INTERGENERATIONAL EQUITY AND THE DISCOUNT RATE FOR COST-BENEFIT ANALYSIS

JEAN-FRANÇOIS MERTENS† AND ANNA RUBINCHIK‡

Abstract. We develop a simple method to evaluate small policy changes affecting several generations, by reducing the dynamic problem to a static one. A necessary condition is time-invariance, which is satisfied by any common solution concept in an overlapping generations model with exogenous growth. The method is applied to derive the discount rate for cost-benefit analysis under two different utilitarian welfare functions: traditional and relative. It is only under relative utilitarianism that the discount rate is well-defined for a heterogeneous society, is corroborated by an independent argument on the value of human life, and equals the growth rate of per capita consumption, thus falling in the range suggested by the U.S. Office of Management and Budget.

Date: February 15, 2008.

2000 *Mathematics Subject Classification.* 91B14, 91B62

J.E.L. *Classification numbers.* D31, D61, D63, E60, H43.

Key words and phrases. Social Welfare Function, Social Welfare Functional, Overlapping Generations, Exogenous Growth, Policy Reform, Intergenerational Fairness, Cost-Benefit Analysis, Social Discount Rate, Utilitarianism, Relative Utilitarianism.

We would like to thank for its hospitality the Center for Rationality in Jerusalem, and for their comments K. Arrow, Cl. d'Aspremont, Ch. de Bartolomé, G. Ponthiere, R. M. Samaniego, and participants of PET'05, "Intergenerational Equity in Climate Negotiations, Overlapping Generations Models and Social Welfare" conference at CORE, International Game Theory Festival 2006 in Stony-Brook, ES Summer 2006 Meetings, SAET Conferences in Vigo and Rhodos, NBER General Equilibrium Conference in Lawrence, KS, European General Equilibrium Conference at Warwick, Conference in honor of E. Kalai in Jerusalem, as well as the seminar participants at Boulder, Brussels, Cornell, Northwestern, Roma, Salerno, Stony-Brook, UPenn, Yale, and the "Séminaire de Jeux" in Paris.

This paper presents research results of the Belgian Program on Interuniversity Poles of Attraction initiated by the Belgian State, Prime Minister's Office, Science Policy Programming. The scientific responsibility is assumed by the authors.

†CORE, Université Catholique de Louvain; 34, Voie du Roman-Pays; B-1348 Louvain-la-Neuve; Belgique. E-mail: `jfm@core.ucl.ac.be`.

‡Dept. of Economics; University of Colorado; Boulder CO 80309; USA.
E-mail: `Anna.Rubinchik@Colorado.edu`.

1. Introduction

1.1. **Motivation.** The usual difficulty in evaluating policies affecting yet-to-be-born generations is two-fold. First, the choice of the relevant criterion is non-trivial, sparking ethical debates on intergenerational equity; second, the evaluation itself — 'comparative dynamics' in infinite economies — is often done using numerical methods,[1] the direct computation being intractable.

The main result is a solution to the last problem, it shows when the evaluation of small policy changes reduces to the commonly-practiced present-value calculation with a fixed discount rate. More precisely, the derivative of a social welfare function with respect to a perturbation of policy is a product of a static and a dynamic component, with the latter being an exponential function of time. The key requirement for the main result — time-invariance — is satisfied, for example, at a locally unique balanced growth equilibrium (BGE) in an overlapping generations economy with exogenous growth (OG).

Rather than entering ethical debates on intergenerational equity and the related choice of policies,[2] we derive a social discount rate, using two different welfare functions, traditional and relative utilitarian. In the latter case individual utilities are 0-1 normalised over the space of feasible policies, and then summed.[3]

Hence the intent is to focus public debate on the choice of the underlying social objectives rather than on relative merits of different policies. This echoes the suggestion of Stern (2006, p. 23) to "go back to the first principles from which the standard marginal results are derived." An additional benefit of doing so is that the methods apply then also to problems involving dynamics and risk in heterogeneous societies. In particular, the main result can be used in a general OG model satisfying the minimal set of assumptions needed for balanced growth, see sect. 3.

1.2. **Difficulties with the traditional approach.** First, consider the standard derivation of the discount rate, closely related to the one suggested by Arrow and Kurz (1970), and used, e.g., in the analysis of Stern (2006). Take a discrete time model where individuals live for just one period, with utility function $U(c_t) = \frac{c_t^{1-\rho}}{1-\rho}$ with $\rho > 0$. The economy is on a balanced growth path with per-capita consumption growing exponentially at rate $\gamma > 0$, production being 'black-boxed'

[1] See de la Croix and Michel (2002) for an overview.

[2] See, e.g., the Stern's review on the economics of climate change, 2006, p. 23: "The breadth, magnitude and nature of impacts imply that several ethical perspectives, such as those focusing on welfare, equity and justice, freedoms and rights, are relevant. [...] Questions of intra- and inter-generational equity are central. Climate change will have serious impacts within the lifetime of most of those alive today. Future generations will be even more strongly affected, yet they lack representation in present-day decisions."

[3] See Dhillon and Mertens (1999) for axiomatisation.

for now. The status-quo per-capita consumption at time t is $c_0 e^{\gamma t}$, with $c_0 > 0$. Consider a policy that involves a variation in aggregate consumption δC_t for each generation t. It is to be evaluated at time 0, using the traditional criterion, $W = \sum_t e^{-\beta t} N_t U(c_t)$, where N_t is the number of individuals at time t. Then the net (social) benefit equals

$$\sum_t e^{-\beta t} N_t U'\big(c_0 e^{\gamma t}\big) \cdot \frac{\delta C_t}{N_t} = \sum_t c_0^{-\rho} e^{-(\rho\gamma+\beta)t} \delta C_t$$

Thus, the traditional approach implies $\beta + \rho\gamma$ as (real) discount rate. This conclusion is robust, see cor. 5. To identify the problems associated with this approach, we start by interpreting each parameter.

γ is the growth of real per-capita GDP, say 2% per year.[4]

Recall, β is the discount rate applied to *utilities*, and the only reason a true utilitarian, wishing to treat all generations equally,[5] would accept $\beta > 0$ is uncertainty about future 'world' existence. Note that, in contrast with the representative agent models, β has nothing to do with the prevailing interest rate, nor with individual intertemporal preferences. It is a purely normative parameter, so there is hardly a way to estimate it from observed behaviour.[6]

The parameter ρ admits three rather distinct interpretations. In the report by Stern (2006) it stands for *the elasticity of the social marginal utility of consumption*. In this case individual utilities are assumed purely ordinal, so that, as in Arrow (1963), the value of this parameter must stem from normative considerations.[7] If one is to adopt a utilitarian approach, then ρ is an individual characteristic, either standing for the *income elasticity of the individual marginal utility of income*, or, as in Harsanyi (1955), for the coefficient of relative risk aversion.[8]

Even if the estimates for this discount rate, $\beta + \rho\gamma$, were to fall into an 'acceptable' range, which is hardly true as shown in sect. 6, a serious conceptual problem still remains. How should one proceed in determining the social discount rate for a society consisting of individuals with different risk attitudes or different income elasticities?

The main result allows to evaluate policies under the traditional utilitarian welfare function in the OG model of sect. 3. All the weight is then put asymptotically on the least risk-averse individuals, cf. cor. 4.

---

[4]For the US, e.g., according to Johnston and Williamson (2007), average till 2006 is 2.1% since 1950, 1.9% since 1900 or 1850, 2% since 1869, the first year where data become reliable (*loc. cit.*), especially for growth computations since by then both colonial expansion and the immediate aftermath of the Civil War are over.

[5]The premise stressed in Sidgwick (1874) and Ramsey (1928).

[6]β is, in principle, identifiable from the policy decisions of an actual government (e.g., cf. the second paragraph of sect. 6).

[7]In this case the social welfare functional is a map from individual preferences to societal preferences, so that individual utilities are purely ordinal. This differs thus from the utilitarian point of view in this paper, where utilities are cardinal.

[8]Estimates for ρ are discussed in sect. 6.

On the other hand, with a relative utilitarian criterion, the discount rate is independent of ρ, even with heterogenous risk attitudes, and is simply $\beta + \gamma$, cor. 8.

In the absence of utility discounting ($\beta = 0$), this implication of relative utilitarianism is corroborated by an independent argument about the value of human life, described in sect. 5.

1.3. **Related literature.** The issue of discounting and, more broadly, intergenerational justice, has been controversial in the literature since, probably, Sidgwick (1874).[9] Ramsey (1928) (p. 543) presents discounting future utility ('enjoyments') as a "practice which is ethically indefensible and arises merely from the weakness of the imagination." He then suggests a way to overcome 'technical' difficulties of constructing a discount-free utilitarian social welfare criterion based on the difference between actual and 'bliss' level of utility. Utility discounting is not required per se in our case either, as the welfare function is used to evaluate *temporary policy changes*, and thus aggregates utility *differences*.

This is also why Ramsey's egalitarianism and strong Pareto can be combined here, avoiding the impossibility results (e.g., Basu and Mitra (2003); Crespo, Núñez, and Rincón-Zapatero (2007)). Related literature in welfare economics and social choice offers diverse ways to construct welfare criteria by weakening one of the two desiderata. Koopmans (1960) axiomatises discounting utilities, or 'social impatience'. d'Aspremont (2006) and Asheim, Mitra, and Tungodden (2006) show existence of welfare functions satisfying some of Koopmans' postulates and principles of intergenerational equity, in particular, Chichilnisky's (1996) axioms of 'sustainable development'. Further, several papers aim at characterising ethically acceptable (just) allocations over time (e.g., Asheim (1991), Fleurbaey and Michel (2003)).

The "need to specify a social welfare function which embodies more definite judgements" in cost-benefit analysis was stressed in Drèze and Stern (1987, p. 49), distinguishing this approach from an examination of "potential improvements"[10] stemming from a project. Formulating a social welfare function, the authors argue, provides greater transparency to the cost-benefit analysis, assures consistency of related choices and avoids a special preference for inaction.

[9] "How far we are to consider the interests of posterity when they seem to conflict with those of existing human beings? It seems, however, clear that the time at which a man exists cannot affect the value of his happiness from a universal point of view; and that the interests of posterity must concern a Utilitarian as much as those of his contemporaries, except in so far as the effect of his actions on posterity — and even the existence of human beings to be affected — must necessarily be more uncertain." (p. 414)

[10] See Mishan (1976) for an in-depth discussion of "potential Pareto improvements" (traced back to Pigou (1932)) and their application to cost-benefit analysis. For a more recent overview of cost-benefit criteria see Coate (2000).

1.4. **Outline of the paper.** Sect. 2.1 presents a simple model, sufficient to prove the main result in sect. 2.2. Sect. 3 describes an overlapping generations economy with exogenous growth, to which the main result is applied in sect. 4. Temporary policy changes are evaluated using two social welfare criteria, traditional and relative utilitarian, with quite different implications for the discount rate. A derivation of the discount rate based on the value of a human life is presented in sect. 5. Merits of the two criteria are then discussed in section 6. Concluding remarks in sect. 7 contain possible steps to evaluate the static component of the derivative of welfare and a reference to the proof of non-vacuity of the main result.

2. Differentiating welfare w.r.t. policy variations

2.1. The basic model.

2.1.1. *Population.* Individuals differ by type $\tau \in \{1, \ldots, \Theta\}$, determining, in particular, their life-length T_τ, and by date of birth, $x \in \mathbb{R}$. Time is continuous (for greater transparency of the model) and note, it starts at $-\infty$ rather than at 0, which is the only way to model fully anticipated policy changes.

2.1.2. *Policies.* A basic policy is a time-independent specification of government decisions. Since already income tax schedules with brackets indexed to per-capita income are in a function space, basic policies belong to a Banach space. A status-quo policy is a basic policy kept constant over time. Any policy (reform) is a temporary deviation from the status-quo, def. 1 (iii).

Definition 1. (i) Let $t_h\colon t \mapsto t+h$ be the translation by h on $\mathbb{R}$; and $\mathbb{S}_h\colon \xi \mapsto \xi \circ t_{-h}$ be the *time-shift* on functions of time.

(ii) K_E is the space of infinitely differentiable functions $\varphi\colon \mathbb{R} \to E$ with compact support. A sequence $\varphi_n \in K_E$ converges to 0 if $\exists h \in \mathbb{R}\colon\ |x| \geq h \Rightarrow \varphi_n(x) = 0$ for all n, and φ_n and its successive derivatives converge uniformly to 0. K_E^* is the space of linear functionals ψ on K_E s.t. $\psi(\varphi_n) \to 0$ when $\varphi_n \to 0$ in K_E.[11]

(iii) $(B, \bar{\pi})$ is the set B of *basic policies*, open in a Banach space E, together with some point $\bar{\pi} \in B$, called the *status-quo* policy. P is the set of *policies* $\pi\colon t \mapsto \pi(t) \in B$, s.t. $\delta\pi = \pi - \bar{\pi} \in K_E$.

Remark 1. $K_E = \{\varphi\colon \mathbb{R} \to E \,|\, \forall f \in E^*,\ f \circ \varphi \in K_\mathbb{R}\}$, with E^* as dual of E.

Proof. Obviously, for $\varphi \in K_E$, $f \circ \varphi \in K_\mathbb{R}$. Conversely, since $f \circ \varphi$ is C^∞ for all $f \in E^*$, φ is C^∞ with values in E (e.g., Edwards, 1965, Ex. 8.14 p. 609). Since further each $f \circ \varphi$ has compact support, it is elementary that φ has compact support. $\square$

[11] $K[= K_\mathbb{R}]$ is defined in Schwartz (1957-59) or Gel'fand and Shilov (1959).

2.1.3. *Utilities.*

Definition 2. A profile v of utility functions v_x^τ, defined on P, is a *valuation* if it is weakly shift-invariant, i.e., $\forall h \in \mathbb{R}$ $\exists$ Lebesgue-measurable $a_h, b_h > 0\colon v(\mathbb{S}_h(\pi)) = a_h + b_h \mathbb{S}_h(v(\pi))$. The profile is a *strict valuation* if it is shift-invariant, i.e., if $a_h = 0, b_h = 1$.

The time-invariance implies, in particular, that a basic policy has to be *unit-free* and *non-discriminatory* in that it does not prescribe special treatment of a particular individual (or generation) and thus can be applied at any time. An example would be a full description of government expenditures, including public goods, regulation and taxation, expressed in per-capita terms, as fractions of per-capita income.

There is a simple translation of a valuation into a strict one:

Lemma 1. *For any valuation v there exist constants ϱ $(= \frac{1}{h}\ln b_h)$ and A s.t. a strict valuation u is obtained after the linear transformation of each agent's utility function v_x^τ to $u_x^\tau = A\frac{e^{-\varrho x}-1}{e^{\varrho}-1} + e^{-\varrho x} v_x^\tau$, where the fraction equals x when $\varrho = 0$, and x is the agent's birth-date.*

Proof. Observe first that either the utility functions are constant and then one can assume $a_h = 0$ and $b_h = 1$, or, by definition of a valuation, $b_{h_1+h_2} = b_{h_1} b_{h_2}$ and $a_{h_1+h_2} = a_{h_2} + a_{h_1} b_{h_2}$, which also hold in the first case. It follows then in a standard way from Lebesgue-measurability that $b_h = e^{\varrho h}$ and $a_h = A\frac{e^{\varrho h}-1}{e^{\varrho}-1}$ for some constants A and ϱ, the ratio being defined by continuity at $\varrho = 0$. Hence the result, by a straightforward verification. □

The next observation delineates the interpretation of basic policies.

Corollary 1. *The strict valuation $v_x^\tau(\pi)$ of a constant policy π, is independent of x, and for a valuation (with $\varrho \neq 0$) it is of the form $e^{\varrho x} v^\tau(\pi) + C$.*

Proof. The first statement follows from the definition of a strict valuation and the second then follows from lemma 1. □

2.1.4. *Social Welfare.* The two different *social welfare functionals* (SWF) used in this paper are based on the same *social welfare aggregator*, the utilitarian one.

Definition 3. A *social welfare aggregator* (SWA) is a real-valued function V_c defined for utility profiles $\tilde{u} \in \mathfrak{U}$, where $\mathfrak{U} \subseteq \mathbb{R}^{\Theta\times\mathbb{R}}$ is shift-invariant, s.t. V_c is weakly translation-invariant: $V_c(\mathbb{S}_h(\tilde{u})) = e^{ch} V_c(\tilde{u})$ for some constant $c \in \mathbb{R}$.

Definition 4. The *utilitarian social welfare aggregator*, S, is a SWA with $c = \nu - \beta$ defined as $S(\tilde{u}) = \int_{-\infty}^{\infty} e^{-\beta t} \sum_\tau N_t^\tau \tilde{u}_t^\tau dt$, where $N_t^\tau dt = N_0^\tau e^{\nu t} dt$ is the number of individuals of type τ born in $(t, t + dt)$.

Clearly, with a continuum of individuals even a utilitarian SWA might not be well defined, whatever be β. However, given the goal to evaluate small temporary policy changes, it is natural to construct a welfare function, W (a real-valued function on the space of policies), as measuring the individual utility *differences* from the status quo.

2.2. **The main result.** Recall a map is Gateaux-differentiable if it has directional derivatives in every direction, which form a continuous linear function of the direction. It is the weakest sense of differentiability.

Theorem 1. *Given a valuation v, let $\varrho = \frac{1}{h}\ln b_h$ (as in lemma 1). Assume the* SWA *V_c is homogeneous of degree r, and let $W(\pi) = V_c(v(\pi) - v(\bar\pi))$ be Gateaux-differentiable on P at $\bar\pi$. Then its differential equals $\int e^{(c+\varrho r)t}\langle q, \delta\pi(t)\rangle dt$ for some $q \in E^*$, the dual of E.*

Proof. By definition of a Gateaux-differential there exists $\mu \in K_E^*$ s.t.

$$DW_{\bar\pi}(\delta\pi) = \lim_{\varepsilon\to 0}\frac{W(\bar\pi + \varepsilon\delta\pi) - W(\bar\pi)}{\varepsilon} = \langle \mu, \delta\pi\rangle \tag{1.1}$$

Start with the particular case $E = \mathbb{R}$, using shortly K for $K_{\mathbb{R}}$.

Since $\mathbb{S}_h\bar\pi = \bar\pi$, $W(\mathbb{S}_h(\pi)) = V_c(v(\mathbb{S}_h\pi) - v(\mathbb{S}_h\bar\pi))$; so, by weak shift invariance of v, lemma 1, homogeneity of V_c and definition of W resp.,

$$\begin{aligned} W(\mathbb{S}_h(\pi)) &= V_c(b_h\mathbb{S}_h(v(\pi) - v(\bar\pi))) = e^{\varrho r h}V_c(\mathbb{S}_h(v(\pi) - v(\bar\pi))) \\ &= e^{(\varrho r + c)h}V_c(v(\pi) - v(\bar\pi)) = e^{(\varrho r + c)h}W(\pi) \end{aligned}$$

Thus $W \circ \mathbb{S}_h = e^{\zeta h}W$, with $\zeta = \varrho r + c$; hence from (1.1),

$$\langle \mu, \mathbb{S}_h(\delta\pi)\rangle = e^{\zeta h}\langle \mu, \delta\pi\rangle$$

Since B is a neighbourhood of $\bar\pi$, every $\varphi \in K$ is a multiple of some $\delta\pi$, hence the following holds for all $h \in \mathbb{R}$ and all $\varphi \in K$:

$$\langle \mu, \varphi - e^{-\zeta h}\mathbb{S}_h\varphi\rangle = 0$$

Dividing by h and taking the limit (in K !) as $h \to 0$ yields

$$\langle \mu, \varphi' + \zeta\varphi\rangle = 0$$

The definition of the derivative of a generalised function, $\mu \in K^*$,

$$\langle \mu', \varphi\rangle = -\langle \mu, \varphi'\rangle, \ \forall\varphi \in K$$

yields then

$$\langle \zeta\mu - \mu', \varphi\rangle = 0, \ \forall\varphi \in K$$

By Gel'fand and Shilov (1959, p. 53) the equation $\zeta\mu - \mu' = 0$ has $\mu = qe^{\zeta t}$ for some $q \in \mathbb{R}$ as only solutions in K^*, so,

$$DW_{\bar\pi}(\delta\pi) = \langle qe^{\zeta t}, \delta\pi\rangle = q\int_{-\infty}^{\infty} e^{\zeta t}\delta\pi_t dt, \ \forall\delta\pi \in K \tag{1.2}$$

Next step is to extend the result to any Banach space E.

Lemma 2. *Any function $\varphi \in K_E$ can be approximated in K_E by functions with finite-dimensional range.*

Proof. Let $D_n = \{\varphi^{(i)}\left(\frac{j}{n!}\right) \mid 0 \le i \le n,\, j \in \mathbb{Z}\}$. D_n is an increasing sequence of finite subsets of E. Let F_n be the subspace spanned by D_n and p_n a projector from E to F_n (i.e., $p_n \colon E \to F_n$ is the identity on F_n). Its existence follows from Hahn-Banach, F_n being finite-dimensional. Then $\varphi_n = p_n \circ \varphi \in K_{F_n}$ and converges in K_E to φ. $\square$

Consider policy variations $\delta\pi \in K_E$ of the form $b\phi\colon t \mapsto b\phi(t)$ with $b \in E$ and $\phi \in K$. By (1.1) and (1.2), $\forall b \in E\ \exists q_b \in \mathbb{R}$ s.t. $\langle \mu, b\phi\rangle = q_b \int_{-\infty}^{\infty} \phi(t)e^{\zeta t}dt\ \forall \phi \in K$. So, for $I_\phi = \int_{-\infty}^{\infty} \phi(t)e^{\zeta t}dt \neq 0$, the map $b \mapsto q_b = \frac{\langle \mu, b\phi\rangle}{I_\phi}$ is in E^*, i.e., $q_b = \langle q, b\rangle$ with $q \in E^*$.

So, for any φ of the form $b\phi$,

$$\langle \mu, \varphi\rangle = \int_{-\infty}^{\infty} \langle q, \varphi(t)\rangle e^{\zeta t}dt \tag{1.3}$$

Since any $\varphi \in K_E$ with finite-dimensional range is a sum of policy variations of the form $b\phi$, (1.3) remains true by linearity for them. They being dense in K_E by lemma 2, (1.3) extends by continuity to K_E. $\square$

Remark 2. By definition, valuations have full domain, but their use in thm. 1 shows that it suffices their domain intersects every straight line through $\bar{\pi}$ in a neighbourhood of $\bar{\pi}$, and this is how they are obtained in fact.[12] One could even further allow them to be a correspondence, interpreting Gateaux-differentiability as that of every selection. However, even in the latter case, it is easy to obtain by the axiom of choice a valuation as defined, which is a selection from the correspondence wherever that one is non-empty. Similarly for strict valuations. So for simplicity we keep our definition above.

Corollary 2. *The statement of the theorem is still true if K_E is replaced in def. 1.iii by a topological vector space F of E-valued functions s.t.:*[13]

- *K_E embeds continuously as a dense subset in F.*
- *$\forall q \in E^\star$, $f \mapsto \int e^{(c+\varrho r)t}\langle q, f(t)\rangle dt$ belongs to $F^\star$.*
- *$\forall f \in F\ \exists \varepsilon_0 > 0\colon |\varepsilon| < \varepsilon_0 \Rightarrow \varepsilon f(t) \in B\ \forall t$.*

The theorem justifies discounting, i.e., shows that the time-component of the derivative of the welfare function is exponential in time, with a time-independent valuation q of the current policy change $\delta\pi(t)$.

It rests on two crucial assumptions. First is the existence of a valuation. The next 2 sections prove this, even in a growing economy, despite the time-invariance requirement in its definition. The other assumption is the differentiability of the welfare function. Verifying this requires an in-depth investigation of regularity (and stability) properties of equilibria in infinite-dimensional economies, and is beyond the scope of this paper. Sect. 7 reports on progress in that direction.

[12] Cf. remark 6.

[13] E.g., the space of continuous functions with compact support with the sup norm.

3. A growing economy

The (strict) valuations with their built-in time-invariance might seem to be confined to stationary economies, however, they naturally arise in an economy with exogenous growth. We impose only the minimal conditions required for existence of balanced growth: homogeneity of utility functions with respect to consumption, constant returns to scale in production, absence of land and natural resources, and labour-saving technological growth.

3.1. **The economy.**

3.1.1. *Consumption and Labour.* Population grows exponentially at a constant rate ν, and the fraction of individuals of any given age and type is constant over time. Instantaneous consumption is a non-negative bundle of n consumption goods and h fractions of total time allocated to h different types of labour. Individual preferences over lifetime streams of consumption and time allocation bundles are represented by a utility function U^τ, homogeneous of degree $1-\rho^\tau$ in consumption. By fixing consumption prices and relative wages, labour income varies linearly with a 'numeraire' wage by the homogeneity, so an individual indirect utility function can be viewed as a function of (labour-)income at those fixed consumption prices and relative wages. It allows thus for two interpretations of the parameter ρ^τ: (1) relative risk aversion coefficient, (2) income elasticity of the marginal utility of income, as mentioned in the introduction.

The fraction of time, $z_i^\tau(s,t)$, devoted at date t to activity i by an agent of type τ and age s is multiplied by a non-negative and integrable efficiency factor $\varepsilon_i^\tau(s)$, to form effective time. Effective time devoted at date t to any activity is multiplied by $e^{\gamma t}$ to form effective labour input, $e^{\gamma t}\varepsilon_i^\tau(s)z_i^\tau(s,t)$, thus representing labour-saving technological progress.

Example. With $\gamma=0$ and $\varepsilon(s)=1$ in the first part of life and zero thereafter the model is a continuous-time reinterpretation of the standard OG model, as presented in Gale (1973), Samuelson (1958).

3.1.2. *Production.* There are m capital goods, and a corresponding investment good for each, linked by the usual capital accumulation equation, $K^{i\prime}(t) = I^i(t) - \delta^i K^i(t)$,[14] with $K^i \geq 0$ capital and I^i investment of type i, and δ^i the depreciation rate. Consumption and investment goods are manufactured instantaneously by production firms from (the services of) capital and effective labour (and, possibly, from investment), with as instantaneous production set a convex cone $Y \subseteq \mathbb{R}^h_- \times \mathbb{R}^m_- \times \mathbb{R}^m \times \mathbb{R}^n$ of production vectors $(-L,-K,I,C)$. Assume free-disposal, and that Y contains no straight line, i.e., the classical

[14]Assumed to hold a.e., and implying the conditions for it to be meaningfull: that K_t^i is assumed locally a Perron primitive and I_t^i locally Perron-integrable.

irreversibility condition. An investment firm of type i acquires capital $K^i(t_0)$ at time t_0, chooses investment flows, rents out accumulated capital to production firms, and sells $K^i(t_1)$ at time $t_1 > t_0$.

Investment goods can be viewed both as outputs and inputs. To model a storable good, for example, one can introduce a corresponding investment good and a capital good ("the good in storage"). A production firm creates the storable investment good, purchased by an intermediary investment ("storage") firm that transforms it into the corresponding capital good, which typically has no use in production. At the time of consumption, the investment firm disinvests and sells the corresponding investment good to a production ("marketing") firm, that transforms it one to one into the corresponding consumption good. So allow all investment firms to disinvest as well as invest in all goods; restrictions on disinvestment are described by Y.

To incorporate consumer durables in this model, introduce the corresponding investment and capital goods. A production firm creates the durable investment good, purchased by an intermediary investment firm, which then rents the corresponding capital good out to a leasing production firm, that produces with this capital the corresponding consumption good (services), purchased by consumers.

The rest of this subsection is to ensure that the production set of the economy (set of feasible input and output paths) is well-defined.

To ensure its boundedness, assume capital can not reproduce itself:

Assumption (No-rabbit economy)**.** $(0, -K, I, C) \in Y \Rightarrow I \leq 0$.

Remark 3. Observe that although production of durables, as described before, involves a production of consumption good with only capital and no labour input, it does not violate our assumption on Y that no *investment good* can be produced without some form of labour input. Similarly production activities (as for storable goods) transforming investment goods one to one into consumption goods, without any capital or labour input, do not violate this assumption.

To see why such restrictions on the instantaneous production set are important consider an example of a "rabbit economy."

Example. Assume a single good, a single type of labour, a CES production function $(AK^\alpha + BL^\alpha)^{1/\alpha}$, $A^{1/\alpha} \geq R$ with $R = \gamma + \nu + \delta$. In order to get an upper bound on capital and investment consider a path with all agents working full-time and consuming nothing. Note that $L_t = L_0 e^{(\gamma+\nu)t}$, so for $D = BL_0^\alpha$, $K'(t) = (AK^\alpha(t) + De^{\alpha(\gamma+\nu)t})^{1/\alpha} - \delta K(t)$; or with $x(t) = K(t)e^{-(\gamma+\nu)t}$, $x'(t) = (Ax^\alpha(t) + D)^{1/\alpha} - Rx(t) \geq D^{1/\alpha} > 0$. Since $x(t) \geq 0$, there is no solution, i.e., the upper bound of $K(t)$ is infinity. And even if $B = 0$, the solutions are $x(t) = Ce^{(A^{1/\alpha} - R)t}$, with $C \geq 0$ arbitrarily large, so $K(t)$ is unbounded in this case too.

As for any differential equation, initial conditions are needed. Their natural form is that the capital stock K_t converges at $-\infty$ to given initial values. Note that such initial values of land and resources[15] are thus part of the description of the technology; any feasible path must converge to the specified values. But, for balanced growth, those initial values must be zero, so land and resources are ruled out:

Assumption (Initial condition)**.** Let $\delta = \min_i \delta^i$. Then $e^{\delta t} K_t$ converges exponentially fast to 0 along some sequence $t \to -\infty$.

Also assume $R \stackrel{\text{def}}{=} \gamma + \nu + \delta > 0$.

Lemma 3. (i) *$K^i(t) = \int_{-\infty}^{t} e^{\delta^i (s-t)} I^i(s) ds$, where the L_1-norms of all feasible integrands are bounded by a constant times $e^{(\gamma+\nu)t}$; in particular, the integral is a Lebesgue integral.*

(ii) *Let $i_t = e^{-(\gamma+\nu)t} I_t$, $k_t = e^{-(\gamma+\nu)t} K_t$. There exists $\bar{K}$ s.t. along any feasible path, $\int_a^b \|i_t\| dt \le \bar{K}(b-a+1)$ for any pair $a \le b$, and (hence) $\|k_t\| \le \bar{K}$.*

Proof. Since the closed convex cone Y is pointed (irreversibility), there exists a linear functional α whose unique maximiser on Y is 0. Then $\langle \alpha, y \rangle \le -\varepsilon \|y\|$ on Y, i.e., by rescaling α, $\langle \alpha, y \rangle \le -\|y\|$. Observe too that free disposal implies $\alpha \gg 0$. Write α as $(\alpha^L, \alpha^K, \alpha^I, \alpha^C)$.

First step is to establish the bound on K sub (ii).

Fix a vector $\bar{L} \in \mathbb{R}^h$ s.t. any feasible vector of labour inputs $L_t \le \bar{L} e^{(\gamma+\nu)t}$ (i.e., to compute a given coordinate of $\bar{L}$, assume all agents spend 100% of their time on that activity).

Allow perfect substitution at rates α^I between all investment goods and between all capital goods: let $F\colon \mathbb{R}_+^2 \to \mathbb{R}_+ : (\bar{\kappa}, \lambda) \mapsto \sup\{\langle \alpha^I, I \rangle \mid \exists K \ge 0, \langle \alpha^I, K \rangle \le \bar{\kappa}, (-\lambda \bar{L}, -K, I, 0) \in Y\}$.

The sup is finite, since $\langle \alpha^I, I \rangle \le \langle \alpha^K, K \rangle + \lambda \langle \alpha^L, \bar{L} \rangle$ and $\langle \alpha^K, K \rangle$ is bounded on the compact set $K \ge 0, \langle \alpha^I, K \rangle \le \bar{\kappa}$ (recall $\alpha \gg 0$). Further the sup is achieved, the sets $\{y \in Y \mid \langle \alpha, y \rangle \ge -M\}$ being compact (since then $\|y\| \le M$), so that the sup is effectively over a compact set. Clearly F is positively homogeneous of degree 1 and concave, and is continuous again because locally everything happens within a compact subset of Y.

Thus by homogeneity $F(\bar{\kappa}, \lambda) = \lambda \varphi(\frac{\bar{\kappa}}{\lambda})$, where $\varphi(x) = F(x, 1)$ is concave, ≥ 0 and continuous. Further by the "no-rabbit" assumption, $F(\bar{\kappa}, 0) = 0$, so, by continuity of F, $\frac{\varphi(x)}{x} \to 0$ at ∞.

For any feasible path (L_t, K_t, I_t, C_t), let $\tilde{\iota}_t = \langle \alpha^I, I_t \rangle$ and $\tilde{\kappa}_t = \langle \alpha^I, K_t \rangle$. Then, since $\delta \le \delta^i$ and $K \ge 0$, the capital-accumulation equation implies $\tilde{\kappa}_t' \le \tilde{\iota}_t - \delta \tilde{\kappa}_t$; further $L_t \le \bar{L} e^{(\gamma+\nu)t}$ implies (free-disposal) $\tilde{\iota}_t \le e^{(\gamma+\nu)t} \varphi(e^{-(\gamma+\nu)t} \tilde{\kappa}_t)$. So with $(l_t, \kappa_t, \iota_t, c_t) = e^{-(\gamma+\nu)t}(L_t, \tilde{\kappa}_t, \tilde{\iota}_t, C_t)$:

$$\kappa_t' \le \varphi(\kappa_t) - R\kappa_t$$

[15]Non-null initial values can occur only for capital goods with $\delta^i = 0$, corresponding to land and resources. Disinvestment is crucial to describe resource extraction.

To bound $\|k_t\|$, it suffices to prove from this that κ_t is bounded by some constant independent of the feasible path, since $\alpha^I \gg 0$.

Also the initial condition yields that $e^{Rt}\kappa_t$ converges exponentially fast to 0 at $-\infty$, i.e., since $R > 0$, there exists ε: $0 < \varepsilon < R$ such that, with $r = R - \varepsilon > 0$, $e^{rt}\kappa_t \to 0$ at $-\infty$ along a subsequence. Since $\frac{\varphi(x)}{x} \to 0$ at ∞, there exists A s.t., $\forall x$, $\varphi(x) \le A + \varepsilon x$; so $\kappa'_t \le A - r\kappa_t$.

Next step is to prove from this that $\kappa_t \le \bar{K}$, with $\bar{K} = \frac{A}{r}$.

Else $\kappa_{t_0} > \bar{K}$ for some t_0; since $\kappa'_t < 0$ for $\kappa_t > \bar{K}$, this implies that $\kappa_t > \bar{K}$ and is decreasing for $t \le t_0$. Define y by $y'_t = A - ry_t$ with the prescribed terminal value κ_{t_0} at t_0. The relations for κ and y are equivalent to $\frac{r\,d}{de^{rt}}(e^{rt}\kappa_t) \le A$ and $\frac{r\,d}{de^{rt}}(e^{rt}y_t) = A$, so, since $\frac{de^{rt}}{rdt} > 0$, $\frac{d}{dt}(e^{rt}\kappa_t) \le \frac{d}{dt}(e^{rt}y_t)$: for $t \le t_0$, $\kappa_t \ge y_t = \frac{A}{r} + (\kappa_{t_0} - \frac{A}{r})e^{r(t_0-t)}$, contradicting that $e^{rt}\kappa_t \to 0$ at $-\infty$ along a subsequence.

Hence the uniform bound on $\|k_t\|$.

Next, $\langle \alpha, y\rangle \le -\|y\|$ yields $\int_a^b \|i_t\|dt \le \int_a^b (\langle \alpha^L, l_t\rangle + \langle \alpha^K, k_t\rangle - \langle \alpha^I, i_t\rangle - \langle \alpha^C, c_t\rangle)dt$. The last inner product is non-negative, and the capital-accumulation equation yields $k_t'^j = i_t^j - R^j k_t^j$ with $R^j = \gamma + \nu + \delta^j$ as before, so that $\int_a^b \langle \alpha^I, i_t\rangle dt = \int_a^b \langle \alpha^I, k'_t\rangle dt + \sum_j \alpha_j^I R^j \int_a^b k_t^j dt = \langle \alpha^I, k_b - k_a\rangle + \sum_j \alpha_j^I R^j \int_a^b k_t^j dt \ge -\langle \alpha^I, k_a\rangle$, since $R^j > 0$. Thus our bounds on k_t and l_t imply $\int_a^b \|i_t\|dt \le \bar{K}(b-a+1)$ for some constant $\bar{K}$.

Thus point (ii). For (i), $e^{-(\gamma+\nu)t}\int_{-\infty}^t e^{\delta^j(s-t)}|I_s^j|ds = \int_0^\infty e^{-R^j x}|i_{t-x}^j|dx \le \sum_{n=0}^\infty e^{-R^j n}\int_n^{n+1}|i_{t-x}^j|dx$ is uniformly bounded by (ii). For $M_t^i = e^{\delta^i t}K_t^i$, the differential equations become $M_t^{i\prime} = h_t^i$, with $h_t^i \stackrel{\text{def}}{=} e^{\delta^i t}I_t^i$, hence, by the integrability, $M_t^i = M_{-\infty}^i + \int_{-\infty}^t h_s^i ds$. And the initial condition yields $\lim_{t\to-\infty} M_t^i = 0$, so $M_{-\infty}^i = 0$, hence (i). □

Remark 4. The initial condition is a bit too stringent conceptually, requiring exponential convergence to 0 instead of just plain convergence. This is not crucial in this paper: land and natural resources being anyway ruled out by the need for balanced growth, it is natural to expect all $\delta^i > 0$, so just K_t bounded at $-\infty$ already ensures exponential convergence to 0.

3.2. **Time-invariant solution concepts.** Solution concepts map an economy and policy pair into allocations. To induce a valuation they have to satisfy time-invariance and be single-valued, in which case they are called outcome functions.

3.2.1. *Isomorphism between Arrow-Debreu economies.* To motivate the definitions below, define isomorphism between two Arrow-Debreu economies with finitely many goods and individuals. They are isomorphic if there is a linear map ζ from the commodity space of one economy to that of the other and there are one-to-one mappings from the sets of individuals and of firms of one to those of the other such that:

(i) the consumption set, preferences, and endowment of any agent in the first economy are mapped by ζ to those of the corresponding agent in the second economy;
(ii) the production set of each firm in one economy is mapped by ζ to that of the corresponding firm in the second;
(iii) shareholdings are preserved.

When consumption sets are the non-negative orthant, ζ is a one-to-one mapping of the commodities' names in the first economy to those in the second, accompanied by appropriate re-scalings (changes of unit).

Another aspect of isomorphism, which is more familiar with a continuum of agents, is to multiply the population measure by a positive constant C. "Shareholdings" refer then for each firm to a probability distribution over the agents; and the one-to-one mapping of agents has to be understood to be measurable as well as its inverse, and such that the induced map on measures maps the first population measure to $\frac{1}{C}$ times the second. Further, the firms' production sets, as well as points therein, are multiplied by C (in addition to the above re-scalings).

When production has constant returns to scale, as here (capital-accumulation equations are linear, and the instantaneous production sets, cones), shareholdings become irrelevant (profits being zero), and multiplication by C maps the production set onto itself.

The whole isomorphism can equivalently be described by a single linear bijection between allocation spaces (product of all consumption sets and production sets) of the 2 economies, provided this map has the required structure. For the isomorphism property, suffices then to check that it maps allocations *to and onto* allocations, endowments to endowments, preserves preferences, and that population measures are mapped to each other by the induced map of agents and the multiplication by C, obtaining C from how the map behaves on production sets as compared to consumption sets.

3.2.2. *Time-invariance in the OG model.* Consider next particular case of such isomorphisms: it maps an agent of type τ born at time t to an agent of the same type born at $t+h$, multiplying his mass by $e^{\nu h}$, and maps any good dated t to the same good dated $t+h$, multiplying non-labour quantities by $e^{\gamma h}$, and labour quantities by 1. Individual time is not a good (not marketed), so the map of sect. 3.2.1 is applied using the equivalent vector of effective time. The map of allocations is thus:

Definition 5. The transformation $\mathbb{T}_h$

(i) applies $\mathbb{S}_h$ to allocations;
(ii) multiplies individual consumption bundles by $e^{\gamma h}$;
(iii) multiplies production plans — aggregate effective labour, capital, investment, and consumption— by $e^{(\gamma+\nu)h}$.

Remark 5. Equivalently, $\mathbb{T}_h$ shifts the origin of time back by h, multiplies population measure and thus all aggregates by $e^{\nu h}$, and divides *units* of all non-labour goods by $e^{\gamma h}$.

Lemma 4. *$\mathbb{T}_h$ is an isomorphism mapping the economy to itself.*

Proof. By (i), the "induced map of agents" maps an individual of type τ born at time t to an individual of the same type born at $t+h$, so the population of the new economy at time t is that of the old at time $t-h$ multiplied by $e^{\nu h}$, hence equals that of the original economy at t: the population measure is preserved. Remains thus only to prove that preferences are preserved and that allocations are mapped to allocations: the one-to-one and onto aspect will then follow from the same property for the inverse $\mathbb{T}_{-h}$. Consumption sets are unchanged: at any t non-negativity constraints are preserved by the re-scalings (ii), besides, time fractions are not re-scaled, so the constraint that the sum of fractions is ≤ 1 is also preserved. Preferences are homogeneous in the consumption goods, so are preserved by re-scaling (ii). As for production, the capital accumulation equation is linear in capital and investment, so is preserved given (iii), as well as the initial condition. And the instantaneous production cone is unchanged under the scaling by $e^{(\gamma+\nu)h}$ (iii). $\square$

Since $\mathbb{T}_h$ maps the economy to itself h units of time later, and since policies are unit-free, the corresponding operation on policies is a pure time-shift, without rescaling. Time-invariance requires thus that solutions for $\mathbb{S}_h\pi$ be obtained via $\mathbb{T}_h$ from the solutions of π. We impose in addition single-valuedness.

Definition 6. An *outcome function* ς is a map from policies to individual allocations, that is invariant under all $\mathbb{T}_h$: $\mathbb{T}_h \circ \varsigma = \varsigma \circ \mathbb{S}_h$.

3.2.3. *Examples of outcome functions.* The first example is the allocation induced by a time-invariant *social welfare function*, say, a utilitarian one, provided the solution to the maximisation problem is unique.

Other examples are equilibrium-based. An outcome function should map a policy perturbation π to a locally unique selection close to that stemming from the status-quo policy, $\bar{\pi}$. For Arrow-Debreu equilibria, the new policy is fully foreseen by the individuals.

One way to model *policy surprises* is to assume the initial contracts signed in anticipation of the status-quo policy can not be changed, so in the wake of an unexpected policy change, individuals sign additional contracts taking their status-quo consumption as new endowment. Given the initial equilibrium is a balanced growth equilibrium, the economy with that endowment also satisfies time-invariance (cf. lemma 5), and the resulting map from policies is again an outcome function if the final allocation is locally unique. This case is particularly simple, as the excess demand under the base-line prices is zero, so the variation in individual utility is just driven by the endowment perturbation under the old prices.

Remark 6. Even if dealing with a situation that does not guarantee local uniqueness, one can choose the closest prices to those in the initial equilibrium in terms of the L_∞ norm: $\sum_i \|\ln p_i(t) - \ln \bar{p}_i(t)\|_\infty$, where $\bar{p}(t)$ is the price vector at the initial equilibrium and $p(t)$ at a perturbed economy. Though the price system does not necessarily specify an equilibrium, it does specify the individual utility levels, which is sufficient for welfare analysis. The logarithms make the distance independent of price normalisation, hence induce a distance between price-rays: for any multiple of $\bar{p}_i$, the minimum, over all multiples of p_i, will be achieved at the corresponding multiple, and the value of the minimum is independent of this multiple, and remains the same when permuting the roles of $\bar{p}_i$ and p_i. Finally, the L_∞ norm being shift-invariant, the selection will be time-invariant. If the set of minimisers is not a singleton, their correspondence can be expected to be sufficiently 'thin' that the axiom of choice (cf. rem. 2) would generically give a differentiable selection. This is the case, e.g., in Mertens and Rubinchik (2007) (discussed in sect. 7).

3.2.4. *A key property of outcome functions.*

Definition 7. A *balanced growth path* is a $\mathbb{T}$-invariant allocation.

On a balanced growth path individuals' time allocation is independent of their birth-date, individual consumption grows at rate γ, and all aggregate inputs and outputs at rate $\gamma + \nu$, as in the standard (one-good, one-consumer-type) definition (e.g., Arrow and Kurz, 1970; King, Plosser, and Rebelo, 2002).

The following sharpens the interpretation of basic policies, see cor. 1:

Lemma 5. *The outcome of a constant policy is a balanced growth path.*

Proof. By def. 6, it is mapped to itself by any $\mathbb{T}_h$. □

4. The discount rate

The discount rate for cost-benefit analysis depends on the social welfare function. We consider both relative and traditional utilitarianism.

Let now $v = U \circ \varsigma$ denote the profile of utility functions on P induced by some outcome function ς and by the profile of utility functions U; and let $\bar{u} = v(\bar{\pi})$.

4.1. **The traditional utilitarian approach.**

Proposition 2. *Assume all types have the same parameter ρ. Then v is a valuation, with $\frac{1}{h} \ln b_h = (1 - \rho)\gamma$.*

Proof. Let $(c,l) \stackrel{\text{def}}{=} \varsigma(\pi)$. Let $\tilde{\pi} = \mathbb{S}_h(\pi)$. By the time-invariance of outcome functions, the corresponding profile under $\tilde{\pi}$ is $\mathbb{T}_h(c,l) = (e^{\gamma h}\mathbb{S}_h(c), \mathbb{S}_h(l))$. Then, by homogeneity of life-time utilities, the profile of individual utilities induced by the outcome function under $\tilde{\pi}$ is $e^{(1-\rho)\gamma h}\mathbb{S}_h(v)$. So, v is a valuation with $a_h = 0$ and $b_h = e^{(1-\rho)\gamma h}$. □

Prop. 2 and thm. 1 imply:

Corollary 3. *Assume all types have the same parameter ρ. If $W = S \circ (v - \bar{u})$ is Gateaux-differentiable on P at $\bar{\pi}$, then its differential equals, for some $q \in E^*$, $\int e^{(\nu-\beta+(1-\rho)\gamma)t}\langle q, \delta\pi(t)\rangle dt$.*

In a society with type-dependent ρ, traditional utilitarianism leads to questionable implications; besides, it invalidates discounting:

Corollary 4. *The weight of the types with the smallest ρ in the social welfare function approaches unity as time goes to $+\infty$.*

Proof. Let $V_c = \sum_\tau V_c^\tau$, where $V_c^{\tau_0}$ with $c = \nu - \beta$ is the social welfare aggregator defined as $\int_{-\infty}^{\infty} e^{-\beta t} \sum_\tau \mathbf{1}_{\tau_0} N_t^\tau \left(v_t^\tau(\pi) - \bar{u}_t^\tau\right) dt$. Applying cor.3 to each V_c^τ, i.e., to the economy in which utilities of all types but τ are identically zero, one obtains the differential of W:

$$\sum_\tau \int_{-\infty}^{\infty} e^{(\nu-\beta+(1-\rho^\tau)\gamma)t}\langle q^\tau, \delta\pi(t)\rangle dt = \int_{-\infty}^{\infty} e^{(\nu-\beta+\gamma)t}\Big\langle \sum_\tau e^{-\rho^\tau\gamma t} q^\tau, \delta\pi(t)\Big\rangle dt$$

Visibly the criterion q^τ of the types with the smallest ρ asymptotically gets all the weight. □

There are other ways to express the same idea; e.g., that along any balanced growth path, in an optimal redistribution of consumption goods (keeping the rest fixed) the fraction allocated to the agents with the smallest ρ converges to 1.

4.1.1. *The discount rate for cost-benefit analysis.* In cost-benefit analysis, the effects of a variation in public policy are traditionally first "monetised", i.e., expressed as an equivalent perturbation of individual endowments of consumption goods, initially 0.

Let thus the set of basic policies B be an open neighbourhood of 0 in the Banach space M of measures[16] on age-groups and types — i.e., on $\cup_\tau(\{\tau\} \times [0, T_\tau])$ — with values in $\mathbb{R}^n$ (space of consumption bundles), with $\bar{\pi} = 0$ as status-quo, where $b \in B$ determines the endowment perturbation $\omega(t) = e^{(\nu+\gamma)t}b$.

Policies are thus arbitrary (smooth) endowment perturbations, representing arbitrary flows of lump-sum real taxes and benefits. Then we get $\beta+\rho\gamma$ as discount rate, confirming our rough calculation in sect. 1.2:

[16]Or L_1, or the measures with continuous densities.

Corollary 5. *Assume all types have the same parameter* ρ*. If* $W = S \circ (v - \bar{u})$ *is Gateaux-differentiable on* P *at* 0*, then its differential equals, for some* $q \in M^*$*,* $\int e^{-(\beta+\rho\gamma)t} \langle q, \omega(t) \rangle dt$*.*

Proof. By construction, $\bar{\pi} = 0$, so $\delta\pi = \pi$ and $\pi(t) = e^{-(\nu+\gamma)t}\omega(t)$. □

Remark 7. Clearly, choosing a different growth rate in the definition of ω would lead to the same corollary with a different discount rate. That statement would however be empty: there can be no outcome function, since it would violate lemma 5.

4.2. **The Relative Utilitarian approach.** As an alternative, we suggest to apply relative utilitarianism (RU),[17] the social welfare functional where individual von Neumann-Morgenstern utilities are normalised between zero and one, and then summed. It is stressed in Dhillon and Mertens (1999) that the RU-normalisation of individual utilities has to be done on some universal set A of acceptable alternatives, not specific to the problem under consideration, and representing the constraints both of feasibility and of justice.

Assumption. The set A of *acceptable* policies is shift-invariant and each individual utility is bounded on A.

The boundedness is a minimal implication of justice; as to the shift-invariance, it is clearly a property of feasibility, but in relation to justice it has a strong meaning, that physical units (like calories per day) are irrelevant (and without it RU might lead to quite different conclusions). But it is straight in the spirit of exogenous growth models that (acceptable) policies affect only the height of the growth path, not the growth rate, and is arguably justified in a world described by such a model; e.g., if calories per-day matter, one would not expect utilities to be homogeneous.

Assume individual utility functions are von Neumann-Morgenstern, and let ς be defined on A and therefore, by the definitions at the beginning of this section, v is so too. Let $\mathbb{M}_A$ denote the RU-*normalisation*, i.e., the transformation of a utility profile such that each individual utility is normalised between zero and one on A.

Definition 8. The relative utilitarian SWF is $S \circ \mathbb{M}_A(u)$.

RU's anonymity axiom implies $\beta = 0$ in the specification of S, def. 4. However, to allow for a richer model incorporating a probability of the world ending tomorrow, β is unrestricted.

4.2.1. *Solutions for* RU. In a growing economy the RU-normalisation yields shift-invariance, hence strict valuations:

Proposition 3. $\mathbb{M}_A(v)$ *is a strict valuation.*

[17]The axiomatisation (Dhillon and Mertens, 1999) is for a finite set of agents.

Proof. Shift-invariance of the set of acceptable policies and $\mathbb{T}$-invariance of ς implies that if some acceptable policy π induces a life-time bundle (c_0, l_0) for an individual born at time 0, then the acceptable policy $\mathbb{S}_x(\pi)$ induces $(e^{\gamma x}c_0, l_0)$ for his clone born at time x. By homogeneity of utilities the worst and the best acceptable consumption for an individual of any type remain the same up to this $e^{\gamma x}$, so their utility difference (which is the normalisation factor) is proportional to $e^{(1-\rho^\tau)\gamma x}$ by homogeneity. By a last use of homogeneity, the normalised utility function $u_x^\tau(c, l)$ equals thus $w^\tau U^\tau(e^{-\gamma x}c, l)$, i.e., is the utility of "normalised consumption", up to a fixed, type-dependent factor.

Then the normalisation factor for consumption of an individual born at time $x + h$ is $e^{-\gamma(x+h)}$. So normalised consumption bundles are just shifted in time by $\mathbb{T}_h$. Hence $\mathbb{T}$-invariance of ς implies that, when ς is applied to the $\mathbb{T}_h$-transformed policy, the normalised consumption and labour bundle of any individual (along with his birthdate) is shifted forward in time by h, and thus the normalised utilities are too. □

Corollary 6. *The* RU*-normalised life-time utility of an agent of type τ on a balanced growth path is independent of his birth-date.*

Proof. Apply prop. 3 to the case where the basic policy space is a singleton, that doesn't do anything, and where the outcome function maps to the chosen balanced growth path. □

Corollary 7. *If $W = S \circ (\mathbb{M}_A(v) - \mathbb{M}_A(\bar{u}))$ is Gateaux-differentiable on P at $\bar{\pi}$, then its differential equals $\int e^{(\nu-\beta)t}\langle q, \delta\pi(t)\rangle dt$ for some $q \in E^*$.*

Proof. Prop. 3 and thm. 1 □

4.2.2. *The discount rate for cost-benefit analysis.* Using cor. 7 one gets, as in sect. 4.1.1, the discount rate $\beta + \gamma$ implied by relative utilitarianism, well-defined even for a population with distinct types ρ^τ:

Corollary 8. *If $W = S \circ (\mathbb{M}_A(v) - \mathbb{M}_A(\bar{u}))$ is Gateaux-differentiable on P at 0, then its differential equals $\int e^{-(\beta+\gamma)t}\langle q, \omega(t)\rangle dt$ for some $q \in M^*$.*

Restricting basic policies b to have all the same distribution over age-groups and types, and setting $\beta = 0$, yields then the main result in Mertens and Rubinchik (2006).

The derived discount rate, γ, differs generically from the interest rate, even at the 'golden rule' equilibrium if ν is non-zero.

5. A Value-of-Life Argument

One touch-stone is the case $\beta = 0$, no discounting of utilities. Do the presciptions of the theory then indeed correspond to the intuitive meaning of treating individuals of different generations equally?

A compelling implication of equal treatment is to give equal weight to individual lives,[18] so, in cost-benefit analysis, to their monetised values.

The monetised value of life, according to any criteria [e.g., each of the four in Mishan's (1971) introduction, or even judicial criteria in assessing damages], is proportional to the individual's life-time income, or to average life-time income at his time.

This is also formally true in the above economic model, when allowing for a variable life-span: individual life-time utility is homogeneous, so willingness to pay to extend life is proportional to life-time income. Indeed, one can evaluate the (relative) increase in utility from augmenting individual life-time, T_τ, assuming that during this additional stretch of time the person's consumption is in fixed proportion to his previous consumption. By homogeneity, an identical increase in utility can be generated by a proportional increase in life-time consumption, i.e., a proportional increase in life-time income.

Note that along a balanced growth path (the outcome of a basic policy), this income is proportional to $e^{\gamma t}$. So, to treat individuals of all generations equally, future incomes have to be discounted *exactly* at rate γ, as implied by relative utilitarianism (corollary 8).

6. The choice of social welfare functional

Since traditional and relative utilitarianism have so different implications for discounting, we wish to discuss some of the underlying principles of equitable treatment of different generations incorporated in each of the criteria. This should provide a better guidance as to which of the two is better suited for evaluating long-term projects.

Interestingly, the implication of relative utilitarianism is consistent with accepted public policy: Circular A-4 of the U.S. Office of Management and Budget (2003) requires that all executive agencies and establishments conduct a "regulatory analysis" for any new proposal, and more specifically (pp. 33–36), a cost-benefit analysis, at the rates of both 3% and 7%.[19] The circular does refer explicitly to the requirement of equity vis-à-vis of future generations, and acknowledges it by requiring, for projects with substantial long-term impact, a further analysis at a "still lower but positive" discount rate (p. 36), but more specific suggestions are hard to find. The rate based on the relative utilitarian criterion, $\approx 2\%$, falls exactly in this range.

[18] See, e.g.: "Morally speaking, there is no difference between current and future risk. Theories which, for example, attempt to discount effects on human health in twenty years to the extent that they are equivalent to only one-tenth of present-day effects in cost-benefit considerations are not acceptable." Wildi, Appel, Buser, Dermange, Eckhardt, Hufschmied, and Keusen (2000)

[19] Both rates are rationalised there as "the interest rate": the first one relative to private savings, the second one relative to capital formation and/or displacement, i.e., as the gross return on capital.

Remarkably, relative utilitarianism is also consistent with the 'balanced generational policy' as presented in Kotlikoff (2002, p. 1905), requiring "[...] that the generational accounts [lifetime net tax burdens] of all future generations are equal, except for a growth adjustment."

The discount rate $\beta + \rho\gamma$ based on traditional utilitarianism (cor. 5) is well-known in the applied literature. Even if individual risk attitudes were identical (far from the empirical findings, see Einav and Cohen (2007)), to get acceptable conclusions, one has to set ρ close to unity (e.g., Stern, 2007, pp. 6–11). So, the choice of a social discount rate involves imposing individual risk preferences and contradicts fully any utilitarian foundation, since only the indifference map is retained as individual characteristic. Without forcing ρ to unity, based on reported estimates (Drèze, 1981; Einav and Cohen, 2007), the implied discount rate would be far above the range suggested by the OMB. Note that forcing ρ to 1 is equivalent to forcing the discount rate implied by RU.

The main reason ρ enters the calculation of the social discount rate under traditional utilitarianism is the presumption that the marginal utility of income is independent of the environment surrounding the individual. In particular, a 1% increase in real income of any of our contemporaries has the same effect as it would 100 years ago for the same individual *with the same real income*.

In contrast, relative utilitarianism, in the context of a growing economy, implies that to compare individual utility differences, the utilities have first to be normalised over the space of feasible policies (consumption paths). As a consequence, even in the presence of economic growth, the social value of a 1% increase in real income of an individual *at a given quantile* of the income distribution is independent of the date. Only if individual utilities are logarithmic, as in Stern (2006), can such an implication can be derived under traditional utilitarianism.

7. Concluding remarks

The problem of evaluating policy changes, i.e., finding the derivative of social welfare with respect to policy variations, is reduced by the above results to the static problem of evaluating q, a linear functional on B, the space of basic policies. If B is finite-dimensional, for example, q can be computed by a finite number of evaluations (direct computations) of the derivative in the direction of each of the $b \in B$ constituting a basis of B. Arguably, the effect of a perturbation in the direction of a *constant* policy should be easier to compute, especially, in the view of the main result assuring the derivative should be of the form $q(b) \int e^{\zeta t}\phi(t)dt$, for some $\zeta \in \mathbb{R}$, where $\phi \in K$ describes the 'intensity' of perturbation away from the status-quo in the direction of b.[20]

[20]Mertens and Rubinchik (2007) contains an example of such a computation, for a one-dimensional basic policy space.

However, applicability of the results hinges on the the differentiability assumption. For the case of endowment perturbations, corollaries 5 and 8, verifying differentiability would be a straight extension of Debreu's 1976 classical generic regularity theorem. There are, however, several aspects that make such an extension highly non-trivial. First, it is well-known that overlapping generations models can give rise to indeterminate equilibria, see e.g., Kehoe and Levine (1984), Geanakoplos and Brown (1985). Next, even if regularity is assured, already for the welfare function to be well-defined, the equilibrium has to be stable: the perturbed equilibrium has to converge sufficiently fast back to the unperturbed solution, both at $+\infty$ and at $-\infty$.

This program was successfully completed for a very particular case in Mertens and Rubinchik (2007), ensuring thus at least non-vacuity of our results. It seems a crucial aspect there too is to start time at $-\infty$.

References

Arrow, K. J. (1963): *Social choice and individual values*, Cowles Foundation for Research in Economics monographs. John Wiley & Sons, New York, London, Sydney, second edn.

Arrow, K. J., and M. Kurz (1970): *Public Investment, the Rate of Return, and Optimal Fiscal Policy.* The Johns Hopkins Press, Baltimore and London.

Asheim, G. B. (1991): "Unjust intergenerational allocations," *Journal of Economic Theory*, 54, 350–371.

Asheim, G. B., T. Mitra, and B. Tungodden (2006): "Sustainable recursive social welfare functions," mimeo.

d'Aspremont, C. (2006): "Formal Welfarism and Intergenerational Equity," in *Intergenerational Equity and Sustainability, Conference Proceedings of the IEA Roundtable Meeting on Intergenerational Equity*, ed. by J. Roemer, and K. Suzumura. To Appear.

Basu, K., and T. Mitra (2003): "Aggregating infinite utility streams with intergenerational equity: the impossibility of being Paretian," *Econometrica*, 71(5), 1557–1563.

Chichilnisky, G. (1996): "An axiomatic approach to sustainable development," *Social Choice and Welfare*, 13, 231–257.

Coate, S. (2000): "An Efficiency Approach to the Evaluation of Policy Changes," *Economic Journal*, 110(463), 437–455.

Crespo, J. A., C. Núñez, and J. P. Rincón-Zapatero (2007): "The impossibility of representing infinite utility streams," Universidad Carlos III de Madrid WP07-55.

de la Croix, D., and P. Michel (2002): *A Theory of Economic Growth: Dynamics and Policy in Overlapping Generations.* Cambridge University Press.

Debreu, G. (1976): "Regular Differentiable Economies," *The American Economic Review*, 66(2), 280–287, Papers and Proceedings of the Eighty-eighth Annual Meeting of the American Economic Association.

Dhillon, A., and J.-F. Mertens (1999): "Relative Utilitarianism," *Econometrica*, 67(3), 471–498.

Drèze, J., and N. Stern (1987): "The Theory of Cost-Benefit Analysis," in *Handbook of Public Economics*, ed. by A. J. Auerbach, and M. Feldstein, vol. II, chap. 14. Elsevier Science Publishers.

Drèze, J. H. (1981): "Inferring Risk Tolerance from Deductibles in Insurance Contracts," *The Geneva Papers on Risk and Insurance*, pp. 48–52.

Edwards, R. E. (1965): *Functional Analysis*. Holt, Rinehart and Winston.

Einav, L., and A. Cohen (2007): "Estimating Risk Preferences from Deductible Choice," *American Economic Review*, 97(3), 745–788.

Fleurbaey, M., and P. Michel (2003): "Intertemporal equity and the extension of the Ramsey criterion," *Journal of Mathematical Economics*, 39, 777–802.

Gale, D. (1973): "Pure Exchange Equilibrium of Dynamic Economic Models," *Journal of Economic Theory*, 6, 12–36.

Geanakoplos, J. D., and D. J. Brown (1985): "Comparative Statics and Local Indeterminacy in OLG Economies: An Application of the Multiplicative Ergodic Theorem," Cowles Discussion Paper 773.

Gel'fand, I. M., and G. E. Shilov (1959): *Obobshennuye funkzii i deistviya nad nimi*. Fizmatgiz, Moscow, 2 edn.

Harsanyi, J. C. (1955): "Cardinal Welfare, Individual Ethics, and Personal Comparisons of Utility," *Journal of Political Economy*, 63(4), 309–321.

Johnston, L. D., and S. H. Williamson (2007): "Source note for US GDP, 1789-Present," Economic History Services, http://www.eh.net/hmit/gdp/.

Kehoe, T. J., and D. K. Levine (1984): "Regularity in overlapping generations exchange economies," *Journal of Mathematical Economics*, 13(1), 69–93.

King, R. G., C. I. Plosser, and S. T. Rebelo (2002): "Production, Growth and Business Cycles: Technical Appendix," *Computational Economics*, 20(1-2), 87–116.

Koopmans, T. C. (1960): "Stationary ordinal utility and impatience," *Econometrica*, 28, 287–309.

Kotlikoff, L. J. (2002): "Generational Policy," in *Handbook of Public Economics*, ed. by A. J. Auerbach, and M. Feldstein, vol. 4, chap. 27, pp. 1873–1932. Elsevier.

Mertens, J.-F., and A. Rubinchik (2006): "Intergenerational equity and the discount rate for cost-benefit analysis," CORE DP 2006/91.

——— (2007): "Validity of Comparative Statics in an overlapping generations model with exogenous growth," .

Mishan, E. J. (1971): "Evaluation of Life and Limb: A Theoretical Approach," *The Journal of Political Economy*, 79(4), 687–705.

——— (1976): *Cost-benefit analysis*. Praeger Publishers.

Pigou, A. C. (1932): *The Economics of Welfare*. Macmillan, 4th edn.

Ramsey, F. P. (1928): "A Mathematical Theory of Saving," *The Economic Journal*, 38(152), 543–559.

Samuelson, P. A. (1958): "An Exact Consumption-Loan Model of Interest with or without the Social Contrivance of Money," *The Journal of Political Economy*, 66(6), 467–482.

Schwartz, L. (1957-59): *Théorie des distributions*, vol. I and II. Hermann, Paris.

Sidgwick, H. (1874): *The Methods of Ethics*. Macmillan and Company, London.

Stern, N. (2006): "Stern Review on the economics of climate change," http://www.hm-treasury.gov.uk/independent_reviews/stern_review_economics_climate_change/stern_review_report.cfm.

——— (2007): "Value judgements, welfare weights and discounting: issues and evidence," http://www.hm-treasury.gov.uk/media/C/8/Paper_B.pdf.

U.S. Office of Management and Budget (2003): "OMB Circular A-4: Regulatory Analysis," .

Wildi, W., D. Appel, M. Buser, F. Dermange, A. Eckhardt, P. Hufschmied, and H.-R. Keusen (2000): "Disposal Concepts for Radioactive Waste," written on behalf of the Federal Department for the Environment, Transport, Energy and Communication of Switzerland.