Incomplete Information and Informative Pricing: Theory and Application

Giacomo Rondina[*†]
Department of Economics
University of California, San Diego

December 2007

Abstract

This paper studies the information contained in the equilibrium aggregate price level of an economy where firms make output price decisions faced with incomplete information about economy-wide disturbances. It is shown that when heterogeneously informed firms are allowed to extract information from the equilibrium aggregate price, the ability of the aggregate price to be a sufficient statistics of the underlying aggregate disturbance depends on the degree of strategic complementarity in firms' pricing strategy. As the incentive to price similarily increases, aggregate price goes from a perfect to an imperfect signal of changes in the aggregate state of the economy. More generally, this paper contributes to the expanding literature on monetary business cycle and incomplete information initiated by Woodford (2003a) in three directions. First, it proposes a set of techniques in the frequency domain that allow for the explicit derivation of individual heterogeneous expectations in a log-linear framework while preserving the tractability of the equilibrium fixed point condition. Second, it develops and solves a stylized model where aggregate price plays a key informational role for imperfectly informed firms of the type advocated by Hayek. Finally, it presents an application to monetary policy in a setting with a simple feedback rule for the supply of money.

[*]Contacts: 9500 Gilman Drive, La Jolla, 92093, CA-USA; phone: 001 858 534 1059, E-mail: grondina@ucsd.edu.

[†]I am especially thankful to Steven Durlauf for his guidance and support. I am also indebted to Buz Brock for his comments and for many discussions on this and related topics. I have benefited from comments by Bruce Hansen, Terry Iverson, Jim Nason, Yongs Shin, Ken West and seminar participants at the macro workshop at the Atlanta Fed, the Boston Fed, Penn State, Pompeu Fabra (CREI), St. Louis Fed, Tufts, UC Davis, UC San Diego, UW Madison. Any errors remain my own. The support of the Alice Gengler Wisconsin Distinguished Graduate Fellowship in Economics is gratefully acknowledged. Part of this work was undertaken while visiting the Ente for Monetary, Banking and Financial Studies "Luigi Einaudi" in Rome, whose hospitality and support is thankfully acknowledged.

"*We must look at the price system as such a mechanism for communicating information if we want to understand its real function*"

Friedrich A. Hayek, "The use of Knowledge in Society"
American Economic Review, 1945, p. 526

1 Introduction

One of the major challenges faced by modern macroeconomic research is the formulation of a model for short-run aggregate supply that is both empirically plausible and theoretically sound. In particular, it has proven difficult to develop a micro-founded model of firm pricing strategies whose theoretical predictions are consistent with existing empirical evidence on the behavior of prices at the aggregate level[1]. Empirical studies emphasize the relevance of past realizations in the determination of current aggregate price dynamics. In order for theoretical models to generate a similar prediction, some sort of frictions in price adjustments have to be assumed rather than arise endogenously. Exogenous adjustment frictions imply that expected future realizations become important for current price dynamics. Persistence of prices does arise endogenously in models with exogenous frictions, but these models appear empirically inferior to those in which the relevance of past realizations is directly postulated. This paper develops an analysis where the relevance of past realizations for current aggregate price dynamics arises endogenously via information asymmetries across agents.

The paper builds on the pioneering work of Woodford (2003a). Subsequent work by Adams (2006), Hellwig (2005), Kawamoto (2004) and Lorenzoni (2006a, 2006b) has further explored the potential for incomplete information to generate price rigidities. While this new literature has uncovered interesting insights, for example on the role of public and private information in explaining the dynamics of aggregate price, there are still important outstanding theoretical questions to be addressed[2]. The formulation and explicit solution of models of incomplete information remains technically difficult because of the heterogeneity in information and the consequent signal extraction problem involved in the formulation of model-consistent expectations on the part of agents. As a consequence, most recent studies on the subject, in order to obtain closed-form solutions, are forced to place certain restrictions on the sources of information available to agents. In particular, agents are usually not allowed to extract information from variables that emerge in equilibrium[3]. This means, for example, that the aggregate price in these models does not convey information, a role that Hayek advocated as key for the price system in general. In addition, this implies that the persistent heterogeneity of agents' information sets is, in essence, built into these models from the outset.

The paper makes three contributions to the new generation of models of aggregate supply under incomplete information. First, I formulate a macroeconomic model where firms price under incomplete information and where the aggregate price conveys information to firms. Using this model, I attempt to address the outstanding theoretical question: is it possible for rational agents to have access to common endogenous information, i.e. prices, and still formulate persistently diverse expectations in equilibrium? When this happens, persistence in the aggregate price dynamics arises endogenously. Second, I propose a set of methods that provide an explicit formulation of individual expectations in heterogeneous incomplete information log-linear environments while

[1] For an up-to-date discussion of firms pricing models and the dynamics of aggregate price and inflation see for example Fhurer (2005) and Galati and Melick (2006).

[2] For a review of the new generation of monetary business cycle models with imperfect information see Hellwig (2006).

[3] Lorenzoni (2006a,b) are important exceptions to this. The contribution of the present paper with respect to Lorenzoni's works, as well as the other works cited, is discussed in Section 4.1.

preserving the tractability of the fixed point problem typical of rational expectations. Third, I show that questions of monetary policy can be addressed in an analytical framework where the aggregate price controlled by the monetary authority plays both an allocative and an informational role.

On a different dimension this paper also relates to the growing literature on global games and information aggregation in equilibrium outcomes such as Morris and Shin (2002), Hellwig (2005), Angeletos and Werning (2006) and Angeletos and Pavan (2007). These papers study the effect of exogenous public information on the equilibrium degree of coordination in global games and evaluate the aggregate outcome with respect to an optimal degree of coordination. The present paper proposes a framework where the only source of public information is endogenous and focuses on characterizing the informational properties of the equilibrium outcome. The result that the precision of endogenous public information depends on the equilibrium degree of coordination has the potential to offer novel insights about the normative results outlined in that strand of literature.

In the model economy, a large number of intermediate product firms compete monopolistically by setting prices to maximize expected profits. Firms are free to adjust their pricing strategies in every period at no cost. Intermediate product firms hire labor from households and combine it with a technology which is subjected to a productivity shock that is specific to the firm. Firm-specific productivity disturbances are thus the primary source of heterogeneity across firms in the model. As one would expect, firms' pricing strategies take the form of a markup over real marginal costs. In turn, real marginal costs are a function of the level of aggregate activity, the aggregate price and the specific productivity of firms. The source of strategic complementarity in firms pricing strategies comes from the sort of real rigidities emphasized by Ball and Romer (1990). Under complete information concerning aggregate disturbances, firms correctly anticipate the level of aggregate activity and of the aggregate price. As a result, individual prices differ across firms solely because of the difference in productivity across firms. In contrast, under incomplete information, firms form expectations of the aggregate activity and price. In this case, the type of information available to firms as they set their price is critical.

As noted above, most existing studies assume that firms can base expectations on the observation of exogenous random variables which are in some measure correlated with the variables to be predicted. We refer to the information set generated by this assumption as *exogenous*. An example would be an information set generated by the history of firm-specific productivity. In this case, to the extent that firm-specific productivity does not move independently from aggregate productivity, firms will condition their prediction of the aggregate variables on their privately observed information. Thus, individual prices will differ because of the direct effect of firm-specific productivity and because of the informational content of such productivity concerning aggregate variables.

In the setting in which firms base predictions on variables that emerge in equilibrium, we refer to the corresponding information set as *endogenous*. An endogenous information set is, for example, one generated by the history of firm-specific productivity and the history of aggregate price. In this case, aggregate price plays a key informational role that is bound to have important consequences for the nature of equilibria. In this setting, individual prices will still differ across firms due to the direct and the informational effect of firm-specific productivity. However, the latter effect is now combined with the informational effect of the history of aggregate price. Not only is aggregate price an endogenous variable, but it is also common knowledge across firms. As a consequence, its observation will tend to homogenize information across firms. Any persistent difference across firms' predictions will depend on the informational content of the aggregate price. In the framework presented below, the informational content of aggregate price is a characteristic emerging in equilibrium.

The non-competitive market structure, typical of models where firms' pricing is explicitly considered, plays a crucial role in shaping the informational properties of the aggregate price. Under the assumption of monopolistic competition, each firm's optimal pricing strategy depends on the average price charged by all other firms. This happens independently of the information available to firms. However, in the presence of incomplete information, the incentive to coordinate behavior (i.e., price adjustments) results in an informational externality that manifests itself when prices are explicitly considered as conveyers of information.

When the incentive to coordinate price adjustments is weak, the individual pricing will be dominated by the direct effect of firm-specific productivity on marginal costs. This implies that aggregate price, being the aggregation of individual prices, will be reactive to aggregate disturbances. By observing its history, firms can recover the history of aggregate disturbances and formulate predictions on the basis of that information. It follows that firms predictions are accurate and the aggregate price is always "close" to the full information benchmark.

On the other hand, when the incentive to coordinate price adjustments is strong, the individual pricing strategies will be dominated entirely by the desire to predict others' behavior. In so doing, individual pricing strategies will react less to firm-specific productivity and more to the information that is common knowledge across firms – the history of aggregate price. Because of the reduced reaction to individual productivity at the firm pricing level, aggregate price will similarly underreact to aggregate productivity. As a consequence, its dynamics will be insufficient to recover exactly the history of disturbances. A rational expectations equilibrium where expectations are conditioned on the history of aggregate price and individual productivity and where the aggregate price are a sufficient statistics for aggregate disturbances will not exist. This non-existence pathology is similar to the one stressed by Futia (1981). Its existence is, however, very sensitive to the representation of the information available to agents. If agents are assumed to condition their expectations also on realized *relative* prices in addition to realized aggregate price, then a rational expectations equilibrium continues to exists and its dynamics are close to the full information benchmark. The key result of the paper, in terms of the simple economy considered, is that the ability of the aggregate price to be a sufficient statistics for aggregate disturbances is directly related to the degree of strategic complementarity in the price setting behavior.

In stationary linear environments, like the one considered in this paper, the ability of the dynamics of the aggregate price to allow the recovery of the aggregate disturbances can be associated with the invertibility of the moving average component of the equilibrium price linear function. If this moving average component is invertible, the equilibrium price is said to be invertible, otherwise it is non-invertible. Whether the aggregate price is invertible or not crucially affects the signal extraction operated by agents in the attempt to formulate optimal predictions.

In order to solve explicitly for a log-linear equilibrium aggregate price, I employ the Wiener-Hopf approach to optimal linear prediction. The power of this method lies in its ability to deal with the covariance structure of the variables in the information sets of agents and those to be predicted by representing them in the domain of square-integrable complex functions. This method allows one to model heterogeneity in information by considering the difference between moving average representations in the space of economic disturbances (structural innovations) and moving average representations in the space of innovations to agents information set (Wold-fundamental innovations).

The important part of this method is the factorization of the covariance generating function of the information set upon which the prediction is based. The form taken by this factorization reflects the amount of information that a given information set can provide at each point in time. We use a factorization procedure initially advanced by Rozanov (1967) and subsequently employed, in different setups, by Townsend (1983) and Kasa (2000).

Applications of the Wiener-Hopf method are often found in macroeconomic time-series analysis. For example, the commonly used Wiener-Kolmogorov prediction formula is a special case of the Wiener-Hopf method. In this paper I argue that the power of the Wiener-Hopf approach to linear prediction extends to heterogeneous information environments which have been only partially explored in time-series macroeconomics. I show that the method is flexible enough to accommodate a wide range of assumptions concerning the distribution of information among agents. I employ the method to formulate individual predictions conditioned on exogenous and endogenous information. The functional forms of this predictions lead to tractable fixed point conditions from which an equilibrium can be analytically computed.

It is important to emphasize that, in terms of the resulting prediction error, the non-invertibility of the moving average component of a linear stochastic process is equivalent to the presence of noise in the observation of the driving process. However, while the latter has to be assumed exogenously, the former can emerge endogenously in equilibrium. This means, in particular, that the amount of information revealed by prices in equilibrium is endogenous and can therefore be modified by policy interventions or shifts in structural parameters. This suggests the application of the framework developed in this paper to questions of optimal monetary policy when aggregate price has both an allocative and an informational role.

In addition to developing basic theory, I present an application to monetary policy. In the model economy, money demand arises from a cash-in-advance constraint faced by households. In equilibrium, the constraint is always binding so the velocity of money remains constant. Monetary policy is specified in terms of a rule for the supply of money. In this framework, the level of nominal activity is proportional to the outstanding quantity of money. Therefore, any rule for the supply of money can be interpreted as a nominal income targeting rule. In our application, we specify a simple feedback rule where money supplied at t depends on past realizations of aggregate price. The objective of this application is to show that, in an environment where aggregate price conveys information to economic agents, the design of good monetary policy must account for both the economic and the informational consequences of changing the dynamics of nominal aggregates. The effort currently undertaken by many central banks to inform economic agents about the state of the economy arises naturally, in the context of this application, as an effective instrument for monetary policy. This suggests an important application of the presented framework to issues of information collection and dissemination on the part of monetary authorities. The issues have been studied in recent work by, for example, Morris and Shin (2005), Amato and Shin (2006) and Adams (2006) in settings where aggregate price conveys information to the central bank but not to agents. The present paper offers an analytical framework where this restrictive assumption can be relaxed and more insights on the informational role of monetary policy can be gained.

The remainder of the paper is organized as follows. In Section 2, the model economy is presented and the resulting equilibrium aggregate price is expressed in terms of higher-order expectations. Section 3 presents the Wiener-Hopf method to provide explicit linear prediction formulas under signal extraction from exogenous and endogenous information sets. Section 4 solves explicitly for the equilibrium aggregate price of the model of section 2 using the methods of section 3. Section 5 presents a simple application to monetary policy. Section 6 concludes and discusses important applications for future research. Most of the proofs are confined to the Appendix.

2 The Model

2.1 Aggregate Demand

The demand side of the economy consists of a large number of infinitely lived households uniformly distributed on the unit interval indexed $i \in [0,1]$. Households rank alternative consumption-labor sequences according to intertemporal preferences representable by

$$\max E_0 \sum_{t=0}^{\infty} \delta^t \left(\log C_t - \frac{N_t(i)^{1+\psi}}{1+\psi} \right), \; \delta \in (0,1), \psi > 0, \tag{1}$$

where δ is the subjective discount factor, C_t is the amount of final good consumed by household i and $N_t(i)$ is the amount of time worked[4]. Household i faces the series of budget constraints

$$P_t C_t + M_t^d + E_t\left[Q_{t|t+1} B_{t+1}(i)\right] = W_t(i) N_t(i) + \Xi_t + M_{t-1}^d + T_t + B_t(i), \;\; t \geq 0, \tag{2}$$

where P_t denotes the price of the final good, M_t^d is the end-of-period money balances, $Q_{t|t+1}$ is the stochastic discount factor related to the evaluation of the non-monetary portfolio B_{t+1} and Ξ_t are the total profits accruing to the household from the production sector. Shares in each firm cannot be traded and each household is assumed to own the same amount of shares in each firm. T_t denotes the nominal lump sum transfer from the monetary authority received by the household at the beginning of period t, and $W_t(i)$ is the nominal wage. It is assumed that labor services are completely specialized so household i can supply its labor exclusively to an intermediate product firm producing an intermediate good of type i. The nominal interest rate R_t is the return associated with a nominal risk-free asset which results in $\frac{1}{1+R_t} = E_t\left[Q_{t|t+1}\right]$.

Agents have access to complete markets to insure against idiosyncratic labor income risk. As a result, even though households receive a different wages and supply different amounts of labor, they face essentially the same sequence of budget constraints (2) and consumes the same amount of the composite good C_t. The timing of the events can be stated by dividing each period in two sub-periods. Consumer i enters period t with money balances M_{t-1}^d carried over from the last period plus the lump-sum transfer T_t. In the first sub-period consumers allocate their portfolio between money and assets. At this stage consumers have full information about the aggregate state at time t and have the same prior over firm-specific productivity, i.e. idiosyncratic uncertainty. It follows that they take the same position over alternative assets (included money). The net transactions in the asset market is financed by the money holding. In the second sub-period consumers undertake transactions in the final goods market with the money available from the first sub-period, supply their labor services and receive monetary payments in terms of wages, dividends and contingent claims which however cannot be used to purchase goods in that period. Because of complete markets, the monetary payoff of contingent claims perfectly compensates the realization of the labor income and each household carry over to the subsequent period the same amount of money.

These set of assumptions lead to a supply of worker-specific labor services that depends positively on the real wage paid by firm i and negatively on the level of consumption C_t. Namely,

$$N_t(i) = \left(\frac{W_t(i)}{P_t C_t} \right)^{\frac{1}{\psi}}, \tag{3}$$

[4] The form of the single period utility function is a specification of a general form known as the King-Plosser-Rebelo form. This specifications ensures that the income effect and the substitution effect generated by a change in the real wage perfectly offset. Kawamoto (2004), Hellwig (2005) and Lorenzoni (2006a,b) all employ this form.

where $\frac{1}{\psi}$ is the Frisch elasticity of labor supply[5].

The demand for money balances arises because of a cash-in-advance constraint on the purchase of final good,

$$M_t^d \geq P_t C_t. \tag{4}$$

Notice that, given our assumption of risk diversification, the demand for money is equal across households as they are ex-ante identical in every period. It is assumed that the nominal interest rate is always positive so that the cash-in-advance constraint is always binding. These set of observations and assumption together with the clearing condition in the money market lead to the relationship $M_t = P_t C_t$ which summarizes the aggregate demand side of the model and where M_t is the process for the supply of money.

Assuming that households can perfectly observe the aggregate state of the economy at period t when formulating their optimal strategies allows one to abstract from issues of incomplete information and financial marekts. This assumption is made in order to isolate the information interactions arising from the supply side of the economy. It is also necessary to assume that firms cannot observe the prices at which assets are traded in financial markets as they would immediately reveal the state of the economy. An analysis of incomplete information both in the supply and in the demand side of the economy is desirable and part of the research program to which the present work aims at contributing.

2.2 Aggregate Supply

The production side of the economy is made up of a monopolistically competitive intermediate goods sector and a perfectly competitive final good sector. The final good sector produces the composite final good Y_t using intermediate goods $Y_t(i)$ according to the Dixit-Stiglitz CES technology

$$Y_t = \left(\int_0^1 Y_t(i)^{\frac{\theta-1}{\theta}} di \right)^{\frac{\theta}{\theta-1}}, \tag{5}$$

where $\theta > 1$. Final good firms maximize profits which, in equilibrium, are identically zero. It follows that the price emerging in the final good sector is

$$P_t = \left(\int_0^1 P_t(i)^{1-\theta} di \right)^{\frac{1}{1-\theta}}. \tag{6}$$

It can be showed that the final sector demand for an intermediate product i is given by

$$Y_t^d(i) = \left(\frac{P_t(i)}{P_t} \right)^{-\theta} Y_t. \tag{7}$$

The intermediate product sector is assumed to be populated by a large number of firms indexed by $i \in [0, 1]$. Each firm produces a specific product variety, also indexed by i, according to a specific labor input technology whose total productivity is proportional to a stochastic factor $A_{it} > 0$,

$$Y_t(i) = \alpha A_{it} N_t(i)^{\frac{1}{\alpha}}, \quad \alpha \geq 1. \tag{8}$$

Firm i is matched with consumer i and acts as a monopsonist in the labor market for its specific labor services. This means that, given an optimally chosen price, $P_t(i)$, according to (7), (8) and the labor supply schedule (3), the nominal wage paid to the specific labor supplier will be a function of aggregate demand, aggregate price and firm-specific productivity.

[5] Empirical studies suggest $\frac{1}{\psi} < 0.5$ which implies that $\psi > 2$.

The pricing problem for the intermediate product firms can then be written as

$$\max_{P_t(i)} E\left[P_t(i)\,Y_t(i) - W_t(i)\,N_t(i)\,|\Omega_{it}\right], \tag{9}$$

where Ω_{it} denotes the information set of firm i at time t. Ownership of intermediate goods firms is distributed equally across consumers and these shares are non-tradable. In addition, markets for risk sharing on idiosyncratic profits uncertainty are assumed to be absent[6]. Firm i sets its price $P_t(i)$ and commits itself to sell any amount of its product demanded by the final good sector at that price. This demand is assumed to remain bounded. In equilibrium the market for the final good clears so that $C_t = Y_t$. It follows that the profit function of firm i can be written as a function of three arguments: the aggregate price level, P_t, the aggregate production, Y_t and firm-specific productivity, A_{it}.

2.3 Equilibrium

In our definition of an aggregate equilibrium for this economy, we follow Hellwig (2005) and Lorenzoni (2006a,b). This means in particular that we focus on symmetric stationary rational expectations equilibria in which the prices of intermediate goods are functions only of intermediate firms' information sets at time t, $\Omega_{it}, i \in [0,1]$. A symmetric stationary equilibrium in this economy is defined by a set of functions $C(\cdot)$, $M^d(\cdot)$, $B(\cdot)$, $W(\cdot)$, $N^s(\cdot)$, $P(\cdot)$ and $P(i)(\cdot)$ such that:

1) $\{C(\cdot), M^d(\cdot), B(\cdot), N^s(\cdot)\}$ solve (1) under (2), (4);

2) $P(\cdot)$ and $P(i)(\cdot)$ let final good producers and intermediate good producers maximize expected profits;

3) all markets clear;

4) individual beliefs are consistent with the distributions emerging in equilibrium.

The symmetry of the equilibrium implies that the functions, $P(i)(\cdot)$, are identical across intermediate good producers.

2.4 Log-Linearized Economy and Higher Order Expectations

It is convenient from this point onward to work with the logarithmic versions of the relationships presented above, under the assumption of a stationary rational expectations equilibrium. The method proposed in this paper applies to linear environments as most of the existing methods dealing with signal extraction.

Letting $a_{it} \equiv \log A_{it}$, it is assumed that firm i specific productivity a_{it} is given by the sum of a common component a_t and of an idiosyncratic component v_{it}, namely

$$a_{it} = a_t + v_{it}. \tag{10}$$

The process for v_{it} is serially uncorrelated normally distributed with zero mean and variance σ_v^2. In addition, the cross sectional distribution of v_{it} is assumed to satisfy $\int_0^1 v_{it}di = 0$. The common component is assumed to be a covariance stationary process given by

$$a_t = a(L)\,u_t. \tag{11}$$

The innovation in aggregate productivity u_t is serially uncorrelated normally distributed with zero mean and variance σ_u^2. In what follows we will refer to the process for u_t as the *structural aggregate innovation* in this

[6] The assumption of incomplete markets on privately held information, admittedly not without loss of generality, is common in the recent literature on imperfect information in monopolistic competitive markets, see Lorenzoni (2006a,b).

economy. This terminology is meant to capture the idea that the current realization for u_t represents the change in productivity experienced, on average, by firms' technologies . Note that, given our assumptions, aggregate productivity is

$$\int_0^1 a_{it} di = a_t. \tag{12}$$

From the linearized first order conditions implied by (9) it can be showed that the optimal pricing function for a generic firm i can be written[7]:

$$p_t(i) - E_t(p_t|\Omega_{it}) = \gamma E_t(y_t|\Omega_{it}) - \gamma a_{it}, \qquad \gamma > 0, \tag{13}$$

where $p_t(i) = \log P_t(i)$, $p_t = \log P_t$, $y_t = \log Y_t$ and

$$\gamma = \frac{\alpha(1+\psi)}{1+\theta(\alpha(1+\psi)-1)}.$$

The optimal relative price for firm i at time t is a decreasing function of its own realized productivity, a_{it}, and an increasing function of aggregate activity, y_t. The latter captures the effect of real aggregate demand on the price of factor inputs necessary to operate firm i's technology and the effect of real aggregate supply in the demand that firm i faces from the final good sector[8]. The parameter γ represents the elasticity of the optimal real price of firm i to its real marginal costs which in part depend on real aggregate activity, this parameter measures the real rigidity in the economy in the sense of Ball and Romer (1990). For low values of γ the firm relative price is insensitive to the conditions on real marginal costs, in other words the supply is very elastic as real rigidity implies that the firm nominal price is always adjusted in order to stay as close as possible to the aggregate nominal price. In this sense a low value for γ is associated to a strong degree of strategic complementarity in nominal price setting.

The equilibrium aggregate price level p_t is approximated, in this linearized environment, by integrating the log-linearized pricing functions (13). This gives[9]

$$p_t = \int_0^1 p_t(i)\, di. \tag{14}$$

Given a process for money supply, m_t, money market clearing implies the canonical quantity equation:

$$m_t = p_t + y_t. \tag{15}$$

Using (15), one can write the optimal pricing strategy (13) as

$$p_t(i) = \gamma E(m_t|\Omega_{it}) + (1-\gamma) E(p_t|\Omega_{it}) - \gamma a_{it}. \tag{16}$$

[7] A detailed derivation is reported in Appendix A.

[8] Under the more standard assumption that firms are wage-takers in the labor market and the technology is linear in labor the optimal relative price is a constant mark-up over real wages

$$\frac{P_t(i)}{P_t} = \frac{\theta}{\theta-1}\frac{W_t}{P_t}\frac{1}{A_{it}}.$$

Taking logs and disregarding the constant terms one has that

$$p_t(i) - E_t(p_t|\Omega_{it}) = E_t(w_t - p_t|\Omega_{it}) - a_{it},$$

where w_t is the log of nominal wages. The structure of the labor market and of the aggregate demand provides a relationship between the real wages and the aggregate activity y_t. Under our assumptions real wages are firm-specific because of heterogeneous productivity, heterogeneous information and firms labor-market power. Taking this into consideration when deriving the optimal pricing leads to (13).

[9] This approximated form for the relationship between the log of the aggregate price and the log of individual prices is standard in the monopolistic pricing literature. Its derivation is reported in the appendix.

The value of the parameter γ determines whether the intermediate products are, respectively, strategic substitutes or strategic complements. In the presence of strategic substitutability ($\gamma > 1$), a generalized increase in the prices charged by other firms triggers a decrease in the individual firm pricing. In contrast, when $\gamma \in (0, 1)$, it triggers an increase in individual pricing. The assumptions made on the structural parameters of the model ensure that firms' pricing strategies are strategic complements[10]. The source of strategic complementarity in the present setting is to be found on the assumption that a common market for input factors does not exist.

Notice that, writing the optimal price $p_t(i)$ as in (16), implies that (14) conveniently summarizes the market clearing condition of this stylized economy. The assumption of rational expectations requires the consistency between the subjective distribution of equilibrium variables used by agents to formulate predictions and the objective distribution generated by the economy in equilibrium. Solving for an equilibrium from (16) is equivalent to imposing the consistency requirement together with (14).

Substituting (16) into (17) implies

$$p_t = \gamma \bar{E}_t(m_t) + (1-\gamma)\bar{E}_t(p_t) - \gamma a_t, \tag{17}$$

where the upper-barred expectational notation stands for the average of individual expectations and it is formally defined as

$$\bar{E}_t(x_t) = \int_0^1 E_t(x_t|\Omega_{it})\, di. \tag{18}$$

Equation (17) illustrates that the equilibrium aggregate price level depends on aggregate productivity, on the average (or "market") expectations of the nominal aggregate demand - the money supply process under our assumptions - and on the price level it self[11].

There exist several approaches to the solution of (17). It is instructive to first solve (17) for p_t using a direct recursive technique as used, for example, by Morris and Shin (2002) and by Bacchetta and Van Wincoop (2005). As each firm forms its expectations rationally over the aggregate price[12],

$$E_t^i(p_t) = \gamma E_t^i \bar{E}_t(m_t) + (1-\gamma) E_t^i \bar{E}_t(p_t) - \gamma E_t^i(a_t). \tag{19}$$

Notice that each firm is implicitly forming expectations over the market expectations of both m_t and p_t. This feature is an example of the phenomenon which Keynes (1936) termed the *beauty contest* effect. As noted by Morris and Shin (2002), heterogeneity of information breaks the law of iterated expectations. That is, $E_t^i \bar{E}_t(m_t) \neq E_t^i(m_t)$. This fact gives rise to *higher order* expectational terms as firms attempt to forecast the average forecast of the market. Averaging (19) across all firms gives an expression for the average market expectations of the price level,

$$\bar{E}_t(p_t) = \gamma \bar{E}_t \bar{E}_t(m_t) + (1-\gamma) \bar{E}_t \bar{E}_t(p_t) - \gamma \bar{E}_t(a_t). \tag{20}$$

[10] The condition for the analysis of the type of strategic interaction is:

$$\frac{\partial p_t(i)}{\partial E(p_t|\Omega_{it})} = (1-\gamma)$$

As remarked by Woodford (2003, Chapter 3) the distinction between strategic substitutability and strategic complementarity in these terms makes sense only if the overall nominal aggregate m_t is kept fixed. For this reason the version (16) of the optimal pricing equation is the appropriate one to evaluate the typology of strategic interaction.

[11] Both (14) and (18) appeal to a law of large numbers for a continuum of random variables. Judd (1985) cautions that, in general, nothing guarantees that when a continuum of agents draw independent draws from the same distribution F , the sample distribution of realized draws coincides with F. However, he also shows that it is always possible to construct probabiliy measures which are extensions of the Kolmogorov measure and which satisfy the law of large numbers in the continuum. In what follows we assume that we work with such measures so any random variable emerging from aggregation is always well defined.

[12] $E_t^i(p_t)$ is short notation for $E_t(p_t|\Omega_{it})$.

Once again the break down of the law of iterated expectations implies that $\bar{E}_t\bar{E}_t(p_t) \neq \bar{E}_t(p_t)$. Substituting (20) into (17) gives an expression for p_t that depends now on the second-order expectational term, $\bar{E}_t\bar{E}_t(p_t)$ (the market expectations of the market expectations). In order to eliminate such terms from the expression one repeats the process above by applying the firm-specific expectational operator, this time to (20), then taking the average across firms. Proceeding in this fashion, the solution for the price level can be finally written as the sum of the expectations of increasing order about aggregate productivity and nominal aggregate demand,

$$p_t = \gamma \sum_{j=1}^{\infty} (1-\gamma)^{j-1} \bar{E}_t^{(j)}(m_t) - \gamma \sum_{j=0}^{\infty} (1-\gamma)^{j} \bar{E}_t^{(j)}(a_t). \tag{21}$$

The notation $\bar{E}_t^{(j)}$ stands for the j-th order expectation, where $\bar{E}_t^{(0)}(a_t) \equiv a_t$.

The aggregate nominal demand relation (15) allows us to express the realized real rate of output as

$$y_t = -\gamma \sum_{j=0}^{\infty} (1-\gamma)^{j} \bar{E}_t^{(j)}(a_t) + m_t - \gamma \sum_{j=1}^{\infty} (1-\gamma)^{j-1} \bar{E}_t^{(j)}(m_t). \tag{22}$$

Equation (22) illustrates that, in addition to the realized aggregate disturbances, a_t and m_t, the entire structure of higher order expectations of both real disturbances and nominal disturbances affects realized output.

2.5 Full Information

The full information rate of output in this context can be computed by assuming that firms have complete information about aggregate productivity and about the process for nominal aggregate demand, m_t. This means $E_t^i(a_t) = a_t$ and $E_t^i(m_t) = m_t$, for every i. Under these informational assumptions, both higher-order expectational structures collapse into a single term, namely $\sum_{j=0}^{\infty}(1-\gamma)^j \bar{E}_t^{(j)}(a_t) = \frac{1}{\gamma}a_t$ and $\sum_{j=1}^{\infty}(1-\gamma)^{j-1} \bar{E}_t^{(j)}(m_t) = \frac{1}{\gamma}m_t$. Substituting these expressions into (21) and (22), one obtains the following expressions for the full information rate of output y_t^* and the associated price level p_t^*,

$$y_t^* = a_t, \tag{23}$$

$$p_t^* = m_t - a_t. \tag{24}$$

Under full information the rate of output coincides with the aggregate productivity a_t. The dynamics of money supply is fully anticipated and entirely allocated to the dynamics of the price level, thereby resulting completely irrelevant for the determination of full information output[13]. The price level corresponding to a full information output also varies over time. This occurs, not only because of the one-to-one effect of the monetary base, but also because productivity has an aggregate effect on the marginal cost structures of the intermediate goods sector. As the productivity of the intermediate good sector increases, the final good becomes cheaper and its production increases. However, the increase in production is obtained without affecting the level of aggregate employment which is $n_t = \alpha(y_t - a_t)$ so that when $y_t = y_t^*$ one has $n_t^* = 0$. Under full information the degree of real rigidity γ does not affect the aggregate dynamic behavior of the system. However, it affects the cross-sectional distribution of prices, production, hours worked and wages. Firm i's optimal price is in fact

$$p_t^*(i) = -a_t - \gamma v_{it}.$$

[13] Uniqueness of an equilibrium realizes because the beliefs off-equilibrium path are specified uniquely by the *form* of the aggregate demand, which is assumed to be common knowledge even when m_t is not perfectly observed.

Firms are heterogeneously productive and under full information the production of each firm is approximately given by

$$y_t^*(i) = a_t + \theta\gamma v_{it},$$

where $\theta\gamma > 1$. To produce this output, firm i requires labor

$$n_t^*(i) = \alpha(\theta\gamma - 1)v_{it},$$

which implies the real wage rate (in terms of the final product),

$$w_t^*(i) - p_t^* = a_t + \psi\alpha(\theta\gamma - 1)v_{it},$$

where $(\theta\gamma - 1) > 0$. It follows that, when the productivity of firm i is higher than aggregate productivity $(v_{it} > 0)$, the real wage for labor of type i is higher than the economy-wide average a_t. Basically, when firm i is more productive than the average it is optimal to reduce its real price, this will increase its demand and therefore its production. The increase in production is obtained with an increase in employment as well as the increased productivity, however an increase in employment is possible only by increaesing real wages as markets for factors are separated. This will limit the reduction in firm i's relative price, hence the real rigidity. The key here is that consumers offer labor according to real wages measured in terms of the final product which they consume. It follows that a decrease in firm i's price will not affect real wages through the price of the final product because of the infinitesimal dimension of the firm. On the other hand, if there is an increase only in aggregate productivity it is optimal for firm i not to change its relative price (i.e. the real rigidity is maximal in this case). To see this suppose that the nominal price of firm i is decreased, then the demand for its product will increase and the increase in production will be satisfied in part by higher employment. In order to increse the hours worked the firm will need to offer a higher real wage. Differently from before, as every firm will be reducing nominal prices, real wages will be increased as a consequence. Nevertheless, production, and thus consumption, in equilibrium will be higher which will decrease the marginal utility of consumption and therefore it will make hours worked less appealing. Overall both effects cancel each other and hours worked do not change, the increase in production will result only from increase in productivity. In addition, as all firms will be changing their nominal price by the same measure the real prices across firms will remain unchanged. It follows that a perfeclty observed change in aggregate productivity has no effect on real prices, as anticipated.

With an expression for the full information rate of output in hand, it is possible to rewrite the optimal pricing strategy of firm i in terms of expectations of differences between full information and realized output

$$p_t(i) - E_t(p_t|\Omega_{it}) = \gamma E_t(y_t - y_t^*|\Omega_{it}) - \gamma E_t(a_{it} - a_t|\Omega_{it}). \tag{25}$$

The pricing equation expressed in this form is known as the notional short run aggregate supply (SRAS) curve[14].

2.6 Incomplete Information

It is instructive to consider at the outset the solution of the model under incomplete information which is commonly shared by firms. As long as this homogeneity of information is common knowledge, price and output

[14] This equation is a generalization of the SRAS derived by Kawamoto (2004) where it is assumed that firms' technologies experience the same productivity shock. The aggregate productivity shock is not observed at the time of pricing, but whatever its realization firms know that $a_{it} = a_t$ so that the first term in (25) is identically equal to zero. In constrast, the assumption that firms observe perfectly their specific productivity eliminates a simple informational channel that would exist independently of the existence of complementarities in firms strategies. In the present model, firms care about aggregate productivity because of the dependence of their optimal strategy on aggregate real activity and not because of the direct effect of productivity on their technology: the latter effect is perfectly observed and it affects the former as a signal.

will deviate from the full information benchmark (23)-(24) but any deviations will be unpredictable.

Incomplete Information on Aggregate Productivity. Suppose that the process for the monetary base is perfectly anticipated by firms. Suppose that aggregate productivity follows a simple $AR(1)$ process,

$$a_t = \rho a_{t-1} + u_t, \ |\rho| < 1,$$

where u_t is an i.i.d. white noise process with zero mean and variance σ_u^2. Information over aggregate productivity is incomplete (though homogeneous) across firms, and it consists in the observation of the history of productivity up to $t-1$. This assumption implies that in setting their prices firms do not observe their specific productivity which forces them to form expectations, which are $E_t(a_{it}|\Omega_{it}) = E_{t-1}(a_t) = \rho a_{t-1}$. Modifying the relevant equations accordingly and noticing that by the law of iterated expectations the structure of higher order expectations collapses to the first order expectation, the realized rate of output and the aggregate price then become

$$y_t = y_t^* - (1-\gamma)[a_t - E_{t-1}(a_t)] = y_t^* - (1-\gamma) u_t \quad (26)$$

$$p_t = p_t^* + (1-\gamma)[a_t - E_{t-1}(a_t)] = p_t^* + (1-\gamma) u_t \quad (27)$$

Output and price deviate from their full information analogues according to the unobserved innovation u_t. In particular, in presence of a positive aggregate shock to productivity, individual prices are not reduced enough and the beneficial effects of the productivity increase do not affect aggregate output as the aggregate demand for final goods does not increase due to the insufficient reduction in aggregate price which ultimately results in a weak demand for labor which justifies a weak aggregate demand. In this equilibrium, the deviation of realized output from full information output is unpredictable,

$$E\left(y_{t+1} - y_{t+1}^*|\Omega_t\right) = 0.$$

Incomplete Information on Money Supply. Suppose now that a_t is perfectly observed and it is common knowledge that the monetary base, m_t, is an exogenous stochastic process with known statistical properties. At the time firms set their prices, m_t is not perfectly observed. However, each firm has the same expectation over its future value, that is $E_t^i(m_t) = E_{t-1}(m_t)$, for every i. Thus, the structure of higher-order expectations reduces to $E_{t-1}(m_t)\frac{1}{\gamma}$ and the realized rate of output and price relative to the full information one is

$$y_t = y_t^* + m_t - E_{t-1}(m_t),$$

$$p_t = p_t^* - (m_t - E_{t-1}(m_t)).$$

In the presence of incomplete homogeneous information, unexpected nominal disturbances affect output. However, as in Lucas (1972), rationality of expectations implies that the effects of these disturbances are unpredictable. Suppose that money supply, and thus aggregate nominal demand, increases unexpectedly. This implies that predetermined individual prices are relatively low given the demand faced by final goods producers. This will result in a higher demand for intermediate goods which will lead to a higher real wage and higher employment thereby justifying the higher aggregate demand. In the subsequent period intermediate firms will align their nominal prices to the quantity of money in circulation and the precedent boost to employment will be reabsorbed.

The upshot of the two examples just presented is that common knowledge of the underlying state tends to result in equilibrium dynamics that are close to the full information benchmark. This happens because in presence of strategic complementarity agents have the incentive to react more to information that is public and less to information that is private. When the precision of public information is full, as in the above example, this incentive has a desirable effect. On the other hand, when information is not precise this incentive can have perverse effects as agents over react to information that is common knowledge and under react to their private information. If the precision of the information that is common knowledge depends on the efficient aggregation of dispersed information one can think of a situation where as the incentive to react to common knowledge information is strong, agents react too weakly to their private information, making the information that is common knowledge an imprecise signal of the state. As the information that is common knowledge becomes imprecise, the incentive for an optimal signal extraction tends to place less weight on that information, contrasting the economic incentive discussed above. Because of this, agents reacts more to their private signals as they are less noisy, but in so doing the information that is common knowledge becomes more informative, triggering back the over reaction discussed above. If the incentives at work do not take the system to a rest point an equilibrium may not exist. If an equilibrium exists, the resulting price process is a noisy signal of the aggregate state of the economy, the aggregate state will not be common knowledge and the economy will stay persistently away from the full information benchmark.

The remaining of the paper aims at making this argument formal first by presenting a set of tools to deal with heterogeneity in information and signal extraction and then by working out illustrative cases for equilibria under two alternative incomplete information sets.

3 Signal Extraction and Frequency Domain Methods

3.1 Wiener-Hopf Approach to Optimal Prediction

In this section we lay out the analytic tools that we will employ in order to write out individual expectations as a linear function of privately observed exogenous information and publicly observed endogenous information. As in most dynamic economic models under uncertainty, the information sets of agents are specified as a sequence of random vectors $\{\mathbf{x}_t\}$ and a filtration $\mathcal{F}$ (i.e. a sequence of σ-algebras Φ satisfying $\Phi_t \subset \Phi_{t+1}$) such that $\mathbf{x}_t$ is Φ_t-measurable for every t. In a linear stationary environment the filtration $\mathcal{F}$ can be equivalently represented as a sequence of Hilbert spaces spanned by the history of realizations of $\mathbf{x}_t$. It follows that for an agent the history of realizations of the $n \times 1$ covariance stationary random vector, $\mathbf{x}_t$, the information set at the beginning of period t, denoted Ω_t, is formally defined by the Hilbert space generated by the random vector $\mathbf{x}$ up to time t. That is,

$$\Omega_t = H_t(\mathbf{x}) = \underset{j=1,n}{\oplus} H_t(x_j), \tag{28}$$

where the operator $\oplus$ denotes the operation of forming the smallest closed subspace containing the elements specified in its argument.

We adopt the Hilbert space description to represent the dynamic flow of information in our model. Agents want to predict a covariance stationary vector $\mathbf{p}_{t+v}$ ($v \in \mathbb{N}$) by minimizing the resulting mean squared error. This allows one to define the expectational operator by the orthogonal projection from the space containing the variable that is to be predicted onto the space spanned by the observed variables. This projection is optimal as it minimizes the square distance between the two spaces. Formally,

$$E(\mathbf{p}_{t+v}|\Omega_t) = \Pi(\mathbf{p}_{t+v}|H_t(\mathbf{x})),$$

where Π denotes the linear least squares projection of $H_{t+v}(\mathbf{p})$ onto $H_t(\mathbf{x})$. A powerful method for solving this type of problems is the widely employed Kalman filter. However, in order to use the Kalman filter one needs to identify a well defined state for the system under study. Unfortunately, in environments of heterogeneous information and higher order expectations, the definition of a state is usually problematic[15]. To overcome this difficulty, we employ an alternative method known as the Wiener-Hopf approach[16]. The main advantage of this method is to express the prediction formula directly in terms of the information set on which the prediction is based, without defining a state variable. An additional advantage, which is discussed below, is the flexibility of the method to accommodate various assumptions on the structural disturbances while still delivering analytical solutions.

The problem of prediction in a linear environment consists therefore in finding the linear combination of the history of observed random variables which minimizes the mean square error $E\left(\hat{\mathbf{p}}_{t+v|t} - \mathbf{p}_{t+v}\right)^2$: formally the sequence of constant matrices Π_j

$$\hat{\mathbf{p}}_{t+v|t} = \sum_{j=0}^{\infty} \Pi_j \mathbf{x}_{t-j}.$$

that minimizes the mean square error . The following result from Whittle (1983) reports the solution to this generic prediction problem.

Theorem 1 *Suppose that $\{\mathbf{x}_t\}$ and $\{\mathbf{p}_t\}$ are covariance stationary random processes with known statistical properties. The linear least squares estimate $\hat{\mathbf{p}}_{t+v|t}$ is then provided by the linear combination of the history of $\mathbf{x}_t$ denoted by $\Pi(z)$ and given by:*

$$\Pi(z) = \left[z^{-v}\mathbf{g}_{\mathbf{px}}(z)\left(\Gamma^*\left(z^{-1}\right)^T\right)^{-1}\right]_+ \Gamma^*(z)^{-1},$$

where $g_{\mathbf{px}}(z)$ is the covariance generating matrix function for $\mathbf{p}$ and $\mathbf{x}$, $\Gamma^*(z)$ is the canonical factorization of the autocovariance generating function for $\mathbf{x}$, denoted by $g_{\mathbf{xx}}(z)$, and the notation $[]_+$ instructs to ignore terms in negative powers of z.

Proof. We sketch the proof for the univariate case with $v = 0$ in order to illustrate the Wiener-Hopf technique. Following Whittle(1983), the univariate version of the prediction problem is

$$\min_{\{\Pi_j\}_{j=0,1,2,\ldots}} E\left(\sum_{j=0}^{\infty} \Pi_j x_{t-j} - p_t\right)^2.$$

Notice that the minimization is taken with respect to the infinite sequence of coefficients Π_j. From the infinite number of first order conditions, one obtains the equations:

$$\sum_{k=0}^{\infty} E\left(x_t x_{t-j+k}\right)\Pi_k = E\left(p_t x_{t-j}\right) \quad \text{for} \quad j = 0, 1, 2, \ldots .$$

Next multiply both sides by z^j and then add over the entire set of j (positive and negative), so that:

$$g_{xx}(z)\,\gamma(z) = g_{px}(z) + h(z). \tag{29}$$

[15] On this point see Sargent (1991) and Pearlman and Sargent (2005). Interestingly, Woodford (2003) is able to employ the Kalman filtering approach as he defines the state as a linear combination of the infinite higher order expectational terms. It is not clear however whether such a representation can be extended to settings where endogenous variables enter agents information set. In fact, Lorenzoni (2006a) considers endogenous information sets but he is forced to truncate the state space in order to compute an equilibrium.

[16] The Wiener-Hopf technique has been applied to economics by Whiteman (1985) and Taub (1989). Taub (1989) shows that the technique can be used to solve simultaneously for an optimization problem and a signal extraction problem when the former is of the linear-quadratic type.

The term $h(z)$ is an unknown series in negative powers of z. All series here are assumed to be convergent for some annulus in the complex domain. The objective here is to eliminate $h(z)$, one can then simply apply the annihilator operator to both sides. However the term $g_{xx}(z)\Pi(z)$ will be transformed in such a way that some components of the series $\Pi(z)$ would be lost as the series $g_{xx}(z)$, being double sided, contains negative powers of z. Here comes the key step: we need to factorize $g_{xx}(z)$ in two components, one with only positive powers for z and the other with only negative powers of z. This factorization of $g_{xx}(z)$ corresponds to the canonical *moving average* representation for x_t[17]. Let the moving average representation in terms of structural innovations for x_t be given by

$$x_t = \Gamma(L)\varepsilon_t, \tag{30}$$

where $\Gamma(z)$ can be represented as the ratio of two finite degree polynomials (i.e. x_t has a rational spectral density)and ε_t is a white noise process. This representation is not necessarily canonical as the zeros of the numerator polynomial are not restricted to be inside the unit circle by the requirement of stationarity for x_t. The key point to notice is that, in economic applications, nothing guarantees that the structural innovation representation (30) is in canonical form. However, one can always find a moving average representation of the form

$$x_t = \Gamma^*(L)w_t,$$

where the zeros of $\Gamma^*(z)$ are all inside the unit circle and where we let $\Gamma^*(0) = \sigma_w$ and $E\left(w_t^2\right) = 1$. Among all these representations in the univariate case, the canonical form is the one where $\Gamma^*(0)^2$ (i.e. the variance of the prediction error) is highest, i.e., the Wold fundamental representation[18]. For convenience let $\Gamma^*(z) \equiv \sigma_w\tilde{\Gamma}^*(z)$. Therefore, the canonical representation is:

$$g_{xx}(z) = \tilde{\Gamma}^*(z)\tilde{\Gamma}^*\left(z^{-1}\right)\sigma_w^2.$$

The next step is to divide both sides of (29) by $\Gamma^*\left(z^{-1}\right)$ so that,

$$\sigma_w^2\tilde{\Gamma}^*(z)\Pi(z) = \frac{g_{px}(z)}{\tilde{\Gamma}^*(z^{-1})} + \frac{h(z)}{\tilde{\Gamma}^*(z^{-1})}.$$

The term on the left hand side now has only positive powers for z. The second element on the right hand side has only negative powers in z (here is key that $\tilde{\Gamma}^*\left(z^{-1}\right)$ is one sided in negative powers of z), thus applying the annihilator operator we see that,

$$\sigma_w^2\tilde{\Gamma}^*(z)\Pi(z) = \left[\frac{g_{px}(z)}{\tilde{\Gamma}^*(z^{-1})} + \frac{h(z)}{\tilde{\Gamma}^*(z^{-1})}\right]_+ = \left[\frac{g_{px}(z)}{\tilde{\Gamma}^*(z^{-1})}\right]_+,$$

and, finally,

$$\Pi(z) = \frac{1}{\sigma_w^2\Gamma^*(z)}\left[\frac{g_{px}(z)}{\tilde{\Gamma}^*(z^{-1})}\right]_+.$$

■

The key step in solving a prediction problem using the Wiener-Hopf approach is the canonical factorization of the covariance generating function $g_{\mathbf{xx}}(z)$. The canonical factorization corresponds to the identification of an innovation representations for a given information set such that the σ-field generated by the current and past observed variables is equivalent to the σ-field generated by the current and past innovations at every point in

[17] Kailath (1981, Lectures on Wiener and Kalman Filtering, Springer-Verlag, New York) shows that the canonical spectral factorization involved in the Wiener-Hopf equation is directly related to the steady state Ricatti equation for the variance-covariance matrix in a Kalman filtering problem.

[18] See Theorem 4.2 in Rozanov (1967), page 60.

time. In the context of our model, the form taken by this factorization depends on the informativeness of the random vector $\mathbf{x}_t$ concerning the underlying aggregate state of the economy. The state of the economy does not have a unique definition. Let the primitive state be given by the history of innovations to aggregate productivity and the supply of money. Although this notion of state is infinite dimensional, under standard informational assumptions it is usually summarized by few past observations of equilibrium variables. However, in the presence of heterogeneity in information the state tends to coincide with the primitive state to the extent that relevant past information does not become common knowledge. It follows that the definition of an appropriate state in an environment of heterogeneous information depends on the informational properties of equilibrium variables. As these properties are part of the equilibrium characterization, a solution method that always works on the primitive state seems advantageous.

The random variables in $\mathbf{x}_t$ might fail to uncover the primitive state, and, as a consequence, they might fail to uncover the state under any alternative definition. The proposed factorization extracts from the information conveyed by $\mathbf{x}_t$ the optimal inference about the primitive state required to formulate an optimal linear prediction.

Let the structural system for the random vector $\mathbf{x}_t$ be

$$\mathbf{x}_t = \underset{m \times n}{\Gamma(L)} \boldsymbol{\varepsilon}_t.$$

The lag polynomial matrix $\Gamma(L)$ is assumed to be regular, i.e., all its elements can be represented as a one sided sequence in L whose coefficients are summable. The number of rows m correspond to the number of observable variables (the dimension of $\mathbf{x}_t$) and the number of columns n correspond to the number of structural disturbances (the dimension of $\boldsymbol{\varepsilon}_t$). It is assumed that the matrix $\Gamma(L)$ is normalized so that the variance-covariance matrix of the random vector of disturbances, ε_t, is the identity matrix, $\underset{n \times n}{I}$. To apply the above result, we look for a matrix $\Gamma^*(z)$ that satisfies

$$\underset{n \times m}{\Gamma(z)} \underset{m \times n}{\Gamma\left(z^{-1}\right)^T} = \underset{n \times n}{\Gamma^*(z)} \underset{n \times n}{\Gamma^*\left(z^{-1}\right)^T}.$$

In addition, it is required that $\det \Gamma^*(z)$ does not present any zeros inside the unit circle. Rozanov (1967) proposes the following procedure that we summarize in a theorem.

Theorem 2. *Let the structural representation of the system be*

$$\mathbf{x}_t = \underset{m \times n}{\Gamma(L)} \boldsymbol{\varepsilon}_t. \tag{31}$$

where $E\left(\boldsymbol{\varepsilon}_t \boldsymbol{\varepsilon}_t^T\right) = I$. *Let* $\hat{\Gamma}(z)$ *denote the principal minor for* $\Gamma(z)$. *Suppose that* $\det \hat{\Gamma}(z)$ *for* $z \in \mathbb{C}$ *vanishes at* $\lambda_k \in \mathbb{C}$, $|\lambda_k| < 1$ *for* $k = 1, ., s$. *Then the fundamental representation for* $\mathbf{x}_t$ *is*

$$\mathbf{x}_t = \underset{n \times n}{\Gamma^*(L)} \mathbf{w}_t, \tag{32}$$

where

$$\Gamma^*(z) = \Gamma(z) \prod_{k=1}^{s} W_{\lambda_k} B_{\lambda_k}(z) \quad \text{and} \quad \mathbf{w}_t = \prod_{k=1}^{s} B_{\lambda_{s+1-k}}\left(z^{-1}\right) W_{\lambda_{s+1-k}}^T. \tag{33}$$

The unitary[19] *matrix* W_{λ_k} *is the right column eigenvector matrix in the singular value decomposition of* $\Gamma(\lambda_k)$ *and the matrix* $B_{\lambda_k}(z)$ *is a Blaschke matrix*[20] *whose form is:*

$$B_{\lambda_k}(z) = \begin{pmatrix} 1 & 0 \\ 0 & \frac{1-\bar{\lambda}_k z}{z-\lambda_k} \frac{\lambda_k}{|\lambda_k|} \end{pmatrix} \tag{34}$$

[19] A square matrix W is said to be unitary if $W^{-1} = W^T$.

[20] Blaschke factors/matrices have been related to macroeconomic time-series analysis in the context of rational expectations models by Hansen and Sargent (1980, 1991). Lippi and Reichlin (1994) present a state-of-the-art formal analysis of nonfundamental representations and Blaschke matrices.

where $\bar{\lambda}_k$ stands for the complex conjugate of λ_k. It follows that $E\left(\mathbf{w}_t\mathbf{w}_t^T\right) = I$ and

$$\Gamma^*(0)\,\Gamma^*(0)^T \geq \Gamma(0)\,\Gamma(0)^T.$$

Proof. See Rozanov (1967, pages 47 and 60). ■

We finally notice that when the problem is one of pure prediction, i.e., one needs to predict the elements of $\mathbf{x}_t$ in the future from the past of $\mathbf{x}_t$ itself, the Wiener-Hopf result returns the prediction formula,

$$\hat{\mathbf{p}}_{t+v|t} = \hat{\mathbf{x}}_{t+v|t} = \left[\Gamma^*(L)\,L^{-v}\right]_+ \mathbf{w}_t. \tag{35}$$

This expression has the form of the familiar Wiener-Kolmogorov prediction formula. However, (35) is more general than the version commonly used in macroeconomic time-series analysis because of the non-standard information set on which the prediction is based. The optimal signal extraction is embedded in the mapping between the pair $\Gamma(L)$ and $\boldsymbol{\varepsilon}_t$ and the pair $\Gamma^*(L)$ and $\mathbf{w}_t$. Usually this mapping is assumed to be a constant. In the present context, because of incomplete heterogeneous information, this mapping is a linear combination of all the elements of the first pair $\Gamma(L)$ and $\boldsymbol{\varepsilon}_t$.

3.2 Signal Extraction from Endogenous Variables

We argue here that when one or more variables in the vector $\mathbf{x}_t$ are endogenous variables emerging in equilibrium, frequency domain methods are useful in solving the fixed point problem which arises under the assumption that predictions be consistent with the economic model. The signal extraction problem in this case is based on law of motions that are themselves the result of the solved signal extraction problem. Let $\mathbf{p}_t$ be a sub-vector of $\mathbf{x}_t$ and let the equilibrium process be of the form

$$\mathbf{p}_t = D(L)\,\mathbf{u}_t = \sum_{j=0}^{\infty} D_j \mathbf{u}_{t-j}. \tag{36}$$

$D(L)$ is said to belong to the space of square-summable sequences ℓ^2 if $\sum_{i=0}^{\infty} D_j^2 < \infty$. The space ℓ^2 is normally referred to as the *time domain.* A powerful result in representation theory (the Riesz-Fischer theorem, see Sargent (1987) chapter XI) shows that the space of square summable sequences and the space of squared integrable analytic functions L^2 are *isometrically isomorphic.* This means that one can work in whichever space is more convenient since there always exists a mapping that allows one to recover a representation in ℓ^2 from an analogous representation in L^2 and viceversa. The space L^2 is also known as the *Hardy* space and, when the analytic functions are evaluate *on* the unit circle (i.e. $z = e^{-i\omega}$ for $\omega \in [-\pi, \pi]$), is normally referred to as the *frequency domain.*

Suppose that $\mathbf{p}_t$ is endogenously defined by the linear equilibrium condition

$$\mathbf{p}_t = \gamma a(L)\,u_t + \gamma \bar{E}(\mathbf{p}_t). \tag{37}$$

Solving for an equilibrium in the time domain in the present context means solving a fixed point problem in the space of sequences of real numbers. This is normally achieved through the method of undetermined coefficients which starts with a specific guess (a finite set of coefficients) of the equilibrium equation and whose solution usually consists of a set of non-linear equations involving the equilibrium coefficients. In contrast, casting the problem in the frequency domain allows one to specify a generic guess (an analytic function) and then to solve a fixed point problem in the space of analytic functions. In problems without signal extraction from endogenous variables, the first method is feasible since the specific guess (i.e. the dimension of the space of state variables belonging to the solution) is normally evident from agents optimal strategies. However, when signal extraction from endogenous variables is required, a specific guess might overlook the existence of equilibria that require a higher dimensional state space (possibly infinite). As a consequence, frequency domain methods seem preferable to time domain methods in such models[21].

Let the guess for the equilibrium process in (37) be the analytic function $D_s(z) \in L^2$ where s stands for the number of zeros of the determinant of $\Gamma(z)$. Then the required fixed point problem is

$$D_s(z) = \gamma a(z) + \gamma T_s(D_s(z)), \tag{38}$$

where $T_s(.)$, described in theorem 1, is the mapping between the structural representation (which involves $D_s(z)$), the canonical fundamental representation and the optimal prediction prediction formula. The fixed point condition (38) is generic since it requires that the number of zeros s involved in the mapping $T_s(.)$ correspond to the number of zeros that the solution $D_s(z)$ forces onto $\Gamma(z)$. Matching the dimension s is still a non-trivial problem. However, the dynamic behavior of the solution $\mathbf{p}_t$ in specific economic models, like the one considered in this study, is drastically different when $s \leq 1$ compared to the case when $s > 1$. The next section shows that this last condition can be checked relatively easily when solving problems of signal extraction from endogenous variables using frequency domain methods.

4 Equilibrium under Incomplete Information

4.1 Woodford meets Hayek: Exogenous and Endogenous Information Sets

In this section we apply the methods of section 3 to derive explicit forms for the expectational terms at the firms' level for the model economy outlined in section 2. We then show how to solve for the fixed point problem under a generic conjecture for the price level. It is useful to assume that the money supply is kept constant so that, $m_t = 0\ \forall t$. This assumption will be relaxed in section 5 where an application to monetary policy is analyzed. At the moment, it is convenient to abstract from nominal uncertainty in order to isolate the independent effect of uncertainty on real disturbances. The optimal pricing strategy form firm i in linear form derived earlier is

$$p_t(i) = (1-\gamma) E(p_t|\Omega_{it}) - \gamma a_{it}. \tag{39}$$

The equilibrium aggregate price is obtained by aggregating individual prices,

$$p_t = \int_0^1 p_t(i)\, di. \tag{40}$$

In section 2 we solved for an equilibrium under perfect common knowledge, which is the same equilibrium that would emerge if dispersed information was pooled so that $\Omega_{it} = \underset{i\in[0,1]}{\oplus} H_t(a_i)$ for every i. In contrast, this section considers two nested information sets that allow for imperfect common knowledge:

(A) *Exogenous Private Information Set (Woodford)*

$$\Omega_{it}^W = H_t(a_i),$$

(B) *Exogenous Private - Endogenous Public Information Set (Hayek)*:

$$\Omega_{it}^H = H_t(a_i) \oplus H_{t-1}(p). \tag{41}$$

[21] See the discussion in Kasa(2000) on time domain versus frequency domain methods.

Firm-specific productivity obeys (10), i.e.,

$$a_{it} = a_t + v_{it}.$$

It is important to note that, unlike models of incomplete information in competitive markets, our assumption that current equilibrium prices do not enter agents' current information set makes a complete knowledge of the current state of the economy impossible, unless the variance of the idiosyncratic component of productivity σ_v^2 is driven to zero. In the equilibrium of our model, prices are set before markets are cleared and only subsequently convey information which, as a consequence, affects only future equilibrium prices[22].

It is instructive to compare information set (41) to the information sets assumed by the literature which is closely related to this paper. Woodford (2003) and Kawamoto (2004) assume that firms can observe only their own private signal which in our notation would correspond to the information set (W)[23]. Under this information set each agent is essentially solving a signal extraction problem which is completely isolated from the rest of the economy. In other words, there is no common component in agents' information sets besides the aggregate process a_t driving a_{it}, which, however, is not common knowledge.

Morris and Shin (2005) and Hellwig (2005) complement the information from the private signal with an exogenous public signal and study the welfare consequences of varying its precision. As this signal is common knowledge across agents it tends to homogenize agents information set and therefore agents strategies, particularly when complementarities are important. The overall beneficial effect of this public signal is ambiguous as it might crowd out part of the private signals thereby taking the equilibrium away from the full information benchmark. The key difference here is that the public signal embedded in the history of aggregate price is *endogenous*. Therefore, in the present framework information set (H) assigns to the aggregate price level that informational role that Hayek (1945) remarked as one of the key functions of the price system.

Lorenzoni (2006a,b) allows agents to extract information from endogenous variables, even though, because of the structure of his model, the flow of information is different from the one considered here. Faced with the problem of solving for a rational expectations equilibrium in presence of higher order expectations he models agents' updating of information using the Kalman filter. He then conjectures a solution for the price process as a linear function of the infinite history of the aggregate disturbances. However, in order to solve for an equilibrium he is forced to truncate the state so to deal with finite dimensional matrices in solving for the steady state Kalman filter.

We anticipate that when information set (H) is considered, the characteristics of the equilibrium aggregate price presents substantial differences when compared to the outcomes from (W). The method of analysis discussed in section 3 conveniently allows one to move across diverse information sets while still obtaining explicit equilibrium solutions.

As already pointed out, the informational content of the aggregate price level in the present setup can be evaluated in terms of the ability of the history of prices to reveal the history of structural economic disturbances. An equilibrium is said to be *invertible* if the Hilbert space generated by the history of prices $H_t(p)$ is equivalent to the Hilbert space generated by the history of innovations on aggregate productivity $H_t(u)$. Conversely, an

[22] The choice of letting agents observe the last period equilibrium prices while natural in a setting where agents are price-setters it represents a non-standard assumption in models of competitive markets. However, Hellwig (1982) analyzes a model of competitive capital markets where agents can condition only on past prices. He shows that the impossibility of an informational efficient market (Grossman and Stiglitz, 1980) in that case tends to disappear as the current private information maintains a positive value in equilibrium and it makes investing in collecting information worthwhile.

[23] To be precise, Woodford (2003) abstracts from real disturbances and allows firms to observe a noisy signal of the current realization of aggregate nominal demand.

equilibrium is said to be *non-invertible* when the history of prices does not allow the recovery of the structural innovations so that $H_t(p) \subset H_t(u)$.

4.2 Price Equilibrium Under Exogenous Private Information

We first consider the prediction problem faced by a firm observing just the history of its own productivity shocks which corresponds to information set Ω_{it}^W. The variable to predict at time t is the price process p_t whose moving average representation in terms of a_t is known to be

$$p_t = \tilde{P}(L)\, a_t. \tag{42}$$

The link between p_t and the individual productivity a_{it} is provided by assumed the relationship between a_t and a_{it} which is

$$a_{it} = a_t + v_{it}, \tag{43}$$

where v_{it} is a white noise process with variance σ_v^2. The covariance generating function of the information set is

$$g_{a_i a_i}(z) = a(z)\, a\left(z^{-1}\right) \sigma_u^2 + \sigma_v^2.$$

The non-trivial step in the Wiener-Hopf approach is the factorization of the covariance generating function of the information set upon which the prediction is based. More precisely one looks for the factor $d(z)$ with $d(0) = 1$ such that

$$\sigma_w^2 d(z)\, d\left(z^{-1}\right) = a(z)\, a\left(z^{-1}\right) \sigma_u^2 + \sigma_v^2.$$

In addition we need the covariance generating function between the price and the state $g_{pa}(z)$. Given the conjectured price function (42) it can be showed that the covariance generating function is

$$g_{pa}(z) = \tilde{P}(z)\, a(z)\, a\left(z^{-1}\right) \sigma_u^2. \tag{44}$$

We first consider a simple white noise process for the state $a_t = u_t$. The factorization is immediately seen as $d(z) = 1$ and $\sigma_w^2 = \sigma_u^2 + \sigma_v^2$. Since the autoregressive structure of the state and the noise coincide the history of observed signals cannot improve the inference on the realizations of the state, just because those signals bear no information about the realization of the state except at the period they are observed. Using (44) the Wiener-Hopf prediction formula returns

$$\begin{aligned} E\left(p_t | \Omega_{it}^W\right) &= \frac{1}{\sigma_u^2 + \sigma_v^2} \left[\tilde{P}(L)\, \sigma_u^2\right]_+ a_{it} \\ &= \frac{\sigma_u^2}{\sigma_u^2 + \sigma_v^2} \tilde{P}(L)\left(u_t + v_{it}\right). \end{aligned}$$

Taking the average across firms, the market expectations of the price level are

$$\bar{E}(p_t) = \frac{\sigma_u^2}{\sigma_u^2 + \sigma_v^2} \tilde{P}(L)\, u_t.$$

Substituting into the market clearing condition gives

$$p_t = (1-\gamma) \frac{\sigma_u^2}{\sigma_u^2 + \sigma_v^2} \tilde{P}(L)\, u_t - \gamma u_t.$$

Given the conjectured form for the price function, the fixed point condition becomes

$$\tilde{P}(z) = (1-\gamma) \frac{\sigma_u^2}{\sigma_u^2 + \sigma_v^2} \tilde{P}(z) - \gamma,$$

which delivers the equilibrium solution

$$p_t = \gamma \frac{\sigma_u^2 + \sigma_v^2}{\gamma \sigma_u^2 + \sigma_v^2} u_t.$$

As one would expect, when productivity is an i.i.d. process only current signals matter. In equilibrium, each agent optimal prediction of the price level is a linear function of the private signal a_{it}, namely $E\left(p_t | \Omega_{it}^W\right) = \frac{\gamma \sigma_u^2}{\gamma \sigma_u^2 + \sigma_v^2} a_{it}$. The relevance of the private signal a_{it} to the prediction of the price process depends on the degree of strategic complementarity γ. Equation (21) shows that γ affects the relative importance of expectations of the higher order in the determination of the price level. The appropriate version of that equation under the current assumption on the monetary base is

$$p_t = -\gamma \sum_{j=0}^{\infty} (1-\gamma)^j \bar{E}_t^{(j)} (a_t) \tag{45}$$

As γ becomes smaller the higher order expectations of the aggregate productivity receive a higher weight. It can be shown that agents rationally downplay the relevance of their private information as the degree of the expectations increases. Consider for example the direct expectation $E_t\left(a_t | \Omega_{it}^W\right)$, using the appropriate version of the Wiener-Hopf prediction formula we can write

$$E\left(a_t | \Omega_{it}^W\right) = \frac{\sigma_u^2}{\sigma_u^2 + \sigma_v^2} a_{it}. \tag{46}$$

The first degree market expectations are

$$\bar{E}_t (a_t) = \frac{\sigma_u^2}{\sigma_u^2 + \sigma_v^2} a_t. \tag{47}$$

The problem of a firm i consists in forming the expectation $E\left(p_t | \Omega_{it}^W\right)$ which, from (45), is equivalent to forming the expectation of the market expectations at every degree, from zero to infinite. The Wiener-Hopf formula is flexible enough to allow the derivation of each of these terms. Let $e_t^1 = \bar{E}_t (a_t)$, then one would simply have

$$\begin{aligned} E\left(\bar{E}_t (a_t) | \Omega_{it}^W\right) &= \frac{1}{\sigma_u^2 + \sigma_v^2} \left[g_{e_t^1 a_i} (z) \right]_+ a_{it} &\quad (48) \\ &= \frac{1}{\sigma_u^2 + \sigma_v^2} \left[\frac{\sigma_u^2}{\sigma_u^2 + \sigma_v^2} \sigma_u^2 \right]_+ a_{it} &\quad (49) \\ &= \left(\frac{\sigma_u^2}{\sigma_u^2 + \sigma_v^2} \right)^2 a_{it}. &\quad (50) \end{aligned}$$

Comparing (46) to (50) the downplaying of private information is evident. What is its rationale? Each firm knows that all the other firms are facing a similar signal extraction problem which leads to expectations of the form (46). At the aggregate level (46) results into (47), which now becomes the object of prediction for each firm. As aggregate productivity is downplayed in the first order average market expectations, firms' estimates of the market expectations will downplay the private prediction of aggregate productivity. This will result in a further downplay of aggregate productivity in the market expectations of the market expectations. Proceeding in this fashion one sees that the terms in (45) take the form

$$\bar{E}_t^{(j)} (a_t) = \left(\frac{\sigma_u^2}{\sigma_u^2 + \sigma_v^2} \right)^j a_t. \tag{51}$$

Substituting and rearranging one can confirm our solution for p_t found by direct computation of $E\left(p_t | \Omega_{it}^W\right)$. From (51) the higher order expectation terms converge to 0, which is the unconditional expectations of the aggregate productivity, the piece of information that is common knowledge across firms. We further note that in the absence of persistence in aggregate productivity the individual optimal prediction is the same across the exogenous and endogenous information sets defined above. In other words, p_{t-1} will contain no useful information about a_t.

We next solve for an equilibrium under a first order autoregressive process for aggregate productivity,

$$a_t = \frac{1}{1-\rho L} u_t.$$

Using Theorem 1 we show the following proposition.

Proposition 1 *Firm i's optimal prediction for current aggregate price level is:*

$$E\left(p_t|\Omega_{it}^W\right) = \frac{(1-\rho L)}{\sigma_w^2 (1-\lambda L)} \left[\frac{L\tilde{P}(L)\,\sigma_u^2}{(1-\rho L)(L-\lambda)} - \frac{\lambda \tilde{P}(\lambda)\,\sigma_u^2}{(1-\rho\lambda)(L-\lambda)}\right] a_{it}, \tag{52}$$

where

$$\lambda = \frac{1}{2}\left[\left(\frac{\sigma_u^2}{\sigma_v^2 \rho}\right) + \left(\frac{1}{\rho}+\rho\right) - \left\{\left[\left(\frac{\sigma_u^2}{\sigma_v^2\rho}\right) + \left(\frac{1}{\rho}+\rho\right)\right]^2 - 4\right\}^{\frac{1}{2}}\right],$$

and:

$$\sigma_w^2 = \frac{\sigma_u^2 + \sigma_v^2 (1-\rho)^2}{(1-\lambda)^2}.$$

Proof. See Appendix. ■

Substituting (52) into the optimal pricing function (39) and aggregating across firms gives the following condition for the equilibrium price function $\tilde{P}(z)$:

$$\tilde{P}(z) = -\left(\frac{\gamma\sigma_w^2 (1-\lambda z)(1-\rho\lambda)(z-\lambda) + (1-\gamma)(1-\rho z)\lambda\tilde{P}(\lambda)\,\sigma_u^2}{(1-\rho\lambda)\left(\sigma_w^2 (1-\lambda z)(z-\lambda) - \sigma_u^2 (1-\gamma) z\right)}\right). \tag{53}$$

This expression is not a closed form solution for the price process because of the term $\tilde{P}(\lambda)$. Evaluating both sides at λ does not allow one to pin down this constant. It follows that in general the solution is not unique. However, applying Whiteman (1983), if one of the roots in the denominator is inside the unit circle (which corresponds to an unstable eigenvalue, or the forward looking part of the solution), then the constant $\tilde{P}(\lambda)$ can be set so to cancel the root and obtain a unique stationary solution[24]. It can be shown that the denominator in (53) always satisfies this condition as its roots are such that one is always the reciprocal of the other. In fact, writing the denominator polynomial as

$$\begin{aligned}\left(\sigma_w^2 (1-\lambda z)(z-\lambda) - \sigma_u^2 (1-\gamma) z\right) &= -\sigma_w^2\lambda\left(1 + z^2 - \frac{\sigma_w^2\left(1+\lambda^2\right) - (1-\gamma)\,\sigma_u^2}{\sigma_w^2\lambda} z\right) \\ &= -\sigma_w^2\lambda(1-\theta_1 z)(1-\theta_2 z),\end{aligned}$$

it follows that

$$\theta_1\theta_2 = 1 \text{ and } \theta_1+\theta_2 = \frac{\sigma_w^2\left(1+\lambda^2\right) - (1-\gamma)\,\sigma_u^2}{\sigma_w^2\lambda}.$$

[24] In stationary equilibria under rational expectations non-uniqueness arises because, in general, a process for the forecast errors satisfying the orthogonality condition and being consistent with the law of motion for the system can be arbitrarily specified. Stationarity implies a unique solution if the the law of motion of the system contains an explosive autoregressive component, which is the case in (53). Under these conditions the process for the forecast errors cannot be arbitrary but it must be a function of the driving process such that the explosive autoregressive component is "succesfully contrasted". The constant $\tilde{P}(\lambda)$ is chosen in order to accomplish this task and deliver a stationary equilibrium that is unique. In models with forward looking rational expectations, uniqueness of a solution is usually ensured by a similar condition (see Beyer and Farmer (2006) and Cochrane (2006)). This is particularly interesting as no forward lookingness per-se is present in firms' pricing equations.

Without loss of generality, we let $\theta_1 = \theta$ where $|\theta| > 1$ so that $\theta_2 = \frac{1}{\theta}$ and $|\theta_2| < 1$. In addition, we note that

$$\sigma_w^2 \left(1 - \lambda \frac{1}{\theta}\right) \left(\frac{1}{\theta} - \lambda\right) = \sigma_u^2 (1-\gamma) \frac{1}{\theta}.$$

The condition to solve for the constant, $\tilde{P}(\lambda)$, can be finally written as:

$$\gamma \frac{1}{\theta} (1 - \rho\lambda) + \left(1 - \rho \frac{1}{\theta}\right) \lambda \tilde{P}(\lambda) = 0$$

which leads to

$$\tilde{P}(\lambda) = -\frac{\gamma}{\lambda} \frac{1 - \rho\lambda}{\theta - \rho}.$$

Substituting back into the solution gives the equilibrium price in closed form,

$$\tilde{P}(z) = -\gamma \frac{\sigma_w^2 (1 - \lambda z)(z - \lambda)(\rho - \theta) + (1 - \gamma)(1 - \rho z) \sigma_u^2}{(\rho - \theta)\left(\sigma_w^2 (1 - \lambda z)(z - \lambda) - \sigma_u^2 (1 - \gamma) z\right)}.$$

The numerator, by construction, contains the factor $(1 - \theta z)$ which cancels with the same factor in the denominator. Thus,

$$p_t = \gamma \left(\frac{\sigma_u^2 (1 - \gamma) - \sigma_w^2 \lambda (\rho - \theta)}{\sigma_w^2 \lambda (\rho - \theta)}\right) \frac{\left(1 - \tilde{\theta} L\right)}{\left(1 - \frac{1}{\theta} L\right)} a_t,$$

where

$$\tilde{\theta} = -\frac{1}{\theta} \frac{\sigma_w^2 \lambda (\rho - \theta)}{\sigma_u^2 (1 - \gamma) - \sigma_w^2 \lambda (\rho - \theta)}. \tag{54}$$

This expression shows that, under incomplete information, the equilibrium price process is an $ARMA(1,1)$ in a_t ($ARMA(2,1)$ in u_t). Incomplete information creates a wedge with respect to the full information benchmark. This wedge consists of a static component, which makes the response of prices to productivity innovation different at impact, and a dynamic component, which makes the response of prices to productivity innovation hump-shaped compared to a monotonically decreasing response under the full information case. These two components are not independent from one another. In particular, from (54), one can see that the dynamic component approaches the full information benchmark (i.e. $\tilde{\theta} \to \frac{1}{\theta}$ so the dynamic term is 1) when the static component also approaches the full information benchmark, i.e., $\frac{\sigma_w^2 \lambda(\rho - \theta)}{\sigma_w^2 \lambda(\rho - \theta) - \sigma_u^2 (1 - \gamma)} \to 1$. By a closer inspection one sees that the degree of strategic complementarity γ is responsible for the existence of the wedge. Incomplete dispersed information would aggregate efficiently and result in the full information dynamics if $\gamma = 1$. **Insert something like equation (3.12) in Walker(2006) to draw the connection between the recovered shock and the underlying shocks.***

We illustrate by a numerical example the properties of the equilibrium aggregate price under incomplete exogenous information.

Example 1. We fix the parameter values

$$\rho = 0.9, \sigma_u = 1, \sigma_v = 0.7$$

while we consider two alternative values for the strategic complementarity parameter $\gamma \in \{.4, .2\}$[25]. In both cases, the full information price is given by:

$$p_t^* = -a_t.$$

[25] The consensus over the appropriate value of the degree of strategic complementarity in price setting is weak. This is mainly due to the fact that the value identified by empirical studies depends strongly on the assumptions about the markets for the factor of production (i.e. the labor market) and on the intertemporal elasticity of substitution in the utility function of households. Given our assumption on specific markets for factors, Table 3.1 at page 172 in Woodford (2003b) suggests a range of values from 0.17 to 0.06. Our choices reflect a conservative reading of those numbers. See Woodford (2003b, pp 171-173) for additional discussion on the appropriate value of γ.

In contrast, under incomplete information,

$$p_t = -0.74 \frac{(1-0.22L)}{(1-0.40L)} a_t \quad \text{for } \gamma = .4,$$

$$p_t = -0.54 \frac{(1-0.19L)}{(1-0.51L)} a_t \quad \text{for } \gamma = .2.$$

The impulse response functions for both examples are reported in Figure (1) and (2). As strategic complementarities increase, aggregate price tends to react less to aggregate innovation at impact (-0.74 against -0.54) and its deviation from the full information counterpart increases in persistence (0.40 against 0.51).

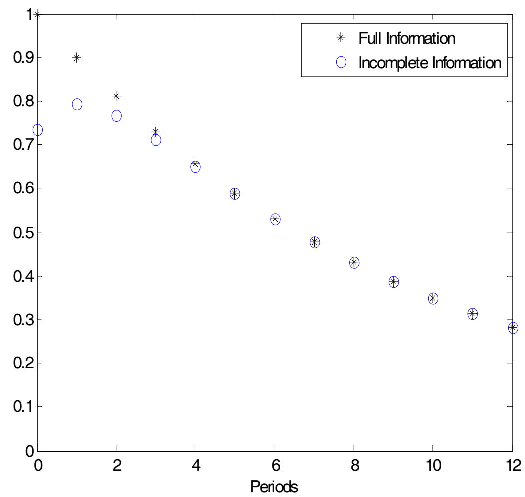

Figure 1, $\gamma = .4$.

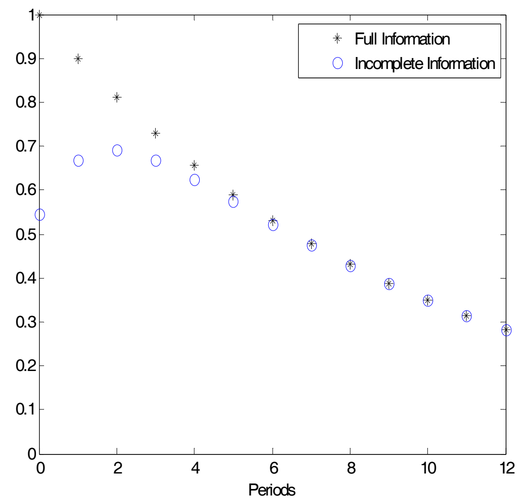

Figure 2, $\gamma = .2$.

Aggregate Price Impulse Response Functions for $\gamma \in \{0.4, 0.2\}$

If one interprets the aggregate price index as the Producer Price Index (PPI) released monthly by the US Bureau of Labor Statistics, the periods in the plots should be considered as months. In both cases the reaction of aggregate price to aggregate innovation is hump-shaped and in both cases the reactions slowly converge to the full information benchmark.

Next we report, for completeness, the impulse response functions for the aggregate price for the lower and upper bound of the range of values for γ proposed by Woodford (2003b) under the set of assumptions on markets for factors that are consistent with the assumptions of our model, i.e. $\gamma \in \{0.17, 0.06\}$. As one would expect, the reaction of the aggregate price is slower than in both previous cases[26].

[26] The processes for aggregate price in the two cases are:

$$p_t = -0.50 \frac{(1-0.18L)}{(1-0.54L)} a_t \quad \text{for} \quad \gamma = 0.17,$$

$$p_t = -0.29 \frac{(1-0.14L)}{(1-0.68L)} a_t \quad \text{for} \quad \gamma = 0.06.$$

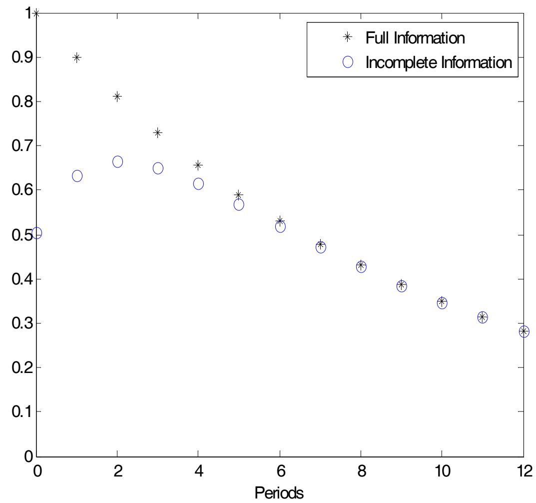

Figure 3, $\gamma = .17$.

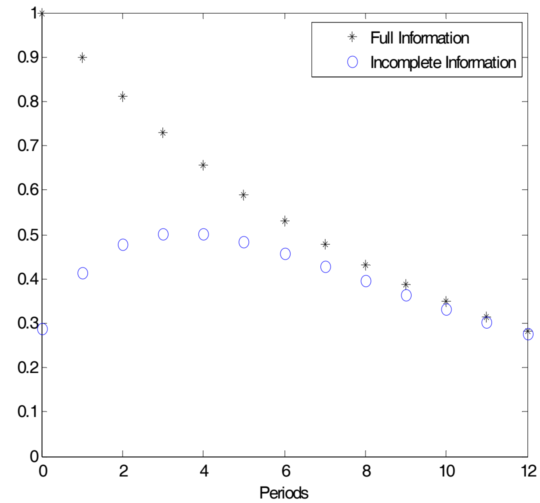

Figure 4, $\gamma = .06$.

Aggregate Price Impulse Response Functions for $\gamma \in \{0.17, 0.06\}$

The informational externality embedded in the slow initial reaction of the aggregate price plays no role under the information set Ω_{it}^W as prices are not included in the information set of agents. In the next section, we investigate the consequences of letting aggregate price signals aggregate disturbances.

4.3 Equilibrium Under Endogenous Information

Under the endogenous information set the representation of the system from the point of view of agent i can be specified as

$$\begin{pmatrix} a_{it} \\ p_{t-1} \end{pmatrix} = \begin{pmatrix} a(L) & 1 \\ P(L)L & 0 \end{pmatrix} \begin{pmatrix} u_t \\ v_{it} \end{pmatrix}, \tag{55}$$

where we assume that $a(0) = 1$. In order to apply Theorem 2 to the appropriate system matrix the following normalization is necessary:

$$\begin{aligned} u_t &= \sigma_u \varepsilon_{ut}, \\ v_{it} &= \sigma_v \varepsilon_{it}, \end{aligned}$$

where ε_{ut} and ε_{it} are i.i.d. white noise processes with mean zero and unit variance. The appropriate system matrix is

$$\Gamma(z) = \begin{pmatrix} \sigma_u a(z) & \sigma_v \\ \sigma_u P(z) z & 0 \end{pmatrix}.$$

According to Theorem 2, the matrix $\Gamma(z)$ is not the canonical spectral factorization for the covariance generating matrix of the information set Ω_{it}^H. The reason being that it cannot be inverted at the point $z = 0$ since $\det \Gamma(0) = 0$. In order to apply the Wiener-Hopf formula, the system matrix needs to be linearly transformed in order to eliminate the problematic root. Following Theorem 2, this is done by post-multiplying it for a constant matrix W_0 obtained from the singular value decomposition of $\Gamma(0)$ and a Blaschke factor $B_0(z)$. Their specific forms in the case under consideration are (see the Appendix for details on their derivation):

$$B_0(z) = \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{z} \end{pmatrix} \text{ and } W_0 = \begin{pmatrix} \frac{\sigma_u}{\sqrt{\sigma_u^2+\sigma_v^2}} & -\frac{\sigma_v}{\sqrt{\sigma_u^2+\sigma_v^2}} \\ \frac{\sigma_v}{\sqrt{\sigma_u^2+\sigma_v^2}} & \frac{\sigma_u}{\sqrt{\sigma_u^2+\sigma_v^2}} \end{pmatrix}.$$

Notice that the determinant of $\Gamma(z)$ is zero at $z = 0$ for any arbitrary price function $P(z)$. However, this is not the only possible point at which the system matrix might be non-invertible. When an invertible price equilibrium is conjectured the only problematic root that has to be removed is the one that the system contains by construction. However, the price function obtained from the fixed point condition must satisfy the initial conjecture by being invertible. If this is not the case, the initial conjecture is falsified and the price function is not part of a rational expectations equilibrium.

Our conjectured price function is a lag polynomial $P(L)$ whose moving average is invertible so that the appropriate canonical spectral factorization is given by:

$$\Gamma^*(z) = \Gamma(z) W_0 B_0(z)$$

This leads to the following proposition.

Proposition 2 *Under an invertible price equilibrium, firm's i optimal prediction for the aggregate price level p_t conditional on the information set Ω_{it}^H is:*

$$E_t\left(p_t|\Omega_{it}^H\right) = \left(P(L) - P_0\right) u_t + P_0 \frac{\sigma_u^2}{\sigma_u^2 + \sigma_v^2}\left(u_t + v_{it}\right). \tag{56}$$

Proof. See Appendix. ■

The individual forecast equation under an invertible price process can be divided into two components. The first component represents the part of the state of the economy that is common knowledge. In fact, any agent, by observing the past price process, can recover the structural disturbances u up to time $t-1$. This results in the first part of the forecast equation $\left(P(L) - P_0\right) u_t$. Next, agents need to optimally forecast the part of the process given by $P_0 u_t$. To understand the information used in this part of the forecast, note that the recovery of the structural disturbances up to time $t-1$ allow agents to recover the aggregate productivity up to the last period, a_{t-1}. It follows that the signal they rely upon to forecast u_t is $a_{it} - a_{t-1} = u_t + v_{it}$. This is essentially a static signal extraction problem whose solution is $\frac{\sigma_u^2}{\sigma_u^2+\sigma_v^2}\left(u_t + v_{it}\right)$. This is then multiplied by P_0. Therefore, in the presence of an invertible price equilibrium the signal extraction problem for agents is divided into two isolated parts: one that is common across agents and concerns the commonly observable history of prices and one that is specific to each agent and static in nature as it concerns the current period innovation.

Aggregating (56) across agents, the agent's specific components v_{it} average out to zero and the market expectation of the price level becomes

$$\bar{E}_t\left(p_t\right) = P(L) u_t - P_0 \frac{\sigma_v^2}{\sigma_u^2 + \sigma_v^2} u_t. \tag{57}$$

This expression shows the effect of the individual signal extraction problem as expectations are aggregated: the lack of perfect correlation between firm-specific productivity and aggregate productivity (which results because $\sigma_v > 0$), prevents the market forecast from perfectly predicting the equilibrium price. Elaborating on this expression it can be confirmed that the law of iterated expectations fails when applied to market expectations. Suppose that we want to evaluate agent i expectations of the market expectations. In order to do this, let $\bar{E}_t\left(p_t\right) = D(L) u_t$ so that the optimal prediction formula can be applied to the covariance generating function involving $D(L)$. This implies

$$E_t\left(\bar{E}_t\left(p_t\right)|\Omega_{it}\right) = \left(D(L) - D_0\right) u_t + D_0 \frac{\sigma_u^2}{\sigma_u^2 + \sigma_v^2}\left(u_t + v_{it}\right),$$

where

$$D_0 = \frac{\sigma_u^2}{\sigma_u^2 + \sigma_v^2} P_0.$$

Rearranging, one gets

$$E_t\left(\bar{E}_t\left(p_t\right)|\Omega_{it}\right)=\left(P\left(L\right)-P_0\right)u_t+\frac{\sigma_u^2}{\sigma_u^2+\sigma_v^2}P_0\frac{\sigma_u^2}{\sigma_u^2+\sigma_v^2}\left(u_t+v_{it}\right). \tag{58}$$

Comparing the individual direct expectations (56) to the individual expectation of the market expectation (58) one can see that agent i's private information (u_t+v_{it}) is rationally given less importance since $\frac{\sigma_u^2}{\sigma_u^2+\sigma_v^2}<1$.

Aggregating (58) across agents one has

$$\bar{E}_t\left(\bar{E}_t\left(p_t\right)\right)=\left(P\left(L\right)-P_0\right)u_t+\left(\frac{\sigma_u^2}{\sigma_u^2+\sigma_v^2}\right)^2P_0u_t, \tag{59}$$

which is different from $\bar{E}_t\left(p_t\right)$ exactly because agents tend to give less importance to their private information in guessing the average market expectations. Repeating the procedure one can see that the j-th higher order market expectation is

$$\bar{E}_t^{(j)}\left(p_t\right)=\left(P\left(L\right)-P_0\right)u_t+\left(\frac{\sigma_u^2}{\sigma_u^2+\sigma_v^2}\right)^jP_0u_t. \tag{60}$$

Taking the limit of this expression one immediately sees that

$$\lim_{j\to\infty}\bar{E}_t^{(j)}\left(p_t\right)=\left(P\left(L\right)-P_0\right)u_t.$$

In the limit, higher order expectational terms are function solely of information that is common knowledge, in the present case $H_{t-1}\left(u\right)$. This result parallels a key result in Morris and Shin (2002).

Proposition 2 and market clearing lead to the following result.

Proposition 3 *There exists a unique stationary rational expectations equilibrium invertible price level for the system (55) if the lag polynomial:*

$$\left(a\left(L\right)-\frac{\left(1-\gamma\right)\sigma_v^2}{\gamma\sigma_u^2+\sigma_v^2}\right),$$

has all its zeros outside the unit circle. The equilibrium invertible price process is:

$$P\left(L\right)=-\left(a\left(L\right)-\frac{\left(1-\gamma\right)\sigma_v^2}{\gamma\sigma_u^2+\sigma_v^2}\right)u_t. \tag{61}$$

Proof. See Appendix. ■

Using (39), (40) and(57), one can write the equilibrium price as

$$p_t=-a\left(L\right)u_t-\frac{\left(1-\gamma\right)}{\gamma}\frac{\sigma_v^2}{\left(\sigma_u^2+\sigma_v^2\right)}P_0u_t.$$

Evaluating $P(L)$ at 0 the equilibrium is uniquely determined by setting

$$P_0=-\frac{\gamma\left(\sigma_u^2+\sigma_v^2\right)}{\gamma\sigma_u^2+\sigma_v^2}. \tag{62}$$

It is interesting to note that, provided the price equilibrium is invertible, it is a *unique* rational expectations equilibrium. Substituting back (62) into the solution one obtains (61).

Consider now the limit when $\sigma_v\to0$. In this case every firm experiences the same productivity process which corresponds to the observability of the current realization of u_t (provided $a(L)$ is itself invertible). In the current setup $\sigma_v\to0$ is equivalent to complete information at time t, and from (61) we see that

$$\lim_{\sigma_v\to0}p_t=p_t^*.$$

Let the process for productivity be $a\left(L\right)=1+aL$ with $|a|<1$, then the unique invertible equilibrium when it exists takes the form

$$p_t=-\left(1-\frac{\left(1-\gamma\right)\sigma_v^2}{\gamma\sigma_u^2+\sigma_v^2}+aL\right)u_t.$$

Under complete information the price process would be:

$$p_t^* = -(1 + aL) u_t.$$

The difference between the two processes arises from the contemporaneous effect of u_t on p_t. In fact one can see that the contemporaneous impact of a unitary variation in u_t is bigger for the complete information case while it is attenuated in the incomplete information case

$$\frac{\Delta p_t}{\Delta u_t} = \frac{\gamma\left(\sigma_u^2 + \sigma_v^2\right)}{\gamma \sigma_u^2 + \sigma_v^2} < 1 = \frac{\Delta p_t^*}{\Delta u_t}$$

At impact, the reaction of aggregate price is similar to the case of an exogenous information set. Nonetheless, the impact of the shock on future realizations of the price level in both cases is the same and equal to a. It follows that, if the contemporaneous impact is attenuated enough because of information incompleteness, the effect of the two impacts might be observationally equivalent to an autoregressive process with completely different realizations for the disturbances. It is at this point that the price level becomes non-invertible in terms of the structural innovations and the validity of the solution just derived breaks down. Interestingly, this can happen even for processes for aggregate productivity a_t with simple dynamics as the following examples show.

Examples

MA(1) Suppose that $a(L) = 1 + aL$ with $|a| < 1$ (so that the aggregate productivity process it self is invertible), then a unique revealing rational expectations equilibrium exists if:

$$\left| a \frac{\gamma \sigma_u^2 + \sigma_v^2}{\gamma\left(\sigma_u^2 + \sigma_v^2\right)} \right| < 1. \tag{63}$$

Substituting $a(z) = 1 + az$ into the pricing function (61) one gets:

$$P(L) = -\left(1 + aL - \frac{(1-\gamma)\sigma_v^2}{\gamma \sigma_u^2 + \sigma_v^2}\right) u_t.$$

Isolating the relevant root one can see that:

$$P(L) = -\frac{\gamma\left(\sigma_u^2 + \sigma_v^2\right)}{\gamma \sigma_u^2 + \sigma_v^2}\left(1 + a \frac{\gamma \sigma_u^2 + \sigma_v^2}{\gamma\left(\sigma_u^2 + \sigma_v^2\right)} L\right) u_t,$$

and condition (63) follows.

AR(1) Suppose, alternatively, that $a(L) = \frac{1}{1-\rho L}$ with $|\rho| < 1$. Then a unique revealing rational expectations equilibrium exists if:

$$\left| \rho \frac{(1-\gamma)\sigma_v^2}{\gamma\left(\sigma_u^2 + \sigma_v^2\right)} \right| < 1. \tag{64}$$

Substituting $a(z) = \frac{1}{1-\rho z}$ in the pricing function and rearranging terms one obtains:

$$\begin{aligned} P(L) &= -\left(\frac{1}{1-\rho L} - \frac{(1-\gamma)\sigma_v^2}{\gamma \sigma_u^2 + \sigma_v^2}\right) u_t \\ &= -\frac{\gamma\left(\sigma_u^2 + \sigma_v^2\right)}{\gamma \sigma_u^2 + \sigma_v^2}\left(\frac{1 + \rho \frac{(1-\gamma)\sigma_v^2}{\gamma\left(\sigma_u^2 + \sigma_v^2\right)} L}{1 - \rho L}\right) u_t \end{aligned}$$

which suggests the condition in (64).

ARMA(1,1) Finally, suppose $a(L) = \frac{1+aL}{1-\rho L}$. Then there exists a revealing equilibrium when:

$$\left|\frac{a\left(\gamma\sigma_u^2+\sigma_v^2\right)+\rho\left(1-\gamma\right)\sigma_v^2}{\gamma\left(\sigma_u^2+\sigma_v^2\right)}\right|<1$$

This is exactly the condition that results from combining the previous two cases.

The above examples illustrate a remarkable property of rational expectations equilibria in models of monopolistic competitive pricing under incomplete information. Namely, if strategic complementarities are strong enough (γ sufficiently close to zero), then it becomes likely, for a large open set in the parameter space, that an invertible-price rational expectations equilibrium conjecture is falsified. Surprisingly, this is the case also when the aggregate productivity process is a simple $AR(1)$ process. This observation motivates the search for an equilibrium that is not-invertible. This work is the next step in this research program.

Numerical Example We consider again the numerical example of section 4.2 where we specified:

$$\rho = 0.9, \sigma_u = 1, \sigma_v = 0.7$$

Let $\gamma = .4$. From proposition 3 the condition for the existence of an invertible equilibrium is:

$$\left|\rho\frac{\left(1-\gamma\right)\sigma_v^2}{\gamma\left(\sigma_u^2+\sigma_v^2\right)}\right| = 0.47.$$

The price process is:

$$p_t = -0.67\left(1+0.47L\right)a_t.$$

Let $\gamma = .2$. The condition for the existence of an invertible equilibrium is:

$$\left|\rho\frac{\left(1-\gamma\right)\sigma_v^2}{\gamma\left(\sigma_u^2+\sigma_v^2\right)}\right| = 1.24.$$

We note that, for $\gamma = .2$, the condition of Proposition 3 is violated.

The impulse response function for the case $\gamma = .4$ is reported in Figure (5). Notice that the only difference from the impulse response for the full information price arises at impact. In particular, when incomplete information and strategic complementarity combine to dampen the effect of individually experienced productivity on individual prices and thereby on aggregate price. From the subsequent period, the effect of the current aggregate innovation is perfectly encoded in prices as the observation of the current price level, although different from the full information one, still allows the recovery of the structural innovations. For the case $\gamma = .2$, the effect at impact of any individual productivity shock is excessively dampened, although rationally so from an individual point of view. However, this creates an informational externality which materializes in the impossibility of conditioning next period's price expectations on the correct value of the current structural innovation. The impulse response reported in Figure (6) is appropriate when one assumes that at period $t+1$ agents are allowed to observe the realization of a_t. In that case, the aggregate price follows the process $p_t = -0.43\left(1+1.24L\right)a_t$. The down-playing of productivity on prices is evident from the weak reaction of aggregate price at the time of impact, $t = 0$.

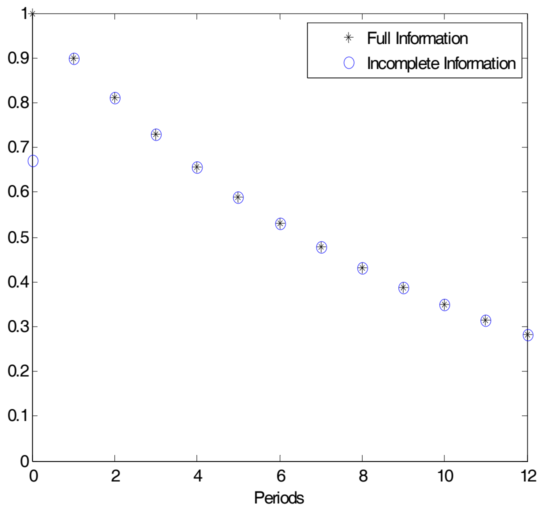

Figure 5

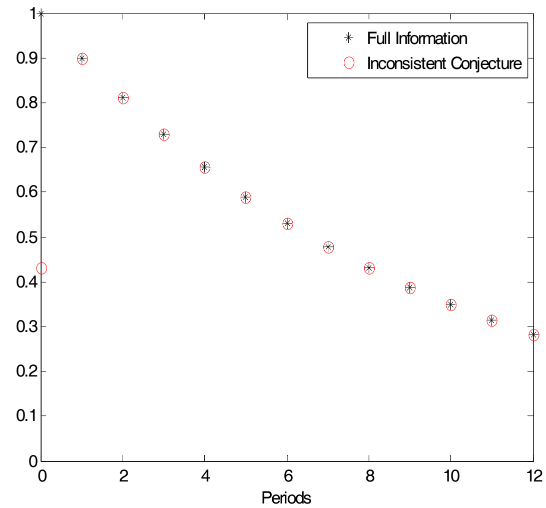

Figure 6

Aggregate Price Impulse Response Functions

Realized Rate of Output It has been assumed for convenience that nominal aggregate demand is always equal to 1 which implies that the log of the realized rate of output is equal to $-p_t$. Although this assumption is somewhat extreme, it is still useful to understand the behavior of the realized rate of output *vis à vis* the natural rate of output defined in Section 2 in the presence of an invertible price equilibrium. From Proposition 2 one immediately sees that:

$$\begin{aligned} y_t &= a(L)\,u_t + \frac{(1-\gamma)\,\sigma_v^2}{\gamma\sigma_u^2+\sigma_v^2}u_t && (65) \\ &= y_t^* + \frac{(1-\gamma)\,\sigma_v^2}{\gamma\sigma_u^2+\sigma_v^2}u_t. && (66) \end{aligned}$$

It follows that, conditional on observing the structural disturbances of the economy, u, the deviation of output from the full information benchmark is unpredictable. Namely:

$$E\left(y_{t+1}|H_t(p)\right) = 0 \tag{67}$$

This result is a version of the non-persistent effects of incomplete information in models like Lucas (1972) and Barro (1976). However, in those models, the unpredictability of deviations from the full information benchmark was due to the assumption that the state of the economy at t was perfectly observed by all the agents at the beginning of period $t+1$. In the present setting, the unpredictability result holds if the observation of $\{p_t\}$ is sufficient to reveal the state of the economy a_t. As our results point out, this is not always the case.

5 Application to Monetary Policy

In this section we briefly explore the effects of a simple monetary policy strategy on the informational properties of the equilibria characterized under endogenous information. The type of policy assumed is a simple feedback rule on the price level of the form

$$m_t = \phi(L)\,p_{t-1}. \tag{68}$$

Provided that agents can observe the history of prices, the level of money supply at time t is common knowledge across agents. This implies that the rule (68) is not affecting the signal extraction problem directly. However, by changing the equilibrium coordination incentive it might take the system to an invertible equilibrium. This is formalized by the following proposition.

Proposition 4 *For any stationary process a_t and parameters values for σ_v,σ_u and $\gamma > 0$ there always exists a feedback monetary policy such that an invertible price equilibrium results.*

The key to design such a policy is to react to the realizations of past prices such that the autoregressive components on fundamental disturbances that are forced into the system eliminate the problematic moving average components. Interestingly, a feedback policy in order to lead to an invertible equilibrium must always satisfy a sort of Taylor principle: namely, the feedback reaction must be more than proportional to the price level. The underlying working of such a policy can be described considering the example of an $AR(1)$ process for aggregate productivity. Suppose the parameters of the model are such that

$$\left|\rho\frac{(1-\gamma)\,\sigma_v^2}{\gamma\left(\sigma_u^2+\sigma_v^2\right)}\right| > 1, \tag{69}$$

then, by Proposition 3, an invertible equilibrium cannot realize. Nevertheless, conjecturing a monetary policy of the form

$$m_t = \phi p_{t-1},$$

it can be shown that the condition in proposition 3 requires that the polynomial

$$P(L) = -\frac{\gamma\left(\sigma_u^2+\sigma_v^2\right)}{\gamma\sigma_u^2+\sigma_v^2}\frac{\left(1+\rho\frac{(1-\gamma)\sigma_v^2}{\gamma(\sigma_u^2+\sigma_v^2)}L\right)}{(1-\rho L)(1-\phi L)},$$

has an invertible moving average component. This condition is met even though (69) applies when:

$$\phi^* = -\rho\frac{(1-\gamma)\,\sigma_v^2}{\gamma\left(\sigma_u^2+\sigma_v^2\right)}.$$

The resulting equilibrium is invertible and given by

$$P(L) = -\frac{\gamma\left(\sigma_u^2+\sigma_v^2\right)}{\gamma\sigma_u^2+\sigma_v^2}a_t.$$

The rate of output under policy ϕ^* becomes:

$$y_t = -\frac{\rho(1-\gamma)\,\sigma_v^2}{\gamma\sigma_u^2+\sigma_v^2}\left(1+\frac{\gamma\left(\sigma_u^2+\sigma_v^2\right)}{\rho(1-\gamma)\,\sigma_v^2}L\right)a_t.$$

Note that the monetary policy rule makes output an $ARMA(1,1)$ process whose MA component is invertible.

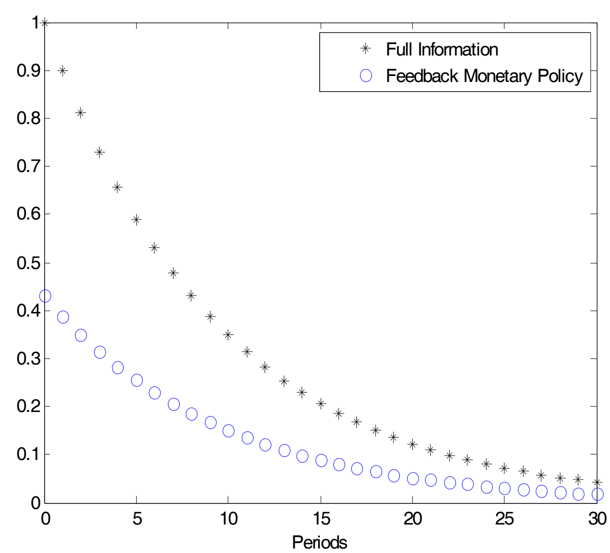

Figure 7

Aggregate Price Impulse Response Function

Figure (7) shows the impulse response function of the equilibrium aggregate price under an invertible-price monetary policy for the simple $AR(1)$ case. It can be seen that invertibility is obtained, but at the cost of taking the price dynamics persistently away from the full information benchmark. This simple example illustrates that invertibility of the price function, i.e., the ability of the aggregate price to be a sufficient statistic of the state of the economy, is not necessarily desirable if obtained by distorting economic incentives. In other words, in an environment where the aggregate price plays an active informational role, the monetary authority faces a trade-off between the distortions implicit in a system that allocates resources under incomplete information and the distortions introduced by monetary policy strategies that force the dynamics of the aggregate price to reveal the economic fundamentals. This type of trade-off could be modified by considering a monetary policy strategy that evaluates the possibility of releasing relevant information on the state of the economy to shape individual incentives so to encode information in prices without the costs implied by a feedback rule. This is, in part, the direction in which central banks currently concentrate their efforts. The model and the methods elaborated in this paper represent an appropriate framework for a first analysis of a monetary policy under an enriched set of instruments: money and information diffusion.

6 Conclusions and Future Research

This paper adapts results from the frequency domain approach to linear prediction theory to explicitly solve for a rational expectations equilibrium in a macroeconomic model in which firms make pricing decisions under incomplete information. This yields a micro-founded aggregate supply that has firms responding to changes in firm-specific and aggregate information. A rational expectations equilibrium is solved for in frequency domain. This approach allows the inclusion of aggregate price in the information set of firms while maintaining the tractability of the fixed point problem typical of rational expectations. Using frequency domain analysis allows me to avoid having to place restrictive assumptions on the information available to agents as in many other models of aggregate supply under incomplete information. Thus, this paper explores a rational expectations equilibrium in which endogenously emerging variables, such as aggregate price, are not excluded from agents' information sets, as is often the case in models of firm pricing faced with incomplete information.

The analytical forms of equilibrium aggregate price offer interesting insights in the mechanisms through which dispersed information diffuses in the economy. Aggregate price acts as an endogenous signal which is common knowledge across firms. It follows that firms' information sets are homogenized, compared to the case where prices are not observed. This fact has ambiguous implications in terms of the information that gets encoded into aggregate price. In the new generation of models of incomplete information, firms are modeled as players in a global game where their pricing strategies are affected by other firms' strategies. Coordination incentives characterize optimal strategies so firms tend to overreact to information that is common knowledge. It follows that firms tend to overreact to information contained in the history of aggregate price. At the same time, firms underreact to their private information which results in less information included in aggregate price. Information sets are thus more homogeneous but, contemporaneously, less informative.

The methods presented in this paper offer a way to understand the informational content of the aggregate price as a property that emerges endogenously in equilibrium. It follows that the way information diffuses across the economy may vary along the business cycle and can be modified by policy interventions. This immediately suggests two interesting applications of the framework developed above. First, persistent aggregate forecast errors that arise endogenously over the business cycle might be regarded as a source of disturbance observationally equivalent to a demand shock. This sets the stage for a theory of endogenous aggregate demand shocks based on rational agents acting under coordination incentives and incomplete information. Second, this paper offers a framework for the evaluation of monetary policies in environments where prices have both an allocative and an informational role. Intuitively, the activities of information collection and dissemination on the part of monetary authorities assume a prominent role in models of incomplete information. From a theoretical point of view, it is still not clear whether public release of information on the state of the economy is desirable in models with coordination incentives. This controversial issue has been recently analyzed, for example, by Morris and Shin (2002, 2005), Hellwig (2005), Amato and Shin (2006) and Adams (2006). In all these studies, the aggregate price conveys information to the central bank but not to economic agents. This paper offers an analytical framework where this restrictive assumption can be relaxed and more insights on the desirability of information diffusion by central banks can be gained. I see these two applications as part of an exciting future research program.

7 Appendix

7.1 Log-Linearization of the Profit Function

The objective of this section is to derive analytically the log-linear approximation to the firm specific pricing function as reported in (13). We begin the analysis from the pricing policy that would result under perfect information. Under this assumption a representative monopolistic competitive firm i sets its price in order to maximize the profit function

$$\Xi\left(P_t\left(i\right)\right) = P_t\left(i\right) Y_t\left(i\right) - W_t\left(i\right) N\left(Y_t\left(i\right), A_{it}\right). \tag{70}$$

$N\left(.\right)$ represents the amount of labor necessary to produce $Y_t\left(i\right)$ under the productivity factor A_{it}. Given the functional form for the production technology assumed in the text this is:

$$N\left(Y_t\left(i\right), A_{it}\right) = \left(\frac{Y_t\left(i\right)}{\alpha A_{it}}\right)^{\alpha}.$$

The demand of intermediate product $Y_t\left(i\right)$, faced by the monopolistic firm i, from the final good sector, given a profit maximizing production of composite final good Y_t, is $\left(\frac{P_t(i)}{P_t}\right)^{-\theta} Y_t$. In equilibrium, aggregate demand for the final product must equal aggregate supply, so we let $C_t = Y_t$. The above profit function can be then written as

$$\Xi\left(P_t\left(i\right)\right) = P_t\left(i\right)\left(\frac{P_t\left(i\right)}{P_t}\right)^{-\theta} Y_t - W_t\left(i\right)\left(\frac{Y_t}{\alpha A_{it}}\left(\frac{P_t\left(i\right)}{P_t}\right)^{-\theta}\right)^{\alpha}.$$

Next, we divide both terms on the RHS by the aggregate price and we substitute $W_t\left(i\right)$ for the schedule of supply of labor of type i necessary to produce $Y_t\left(i\right) = \left(\frac{P_t(i)}{P_t}\right)^{-\theta} Y_t$. This is

$$\left(\frac{W_t\left(i\right)}{P_t Y_t}\right)^{\frac{1}{\psi}} = \left(\frac{Y_t}{\alpha A_{it}}\left(\frac{P_t\left(i\right)}{P_t}\right)^{-\theta}\right)^{\alpha} \rightarrow \frac{W_t\left(i\right)}{P_t} = Y_t\left(\frac{Y_t}{\alpha A_{it}}\left(\frac{P_t\left(i\right)}{P_t}\right)^{-\theta}\right)^{\alpha\psi},$$

so that real profits, $\hat{\Xi}\left(P_t\left(i\right)\right)$, are

$$\hat{\Xi}\left(P_t\left(i\right)\right) = \left(\frac{P_t\left(i\right)}{P_t}\right)^{1-\theta} Y_t - Y_t\left(\frac{Y_t}{\alpha A_{it}}\left(\frac{P_t\left(i\right)}{P_t}\right)^{-\theta}\right)^{\alpha(1+\psi)}.$$

The first order condition for optimal pricing is

$$(1-\theta)\left(\frac{P_t(i)}{P_t}\right)^{-\theta}\frac{1}{P_t}+\theta\alpha(1+\psi)\left(\frac{Y_t}{\alpha A_{it}}\right)^{\alpha(1+\psi)}\left(\frac{P_t(i)}{P_t}\right)^{-\theta\alpha(1+\psi)-1}\frac{1}{P_t}=0.$$

Excluding the trivial solution $P_t(i)=0$ (which automatically excludes $P_t=0$), we can multiply both sides by $P_t\left(\frac{P_t(i)}{P_t}\right)^{\theta}$, rearranging

$$(1-\theta)+\theta\alpha(1+\psi)\left(\frac{Y_t}{\alpha A_{it}}\right)^{\alpha(1+\psi)}\left(\frac{P_t(i)}{P_t}\right)^{-\theta\alpha(1+\psi)-1+\theta} = 0 \tag{71}$$

$$\left(\frac{Y_t}{\alpha A_{it}}\right)^{\alpha(1+\psi)}\left(\frac{P_t(i)}{P_t}\right)^{-\theta\alpha(1+\psi)-1+\theta} = \frac{\theta-1}{\theta\alpha(1+\psi)}. \tag{72}$$

Let

$$A_{it}=Ae^{a_{it}}$$

Taking logs on both sides of (72),

$$\begin{aligned} p_t(i)-p_t &= \frac{\ln\left(\frac{(\theta-1)}{\theta\alpha(1+\psi)}\right)}{1-\theta+\theta\alpha(1+\psi)}+\frac{\alpha(1+\psi)}{1-\theta+\theta\alpha(1+\psi)}y_t-\frac{\alpha(1+\psi)}{1-\theta+\theta\alpha(1+\psi)}a_{it} \\ &\quad -\frac{\alpha(1+\psi)}{1-\theta+\theta\alpha(1+\psi)}\ln A-\frac{\alpha(1+\psi)}{1-\theta+\theta\alpha(1+\psi)}\ln\alpha. \end{aligned}$$

Letting

$$\gamma=\frac{\alpha(1+\psi)}{1+\theta(\alpha(1+\psi)-1)},$$

one can write

$$p_t(i)-p_t=\gamma y_t-\gamma a_{it}.$$

The constant term can be normalized to zero by an appropriate choice of A, without loss of generality, as it will not play any role in the dynamics. Under uncertainty over aggregate disturbances, one can log-linearize and take a quadratic approximation of expected profits. The first order condition of the quadratic approximation under uncertainty would then result in

$$p_t(i)-E(p_t|\Omega_{it})=\gamma E(y_t|\Omega_{it})-\gamma a_{it}, \tag{73}$$

which is the form reported in the text[27]. Note that, under the assumption $\psi>0$, $\alpha>1$ and $\theta>1$, one has $\gamma\in(0,1)$. In other words, firms pricing strategies are strategic complements. This justifies the special attention given in this paper to the case where $\gamma\in(0,1)$.

Log-Linear Approximation of the Price Index The appropriate price index in the economy considered in the paper is given by:

$$P_t=\left(\int_0^1 P_t(i)^{1-\theta}\,di\right)^{\frac{1}{1-\theta}}.$$

Letting $P_t(i)=e^{p_t(i)}$, $P_t=e^{p_t}$ and taking logs, we can write

$$p_t=\frac{1}{1-\theta}\log\left(\int_0^1 e^{p_t(i)(1-\theta)}di\right).$$

[27] The assumption that the certainty equivalence principle holds implies that agents act as if they are maximizing a quadratic version of the profit function. Alternatively, under the assumption of log-normal distributions for the variables involved in the maximization problem, the first order condition under uncertainty (73) holds exactly.

We know that, in steady state, firms set the same price. Without loss of generality, let this price be $P(i) = 1$. It follows that $p(i) = 0$ and

$$\log\left(\int_0^1 e^{p_t(i)(1-\theta)} di\right) = \log\left(\int_0^1 1di\right) = 0.$$

Perturbing each single price and taking the first order Taylor approximation around the perturbed point gives

$$p_t \simeq \frac{1}{1-\theta} \frac{(1-\theta)\, e^{p(i)(1-\theta)}}{\left(\int_0^1 e^{p(i)(1-\theta)} di\right)} \int_0^1 (p_t(i) - p(i))\, di = \int_0^1 p_t(i)\, di,$$

which is the aggregation condition used throughout the paper.

7.2 Proof of Proposition 1: Exogenous Private Information Equilibrium

The auto-covariance generating function for a_{it} is given by:

$$g_{a_i a_i}(z) = \sum_{j=-\infty}^{\infty} z^j cov(a_{it}, a_{it-j}).$$

Given the assumption (43) one has that

$$\begin{aligned} cov(a_{it}, a_{it-j}) &= E[(a_t + \varepsilon_{it})(a_{t-j} + \varepsilon_{it-j})] \\ &= E[a_t a_{t-j}] \\ &= cov(a_t, a_{t-j}), \end{aligned}$$

for $j \neq 0$. For $j = 0$, one has that the two covariances differ just for the variance of the noise term,

$$cov(a_{it}, a_{it}) = cov(a_t, a_{t-j}) + \sigma_i^2.$$

This results in

$$g_{a_i a_i}(z) = g_{aa}(z) + \sigma_i^2.$$

The next step is to consider the covariance generating function between the variable to be predicted, p_t, and the observed variable,

$$g_{pa_i}(z) = \sum_{j=-\infty}^{\infty} z^j cov(p_t, a_{it-j}).$$

Note that the covariance terms can be written as

$$\begin{aligned} cov(p_t, a_{it-j}) &= E[p_t, a_{it-j}] \\ &= E[p_t, a_{t-j} + \varepsilon_{it-j}] \\ &= E[p_t, a_{t-j}] + E[p_t, \varepsilon_{it-j}] \\ &= cov(p_t, a_{t-j}), \end{aligned}$$

where the condition $E[p_t, \varepsilon_{it-j}] = 0$ holds because of the assumption of independence of the ε_{it-j}'s from the aggregates. It follows that

$$g_{pa_i}(z) = g_{pa}(z).$$

The term $g_{pa}(z)$ can be easily computed using Sargent (1987, page 268, eqs 46-47). This gives,

$$g_{pa}(z) = \tilde{P}(z)\, g_{aa}(z).$$

Let the moving average representation for a_t be

$$a_t = a(L)\,\sigma_u \varepsilon_{ut},$$

where ε_{ut} is a white noise process with unit variance. It follows that the autocovariance generating function for a_t is

$$g_{aa}(z) = a(z)\,a\left(z^{-1}\right)\sigma_u^2.$$

We can finally write

$$g_{a_i a_i}(z) = a(z)\,a\left(z^{-1}\right)\sigma_u^2 + \sigma_v^2,$$

$$g_{pa_i}(z) = \tilde{P}(z)\,a(z)\,a\left(z^{-1}\right)\sigma_u^2.$$

The result of Theorem 1 stated that the optimal prediction coefficients $\pi(z)$ are given by:

$$\Pi(z) = \frac{1}{\sigma_w^2 \Gamma^*(z)}\left[\frac{g_{pa_i}(z)}{\Gamma^*\left(z^{-1}\right)}\right]_+,$$

where $\Gamma^*(z)$ and σ_w^2 come from the canonical factorization of $g_{a_i a_i}(z)$.

Since we work under the specification

$$a_t = \frac{1}{1-\rho L} u_t,$$

we can write

$$g_{a_i a_i}(z) = \frac{\sigma_u^2}{(1-\rho z)\left(1-\rho z^{-1}\right)} + \sigma_v^2,$$

$$g_{pa_i}(z) = \frac{\tilde{P}(z)}{(1-\rho z)\left(1-\rho z^{-1}\right)}\sigma_u^2.$$

The canonical factorization for the autocovariance of a_{it}, following Whittle (1963, page 31) and Sargent (1987, page 300-302), can be shown to be

$$\Gamma^*(z) = \frac{1-\lambda z}{1-\rho z},$$

where $|\lambda| < 1$,

$$\lambda = \frac{1}{2}\left[\left(\frac{\sigma_u^2}{\sigma_v^2 \rho}\right) + \left(\frac{1}{\rho} + \rho\right) - \left\{\left[\left(\frac{\sigma_u^2}{\sigma_v^2 \rho}\right) + \left(\frac{1}{\rho} + \rho\right)\right]^2 - 4\right\}^{\frac{1}{2}}\right],$$

and

$$\sigma_w^2 = \frac{\sigma_u^2 + \sigma_v^2 (1-\rho)^2}{(1-\lambda)^2}.$$

It follows that

$$a_{it} = \left(\frac{1-\lambda L}{1-\rho L}\right) w_{it},$$

where w_{it} is a white noise with variance σ_w^2 recoverable from $\{a_{it}\}$. In other words $\{w_{it}\}$ are the Wold fundamental innovations for a_{it}: the innovations to agent i information set induced by the observation of its own productivity process.

Applying the Wiener-Hopf formula,

$$\begin{aligned}
\Pi(z) &= \frac{1}{\sigma_w^2 \Gamma^*(z)}\left[\frac{g_{pa_i}(z)}{\Gamma^*\left(z^{-1}\right)}\right]_+ \\
&= \frac{(1-\rho z)}{\sigma_w^2 (1-\lambda z)}\left[\frac{\tilde{P}(z)\,\sigma_u^2}{(1-\rho z)\left(1-\rho z^{-1}\right)}\frac{\left(1-\rho z^{-1}\right)}{\left(1-\lambda z^{-1}\right)}\right]_+ \\
&= \frac{(1-\rho z)}{\sigma_w^2 (1-\lambda z)}\left[\frac{\tilde{P}(z)\,\sigma_u^2}{(1-\rho z)\left(1-\lambda z^{-1}\right)}\right]_+.
\end{aligned}$$

The optimal prediction can be then written as

$$E\left(p_t|\Omega_{it}^W\right) = \frac{(1-\rho L)}{\sigma_w^2 (1-\lambda L)} \left[\frac{\tilde{P}(z)\,\sigma_u^2}{(1-\rho L)(1-\lambda L^{-1})}\right]_+ a_{it}.$$

$\tilde{P}(z)$ has been assumed to be a polynomial in non-negative powers for z, where $\tilde{P}(0) \neq 0$. One can always write

$$\left[\frac{\tilde{P}(z)\,\sigma_u^2}{(1-\rho z)(1-\lambda z^{-1})}\right]_+ = \left[\frac{z\tilde{P}(z)\,\sigma_u^2}{(1-\rho z)(z-\lambda)}\right]_+ .$$

The annihilator operation can be solved using the Lemma of Hansen and Sargent (1980, Lemma on page 39). This lemma states that the annihilate is the annihiland less the principal parts of its laurent series expansion about the singularities inside the unit circle. Let $A(z)$ be regular inside the unit circle, with an isolated singularity at $z = \lambda$, then in its Laurent series expansion around λ the residue is

$$R_{A(z)}(\lambda) = \lim_{z\to\lambda} (z-\lambda)\,A(z).$$

The singularity in our case is at $z = \lambda$, hence the residue can be written as

$$R(\lambda) = \lim_{z\to\lambda} (z-\lambda)\,\frac{z\tilde{P}(z)\,\sigma_u^2}{(1-\rho z)(z-\lambda)} = \frac{\lambda\tilde{P}(\lambda)\,\sigma_u^2}{(1-\rho\lambda)}.$$

Therefore, the principal part of the Laurent series expansion is

$$\frac{\lambda\tilde{P}(\lambda)\,\sigma_u^2}{(1-\rho\lambda)(z-\lambda)}.$$

This leads to

$$\left[\frac{z\tilde{P}(z)\,\sigma_u^2}{(1-\rho z)(z-\lambda)}\right]_+ = \frac{z\tilde{P}(z)\,\sigma_u^2}{(1-\rho z)(z-\lambda)} - \frac{\lambda\tilde{P}(\lambda)\,\sigma_u^2}{(1-\rho\lambda)(z-\lambda)},$$

which completes the proof of the proposition.

7.3 Derivation of the Constant Matrix W_λ

In the procedure to reach the canonical factorization of the spectral density proposed by Rozanov (1967), a key role is played by the unitary matrix W_λ. A square matrix is said to be unitary if it obeys the property $W_\lambda^{-1} = W_\lambda^T$ which implies $W_\lambda^T W_\lambda = W_\lambda W_\lambda^T = 1$. The unitary matrix involved in the factorization is the matrix whose columns are the left singular vectors of the system matrix evaluated at the singular point λ. Let the system matrix be $\Gamma(z)$, at the point $z = \lambda$ the matrix is singular, i.e. it cannot be inverted. The pseudo inverse $\Gamma(\lambda)^+$ (or Moore-Penrose inverse) always exists and it is given by:

$$\Gamma(\lambda)^+ = V\Sigma^+ U^T,$$

where $U\Sigma V^T$ is the singular value decomposition (SVD) of the matrix $\Gamma(\lambda)$. The pseudo inverse is a widely used object, especially when a system matrix cannot be inverted because not square. An example of a pseudo inverse is the least squares operator $(X'X)^{-1} X'$ [28]. The unitary matrix W_λ is the matrix V in the singular value decomposition of $\Gamma(\lambda)$. For the main cases considered in this paper the system matrix evaluated at a zero λ has always the form

[28] Skogestand and Posteltwaite (1996), Appendix A, contains an excellent expository treatment of the tight connection between pseudo inverses and least squares problems.

$$\Gamma(\lambda) = \begin{pmatrix} c & d \\ 0 & 0 \end{pmatrix}.$$

It can be showed (details available from the author on request) that the matrix V in the singular value decomposition of such a matrix is

$$W_\lambda = \frac{1}{\sqrt{c^2+d^2}} \begin{pmatrix} \frac{cd}{|d|} & -\frac{cd}{|c|} \\ \frac{d^2}{|d|} & \frac{c^2}{|c|} \end{pmatrix}.$$

Consider the case of the system with only one zero at $\lambda = 0$. This is

$$\Gamma(z) = \begin{pmatrix} a(z)\sigma_u & \sigma_v \\ zP(z)\sigma_u & 0 \end{pmatrix}.$$

Applying the formula above, it is straightforward to see that

$$W_0 = \frac{1}{\sqrt{\sigma_u^2+\sigma_v^2}} \begin{pmatrix} \sigma_u & -\sigma_v \\ \sigma_v & \sigma_u \end{pmatrix}, \tag{74}$$

which is used in proving proposition 2.

7.4 Proof of Propositions 2 and 3: Invertible Price Equilibrium

The first step of the proof consists in deriving the Wold fundamental representation for the observer system (which corresponds to the canonical factorization of its spectral density). The key assumption of the proposition is that the resulting moving average representation for the price process is invertible. Following the results of section 3 this means that in order to derive the fundamental representation for an arbitrary price process $P(L)$ the removal of only one zero is necessary. Let the fundamental representation be denoted by

$$\begin{pmatrix} a_{it} \\ p_{t-1} \end{pmatrix} = \Gamma^*(z) \begin{pmatrix} v_{1t} \\ v_{2t} \end{pmatrix}. \tag{75}$$

Making use of (74) the matrix $\Gamma^*(z)$ takes the following form

$$\Gamma^*(z) = \Gamma(z) W_0 B_0(z) = \begin{pmatrix} \frac{\sigma_u^2 a(z)}{\sqrt{\sigma_u^2+\sigma_v^2}} + \frac{\sigma_v^2}{\sqrt{\sigma_u^2+\sigma_v^2}} & -\frac{\sigma_v\sigma_u a(z)}{\sqrt{\sigma_u^2+\sigma_v^2}}\frac{1}{z} + \frac{\sigma_v\sigma_u}{\sqrt{\sigma_u^2+\sigma_v^2}}\frac{1}{z} \\ zP(z)\frac{\sigma_u^2}{\sqrt{\sigma_u^2+\sigma_v^2}} & -\frac{\sigma_v\sigma_u}{\sqrt{\sigma_u^2+\sigma_v^2}}P(z) \end{pmatrix} \tag{76}$$

$$= \begin{pmatrix} \frac{\sigma_u^2 a(z)+\sigma_v^2}{\sqrt{\sigma_u^2+\sigma_v^2}} & \frac{1}{z}\frac{\sigma_v\sigma_u}{\sqrt{\sigma_u^2+\sigma_v^2}}(1-a(z)) \\ z\frac{\sigma_u^2}{\sqrt{\sigma_u^2+\sigma_v^2}}P(z) & -\frac{\sigma_v\sigma_u}{\sqrt{\sigma_u^2+\sigma_v^2}}P(z) \end{pmatrix}. \tag{77}$$

The Wold fundamental disturbances are

$$\begin{pmatrix} v_{1t} \\ v_{2t} \end{pmatrix} = B\left(L^{-1}\right)^T W_0^T \begin{pmatrix} \varepsilon_{ut} \\ \varepsilon_{it} \end{pmatrix} \tag{78}$$

$$= \begin{pmatrix} \frac{\sigma_u}{\sqrt{\sigma_u^2+\sigma_v^2}} & \frac{\sigma_v}{\sqrt{\sigma_u^2+\sigma_v^2}} \\ -\frac{\sigma_i}{\sqrt{\sigma_u^2+\sigma_v^2}}L & \frac{\sigma_u}{\sqrt{\sigma_u^2+\sigma_v^2}}L \end{pmatrix} \begin{pmatrix} \varepsilon_{ut} \\ \varepsilon_{it} \end{pmatrix} \tag{79}$$

$$= \begin{pmatrix} \frac{\sigma_u}{\sqrt{\sigma_u^2+\sigma_v^2}}\varepsilon_{ut} + \frac{\sigma_v}{\sqrt{\sigma_u^2+\sigma_v^2}}\varepsilon_{it} \\ -\frac{\sigma_v}{\sqrt{\sigma_u^2+\sigma_v^2}}\varepsilon_{ut-1} + \frac{\sigma_u}{\sqrt{\sigma_u^2+\sigma_v^2}}\varepsilon_{it-1} \end{pmatrix}. \tag{80}$$

The firm specific expectations of the price level can be expressed in terms of the Wiener-Kolmogorov prediction formula applied to the fundamental representation just derived

$$E_t\left(p_t|\Omega_{it}^H\right) = \left[\Gamma_{21}^*(L)L^{-1}\right]_+ v_{1t} + \left[\Gamma_{22}^*(L)L^{-1}\right]_+ v_{2t}.$$

Substituting the expressions for $\Gamma^*(L)$ and v_t one sees that

$$\begin{aligned}
E_t\left(p_t|\Omega_{it}^H\right) &= \left[\Gamma_{12}^*(L)L^{-1}\right]_+ v_{1t} + \left[\Gamma_{22}^*(L)L^{-1}\right]_+ v_{2t} &(81)\\
&= \frac{\sigma_u^2}{\sqrt{\sigma_u^2+\sigma_v^2}}P(L)\left[\frac{\sigma_u}{\sqrt{\sigma_u^2+\sigma_v^2}}\varepsilon_{ut} + \frac{\sigma_v}{\sqrt{\sigma_u^2+\sigma_v^2}}\varepsilon_{it}\right] &(82)\\
&\quad -\frac{\sigma_u\sigma_v}{\sqrt{\sigma_u^2+\sigma_v^2}}\left(P(L)-P_0\right)\left[-\frac{\sigma_v}{\sqrt{\sigma_u^2+\sigma_v^2}}\varepsilon_{ut} + \frac{\sigma_u}{\sqrt{\sigma_u^2+\sigma_v^2}}\varepsilon_{it}\right]\\
&= \frac{\sigma_u^2}{\sigma_u^2+\sigma_v^2}P(L)\left[\sigma_u\varepsilon_{ut}+\sigma_v\varepsilon_{it}\right] + \frac{\sigma_u\sigma_v}{\sigma_u^2+\sigma_v^2}\left(P(L)-P_0\right)\left[\sigma_v\varepsilon_{ut}-\sigma_u\varepsilon_{it}\right]\\
&= \frac{\sigma_u^2}{\sigma_u^2+\sigma_v^2}P(L)\left[\sigma_u\varepsilon_{ut}+\sigma_v\varepsilon_{it}\right] + \frac{\sigma_u\sigma_v}{\sqrt{\sigma_u^2+\sigma_v^2}}P(L)\left[\sigma_v\varepsilon_{ut}-\sigma_u\varepsilon_{it}\right] - \frac{\sigma_u\sigma_v}{\sqrt{\sigma_u^2+\sigma_v^2}}P_0\left[\sigma_v\varepsilon_{ut}-\sigma_u\varepsilon_{it}\right]\\
&= \sigma_u P(L)\varepsilon_{ut} - \frac{\sigma_u\sigma_v}{\sigma_u^2+\sigma_v^2}P_0\left[\sigma_v\varepsilon_{ut}-\sigma_u\varepsilon_{it}\right]\\
&= \sigma_u P(L)\varepsilon_{ut} - \frac{\sigma_u\sigma_v^2}{\sigma_u^2+\sigma_v^2}P_0\varepsilon_{ut} + \frac{\sigma_v\sigma_u^2}{\sigma_u^2+\sigma_v^2}P_0\varepsilon_{it}\\
&= \sigma_u P(L)\varepsilon_{ut} - \frac{\sigma_v^2}{\sigma_u^2+\sigma_v^2}\sigma_u P_0\varepsilon_{ut} - \frac{\sigma_u^2}{\sigma_u^2+\sigma_v^2}\sigma_u P_0\varepsilon_{ut} + \frac{\sigma_u^2}{\sigma_u^2+\sigma_v^2}\sigma_u P_0\varepsilon_{ut} + \frac{\sigma_v\sigma_u^2}{\sigma_u^2+\sigma_v^2}P_0\varepsilon_{it}\\
&= \sigma_u\left(P(L)-P_0\right)\varepsilon_{ut} + P_0\frac{\sigma_u^2}{\sigma_u^2+\sigma_v^2}\left(\sigma_u\varepsilon_{ut}+\sigma_v\varepsilon_{it}\right). &(83)
\end{aligned}$$

As a check on the correctedness of the method we consider the correlation between the individual prediction error and the information set . From the sixth line above the prediction error can be written as

$$p_t - E_t\left(p_t|\Omega_{it}^H\right) = \frac{\sigma_u\sigma_v}{\sigma_u^2+\sigma_v^2}P_0\left[\sigma_v\varepsilon_{ut}-\sigma_u\varepsilon_{it}\right]. \qquad (84)$$

It follows that one just needs to check that the current observation a_{it} is uncorrelated with the prediction error, in fact

$$\begin{aligned}
E\left(\left(p_t - E_t\left(p_t|\Omega_{it}^H\right)\right)a_{it}\right) &= E\left(\frac{P_0}{\sigma_u^2+\sigma_v^2}\left[\sigma_v^2 u_t - \sigma_u^2 v_{it}\right]\left(a(L)u_t + v_{it}\right)\right)\\
&= \frac{P_0}{\sigma_u^2+\sigma_v^2}\left(\sigma_v^2 E u_t^2 - \sigma_u^2 E v_{it}^2\right)\\
&= 0.
\end{aligned}$$

Substituting the individual expectation expression into firms optimal pricing formula one has

$$p_t(i) = (1-\gamma)\left(\sigma_u\left(P(L)-P_0\right)\varepsilon_{ut} + P_0\frac{\sigma_u^2}{\sigma_u^2+\sigma_v^2}\left(\sigma_u\varepsilon_{ut}+\sigma_v\varepsilon_{it}\right)\right) - \gamma a(L)\sigma_u\varepsilon_{ut} - \gamma\sigma_v\varepsilon_{it}.$$

Aggregating across firms the individual specific components ε_{it} average out to zero and the expression for the equilibrium price level becomes

$$\begin{aligned}
p_t &= \int_0^1\left((1-\gamma)\left(\sigma_u\left(P(L)-P_0\right)\varepsilon_{ut} + P_0\frac{\sigma_u^2}{\sigma_u^2+\sigma_v^2}\left(\sigma_u\varepsilon_{ut}+\sigma_v\varepsilon_{it}\right)\right) - \gamma a(L)\sigma_u\varepsilon_{ut} - \gamma\sigma_v\varepsilon_{it}\right)di\\
&= (1-\gamma)\left(P(L)\sigma_u\varepsilon_{ut} - \frac{\sigma_v^2}{\sigma_u^2+\sigma_v^2}P_0\sigma_u\varepsilon_{ut}\right) - \gamma\sigma_u a(L)\varepsilon_{ut}.
\end{aligned}$$

Instead of proceeding using the method of undetermined coefficients we follow Kasa (2000) and Kasa, Walker and Whiteman (2006) and we exploit the Riesz-Fischer theorem by stating the fixed point condition implied by the above expression in the frequency domain

$$P(z) = (1-\gamma) P(z) - (1-\gamma) \frac{\sigma_v^2}{\sigma_u^2 + \sigma_v^2} P_0 - \gamma a(z).$$

If the agents make optimal forecasts based on the model of the economy the above condition needs to hold. Rearranging, one gets

$$P(z) = -a(z) - \frac{(1-\gamma)}{\gamma} \frac{\sigma_v^2}{(\sigma_u^2 + \sigma_v^2)} P_0. \tag{85}$$

The P_0 element is then obtained evaluating this expression at $z = 0$ and considering that we assumed $a(0) = 1$ one gets

$$P_0 = -1 - \frac{(1-\gamma)}{\gamma} \frac{\sigma_v^2}{(\sigma_u^2 + \sigma_v^2)} P_0.$$

It follows that

$$\begin{aligned} P_0 \left(1 + \frac{(1-\gamma)}{\gamma} \frac{\sigma_v^2}{(\sigma_u^2 + \sigma_v^2)}\right) &= -1 \\ P_0 \left(\frac{\gamma(\sigma_u^2 + \sigma_v^2) + (1-\gamma)\sigma_v^2}{\gamma(\sigma_u^2 + \sigma_v^2)}\right) &= -1 \\ P_0 \left(\frac{\gamma\sigma_u^2 + \sigma_v^2}{\gamma(\sigma_u^2 + \sigma_v^2)}\right) &= -1. \end{aligned}$$

Finally, we have

$$P_0 = -\frac{\gamma(\sigma_u^2 + \sigma_v^2)}{\gamma\sigma_u^2 + \sigma_v^2}. \tag{86}$$

Substituting into (85) and rearranging terms one obtains

$$P(L) = -\left(a(L) - \frac{(1-\gamma)\sigma_v^2}{\gamma\sigma_u^2 + \sigma_v^2}\right) u_t. \tag{87}$$

In order for this representation to satisfy our initial invertibility conjecture the polynomial $\left(a(L) - \frac{(1-\gamma)\sigma_v^2}{\gamma\sigma_u^2+\sigma_v^2}\right)$ must have a one-sided inverse in non-negative powers for L. When such a condition is satisfied by the parameters in $a(L)$, γ, σ_u and σ_v the expression (87) is an equilibrium of this economy.

From (84) and (86) the variance of the prediction error (the minimized objective function in the signal extraction problem for firms), is:

$$Var\left(p_t - E_t\left(p_t|\Omega_{it}^H\right)\right) = 2\left(\frac{\gamma\sigma_u^2\sigma_v^2}{\gamma\sigma_u^2 + \sigma_v^2}\right)^2$$

Provided the conditions for an invertible equilibrium are satisfied, the precision of firms prediction increases when complementarities are stronger since information that is common knowledge tends to drive most of the price dynamics.

7.5 Proof of Proposition 4: Monetary Policy

Since the feedback monetary policy is common knowledge it does not affect the informational properties of the conjectured price function. It follows that one can apply the same steps as in the proof for proposition 3 to get the expression for agents specific expectations. The fixed point condition however will be different, namely,

$$\begin{aligned} P(L)\varepsilon_{ut} &= \int_0^1 \begin{pmatrix} (1-\gamma)\left((P(L) - P_0)\varepsilon_{ut} + P_0 \frac{\sigma_u}{\sigma_u^2+\sigma_v^2}(\sigma_u\varepsilon_{ut} + \sigma_v\varepsilon_{it})\right) \\ -\gamma a(L)\sigma_u\varepsilon_{ut} - \gamma\sigma_v\varepsilon_{it} + \gamma\phi(L) P(L) L\varepsilon_{ut} \end{pmatrix} di \\ &= (1-\gamma)\left(P(L)\varepsilon_{ut} - \frac{\sigma_v^2}{\sigma_u^2 + \sigma_v^2} P_0\varepsilon_{ut}\right) - \gamma\sigma_u a(L)\varepsilon_{ut} + \gamma\phi(L) P(L) L\varepsilon_{ut}. \end{aligned}$$

The fixed point condition in the frequency domain is

$$
\begin{aligned}
P(z) &= (1-\gamma)\left(P(z) - \frac{\sigma_v^2}{\sigma_u^2+\sigma_v^2}P_0\right) - \gamma\sigma_u a(z) + \gamma\phi(z)P(z)z & (88) \\
\gamma P(z)(1-\phi(z)z) &= -(1-\gamma)\frac{\sigma_v^2}{\sigma_u^2+\sigma_v^2}P_0 - \gamma\sigma_u a(z). & (89)
\end{aligned}
$$

The feedback form of the monetary policy does not affect the reaction of prices at impact but only the subsequent dynamics, in fact evaluating the above condition at $L = 0$ one gets (86). Substituting in (89) and rearranging one obtains:

$$
P(z) = -\sigma_u\left(a(z) - \frac{(1-\gamma)\sigma_v^2}{\gamma\sigma_u^2+\sigma_v^2}\right)\frac{1}{(1-\phi(z)z)}.
$$

It is straightforward to notice from this expression that, given $a(z)$, one can always find a function $\phi(z)$ that neutralizes the non-invertible moving average components, making an invertible price conjecture always verified.

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