When Does Determinacy Imply Expectational Stability?

James Bullard*
Federal Reserve Bank of St. Louis

Stefano Eusepi†
Federal Reserve Bank of New York

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Abstract

We study the connections between determinacy of rational expectations equilibrium, and expectational stability or learnability of that equilibrium, in a relatively general New Keynesian model. Adoption of policies that induce both determinacy and learnability of equilibrium has been considered fundamental to successful policy in the literature. We ask what types of economic assumptions drive differences in the necessary and sufficient conditions for the two criteria. Our framework is sufficiently flexible to encompass lags in information, alternative pricing assumptions, a cost channel for monetary policy, and either Euler equation or infinite horizon approaches to learning. We are able to isolate conditions under which determinacy does and does not imply learnability, and also conditions under which long horizon forecasts make a clear difference to conclusions about expectational stability. The sharpest result is that informational delays break equivalence connections between determinacy and learnability.

Keywords: Long horizon expectations, multiple equilibria, learnability of equilibrium, E-stability.

JEL codes: E3, E4, D8.

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*Email: bullard@stls.frb.org. Any views expressed are those of the authors and do not necessarily reflect the views of the Federal Reserve Bank of St. Louis, the Federal Reserve Bank of New York, or the Federal Reserve System.

†Email: stefano.eusepi@ny.frb.org

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1 Introduction

1.1 Overview

In the recent literature on monetary economics and optimal monetary policy, there has been considerable interest in the promotion of policies that can deliver both a determinate, or unique, rational expectations equilibrium, and also learnability, or stability, of that equilibrium. Policies that generate equilibria which are both unique and stable are viewed as preferable to those that might allow either a multiplicity of equilibria, or instability, or both. The determinacy criterion is defined according to Blanchard and Kahn (1980), and the learnability, or expectational stability,[1] criterion is defined according to Evans and Honkapohja (2001).

In some parts of this literature, there appears to be a tight connection between determinacy and learnability. The discussion in Woodford (2003a, 2003b) as well as in Bullard and Mitra (2002) highlights cases where the conditions for determinacy of equilibrium are the same as the conditions for expectational stability. Woodford (2003a, p. 1180) states, "Thus both criteria ... amount in this case to a property of the eigenvalues of [a matrix] A, and the conditions required for satisfaction of both criteria are related, though not identical." McCallum (2007a) presents results suggesting that, for a large class of models of interest for macroeconomists, expressed as linear systems of expectational difference equations, determinacy implies E-stability.[2] Yet, an examination of the "general linear model" case in Evans and Honkapohja (2001) makes it plain that determinacy does not imply learnability. And, all of the above authors stress that there is no general presumption that determinacy implies learnability.[3] Still, there seems to

[1]We use the terms *expectational stability*, *E-stability*, and *learnability* interchangably in this paper. The connections between the expectational stability condition and local convergence of systems under real time recursive learning are discussed extensively in Evans and Honkapohja (2001).

[2]Recently, McCallum (2007b) has argued that "well-formulated" models, which meet a certain technical condition, the learnable rational expectations equilibrium is always unique, even when the rational expectations equilibrium itself may not be unique.

[3]Bullard and Mitra (2002) in particular provide one example where the conditions re-

be a close relationship between the criteria in many applications, and this is a puzzle we would like to help resolve. In particular, we would like to better understand the nature of the relationship between determinacy and learnability in economic terms.

Meanwhile, Preston (2005, 2006) and Woodford (2003b) have suggested that introducing learning into certain microfounded environments creates situations where long-horizon forecasts can matter for learning dynamics.[4] In the *infinite horizon approach* to learning,[5] the fundamental equations describing the evolution of the state of the economy are altered under learning relative to the case under rational expectations. Appropriately taking those changes in the dynamic system into account can, but does not always, change the conclusions one would draw concerning the learnability of a particular rational expectations equilibrium. For the baseline Bullard and Mitra (2002) findings, the infinite horizon approach yields the same conclusions, and Preston (2005, p. 86) notes, "That these results concur with the results [under the infinite horizon approach] is not necessarily to be expected." Indeed, in subsequent work, such as Preston (2006), the learnability conditions differ under the two approaches. We would like to understand more about the economics of the relationship between determinacy and learnability in the infinite horizon learning environment as well.

The theme of this paper, then, is to try to understand the sense in which meaningful economic additions to a standard and widely-studied macroeconomic model might cause learnability and determinacy to be governed by differing conditions. We allow for both Euler equation and infinite horizon approaches to learning. It is already known that determinacy does not imply E-stability in general; what is not well understood is the nature of economic models in which the two sets of conditions diverge. To understand the economic aspects we need to perform our analysis inside of a

quired for the two criteria do not coincide.

[4] Also see Marcet and Sargent (1989).

[5] Contrasting with the *one-step-ahead* (a.k.a, *Euler equation*) approach to learning standard in Evans and Honkapohja (2001).

known model framework, which is why we commit to using a fairly general version of the New Keynesian (NK) macroeconomic model. This also facilitates our inclusion of the infinite horizon approach, which is inherently an issue that depends on the microfoundations.

1.2 What we do

We study a generalized NK macroeconomic model which includes certain features which will help us delineate between the conditions for determinacy and those for learnability in a variety of circumstances. We allow for informational delays, as explained below, and also we allow for the cost channel of monetary policy as analyzed recently by Ravenna and Walsh (2006). Another feature of the model is that we nest two possible pricing models, which depend on the information available to producers when setting their prices. With the model environment in place, we turn to calculating determinacy and expectational stability conditions. We characterize situations in which determinacy implies E-stability and situations in which it does not, under both Euler equation and infinite horizon learning.

1.3 Main findings

We first present two propositions. Proposition 1 isolates conditions under which the Euler equation and infinite horizon approaches to learning yield the same expectational stability conditions. Proposition 2 provides conditions under which determinacy implies expectational stability. Armed with these two results, we proceed to illustrate four example economies, each of which are special cases of the general framework, and each of which illustrates a different aspect of the determinacy-learnability nexus.

Proposition 3 restricts attention to one specific pricing assumption (which includes some elements of rational expectations) and assumes that the participants in the economy know the monetary policy rule in place. In this baseline case, the expectational stability conditions for the Euler equation approach and the infinite horizon approach are identical, independently of the monetary policy rule and independently of any information lags. If in addition there are no information delays, then determinacy implies E-stability. This is an important result, as it means that in a very interesting baseline class of models, determinacy implies E-stability and the nature of the learning analysis does not matter for E-stability conditions.

Proposition 4 constructs a case in which differences between Euler equation learning and infinite horizon learning become apparent. To obtain this case, we consider a model with no cost channel and no informational lags. In this environment, if the policy rule includes expectations of variables for the next period, then determinacy implies expectational stability in the case of Euler equation learning, but not in the case of infinite horizon learning. This is the case is considered in Preston (2007). This class of models, again certainly quite plausible in the context of current research, provides one clear cut case where the nature of the learning analysis will make a difference for the conclusions one draws from the E-stability analysis of unique equilibria. The key aspect of this example is that the agents must independently forecast the interest rate.

In Proposition 5, we do not require the participants in the economy to independently forecast the interest rate—instead, they know the interest rate rule in place. We assume that the cost channel is operative, that there are no informational delays, that the policy rule is forward-looking, and in addition that the policymaker is a strict inflation targeter. In this situation determinacy again implies expectational stability, regardless of the whether one adopts the Euler equation approach or the infinite horizon approach to the analysis of learning. In the indeterminate region, however, there can still be differences between the two approaches to learning. Under certain conditions made explicit in the statement of the proposition, the indeterminate (MSV) equilibria are always expectationally unstable under Euler equation learning but may be expectationally stable under infinite horizon learning. Hence, the E-stability and determinacy criteria select different classes of equilibria.

So far, we have not said too much about the role of informational delays. In the second class of models analyzed in Bullard and Mitra (2002), determinacy did not imply E-stability, and this case involved a certain information lag. Propositions 6 and 7 establish that if the policy rule calls for the monetary authority to react to contemporaneous information, then determinacy implies E-stability when there are no information delays, but determinacy does not imply E-stability when there are information delays. Furthermore, Euler and infinite horizon learning yield different stability conditions. This sharp result suggests that for models that have some type of informational delay, a wedge will be driven between the determinacy conditions and the learnability conditions. Since information delays of some type are probably the most realistic case, this is the most general conclusion of the paper.

1.4 Organization

In the next section we develop the New Keynesian model we wish to study, which is standard but which also encompasses a variety of features that have been discussed in the literature. We then turn to calculations indicating when determinacy and learnability conditions will coincide, and when they will not. We summarize our findings in the concluding section; there are also two appendices to the paper which contain some of the development of the model.

2 Environment

2.1 Informational delays

A key aspect of our approach is that we allow for differing information sets to be available to agents at the time expectations are formed and decisions are made as we consider the microfoundations of the model. This means that date t expectations may be formed either with information available at date t, or with information available at date $t-1$, and that the equations we derive to represent the equilibrium dynamics will be equally valid in either case. To accomplish this, we use the operator $\hat{E}_{t-\ell}$, where $\ell \in \{0,1\}$, which we think of as an information lag or a "delay" when $\ell = 1$, and which is just the standard $\hat{E}_t$ when $\ell = 0$. The hat indicates that expectations may not initially be rational.

Many models in macroeconomics assume rational expectations and have $t-$dating of the expectations operator, E_t, and this has come to be thought of as the standard case. For many purposes under the rational expectations assumption it may not be too important, although it is rarely analyzed in the literature. In an environment with learning, the dating of the expectations operator may be more critical, and Evans and Honkapohja (2001) have suggested that the $t-1$ dating of the expectations operator may be more natural when learning is explicitly considered. This is because the general equilibrium has date t quantities and prices being determined at date t, but the agents are supposed to be forming expectations using the date $t-1$ data in their recursive algorithms. This type of simultaneity is a constant companion in standard economic theory but does not make very much sense if we think more explicitly about the microfoundations of how the expectations are being formed. Even under the rational expectations assumption, the case for information lags has been made. McCallum (1999) has argued that actual policymakers rarely have contemporaneous information available when making decisions, especially concerning variables like GDP, and that operational interest rate rules would involve a reaction to readings on endogenous variables at least one period in the past. In an influential paper, Rotemberg and Woodford (1999) estimated a DSGE model with rational expectations but dated their expectations operator at $t-2$ to provide a better fit to the data.

2.2 Households

There is a continuum of household that consume, save and supply labor in a homogeneous labor market. A household i maximizes utility over an infinite horizon

$$\hat{E}^i_{t-\ell} \sum_{T=t}^{\infty} \beta^{T-t} \left[U\left(C^i_T\right) - V\left(h^i_T\right) \right], \tag{1}$$

where the parameter $\beta \in (0,1)$ is the discount factor and the period utility functions $U(C_t)$ and $V(h_t)$ have standard properties. For tractability we assume that in the case of delays, $\ell = 1$, *only consumption decisions are predetermined*. This means that labor supply decisions and saving decisions are taken with date t information, independently of the value of ℓ. Consumption C_t is given as a composite of all goods produced in the economy,

$$C^i_t \equiv \left[\int_0^1 c^i_t(j)\, dj \right]^{\frac{\theta}{\theta-1}} \tag{2}$$

and has an associated price index

$$P_t \equiv \left[\int_0^1 p_t(j)^{1-\theta}\, dj \right]^{\frac{1}{1-\theta}}. \tag{3}$$

Households hold money that can be used to purchase the consumption good and can also be deposited at a financial intermediary. The budget constraint of household i is given by

$$M^i_{t+1} + P_t C^i_t \leq M^i_t - D^i_t + (1+i_t) D^i_t + W_t h^i_t + P_t \int \phi^f_t(j) dj + P_t \phi^M, \tag{4}$$

where M^i_t denotes money holdings at the beginning of the period and D_t denotes the amount on deposit at the financial intermediary, which pays the gross nominal interest rate $(1+i_t)$.[6] The variable W_t is the economy-wide nominal wage determined in a perfectly competitive labor market, and $\phi_t(j)$ and ϕ^M denote real profits from firms and the financial intermediary. Each agent i is assumed to have an equal share of each firm j. These assumptions guarantee that the households income profiles are identical, even in the case of incomplete markets. The household also faces the cash-in-advance constraint

$$P_t C^i_t \leq M^i_t + W_t h^i_t - D_t, \tag{5}$$

[6] We assume that money bears no interest.

which takes this form because households receive their wages at the beginning of the period.

The first order condition for consumption yields

$$U_c\left(C_t^i\right) = \beta \hat{E}_{t-\ell}\left[\frac{(1+i_t)\,U_c\left(C_{t+1}^i\right)}{\Pi_{t+1}}\right]. \tag{6}$$

Log-linearizing equation (6) and solving backward we obtain

$$\hat{C}_t^i = \hat{E}_{t-\ell}\hat{C}_T^i + \sigma\hat{E}_{t-\ell}^i\sum_{T=t}^{T-1}(i_t - \pi_{t+1}). \tag{7}$$

Assuming initial net wealth equal to zero, the intertemporal budget constraint of the household can be expressed in terms of log-deviations from the deterministic steady state,

$$\hat{E}_{t-\ell}^i\sum_{T=t}^{\infty}\beta^{T-t}\hat{C}_T^i = \hat{E}_{t-\ell}^i\sum_{T=t}^{\infty}\beta^{T-t}\hat{Y}_T^i, \tag{8}$$

and substituting for $\hat{E}_{t-\ell}^i\hat{C}_T$ yields the optimal consumption rule for each agent i,

$$\hat{C}_t^i = \hat{E}_{t-\ell}^i\sum_{T=t}^{\infty}\beta^{T-t}\left[(1-\beta)\hat{Y}_T^i - \beta\sigma(i_T - \pi_{T+1})\right], \tag{9}$$

where $\hat{Y}_t$ denotes real income.[7] In an equilibrium with a positive nominal interest rate, the cash in advance constraint will bind as

$$P_tC_t^i = M_t^i + W_th_t^i - D_t. \tag{10}$$

2.3 Firms

2.3.1 Nature of the firms' problem

Each firm j produces a differentiated good and has market power. They face a demand for their output given by

$$Y_t^D(j) = \left(\frac{p_t(j)}{P_t}\right)^{-\theta}C_t \tag{11}$$

[7]Following Preston (2002), for simplicity we assume that each agent forecasts her entire income and not its single components. More details are available in the Appendix.

where C_t denotes aggregate consumption. Labor is the only input in the production function, which is assumed to be linear in order to simplify the analysis,

$$Y_t(j) = A_t h_t(j). \tag{12}$$

The labor market is viewed as economywide and perfectly competitive. Firms are subject to a cash constraint, which gives rise to a *cost channel* for monetary policy.[8] They have to anticipate their wage bill to the workers and therefore have to borrow funds from the financial intermediary in the amount corresponding to $h_t(j) W_t$.

We study a standard model of nominal pricing rigidities a là Calvo. Firms maximize their expected profits,[9]

$$\hat{E}_{t-\ell} \sum_{T=t}^{\infty} \alpha^{T-t} Q_{t,T} \phi_T(j), \tag{13}$$

where $Q_{t,T}$ is the stochastic discount factor and where real flow profits are

$$\phi_t(j) = \frac{p_t(j)}{P_t} Y_t^D(j) - (1+i_t) \frac{w_t}{P_t} h_t(j).$$

Financial intermediaries operate in a perfectly competitive markets for funds. Therefore the cost of borrowing for each firm is

$$i_t \frac{W_t}{P_t} h_t(j). \tag{14}$$

Finally, financial intermediaries make a profit of

$$P_t \phi_t^M = (1+i_t) \int W_t h(j) dj - (1+i_t) \int D_t^i di = (1+i_t) T_t \tag{15}$$

where T_t is a cash injection from the government to be defined below.

[8]For more detailed discussion of the cost channel of monetary policy, see Ravenna and Walsh (2006), Schmitt-Grohe and Uribe (2006), and Woodford (2003b).

[9]Discounted with the average household stochastic discount factor, which turns out to be the correct way of discounting the future stream of profits because agents are identical.

2.3.2 Optimal pricing

In order to simplify the analysis we assume that, independently of informational delays, firms can observe the current aggregate price level when deciding their optimal price.[10] When there are informational delays the optimal relative price depends on the expected current and future marginal cost. Similarly to households, firms choose their labor input (and the amount of funds to borrow) using current information—delays can occur only at the pricing stage. The first order condition for the optimal price becomes

$$\hat{E}_{t-\ell} \sum_{T=t}^{\infty} \alpha^{T-t} Q_{t,T} Y_T^D \left(P_T\right)^{\theta} \left[P_t^* \left(j\right) - \frac{\theta - 1}{\theta} P_T s_T\right] = 0, \tag{16}$$

where

$$s_t = \frac{w_t}{P_t A_t} \left(1 + i_t\right), \tag{17}$$

denotes the real marginal cost of production, which is a function of the real wage and the opportunity cost of holding cash. Log-linearization leads to

$$\hat{p}_t^*(j) = \hat{E}_{t-\ell} \sum_{T=t}^{\infty} \left(\alpha\beta\right)^{T-t} \left[\left(1 - \alpha\beta\right) \hat{s}_T + \alpha\beta\pi_{T+1}\right] \tag{18}$$

where $\hat{p}_t^* = \ln\left(P_t^* / P_t\right)$. Given homogeneous factor markets implies that each firm that chooses the optimal price will choose the same price $\hat{p}_t^*(j) = \hat{p}_t^*$. While Calvo pricing implies a distribution of prices across firms, nevertheless the model specification leads to a simple log-linear relation between the optimal price and inflation

$$\pi_t = \frac{1 - \alpha}{\alpha} \hat{p}_t^*. \tag{19}$$

[10]The instability results in this paper do not depend on this assumption. Eusepi and Preston (2007a) consider the case where firms no not observe the current price level when deciding their price, in a simpler model.

2.4 Marginal cost and the output gap

From the households' first order conditions, the labor supply is[11]

$$\frac{V_h\left(h_t^i\right)}{u_c\left(C_t^i\right)} = \frac{w_t}{P_t}. \tag{20}$$

Log-linearizing (20) and (17) and combining them gives a relation between output and the real marginal cost as

$$\hat{s}_t = \left(\sigma^{-1} + \gamma\right)\hat{y}_t - (1+\gamma)\hat{a}_t + \hat{\imath}_t, \tag{21}$$

where γ is the inverse of the elasticity of the labor supply, which depends on the disutility of hours worked, and σ is the intertemporal elasticity of substitution in consumption. The marginal cost also depends on the nominal interest rate, depending on the wage bill that needs to be anticipated by the firm.

Next, we define $\hat{y}_t^n$ as the natural level of output, that is, the equilibrium level of output under flexible prices and no information delays, which can be shown to be

$$y_t^n = \frac{(1+\gamma)}{(\sigma^{-1}+\gamma)}\hat{a}_t - \left(\sigma^{-1}+\gamma\right)^{-1}\hat{\imath}_t^n. \tag{22}$$

Using this expression we can rearrange the marginal cost equation to obtain

$$\hat{s}_t = \left(\sigma^{-1}+\gamma\right)\left(y_t - y_t^n\right) + \left(\hat{\imath}_t - \hat{\imath}_t^n\right). \tag{23}$$

We assume that in the flexible price equilibrium the nominal interest rate is set to zero, eliminating the labor supply distortion, as in Ravenna and Walsh (2006). Hence in the remainder of the paper we set $\hat{\imath}_t^n = 0$.

2.5 Equilibrium

Equilibrium in the goods market requires

$$y_t(j) = \left(\frac{p_t(j)}{P_t}\right)^{-\theta} C_t \tag{24}$$

[11] We stress the assumption that households take their labor supply decisions using current information about real wages.

for every j, and equilibrium in the labor market gives

$$\int h^i di = h_t = \int h(j)dj, \tag{25}$$

where $h_t^i = h_t$. We also have a market clearing condition for loanable funds of financial intermediaries,

$$W_t \int h(j)dj = \int D_t^i di + T_t, \tag{26}$$

where T_t denotes a cash injection from the government which is equal to

$$T_t = M_{t+1} - M_t. \tag{27}$$

2.6 Monetary and fiscal policies

We assume that the fiscal authority operates a zero debt, zero spending fiscal policy. Monetary policy is described by a simple Taylor-type rule of the (log-linear) form

$$\hat{\imath}_t = \hat{\imath}_t^* + \phi \left[\hat{E}_{t-\ell}\pi_{t+j} + \bar{\lambda}\hat{E}_{t-\ell}x_{t+j} \right] \tag{28}$$

for $j = 0, 1$, where the monetary authority reacts to *private sector* expectations in the case of information delays or in the case of a forward-looking policy rule. For generality, the intercept $\hat{\imath}_t^*$ is time-varying.[12] The term in brackets defines the linear combination between expected output and inflation corresponding to the optimal discretionary targeting rule under rational expectations, where the coefficient $\bar{\lambda}$ on the output gap can be interpreted as a function of the central bank's relative preference for output stabilization.[13]

[12]Under optimal discretionary policy $\hat{\imath}_t^*$ is a linear combination of the underlying shocks. Alternatively, it can be interpreted as a monetary policy shock. The specific form of the intercept does not affect the stability and determinacy results.

[13]Details are provided in the appendix.

2.7 Evolution of aggregate variables

2.7.1 The output gap

Combining the consumption decision rule (9), the equilibrium conditions, and the definition of natural output we obtain

$$x_t = \hat{E}_{t-\ell} \sum_{T=t}^{\infty} \beta^{T-t} \left[(1-\beta) x_T - \sigma\beta(\hat{\imath}_T - \pi_{T+1}) + \beta\sigma r_T^n \right], \tag{29}$$

where x_t is the output gap, i_t is the deviation of the one-period nominal interest rate from the value consistent with inflation at target and output at potential, π_t is the deviation of inflation from target, and r_t^n is a disturbance term[14] describing the natural rate of interest in the economy. We assume this term is governed by the stochastic process

$$r_t^n = \rho r_{t-1}^n + \epsilon_t$$

where $0 < \rho < 1$ and $\epsilon_t \sim N(0, \sigma_\epsilon)$. We normalize the inflation target to zero. The parameter β is the discount factor of the representative household and, again, the parameter σ is the intertemporal elasticity of substitution in consumption. Quasi-differencing this expression we obtain the familiar forward-looking IS curve, including one extra term

$$x_t = (1-\beta)\hat{E}_{t-\ell} x_t - \beta\sigma \hat{E}_{t-\ell} \left(i_t + \pi_{t+1} - \hat{r}_t^n \right) + \beta \hat{E}_{t-\ell} x_{t+1}. \tag{30}$$

2.7.2 Inflation

We can express the inflation equation as

$$\pi_t = \hat{E}_{t-\ell} \sum_{T=t}^{\infty} (\alpha\beta)^{T-t} \left\{ \xi \left[(\sigma^{-1} + \gamma) x_T + \hat{\imath}_T \right] + \beta(1-\alpha)\pi_{T+1} \right\}. \tag{31}$$

where $\xi = \frac{(1-\alpha\beta)(1-\alpha)}{\alpha} > 0$. Quasi-differencing and imposing rational expectations we obtain

$$\pi_t = \beta \hat{E}_{t-\ell} \pi_{t+1} + \xi \hat{E}_{t-\ell} \left[\left(\sigma^{-1} + \gamma \right) x_t + \hat{\imath}_t \right], \tag{32}$$

[14] This is a function of the only exogenous shock A_t.

which depends only on one-period-ahead forecasts. We also consider a third specification, which we define as the long horizon (LH) pricing specification.[15] This is obtained by iterating forward equation (32) and imposing a transversality condition, which yields

$$\pi_t = \hat{E}_{t-\ell} \sum_{T=t}^{\infty} \beta^{T-t} \left\{ \xi \left[(\sigma^{-1} + \gamma) x_T + \hat{\imath}_T \right] \right\}. \tag{33}$$

We stress that this formulation also *uses rational expectations* since it is derived from (32). Firms do not have to forecast the expected evolution of prices when setting prices. We can express the inflation equation in a nested form, which includes both the Calvo and the LH specifications, as

$$\pi_t = \hat{E}_{t-\ell} \sum_{T=t}^{\infty} (\vartheta\beta)^{T-t} \left\{ \xi \left[(\sigma^{-1} + \gamma) x_T + \hat{\imath}_T \right] + \beta(1-\vartheta)\pi_{T+1} \right\}. \tag{34}$$

Here we define a new parameter, $\vartheta \in \{\alpha, 1\}$. If $\vartheta = \alpha$, we obtain Calvo pricing, and if $\vartheta = 1$ we obtain the LH pricing. By quasi-differencing (34) we obtain (32) independently of the pricing assumptions (for any admissable ϑ). Hence, Euler learning delivers the same result, independently of the pricing assumptions. We stress that, *crucially, the main difference between Calvo and LH pricing is that firms are effectively assumed to discount the future at different rates*. With Calvo pricing firms discount the future more heavily as indicated by the effective discount factor $\vartheta\beta = \alpha\beta$ in equation (31) for that case, while with LH pricing $\vartheta\beta = \beta$.

In much of the remainder of the analysis we assume that the agents know the monetary policy rule, so that, knowing the relation between the nominal interest, the output gap and inflation, in order to take consumption and pricing decisions they only need to forecast the future evolution of inflation and output. There is one exception to this assumption which is discussed below.

[15]We adopt this terminology to keep the label for this pricing assumption, "long-horizon," distinct from the label for alternative approach to learning, "infinite-horizon."

2.8 Matrix notation

The general class of models we have described can be expressed in matrix notation as

$$A_0 Y_t = A_1 \hat{E}_{t-\ell} Y_t + \sum_{J=1}^{N} A_{2,J} \hat{E}_{t-\ell} \sum_{T=t}^{\infty} \beta_J^{T-t} Y_{T+1} + A_3 X_t, \tag{35}$$

where Y_t denotes a $n-$dimensional vector of endogenous variables, β_J denotes discount factors associated with different decision rules, X_t denotes a $k-$dimensional vector of shocks which are assumed to be $AR(1)$, with normally distributed disturbances, and all matrices A are conformable. Finally, the matrix H denotes a diagonal matrix containing the aucorrelation coefficients of the shocks, assumed to be between zero and one.

3 Learning dynamics

3.1 Local stability under infinite horizon decision rules

We begin our discussion of learning dynamics with the infinite horizon approach. In the analysis below we relate the quantities obtained under this approach to those obtained under Euler equation learning.

We begin with the assignment of a perceived law of motion, or PLM, which is consistent with the actual law of motion in the rational expectations equilibrium. The PLM is

$$Y_t = a + bX_{T-\ell}, \tag{36}$$

where a denotes the intercept vector and b denotes a $n \times k$ matrix of coefficients. The discounted infinite sum in equation (35) can then be expressed as

$$\begin{aligned} \hat{E}_{t-\ell} \sum_{T=t}^{\infty} \beta_J^{T-t} Y_{T+1} &= \sum_{T=t}^{\infty} \beta_J^{T-t} a + \sum_{T=t}^{\infty} b \beta_J^{T-t} H^{(T-t+1)} X_{T-\ell} && (37) \\ &= \frac{1}{1-\beta_J} a + b \left(1 - \beta_J H\right)^{-1} H X_{t-\ell}. && (38) \end{aligned}$$

This perceived law of motion induces an actual law of motion, or ALM, given by

$$Y_t = \tilde{A}_1 \left(a + bX_{t-\ell}\right) + \sum_{J=1}^{N} \tilde{A}_{2,J} \left(\frac{1}{1-\beta_J} a + b \left(1 - \beta_J H\right)^{-1} H X_{t-\ell} \right) + \tilde{A}_3 X_t, \quad (39)$$

where $\tilde{A}_i = A_0^{-1} A_i$ for $i = 1, 2,$ and 3. The associated ordinary differential equation is given by, for the intercept vector,

$$\dot{a} = T(a) - a \quad (40)$$

where

$$T(a) = \left[\tilde{A}_1 + \sum_{J=1}^{N} \tilde{A}_{2,J} \frac{1}{1-\beta_J} \right] a, \quad (41)$$

and for the coefficient matrix,

$$\dot{b} = T(b) - b, \quad (42)$$

where

$$T(b) = \tilde{A}_1 b + \sum_{J=1}^{N} \tilde{A}_{2,J} b \left(1 - \beta_J H\right)^{-1} H. \quad (43)$$

Under the infinite horizon approach to learning, the intercept matrix governing expectational stability can be written as

$$M^{IH}(a) = \tilde{A}_1 + \sum_{J=1}^{N} \tilde{A}_{2,J} \frac{1}{1-\beta_J} - I_n. \quad (44)$$

Considering the coefficients b, after vectorizing $T(b)$ we obtain

$$M^{IH}(b) = \left(I_k \otimes \tilde{A}_1\right) + \sum_{J=1}^{N} \left\{ \left[\left(I_k - \beta_J H\right)^{-1} H \right]' \otimes \tilde{A}_{2,J} \right\} - I_k \otimes I_n. \quad (45)$$

We stress that to obtain the case of contemporaneous expectations all that is needed is to re-define the matrix A_0 and set $\tilde{A}_1 = 0$. In the next section we use (45) to establish that E-Stability conditions need not be the same as determinacy conditions.

3.2 An equivalence result

We first consider the connections between infinite horizon (IH) and finite horizon (FH) (*a.k.a.* one-period-ahead or Euler equation learning). The following proposition states the *sufficient* condition to have equivalence in the E-stability conditions under different learning approaches.

Proposition 1 *Consider the model (35) with $J = 1$. Then finite horizon and infinite horizon learning approaches deliver the same expectational stability conditions.*

Proof. The model in matrix representation becomes

$$Y_t = \tilde{A}_1 \hat{E}_{t-\ell} Y_t + \tilde{A}_{2,1} \hat{E}_{t-\ell} \sum_{T=t}^{\infty} \beta^{T-t} Y_{T+1} + \tilde{A}_3 X_t \tag{46}$$

where it is assumed that all decision rules use the same[16] discount rate. We now proceed to quasi-difference (46). First we forward (46) one period and taking expectations

$$E_{t-\ell} Y_{t+1} = \tilde{A}_1 \hat{E}_{t-\ell} Y_{t+1} + \tilde{A}_{2,1} \hat{E}_{t-\ell} \sum_{T=t+1}^{\infty} \beta^{T-t-1} Y_{T+1} + \hat{E}_{t-\ell} \tilde{A}_3 X_{t+1}. \tag{47}$$

Second, we rewrite (46) as

$$Y_t = \tilde{A}_1 \hat{E}_{t-\ell} Y_t + \tilde{A}_{2,1} \hat{E}_{t-\ell} Y_{T+1} + \tilde{A}_{2,1} \hat{E}_{t-\ell} \sum_{T=t+1}^{\infty} \beta^{T-t} Y_{T+1} + \tilde{A}_3 X_t \tag{48}$$

and combining the two expressions we get,

$$\begin{aligned} Y_t = \tilde{A}_1 \hat{E}_{t-\ell} Y_t + \tilde{A}_{2,1} \hat{E}_{t-\ell} Y_{t+1} + \beta \hat{E}_{t-\ell} Y_{t+1} \\ - \beta \tilde{A}_1 \hat{E}_{t-\ell} Y_{t+1} + \tilde{A}_3 \left(H - I_k \right) X_t. \end{aligned} \tag{49}$$

Re-arranging yields

$$Y_t = \tilde{A}_1 \hat{E}_{t-\ell} Y_t + \left[\tilde{A}_{2,1} + \beta \left(I_n - \tilde{A}_1 \right) \right] \hat{E}_{t-\ell} Y_{t+1} + \tilde{A}_3 \left(H - I_k \right) X_t, \tag{50}$$

[16] This implies that every row in $\tilde{A}_{2,1}$ must constain at least one non-zero element. Otherwise, the corresponding equation would imply $\beta = 0$.

which defines the finite horizon representation of the model. E-stability can be evaluated following Evans and Honkapohja (2001), which gives the following ALM maps

$$
\begin{aligned}
T_{FH}(a) &= \left[\tilde{A}_1 + \tilde{A}_{2,1} + \beta\left(I_n - \tilde{A}_1\right)\right] a && (51)\\
&= \left[(1-\beta)\,\tilde{A}_1 + \tilde{A}_{2,1} + \beta I_n\right] a && (52)
\end{aligned}
$$

$$
\begin{aligned}
T_{FH}(b) &= \tilde{A}_1 b + \left[\tilde{A}_{2,1} + \beta\left(I_n - \tilde{A}_1\right)\right] bH && (53)\\
&= \tilde{A}_1 b - \tilde{A}_1 b\beta H + \tilde{A}_{2,1} bH + \beta bH && (54)
\end{aligned}
$$

and the following associated Jacobian matrices, for the constant

$$
\begin{aligned}
M^{FH}(a) &= (1-\beta)\tilde{A}_1 + \tilde{A}_{2,1} - (1-\beta)\,I_n && (55)\\
M^{FH}(a) &= (1-\beta)M^{IH} && (56)
\end{aligned}
$$

where the last equality follows from (44), imposing $J = 1$. For the disturbance coefficients we obtain (after vectorization)

$$
\begin{aligned}
M^{FH}(b) &= I_k \otimes \tilde{A}_1 - \beta H \otimes \tilde{A}_1 + \beta H \otimes I_n - I_k \otimes I_n && (57)\\
&= (I_k - \beta H) \otimes \tilde{A}_1 + H \otimes \tilde{A}_{2,1} - (I_k - \beta H) \otimes I_n && (58)\\
M^{FH}(b) &= (I_k - \beta H) \otimes M^{IH} && (59)
\end{aligned}
$$

where the first equality follows from the properties of the Kroneker product and the second equality follows from setting $J = 1$ in (45) and again using the properties of the Kroneker product (see for example Dhrymes (2000)). ∎

3.3 When determinacy implies E-stability

The next proposition states the sufficient conditions under which determinacy implies expectational stability.

Proposition 2 *Consider the model (35) with $J = 1$. If $\tilde{A}_1 = \mathbf{0}$ determinacy implies E-stability.*

Proof. By setting $\tilde{A}_1 = 0$ local determinacy depends on the eigenvalues of the matrix

$$M^D = \tilde{A}_{2,1} + \beta I_n. \tag{60}$$

In order to yield local determinacy all eigenvalues of M^D must be inside the unit circle. We have

$$M^D - I = M^{FH}(c) = M^{IH}(c) \tag{61}$$

where the second equality comes from Proposition 1. Hence, if the eigenvalues of M^D are inside the unit circle, E-Stability is verified. Concerning the Jacobian for the shock coefficients, we have

$$\begin{aligned} eig\left(M^{FH}(s)\right) &= eig\left(H \otimes \tilde{A}_{2,1} - (I_k - \beta H) \otimes I_n\right) & (62) \\ &= eig\left(H \otimes \tilde{A}_{2,1}\right) - eig\left((I_k - \beta H) \otimes I_n\right) & (63) \end{aligned}$$

which implies that $M^{FH}(s)$ has eigenvalues with real parts less than one if $M^{FH}(c)$ has real eigenvalues less than one. ■

4 Four economies

4.1 Baseline case

We now consider the monetary model described in Section 2. The first proposition states the conditions for determinacy to imply learnability.

Proposition 3 *Consider the model with long-horizon pricing ($\vartheta = 1$) and assume the agents understand the policy rule (28).*

1) The E-stability conditions under infinite horizon and finite horizon learning are identical, independently of the timing assumptions and the monetary policy rule.

2) In the case of no delays ($\ell = 0$), determinacy implies E-stability.

Proof. Set $\vartheta = 1$ and substitute the policy rule (28) for the nominal interest rate in both the consumption and the pricing decision rules. Then the model can be expressed in form (46). In the case of contemporaneous expectations set $A_0 = \left(I - \tilde{A}_1\right)^{-1}$. The result then follows by direct application of the first two propositions. ■

This result demonstrates that, in this *specific class* of models, checking for determinacy is sufficient to determine whether the equilibrium is locally expectationally stable under a process of recursive learning.

4.2 Interest rate forecasts and learnability

We now consider the case where agents also forecast the nominal interest rate, as in Preston (2006, 2007) and Eusepi and Preston (2007a,b). Here we consider the example in Preston (2007), where the policy rule considered is forward-looking. This model does not satisfy the conditions of the first two propositions, as we explain below.

Proposition 4 *Preston (2007). Consider the model with no cost channel, no delays ($\ell = 0$) and the forward-looking policy rule ($j = 1$). Then determinacy implies E-stability under finite horizon learning but not under infinite horizon learning.*

It is easy to see how this version of the model violates the conditions in Proposition 1. The model can be written in matrix notation as

$$A_0 \tilde{Y}_t = A_1 \hat{E}_{t-\ell} \tilde{Y}_t + \sum_{J=1}^{3} A_{2,J} \hat{E}_{t-\ell} \sum_{T=t}^{\infty} \beta_J^{T-t} \tilde{Y}_{T+1} + A_3 X_t. \tag{64}$$

where $\tilde{Y}_t = (x_t, \pi_t, \hat{\imath}_t)'$. Given that the policy rule does not respond to infinite horizon forecasts, the discount rate associated with the policy rule is $\beta_3 = 0$. In fact, Preston (2007) shows that E-Stability conditions are more stringent than the determinacy conditions, while Bullard and Mitra (2002) show that in the case of FH learning determinacy implies E-stability. This example illustrates that even in the simplest environment determinacy and E-Stability need not coincide.[17]

The next example shows how different discount factors can matter.

[17] Eusepi and Preston (2007b) also show that in an economy with non-zero government bonds with long average maturity determinacy does not imply learnability, even in the case where the policy rule is perfectly understood by the public.

4.3 Indeterminacy and long-horizon forecasts

Here, as in the rest of the paper, we assume that agents do not have to independently forecast the nominal interest rate. The following proposition shows that different versions of the model can have different implications in terms of E-stability, depending on the agent's decision rules and the learning approach. In order to simplify the analysis in this particular example we assume $\sigma = 1$ and $\gamma = 0$**.**

Proposition 5 *Consider the model with strict inflation targeting* $\left(\bar{\lambda} = 0\right)$, *no delays* $(\ell = 0)$ *, and the forward-looking policy rule* $(j = 1)$*. Then*

1) Determinacy implies E-stability.

Assume in addition that $\xi > 1 - \beta$*:*

2) Under FH learning and IH learning with $\vartheta = 1$ *indeterminate equilibria are E-unstable.*

3) Under IH learning with Calvo pricing ($\vartheta = \alpha$*), indeterminate equilibria can be E-stable.*

Proof. Llosa and Tuesta (2007) show that determinacy[18] obtains for this model if and only if

$$1 < \phi < \frac{1-\beta}{\xi}. \tag{65}$$

The E-stability conditions for infinite horizon learning imply the following restrictions on the parameters

$$\phi > 1 \tag{66}$$

and[19]

$$-1 - \frac{\xi(\phi - 1)}{(1-\beta)} + \frac{(\beta(1-\vartheta) + \xi\phi)}{(1-\vartheta\beta)} < 0. \tag{67}$$

First, simple algebra shows that for any $\vartheta \in (0, 1]$, inequality (65) implies the restrictions are verified, and hence determinacy implies expectational

[18] See Llosa and Tuesta (2007), Proposition 3, pp 11-12.

[19] The first retraction requires that the determinat of the relevant matrix is positive, while the second restriction requires that the trace is negative. Details about the matrix and the computation can be obtained by inspecting the file analytic_current.m.

stability. Second, imposing $\vartheta = 1$, it can be shown that $\xi > 1 - \beta$ is the condition to have expectational instability. From Proposition 1 it follows that under IH learning with the LH pricing rule and under FH learning $\xi > 1 - \beta$ implies instability. Third, in the case of $\vartheta = \alpha < 1$ it is possible to find parameter values for which the equilibrium is expectationally stable under the assumption $\xi > 1 - \beta$. ■

As an example, consider the standard parameter values $\beta = 0.99$, $\xi = 0.024$ and $\phi = 1.5$. Then REE are indeterminate and E-unstable in the case of FH learning and in the case of LH pricing, but REE are E-stable under IH learning with Calvo pricing. This example and the last example in the next section show that under learning, the specification of pricing behavior matters for stability analysis.

4.4 The role of information delays

We now consider the crucial role of information delays. We first consider the model with contemporaneous information and a standard Taylor rule and compare the results with the case of information delays. In the case of contemporaneous expectations we obtain the following proposition.

Proposition 6 *In the model with contemporaneous expectations* ($\ell = 0$, $j = 0$) *determinacy implies E-stability.*

Proof. *Part 1.* We first show the parameter restrictions that ensure expectational stability under infinite horizon learning. E-stability in this case depends on the eigenvalues of the matrix

$$M^{IH}(a) = (1-\beta)^{-1} \left(I_2 - \tilde{A}_1\right)^{-1} \tilde{A}_{2,1} + (1-\vartheta\beta)^{-1} \left(I_2 - \tilde{A}_1\right)^{-1} \tilde{A}_{2,2} - I_2. \tag{68}$$

where $\vartheta \in (0, 1]$. Local stability obtains if the trace of the matrix is negative and the determinant is positive. Consider the determinant: stability obtains if

$$\frac{[(\phi-1) + (\phi-1)\,\gamma\sigma]\,\xi + \sigma\phi\bar{\lambda}\,[(1-\beta)-\xi]}{\left(1+\phi\gamma\sigma\xi+\sigma\phi\bar{\lambda}\right)(\beta-1)(\vartheta\beta-1)} > 0. \tag{69}$$

If $(1-\beta)-\xi > 0$, this condition is satisfied for any other parameter values. If $(1-\beta)-\xi < 0$, the determinant is positive provided

$$0 \leq \bar{\lambda} < \bar{\lambda}^* = \frac{\phi-1}{\phi}\frac{\xi\left(\sigma^{-1}+\gamma\right)}{\xi-1+\beta}. \tag{70}$$

Now consider the trace. We can write it as

$$\frac{Tr(\vartheta,\beta)}{\left(1+\phi\gamma\sigma\xi+\sigma\phi\bar{\lambda}\right)(\beta-1)(\vartheta\beta-1)} \tag{71}$$

where

$$\begin{aligned} Tr(\vartheta,\beta) = -1 - \phi\xi - 2\sigma\phi\bar{\lambda} - \xi\gamma\sigma\vartheta\beta - \xi\vartheta\beta + \\ \phi\xi\vartheta\beta\gamma\sigma + \phi\xi\gamma\sigma\beta - \beta^2 + \sigma\phi\bar{\lambda}\vartheta\beta + \\ \left(-\sigma\phi\bar{\lambda}\beta^2\right) - \xi\vartheta\beta\bar{\lambda}\phi\sigma + \xi - 2\phi\xi\gamma\sigma + \\ \xi\phi\bar{\lambda}\sigma + 2\beta + 2\beta\sigma\phi\bar{\lambda} + \xi\gamma\sigma + \phi\xi\vartheta\beta. \end{aligned} \tag{72}$$

First, it is straightforward to show that

$$\begin{aligned} & Tr(1,\beta) = \\ & -\left[\xi\phi\gamma\sigma + (1-\beta) + (\phi-1)\,\xi + (\phi-1)\,\xi\gamma\sigma + ((2-\beta)-\xi)\,\sigma\phi\bar{\lambda}\right] < 0 \end{aligned} \tag{73}$$

provided the conditions for a positive determinant are satisfied. Second, consider

$$Tr(0,\beta) = -1 - \phi\xi\gamma\sigma\,(1-\beta) - 2\,(1-\beta)\,\sigma\phi\bar{\lambda} + \Psi\,(\beta) \tag{74}$$

where

$$\begin{aligned} \Psi\,(\beta) \equiv -\left(1+\sigma\phi\bar{\lambda}\right)\beta^2 + 2\beta + \xi\phi(1-\beta) \\ - \left[(\phi-1) + (\phi-1)\,\gamma\sigma\right]\xi + \xi\phi\bar{\lambda}\sigma. \end{aligned} \tag{75}$$

It is easy to verify that $\Psi\,(0) < 0$ and $\Psi\,(1) < 0$ (provided the conditions for a positive determinant are satisfied). This implies $\Psi\,(\beta) < 0$. Finally, given that $Tr(\vartheta,\beta)$ is linear in ϑ, we can conclude that $Tr(\vartheta,\beta) < 0\ \forall\vartheta \in$

$(0,1]$. The above results imply that both the Calvo and LH pricing model are consistent with E-Stability if and only if (70) is satisfied.

Part 2. Using Proposition 1, under the FH approach to learning we obtain the same stability conditions as under the LH pricing approach.

Part 3. Using Proposition 2, we conclude that local determinacy implies expectational stability. We stress that the reverse does not hold: indeterminate equilibria can be E-stable in this model.[20] ■

We now prove the second proposition concerning the model with delays.

Proposition 7 *In the model with delays:*

1) Local determinacy does not imply expectational stability.

2) Under IH learning, the E-stability conditions depend on the firms' pricing rule.

Proof. The determinant of $M^{IH}_{\ell=1}$ is

$$\frac{\beta([(\phi-1)+(\phi-1)\,\gamma\sigma]\,\xi+\sigma\phi\bar{\lambda}\,[(1-\beta)-\xi])}{(1-\beta)\,(1-\vartheta\beta)}>0 \tag{76}$$

provided (70) holds. The trace is

$$\frac{\beta-1+\xi\phi}{1-\vartheta\beta}-\frac{\beta\sigma\phi\bar{\lambda}}{1-\beta}, \tag{77}$$

and it is negative provided

$$\bar{\lambda}>\bar{\lambda}_{*}=\frac{1-\beta}{1-\vartheta\beta}\frac{[\xi\phi-1+\beta]}{\beta\sigma\phi} \tag{78}$$

which adds an extra restriction on the parameters with respect to the one that guarantees determinacy. The extra restriction does not vanish once we set $\vartheta=1$, but it is clear that the restriction is less stringent in this case. Also, from Proposition 1, inequality (78) holds also under the Euler approach

[20] A similar result is also documented by Bullard and Mitra (2002).

(equivalent to the case where $\vartheta = 1$). Considering the coefficients on the shocks, the trace is

$$-\frac{\beta(1+\sigma\phi\bar{\lambda}-\rho)}{(1-\rho\beta)}+\frac{(\xi\phi+\rho\beta-1)}{(1-\vartheta\beta\rho)} \tag{79}$$

which is negative if the condition for the constant is satisfied. The determinant is

$$-\frac{\beta(\xi\phi+\rho\beta-1+\sigma\phi\bar{\lambda}\rho\beta-\sigma\phi\bar{\lambda}-\xi\phi\rho}{(1-\rho\beta)(1-\vartheta\beta\rho)} \\ \frac{-\rho^2\beta+\rho-\xi\phi\gamma\sigma-\xi\phi+\xi\rho\gamma\sigma+\xi\rho+\xi\rho\phi\bar{\lambda}\sigma)}{(1-\rho\beta)(1-\vartheta\beta\rho)} \tag{80}$$

which can be simplified to

$$\beta\left[\xi\left(\phi-\rho\right)+\xi\gamma\sigma\left(\phi-\rho\right)+\sigma\phi\bar{\lambda}\left(1-\rho\beta-\xi\rho\right)\right] \\ +\beta\left(1-\rho\right)\left(1-\beta\rho-\xi\phi\right)>0 \tag{81}$$

provided the condition for the constant is verified. ■

4.5 An Example

We now consider a calibrated version of the simple monetary model, under differing assumptions concerning the pricing rule imposed on firms. We set parameters to values that are commonly used in the literature. In particular, we set $\beta = 0.99$, $\gamma = 0$ (infinitely elastic labor supply), $\sigma = 1$ and $\phi = 1.5$. We wish to show how determinacy and E-stability conditions change as we vary the degree of price rigidity. We measure the degree of price rigidity with α, the probability of not having an opportunity to set the price under Calvo pricing. We then calibrate ξ consistently with a given value of α.

The parameter $\bar{\lambda}$ governs the policymaker response to the output gap in the monetary policy rule. Figure 1 shows how $\bar{\lambda}^*$ from (70) and $\bar{\lambda}_*$ from (78) change as the degree of price rigidity changes. We plot both $\bar{\lambda}_*^{\vartheta=1}$, for the LH pricing case, and $\bar{\lambda}_*^{\vartheta=\alpha}$ for the Calvo pricing case. According to the above discussion, for a given α, expectational stability obtains for $\bar{\lambda}_* < \bar{\lambda} < \bar{\lambda}^*$.

The Figure shows that for the Calvo pricing case, the interval for $\bar{\lambda}$ consistent with determinacy and expectational stability is roughly constant as the degree of price flexibility increases, perhaps with some exception toward the right in the diagram. However, for the LH pricing case, the interval for $\bar{\lambda}$ consistent with determinacy and expectational stability is generally smaller, and there would be no interval at all for a sufficiently high degree of price stickiness, that is, toward the left in the diagram. The unlabelled black lines in the diagram indicate the additional area consistent with expectational stability under Calvo pricing.

5 Conclusions

We have studied a New Keynesian model generalized on certain dimensions in an attempt to delineate the differences between conditions for equilibrium determinacy and conditions for equilibrium learnability in terms of meaningful economic assumptions. One of the sharpest findings is that in models with informational lags, the connections between determinacy and learnability conditions are broken. In other situations, and certainly in some interesting baseline cases, we are able to show that determinacy does imply learnability and thus that in some practical settings, it is not necessary to analyze the two conditions separately. We are also able to illustrate some interesting cases where there are and are not differences between the Euler equation and infinite horizon approaches to learnability.

An important caveat to this analysis is that there are no natural lags in the NK framework as we have analyzed it here, such as those that might come from time-to-build technologies or related concepts. The baseline NK model is entirely forward-looking. Given that we have found that information lags are one important source of differences in conditions for determinacy versus learnability, one might suspect that in models with more realistic lag structures, conditions for determinacy will in general not be the same as conditions for learnability. We leave this as an interesting topic for future research.

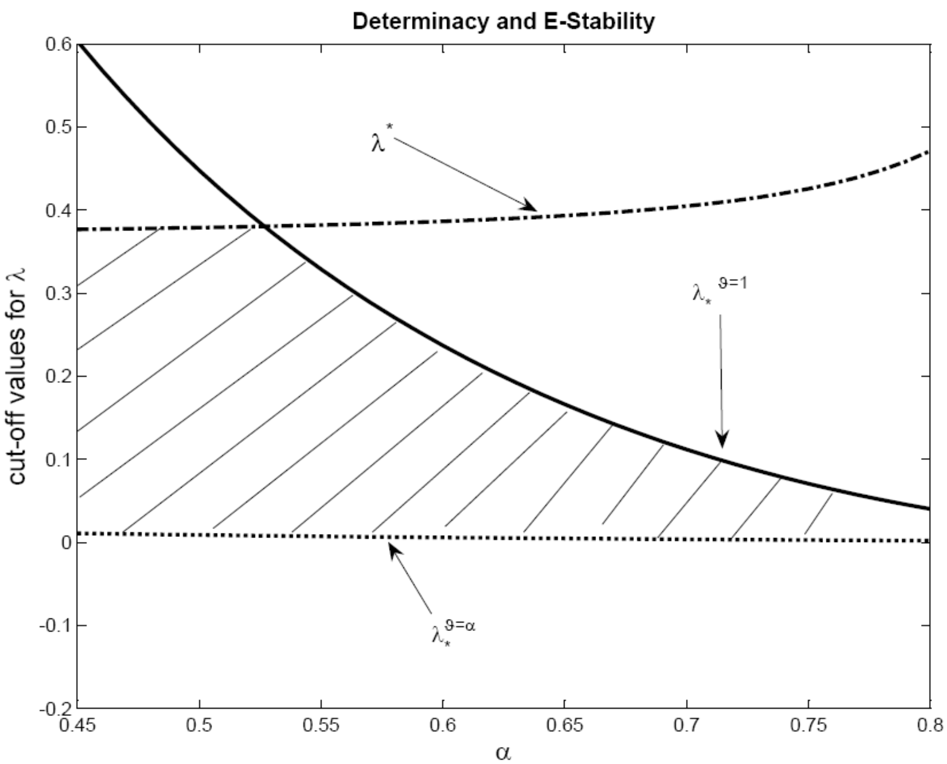

Figure 1: The relationship between determinacy and E-stability as a function of the degree of price stickiness α and the policymaker response to the output gap λ. Expectational stability requires a value of λ between λ_* and λ^*. Calvo pricing, $\vartheta = \alpha$, creates a larger region than LH pricing, $\vartheta = 1$.

References

[1] Blanchard, O., and C. Kahn. 1980. The solution of linear difference models under rational expectations. *Econometrica* 48(5): 1305-1311.

[2] Bullard, J., and K. Mitra. 2002. Learning about monetary policy rules. *Journal of Monetary Economics* 49(6): 1105-1129.

[3] Dhrymes, P. 2000. *Mathematics for Econometrics*. New York: Springer. Third Edition.

[4] Evans, G., and S. Honkapohja. 2001. *Learning and Expectations in Macroeconomics.* Princeton University Press.

[5] Evans, G., and S. Honkapohja. 2003. Expectations and the stability problem for optimal monetary policies. *Review of Economic Studies* 70(4): 807-824.

[6] Eusepi, S., and B. Preston. 2007a. Central bank communication and expectations stabilization. Manuscript, Federal Reserve Bank of New York and Columbia University.

[7] Eusepi, S., and B. Preston. 2007b. Expectations-driven business cycles and stabilization policy. Manuscript, Federal Reserve Bank of New York and Columbia University.

[8] Llosa, G., and V. Tuesta. 2007. E-stability of monetary policy when the cost channel matters. Manuscript, UCLA and Central Reserve Bank of Peru.

[9] Marcet, A., and T. Sargent. 1989. Convergence of least squares learning mechanisms in self-referential linear stochastic models. *Journal of Economic Theory* 48(2): 337-368.

[10] McCallum, B. 1999. Issues in the design of monetary policy rules. In: J. Taylor and M. Woodford, (Eds.), *Handbook of Macroeconomics*. Amsterdam: North-Holland.

[11] McCallum, B. 2007a. E-stability vis-a-vis determinacy results for a broad class of linear rational expectations models. *Journal of Economic Dynamics and Control* 31(4): 1376-1391.

[12] McCallum, B. 2007b. Determinacy and learnability in monetary policy analysis: additional results. Manuscript, Carnegie-Mellon.

[13] Preston, B. 2005. Learning about monetary policy rules when long-horizon expectations matter. *International Journal of Central Banking* 1(2): 81-126.

[14] Preston, B. 2006. Adaptive learning, forecast-based instrument rules and monetary policy. *Journal of Monetary Economics* 53(3): 507-535.

[15] Preston, B. 2007. Adaptive learning and the use of forecasts in monetary policy. Mimeo.

[16] Ravenna, F., and C. Walsh. 2006. Optimal monetary policy with the cost channel. *Journal of Monetary Economics* 53(2): 199-216.

[17] Schmitt-Grohe, S., and M. Uribe. 2006. Optimal simple and implementable monetary and fiscal rules. *Journal of Monetary Economics*, forthcoming.

[18] Woodford, M. 2003a. Comment on: Multiple-solution indeterminacies in monetary policy analysis. *Journal of Monetary Economics* 50(5): 1177-1188.

[19] Woodford, M. 2003b. *Interest and Prices: Foundations of a Theory of Monetary Policy*. Princeton Unversity Press.

APPENDICES

A Derivation of the consumption rule

Combining the household's budget constraint and the cash-in-advance constraint (which holds with equality at a positive nominal interest rate), we obtain

$$M_{t+1}^{i} \geq (1+i_t)\, M_t^{i} - (1+i_t)\, P_t C_t^{i} + (1+i_t)\, W_t h_t^{i} + \Phi_t^{i}, \tag{82}$$

where Φ_t^i denotes the household shares in the profits of firms and financial intermediaries. Solving forward and making use of the tranversality condition we obtain

$$M_t^{i} \geq \sum_{j=0}^{\infty} R_{t,t+j} \left[P_{t+j} C_{t+j} + \left(i_{t+j} / \left(1+i_{t+j}\right)\right) M_{t+j+1}^{i} - T_{t+j}^{i} - P_{t+j} Y_{t+j}^{i} \right], \tag{83}$$

which must hold in all states and dates, where

$$R_{t,t+j} = \prod_{k=1}^{j} (1+i_{t+k-1})^{-1} \tag{84}$$

and where $P_t Y_t^i = W_t h_t^i + P_t \int \left[\phi_t^f(j) dj + i_t W_t h_t \right] dj$. To obtain the expression above we use[21]

$$\frac{(1+i_t)\, W_t h_t^i + \Phi_t^i}{(1+i_t)} = \left[W_t h_t^i + P_t \frac{\int \phi_t^f(j) dj}{1+i_t} + \left(i_t / \left(1+i_t\right)\right) \int P_t C_t dj + T_t \right] - \left(i_t / \left(1+i_t\right)\right) M_{t+1}^i, \tag{85}$$

where the second and third terms in brackets take into account that wages are anticipated by households at the beginning of the period, while revenues are transferred at the end of the period. Furthermore, for simplicity

[21] The last expression can be simplified by using $P_t C_t = M_{t+1}$ in every period along with the condition that aggregate sales must be equal to aggregate consumption. Also notice that $C_t^i = C_t$ and $M_t^i = M_t$ in every period and for every household i.

we assume each household understands that the present value of government tranfers must be equal to the present value of government liabilities held net of seigniorage,[22] so that the intertemporal budget constraint simplifies to

$$0 = \sum_{j=0}^{\infty} R_{t,t+j} \left[P_{t+j} C_{t+j} - P_{t+j} Y_{t+j} \right], \tag{86}$$

where for simplicity we assume each household has zero initial net wealth. Log-linearizing the intertemporal budget constraint we obtain

$$0 = \sum_{j=0}^{\infty} \beta^t \hat{C}_{t+j} - \sum_{j=0}^{\infty} \beta^t \hat{Y}_{t+j}. \tag{87}$$

Linearization of the Euler equation yields

$$\hat{C}_t = \hat{E}_{t-\ell} \hat{C}_{t+1} - \sigma \hat{E}_{t-\ell} (\hat{\imath}_t - \pi_{t+1}). \tag{88}$$

Following the same logic as above

$$\hat{C}_t = \hat{E}_{t-\ell} \hat{C}_T + \sigma \hat{E}_{t-\ell} \sum_{T=t}^{T-1} (i_t - \pi_{t+1}). \tag{89}$$

Taking the expectation of the budget constraint yields

$$\hat{E}_{t-\ell} \sum_{T=t}^{\infty} \beta^{T-t} \hat{C}_T = \hat{E}_{t-\ell} \sum_{T=t}^{\infty} \beta^{T-t} \hat{Y}_T, \tag{90}$$

and substituting for $E_{t-1} \hat{C}_T$ we obtain

$$\hat{E}_{t-\ell} \sum_{T=t}^{\infty} \beta^{T-t} \left[\hat{C}_t - \sigma \sum_{T=t}^{T-1} (i_t - \pi_{t+1}) \right] = \hat{E}_{t-\ell} \sum_{T=t}^{\infty} \beta^{T-t} \hat{Y}_T \tag{91}$$

which implies

$$\hat{C}_t = \hat{E}_{t-\ell} \sum_{T=t}^{\infty} \beta^{T-t} \left[(1-\beta) \hat{Y}_T - \beta \sigma (i_T - \pi_{T+1}) \right]. \tag{92}$$

[22] Eusepi and Preston (2007b) relax this assumption and study the implications of learning when agents have to forecast fiscal variables.

B Optimal policy under discretion

In this appendix we show that the monetary policy rule in the main text is of the same form as if the policymaker was pursuing a certain optimal policy. We consider optimal policy under *rational expectations*,[23] meaning that the central bank is assuming rational expectations of the private sector when it is deciding upon an optimal policy. The behavioral equations for output gap and inflation can be reduced to

$$x_t = -\sigma E_{t-\ell} \left(\hat{\imath}_t - \pi_{t+1}\right) + E_{t-\ell} x_{t+1} + r_t^n \tag{93}$$

and

$$\pi_t = \beta E_{t-\ell} \pi_{t+1} + E_{t-\ell} \xi \left(\left(\sigma^{-1} + \gamma \right) x_t + \hat{\imath}_t \right) \tag{94}$$

where $\xi = \frac{(1-\alpha\beta)(1-\alpha)}{\alpha}$. The central bank maximizes a quadratic loss function in inflation and output gap

$$L = \pi_t^2 + \lambda x_t^2 \tag{95}$$

which is consistent with the model's microfoundations. We consider the general case where λ is arbitrary.[24] The central bank chooses optimal policy to maximize

$$E_{t-\ell} \sum_{t=0}^{\infty} \beta^t L_t \tag{96}$$

$$E_{t-\ell} \sum_{T=t}^{\infty} \beta^t \left(\pi_T^2 + \lambda x_T^2\right) \tag{97}$$

subject to (93) and (94). The first order conditions are, for x_t:

$$-\lambda x_t - \xi \left(\sigma^{-1} + \gamma \right) E_{t-\ell} \psi_{2,t} + E_{t-\ell} \psi_{1,t} = 0, \tag{98}$$

for π:

$$-\pi_t + E_{t-\ell} \psi_{2,t} = 0, \tag{99}$$

[23]We could do this under learning but it would complicate the analysis a lot without making any difference for the issues we are discussing.

[24]This because we are going to show that the instability result is not only obtained in the case where the central bank maximizes the representative consumer's welfare function.

and for i_t:

$$-\xi\zeta E_{t-\ell}\psi_{2,t}+\sigma E_{t-\ell}\psi_{1,t}=0. \tag{100}$$

Eliminating the multipliers, we obtain two conditions

$$-\lambda E_{t-\ell}x_t-\xi\left(1+\gamma\right)E_{t-\ell}\pi_t+\xi E_{t-\ell}\pi_t = 0, \tag{101}$$

$$\bar{\lambda}E_{t-\ell}x_t+E_{t-\ell}\pi_t = 0, \tag{102}$$

where $\bar{\lambda}=\frac{\lambda}{\xi\gamma}$. The central bank is responding to private sector forecasts under the assumption that the private sector forecasts are rational. This policy rule is of the same form as the one employed in the main text, under the assumption that the targeting rule is implemented using a Taylor-type rule.