Ideology and Competence in Alternative Electoral Systems.*

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Abstract

We develop a model of elections in proportional (PR) and majoritarian (FPTP) electoral systems. The model allows for an endogenous number of candidates, differentiation of candidates in a private value dimension, or ideology, and a common value dimension, which we interpret broadly as quality. Voters are fully rational and strategic. We show that the quality of the candidates is always at least as high in majoritarian electoral systems than in proportional electoral systems, and that the number of candidates is always at least as large in PR as in FPTP (where exactly two candidates run). Moreover, we provide conditions under which the rankings are strict. The diversity of ideological positions represented in elections can be larger in PR or FPTP elections. In the most efficient equilibrium, however, FPTP implies perfect convergence of candidates in the median. Since a convergent equilibrium can never occur in PR, and voters are risk averse, the most efficient equilibrium in FPTP dominates in terms of voters' welfare the most efficient equilirbium in PR.

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1 Introduction

Electoral systems translate votes cast in elections to the number of seats won by each party in the national legislature. By affecting how voters' preferences are ultimately mapped into policy outcomes, they are one of the fundamental institutions in representative democracies. They also influence indirectly, through the choices they induce in voters and politicians, most key features of modern political systems: from the number of alternatives faced by voters and the diversity of ideological positions these represent, to the quality of the candidates, of their staff and of their platforms. These outcomes are themselves naturally intertwined: the less diverse are the policy positions represented by candidates running for office, the larger the number of voters that will be swayed by a quality differential between these alternatives; the more diverse the ideological positions represented by candidates running for office, the larger the incentive for a new candidate to run representing an intermediate ideological alternative.

The approach and contribution of this paper is to tackle jointly the effect of alternative electoral systems on the number of candidates running for office and the quality and ideological diversity of their platforms. To do so, we develop a model of electoral competition that integrates three different approaches in formal models of elections. We allow free entry of candidates, we allow differentiation in a private value dimension, or *ideology*, and we allow differentiation along a common value dimension, which we interpret throughout as *quality*. The main result of the paper is that the quality of candidates is always at least as high in First Past the Post (FPTP) electoral systems than in Proportional Representation (PR) electoral systems, and that consistent with the Duvergerian predictions, the number of candidates is always at least as large in PR as in FPTP (where exactly two candidates run). We show, moreover, that under mild conditions, these rankings are in fact strict; i.e, we show that for a relatively large set of parameters, PR leads to more parties, with strictly less quality on average, than in FPTP. The diversity and polarization of the ideological positions represented in the election can in general be larger or smaller in PR than in FPTP. In the most efficient equilibrium, however, FPTP implies perfect convergence of candidates to the median. Since a convergent equilibrium can never occur in PR, and voters are risk averse, the most efficient equilibrium in FPTP dominates the most efficient equilibrium in PR in terms of voters' welfare.

In the model, each potential candidate is endowed with an ideological position that can credibly implement if he chooses to run and gets elected. While in principle all positions in the ideological space can be represented in an election, running for office is costly, and therefore in equilibrium only some positions will be championed. With the field of competitors given, candidates running for office can then invest (time, money, effort) to improve the quality of the alternative they represent. This differentiation along a common-values dimension can in fact have several interpretations. To fix ideas, we invite the reader to think of candidates as investing valuable resources (money, time, or effort) to increase the probability that their staff is competent or non-corrupt. We assume that in deciding whether to run for office or not, each potential candidate only cares about the spoils he can appropriate from being in office. Voters are risk averse and fully rational, and therefore vote strategically.

The incentives of voters and politicians are shaped by the electoral system under consideration. Here we focus on proportional and majoritarian electoral systems. The rationale for this is twofold. On the one hand, PR and FPTP are two of the most commonly employed electoral systems in current democracies around the world.[1] On the other hand, proportional and majoritarian systems represent *ideal* entities at the opposite side of the spectrum of what is possibly the main attribute of electoral systems: how each system maps votes into seats. In its pure form, PR implies that the share of votes obtained by each party in the election translates to an equal share of seats for the party in the legislature. FPTP, instead, gives all the control to the candidate obtaining a plurality of votes.[2] If we extend the comparison to the broader link between votes and *policy outcomes*, this basic difference in how these alternative electoral systems transform votes (in the electorate) to patterns of representation (in government) induces in turn a second level in which proportional and majoritarian systems typically differ. As Lizzeri and Persico (2001) note: “Proportional systems are usually associated with many parties having an influence on policymaking, through the process of post-election bargaining [...] Majoritarian systems are thought to favor the party with the highest share of the vote, in the sense that more power of policy setting is conferred to that party [...].” Together, these two channels generate two main differences between proportional and majoritarian systems. First, proportional and majoritarian systems differ in how they ultimately reward vote shares: vote shares are rewarded

[1]Around one fourth of countries employing FPTP electoral systems, and around one third of all countries employing PR systems.These proportions change from 23.6 percent to 32.4 per cent for FPTP, and from 35 percent to 30.9 percent for PR when the universe is the class of established democracies, see IDEA (2005).

[2]This is, of course, a very stylized representation of a complex and diverse array of electoral institutions. As Cox (1997) argues, however, “much of the variance in two of the major variables that electoral systems are thought to influence - namely, the level of disproportionality between each party’s vote and seat shares, and the frequency with which a single party is able to win a majority of seats in the national legislatuure - is explained by this distinction.”

more disproportionally in majoritarian systems, and more proportionally in PR systyems. Second, given a voting outcome in the electorate, the process of post-election bargaining in PR introduces more uncertainty for voters in the selection of the policy outcome. To capture these two fundamental differences in a stylized manner within our model, we assume that in FPTP the candidate with a plurality of votes appropriates all rents from office and implements the policy he represents, while in PR systems the policy outcome is a lottery between the ideologies championed by the candidates participating in the election, with weights equal to their vote shares in the election (or seat share in the assembly).

To prove our main result, we begin by characterizing equilibria of the model for FPTP. We show that in any electoral equilibrium in FPTP only two candidates compete for office, these candidates are symmetrically located around the median voter in the policy space, and choose maximal quality. The characterization of equilibria with two candidates builds on previous work in the literature and might be familiar to the reader: given the winner-takes-all nature of FPTP elections, candidates will choose to run for office only if they have a positive chance of collecting a plurality, and must therefore tie in equilibrium; from this it follows almost immediately that voters must vote sincerely and candidates must be choosing maximal quality. Candidates must then be located in the policy space so that they obtain the same number of votes. But given that voters' ideal policies are uniformly distributed in the policy space, this implies that candidates running for office must be located symmetrically with respect to the median ideological position. In the paper, we show that given the previous constraints that any equilibria must satisfy, the Duvergerian result that exactly two candidates will run for office in FPTP follows from our assumption that voters are risk averse. Existence of a two candidate equilibrium along the lines described above relies heavily on strategic voting. To see how strategic voting operates in this context, suppose that even after entry of a third candidate, all voters vote for their preferred candidate among the two equilibrium candidates. Then any deviation from sincere voting by any voter would only cause her least preferred equilibrium candidate to win the race for sure. These off-equilibium beliefs rule out entry in any two-candidate race.

In PR systems instead, multi-candidate symmetric equilibria can be supported as long as candidates represent policies which are sufficiently differentiated from one another. Consider for example perfect convergence (i.e., only centrist candidates run for office). It is not difficult to show that perfect convergence, which is attainable in equilibrium in FPTP, cannot be supported in PR. The reason is that, if candidates are all centrist, it is always possible for a candidate with ideology close to the median to run capturing almost half of the votes. Since the converging candidates were making non-negative rents in the proposed equilibrium, the entrant's expected payoff from running is always positive, and there is no way to deter his entry. The driving force behind this result is that strategic voting in PR is sincere, on and off the equilibrium path; i.e., given any field of candidates running for office, voting for the preferred candidate in the set is strictly dominant for any voter.

The previous argument can be extended to show that, in any symmetric equilibrium in PR, candidates must be sufficiently "spaced" with respect to their ideological positions, and as a consequence they will not be perfect substitutes in the voters' eyes. The reason for this follows from the tension that emerges in our model between differentiation in policies and quality. The closer are the candidates in terms of their ideological position, the larger is the number of voters that can be attracted by a given increase in quality by one of the candidates. This implies that if candidates are sufficiently close to one another, the game of quality competition will be fierce and candidates will compete away their rents. If instead candidates are sufficiently differentiated on the ideological dimension, they will also be worst substitutes for voters. Hence, quality competition is relaxed and candidates running for office can choose an optimal non-maximal quality and stil get a positive share of office rents. The limit for this is entry: candidates cannot be too much differentiated in PR elections, for otherwise - given that strategic voting is sincere - this will induce entry. In the paper we show that we can support electoral equilibria in PR with more than two candidates in which the average quality of candidates is non-maximal. We also show, moreover, that we can support equilibria of this class with an increasingly larger number of parties given a sufficiently lower cost of running for office, and cost of quality, and a sufficiently larger "ideological focus" of voters (Stokes (1963)) (equivalently, a sufficiently smaller responsiveness of voters to quality).

We also consider a modified version of the problem in which the majority candidate captures all the rents from office with a probability more than proportional to his vote share. In particular, we consider a variant of this problem in which the candidate obtaining a plurality of votes gets a bonus $\gamma \in (0, 1)$ of office rents. We assume that this majority bonus is also reflected in the probability with which the policy he represents is selected as outcome. We call this new environment, which can be thought as an intermediate electoral system between PR, (when $\gamma = 0$), and FPTP (when $\gamma = 1$) PR-plus (PRP). We show that symmetric two candidate equilibria in which candidates choose maximal quality can be supported in PRP both for large majority bonus and, given the bonus, for large electorates. Moreover, we show that in equilibria in large electorates either two parties tie with maximal quality or a single candidate appropriates the majority bonus.

Our paper is related to three strands of literature. The first strand focuses on the relation between electoral systems and the number of political parties. This literature provides several formalizations of the well-known *Duvergerian* predictions, namely that the number of parties tend to be larger in proportional electoral systems than in majoritarian systems (where only two parties should run). This literature, contrary to our paper, considers only "horizontal" or ideological differentiation between parties. Examples are Osborne and Slivinski (1996) for a comparison between plurality and plurality with runoff undere sincere voting, and Morelli (2004) for a comparison between majoritarian and proportional electoral systems under strategic voting. Both of these papers show that the comparison crucially depends on the initial distribution of preferences among voters. See also Cox (1997) for an empirical discussion of the Duvergerian predictions.[3]

A second strand of literature analyzes how variations in the electoral system affect policy outcomes. Myerson (1993) focuses entirely on how the nature of electoral competition affects promises of redistribution by candidates in the election. Building on this work, Lizzeri and Persico (2001) consider redistribution and provision of public goods in PR and FPTP electoral systems. In both of these papers, the emphasis is on the vote-buying strategies of the candidates. Austen-Smith and Banks (1988) and Baron and Diermeier (2001) consider models of elections *and* legislative outcomes in PR, were rational voters the effect of their vote on the bargaining game between parties in the elected legislature. Finally, Austen-Smith (2000), Persson, Roland, and Tabellini (2003)), and Persson, Tabellini, and Trebbi (2006), consider alternative electoral systems with strategic voting, when the relevant policy outcome is not bargaining over a fixed prize, but instead taxation and redistribution (Austen-Smith (2000), Persson, Roland, and Tabellini (2003)), or corruption (Persson, Tabellini, and Trebbi (2006)).

Finally, our paper is also related to the large literature that, following Stokes (1963)'s original critique to the Downsian model, incorporates competition in *valence* issues, typically within FPTP, and with a given number of candidates (two). For recent papers see Ashworth and Bueno de Mesquita (2007), Callander (2008), Carrillo and Castanheira (2006), Eyster and Kittsteiner (2007), Herrera, Levine, and Martinelli (2005), and Meirowitz (2007).[4] Of these, the closest paper to ours is Ashworth and Bueno de Mesquita (2007),

[3] For papers that study entry in FP elections under the assumption of sincere voting see, e.g., Palfrey (1984), and Greenberg and Shepsle (1987). For papers that study entry in FP under strategic voting see, e.g., Feddersen, Sened, and Wright (1990), and Besley and Coate (1997). For models of differentiation and entry in industrial organization, see d'Aspremont, Gabszewicz, and Thisse (1979), Shaked and Sutton (1982, 1987), and Perloff and Salop (1985)

[4] See also Groseclose (2001), Aragones and Palfrey (2002), Schofield (2004), and Kartik and McAfee

who following the insight of d'Aspremont, Gabszewicz, and Thisse (1979) in the context of firms, obtain that in FPTP elections with two candidates, candidates have an incentive to "diverge" in order to soften valence competition. It is useful to clarify, however, that while the same underlying mechanism linking policy and quality differentiation is present in our model, the centrifugal force does not exist, since in our case an endogenous field of candidates is endowed with fixed policy positions and voters are strategic.

The rest of the paper is organized as follows. Section 2 presents the model. Section 3 presents the main results. Section 4 presents the results for the PR-plus regime. Section 5 concludes. All proofs are in the appendix.

2 The Model

Let $X = [0, 1]$ be the ideology space. In any $x \in X$ there is a set of potential candidates, each of whom will perfectly represent policy x if elected. There are three stages in the game. In the first stage, all potential candidates simultaneously decide whether or not to run for office. Potential candidates only care about the spoils they can appropriate from being in office, and to run a candidate must pay a fixed cost F. We denote the set of candidates running for office at the end of the first stage by $\mathcal{K} = \{1, \ldots, K\}$. In a second stage, all candidates running for office simultaneously invest in quality $\theta_k \in [0, 1]$. Candidates can acquire quality θ_k at a cost $C(\theta_k)$, $C(\cdot)$ increasing and convex, and we let $C(1) \equiv \bar{c}$ and assume that $F + \bar{c} \leq \frac{1}{2}$.[5] In the third stage, N fully strategic voters vote in an election, where we think as N being a large finite number. A voter i with ideal point $z^i \in X$ ranks candidates according to the utility function $u(\cdot; z^i)$, which assigns to candidate k with characteristics (θ_k, x_k) the payoff $u(\theta_k, x_k; z^i) \equiv 2\alpha v(\theta_k) - (x_k - z^i)^2$, with v increasing and concave. The ideal point of each voter is drawn at the beginning of the third stage from a uniform distribution with support in $[0, 1]$.

The electoral system determines the mapping from voting profiles to policy outcomes and the allocation of rents. In FPTP the candidate with a plurality of votes appropriates all rents from office and implements the policy he represents. In PR systems, instead, the policy outcome is a lottery between the ideologies championed by the candidates participating in the election, with weights equal to their vote shares in the election (or seat share in the assembly). The (expected) share of rents captured by each candidate is also proportional

(2007) for models where one candidate has an exogenous valence advantage.

[5] Without this assumtpion the model is unniteresting because it only admits dictatorial equilibria, i.e., with only one candidate running for office

to his vote share in the election. Letting s_k denote the proportion of votes for party k, letting m_k denote the proportion of rents captured by party k, letting $\theta_{\mathcal{K}} \equiv \{\theta_k\}_{k\in\mathcal{K}}$ and $x_{\mathcal{K}} \equiv \{x_k\}_{k\in\mathcal{K}}$ the quality and policy positions of the candidates running for office, and normalizing the payoff of a candidate not running to zero, the expected payoff of a candidate k running for office in electoral system j can be written as

$$\Pi_k^j(\mathcal{K}, x_{\mathcal{K}}, \theta_{\mathcal{K}}) = m_k^j(\theta_{\mathcal{K}}, x_{\mathcal{K}}) - C(\theta_k) - F. \tag{1}$$

We assume that $m_k^{PR}(\theta_{\mathcal{K}}, x_{\mathcal{K}}) = s_k(\theta_{\mathcal{K}}, x_{\mathcal{K}})$, and that under FP ties are broken by the toss of a fair coin, so that letting $H \equiv \{h \in \mathcal{K} : s_k = s_h\}$,

$$m_k^{FP}(\theta_{\mathcal{K}}, x_{\mathcal{K}}) = \begin{cases} \frac{1}{|H|} & \text{if } s_k \geq \max_{j\neq k}\{s_j\} \\ 0 & \text{o.w.} \end{cases}$$

A strategy for candidate k is composed of a running decision $e_k \in \{0, 1\}$, where $e_k = 1$ indicates the choice of running for office, and a quality decision $\theta_k(\mathcal{K}, x_{\mathcal{K}}) \in [0, 1]$. A strategy for voter i is a function $\sigma_i(\mathcal{K}, x_{\mathcal{K}}, \theta_{\mathcal{K}}) \in \mathcal{K}$, where $\sigma_i(\mathcal{K}, x_{\mathcal{K}}, \theta_{\mathcal{K}}) = k$ indicates that the choice of voting for candidate k, and $\sigma = \{\sigma_1(\cdot), \ldots, \sigma_N(\cdot)\}$ denotes a voting strategy profile. A *short run electoral equilibrium* is a set of candidates running for office $\mathcal{K}^*$, policy positions $x^*_{\mathcal{K}^*}$, quality choices $\theta^*_{\mathcal{K}^*}$, and a voting profile σ^* such that: *(i)* θ_k^* is optimal for k given $\{\theta^*_{\mathcal{K}^*\setminus k}(\mathcal{K}^*, x^*_{\mathcal{K}^*}), x^*_{\mathcal{K}^*}, \sigma(\mathcal{K}^*, x^*_{\mathcal{K}^*}, \theta^*_{\mathcal{K}^*\setminus k}, \theta^*_k)\}$; i.e., $\theta^*_{\mathcal{K}^*}$ is a (pure Nash) equilibrium of the continuation game $\Gamma_{\mathcal{K}^*}$, and *(ii)* if $k \in \mathcal{K}^*$, then $\Pi_k(\mathcal{K}^*, x^*_{\mathcal{K}^*}, \theta^*_{\mathcal{K}^*}, \sigma^*(\mathcal{K}^*, x^*_{\mathcal{K}^*}, \theta^*_{\mathcal{K}^*})) \geq 0$ (no exit). An *electoral equilibrium* is a Subgame Perfect Nash Equilibrium in pure strategies of the game of electoral competition; i.e., a short run electoral equilibrium with the additional condition of non-profitable entry: *(iii)* if $k \notin \mathcal{K}^*$, then $\Pi_k(\mathcal{K}^* \cup k, x_k, x^*_{\mathcal{K}^*}, \theta^*_k, \theta^*_{\mathcal{K}^*}, \sigma^*(\mathcal{K}^* \cup k, x_k, x^*_{\mathcal{K}^*}, \theta^*_k, \theta^*_{\mathcal{K}^*})) < 0$ in an equilibrium of the continuation game.

An *outcome* of the game is a set of candidates running for office $\mathcal{K}$, policy positions $x_{\mathcal{K}}$, and quality choices $\theta_{\mathcal{K}}$. A *polity* is a triplet $(\alpha, \overline{c}, F) \in \Re^3_+$. We say that the model *admits* an electoral equilibrium with outcome $(\mathcal{K}, x_{\mathcal{K}}, \theta_{\mathcal{K}})$ if there exist a set of polities $P \subseteq \Re^3_+$ with positive measure such that if a polity $p \in P$ then there exists an electoral equilibrium with outcome $(\mathcal{K}, x_{\mathcal{K}}, \theta_{\mathcal{K}})$.

3 Main Results

In this section we state our main result regarding the comparison between alternative electoral systems (Theorem 3). In particular, we show that average quality among all candidates running for office is always (weakly) higher under majoritarian elections than under proportional elections, and that the number of candidates running for office is always (weakly) smaller under majoritarian electoral systems. We show, moreover, that under very natural conditions, these rankings are in fact strict; i.e, we show that for a relatively large set of parameters, PR leads to more parties, with strictly less quality on average, than in FPTP.

We begin our analysis by considering the case of majoritarian or First-Past-the-Post (FPTP) electoral systems. We show that FPTP elections have a powerful impact on behavior, in the sense that in equilibrium the number of candidates running for office and their choice of quality is uniquely determined. Moreover, although there are multiple equilibria (in fact a continuum), they can be ranked according to a utilitarian social welfare function. The following theorem charactrizes equilibria in FPTP elections.

Theorem 1 *Consider elections in FPTP electoral systems. An electoral equilibrium always exists. In any equilibrium: (i) exactly two candidates compete for office, (ii) candidates are symmetrically located around the median in the policy space (i.e., $x_1 = 1 - x_2$), and (iii) they choose maximal quality (i.e., $\theta_1^* = \theta_2^* = 1$).*

To see the intuition for the result, note that in FPTP elections all candidates that are running for office must tie in equilibrium. In fact, given the Winner-Takes-All nature of FPTP elections, candidates will choose to run for office (which is costly) only if they have a positive chance of collecting a plurality. From the fact that in any equilibrium all candidates are tying, it follows that *(a)* voters must vote sincerely and that *(b)* candidates must be choosing maximal quality. Given that voters are uniformly distributed in X, these facts also imply that *(c)* in any equilibrium the set of candidates running for office must be located symmetrically with respect to the median ideological position.

But then there cannot be an electoral equilibrium with $K > 2$ candidates running for office. In fact, if this were the case, *(a)* and *(b)* imply that by deviating and voting for any candidate j other than her preferred candidate, a voter could get candidate j elected with probability one. But then equilibrium implies that this voter must prefer the lottery among all $\mathcal{K}^*$ candidates running for office to having j elected for sure. However, voters' risk-aversion and *(c)* imply that any voter must prefer a centrist candidate (i.e., located at the median) to the lottery. As a result any voter must also prefer a centrist candidate to any other candidate which is not her most preferred choice, in particular a candidate with an ideological position which is between the median and her most preferred ideological position. But this leads to a violation of single-peakedness, which is not consistent with the assumption of a strictly concave utility function.[6] As a result, the only possible equilibrium must have exactly two symmetrically located candidates chosing maximal quality. In the proof we show that such an equilibrium exists, and in fact that there is a continuum of two-candidates symmetric equilibria, with candidates chosing maximal quality.

Two remarks are in order. First, the reason why any two-candidates symmetric configuration can be supported in equilibrium follows from two assumptions: *(i)* strategic voting and *(ii)* the fact that each potential candidate, when deciding whether to run or not, only cares about the spoils he can appropriate from being in office. To see how strategic voting operates in this context, suppose that all voters vote for their preferred candidate among the two equilibrium candidates, even after entry of a third candidate. Then any deviation from sincere voting by any voter would only cause her least preferred equilibrium candidate to win the race for sure, and therefore entry is never profitable in any two-candidate race. Assumption *(ii)* rules out situations where potential candidates may choose not to run for office simply because another candidate championing a "close" ideological position is already running. Note that without this assumption perfect convergence in FPTP elections cannot be supported in equilibrium.[7] Stated differently, assuming policy-motivated candidates immediately buys some policy divergence in equilibrium. Although we believe that both policy and office motivation are important in the ultimate decision whether to run for office or not, by assuming *(ii)* we can identify the specific effect of the electoral system on the degree of differentiation between ideological positions championed in equilibrium. Indeed, as it will be apparent later, even assuming *(ii)* convergence cannot be supported in equilibrium in PR elections.

The second remark is that two-candidates symmetric equilibria can be easily ranked in terms of aggregate voters' welfare. In particular the equilibrium that displays perfect convergence, i.e., where $x_1 = x_2 = \frac{1}{2}$ and $\theta_1 = \theta_2 = 1$, is the most efficient in the sense that it maximizes aggregate welfare. To see this notice that when $x_1 = 1 - x_2 = \frac{1}{2} - y$ for $y \in [0, \frac{1}{2}]$ and $\theta_1 = \theta_2 = 1$, aggregate welfare equals

$$\int_0^1 \left(\frac{1}{2} u \left(1, \frac{1}{2} + y; z_i\right) + \frac{1}{2} u \left(1, \frac{1}{2} - y; z_i\right)\right) dz_i = \alpha v(1) - \left(y^2 + \frac{1}{12}\right),$$

which is decreasing in y and maximized at $y = 0$ or $x_1 = x_2 = \frac{1}{2}$.

We consider next the case of proportional (PR) electoral systems. The characterization of the equilbrium in this case leads to dramatically different results with respect to the

[6]A similar argument can be found in Feddersen, Sened, and Wright (1990) in the context of pure private value model.

[7]See, e.g., the "citizen-candidate" models of Osborne and Slivinski (1996) and Besley and Coate (1997).

FTPT case. Not only we can now support multi-candidate equilibria with non-maximal quality choice, but complete convergence (in the sense of all candidates running for office representing the same ideological position) cannot be an equilibrium. Before proving these results we state a lemma that will be useful in order to provide an intuition for the next theorem.

Lemma 1 *In any electoral equilibrium in PR elections with $K \geq 2$ candidates running for office, voters vote sincerely.*

By showing that in PR elections sophisticated voting is in fact equivalent to sincere voting, this lemma greatly simplifies the characterization of equilibria. To get an intuition for this result, recall that we are assuming that in PR each candidate running for office is elected and implements his ideology with a probability proportional to the share of votes received in the election. As a consequence, the mapping from votes in favor of a particular candidates to his probability of being the ruler is strictly increasing. In particular, when a voter votes for a certain candidate, she is affecting the lottery among all candidates running for office by increasing the weight on that particular candidate's position. But this implies that voting for a candidate which is not the most preferred one is always a strictly dominated strategy. In fact, by switching his vote to his most preferred candidate a voter only affects the lottery's weights of exactly two candidates. But we know that with two alternatives strategic voting and sincere voting coincide.

We are now ready to state our second major result, which characterizes equilbrium outcomes in PR elections. Let Γ^{FP} and Γ^{PR} denote electoral equilibria in FPTP and PR systems respectively, and let $\overline{\theta}(\Gamma^j)$ denote the average quality among all candidates running for office in Γ^j, for $j \in \{FP, PR\}$. Then, we have the following result:

Theorem 2 *Consider elections in PR*

(1) The model admits an electoral equilibrium in PR, Γ^{PR}, such that $\overline{\theta}(\Gamma^{PR}) < 1$. Moreover, there exists a decreasing sequence of polities $\mathcal{P} \equiv \{\alpha_K, \overline{c}_K, F_K\}_{K \geq 3}$ such that for any polity $(\alpha_K, \overline{c}_K, F_K) \in \mathcal{P}$ we can support an electoral equilibrium Γ_K^{PR} with $K \geq 3$ candidates running for office and $\overline{\theta}(\Gamma_K^{PR}) < 1$.

(2) The model does not admit an electoral equilibrium in PR in which all candidates represent the same policy.

One important lesson that comes from Theorem 2 is that in PR elections candidates running for office tend to be sufficiently differentiated in the ideological dimension. Too see why this must be true, consider first the case of perfect convergence (i.e., only centrist candidates run). If candidates are all centrist, they must all choose maximal quality since otherwise the candidate with highest quality will get all votes. But now a candidate with ideology sufficiently close to the center can choose to run and, given sincere voting - which by Lemma 1 is the unique equilibrium in the voting stage in PR - capture almost half of the votes. Since the converging candidates were making non-negative rents in the proposed equilibrium, the entrant's expected payoff from running is always positive, and there is no way to deter his entry.

The previous argument can be extended to show that, in any symmetric equilibrium in PR, candidates must be sufficiently "spaced" with respect to their ideological positions, and as a consequence they will not be perfect substitutes in the voters' eyes. The reason for this follows from the tension that emerges in our model between differentiation in policies and quality. The closer are the candidates in terms of their ideological position, the larger is the number of voters that can be attracted by a given increase in quality by one of the candidates. This implies that if candidates are sufficiently close to one another, the game of quality competition will be fierce and candidates will compete away their rents. If instead candidates are sufficiently differentiated on the ideological dimension, they will also be worst substitutes for voters. Hence, quality competition is relaxed and candidates running for office can choose an optimal non-maximal quality and stil get a positive share of office rents. The limit for this is entry: candidates cannot be too much differentiated in PR elections, for otherwise - given that strategic voting is sincere - this will induce entry.

We exploit the above properties in the proof of Theorem 2 to contruct a particular class of equilibria called Location Symmetric Equilibria (LSE), where candidates are equally spaced in the ideological dimension. We show that *(i)* if voters are not too responsive to quality (α is sufficiently low), *(ii)* the cost of attaining maximal quality, $\overline{c}$, is sufficiently low, and *(iii)* there is a sufficiently large wedge between fixed and variable costs, F and $\overline{c}$, then we can sustain an LSE with $K \geq 3$ candidates choosing non-maximal quality.

To understand why we need these particular restictions on paramenters, note that when α is small it means that voters have a "strong ideological focus"(see Stokes (1963)) in the sense that the number of voters that will be swayed by a quality differential between candidates is relatively small. As a consequence, incentives to invest in quality are not particularly strong. At the same time, when $\overline{c}$ is relatively small, *and* there is a sufficiently large wedge between fixed and variable costs, it is easier to deter a potential entry. In fact, when this is the case, even if candidates k and $k+1$ are choosing non-maximal quality on the equilibirum path, we can support an "armed response", i.e., $\theta_k = \theta_{k+1} = 1$, as an equilibrium of the continuation game following the entry of a candidate in the interval (x_k, x_{k+1}). We also show, moreover, that we can support equilibria of this class with an increasingly larger number of candidates given sufficiently lower costs $\bar{c}$ and F, and a sufficiently larger "ideological focus" of voters (Stokes (1963)).

Combining the results of Theorem 2 together with our earlier results in Theorem 1, we can conclude that average quality is always weakly higher, and the number of candidates weakly smaller under FPTP systems than under PR systems. Moreover, we can find a full measure set of parameters for which the inequalities are strict. The next theorem, which follows as an immediate corollary of Theorem 1 and 2 summarizes the comparison result.

Theorem 3 (Corollary)

(1) In any electoral equilibrium under FPTP, Γ^{FP}, the average quality of candidates running for office is weakly higher and their number weakly smaller than under any electoral equilibrium under PR, Γ^{PR}.

(2) There exists a decreasing sequence of polities $\mathcal{P} \equiv \{\alpha_K, C_K(B), F_K\}_{K\geq 3}$ such that, for any polity $(\alpha_K, C_K(B), F_K) \in \mathcal{P}$, we can support under PR an electoral equilibrium with K candidates running for office, Γ_K^{PR}, where $\overline{\theta}(\Gamma^{FP}) > \overline{\theta}(\Gamma_K^{PR})$ for any electoral equilibrium Γ^{FP} under FPTP.

We conclude this section by suggesting a possible welfare comparison between electoral systems. Given the multiplicity of equilibria under both systems, we confine our comparison to be between the most efficient equilibrium in terms of aggregate voters' welfare in FPTP, which we label $\tilde{\Gamma}^{FP}$, and the most efficient equilibrium in PR, $\tilde{\Gamma}^{PR}$. Then we have:

Proposition 1 *$\tilde{\Gamma}^{FP}$ dominates $\tilde{\Gamma}^{PR}$ in terms of aggregate voters' welfare.*

Note that if we consider the class of LSE under PR, the welfare comparison comes as an immediate corollary of our previous results. In fact, we already know that it is not possible to have convergence in PR elections and, given the same level of quality, concavity of voters' preferences implies that any voter strictly prefers the *expected candidate* with ideological position corresponding to the expected value of the equilibrium lottery to the lottery itself. The result follows from noticing that in any LSE the expected candidate is in fact centrist, and in the most efficient equilibrium in FPTP all candidates running for office are centrists. However, the result of Proposition 1 holds more generally for any electoral equilibirum in PR. To see this note that first that for *any* equilibrium in PR, any voter prefers the expected candidate of the equilibrium lottery to the lottery itself. If this expected candidate is centrist, we are done. Otherwise, by concavity of voters' preferences, a centrist candidate (located at the median) will always be preferred by a majority of voters to the expected candidate.

4 Majority Bonus

We have assumed up to now that in PR elections each candidate running for office captures a proportion of office rents equal to his share of votes in the election. In several systems, however, it might be reasonable to expect that the majority party can obtain an additional reward over and above his share of votes in the election. In several parliamentary democracies adopting some form of PR, for instance, the *formateur* is typically the head of the majority.[8] To capture this feature, we consider next a modified version of the problem, in which the majority candidate is elected and captures all the rents from office with a probability more than proportional to his votes' share. Given that in our model candidates are risk neutral this corresponds to have the majority candidate getting a bonus $\gamma \in (0, 1)$ of office rents. We call this new environment PR-plus (PRP) and it can be thought as an intermediate electoral system between PR, when $\gamma = 0$, and FPTP, when $\gamma = 1$. In PRP, letting as before $H \equiv \{h \in \mathcal{K} : s_k = s_h\}$, k's proportion of office's rents after the election, m_k, is given by

$$m_k = \begin{cases} s_k(1-\gamma) + \frac{\gamma}{|H|} & \text{if } s_k \geq \max_{j \neq k}\{s_j\}, \\ s_k(1-\gamma) & \text{o.w.} \end{cases} \tag{2}$$

In order to characterize electoral equilibria in PRP we first start proving some useful intermediate results.

Lemma 2

(i) There exists a $\overline{\gamma}(N) \in (0, 1)$ such that if $\gamma > \overline{\gamma}(N)$ the model admits an electoral equilibrium with 2 candidates running for office symmetrically located and choosing maximal quality.

[8]In Greece, for example, the fact that the *formateur* has to be the head of the majority is mandated by law.

(ii) There exists an $\overline{N}$ such that if $N > \overline{N}$ for any $\gamma \in (0, 1)$ the model admits an electoral equilibrium with 2 candidates running for office symmetrically located and choosing maximal quality.

Proof. The proof is not included. ■

We can also show the following result.

Proposition 2 *Given a size of the electorate N_0, consider any fixed strategy profile by candidates and voters in which more than two parties tie. There exists $\overline{N}$ such that if $N \geq \overline{N}$, there is no voting strategy profile for which this strategy profile of candidates can be part of an electoral equilibrium.*

5 Conclusion

By affecting how voters' preferences are ultimately mapped into policy outcomes, electoral systems are one of the fundamental institutions in representative democracies. They also influence indirectly, through the choices they induce in voters and politicians, most key features of modern political systems. The approach and contribution of this paper is to tackle jointly the effect of alternative electoral systems on the number of candidates running for office and the quality and ideological diversity of their platforms. To do so, we develop a model of electoral competition that integrates three different approaches in formal models of elections. We allow free entry of candidates, we allow differentiation in a private value dimension, or *ideology*, and we allow differentiation along a common value dimension, which we interpret throughout as *quality*. The main result of the paper is that the quality of candidates is always at least as high in First Past the Post (FPTP) electoral systems than in Proportional Representation (PR) electoral systems, and that consistent with the Duvergerian predictions, the number of candidates is always at least as large in PR as in FPTP (where exactly two candidates run). We show, moreover, that under mild conditions, these rankings are in fact strict; i.e, we show that for a relatively large set of parameters, PR leads to more parties, with strictly less quality on average, than in FPTP. The diversity and polarization of the ideological positions represented in the election can in general be larger or smaller in PR than in FPTP. In the most efficient equilibrium, however, FPTP implies perfect convergence of candidates to the median. Since a convergent equilibrium can never occur in PR, and voters are risk averse, the most efficient equilibrium in FPTP dominates the most efficient equilibrium in PR in terms of voters' welfare.

6 Appendix

Proof of Theorem 1. Note first that in any equilibrium all candidates that are running for office must tie, since otherwise there would be at least one candidate who would lose for sure and - given the fixed cost of running for office $F > 0$ - would prefer not to run. Since candidates are tying, in equilibrium voters must vote sincerely. If this were not the case, there would exist some voter who is not voting for her most preferred candidate in equilibrium but who could have this candidate winning with probability one by deviating to voting sincerely. Third, note that in any equilibrium it must be that $\theta_k^* = 1$ for all $k \in \mathcal{K}^*$. In fact, since all candidates that are running for office must tie in equilibrium, if $\theta_h^* < 1$ for some $h \in \mathcal{K}^*$, candidate h can profitably deviate by choosing $\tilde{\theta}_h = \theta_h^* + \nu$, for some sufficiently small $\nu > 0$ (winning the election with probability one). The previous results and the assumption that voters' preferences are uniformly distributed in X imply that in any equilibrium the set of candidates running for office must be located symmetrically with respect to $\frac{1}{2}$. We have then established that in any equilibrium *(i)* running candidates must tie, *(ii)* voting is sincere, and *(iii)* $\theta_k^* = 1$ for all $k \in \mathcal{K}^*$, and that *(iv)* running candidates must be symmetrically located.

We show next that there cannot be an electoral equilibrium with $K > 2$ running candidates. If this were is the case, *(i)* and *(iii)* imply that by deviating and voting for any candidate j other than her preferred candidate, a voter could get candidate j elected with probability one. But then equilibrium implies that this voter must prefer the lottery among all $\mathcal{K}^*$ running candidates to having j elected for sure. This implies, in particular, that

$$\frac{1}{|\mathcal{K}^*|} \sum_{k \in \mathcal{K}^*} u(1, x_k^*; z^i) \geq u(1, x_{K-1}^*; z^i) \tag{3}$$

for all voters such that $z^i > \frac{x_{K-1}^* + x_K^*}{2}$, i.e., all voters whose most preferred winning candidate is $k = K$ and next most preferred winning candidate is $k = K - 1$. On the other hand, strict concavity of $u(\cdots; z^i)$ with respect to policy and *(i)*, *(iii)*, and *(iv)* imply that for all z^i

$$u(1, \frac{1}{2}; z^i) > \frac{1}{|\mathcal{K}^*|} \sum_{k \in \mathcal{K}^*} u(1, x_k^*; z^i). \tag{4}$$

Combining (3) and (4), we obtain

$$u(1, \frac{1}{2}; z^i) > u(1, x_{K-1}^*; z^i)$$

for all voters such that $z^i > \frac{x^*_{K-1}+x^*_K}{2}$. But $K > 2$ and *(iv)* imply that $\frac{1}{2} \leq x^*_{K-1}$. Hence $u(\cdots; z^i)$ cannot be single-peaked for all voters such that $z^i > \frac{x^*_{K-1}+x^*_K}{2}$, contradicting the strict concavity of $u(\cdots; z^i)$.[9]

Finally, note that $\overline{c} + F \leq \frac{1}{2}$ implies that a unique candidate equilibrium cannot be supported, since otherwise a second candidate, symmetrically located with respect to the median, will always find it profitable to run. As a result, the only possible equilibrium must have exactly two symmetrically located candidates chosing maximal quality. We are only left to show that such an equilibrium exists. Given that $\overline{c} + F \leq \frac{1}{2}$, equilibrium rents of the two running candidates are always non-negative. Suppose now that a third candidate decides to enter at any $x \in [0, 1]$, and consider the following continuation play: $(x^*_1, \theta_1 = 1)$, $(x^*_2 = 1 - x^*_1, \theta_2 = 1)$, $(x_3, \theta_3 = 0)$ and the strategy profile of voters is such that for all $(\theta_1, \theta_2, \theta_3)$ for which $\max\{\theta_1, \theta_2\} = 1$, voters vote sincerely between $k = 1$ and $k = 2$ and, if indifferent, they split equally between $k = 1$ and $k = 2$. Given these strategies, there is no voter which can benefit from a deviation. In fact, since candidates 1 and 2 are tying, any deviation from sincere voting between candidate 1 and candidate 2 in order to support the entrant will either determine a victory of the least preferred candidate instead of having a lottery between $k = 1$ and $k = 2$, or will have no effect (in the case of $x^*_1 = x^*_2 = \frac{1}{2}$). But then candidates' behavior is optimal. In particular, the entrant cannot gain by choosing $\theta_3 > 0$. ■

Proof of lemma 1. Suppose voter i's preferred candidate is $k^*(i) \in \mathcal{K}$, and that $\tilde{k} \in \mathcal{K}$ and $\tilde{k} \neq k^*(i)$. Let $t_k(\sigma^v_{-i})$ denote the number of votes for candidate k given a voting strategy profile σ^v_{-i} for all voters other than i. The payoff for i of voting for $\tilde{k}$ given σ^v_{-i} is

$$U(\tilde{k}; \sigma^v_{-i}) = \sum_{k \neq \tilde{k}, k^*(i) \in \mathcal{K}} \frac{t_k(\sigma^v_{-i})}{N} u(x_k; z^i) + \frac{[t_{\tilde{k}}(\sigma^v_{-i}) + 1]}{N} u(x_{\tilde{k}}; z^i) + \frac{t_{k^*(i)}(\sigma^v_{-i})}{N} u(x_{k^*(i)}; z^i).$$

Similarly, the payoff for i of voting for $k^*(i)$ given σ^v_{-i} is

$$U(k^*(i); \sigma^v_{-i}) = \sum_{k \neq \tilde{k}, k^*(i) \in \mathcal{K}} \frac{t_k(\sigma^v_{-i})}{N} u(x_k; z^i) + \frac{t_{\tilde{k}}(\sigma^v_{-i})}{N} u(x_{\tilde{k}}; z^i) + \frac{[t_{k^*(i)}(\sigma^v_{-i}) + 1]}{N} u(x_{k^*(i)}; z^i).$$

Thus

$$U(k^*(i); \sigma^v_{-i}) - U(\tilde{k}; \sigma^v_{-i}) = \frac{1}{N}[u(x_{k^*(i)}; z^i) - u(x_{\tilde{k}}; z^i)],$$

[9]It should be noted that property *(iv)*, which follows from the assumption that voters' preferences are uniformly distributed, is in fact not needed to show that an equilibrium with more than two candidates cannot exist. Indeed, the argument can be slightly modified in order to account for a general continuous distribution F of voters' preferences.

which is positive by definition of $k^*(i)$. Since σ^v_{-i} was arbitrary, this shows that voting sincerely strictly dominates voting for any other available candidate and is thus a dominant strategy for voter i. It follows that in all Nash equilibria in the voting stage voters vote sincerely among running candidates. ■

Proof of Theorem 2.

Proof of Part 1. Step 1. The first step towards proving the result is to provide conditions for the existence of electoral equilibria of a simple class, which we call location symmetric equilibria (LSE). In equilibria of this class, all candidates running for office are located at the same distance to their closest neighbors; i.e., $x_{k+1} - x_k = \Delta$ for all $k = 1, \ldots, K-1$, $x_1 = 1 - x_K = \Delta_0$, and all interior candidates $k = 2, \ldots, K-1$ choose the same level of quality. Within this class, the relevant competitors for any candidate k's decision problem are k's neighbors, $k+1$ and $k-1$. This is enough to show that payoff functions are twice differentiable in the relevant set (non-differentiabiliies can only arise for quality choices that are not optimal), and that whenever rents cover variable costs, first order conditions in the quality subgame completely characterize best response correspondences (see Iaryczower and Mattozzi (2008) for more details).

With this in mind, consider then two candidates k and $j > k$ with policy positions x_k and $x_j > x_k$, and choosing quality levels θ_k and θ_j, and let $\tilde{x}_{k,j} \in \mathcal{R}$ denote the (unique) value of x for which $u(\theta_k, x_k; x) = u(\theta_j, x_j; x)$, so that $u(\theta_k, x_k; z^i) > u(\theta_j, x_j; z^i)$ if and only if $z^i > \tilde{x}_{k,j}$. This always exists, and is given by

$$\tilde{x}_{k,j} = \frac{x_k + x_j}{2} + \alpha \frac{[v(\theta_k) - v(\theta_j)]}{|x_j - x_k|}. \tag{5}$$

Note then that provided that all cutpoints $\tilde{x}_{k,k+1}$ are in $(0,1)$ for $k = 1, \ldots, K-1$, k's vote share given $\{(\theta_\ell, x_\ell)\}_{\ell \in \mathcal{K}}$ is $s_k(\theta_k; \theta_{-k}, x) = \tilde{x}_{k,k+1} - \tilde{x}_{k-1,k}$, and therefore from (1) for PR, the payoff for an interior candidate $k = 2, \ldots, K-1$ is

$$\Pi_k(\theta_\mathcal{K}, x_\mathcal{K}, \mathcal{K}) = \Delta + \alpha \left[\frac{v(\theta_k) - v(\theta_{k+1})}{\Delta} + \frac{v(\theta_k) - v(\theta_{k-1})}{\Delta} \right] - C(\theta_k) - F.$$

Defining $\Psi(\theta) \equiv \frac{v'(\theta_k)}{C'(\theta_k)}$, and noting that this is a decreasing function, k's best response is then

$$\theta_k^* = \begin{cases} \Psi^{-1}\left(\frac{\Delta}{2\alpha}\right) & \text{if } \Psi^{-1}\left(\frac{\Delta}{2\alpha}\right) \leq 1, \\ 1 & \text{if } \Psi^{-1}\left(\frac{\Delta}{2\alpha}\right) > 1. \end{cases} \tag{6}$$

Similarly, for an extremal candidate (say $k = 1$),

$$\Pi_1(\theta_{\mathcal{K}}, x_{\mathcal{K}}, \mathcal{K}) = \Delta_0 + \frac{\Delta}{2} + \alpha \left[\frac{v(\theta_1) - v(\theta_2)}{\Delta}\right] - C(\theta_1) - F,$$

and $k = 1$'s best response is then

$$\theta_1^* = \begin{cases} \Psi^{-1}\left(\frac{\Delta}{\alpha}\right) & \text{if } \Psi^{-1}\left(\frac{\Delta}{\alpha}\right) \leq 1, \\ 1 & \text{if } \Psi^{-1}\left(\frac{\Delta}{\alpha}\right) > 1. \end{cases} \tag{7}$$

We can now state and prove two intermediate results. For convenience, we define $L \equiv \max\{2\bar{c}, \bar{c} + F, \frac{1-2F}{K-1}\}$ and $U \equiv \min\{2(\bar{c} + F), \frac{1}{K}\}$. We will show that

(a) If $\max\{\alpha\Psi(1), L\} < \min\{U, 2\alpha\Psi(1)\}$*, then there exists a LSE with* K *candidates running for office in which all interior candidates choose maximal quality, and extremal candidates choose non-maximal quality.*

(b) If $\max\{2\alpha\Psi(1), L\} < U$*, then there exists a LSE with* K *candidates running for office and choosing a non-maximal quality.*

Results *(a)* and *(b)* then imply in particular that if $\max\{\alpha\Psi(1), L\} < U$, there exists a LSE, Γ^{PR}, in which $\overline{\theta}(\Gamma^{PR}) < 1$.

Proof of *(a)*. Suppose that in equilibrium $\theta_k^* = 1$ for $k = 2, \ldots, K-1$. For $\theta_k = 1$ to be optimal for k given θ_{-k} it must be that the marginal vote share given that rivals are also choosing maximal quality is higher than the marginal cost at $\theta_k = 1$; i.e., $\frac{2\alpha}{\Delta}v'(1) \geq C'(1)$, or equivalently $\Delta \leq 2\alpha\Psi(1)$. For non-negative rents we must have $\Pi_k^* = \Delta - (\bar{c} + F) \geq 0$; i.e, $\Delta \geq \bar{c} + F$. For extremal candidates to choose non-maximal quality, i.e., $\theta_1^* = \theta_K^* = \Psi^{-1}(\frac{\Delta}{\alpha})$, it must be that $\Delta > \alpha\Psi(1)$. For non-negative rents we need

$$\Pi_1^* = \Pi_K^* = \Delta_0 + \frac{\Delta}{2} - \frac{\alpha}{\Delta}[v(1) - v(\theta_1^*)] - C(\theta_1^*) - F \geq 0.$$

Since θ_1^* maximizes $\Pi_1(\theta_1)$, then $\Pi_1^* \equiv \Pi_1(\theta_1^*) \geq \Pi_1(\theta_1)$ for all $\theta_1 \neq \theta_1^*$, and thus it is enough to show that $\Pi_1(1) \geq 0$. But this is $\Delta_0 + \frac{\Delta}{2} \geq \bar{c} + F$, which holds whenever $\Delta_0 \geq \frac{\Delta}{2}$ since $\Delta \geq \bar{c} + F$. Now consider entry by j at $x_j \in (x_k, x_{k+1})$ for $k = 1, \ldots, K-1$. Note that we can always sustain in the continuation game an equilibrium such that $\hat{\theta}_j = \hat{\theta}_k = 1$ for all $k = 1, \ldots, K$, provided that $\frac{\Delta}{2} \geq \bar{c}$ which, given the previous conditions, also implies that $\Delta_0 \geq \bar{c}$. Since for equilibrium this type of entry must be nonprofitable, it must be that $\hat{\Pi}_j = \frac{\Delta}{2} - \bar{c} - F < 0$, or equivalently $\Delta < 2[\bar{c} + F]$. For no entry at the extremes (i.e., in $[0, x_1)$ and $(x_K, 1]$), a sufficient condition is that $\Delta_0 \leq F$ and $2\bar{c} \leq \Delta$ (this last inequality guarantees that extreme candidates will not drop out in the continuation game). Substituting $\Delta_0 = \Delta_K = \frac{1-(K-1)\Delta}{2}$, we have that a sufficient condition for existence of a LSE with K candidates running for office in which all interior candidates choose maximal quality and extremal candidates choose non-maximal quality is $\Delta \in (\max\{L, \alpha\Psi(1)\}, \min\{U, 2\alpha\Psi(1)\})$.

Proof of *(b)*. Consider first the interior candidates $k = 2, \ldots, K-1$. If $\theta_j^* = \theta_r^* < 1$ for all $j, r \neq k$, then k's marginal vote share is differentiable, and k's FOC is given by $\frac{2\alpha}{\Delta} v'(\theta_k^*) = C'(\theta_k^*)$. Therefore,

$$\theta_k^* = \theta^* = \Psi^{-1}\left(\frac{\Delta}{2\alpha}\right) \quad \text{for all } k = 2, \ldots, K-1.$$

Moreover, since $\theta^* < 1$, it must be that $\Delta > 2\alpha\Psi(1)$. Non-negative rents for interior candidates requires that $\Pi_k^* = \Delta - C(\theta^*) - F \geq 0$, or equivalently $\theta^* \leq C^{-1}(\Delta - F)$. Substituting θ^* we get $\Delta \geq 2\alpha\Psi(C^{-1}(\Delta - F))$. Note that $2\alpha\Psi(1) \geq 2\alpha\Psi(C^{-1}(\Delta - F))$ if and only if $\Delta \geq \overline{c} + F$. Then, as long as in equilibrium $\Delta \geq \overline{c} + F$ (i.e., $\Pi_k(1) \geq 0$ for $k = 2, \ldots, K-1$), $\Delta \geq 2\alpha\Psi(1)$ implies $\Delta \geq 2\alpha\Psi(C^{-1}(\Delta - F))$; i.e., if interior candidates are choosing (the same) non-maximal quality, they obtain non-negative rents. It will be sufficient for our result to look for equilibria in which $\Delta \geq \overline{c} + F$, and therefore we require that

$$\max\{\overline{c} + F, 2\alpha\Psi(1)\} < \Delta. \tag{8}$$

Next, we consider the possibility of entry. First, we require that all candidates that all equilibrium candidates have an incentive not to drop from the competition in any continuation game. For this it is sufficient that $\max\{\Delta_0, \frac{\Delta}{2}\} \geq \overline{c}$. Since $2\Delta_0 + (K-2)\Delta = 1$, then $\Delta_0 = \frac{1-(K-2)\Delta}{2}$, and the previous condition can be written as

$$2\overline{c} \leq \Delta \leq \frac{1 - 2\overline{c}}{K-2}. \tag{9}$$

Suppose now that j enters at $x_j \in (x_k, x_{k+1})$ for $k = 1, \ldots, K-1$, and define $\delta_j^r \equiv \frac{x_{k+1} - x_j}{\Delta}$. Suppose first that in the continuation $\hat{\theta}_k = \hat{\theta}_{k+1} = \hat{\theta}_j = 1$. Then it must be that

$$\alpha v'(1)\left[\frac{1}{\delta_j^r \Delta} + \frac{1}{\Delta}\right] \geq C'(1),$$

$$\alpha v'(1)\left[\frac{1}{(1-\delta_j^r)\Delta} + \frac{1}{\Delta}\right] \geq C'(1).$$

Then if $\delta_j^r \geq \frac{1}{2}$ (j enters in (x_k, x_{k+1}) closer to x_k than to x_{k+1}) the first two inequalities above hold if and only if $\Delta \leq \alpha\Psi(1)\left[1 + \frac{1}{\delta_j^r}\right]$, or $\delta_j^r \leq \frac{\alpha\Psi(1)}{\Delta - \alpha\Psi(1)}$. Thus, the continuation strategy profile is a Nash equilibrium for $\frac{1}{2} \leq \delta_j^r \leq \frac{\alpha\Psi(1)}{\Delta - \alpha\Psi(1)}$, which is feasible if and only if $\Delta \leq 3\alpha\Psi(1)$. When instead $\delta_j^r \leq \frac{1}{2}$ (j enters closer to x_k) then we need $\Delta \leq \alpha\Psi(1)\left[1 + \frac{1}{(1-\delta_j^r)}\right]$, or $\delta_j^r \geq \frac{\Delta - 2\alpha\Psi(1)}{\Delta - \alpha\Psi(1)}$. Thus, the continuation strategy profile is a Nash equilibrium for $\frac{\Delta - 2\alpha\Psi(1)}{\Delta - \alpha\Psi(1)} \leq \delta_j^r \leq \frac{1}{2}$, which is feasible if and only if $\Delta \leq 3\alpha\Psi(1)$. Therefore, the strategy profile $\hat{\theta}_k = \hat{\theta}_{k+1} = \hat{\theta}_j = 1$ is a Nash equilibrium in the continuation for entrants such that

$$\frac{\Delta - 2\alpha\Psi(1)}{\Delta - \alpha\Psi(1)} \leq \delta_j^r \leq \frac{\alpha\Psi(1)}{\Delta - \alpha\Psi(1)}, \tag{10}$$

where $2\alpha\Psi(1) < \Delta \leq 3\alpha\Psi(1)$. Since the entrant in this case obtains $\hat{\Pi}_j = \frac{\Delta}{2} - [\bar{c} + F]$, then as long as in equilibrium

$$\Delta < 2[\bar{c} + F], \tag{11}$$

entry in an "interior" region as in (10) is not profitable. And it should be clear that this rules out "interior" entrants only, since $2\alpha\Psi(1) < \Delta$ from (8) implies with (10) that $\delta_j^r \in (0, 1)$.

Consider then $\delta_j^r > \frac{\alpha\Psi(1)}{\Delta - \alpha\Psi(1)}$ (j enters close to x_k; the other case is symmetric). Consider the continuation $\hat{\theta}_k = \hat{\theta}_j = 1$, $\hat{\theta}_{k+1} = \Psi^{-1}(\frac{\delta_j^r}{1+\delta_j^r}\frac{\Delta}{\alpha}) < 1$. This is clearly an equilibrium in the continuation (j and k have even a greater incentive to choose 1 than in the previous case since they are now closer substitutes). For entry not to be profitable, we need

$$\hat{\Pi}_j = \frac{\Delta}{2} + \frac{\alpha}{\delta_j^r \Delta}[v(1) - v(\hat{\theta}_{k+1})] - [\bar{c} + F] < 0,$$

and a sufficient condition for the above inequality to be true is

$$\Delta \leq 2F. \tag{12}$$

To see this, suppose that the division of the electorate between k and j were fixed, with cutpoint $\tilde{x}_{kj} = \frac{x_k + x_j}{2}$. Then j would optimally choose $\tilde{\theta}_j = \Psi^{-1}(\frac{\delta_j^r \Delta}{\alpha}) < \hat{\theta}_{k+1}$, and we have that

$$\hat{\Pi}_j \leq \frac{\Delta}{2} - \frac{\alpha}{\delta_j^r \Delta}[v(\hat{\theta}_{k+1}) - v(\tilde{\theta}_j)] - [C(\tilde{\theta}_j) + F] < \frac{\Delta}{2} - [C(\tilde{\theta}_j) + F].$$

Consider next optimality and non-negative rents for extremal candidates, and no-entry conditions at the extremes. Note first that given that interior candidates are choosing non-maximal quality, then optimal quality by extremal candidates must be non-maximal as well. In particular, it must be that $\theta_1^* = \theta_K^* = \Psi^{-1}(\frac{\Delta}{\alpha})$. For no entry at the extremes it is sufficient as before that $\Delta_0 < F$, and since $\Delta_0 = \frac{1-(K-1)\Delta}{2}$ this can be written as

$$\frac{1 - 2F}{K - 1} < \Delta. \tag{13}$$

For non-negative rents we need $\Pi_1^* = \Delta_0 + \frac{\Delta}{2} - \frac{\alpha}{\Delta}[v(\theta^*) - v(\theta_1^*)] - C(\theta_1^*) - F \geq 0$. Since Π_1^* is maximized at θ_1^*, then $\Pi_1(\theta_1^*) \geq \Pi_1(\theta_1)$ for all $\theta_1 \neq \theta_1^*$ and, as a result, it suffices to show that $\Pi_1(\theta^*) > 0$, or equivalently, $\frac{(K-2)}{2}\Delta + [C(\theta^*) + F] \leq \frac{1}{2}$. But since in equilibrium $\Delta \geq C(\theta^*) + F$, then it is sufficient that

$$\Delta \leq \frac{1}{K}. \tag{14}$$

We have then shown that the strategy profile specified above is an electoral equilibrium (in which all candidates choose non-maximal quality) if Δ satisfies conditions (8) - (14). Now, (8) and (12) imply that for this to be feasible it is necessary that $\bar{c} < F$ $(*)$. From $(*)$, $\bar{c} + F < \Delta$ in (8) and (14) imply (9), and (11) implies (12). The relevant conditions on the degree of policy differentiation, Δ, can then be written as $\max\{2\alpha\Psi(1), L\} \leq \Delta < U$, as we wanted to show.

Proof of Part 1. Step 2. Now let $L_K \equiv \max\{2\bar{c}_K, \bar{c}_K + F_K, \frac{1-2F_K}{K-1}\}$ and $U_K \equiv \min\{2(\bar{c}_K + F_K), \frac{1}{K}\}$. Result *(a)* above shows that if $\max\{\alpha_K\Psi(1), L_K\} < \min\{U_K, 2\alpha_K\Psi(1)\}$, then there exists a LSE in which extremal candidates choose interior quality. Result *(b)* above shows that if $\max\{2\alpha_K\Psi(1), L_K\} < U_K$, then there exists a LSE in which all candidates running for office choose interior quality. Thus, if $\max\{\alpha_K\Psi(1), L_K\} < U_K$, there exists a LSE, Γ_K^{PR}, in which $\bar{\theta}(\Gamma_K^{PR}) < 1$. After some algebra, [10] we note that $\max\{\alpha_K\Psi(1), L_K\} < U_K$ is implied by the following three inequalities

$$\bar{c}_K < F_K, \tag{15}$$

$$\frac{1}{2K} \leq F_K \leq \frac{1}{K} - \bar{c}_K, \tag{16}$$

and

$$\alpha_K < \frac{1}{\Psi(1)K}. \tag{17}$$

Now let $\bar{c}_K = \frac{1}{2K}(1-\varepsilon)$ for $\varepsilon \in (0,1)$. Then there exists F_K satisfying (16). Choose $F_K = \frac{(1+\varepsilon)}{2}\frac{\gamma}{K}$ for $\frac{1}{1+\varepsilon} < \gamma < 1$. Finally, let $\alpha_K = \frac{\eta}{\Psi(1)K}$ for some $\eta \in (0,1)$. This triple $(\bar{c}_K, F_K, \alpha_K)$ satisfy (15), (16) and (17) for any $K \geq 3$. This concludes the proof of part 1 of the Theorem.

[10] The condition $\max\{\alpha_K\Psi(1), L_K\} < U_K$ embodies six relevant inequalities: *(a)* $\alpha_K\Psi(1) < 2[\bar{c}_K + F_K]$, *(b)* $\alpha_K\Psi(1) < 1/K$, *(c)* $2\bar{c}_K < 1/K$, *(d)* $(\bar{c}_K + F_K) < 1/K$, *(e)* $\frac{1-2F_K}{K-1} < 1/K$ and *(f)* $\frac{1-2F_K}{K-1} < 2[\bar{c}_K + F_K]$. Note that *(e)* can be written as $F_K > \frac{1}{2K}$, and *(f)* as $F_K > \frac{1}{2K} - \frac{K-1}{K}\bar{c}_K$. Thus *(e)* implies *(f)*. Moreover, from this it follows that $\frac{1}{K} < 2[\bar{c}_K + F_K]$, and that therefore *(b)* implies *(a)*. Finally, given (15), *(d)* implies *(c)*. Inequalities *(d)* and *(f)* give (16).

Proof of Part 2. Consider first the case of $K = 2$. Note that since identically located candidates are perfect substitutes, in equilibrium quality must be maximal. Otherwise candidate k can increase rents discretely (in fact capturing all votes) by increasing θ_k (and costs) only marginally. The rents of candidates are non-negative if and only if $\frac{1}{2} - \overline{c} \geq F$. To show that an equilibrium cannot exist it is enough to show that there exists a small positive ν such that entry of a third candidate at $x' = \frac{1}{2} - \nu$ is always profitable. Note that if a third candidate j enters at x' with $\theta_j = 1$ either $\hat{\theta}_k = 1$ for $k = 1, 2$, or $\hat{\theta}_k = 1$ and $\hat{\theta}_{-k} = 0$, $k = 1, 2$ ($\frac{1}{2} - \overline{c} \geq F$ implies that the case $\hat{\theta}_k = 0$, $k = 1, 2$ can never happen). If $\frac{1}{2}(1 - \frac{x'+1/2}{2}) - \overline{c} = \frac{3-2x'}{8} - \overline{c} \geq 0$, we have that in the continuation game $\hat{\theta}_k = 1$, $k = 1, 2$, and to deter entry at $\tilde{x}$ we need $\frac{x'+\frac{1}{2}}{2} - \overline{c} < F$. When $\nu \to 0$ the two last inequalities become $\frac{1}{2} - \overline{c} \in \left[\frac{1}{4}, F\right]$. Together with the above condition for non-negative rents for candidates, the last expression implies that a two candidate equilibrium with perfectly centrist candidates exists if and only if $F \geq \frac{1}{4}$ and $\frac{1}{2} - \overline{c} = F$. If instead $\frac{3-2\tilde{x}}{8} - \overline{c} < 0$, we have that in the continuation game one of the two running candidates will drop, i.e., $\hat{\theta}_k = 1$, and $\hat{\theta}_{-k} = 0$, $k = 1, 2$. Since to deter entry at $\tilde{x}$ it must be that $\frac{x'+\frac{1}{2}}{2} - \overline{c} < F$, in this case when $\nu \to 0$ we need $\frac{1}{2} - \overline{c} \leq \min\left\{\frac{1}{4}, F\right\}$. Once again combining the last expression with the above condition for non-negative rents for candidates we get that a two candidate equilibrium with perfectly centrist candidates exists if and only if $F \leq \frac{1}{4}$ and $\frac{1}{2} - \overline{c} = F$. If $K > 2$ we need $\frac{x'+\frac{1}{2}}{2} - \overline{c} < F$ and $\frac{1}{K} - \overline{c} \geq F$, which leads to a contradiction when $\nu \to 0$. ■

Proof of Proposition 2. First, note that if a set of candidates $W \in \mathcal{K}$ tie for the win, then $\theta_k = 1$ for all $k \in W$, for otherwise there exists a candidate $\ell \in W$ with $\theta_\ell < 1$, who would gain from deviating to $\theta'_\ell = \theta_\ell + \eta$ for sufficiently small $\eta > 0$.

(a) So suppose first that in equilibrium all K candidates in $\mathcal{K}$ tie, with $\theta_k = 1$ for all k, and let $k^*(i)$ denote i's preferred candidate in $\mathcal{K}$. It is immediate that here all voters $i \in N$ must vote sincerely, for otherwise any voter not voting sincerely would induce a strictly preferred lottery over outcomes by voting for her preferred candidate $k^*(i)$. Since all candidates are tying choosing maximal quality and voting is sincere, candidates must be equally spaced. Next, note that equilibrium implies that all voters $i \in N$ must prefer the equal probability lottery among all $k \in W$ induced in equilibrium to the lottery that is implied after a deviation to any candidate $\ell \neq k^*(i)$. Moreover, if this is an electoral equilibrium for all $N \geq \overline{N}$, it must be that all voters $i \in N$ must prefer the equal probability lottery among all $k \in W$ induced in equilibrium to the degenerate lottery in which they get any candidate $\ell \neq k^*(i)$ for sure. To see this, note that i's equilibrium payoff, voting for $k^*(i)$, is

$$U(k^*(i);\sigma^v_{-i}) = \sum_{k\in\mathcal{K}}\left[\frac{1}{K}\frac{N-1}{N}(1-\gamma)+\frac{\gamma}{K}\right]u(x_k;z_i)+\frac{1}{N}(1-\gamma)u(x_{k^*(i)};z_i).$$

Deviating and voting for $\ell \neq k^*(i)$, i obtains

$$U(\ell;\sigma^v_{-i}) = \sum_{k\in\mathcal{K}}\left[\frac{1}{K}\frac{N-1}{N}(1-\gamma)\right]u(x_k;z_i)+\left[\frac{1}{N}(1-\gamma)+\gamma\right]u(x_\ell;z_i).$$

The deviation gain $U(\ell;\sigma^v_{-i}) - U(k^*(i);\sigma^v_{-i}) < 0$ implies then that

$$u(x_\ell;z_i) - \frac{1}{K}\sum_{k\in\mathcal{K}}u(x_k;z_i) < \frac{1}{N}\frac{(1-\gamma)}{\gamma}\left[u(x_{k^*(i)};z_i)-u(x_\ell;z_i)\right],$$

but since this holds for all $N > \overline{N}$, it must be that $u(x_\ell;z_i) < \frac{1}{K}\sum_{k\in\mathcal{K}}u(x_k;z_i)$, for otherwise, we can always find an N that would reverse this inequality. Thus if this is an equilibrium for all $N > \overline{N}$, it must be that all voters $i \in N$ must prefer the equal probability lottery among all $k \in W$ induced in equilibrium to the degenerate lottery in which they get any candidate $\ell \neq k^*(i)$ for sure. But then the same argument as in Theorem 1 shows that this can't be an equilibrium.

(b) Next suppose that $2 \leq |W| < K$ candidates tie for the win in equilibrium, and let W denote the set of winning candidates and L the set of losing candidates. This can't be an equilibrium either for sufficiently large N, since otherwise a voter i voting for one of the losing candidates $\ell_0 \in L$ could gain by breaking the tie among the candidates in W in favor of her favorite candidate among W, w_0. To see this, denote the fraction of votes obtained by candidate in W by ω, and note that i's equilibrium payoff, voting for $\ell_0 \in L$, is

$$U(\ell_0;\sigma^v_{-i}) = \sum_{w\in W}\left[\omega(1-\gamma)+\frac{\gamma}{|W|}\right]u(x_w;z_i)+\sum_{\ell\in L}\frac{t_\ell}{N}(1-\gamma)u(x_\ell;z^i)$$

The expected payoff of deviating and voting for $w_0 \in W$ is instead

$$U(w_0;\sigma^v_{-i}) = \sum_{w\in W}\omega(1-\gamma)u(x_w;z^i)+\left[\frac{1}{N}(1-\gamma)+\gamma\right]u(x_{w_0};z^i)+$$
$$\sum_{\ell\neq\ell_0\in L}\frac{t_\ell}{N}(1-\gamma)u(x_\ell;z^i)+\frac{(t_{\ell_0}-1)}{N}(1-\gamma)u(x_{\ell_0};z^i)$$

But then $U(w_0;\sigma^v_{-i}) - U(\ell_0;\sigma^v_{-i}) > 0$ if and only if

$$\frac{\gamma}{1-\gamma} > \frac{1}{N} \frac{[u(x_{\ell_0}; z^i) - u(x_{w_0}; z^i)]}{\left[u(x_{w_0}; z^i) - \frac{1}{|W|} \sum_{w \in W} u(x_w; z_i)\right]},$$

which holds for sufficiently large N. ■

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