized today, and the signal that will be realized tomorrow (and whether that signal will be revealed.) Since the investor cannot purchase securities that payoff in each state-signal pair, the first order conditions are given by

$$q_{t+1}(\cdot|s^t, \theta^t) = \beta \pi_t^{\mathcal{Z},E}(\theta^t) \circ E\mathbf{u}_t^E(\theta^t) \tag{7}$$

where the expectation operator is over the signal s_{t+1} that is to be realized at time $t+1$. If in equilibrium the signal will not be revealed in any payoff state z at time $t+1$ then $\mathbf{u}_t^n(\theta^t)$ will be constant across signals and the expectations operator can be removed. In general however, because of the incomplete markets in this economy, consumption and thus marginal utility can vary across signals next period when these signals are revealed. These conditions will be used to identify when partial revelation will occur in the economy.

Assume that the partial histories θ^t and θ'^t differ only in the signal that has been received by some ambiguity averse investor labelled A at time t and denote the corresponding joint signals s^t and s'^t respectively. The optimality conditions for the ambiguity averse investor A after θ^t are

$$q_{t+1}(\cdot|s^t, \theta^t) \in \beta E\mathbf{u}_t^A(\theta^t) \circ \Gamma_t^{\mathcal{Z},A}(\theta^t) \tag{8}$$

Notice that since there is no ambiguity about signals, these first order conditions differ from those of an expected utility maximizing investor only in the presence of the set $\Gamma_t^{\mathcal{Z},A}(\theta^t)$ which represents the investor's ambiguity over the distribution of states z next period. The expectation of the vector of marginal utility $\mathbf{u}_t^i(\cdot)$ in (8) and (7) is taken with respect to the true marginal distribution over signals.

The following result is a consequence of the analysis in Condie and Ganguli (2008).

Proposition 1. *Prices in this economy will be robustly partially-revealing at time t if*

$$\begin{aligned} q_{t+1}(\cdot|s^t) = q_{t+1}(\cdot|s'^t) &\in \beta E\mathbf{u}_t^A(\theta^t) \circ \Gamma_t^{\mathcal{Z},A}(\theta^t) \cap \Gamma_t^{\mathcal{Z},A}(\theta'^t) \\ &= \beta E\mathbf{u}_t^A(\theta'^t) \circ \Gamma_t^{\mathcal{Z},A}(\theta^t) \cap \Gamma_t^{\mathcal{Z},A}(\theta'^t) \end{aligned} \tag{9}$$

When this condition holds the the demand for Arrow securities by the ambiguity averse investor will not differ across the signals s_t and s'_t and therefore prices will be uninformative. In order to understand the requirements that this condition places on the primitives of the model, one must apply the remaining equilibrium conditions.

The Euler equations for an expected utility maximizing investor n are given by (7). Subsituting this equation into the inclusion given in (9) shows that partial revelation will occur at t if

$$\begin{aligned}\pi_t^{\mathcal{Z},n}(\{\theta^t,\theta'^t\})\circ E\mathbf{u}_t^n(\theta^t) &\in E\mathbf{u}_t^A(\theta^t)\circ\Gamma_t^{\mathcal{Z},A}(\theta^t)\cap\Gamma_t^{\mathcal{Z},A}(\theta'^t)\\ &= E\mathbf{u}_t^A(\theta'^t)\circ\Gamma_t^{\mathcal{Z},A}(\theta^t)\cap\Gamma_t^{\mathcal{Z},A}(\theta'^t)\end{aligned} \tag{10}$$

4 Endogenous partial revelation

We now examine conditions under which the partial-revelation property of equilibrium prices occurs endogeneously in our setting. In particular, we characterize the observable implications of prices when they are fully- and partially-revealing. We will do so in the context of a simplified model.

Suppose that $\mathcal{Z}=\{1,2\}$. There are two types of investors labelled E for expected utility and A for ambiguity averse. Only investors of type A receive a signal in each period, i.e. $\mathcal{S}^A=\{s_1,s_2\}$ and $\mathcal{S}^E=\emptyset$.[2] The aggregate endowment is i.i.d. conditional on each signal realization with $e(1|s)=e_1$ and $e(2|s)=e_2$ for $s\in\{s_1,s_2\}$. We will drop the time subscripts and superscripts to ease explication unless otherwise indicated.

At time t we consider security prices for consumption at $t+1$. The equilibrium will be partially revealing if $q_{t+1}(\cdot|s_1)=q_{t+1}(\cdot|s_2)$.

Let $\bar{\Gamma}^{\mathcal{Z}}=\Gamma^{\mathcal{Z},A}(s_1)\cap\Gamma^{\mathcal{Z},A}(s_2)$, where $\Gamma^{\mathcal{Z},A}(s)$ denotes A's beliefs over $\mathcal{Z}$ given signal $s\in\mathcal{S}^A$. Given that $\bar{\Gamma}^{\mathcal{Z}}\subseteq(\Delta(\mathcal{Z}))$, it can be parametrized by the set $[\underline{\pi},\overline{\pi}]$ where $\pi^{1,A}\in[\underline{\pi},\overline{\pi}]$ represents the probability of state 1 occuring. Let $\pi^{1,*}=\pi^{\mathcal{Z},E}(1|\{s_1,s_2\})$ denote E's belief of the probability of state 1 occuring

[2] While this does not conform to the general model described above, we adopt this structure to simplify the notation. One could carry out the same exercise with $\mathcal{S}^E=\{\bar{s}\}$ for some $\bar{s}$ with some additional notation involved.

if prices do not reveal which signal has been received by A. Throughout this example we will maintain the "minimum consensus" assumption that $\pi^{1,*} \in (\underline{\pi}, \overline{\pi})$ and that $\pi^* = \rho(1|\{s_1, s_2\})$, i.e., the unconditional beliefs of investor E are correct and the ambiguity averse investor believes that this probability distribution is possible.

In the context of these restrictions, partial revelation will occur if and only if

$$E\left(\frac{u(x^A_{t+1}(\theta^t 1s))}{u(x^A_t(\theta^t))}\right) = E\left(\frac{u(x^A_{t+1}(\theta^t 2s))}{u(x^A_t(\theta^t))}\right) \tag{11}$$

and prices satisfy

$$\begin{pmatrix} q_{t+1}(1|s_1) \\ q_{t+1}(2|s_1) \end{pmatrix} = \begin{pmatrix} q_{t+1}(1|s_2) \\ q_{t+1}(2|s_2) \end{pmatrix} \in \beta \begin{pmatrix} \pi E\left(\frac{u'(x^A_{t+1}(\theta^t 1s))}{u'(x^A_t(\theta^t))}\right) \\ (1-\pi)E\left(\frac{u'(x^A_{t+1}(\theta^t 2s))}{u'(x^A_t(\theta^t))}\right) \end{pmatrix} \tag{12}$$

for some $\pi \in (\underline{\pi}, \overline{\pi})$.

Together, conditions (11) and (12) imply that

$$\frac{q_{t+1}(1|s)}{q_{t+1}(1|s) + q_{t+1}(2|s)} \in [\underline{\pi}, \overline{\pi}]. \tag{13}$$

which can be be rewritten as

$$\frac{\underline{\pi}}{1-\underline{\pi}} \leq \frac{q_{t+1}(1|s)}{q_{t+1}(2|s)} \leq \frac{\overline{\pi}}{1-\overline{\pi}}. \tag{14}$$

From the point of view of an external observer, these inequalities demonstrate that the observation of market prices (without the observation of signals or consumption) can be enough to determine if the privately held information of the ambiguity averse investor is being revealed. This inference does rely on knowledge of the interval $(\underline{\pi}, \overline{\pi})$. We will use equation (14) to understand the nature of consumption for the investors in this economy when information is partially and fully revealed.

Inserting the Euler conditions for the expected utility maximizing investor E, equation (14) becomes

$$\frac{\underline{\pi}}{1-\underline{\pi}} \leq \frac{\pi^* Eu'(x^E(\theta^t 1s))}{(1-\pi^*)Eu'(x^E(\theta^t 2s))} \leq \frac{\overline{\pi}}{1-\overline{\pi}}. \tag{15}$$

Lemma 4.1. *There exists an $\underline{x}$ such that if along the path θ, $x^E(\theta^t) < \underline{x}$ then prices at θ^{t-1} will be fully-revealing.*

Proof. By the assumptions placed on $u(\cdot)$, $Eu'(x^E(\theta^t zs))$ is bounded below by $u'(e_z)$ but increases without bound as x^E goes to zero. This observation implies that there exists an x_1^E such that if $x^E(\theta^t 1s) < x_1^E$ then

$$\frac{\pi^* Eu'(x^E(\theta^t 1s))}{(1-\pi^*)Eu'(x^E(\theta^t 2s))} > \frac{\overline{\pi}}{1-\overline{\pi}} \tag{16}$$

and consumption amount x_2^E such that if $x^E(\theta^t 2s) < x_2^E$ then

$$\frac{\pi^* Eu'(x^E(\theta^t 1s))}{(1-\pi^*)Eu'(x^E(\theta^t 2s))} < \frac{\underline{\pi}}{1-\underline{\pi}}. \tag{17}$$

Defining $\underline{x} = \min\{x_1^E, x_2^E\}$ gives the result. [**THIS RESULT CAN BE SHARPENED A BIT IF WE CONSIDER A SEPARATE BOUND FOR EACH STATE SIGNAL PAIR IN ANY PERIOD.**] ■

This lemma demonstrates that a sufficient condition for full-revelation in this market is that the expected utility maximizing investor have low enough consumption. Put another way, during periods in which the consumption of the ambiguity averse investor is sufficiently high, this investor's information will be revealed in the market. As the ambiguity averse investor is able to finance higher an higher consumption, her attempts to fully-insure across market risk become increasingly expensive. Eventually, this increase in consumption must cause the price of consumption in one of the states to be so high that it is no longer utility maximizing for the ambiguity averse investor to insure across all states. Once this happens, the act of trading away from the full insurance allocation reveals the ambiguity averse investor's information.

4.1 The general case

As defined earlier, an equilibrium will be partially-revealing at θ_t if there exist $(\theta', \theta'') \in \theta_{t-1}^2$ that differ in the signal recieved at time t and the equilibrium price process satisfies $q_t(\cdot|\theta') = q_t(\cdot|\theta'')$. Suppose that the joint signal received at time t in θ and θ' differ in the private signal received by an ambiguity averse investor j. Label these joint signals as s and s' respectively and assume $\Gamma(s) \cap \Gamma(s') \subsetneq \Delta^Z$.

A sufficient condition for j's private information to not be revealed at θ_t is that j plan for constant consumption across all states under both signals s and s' at time t. Since under full insurance the vector $f_t^j(\theta)$ is constant across all states. This will occur if and only if

$$\frac{1}{\sum_z q(z|\theta_t)} q(\cdot|\theta_t) = \frac{1}{\sum_z q(z|\theta_t')} q(\cdot|\theta_t') \in \Gamma(s) \cap \Gamma(s') \tag{18}$$

Substituting into this equation the first order conditions for any expected utility maximizing investor k gives

$$\frac{1}{\sum_z \pi_t^k(z|\{\theta,\theta'\}) u'(x_{t+1}(z,\theta_t))} \begin{pmatrix} \pi_1 u'(x_{t+1}(z_1|\theta_t)) \\ \vdots \\ \pi_Z u'(x_{t+1}(z_Z|\theta_t)) \end{pmatrix} \in \Gamma(s) \cap \Gamma(s'). \tag{19}$$

Under the market clearing conditions of equilibrium, we have that $x_{t+1}^k(z_1|\theta_t) + x_{t+1}^j(z_1\theta_t) + \sum_{i\neq j,k} x_{t+1}^i(z_1|\theta_t) = e_{t+1}(z_1|\theta_t)$. Plugging these into k's first order conditions gives

$$\frac{1}{\sum_z \pi_z u'(x^k_{t+1}(z|\theta_t))}\begin{pmatrix} \pi_1 u'(e_{t+1}(z_1,\theta_t) - x^j_{t+1}(\theta_t) - \sum_{i\neq j,k} x^i_{t+1}(z_1|\theta_t)) \\ \vdots \\ \pi_Z u'(e_{t+1}(z_Z,\theta_t) - x^j_{t+1}(\theta_t) - \sum_{i\neq j,k} x^i_{t+1}(z_Z|\theta_t)) \end{pmatrix} \in \Gamma(s)\cap\Gamma(s'). \tag{20}$$

Without loss of generality we make the assumption that $e(1) < e(z')$ for all $z' \neq 1$. Consider the fraction

$$\frac{\pi_z u'(e_{t+1}(z,\theta_t) - x^j_{t+1}(\theta_t) - \sum_{i\neq j,k} x^i_{t+1}(z|\theta_t))}{\sum_z \pi_z u'(x^k_{t+1}(z|\theta_t))} \tag{21}$$

which in equilibrium is the weight of state z in the state-price density. Under the assumption that $u'(x)$ is decreasing in x and satisfies an Inada condition at 0, then when $z = z_1$,

$$\lim_{x^j_{t+1}\to e_1} \frac{\pi_z u'(e_{t+1}(z_1,\theta_t) - x^j_{t+1}(\theta_t) - \sum_{i\neq j,k} x^i_{t+1}(z|\theta_t))}{\sum_z \pi_z u'(x^k_{t+1}(z|\theta_t))} = 1 \tag{22}$$

Since $\Gamma(s)\cap\Gamma(s') \subsetneq \Delta^Z$, as x^j_{t+1} increases, consumption in state z_1 becomes so scarce that the relative price of consumption in this state becomes higher than all possible probabilities in $\Gamma(s)\cap\Gamma(s')$. Of course, if $\Gamma(s)\cap\Gamma(s') = \Delta^Z$ (i.e. the ambiguity averse investor can gain no information that helps her to reduce the ambiguity in her preferences) then this investor will always fully-insure and her information will never be revealed.

Suppose now that $z \neq 1$. Then since $e(z) > e(1)$, if the ambiguity averse investor is fully-insuring at θ_t then

$$\lim_{x^j_{t+1}\to e(1)} e(z) - x^j - \sum_{i\neq j,k} x^i_{t+1}(z|\theta_t)) \tag{23}$$

will be strictly positive. On the other hand the denominator of equation (19) goes to infinity as $x^j_{t+1}(\theta_t)$ converges to $e(z_1)$ implying that for $z \neq z_1$, equation (19) goes to zero. As before, since $\Gamma(s) \cap \Gamma(s') \subsetneq \Delta^Z$, eventually the necessary condition for full insurance (and hence partial revelation) will not hold. This means that along paths where the ambiguity averse investor's consumption is high, there will be full revelation of information.

From the above calculations it can be seen that prices will reveal investor j's signal exactly when the pressure that the desire for full insurance of investor j puts on prices is sufficient to push the state-price density outside of the set $\Gamma(s) \cap \Gamma(s')$.

5 Prices and allocations in the long-run

The model used in this section is identical to that presented previously except that now $T = \infty$. As highlighted in Condie and Ganguli (2008) and in this paper, the existence of partially-revealing REE often coincides with the existence of fully-revealing REE. This section investigates the long-run implications of these two different equilibria and in particular addresses the question of market selection under asymmetric information (Mailath and Sandroni (2003).[3]

There are three groups of investors in this economy labelled **C**, **U** and **A**. Type **A** investors are ambiguity averse and to maintain parsimony, these are the only investors that recieve a private signal.[4]

The first group (labelled **C**-investors) has correct *conditional* beliefs about the payoff state to be realized in each period. That is, if **C**-investors observe **A**'s signal (through prices) then they form correct beliefs about the probabilities of payoff states in the following period. However, if prices do not reveal the

[3]The market selection hypothesis under symmetric information and (dynamically) complete markets is addressed by Blume and Easley (2006) and Sandroni (2000).

[4]This restriction is not essential, but is the most parsimonious signal structure that allows for the results that we will demonstrate here. If other investors receive signals similar phenomena can occur.

signal then the beliefs of these investors are incorrect. The third group (labelled **U**-investors has correct *unconditional* beliefs but incorrect conditional beliefs. Thus, **U**-investors hold correct beliefs about the probabilities of states before the signal is received. However, if they observe the signal (through prices) they do not correctly interpret the meaining of the signal and assign incorrect beliefs to the realization of payoff states.

Both **U**- and **C**-investors have preferences that can be represented by a utility function of the form

$$U^n(x) = \mathbb{E}^n \sum_{t=0}^{\infty} \beta^t u(x_t), n \in \{\mathbf{C}, \mathbf{U}\}. \tag{24}$$

The discount factor β and the Bernoulli utility u assumed to be the same across investors. The only way that these investors differ is in their beliefs over the payoffs of assets.

If the ambiguity averse investor has rectangular beliefs and is always sufficiently ambiguity averse then there will exist a partially-revealing REE in which **A**'s signal is never revealed. In this equilibrium, since the signal is never revealed, the economy behaves like one with i.i.d. payoffs and with the probability of the state in each period given by the unconditional probability distribution over states. Since **U**-type investors have correct unconditional beliefs, applying the results in Condie (2008) shows that with probability one, the consumption of investor **C** and investor **A** will go to 0 in the limit. Thus, the only investor that can survive in this market will be the investor of type **U**.

Now, consider an equilibrium that is fully-revealing after every partial history. Such an equilibrium can be pictured if one imagines maintaining the beliefs of the ambiguity averse investor from the partially-revealing REE and assuming that following every partial history, the signal that has been received by **A** is simply told to all market participants.

That such an equilibrium exists is not at all obvious. We do not attempt to demonstrate that such an equilibrium exists, but merely show properties that this equilibrium must have if it were to exist. This equilibrium has a countable signal space, and establishing its existence involves extending the existence results in Podczeck and Yannelis (2008).

In this market, one may again apply the results of Condie (2008). Since the signal is revealed in every period, the relative entropy of investor **C**'s beliefs will always be lower than those of investor **U**. As such, investor **C**'s consumption will almost surely go to zero in the limit. If the economy exhibits aggregate uncertainty in every period, then investor **A**'s wealth will also go to zero in the limit.

Thus, the long-run predictions of the partially-revealing and fully-revealing equilibria are quite different. The set of investors that have non-vanishing consumption in the long-run is different. Furthermore, if a fully-revealing equilibrium exists, then prices must be fully-revealing in the limit. Thus prices will also be fundamentally different in these two economies. Our preliminary results suggest that this framework is a very tractable one to examine the market selection hypothesis in greater generality.

References

Blume, L., and D. Easley (2006): "If you're so smart, why aren't you rich? Belief selection in complete and incomplete markets," *Econometrica*, 74(4), 929–966.

Chateauneuf, A., R.-A. Dana, and J.-M. Tallon (2000): "Optimal risk-sharing rules and equilibria with Choquet-expected-utility," *Journal of Mathematical Economics*, 34, 191–214.

Condie, S. (2008): "Living with ambiguity: prices and survival when investors have heterogeneous preferences for ambiguity," *Economic Theory*, 36, 81–108.

Condie, S., and J. V. Ganguli (2008): "Ambiguity and rational expectations equilibria," mimeo.

Epstein, L. G., and M. Schneider (2003): "Recursive multiple-priors," *Journal of Economic Theory*, 113, 1–31.

——— (2007): "Learning under ambiguity," *Review of Economic Studies*, 74, 1275–1303.

Mailath, G., and A. Sandroni (2003): "Market selection and asymmetric information," *Review of Economic Studies*, 70, 343–368.

Podczeck, K., and N. C. Yannelis (2008): "Equilibrium theory in differential information economies with infinitely many commodities," *Journal of Economic Theory*, 141, 152–183.

Radner, R. (1979): "Rational expectations equilibrium: Generic existence and the information revealed by prices," *Econometrica*, 47(3), 655–678.

Sandroni, A. (2000): "Do markets favor agents able to make accurate predictions?," *Econometrica*, 68, 1303–1341.

Tirole, J. (1982): "On the possibility of speculation under rational expectations," *Econometrica*, 50(5), 1163–1181.