# International Borrowing, Investment and Default*

Veronica Guerrieri, Guido Lorenzoni and Fabrizio Perri

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**Abstract**

Models of international borrowing and default have emphasized intertemporal consumption smoothing as the main driver of borrowing. This leads these models to predict a very low frequency of default, as countries tend to accumulate foreign assets and avoid going into debt for protracted periods of time. This makes it difficult to replicate the frequency of default and average levels of borrowing observed in the data.

In this paper, we present a tractable model of a small open economy where the main driver of international borrowing is investment. Debt is non-state-contingent and the choice of default is endogenous, as in Eaton and Gersovitz (1981). By introducing a simple AK technology, we obtain a setup where countries can be permanent debtors (or permanent creditors). In particular, a country will be a net borrower when the expected productivity of capital, adjusted for risk, is higher than the world interest rate. Moreover, we can derive analytically the equilibrium level of debt and the probability of default, and look at the effects of expected productivity on both. The model can deliver higher default frequencies and higher borrowing levels than consumption smoothing models.

During the last two decades, emerging economies have experienced periods of growing investment, financed by international borrowing. These periods have been followed, in some cases, by episodes of financial distress and default. Understanding the dynamics of investment, foreign borrowing and default in these countries is essential in evaluating potential inefficiencies of international financial markets, and in designing policies to alleviate them.

The objective of this project is to propose a tractable model of investment and default, and to evaluate its ability to replicate basic quantitative features of observed

*[VERY PRELIMINARY AND INCOMPLETE]

episodes of financial crises. The approach is to consider an emerging economy with access to a risky domestic investment technology, who can finance this investment by issuing defaultable bonds. The risk in the investment payoff is due to exogenous productivity shocks. Default is an endogenous decision by the country. In this environment, default has benefits, as it allows the country to make its payments state-contingent, but it is also has costs, as countries are temporarily excluded from financial markets after a default episode. By making simplifying assumptions on preferences and technology, we are able to derive in closed form the optimal investment rule and a simple cutoff rule for the default decision. The cutoff rule has a natural interpretation: countries default for low realizations of the productivity shock, the cutoff is higher when the shock is more persistent.

The current literature finds it hard to replicate the observed frequency of default episodes and the size of the current account deficits preceding default. This, in part, may be due to the emphasis on consumption smoothing, since countries prefer to reduce their consumption when they reach a high level of foreign debt, and, hence, to stay away from default. By emphasizing the investment side of the model, we hope to overcome some of these quantitative problems.

# 1 The Model

Consider a small open economy with an infinitely lived representative consumer whose preferences are represented by the utility function

$$(1-\beta)\,\mathbb{E}\left[\sum_{t=0}^{\infty}\beta^t u\left(c_t\right)\right],$$

where $c_t$ is consumption, $\beta \in (0,1)$ is a constant discount factor, and $u\left(c_t\right) = \log c_t$. The consumer has access to the linear technology

$$y_t = a_t k_t,$$

where $a_t$ is an exogenous stochastic productivity parameter and $k_t$ is capital invested at date $t-1$. Each period $a_t$ is drawn from a continuous distribution with cumulative distribution function $F(a)$ and support $[\underline{a}, \overline{a}]$, with $\underline{a} > 0$. The rest of the world is represented by a risk neutral international investor with preferences $(1-\delta)\,\mathbb{E}\left[\sum_{t=0}^{\infty}\delta^t c_t^*\right]$ who receives a large endowment of consumption goods each period. The international discount factor $\delta$ can be different from the domestic one $\beta$.

The country only trades one-period non-state-contingent bonds with the rest of the world. A bond issued at date $t$ is a promise to repay one unit of consumption good in the following period. The stock of bonds issued by the country in period $t$ is denoted by $d_t$. When $d_t < 0$ the country is a net lender in international capital markets. The bonds trade at the price $p_t$. Each period $t$, the country decides whether to repay the old bonds, if $d_t > 0$, how many new bonds $d_{t+1}$ to issue, how much to invest in physical capital, $k_{t+1}$, and how much to consume. If the country ever defaults, we assume that the punishment is autarky, i.e., the international investors stop trading with the country.

We can then write the country's budget constraint as follows. If the country has not defaulted up to period $t$, the budget constraint is

$$c_t + k_{t+1} + p_t d_{t+1} \leq a_t k_t - \chi_t d_t, \tag{1}$$

where $\chi_t \in \{0, 1\}$ reflects the choice to default at time $t$. If the country has defaulted in the past, the budget constraint is simply

$$c_t + k_{t+1} \leq a_t k_t.$$

We assume that the country begins with some $k_0 > 0$ and no debt.

We make the following additional assumptions.

**Assumption 1** *Domestic capital dominates international riskless bonds in expected returns:*

$$\delta \mathbb{E}\left[a\right] > 1. \tag{2}$$

**Assumption 2** *The cumulative distribution function $F\left(.\right)$ satisfies the following condition for some scalar $\eta < 1$:*

$$\delta\left(1 - F\left(a\right)\right) a \leq \eta \text{ for all } a. \tag{3}$$

**Assumption 3** *The density $f\left(.\right)$ is continuous and satisfies $f\left(\underline{a}\right) = 0$.*

The first assumption is needed to ensure that the country invests a positive fraction of its wealth in domestic capital. The second is needed to ensure that borrowing and investment by the country are finite. The last assumption is more technical and is used to ensure that the borrowing country's objective function is continuously differentiable in some relevant range.

We focus on Markov equilibria where the state variables are $k_t, d_t$ and $S_t \in \{N, D\}$. $S_t$ is a state variable which starts at $N$ (no-default) and switches to $D$ as soon as the country defaults. The country's strategy is described by three maps $\left(\chi, \sigma^N, \sigma^D\right)$. The first is a map $\chi : [\underline{a}, \overline{a}] \times R_+ \times R \rightarrow \{0, 1\}$, which gives us the repayment decision $\chi_t$ for a given triple $(a_t, k_t, d_t)$, the second $\sigma^N : [\underline{a}, \overline{a}] \times R_+ \times R \rightarrow R_+ \times R$ gives us the investment and borrowing decisions $(k_{t+1}, d_{t+1})$ after the country chooses not to default, and the third, $\sigma^N : [\underline{a}, \overline{a}] \times R_+ \rightarrow R_+$, gives us the investment decision $k_{t+1}$ if default happened in the current or in any previous period.

The price of international bonds is given by the function $\mathcal{P}$:

$$p_t = \mathcal{P}\left(k_{t+1}, d_{t+1}\right).$$

**Definition 1** *A Markov equilibrium is given by a strategy* $\left(\chi, \sigma^N, \sigma^D\right)$*, and a price function* $\mathcal{P}$*, such that:*

**(i)** *the country's strategy is optimal given* $\mathcal{P}$*;*

**(ii)** $\mathcal{P}$ *is consistent with rational expectations:*

$$\mathcal{P}\left(k_{t+1}, d_{t+1}\right) = \delta \int_{\underline{a}}^{\overline{a}} \chi\left(a_{t+1}, k_{t+1}, d_{t+1}\right) dF\left(a_{t+1}\right).$$

# 2 Equilibrium

We will look for equilibria where the function $\mathcal{P}$ takes the following form:

$$\mathcal{P}(k, d) = \begin{cases} \delta\left(1 - F\left(\theta \frac{d}{k}\right)\right) & \text{if } k > 0 \text{ and } d \geq 0, \\ 0 & \text{if } k = 0 \text{ and } d > 0, \\ \delta & \text{if } k = 0 \text{ and } d \leq 0, \end{cases}, \tag{4}$$

for some constant $\theta \geq 1$ to be determined. The reason why we focus on this functional form will be clear when we analyze the country's problem.

Notice that, given (4), Assumption 2 and $\theta \geq 1$ imply that

$$\mathcal{P}(k, d)\, d/k = \beta\left(1 - F\left(\theta d/k\right)\right) d/k \leq \beta\left(1 - F\left(\theta d/k\right)\right) \theta d/k \leq \eta,$$

for all $d$ and all $k > 0$, where $\eta$ is the constant defined in Assumption 2. Moreover, if $k = 0$, we immediately have $\mathcal{P}(k, d)\, d \leq 0$. These results can be summarized in the following inequality

$$\mathcal{P}(k, d)\, d \leq \eta k \text{ for all } k, d, \tag{5}$$

which says that the value of bonds issued at any point in time has an upper bound which is proportional to the capital stock of the country. In other words, this property says that the debt-to-capital ratio $p_t d_{t+1}/k_{t+1}$ has an upper bound $\eta$ smaller than one. This property will ensure that the country's portfolio problem has a finite solution.

## 2.1 Optimal borrowing

First, let us analyze the optimal behavior of the borrowing country, taking as given the pricing function (4) and a given $\theta \geq 1$. We use a recursive notation, letting $a, k, d$ denote the current values of the variables and $a', k', d'$ denote their value in the next period.

Suppose first that the country has defaulted at some point in the past, and let $V^D(a,k)$ denote its continuation utility. Notice first that if the country has no access to international capital markets and $k = 0$, the country gets zero consumption in the current period and in all future periods. Therefore, in this case $V^D(a,k) = -\infty$. When $k > 0$, the value function $V^D(a,k)$ is characterized by the following Bellman equation

$$\begin{aligned} V^D(a,k) = \max_{c,k'} \quad & (1-\beta)\log c + \beta \int_{\underline{a}}^{\overline{a}} V^D(a',k')\, dF(a') \\ s.t. \quad & c + k' = ak. \end{aligned} \tag{6}$$

Suppose next that the country has never defaulted in the past, and let $V(a,k,d)$ denote its continuation utility. Let us rewrite the budget constraint (1) as

$$c + k' - \mathcal{P}(k',d')\, d' \leq ak - d. \tag{7}$$

If $ak - d < 0$, the country simply cannot repay its current debt, because if it tries to repay it faces an empty constraint set. To prove this claim, notice that (5) implies that $k' - \mathcal{P}(k',d')\, d' \geq (1-\eta)\, k' \geq 0$, which, together with (7), implies that $c < 0$ for any choice of $k'$ and $d'$. Therefore, if $ak - d < 0$ the country is forced to default and

$$V(a,k,d) = V^D(a,k).$$

Consider next the case where the state variables $k, d$ and the current shocks are such that $ak - d \geq 0$. Now the country has a choice between defaulting and not defaulting. Let $V^N(a,k,d)$ denote the optimal value the country can achieve after not defaulting. If $ak - d = 0$, (7) and (5) imply that the only feasible consumption is $c = 0$, so $V^N(a,k,d) = -\infty$. When $ak - d > 0$ the value function $V^N(a,k,d)$ is characterized by the Bellman equation

$$\begin{aligned} V^N(a,k,d) = \max_{c,k',d'} \quad & (1-\beta)\log c + \beta \int_{\underline{a}}^{\overline{a}} V(a',k',d')\, dF(a') \\ s.t. \quad & c + k' - \mathcal{P}(k',d')\, d' = ak - d. \end{aligned} \tag{8}$$

Therefore, when $ak - d \geq 0$, the country will decide whether to default or not by comparing $V^D(a,k)$ and $V^N(a,k,d)$. In this case, the value function satisfies

$$V(a,k,d) = \max\left\{V^N(a,k,d), V^D(a,k)\right\}.$$

### 2.1.1 Finding the value functions

To characterize the country's optimization problem, we will now derive explicitly the value functions $V^D$ and $V^N$, by guessing and verifying. Our guess is that the value functions take the form

$$V^D(a,k) = v^D + \log(ak), \tag{9}$$
$$V^N(a,k,d) = v^N + \log(ak-d), \tag{10}$$

and that

$$\Delta \equiv v^D - v^N \geq 0.$$

We now proceed to verify (9) and (10), and, in the process, derive the equilibrium values of the constant terms $v^D$ and $v^N$. The solution for the value function (9) is completely standard, and is summarized in the following proposition.

**Proposition 2** *The value function after default takes the form (9), with*

$$v^D = \frac{1}{1-\beta}\left[(1-\beta)\log(1-\beta) + \beta\log(\beta) + \beta\mathbb{E}[\log a]\right]. \tag{11}$$

*The optimal policy for consumption and investment after default is*

$$\begin{aligned} c &= (1-\beta)\, ak, \\ k' &= \beta ak. \end{aligned}$$

We can now turn to the function $V^N(a,k,d)$. To do so consider the maximization problem in (8). We begin by deriving an expression for $\int_{\underline{a}}^{\overline{a}} V(a',k',d')\, dF(a')$ in the objective function. In the following derivations we assume $k' > 0$, in the proof of Lemma 3 we show that $k' = 0$ is never optimal.

Let us show that default in the next period will occur if and only if $a' < \hat{a}$, where the cutoff $\hat{a}$ is given by

$$\hat{a} = \frac{e^{\Delta}}{e^{\Delta} - 1} \frac{d'}{k'}. \tag{12}$$

To prove this claim, notice that if $a'k' - d' \geq 0$ we have

$$\begin{aligned} V^N(a', k', d') - V^D(k', d') &= v^N + \log(a'k' - d') - v^D - \log a'k' = \\ &= v^N + \log\left(a' - \frac{d'}{k'}\right) - v^D - \log a', \end{aligned}$$

and this expression is greater than or equal to zero whenever $a' \geq \hat{a}$. Notice that the default cutoff defined in (12) also covers the case $a'k' - d' < 0$: in that case $a' < \hat{a}$ follows from $a' < d'/k' \leq \hat{a}$, where the second inequality comes from $\Delta \geq 0$. Notice also that the default cutoff in (12) depends linearly on the debt-to-capital ratio $d'/k'$, which justifies our conjecture that the equilibrium pricing function takes the form in (4). We will return to this relation below.

The cutoff $\hat{a}$ fully characterizes the default region and we can write

$$\begin{aligned} \int_{\underline{a}}^{\overline{a}} V(a', k', d')\, dF(a') &= \int_{\underline{a}}^{\max\{\hat{a}, \underline{a}\}} \left(v^D + \log a'\right) dF(a') + \\ &+ \int_{\min\{\hat{a}, \overline{a}\}}^{\overline{a}} \left(v^N + \log\left(a' - \frac{d'}{k'}\right)\right) dF(a') + \log k' \end{aligned} \tag{13}$$

Moreover, the budget constraint (7) can be rewritten as

$$k' = \frac{ak - d - c}{1 - \mathcal{P}\left(1, \frac{d'}{k'}\right) \frac{d'}{k'}}, \tag{14}$$

given that $\mathcal{P}(1, d'/k') = \mathcal{P}(k', d')$, from (4), and given that (5) implies that $1 - \mathcal{P}(1, d'/k')\, d'/k' > 0$. This expression can be interpreted in the following way: given the initial net wealth $ak - d$, the country decides to consume $c$ and invest the rest. Since the country has access to foreign debt, it can leverage its investment by raising funds from foreign investors and obtain $\mathcal{P}(1, d'/k')\, d'/k'$ per dollar invested. Therefore, $1/(1 - \mathcal{P}(1, d'/k')\, d'/k') > 1$ represents the country's leverage ratio, that is, the ratio between total capital and net wealth invested at the end of the period.

We can now substitute (14) in (13) and (13) in the objective function of problem (8). Letting $x$ denote the debt-to-capital ratio $d'/k'$, the country's problem is then to choose $c$ and $x$ that maximize

$$(1 - \beta) \log c + \beta \log(ak - d - c) + \beta v^D + \beta \Phi(x, \Delta),$$

where $\Phi(x, \Delta)$ is defined as

$$\Phi(x, \Delta) \equiv \int_{\underline{a}}^{A(x)} \log(a') \, dF(a') + \int_{A(x)}^{\overline{a}} (\Delta + \log(a' - x)) \, dF(a') - \log(1 - \mathcal{P}(1, x) x), \tag{15}$$

with

$$A(x) \equiv \min\left\{\max\left\{\frac{e^{\Delta}}{e^{\Delta} - 1} x, \underline{a}\right\}, \overline{a}\right\}.$$

This shows that the country's problem can be split in two pieces: choose $c$ to maximize $(1 - \beta) \log c + \beta \log(ak - d - c)$, and choose $x$ to maximize $\Phi(x, \Delta)$. The first maximization gives the standard result

$$c = (1 - \beta)(ak - d), \tag{16}$$

and the budget constraint implies that $k'$ and $d'$ are given by

$$k' = \frac{\beta(ak - d)}{1 - \mathcal{P}(1, x) x}, \tag{17}$$

$$d' = x \frac{\beta(ak - d)}{1 - \mathcal{P}(1, x) x}. \tag{18}$$

To complete our characterization of the optimization problem in (8), it remains to show that the function $\Phi(x, \Delta)$ has a well defined maximum with respect to $x$. This is shown in the appendix, and, together with the derivations above, proves the following lemma.

**Lemma 3** *Suppose the value functions take the form (9) and (10) for some pair $v^D, v^N$ such that $\Delta = v^D - v^N \geq 0$. Then, the maximization problem in (8) has a solution with consumption, capital and debt given by (16)-(18), where*

$$x \in \arg\max_{\tilde{x}} \Phi(\tilde{x}, \Delta).$$

Substituting the optimal values of $c$ and $x$ in the objective function, we then obtain

$$(1 - \beta) \log(1 - \beta) + \beta \log \beta + \log(ak - d) + \beta v^D + \beta \tilde{\Phi}(\Delta),$$

where

$$\tilde{\Phi}(\Delta) \equiv \max_{\tilde{x}} \Phi(\tilde{x}, \Delta).$$

This confirms our initial conjecture (10) if and only if $v^N$ satisfies the condition

$$v^N = (1 - \beta) \log(1 - \beta) + \beta \log \beta + \beta v^D + \beta \tilde{\Phi}(\Delta).$$

Subtracting $v^D$ on both sides and substituting (11), we can express this condition in terms of $\Delta$ as

$$\Delta = \beta(\tilde{\Phi}(\Delta) - \mathbb{E}[\log a]). \tag{19}$$

In the following proposition, we show that this equation has a unique solution in $\Delta$, and this completes our characterization of the country's value function under no default. Recall that this characterization is made under the assumption that the pricing function $\mathcal{P}(k,d)$ takes the form (4) for a given value of $\theta$. We use the notation $\Delta(\theta)$, to denote the solution of (19) associated to a given $\theta$.

**Proposition 4** *Suppose the pricing function $\mathcal{P}(k,d)$ takes the form (4) for some $\theta \geq 1$. Then, the value function under no default takes the form (10), with $v^N = v^D + \Delta(\theta)$, where $\Delta(\theta)$ is the unique solution of (19).*

## 2.2 Equilibria with borrowing and default

So far we have characterized the optimal country's behavior for a given pricing function, $\mathcal{P}(k,d)$. To complete our equilibrium characterization, we need to check that $\mathcal{P}(k,d)$ satisfies rational expectations. As we just showed, a country with debt $d > 0$ and a capital stock $k$ will default if and only if the current realization of the productivity shock $a$ is lower than the cutoff $\hat{a} = e^{\Delta}/\left(e^{\Delta}-1\right)d/k$. This implies that the probability of default is equal to $1 - F(\hat{a})$ and the bond price must be equal to $\delta(1 - F(\hat{a}))$. Therefore, $\mathcal{P}(k,d)$ in (4) is consistent with rational expectations if and only if

$$\theta = \frac{e^{\Delta(\theta)}}{e^{\Delta(\theta)} - 1}. \tag{20}$$

This shows that all Markov equilibria with prices given by (4) can be found by finding the values of $\theta$ which solve equation (20). The following proposition shows that such an equilibrium always exists. In particular, there is always a trivial equilibrium with $\theta = \infty$ and $\Delta = 0$, where the country never borrows. More interestingly, if investment in domestic capital is sufficiently profitable relative to the world interest rate, there is also another equilibrium with $\theta < \infty$ and $\Delta > 0$ where the country borrows positive amounts.

**Proposition 5** *There always exists an equilibrium with zero borrowing. If*

$$\delta\left(\mathbb{E}\left[a^{-1}\right]\right)^{-1} > 1, \tag{21}$$

*then there also exists an equilibrium with positive borrowing.*

Notice that condition (21) is stronger than condition (2) in Assumption 1 (by Jensen's inequality). The latter only ensures that the country will invest positive amounts of physical capital. But, in general, the country could choose to allocate its wealth between domestic capital and *positive* amounts of international bonds, that is, the country could be a net lender. Condition (21) ensures that the rate of return on domestic capital is sufficiently high that the country always chooses to be a net borrower.

Since $\underline{a} > 0$, a country that goes into debt may never default. This happens if, in equilibrium, the country chooses a debt-to-capital ratio $x$ which satisfies $\theta x \in (0, \underline{a})$. In that case, no default is always preferred to default. Therefore, it is interesting to study under what conditions there are equilibria with a positive probability of default. The next proposition provides a sufficient condition.

**Proposition 6** *If*

$$\delta \left( \mathbb{E} \left[ [a - \underline{a}]^{-1} \right] \right)^{-1} > 1, \tag{22}$$

*then in all the equilibria with positive borrowing the country defaults with positive probability.*

Notice that condition (22) is stronger than (21). That is, the return on domestic capital must be even higher to induce the country to borrow amounts large enough that may trigger default in the following period.

Let us now turn to the equilibrium dynamics of investment and borrowing. Let $\Delta^*$ and $\theta^*$ denote the equilibrium values of $\Delta$ and $\theta$. Except in non typical cases, there is a unique $x^*$ that maximizes $\Phi(x, \Delta^*)$ in equilibrium. Let $a^* = \theta^* x^*$ and

$$\alpha = \frac{1}{1 - \delta (1 - F(a^*)) x^*} > 1,$$

denote the the equilibrium leverage ratio, the dynamics of the capital stock are then given by

$$k_{t+1} = \alpha\beta (a_t k_t - d_t) = \alpha\beta (a_t - x^*) k_t,$$

as long as $a_t \geq a^*$. The first time that $a_t < a^*$, the country defaults and from then on the capital stock follows

$$k_{t+1} = \beta a_t k_t.$$

# 3 Appendix

**Proof of Proposition 2.** Rewrite the problem as

$$\max_c (1 - \beta) \log c + \beta v^D + \beta \mathbb{E} [\log a' + \log (ak - c)] .$$

The first-order condition is

$$\frac{1-\beta}{c} = \frac{\beta}{ak-c},$$

which gives the optimal consumption policy. Substituting in the objective function gives

$$(1-\beta)\log(1-\beta) + \beta\log(\beta) + \beta v^D + \beta\mathbb{E}[\log a'] + \log(ak),$$

which confirms our guess. ■

**Proof of Lemma 3.** The main derivations are in the text. Here, we prove two additional steps which complete the argument. First, we prove that $k' = 0$ is never optimal. Second, we prove that the function $\Phi(x, \Delta)$ has a maximum with respect to $x$.

*Step 1.* First, notice that if $k' = 0$ it can never be optimal to choose $d' \leq 0$, otherwise the country would either be forced to default or would have zero wealth in the next period, obtaining zero consumption from then on and expected utility equal to $-\infty$. Suppose then that it is optimal to choose $k' = 0$ and $d' < 0$. Then the country will never default in the next period, the price is $\mathcal{P}(0, d') = \delta$ and expected utility is given by

$$(1-\beta)\log(ak - d + \delta d') + \beta v^N + \beta\log(-d').$$

By choosing $\hat{k}' = \epsilon$ and $\hat{d}' = d' - \epsilon/\delta$ for some small positive $\epsilon$, the country is still in the region $\hat{d}' > 0$ and obtains utility

$$(1-\beta)\log(ak - d + \delta d') + \beta v^N + \beta\mathbb{E}[\log(a'\epsilon - d' - \epsilon/\delta)].$$

Differentiating this expression with respect to $\epsilon$ at $\epsilon = 0$ gives $\mathbb{E}[a' - 1/\delta]/(-d')$ which is strictly positive by Assumption 1, so the initial allocation cannot be an optimum.

*Step 2.* For ease of notation, in this proof we use $\Phi(x)$ in place of $\Phi(x, \Delta)$. To prove that $\Phi(x)$ has a maximum on $(-\infty, \infty)$, we first prove that it has a maximum on $(-\infty, 0]$ and then that it has a maximum on $[0, \infty)$. In the region, $(-\infty, 0]$ no default occurs and the bond's price is constant and equal to $\delta$, therefore, in this region, the problem of maximizing $\Phi(x)$ is analogous to a standard portfolio problem with one risky and one riskless asset. In particular, we can show that $\Phi(x)$ is quasi-concave on $(-\infty, 0]$. Notice that if $x \leq 0$ we have

$$\Delta + \log(a - x) \geq \log a$$

for all $a$ and $\mathcal{P}(1,x)=\delta$, from (4). Therefore, in this region we have

$$\begin{aligned}\Phi(x) &= \Delta+\mathbb{E}[\log(a-x)]-\log(1-\delta x)= \\ &= \Delta+\mathbb{E}\left[\log\left(\frac{a-x}{1-\delta x}\right)\right], \\ &= \Delta+\mathbb{E}\left[\log\left(ah(x)+\frac{1}{\delta}(1-h(x))\right)\right],\end{aligned} \tag{23}$$

where $h(x)=1/(1-\delta x)$. Since $h(x)$ is increasing on $(-\infty,0]$ and $\log\left(ah+\frac{1}{\delta}(1-h)\right)$ is concave in $h$, it follows that $\Phi(x)$ is quasi-concave on $(-\infty,0]$. Moreover, Assumption 3 implies that $\Phi(x)$ is continuously differentiable on $(-\infty,\overline{a}/\theta)$. To show that the function has a maximum on $(-\infty,0]$, it is then sufficient to show that $\Phi'(\hat{x})>0$ for some $\hat{x}\in(-\infty,0]$. To prove this claim, differentiate $\Phi(x)$ to obtain

$$\Phi'(x)=\mathbb{E}\left[\left(a-\frac{1}{\delta}\right)\frac{1}{ah(x)+\frac{1}{\delta}(1-h(x))}\right]h'(x),$$

and notice that $\lim_{x\to-\infty}h(x)=0$ and $\mathbb{E}\left[a-\frac{1}{\delta}\right]>0$ from Assumption 1. This implies that we can make the term in expectation positive for some $\hat{x}$ small enough. The result follows given that $h'(x)>0$ for $x\in(-\infty,0]$.

Let us now turn to the region $[0,\infty)$ for $x$. For all $x\geq 0$, we have

$$\int_{\underline{a}}^{A(x)}(\log a')\,dF(a')+\int_{A(x)}^{\overline{a}}(\Delta+\log(a'-x))\,dF(a')\leq\Delta+\mathbb{E}[\log a'],$$

and, by property (5),

$$-\log(1-\mathcal{P}(1,x)x)\leq-\log(1-\eta).$$

So the function $\Phi(x)$ is bounded on $[0,\infty)$. Define

$$\bar{x}\equiv\max\left\{\frac{e^{\Delta}-1}{e^{\Delta}}\overline{a},\frac{1}{\theta}\overline{a}\right\},$$

and notice that for all $x\geq\bar{x}$ the function $\Phi(x)$ is constant and equal to $\mathbb{E}[\log a']$, given that $A(x)=\overline{a}$ and $\mathcal{P}(1,x)=0$. Since the function is continuous and bounded on $[0,\bar{x}]$ and constant for all $x\geq\bar{x}$, a maximum exists. ■

**Proof of Proposition 4.** Take any two values $\Delta''\geq\Delta'\geq 0$. Take an $x'$ which maximizes $\Phi(x,\Delta')$. Then, we have, by definition,

$$\tilde{\Phi}(\Delta'')-\tilde{\Phi}(\Delta')\leq\Phi(x',\Delta'')-\Phi(x',\Delta').$$

Moreover, $\Phi(x,\Delta)$ is differentiable in $\Delta$ with $\partial\Phi(x,\Delta)/\partial\Delta = 1 - F(A(x)) \leq 1$. It follows that

$$\tilde{\Phi}(\Delta'') - \tilde{\Phi}(\Delta') \leq \Delta'' - \Delta'$$

for all $\Delta'' \geq \Delta' \geq 0$. It is easy to show that $\tilde{\Phi}(\Delta)$ is non-decreasing, so we have

$$\left|\tilde{\Phi}(\Delta'') - \tilde{\Phi}(\Delta')\right| \leq |\Delta'' - \Delta'|,$$

for all pairs $\Delta', \Delta'' \geq 0$. Moreover, for all $\Delta \geq 0$ we have

$$\tilde{\Phi}(\Delta) \geq \Phi(0,\Delta) = \Delta + \mathbb{E}[\log a],$$

which implies that $\beta(\tilde{\Phi}(\Delta) - \mathbb{E}[\log a]) \geq 0$. These results together show that the function on the right-hand side of (19) is a contraction mapping on $[0,\infty)$, and thus it has a unique fixed point which corresponds to the unique solution of (19). ■

**Proof of Proposition 5.** We will first characterize optimal borrowing for any $\theta \in [1,\infty)$, then we will show that under condition (21) there is a $\theta \in (1,\infty)$ which solves (20). Assumption 3 ensures that $\Phi(x,\Delta)$ is continuously differentiable in $x$, on $(-\infty, \overline{a}/\theta)$, with

$$\frac{\partial\Phi(x,\Delta)}{\partial x} = -\int_{A(x)}^{\overline{a}} \frac{1}{a-x} dF(a) + \frac{\delta(1-F(\theta x)) - \delta\theta f(\theta x)x}{1-\delta(1-F(\theta x))x}. \tag{24}$$

Moreover, as shown in the proof of Lemma 3, $\Phi(x,\Delta)$ is quasi-concave in $x$, on $(-\infty, 0]$. Therefore, the optimum $x$ is strictly positive as long as $\partial\Phi(x,\Delta)/\partial x|_{x=0} > 0$. To prove the latter inequality, substitute in (24) and use assumption (21), to get

$$\left.\frac{\partial\Phi(x,\Delta)}{\partial x}\right|_{x=0} = -\int_{\underline{a}}^{\overline{a}} \frac{1}{a} dF(a) + \delta > 0.$$

This also implies that there exists an $\hat{x} \in (0, \underline{a})$ such that

$$\varepsilon \equiv \Phi(\hat{x},\Delta) - \Phi(0,\Delta) = \mathbb{E}[\log(a-\hat{x})] - \log(1-\delta\hat{x}) - \mathbb{E}[\log a] > 0,$$

where $\varepsilon$ is independent of $\Delta$. Moreover,

$$\Phi(0,\Delta) = \Delta + \mathbb{E}[\log a].$$

We then have

$$\tilde{\Phi}(\Delta) \geq \Phi(\hat{x},\Delta) - \Phi(0,\Delta) + \Phi(0,\Delta) = \Delta + \mathbb{E}[\log a] + \varepsilon, \tag{25}$$

which implies that $\beta(\tilde{\Phi}(\Delta) - \mathbb{E}[\log a]) \geq \beta(\Delta + \varepsilon)$. The last inequality together with the definition of $\Delta(\theta)$, implies that

$$\Delta(\theta) \geq \frac{\beta}{1-\beta}\varepsilon > 0, \tag{26}$$

for any $\theta \in [1, \infty)$. Consider now the function

$$f(\theta) = \frac{e^{\Delta(\theta)}}{e^{\Delta(\theta)} - 1} - \theta.$$

We then have

$$f(1) = \frac{e^{\Delta(1)}}{e^{\Delta(1)} - 1} - 1 > 0$$

since $\Delta(1)$ is finite and 26 implies that

$$f\left(\bar{\theta}\right) = \frac{e^{\Delta\left(\bar{\theta}\right)}}{e^{\Delta\left(\bar{\theta}\right)} - 1} - \bar{\theta} \leq 0,$$

where

$$\bar{\theta} \equiv \frac{e^{\frac{\beta}{1-\beta}\varepsilon}}{e^{\frac{\beta}{1-\beta}\varepsilon} - 1}.$$

It can also be shown that $f(.)$ is continuous. Therefore, $f(.)$ must have a zero in $(1, \bar{\theta}]$, completing the proof. ■

**Proof of Proposition 6.** Proceeding as in the proof of Lemma 3 we can show that $\Phi(x, \Delta)$ is quasi-concave in $x$, on $(-\infty, \underline{a}/\theta]$. Assumption 3 ensures that $\Phi(x, \Delta)$ is continuously differentiable in $x$, on $(-\infty, \overline{a}/\theta)$, therefore, the optimum $x$ has to be strictly greater than $\underline{a}/\theta$ if $\partial\Phi(x, \Delta)/\partial x|_{x=\underline{a}/\theta} > 0$. To prove this inequality notice that (24) gives

$$\left.\frac{\partial\Phi(x, \Delta)}{\partial x}\right|_{x=\underline{a}/\theta} = -\int_{\underline{a}}^{\overline{a}} \frac{1}{a - \underline{a}/\theta} dF(a) + \frac{\delta}{1 - \delta\underline{a}/\theta} \geq -\int_{\underline{a}}^{\overline{a}} \frac{1}{a - \underline{a}} dF(a) + \delta > 0,$$

where the first inequality follows since $\theta \in (1, \infty)$ and the second inequality follows from (22). ■