Learning Options in Coordination Problems*

(Preliminary and incomplete)

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Abstract

We study intertemporal tradeoffs of players who fear a potential crisis in an environment with strategic complementarities. Such players may wish to delay their investment decisions in order to gather additional information. Drawing on the global game framework, we characterize the effects of the learning option on the coordination outcome. The option can either enhance or hamper efficient coordination, and we are able to determine which effect arises based only on qualitative but not quantitative features of the payoff functions. This allows to draw normative conclusions for the provision of the option.

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1 Introduction

Many economic interactions exhibit strategic complementarities and multiple equilibria. For instance, some projects succeed only if enough agents choose risky investment instead of a safe outside option, and hence such projects may be prone to crises that stem from inefficient coordination. An agent at an outset of the crisis is uncertain about the outcome of the crisis, and hence chooses her action under strategic uncertainty over the fellow-players' actions. The crisis occurs if the safe but inefficient action, such as run on bank, is the best response under the strategic uncertainty. This intuition is formalized in the theory of global games introduced by Carlsson and van Damme (1993) and extended by Morris and Shin (2003). Global game is a static game of incomplete information in which players receive private noisy signals about an underlying economic fundamental. The game has a unique equilibrium characterized by a threshold signal such that players choose risky action for signals above the threshold and safe action below the threshold. The agent at the outset of the crisis corresponds to the threshold type in the global game, as the type observing the threshold signal is uncertain about the proportion of the fellow-players with signals above the threshold and hence uncertain about the fellow-players' actions.

The economic fundamental influences the payoff of each player not only directly but also through fellow-players' actions via the correlation the fundamental with the private information. Thus, the agent at the outset of the crisis would find an additional information about the economic fundamental very valuable, mostly because it would reduce her strategic uncertainty. Therefore, the agent is willing to sacrifice resources or investment opportunities for the additional information.

Our model focuses on the strategic consequences of this information-hunger during crises. We allow players in the global game to acquire additional information beyond their first private signals, and we study effects of this learning option on the equilibrium behavior. Our dynamic game is a simple extension of the standard static global game: It has two rounds and a continuum of players. Each player receives a private signal about economic fundamental in the first round and can play an irreversible action immediately or postpone the decision until round 2 at the beginning of which she receives the additional information. The payoff of the player depends on the economic fundamental and on the action profiles in each round.

While most types of players in the static global game would not find the additional information too useful, the types close to the threshold would find it very useful, as these types suffer the most from the strategic uncertainty. Hence the provision of the learning option may change the occurrence of the crisis, because it modifies the best responses of the agents under the strategic uncertainty. In the formal model, the provision of the learning option changes the incentives of the threshold type in round 1, and so the equilibrium threshold satisfying the indifference condition may change.

Provision of such learning option induces two effects. First, there is a direct (non-strategic) effect that additional information in round 2 reduces the risk associated with the reversible action. Second, there is an indirect (strategic) effect that some players may indeed reverse their actions after obtaining the additional information. Despite that these two effect have opposite impacts we can characterize the total effect.

We study the effect of the learning option under a rich set of setups and we split them based on the outcomes into three categories: (i) the provision of the option has no effect on the equilibrium outcome; (ii) the provision of the option enhances efficient coordination; and (iii) the provision of the option hampers efficient coordination. The setups are split into the three categories based only on *qualitative* properties of the game which may be observed by outside observers.

Our results on the effect of the learning option have normative and positive implications. First, as the provision or the lack of the provision of the learning option may enhance efficient coordination, then a policy maker may enhance welfare by subsidizing or penalizing the learning. Second, the static global game that is often used as an applied modelling tool abstracts from the learning, and hence it is useful to understand when and how the provision of the learning option distorts the static result.

The equilibrium analysis is centered around the indifference condition(s) imposed on the threshold type(s). This condition has a simple form in the static game because the threshold type in the static global game has *uniform* belief over fellow-players' actions, see Morris and Shin (2003). To emphasize the connection to Laplace's principle of insufficient reason, Morris and Shin dub this belief as *Laplacian*.

Our equilibrium analysis of the dynamic game also focuses on the indifference conditions of the threshold types, one for each round, but in order to to successfully analyze the conditions, we need to formulate the expected payoffs of the threshold types in a simple way. A direct evaluation of the expected payoff of the threshold type in round 1 of the dynamic game is complicated: the threshold type is deciding whether to do an irreversible decision now or whether to postpone the decision, and hence she needs to evaluate the expected *option* value. The threshold type suffers from the strategic uncertainty, which further complicates the evaluation of the option as its value depends on the actions of the fellow-players. This strategic nature of the environment precludes the application of tools developed for analysis of options in non-strategic environments such as that in Dixit (1989). Despite of these complications we find that the threshold type in round 1 of the dynamic game forms payoff expectations *as if she had simple Laplacian beliefs* over fellow-players actions in *both* rounds, and this finding substantially simplifies the equilibrium analysis.

Instead of a direct equilibrium characterization in the dynamic game, we characterize the effect of the learning option indirectly. To avoid the evaluation of the option value in the strategic environment, we map the dynamic game to a *virtual static* game and then solve the virtual static game by existing static global game tools. Such mapping simplifies the analysis — the virtual static game is simpler as the players do no have any option to evaluate. But the virtual static game will be informative about the original dynamic game only if the mapping is not too distortive. We find a mapping which assures that the threshold types in the original dynamic and the virtual static games face environments that are identical in all aspects relevant for the equilibrium analysis. In particular, to preserve the expected payoffs of the threshold types across the original dynamic and the virtual static games, we need to compensate the players in the virtual static game for their loss of the reversibility (learning) option. We achieve this by finding a virtual signal which is superior to the original signal, and its advantage is equal to the advantage of the option in the original dynamic game. This transforms the advantage of the option to the advantage of the superior signal and the latter is tractable by the existing global game tools.

Intertemporal tradeoffs and learning are recognized and emphasized as central topics in a growing piece of literature on dynamic global games. One stream of this literature studies repeated coordination in an evolving environment. In Burdzy, Frankel, and Pauzner (2001) economic fundamental evolves according to a random process and players, who repeatedly interact in coordination problems, can reverse their actions only at randomly arising opportunities. Levin (2001) studies an overlapping generation global game in a evolving environment. In Morris and Shin (1999), players repeatedly play a global game with evolving fundamental. In these models, history of fundamentals serves as a public signal about the current fundamental, and this may induce history-dependent equilibrium thresholds.

Another stream of the dynamic global game literature focuses on equilibrium multiplicity induced by public learning in an environment with a static fundamental. In Angeletos, Hellwig, and Pavan (2007a) players repeatedly interact in a dynamic coordination game of regime change and in later rounds publicly observe the (lack of) success of early attacks. In Angeletos and Werning (2006) information is publicly revealed through prices. In Angeletos, Hellwig, and Pavan (2006), the role of public signal is fulfilled by policy chosen in equilibrium by a policy maker prior to the actions of agents. As emphasized in Angeletos, Hellwig, and Pavan (2007b), these models maintain predictive power despite of the multiple equilibria induced by public learning, as some of the predictions hold across the equilibria.

Our paper is related to a third stream of the literature. Dasgupta (2007), and Heidhues and Melissas (2006) allow players to engage in social learning by imperfectly observing aggregate statistics of early actions.[1] The observations are assumed to be private, and hence, unlike in the second stream of the literature, equilibrium uniqueness may be preserved. The papers focus on the effects of the provision of the learning option. Dasgupta draws unambiguous conclusions about the welfare effect of the provision of the option in his setup. Heidhues and Melissas delineate conditions for equilibrium uniqueness in their setups. Both papers solve the games under specific simple payoff functions by methods that are not readily

[1]Dasgupta, Steiner, and Stewart (2007) study a dynamic model with reversible safe action and learning. They provide an asymptotic result that the coordination outcome is approximately efficient in environments with gradual learning and many investment opportunities.

extensible to general payoffs. Our method, based on the mapping of the dynamic problem to the virtual static one, is indirect and more general. It allows to asses effects of the learning option across large sets of payoff functions.

The organization of the paper is the following. In Section 2 we informally describe two illustrative applied problems which we then resolve by our framework at the end of the paper in Section 8. Section 3 introduces the model. The central technical result describing the Laplacian property in dynamic games is formulated in Section 4. The Laplacian Property is particularly useful in global games because they have unique equilibria in threshold strategies as described in Section 5. Section 6 contains main comparative result driven by the Laplacian property. Section 7 contains several modifications and generalizations of our basic dynamic global game.

2 Applied questions

The message of the paper at hand is methodological; it proposes a new general method by which modelers can incorporate the informational and intertemporal tradeoffs into the standard static global game framework. We illustrate the method on two applied examples. In this section, we discuss the informational and intertemporal tradeoffs arising in two leading global game applications. We resolve the proposed questions by applying our results in Section 8.

2.1 Debt crises

Morris and Shin (2004) use the static global game framework to study debt crises. They consider a group of creditors who have invested into a project. The model focuses on the interim stage of the project — after the creditors have invested and before the completion of the project. Each creditor has an opportunity to withdraw and seize a collateral in the interim stage. The project succeeds if the mass of creditors who stay in the project exceeds a critical mass that depends on economic fundamental. If the project succeeds the creditors who have stayed receive a face value of the repayment which exceeds the collateral value, and if the project fails they receive a payoff below the collateral.

The creditors face a coordination problem as the incentive to withdraw increases with the mass of the withdrawing creditors. The equilibrium uniqueness assured by the global game assumptions allows for comparative statics that gives a better fit to the empirical data than some other less strategic models, such as Merton (1974).

The prediction of the inefficient debt crises begs for normative analysis of the provision of the exit option. Should we prevent the debt crises by penalizing the exit decision? Such an analysis cannot be performed within the static model as any change in the exit option would induce changes in entry decisions and these are exogenous in the static model.

Our dynamic framework allows to treat the entry decisions endogenously, and hence

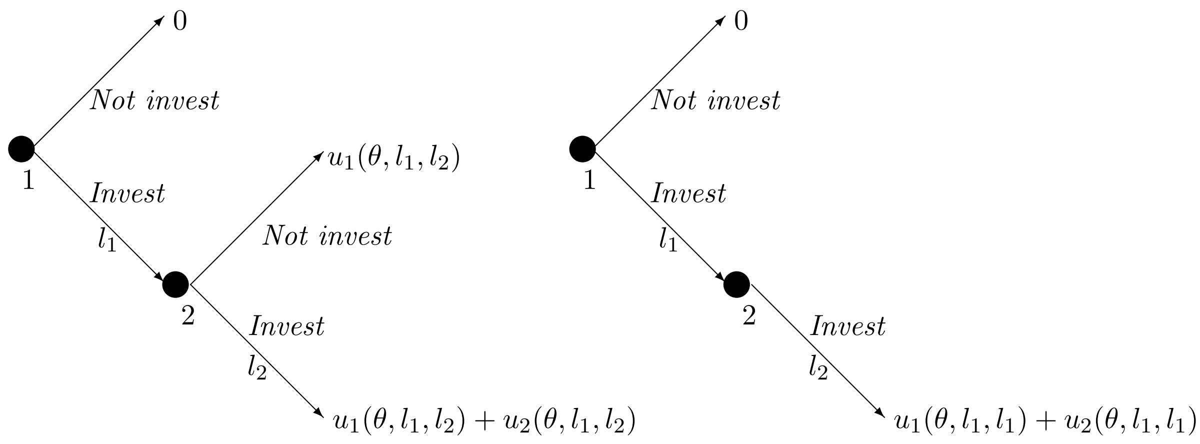

Figure 1: Dynamic game Γ_{dyn}^{risky} (left) and the benchmark static game Γ_{st} (right).

it allows for the normative analysis of the exit option. An exit penalty will influence the entry decision in two opposing ways: (i) the exit penalty decreases the incentive to enter by the direct effect as player plans to exit in some contingencies, but (ii) it also increases the incentive to enter as it decreases the incidence of the coordination failures. Our framework unambiguously predicts which of the effects prevails, see Section 8.2 for details.

2.2 Currency attacks

Morris and Shin (1998) use the global games framework to analyze self-fulfilling currency attacks. They consider a group of speculators who simultaneously decide whether to attack a currency by short-selling it. The attack succeeds if its size exceeds a critical attack size that increases with the quality of the economy, and players receive heterogenous private signals about the quality. The equilibrium uniqueness assured by the information heterogeneity and by other global game assumptions allows for an unambiguous comparative statics. This in turn allows to test various normative recommendations such as the "sand in the wheels" suggestion by Eichengreen, Tobin, and Wyplosz (1995).

The simplification of the static model by Morris and Shin is that it does not allow the speculators to acquire additional information. This static nature of the model is particularly questionable at the outset of the attacks during which the potential speculators are highly motivated to acquire the additional information even if it is costly. Our dynamic framework allows to examine the strategic consequences of the provision of the learning option; see Section 8.1 for details.

3 Model

The *dynamic* game Γ_{dyn}^{risky} with *reversible risky* action has two rounds. There is a continuum of players indexed by $i \in [0, 1]$ who decide whether to invest into a project. At the beginning of round 1 all players decide simultaneously whether to play *invest* or *not invest*. By not investing the player reaches her final node and receives payoff that is normalized to 0. All players that have invested in round 1, decide then simultaneously whether to *invest* or *not invest* in round 2. Players who do not invest in round 2 receive "partial payoff" $u_1(\theta, l_1, l_2)$ for the participation in the first stage of the project, and those investing in round 2 receive "full payoff" $u_1(\theta, l_1, l_2) + u_2(\theta, l_1, l_2)$ for the participation in the first and the second stages of the project. Letter θ denotes a payoff parameter and l_t (where $t = 1, 2$) denotes the *investment level* in the project at the end of round t; that is l_1 is the measure of players choosing *invest* in round 1, and l_2 is the measure of players choosing *invest* in both rounds. See Figure 1 for illustration. Functions u_1 and u_2 are real-valued and are defined on the domain $\{(\theta, l_1, l_2) \in \mathbb{R} \times [0, 1] \times [0, 1] : l_2 \leq l_1\}$.

Note that the risky action labelled as *invest* is reversible and the safe action *not invest* is irreversible. Later, in Section 7.1, we discuss a related game Γ_{dyn}^{safe} in which the safe action is reversible and the risky action is the irreversible one.

We assume heterogeneity in private information along the lines of the global games literature. Nature draws common economic fundamental θ from improper[2] uniform distribution on $\mathbb{R}$. Each player i observes her first private signal

$$x_1^i = \theta + \sigma\varepsilon_1^i + \sigma\varepsilon_2^i$$

at the beginning of the first round. In the second round, each player i observes the second private signal

$$x_2^i = \theta + \sigma\varepsilon_2^i.$$

By the assumption on the error structure, x_2^i is a sufficient statistics for θ over the data set (x_1^i, x_2^i).[3] The state of the world is $\omega = (\theta, (x_1^i, x_2^i)_i)$.

The errors ε_t^i, $(t = 1, 2)$ are independent across investors and rounds and independent of θ. The cumulative distribution function of the errors ε_t^i, $(t = 1, 2)$ are denoted as $F_t(\cdot)$. The cumulative distribution function of $\varepsilon_1^i + \varepsilon_2^i$ is denoted by $F(\cdot)$. Both ε_1^i, ε_2^i are continuous random variables and their supports are finite or infinite intervals. The parameter σ is used to scale the noise; some of our results are formulated in the limit of precise signals, which is formally the limit $\sigma \to 0$.

A particularly simple and relevant class of strategies are *threshold strategies*. Such a strategy of player i is characterized by threshold pair (x_1^*, x_2^*) such that i plays *invest* in round t if and only if $x_t^i > x_t^*$.

We will compare the dynamic game Γ_{dyn}^{risky} with a benchmark *static* game Γ_{st} which differs from Γ_{dyn}^{risky} only in the lack of the reversibility option — each player can move only in round 1; once a player invests in round 1, she must automatically invest in round 2;

[2]The use of such improper distribution is appropriate, because we work only with probabilities conditional on private signals and these are well defined; see Morris and Shin (2003) and Hartigan (1983) for more details on this issue.

[3]Similar information structure with sufficient statistics is used in Heidhues and Melissas (2006).

see Figure 1. Threshold strategy of player i in the static game is characterized by a single threshold $x^*_{1,st}$ such that i invests if and only if $x^i_1 > x^*_{1,st}$. To facilitate comparison with the dynamic game, we keep the lower index 1 when describing the signal in the static game despite that it has only one non-trivial round.

3.1 Discussion of the setup

We have made several simplifying assumptions in the setup of the model. First, we assume uninformative prior. While an informative prior would have been more realistic, it is well understood in the global game literature that results obtained for the uninformative prior are approximately valid for any prior if the private signals are sufficiently precise.

Second, the existence of a sufficient statistics[4] in round 2 assures that the information in round 2 can be summarized by a one-dimensional signal, and hence some of the standard one-dimensional global game tools apply. We conjecture that our results would continue to hold without the sufficient statistics assumption in the limit in which the second signal is increasingly more precise than the first signal, because then players would essentially ignore the first signal in round 2.

Third, we assume that the source of the additional information in round 2 is exogenous. This source can be endogenized by using Dasgupta's (2007) approach to social learning, where the players observe l_1 imprecisely in round 2. In particular, player i receives private signal

$$y^i = F^{-1}(l_1) + \varepsilon^i_2 \tag{1}$$

in round 2.[5] If the equilibrium is symmetric and involves threshold strategies, then information about the fundamental θ conveyed in y^i is equivalent to our exogenous information. This is because $l_1 = F\left(\frac{\theta - x^*_1}{\sigma}\right)$ in a threshold equilibrium and so x^i_2 defined as $x^i_2 = \sigma y^i + x^*_1$ is a private signal about θ with an error $\sigma\varepsilon^i_2$; $x^i_2 = \theta + \sigma\varepsilon^i_2$. We will occasionally use this interpretation of learning in the discussion of our results, but we keep the source of the additional information in round 2 to be exogenous in the main version of the model, because the source of the information is orthogonal to our results.

For instance, the social learning interpretation of the model is useful to gain intuition for the strategic relevance of the additional information in round 2. To that end, consider the limit of precise signals $\sigma \to 0$. At the first sight, and especially in games with very precise first signals, the additional information received in round 2 may not seem very relevant, and one may be tempted to conclude that the provision of the option would not make a difference. But this intuition fails for the threshold type in round 1 in an equilibrium in threshold strategies. The type observing the threshold signal will be uncertain about the actions of fellow-players even if her first signal is very precise. Such type will find the

[4] We seemingly assumed more than the existence: player observes the sufficient statistics directly in round 2. This can be assumed without loss of generality whenever the sufficient statistics exists.

[5] This modelling technique is also used by Angeletos and Werning (2006).

additional information in round 2 very valuable because it will provide her information about fellow-players' early actions, and in the social learning interpretation she simply imprecisely observes fellow-players' early actions. The threshold type plays a focal role in the analysis, and hence the learning option is strategically relevant even in the limit $\sigma \to 0$.

4 Laplacian beliefs

In this section, we analyze symmetric equilibria in threshold strategies. Such equilibria are completely characterized by *indifference* conditions on the threshold types: types $x_t^i > x_t^*$ prefer investing and hence they must expect non-negative payoff for investing, types $x_t^i < x_t^*$ expect non-positive payoff for investing, and by the continuity argument the threshold type must expect 0 payoff for investing.

The *beliefs* of the threshold types play a central role in the formulation of the indifference conditions. Morris and Shin (2003) find that threshold type's belief over l is uniform on $[0, 1]$ in the static game, and they refer to such belief as being *Laplacian.* The central technical finding of this paper is that the threshold type x_1^* in the dynamic game forms payoff expectations *as if* she had Laplacian beliefs; see Section 4.3.

Unless we impose some monotonicity assumptions on the payoff functions, the symmetric equilibrium in threshold strategies may fail to exist which would make the Laplacian property vacuous. We impose monotonicity assumptions that are sufficient for the existence and uniqueness of the threshold equilibria only in Section 5. This organization of the paper highlights that the Laplacian beliefs are driven solely by the structure of the threshold strategies and by the information structure. The monotonicity properties of the payoffs assumed later will enter the argument only indirectly by assuring the existence of the threshold equilibria.

We first review the existing result on Laplacian beliefs in static games in Section 4.1, and then we prove the Laplacian property in the dynamic game in Section 4.3.

4.1 Laplacian beliefs in the static game

We start by reviewing briefly the result of Morris and Shin (2003) for the static game.

Threshold x_1^* induces a function

$$l_1(\theta; x_1^*) = \Pr\left(x_1^i > x_1^* \mid \theta\right) \tag{2}$$

that specifies the investment level as a function of the realized fundamental θ. To simplify notation, we will often skip the second argument and write only $l_1(\theta)$. Note that l_1 is non-decreasing in θ and non-increasing in x_1^*. We denote the inverse function of $l_1(\theta)$ as $\vartheta_1(l_1; x_1^*)$, and often simply write $\vartheta_1(l_1)$.

The Laplacian property in the static game is formalized by the following Lemma:

Lemma 1. (Morris and Shin, 2003) *Conditional distribution* $l_1(\theta; x_1^*) \mid x_1^i = x_1^*$ *is uniform on* $[0,1]$.

The Laplacian property is useful in the formulation of the indifference condition on the threshold type because the condition simplifies to

$$\mathrm{E}\left[u(\theta, l_1(\theta)) \mid x_1^i = x_1^*\right] = \int_0^1 u(\vartheta_1(l_1; x_1^*), l_1)\, dl_1 = 0, \tag{3}$$

where u denotes the total payoff in the static game, $u = u_1 + u_2$.

The intuition behind the Laplacian property in the static game is the following. The threshold type knows that players who invest are those who received a higher signal than her own signal $x_1^i = x_1^*$. Therefore, the investment level is determined by the rank of her signal within the signals of the whole population of players. The only information the player has is her own signal, which is entirely uninformative over the rank of her own signal. The player therefore finds any rank equally probable, which translates into the uniform distribution over l_1. For future exposition, we emphasize that Laplacian property holds for any noise distribution, if the noise is independent of θ.

4.2 Preliminaries for the dynamic game

The pair of thresholds x_1^*, x_2^* induces a pair of functions $l_1(\theta; x_1^*)$, $l_2(\theta; x_1^*, x_2^*)$ that specify investment levels in round $t = 1, 2$. The function l_1 is the one defined by (2) as in the static game and[6]

$$l_2(\theta; x_1^*, x_2^*) = \Pr\left(x_1^i > x_1^* \text{ and } x_2^i > x_2^* \mid \theta\right). \tag{4}$$

Both l_1 and l_2 are non-decreasing in θ and non-increasing in x_1^*; l_2 is also non-increasing in x_2^*. Again, we will often simply write $l_2(\theta)$. The inverse function to $l_2(\theta; x_1^*, x_2^*)$ is denoted as $\vartheta_2(l_2; x_1^*, x_2^*)$. We will also need to express l_1 as a function of l_2 for which purpose we introduce $\lambda_1(l_2; x_1^*, x_2^*) = l_1(\vartheta_2(l_2; x_1^*, x_2^*); x_1^*)$, and similarly $\lambda_2(l_1; x_1^*, x_2^*) = l_2(\vartheta_1(l_1; x_1^*); x_1^*, x_2^*)$. To summarize, out of the triple of variables θ, l_1, l_2 related by (2) and (4), we can choose any one as the independent one and express the remaining two variables as its non-decreasing functions.[7]

The pair of thresholds x_1^*, x_2^* is characterized by a pair of indifference conditions. Let us start with the indifference condition on the threshold type x_2^* in round 2:

$$\mathrm{E}[u_2(\theta,\, l_1,\, l_2) \mid x_2^i = x_2^*] = 0. \tag{5}$$

Note, that since x_2^i is sufficient statistics for θ, we can omit conditioning on the first-round

[6] The values of l_1 and l_2 can also be expressed as integrals over densities of errors ε_t. For clarity we omit these expressions here, but provide them in the proof of Lemma 3.

[7] For the ease of notation, instead of l we use its Greek equivalent λ to denote the corresponding function. Similarly, the function corresponding to θ, is denoted by ϑ.

signal x_1^i.

We now move to the indifference condition in round 1. We write the expected payoff of the threshold type x_1^* in two parts. We introduce function $D_1(x_1^*, x_2^*)$ associated with the first stage of the project:

$$D_1 = \mathrm{E}[u_1(\theta, l_1, l_2) \mid x_1^i = x_1^*],$$

and function $D_2(x_1^*, x_2^*)$ associated with the second stage of the project:

$$D_2 = \mathrm{E}\left[\max\left\{0,\, \mathrm{E}[u_2\left(\theta,\, l_1,\, l_2\right) \vert\, x_2^i]\right\} \mid x_1^i = x_1^*\right]. \tag{6}$$

The function D_2 reflects the value of the reversibility option from the perspective of round 1: player will do an optimal decision in round 2 based on her signal x_2^i which is yet unknown in round 1.

The indifference condition in round 1 is

$$D_1 + D_2 = 0. \tag{7}$$

4.3 Laplacian beliefs in the dynamic game

For any θ and x_1^*, the investment level $l_1(\theta; x_1^*)$ is identical across the static and the dynamic games. Therefore, Lemma 1 applies, $l_1 \mid x_1^i = x_1^*$ is uniform and

$$D_1 = \int_0^1 u_1\left(\vartheta_1(l_1), l_1, \lambda_2(l_1)\right) dl_1. \tag{8}$$

The following Lemma, central for the paper, states that D_2 can also be written as an expectation based on Laplacian belief over l_2.

Lemma 2. (Central Lemma) *If* (x_1^*, x_2^*) *satisfies the indifference condition* (5) *in round 2 then*

$$D_2 = \int_0^1 u_2\left(\vartheta_2(l_2),\, \lambda_1(l_2),\, l_2\right) dl_2. \tag{9}$$

Before proceeding with the proof of Lemma 2, let us try to obtain an expression for D_2 directly. We can write D_2 as

$$D_2 = \int_0^1 u_2\left(\vartheta_2(l_2),\, \lambda_1(l_2),\, l_2\right) dP(l_2), \tag{10}$$

where

$$P(l) = \Pr(l_2 < l \mid x_1^i = x_1^* \text{ and } x_2^i > x_2^*) \cdot \Pr(x_2^i > x_2^* \mid x_1^i = x_1^*).$$

The logic behind (10) is the following: The threshold type $x_1^i = x_1^*$ first conditions on her playing *invest* in round 2, and then, conditional on investing she computes distribution of

l_2. Note that $P(\cdot)$ is *not* c.d.f. of uniform distribution.[8] Despite that, Lemma 2 claims that D_2 is the expectation based on *uniform* distribution of l_2.

The direct characterization of D_2 by (10) is cumbersome, because the function $P(\cdot)$ is a complicated object reflecting distributional assumptions on errors and the relative positions of the thresholds x_1^*, x_2^*. The advantage of Lemma 2 is that it circumvents the function $P(\cdot)$; the simple Laplacian integral in (9) gives the correct value of D_2 despite that the relevant belief of the threshold type x_1^* is *not* uniform.

Note also that we can unify the formulas for D_1 and D_2 by rewriting them as *Riemann-Stieltjes integrals* with respect to functions $l_1(\theta)$ and $l_2(\theta)$. More precisely, for $t = 1, 2$, we obtain

$$D_t = \int_{-\infty}^{+\infty} u_t(\theta, l_1(\theta), l_2(\theta))\, dl_t(\theta). \tag{11}$$

Although we will mostly use the expressions involving uniform distribution, this form will become useful for some results.

Proof of Lemma 2. The proof is based on a mapping to a virtual static game. We will proceed it three steps. First, we define the mapping and list its desirable properties. Second, we prove (9) under assumption that the mapping with the properties exist. Third, we find a mapping that satisfies those properties.

The first step is as follows. Let us fix the thresholds x_1^* and x_2^*. We consider a mapping $d : \mathbb{R}^2 \to \mathbb{R}$ and let $\tilde{x}^i = d(x_1^i, x_2^i)$. The mapping d induces the following *virtual static* game: Each of the continuum of players indexed by $i \in [0, 1]$ simultaneously chooses action from the set $\{$*invest*, *not invest*$\}$. The payoff function is

$$\tilde{u}(\theta, \tilde{l}) = u_2(\theta, \lambda_1(\tilde{l}; x_1^*, x_2^*), \tilde{l}),$$

where $\tilde{l}$ is the measure of players who choose action *invest*, and the function $\lambda_1(\cdot; x_1^*, x_2^*)$ is defined in the original dynamic game. The state of the world is $\tilde{\omega} = (\theta, (\tilde{x}^i)_i)$ with $\tilde{x}^i = d(x_1^i, x_2^i)$ for all i and each player i observes only $\tilde{x}^i$. The probability distribution over $\tilde{\omega}$ is the one induced by the probability distribution in the original dynamic game, and by the mapping d. Note that the economic fundamental θ in the virtual static game has again the improper uniform distribution on $\mathbb{R}$.

We define $\tilde{x}^* = d(x_1^*, x_2^*)$ and interpret it as a virtual threshold. We consider a strategy profile in which all players in the virtual game play according to a symmetric threshold profile with threshold $\tilde{x}^*$, so that $\tilde{l}(\theta) = \Pr(\tilde{x}^i > \tilde{x}^* \mid \theta)$.

Below we construct a mapping $d(x_1^i, x_2^i)$ with the following three properties:

(i) $D_2 = \mathrm{E}\left[\tilde{u}(\theta, \tilde{l}) \mid \tilde{x}^i = \tilde{x}^*\right]$.

[8] This is easy to see in a case when the second signal is much more precise than the first one. Then player in round 1 believes that if she will receive $x_2^i > x_2^*$ in round 2, most of other players will receive $x_2^j > x_2^*$ as well. Hence then $P(\cdot)$ assigns most of the weight to the neighborhood of 1.

(ii) ($x_1^i > x_1^*$ and $x_2^i > x_2^*$) if and only if $\tilde{x}^i > \tilde{x}^*$.

(iii) Virtual error $\tilde{\varepsilon}^i$, defined as $\tilde{\varepsilon}^i = \tilde{x}^i - \theta$, is a random variable independent of θ.

In the second step we prove (9). The three properties have the following implications: property (iii) assures that Laplacian property holds in the virtual static game; see Lemma 1. Therefore, the conditional random variable $\tilde{l}(\theta) \mid \tilde{x}^i = \tilde{x}^*$ is uniform on $[0,1]$. Note that the particular distribution of the virtual error $\tilde{\varepsilon}^i$ is irrelevant for the argument because the Laplacian property is error-independent. The Laplacian property implies that virtual threshold type $\tilde{x}^i = \tilde{x}^*$ has the following expected payoff for investing:

$$\int_0^1 \tilde{u}(\tilde{\vartheta}(\tilde{l}), \tilde{l}) d\tilde{l},$$

where $\tilde{\vartheta}(\tilde{l})$ is the inverse function of $\tilde{l}(\theta)$.

Property (ii) implies that type $\tilde{x}^i = d(x_1^i, x_2^i)$ chooses to invest in the virtual static game if and only if player i with signals x_1^i, x_2^i invests in the both rounds of the original dynamic game. Hence $\tilde{l}(\theta) = l_2(\theta)$. Then their inverse functions are equal as well, $\tilde{\vartheta}(l) = \vartheta_2(l)$ and the expected payoff of the virtual threshold type $\tilde{x}^*$ is

$$\int_0^1 \tilde{u}(\vartheta_2(l_2), l_2) dl_2 = \int_0^1 u_2\left(\vartheta_2(l_2),\ \lambda_1(l_2),\ l_2\right) dl_2.$$

The right hand side is the expectation based on the Laplacian belief over l_2 in the original dynamic game as stated in Lemma, and by property (i) it is equal to D_2.

To complete the proof, we proceed with the third step. We will demonstrate that the mapping

$$d(x_1^i, x_2^i) = \min\{x_1^i - x_1^*, x_2^i - x_2^*\}$$

satisfies properties (i)–(iii). Note that the virtual threshold $\tilde{x}^* = d(x_1^*, x_2^*)$ equals 0 under this mapping.

Mapping d satisfies property (ii) by definition; see Figure 2 for illustration. It also satisfies property (iii), as follows from this computation

$$\begin{aligned}\tilde{\varepsilon}^i = \tilde{x}^i - \theta = \min\{x_1^i - x_1^*, x_2^i - x_2^*\} - \theta = \min\{\theta + \varepsilon_1^i + \varepsilon_2^i - x_1^*, \theta + \varepsilon_2^i - x_2^*\} - \theta = \\ = \theta + \varepsilon_2^i + \min\{\varepsilon_1^i - x_1^*, -x_2^*\} - \theta = \varepsilon_2^i + \min\{\varepsilon_1^i, x_1^* - x_2^*\} - x_1^*\end{aligned}$$

and from the fact that the last expression is independent of θ, because ε_1^i and ε_2^i are assumed to be independent of θ.

In order to show that the mapping d satisfies property (i), we compare the payoff of the threshold types in the original dynamic game and in the virtual static game. First, consider the threshold type $x_1^i = x_1^*$ in round 1 of the original dynamic game. If Nature chose state of the world such that the error $\varepsilon_1^i > x_2^* - x_1^*$ then i will receive signal $x_2^i < x_2^*$ in round

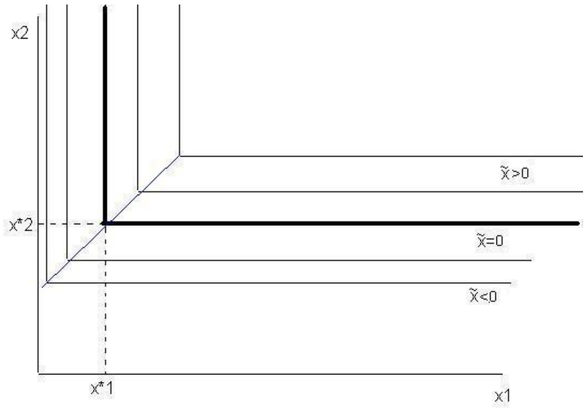

Figure 2: Mapping d. The Leontief-like curves are sets of pairs of original signals that map to a fixed virtual signal.

2. Hence she will not invest in round 2 and will receive payoff 0. If $\varepsilon_1^i \leq x_2^* - x_1^*$, player i will receive signal $x_2^i \geq x_2^*$, will invest and will receive a non-negative expected payoff conditional on $x_2^i = x_1^* - \varepsilon_1^i$.

Now consider the threshold type $\tilde{x}^i = \tilde{x}^*$ in the virtual static game. If Nature chose a state of the world with $\varepsilon_1^i > x_2^* - x_1^*$ then $x_2^i - x_2^* < x_1^i - x_1^*$. Hence $\tilde{x}^i = \min\{x_1^i - x_1^*, x_2^i - x_2^*\} = x_2^i - x_2^*$, and as $\tilde{x}^i = \tilde{x}^* = 0$, it must be that $x_2^i = x_2^*$ in this state of the world. Thus, the expected payoff for investing is 0 in the contingencies where $\varepsilon_1^i > x_2^* - x_1^*$, because x_2^* satisfies the indifference condition (5) in round 2 of the original dynamic game. Hence, in the contingencies where $\varepsilon_1^i > x_2^* - x_1^*$, the expected payoff is identical to the expected payoff of the type x_1^* of the original dynamic game.

On the other hand, if Nature chose a state of the world with $\varepsilon_1^i \leq x_2^* - x_1^*$, then $x_2^i - x_2^* \geq x_1^i - x_1^*$. Hence, $\tilde{x}^i = \min\{x_1^i - x_1^*, x_2^i - x_2^*\} = x_1^i - x_1^*$, and as $\tilde{x}^i = \tilde{x}^* = 0$, it must be that $x_1^i = x_1^*$ and $x_2^i = x_1^* - \varepsilon_1^i \geq x_2^*$ in this state of the world. Hence, the player receives a non-negative expected payoff conditional on $x_2^i = x_1^* - \varepsilon_1^i$. Thus again, also in these contingencies, expected payoffs of type $\tilde{x}^*$ in the virtual game is identical to the payoff of type x_1^* in the original game.

Hence, the expected payoffs of the virtual type $\tilde{x}^*$ and of the original type x_1^* are identical in the all contingencies. □

Let us now informally discuss the proof of Lemma 2. We find it helpful to personify the mapping $\tilde{x}^i = d(x_1^i, x_2^i)$ as an intervention by a "devil". At the ex ante stage, before observing the signals, each player i pacts with the devil. Each player i foregoes her reversibility option in round 2 in exchange for the following devil's service. The devil has access to both signals x_1^i, x_2^i at the ex ante stage, and instead of announcing signal x_1^i to i in round 1, the devil announces $\tilde{x}^i = d(x_1^i, x_2^i)$ which combines information from the both signals.

The interaction of players under the pact can be easily analyzed as a static game, and the results are informative about the original dynamic game, but only if *the pact is not too* *distortive.* The properties (i)–(iii) in the proof state that the pact preserves certain features of the original dynamic game and of the original threshold strategy profile. In particular, it preserves the expected payoff of the threshold type in round 1 (property (i)). Let us explain on an intuitive level why this is so. The advantage of the reversibility option is that if player is "unpleasantly surprised" in round 2 by a signal $x_2^i \ll x_1^i$ then she can reverse her investment. The player who pacts with the devil gives up this advantage, as she gives up the reversibility option, but she is compensated for this by the devil's service. The mapping $d(x_1^i, x_2^i)$ is such that player receives $\tilde{x}^i = x_1^i$ if x_1^i is not too larger than x_2^i, but if $x_2^i \ll x_1^i$, so that player without the pact would be "unpleasantly surprised" in round 2 of the original game, then player receives $\tilde{x}^i = x_2^i$. The proof consists of showing that this way of avoiding the 'unpleasant surprise" has the same expected payoff advantage for the threshold type as the reversibility option in the original game.[9]

5 Global game assumptions

We now introduce monotonicity assumptions that assure the existence and uniqueness of the threshold equilibria. These can then be characterized by the technique from the previous section. The following assumptions are maintained throughout the rest of the paper.

The first assumption states that higher parameter θ is associated with *ceteris paribus* more profitable projects:

A1 Weak State Monotonicity: Both $u_1(\theta, l_1, l_2)$ and $u_2(\theta, l_1, l_2)$ are non-decreasing in θ.

Next assumption imposes rich strategic complementarities. It assures that investing by any player in any round increases incentive to invest of any other player in any round.

A2 Weak Action Monotonicity: Both $u_1(\theta, l_1, l_2)$ and $u_2(\theta, l_1, l_2)$ are non-decreasing in l_1 and l_2.

Next we assume dominance regions in the static game and in each round of the dynamic game: Assumption A3a states that there exist dominance regions for the project as a whole, A3b states the existence of dominance regions for the second stage of the project, and A3c states the existence of lower dominance region for the first stage of the project.[10]

A3 Dominance Regions:

A3a (static game): There exists $\underline{\theta}$, $\overline{\theta}$ such that $u_1(\theta, l_1, l_2) + u_2(\theta, l_1, l_2) < 0$ for all $\theta < \underline{\theta}$, and all $l_1, l_2 \in [0, 1]$, $l_2 \leq l_1$, and $u_1(\theta, l_1, l_2) + u_2(\theta, l_1, l_2) > 0$ for all $\theta > \overline{\theta}$ and all $l_1, l_2 \in [0, 1]$, $l_2 \leq l_1$.

[9]Note that we need to assure only preservation of expected payoffs of the threshold types. Indeed, expected payoffs of any other types are distorted by the mapping.

[10]Note that the existence of the upper dominance region in which investing is dominant in the round 1 of the dynamic game is implied by A3a. However A3a alone does not imply the existence of the lower dominance region in round 1, which we assume separately in A3c.

A3b (round 2): There exists $\underline{\theta}_2$, $\overline{\theta}_2$ such that $u_2(\theta, l_1, l_2) < 0$ for all $\theta < \underline{\theta}_2$ and all $l_1, l_2 \in [0,1]$, $l_2 \leq l_1$, and $u_2(\theta, l_1, l_2) > 0$ for all $\theta > \overline{\theta}_2$ and all $l_1, l_2 \in [0,1]$, $l_2 \leq l_1$.

A3c (round 1): There exists $\underline{\theta}_1$ such that $u_1(\theta, l_1, l_2) < 0$ for all $\theta < \underline{\theta}_1$ and all $l_1, l_2 \in [0,1]$, $l_2 \leq l_1$.

While the weak monotonicities assumed in A1 and A2 do not suffice to ensure equilibrium uniqueness, any of the last two assumptions A4a, A4b is sufficient for equilibrium uniqueness:

A4 Strengthened Monotonicity: At least one of the following assumptions holds:

A4a: $u_2(\theta, l_1, l_2)$ is (strictly) increasing in θ, or

A4b: error ε_2^i has support on the whole real line.

For some of our limit results we require an additional continuity property:

A5 Continuity: Each of the three integrals $\int_0^1 u_1(x, l, l)dl$, $\int_0^1 u_2(x, l, l)dl$, and $\int_0^1 u_2(x, 1, l_2)dl_2$ is continuous in x.

5.1 Existence and uniqueness of equilibria

The statements in the following two propositions are standard in the global game literature. By argument of Milgrom and Roberts (1990), the supermodularity implies existence of the largest and the smallest strategy surviving iterated elimination of dominated strategies (IEDS), and that these constitute equilibria. The state monotonicity implies that these strategies are threshold strategies. Thus, if there is a unique threshold strategy equilibrium, then the largest and the smallest strategy surviving IEDS are identical. The weak monotonicity assumptions A1 and A2 are not alone sufficient for the uniqueness of the threshold strategy equilibrium, because the expected threshold payoff as a function of the threshold could have "flat parts" and hence multiple roots. To avoid these non-generic cases we imposed assumption A4 which assures that, controlling for player i's belief over fellow-players' actions, i's expected payoff for investing in round t strictly increases in x_t^i. This rules out the "flat parts" and assures unique roots.

Proposition 1. (Morris and Shin, 2003) *The essentially unique strategy surviving iterated deletion of strictly dominated strategies in the static game is the threshold strategy with the unique threshold $x^*_{1,st}$ satisfying the indifference condition* (3)*.*

Proposition 2. *The following statements hold:*

(i) *The essentially unique strategy surviving iterated deletion of strictly dominated strategies in the dynamic game Γ^{risky}_{dyn} is the threshold strategy with the unique threshold pair $(x^*_{1,dyn}, x^*_{2,dyn})$ satisfying the indifference conditions* (7) *in round 1 and* (5) *in round 2.*

(ii) *For each value of the threshold* x_1^* *there exists a unique* $x_2^*(x_1^*)$ *satisfying the indifference condition* (5) *in round 2, and the difference* $x_2^*(x_1^*) - x_1^*$ *is non-decreasing in* x_1^*.

Proof of Proposition 2. is an adaptation of the translation argument in Frankel, Morris, and Pauzner (2003); see Appendix. □

6 Comparative results

6.1 Preliminaries

In this section we compare the equilibrium thresholds across the static and the dynamic games. In what follows, we focus on the equilibrium threshold x_1^* in round 1 of the dynamic game, because x_2^* induced by given x_1^* will be close to x_1^* for small σ, whenever the game is not trivial. Note that if x_2^* is close to x_1^* and σ is small, then an outside observer observes the runs in round 2 only in a small set of realizations with $\theta \approx x_1^* \approx x_2^*$.

To make precise what we mean by the non-trivial games, let us introduce two non-triviality conditions. Some of the statements below hold without the non-triviality conditions, so these will be assumed only if we explicitly state so.

Definition 1. We say that

(i) the payoffs satisfy the *upper non-triviality condition* at point x, if

$$\int_0^1 u_2(x, 1, l_2) dl_2 > 0.$$

(ii) the payoffs satisfy the *lower non-triviality condition* at point x, if $u_2(x, 0, 0) < 0$.

To get an informal and intuitive interpretation of the non-triviality conditions, let us resort in this paragraph to the version of our model with the social learning. In the limit $\sigma \to 0$, the threshold type $x_1^i = x_1^*$ is quite certain that the fundamental θ is close to x_1^*, but she is uncertain over the measure l_1 of fellow-players investing in round 1. The measure l_1 influences the success of the project in the second stage. The players will invest in round 2 only if they believe l_1 is large enough, otherwise they run in round 2. If the non-triviality conditions fail, each player is virtually sure *already in round 1* whether there will be run-out in round 2: If the upper non-triviality condition at x_1^* fails then even under the best possible contingency in which all players invest in round 1 and $l_1 = 1$, the global game criterium selects the run in round 2, as the Laplacian integral $\int_0^1 u_2(x_1^*, 1, l_2) dl_2 < 0$ is negative. Hence, round 2 is trivial; each player understands already in round 1 that everybody, including herself, will run in round 2, and the dynamic game collapses to a static game with payoff u_1. If the lower non-triviality condition at x_1^* fails then even under the worst possible contingency in which no fellow-players invest in round 1 and $l_1 = 0$, the payoff for investing $u_2(x_1^*, 0, 0) > 0$ is positive. Hence, each player understands already in round 1 that everybody, including herself, will invest in round 2 (provided they invest in round 1), and the dynamic game collapses to a static game with payoff $u_1 + u_2$.

Let $h^\sigma(x_1^*) = \frac{x_2^*(x_1^*) - x_1^*}{\sigma}$ denote the normalized difference of the thresholds; where $x_2^*(x_1^*)$ satisfies the indifference condition (5) in round 2 for given x_1^*.

Lemma 3. *The following statements hold:*

(i) *If the payoffs satisfy the upper non-triviality condition at point x_1^*, then $h^\sigma(x_1^*)$ is bounded from above at $(0, \overline{\sigma}]$ for some $\overline{\sigma} > 0$.*

(ii) *If the payoffs satisfy the upper non-triviality condition at point x_1^*, then $h^\sigma(x_1^*)$ is bounded from below at $(0, \overline{\sigma}]$ for some $\overline{\sigma} > 0$.*

Proof of Lemma 3. In Appendix. □

6.2 Main comparative results

The main aim of this section is to compare the equilibrium threshold $x_{1,st}^*$ in the static game with the equilibrium threshold $x_{1,dyn}^*$ in the first round of the dynamic game. These thresholds are characterized by the indifference conditions (3) and (7) and hence we need to examine the expected payoffs of the threshold types as functions of the assumed thresholds x_1^*. For any x_1^*:

(i) let $m_{st}(x_1^*)$ denote the expected payoff for investing of the threshold type $x_1^i = x_1^*$ in the static game under the symmetric threshold strategy profile with threshold x_1^*,

(ii) let $m_{dyn}(x_1^*)$ denote the expected payoff for investing of the threshold type $x_1^i = x_1^*$ in the round 1 of the dynamic game under the symmetric threshold strategy profile with thresholds x_1^*, $x_2^*(x_1^*)$, where $x_2^*(x_1^*)$ satisfies indifference condition (5) in round 2.

In terms of the previous notation, $m_{st}(x_1^*) = \mathrm{E}\left[u\left(\theta, l_1(\theta)\right) \mid x_1^i = x_1^*\right]$ and $m_{dyn}(x_1^*) = D_1(x_1^*, x_2^*(x_1^*)) + D_2(x_1^*, x_2^*(x_1^*))$.

Lemma 4. *Functions $m_{st}(x_1^*)$ and $m_{dyn}(x_1^*)$ are well defined, strictly increasing, and have unique roots.*

Proof of Lemma 4. In Appendix. □

The equilibrium thresholds $x_{1,st}^*$ and $x_{1,dyn}^*$ are the unique roots of m_{st} and m_{dyn}. Thanks to the monotonicity of the functions m_{st} and m_{dyn} we can compare the roots without fully specifying them. It suffices to prove that one of the two functions exceeds the other for all values x_1^*; see Figure 3.

To facilitate comparison of $m_{st}(\cdot)$ and $m_{dyn}(\cdot)$, let us decompose m_{st} as sum $S_1 + S_2$, where

$$S_t(x_1^*) = \mathrm{E}[u_t(\theta, l_1, l_1) \mid x_1^i = x_1^*]$$

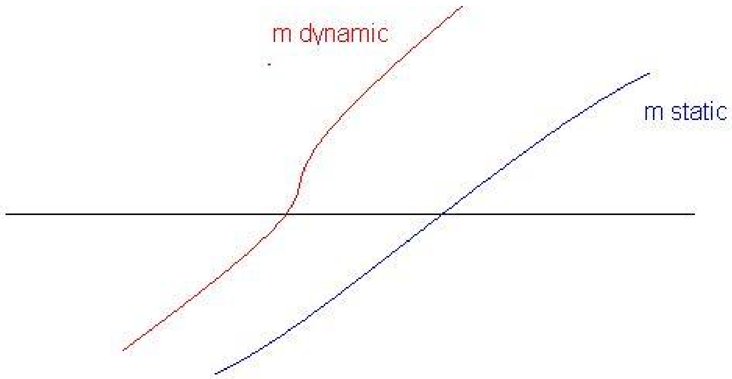

Figure 3: Comparison of $m_{st}(x_1^*)$ and $m_{dyn}(x_1^*)$.

is the expected payoff of the threshold type in the static game payoff associated with u_t.

We are now ready to compare m_{st} and m_{dyn} by parts:

Lemma 5. *The following inequalities hold:*

(i) $S_1 \geq D_1$,

(ii) $S_2 \leq D_2$.

Proof of Lemma 5.

Inequality (i): We can express both S_1 and D_1 as Laplacian integrals and these differ only in the third argument of u_1:

$$S_1 = \int_0^1 u_1(\vartheta_1(l_1), l_1, l_1)dl_1 \geq \int_0^1 u_1(\vartheta_1(l_1), l_1, \lambda_2(l_1))dl_1 = D_1. \tag{12}$$

The inequality in (12) follows from the fact that $\lambda_2(l_1) = l_2$ is weakly smaller than l_1 and that the payoff function u_1 is non-decreasing in the third argument.

Inequality (ii): Again we can express both S_2 and D_2 as Laplacian integrals, but in this case these differ both in the first and the second argument of u_2:

$$S_2 = \int_0^1 u_2(\vartheta_1(l_2), l_2, l_2)dl_2 \leq \int_0^1 u_2(\vartheta_2(l_2), \lambda_1(l_2), l_2)dl_2 = D_2. \tag{13}$$

The inequality in (13) follows from the facts $\vartheta_1(l_2) \leq \vartheta_2(l_2)$ and $l_2 \leq l_1 = \lambda_1(l_2)$. The first fact holds because $\vartheta_1(l)$ is the inverse function to $l_1(\theta)$, $\vartheta_2(l)$ inverse function to $l_2(\theta)$, both l_1 and l_2 are increasing and $l_1(\theta) \geq l_2(\theta)$ for all θ, so the opposite ordering holds for the inverse functions. The second fact holds by the same argument as in the proof of the inequality (i). $\square$

As the inequalities between S_1 and D_1 and between S_2 and D_2 have opposite signs we cannot compare the functions m_{st} and m_{dyn} in a general case. Yet, we are able to compare them in relevant particular cases, based on intertemporal characteristics of payoffs:

Definition 2. We say that

(i) the payoffs do *not* exhibit *forward spillovers* if $u_2(\theta, l_1, l_2)$ does not depend on l_1.

(ii) the payoffs do *not* exhibit *backward spillovers* if $u_1(\theta, l_1, l_2)$ does not depend on l_2.

We start with the latter case, when the payoffs do not exhibit backward spillovers. In other words, this means that the payoff from investing in round 1 does not depend on future events. It turns out that in such case, u_1 plays no role in comparison of round 1 thresholds in the static and the dynamic game. Moreover, this result holds also when the signals are not precise (i.e., when $\sigma > 0$).

Lemma 6. *If the payoffs do not exhibit backward spillovers then $S_1 = D_1$.*

Proof of Lemma 6. If $u_1(\theta, l_1, l_2)$ does not depend on l_2 then the Laplacian integrals characterizing S_1 and D_1 are identical. □

Next we establish that if the payoffs do not exhibit backward spillovers then $S_2 = D_2$, though only in the limit of precise signals and under the upper non-triviality condition. The complication is that the Laplacian integrals characterizing S_2 and D_2 differ also in the first argument; see inequality (13): fundamental θ associated with a particular investment level l_2 is $\vartheta_1(l_2)$ in the static game, while it is $\vartheta_2(l_2)$ in the dynamic game. Intuitively, this difference disappears in the limit of precise signals, as then θ is close to the private signal $x_1^i = x_1^*$ with probability approaching 1. However, this intuition fails if $x_2^* \gg x_1^*$ compared to the scaling parameter σ. In such a case, the type $x_1^i = x_1^*$ expects, *conditional on investing in round 2*, θ to be close to x_2^* rather than to x_1^*. While $x_2^* \gg x_1^*$ can be an outcome of the dynamic game, we established in Lemma 3 that it can happen only in cases that violate the upper non-triviality condition.

Let us denote $S_2^*(x_1^*) = \lim_{\sigma \to 0} S_2(x_1^*)$ and $D_2^*(x_1^*) = \lim_{\sigma \to 0} D_2(x_1^*, x_2^*(x_2^*))$.

Lemma 7. *If the payoffs do not exhibit forward spillovers and the payoffs satisfy the upper non-triviality condition at x_1^* then $S_2^*(x_1^*) = D_2^*(x_1^*)$.*

Proof of Lemma 7. In Appendix. □

Using Lemmas 6 and 7 we can characterize the effect of the reversibility option if the payoffs do not exhibit at least one of the spillovers because then at least one of the weak inequalities in Lemma 5 holds with equality. Let us denote $m_{st}^*(x) = \int_0^1 (u_1(x, l, l) + u_2(x, l, l))\, dl$.

Proposition 3. *The following statement holds:*

(i) *If the payoffs do not exhibit backward spillovers, then for all* $\sigma > 0$*:*

$$x^*_{1,dyn}(\sigma) \leq x^*_{1,st}(\sigma).$$

Suppose, in addition, that the equation $m^*_{st}(x) = 0$ *has a unique solution and denote it* $x^*_{1,st}(0)$*. Then, the following statements hold:*

(ii) *If the payoffs do not exhibit forward spillovers and satisfy the upper non-triviality condition at* $x^*_{1,st}(0)$*, then for any* $\varepsilon > 0$ *there exists* $\overline{\sigma} > 0$ *such that for all* $\sigma \in (0, \overline{\sigma}]$*:*

$$x^*_{1,st}(\sigma) - \varepsilon \leq x^*_{1,dyn}(\sigma).$$

(iii) *If the payoffs do not exhibit forward nor backward spillovers and satisfy the upper non-triviality condition at* $x^*_{1,st}(0)$*, then for any* $\varepsilon > 0$ *there exists* $\overline{\sigma} > 0$ *such that for all* $\sigma \in (0, \overline{\sigma}]$*:*

$$x^*_{1,st}(\sigma) - \varepsilon \leq x^*_{1,dyn}(\sigma) \leq x^*_{1,st}(\sigma).$$

Proof of Proposition 3. In Appendix. □

Proposition 3 states that the provision of the reversibility option:

(i) enhances efficient coordination in projects without backward spillovers,

(ii) hampers efficient coordination in projects without forward spillovers,

(iii) and has no effect in projects without both spillovers.

While the first result holds even under a large noise, the latter two results hold only in the limit of precise signals.

6.3 Example

The advantage of our indirect mapping method is that it compares the equilibrium thresholds across the static and the dynamic games without characterizing the equilibrium functions $\lambda_1(l_2)$ and $\lambda_2(l_1)$ whose characterization may be complex in general cases. Yet, the characterization of the λ functions is simple in the double limit with $\sigma \to 0$ and with the other signal becoming increasingly more precise than the first signal. We can explicitly specify the equilibrium threshold under those specific circumstances. We will informally review the results under the double limit on the following simple example.

Consider the dynamic game Γ^{risky}_{dyn} with payoffs

$$u_1(\theta, l_1, l_2) = \theta - 1 + l_1, \qquad u_2(\theta, l_1, l_2) = \theta - 1 + \frac{l_1 + l_2}{2},$$

and with normally distributed signals $x_1^i \sim N(\theta, \sigma_1^2)$, $x_2^i \sim N(\theta, \sigma_2^2)$. We examine the ordered limit $\sigma_1 \to 0$, $\frac{\sigma_1}{\sigma_2} \to 0$.

Let us construct the function $m_{dyn}(x_1^*)$. For given x_1^* and induced $x_2^*(x_1^*)$, define $l_1^*(x_1^*) = 1 - F\left(\frac{x_1^* - x_2^*(x_1^*)}{\sigma_1}\right)$. Using the social learning version of the model, we informally interpret l_1^* in the limit of precise signals as a critical measure of early investment such that players in round 2 coordinate on investing for $l_1 > l_1^*$ and on run if $l_1 < l_1^*$.

The indifference condition (5) for x_2^* and hence for l_1^* is simple in the double limit. We will examine the condition (5) at x_1^* at which both the upper and the lower non-triviality conditions are satisfied. The threshold type x_2^* assigns probability approaching 1 to $\theta \approx x_2^* \approx x_1^*$ and to $l_1 \approx l_1^*$ but she holds Laplacian beliefs over l_2; distribution of l_2 conditional on $x_2^i = x_2^*$ is uniform on $[0, l_1^*]$. Hence the indifference condition for l_1^* is $\frac{1}{l_1^*}\int_0^{l_1^*} u_2(x_1^*, l_1^*, l_2) dl_2 = 0$,[11] or equivalently, $x_1^* - 1 + \frac{3}{4} l_1^* = 0$. Thus,[12]

$$l^* = \frac{4}{3}(1 - x_1^*).$$

For $l_1^* \in (0,1)$ we have in the double limit:

$$l_2 = \lambda_2(l_1) = \begin{cases} 0 & \text{if } l_1 < l_1^*, \\ l_1 & \text{if } l_1 > l_1^*, \end{cases} \qquad l_1 = \lambda_1(l_2) = \begin{cases} l_1^* & \text{if } l_2 < l_1^*, \\ l_2 & \text{if } l_2 > l_1^*. \end{cases}$$

Having specified the λ functions, we can explicitly compute the Laplacian integrals in the expressions for D_1 and D_2:

$$\begin{aligned} D_1 &= \int_0^1 u_1\left(x_1^*, l_1, \lambda_2(l_1)\right) dl_1 = x_1^* - \frac{1}{2}, \\ D_2 &= \int_0^1 u_2(x_1^*, \lambda_1(l_2), l_2) dl_2 = x_1^* - 1 + \frac{\int_0^1 \lambda_1(l_2) dl_2 + \frac{1}{2}}{2} = x_1^* - \frac{1}{2} + \frac{(l_1^*)^2}{4}. \end{aligned}$$

Hence, using the expression for l_1^*, we obtain $m_{dyn}(x_1^*) = D_1 + D_2 = \frac{1}{9}[4(x_1^*)^2 + 10x_1^* - 5]$. The solution of $m_{dyn}(x_1^*) = 0$ is $x_{1,dyn}^* \approx 0.43$ and $l_1^* \approx 0.76$.[13]

Solution of the static game is simple. For $t = 1, 2$ we have

$$S_t = x_1^* - 1 + \frac{1}{2}$$

and the root of $m_{st}(x_{1,st}^*) = S_1 + S_2 = 0$ is $x_{st}^* = \frac{1}{2}$.

Let us emphasize that the strategic effect of learning option *does not disappear* in the limit of precise signals $\sigma \to 0$ as illustrated in this example with limit thresholds $x_{1,dyn}^* <$

[11] When upper or lower non-triviality condition fails, then $\frac{2}{3}(1 - x_1^*)$ falls out of $[0,1]$ interval, and l_1^* is 1 or 0 respectively.

[12] The values of x_1^* at which this expression does not belong to $(0,1)$ are those that violate the non-triviality conditions.

[13] The upper and lower non-triviality condition is satisfied at the root, thereby the application of the indifference condition on l_2^* which holds only under the non-triviality conditions is confirmed to be correct.

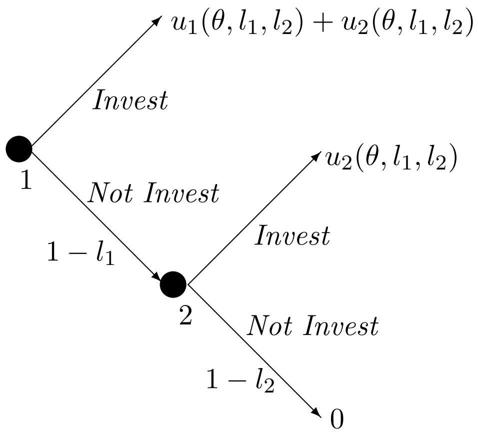

Figure 4: Dynamic game Γ_{dyn}^{safe} with reversible *safe* action.

$x_{1,st}^*$. This is because the types x_1^i in a distance to the threshold x_1^* comparable to σ suffer from the strategic uncertainty even for small σ, and the fact that these types become "rare" in the limit is not relevant for the indifference condition.

7 Modifications and generalizations

7.1 Dynamic game with reversible *safe* action

In the game Γ_{dyn}^{risky} studied until now, the risky action (invest) was reversible, and the safe action (not invest) was irreversible. We now examine closely related game Γ_{dyn}^{safe} which differs from Γ_{dyn}^{risky} only in the reversibility of the actions.

Each player decides whether to *invest* or *not invest* in round 1. By investing player reaches her final node and receives payoff $u_1(\theta, l_1, l_2) + u_2(\theta, l_1, l_2)$ which reflects the player's participation in both stages of the investment project. If player does not invest in round 1, she decides whether to *invest* or *not invest* in round 2. Players who invest in round 2 receive $u_2(\theta, l_1, l_2)$ reflecting the participation in the second stage of the project, and those who do not invest in round 1 nor 2 receive the outside option payoff 0. Variables l_t $(t = 1, 2)$ denote again the investment level in round t; that is l_1 is the measure of players choosing to invest in round 1, and l_2 is the measure of players choosing to invest in either of the rounds.

We keep the information structure identical to the one in the game Γ_{dyn}^{risky} and we also maintain the assumptions A1–A5, except that we modify assumption A3c to A3c':[14]

A3c': There exists $\overline{\theta}_1$ such that $u_1(\theta, l_1, l_2) > 0$ for all $\theta < \overline{\theta}_1$ and all $l_1, l_2 \in [0, 1]$, $l_2 \geq l_1$.

The analysis of the game Γ_{dyn}^{safe} is identical to the analysis of Γ_{dyn}^{risky} with one important difference: In the game Γ_{dyn}^{risky}, $l_2 \leq l_1$ because some players who have invested in round 1

[14]Assumption A3c in the case of game Γ_{dyn}^{risky} assures that exiting is dominant in round 1 for low θ. Assumption A3c' in the case of game Γ_{dyn}^{safe} assures that investing is dominant for high θ. The existence of region in which the opposite action is dominant is assured in both games by A3a.

may have not invested (exited) in round 2. In Γ^{safe}_{dyn} the opposite inequality holds, $l_1 \leq l_2$, because some players who have not invested in round 1 may join the project in round 2. Because of this opposite inequality, all the effects are reversed in Γ^{safe}_{dyn} compared to the effects in Γ^{risky}_{dyn}.

Proposition 4. *The following statements hold:*

(i) *The essentially unique strategy surviving iterated deletion of strictly dominated strategies in the game Γ^{safe}_{dyn} is a threshold strategy with a unique threshold pair $x^*_{1,dyn}$, $x^*_{2,dyn}$.*

(ii) *Proposition 3 applies but with reverse inequalities.*

Proof of Proposition 4. See proofs of Propositions 2 and 3. □

7.2 Games with many rounds

The results obtained for games with 2 rounds generalize to games with many rounds. Consider the game Γ^{risky}_{dyn} with $T \geq 2$ rounds. Players decide in each round between actions *invest* and *not invest* where *invest* is reversible and *not invest* irreversible. Variable l_t denotes the investment level in the project after t rounds. The payoff of a player who played invest until (including) round t is

$$\sum_{\tau=1}^{t} u_\tau(\theta, l_1, \dots, l_T).$$

The payoff for not investing in round 1 is 0. The Laplacian property from the Central Lemma 2 generalizes in the following manner:

Lemma 8. *Consider a threshold profile $(x^*_t)^T_{t=1}$ which satisfies indifference conditions in rounds $2, \dots, T$. The threshold type $x^i_1 = x^*_1$ in round 1 expects payoff for investing equal to $\sum_{t=1}^T D_t$ with*

$$D_t = \int_0^1 u_t(\theta, l_1, l_2, \dots, l_t, \dots, l_T) dl_t,$$

where θ and $l_{t'}$, $t' \neq t$, are treated as functions of l_t induced by the profile $(x^*_t)^T_{t=1}$.

Proof of Lemma 8. The proof in Appendix inductively applies the argument from the proof of the Central Lemma 2. □

The existence and uniqueness results in Proposition 2 generalizes to games with many rounds as well, see the proof of Proposition 2 which is formulated for games with T rounds. After extending the definitions of cases when payoffs do not exhibit backward and/or forward spillowers, the comparative results from Section 6 also generalize to setups with many rounds, as they are based on the Laplacian property.

7.3 Penalties and subsidies for action reversions

Proposition 3 compares equilibrium thresholds across two extreme setups — one with reversible risky action and other with fully irreversible risky action. In this section we examine more continuous comparative statics. We will keep the risky action reversible and vary the cost of reversion.

We start with the dynamic game Γ_{dyn}^{risky} with payoff functions u_1 and u_2 and with *invest* being reversible action, as introduced in Section 3. We modify this game by introducing a reversion "penalty": We set the payoff for action history (*invest*, *not invest*) to $u_1 - c$ and we keep the payoff for action history (*invest*, *invest*) to be $u_1 + u_2$.[15] The "penalty" c may be positive or negative; in the latter case we interpret it as a reversion subsidy. We assume that $|c|$ is sufficiently small so Assumptions A3b and A3c on the dominance regions still hold.

The following Proposition, analogous to Proposition 3, describes comparative statics with respect to c.

Disclaimer: *This Proposition is still a work in progress. We have not clarified yet the precise technical formulation and all details of the proof.*

Proposition 5. *The following statements hold:*

(i) *If the payoffs do not exhibit backward spillovers, then the equilibrium threshold $x^*_{1,dyn}(c;\sigma)$ is non-decreasing in c for all $\sigma > 0$.*

(ii) *If the payoffs do not exhibit forward spillovers, and the upper non-triviality condition is satisfied then the equilibrium threshold $x^*_{1,dyn}(c)$ is non-increasing in c in the limit $\sigma \to 0$.*

(iii) *If the payoffs do not exhibit forward nor backward spillovers, and the upper non-triviality condition is satisfied then the equilibrium threshold $x^*_{1,dyn}(c)$ does not depend on c in the limit $\sigma \to 0$.*

Proof of Proposition 5. To be added. □

Proof of Proposition 5 is based on the observation that the "penalty" c can be captured within our dynamic game Γ_{dyn}^{risky} with modified payoff functions

$$\hat{u}_1(\theta, l_1, l_2; c) = u_1(\theta, l_1, l_2) - c, \qquad \hat{u}_2(\theta, l_1, l_2; c) = u_2(\theta, l_1, l_2) + c.$$

We can study the comparative statics with respect to c by studying the effects of c on $\hat{D}_1 = \int_0^1 \hat{u}_1 dl_1$ and $\hat{D}_2 = \int_0^1 \hat{u}_2 dl_2$. Observe that the direct effects cancel out, and thus only the indirect effects through the comparative statics of the threshold difference $x_2^*(x_1^*) - x_1^*$ with respect to c remain, as now the indifference condition in round 2 becomes $\mathrm{E}[u_2(\theta, l_1, l_2) \mid$

[15] Payoff from not investing in round 1 is kept 0.

$x_2^i = x_2^*] = -c$. We establish that $\hat{D}_1(c)$ is non-decreasing in c and $\hat{D}_2(c)$ is non-increasing in c which then leads to Proposition 5 similarly as Lemma 5 leads to Proposition 3. The interpretation of the above proposition is also similar to Proposition 3. An increase in the penalty for action reversion enhances efficient coordination when the payoffs do not exhibit backward spillovers, hampers efficient coordination when the payoffs do not exhibit backward spillovers, and has no effect when the payoffs do not exhibit any spillovers. Note that Proposition 3 can be interpreted as a special case of Proposition 5 if we interpret the static game as a dynamic game with infinite reversion penalty.

7.4 Sequential investment under constant returns to scale

In this section we discuss another particular comparative statics result which is related but not implied by the main comparative result in Proposition 3.

We study a particular dynamic game Γ_{dyn}^{risky} with reversible risky investment, but we interpret the investing actions in rounds 1 and 2 in a particular way. Action *invest* in round 1 is interpreted as investment of monetary amount γ where γ is a parameter in $[0, 1]$. Action *invest* in round 2 is interpreted in the following way: players who have invested γ in round 1 have an opportunity to increase their investment to amount 1 (or to exit the project). We assume that the payoffs depend only on the total amount of aggregate investment denoted by k and not by the timing of the investment actions. Furthermore, we assume that payoffs are proportional to players' investments, i.e., the payoff for the action history (*invest*, *not invest*) is $\gamma \cdot u(\theta, k)$ and the payoff for the action history (*invest*, *invest*) is $u(\theta, k)$.[16]

Note that if l_1 and l_2 are the measures of players who play *invest* in rounds 1 and 2, then the total investment level is $k = \gamma(l_1 - l_2) + l_2 = \gamma l_1 + (1-\gamma)l_2$. We are interested in the comparative statics with respect to the parameter γ. In particular, when $\gamma = 1$, then $k = l_1$ and thus the game is equivalent to the static game.

Proposition 6. *The equilibrium threshold $x_{1,dyn}^*(\gamma;\sigma)$ is non-decreasing in γ.*

Proof of Proposition 6. In Appendix. □

The proof of Proposition 6 is based on the observation that the studied game can be captured as a game Γ_{dyn}^{risky} with payoffs

$$u_1(\theta, l_1, l_2) = \gamma\, u(\theta, \gamma l_1 + (1-\gamma)l_2), \qquad u_2(\theta, l_1, l_2) = (1-\gamma)\, u(\theta, \gamma l_1 + (1-\gamma)l_2)$$

and that k is non-decreasing in γ.

[16] Payoff from not investing in round 1 is kept 0.

8 Applied answers

In this section we utilize our dynamic framework to resolve the illustrative applied questions from Section 2.

8.1 Currency attacks

The model of Morris and Shin (1998) is a static global game with a continuum of players and with a binary action set $\{attack,\ not\ attack\}$ where the payoff for the risky action *attack* is $u_{MS}(\theta, l)$ with l being the measure of the attacking players. The payoff for the safe action *not attack* is 0. Players choose their actions simultaneously based on their private signals about θ.[17]

To examine the strategic consequences of the learning option we study the following dynamic extension: After the players receive their (first) private signals they may either attack immediately, or they may postpone their decisions; however as a consequence of the delay, they lose part of their their investment (attacking) opportunities.

We capture this extension by our framework. Players in round 1 choose from action set $\{attack, not\ attack\}$ based on their private signals x_1^i. Those who have attacked in round 1 get into their final node and receive payoff $u_{MS}(\theta, l_2)$, and those who have not attacked continue into round 2 in which they choose from the action set $\{attack, not\ attack\}$ based on the updated signals x_2^i. Payoff for attacking in round 1 is $u_{MS}(\theta, l_2)$, payoff for action history $(not\ attack, attack)$ is $\delta u_{MS}(\theta, l_2)$, and payoff for action history $(not\ attack, not\ attack)$ is 0. The parameter $\delta > 0$ does not model impatience, rather it is a reduced form of modelling the loss of investment (attacking) opportunity by delay. The variable l_2 denotes the total size of the attack; it is a measure of players who attack in either of the rounds.

This game falls into the category Γ_{dyn}^{safe} — the game with reversible safe option. We can match the assumed payoffs by our payoff structure with payoff functions

$$u_1(\theta, l_1, l_2) = (1 - \delta)u_{MS}(\theta, l_2), \qquad u_2(\theta, l_1, l_2) = \delta u_{MS}(\theta, l_2).$$

Hence, the payoffs exhibit backward but not forward spillovers, and by Proposition 4, the provision of the delay option enhances efficient coordination compared with the game without the option, see Section 7.1.[18]

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[17] The parameter θ must be interpreted as a *lack* of quality of the attacked economy, because the lack of quality increases the chances for the success of the attack.

[18] This example is closely related to Dasgupta (2007). Dasgupta's game, although it has slightly different cost of delay than this example, also falls into the category of Γ_{dyn}^{safe} with reversible safe option. Also, as in this example, the payoffs in Dasgupta depend only on l_2 and not on l_1. Hence his game exhibits backward but not forward payoff spillovers, and so by our Proposition 4 the provision of the delay option enhances efficient coordination.

8.2 Debt crises

We capture the problem of debt crises with endogenous entry from section 2.1 as our two-stage dynamic global game. Players decide in round 1 between entering a project and the outside option which pays 0. Those who have entered may exit or stay in round 2. Those who have entered and exited receive payoff u_1 which includes the collateral and any net profit from the first stage of the project. Those who have entered and stayed receive payoff $u_1 + u_2$ where u_2 is the profit from the second stage of the project net of the collateral.

The reversible action in this setup is the risky action, and hence the problem falls into the category of games Γ_{dyn}^{risky}. Modeler's choice is to specify the structure of payoff spillovers.[19] The assumption of the forward payoff spillovers, that is dependence of $u_2(\theta, l_1, l_2)$ on l_1, is realistic as the profit in the second stage of the project may be increasing in the first stage investment l_1 due to inertia in the production process. The assumption of the backward payoff spillovers, that is dependence of $u_1(\theta, l_1, l_2)$ on l_2, is unrealistic because the instantaneous profit in the first stage cannot be causally influenced by the later investment level l_2.

Given this modeling choice, we can unambiguously specify the effect of the exit penalties. By Proposition 5 $x^*_{1,dyn}$ increases with the exit penalty, and hence the normative recommendation is to avoid the exit penalties or even to subsidize the exit.

9 Conclusion

Economically relevant coordination problems are rarely simultaneous move games. Typically, they are dynamic processes in which economic agents can postpone their irreversible decisions and acquire additional information. We develope a modelling framework that incorporates learning without compromising the analytical tractability.

The framework allows for a qualitative assessment of the provision of the reversibility option based only on two features observable by an outside modeler. The first relevant feature is the (ir)reversibility of actions available to the economic agents, and the second feature is the structure of payoff spillovers in between different stages of the coordination process. Based on these two features, the modeler can assess the effects of the reversibility option, and specify the normative recommendation as summarized in Table 1.

The solution of the dynamic global game with the reversibility option is based on a new technical insight. Rather than solving the dynamic game directly, we map it to a virtual static game with a mapping that preserves relevant characteristics of the game and of the equilibrium. In particular, the mapping transforms the advantage of the option to an advantage of an improved private signal, by which it transforms the original dynamic game to a virtual game tractable by existing static global game tools.

[19] Modeler's choice of the payoff functions is constrained by the assumptions A1–A5. The monotonicity assumptions are reasonable in this application with strategic complementarities.

payoff spillovers		safe action irreversible	risky action irreversible
backward	forward	risky action reversible	safe action reversible
✓	×	subsidize delay	penalize delay
×	✓	penalize delay	subsidize delay
×	×	irrelevance result	

Table 1: Summary of normative recommendations.

We conduct the analysis under certain simplifying assumptions: we assume the existence of sufficient statistics for the history of private signals, we avoid public learning, and we used a standard but unrealistic setup of social learning. Generalization of our mapping method to environments without these simplifications is a promising opportunity for a future research.

A Appendix

Proof of Proposition 2. (based on Frankel, Morris, and Pauzner, 2003)

Statement (i): The existence of the upper dominance region in round 1 (A3a) and in round 2 (A3b) assure that there exist thresholds $\overline{x}_1^{(1)}$ and $\overline{x}_2^{(1)}$ such that exiting is dominated for types $x_t^i > x_t^{(1)}$, $t = 1, 2$. The monotonicity properties A1 and A2 assure that the best response to the threshold strategy with $\overline{x}_1^{(1)}$ and $\overline{x}_2^{(1)}$ is again a threshold strategy with thresholds $\overline{x}_1^{(2)}$ and $\overline{x}_2^{(2)}$ with $\overline{x}_t^{(2)} \leq \overline{x}_t^{(1)}$.[20] Iteration of the best response mapping defines a pair of non-increasing sequences $\left(\overline{x}_1^{(k)}\right)_k$, $\left(\overline{x}_2^{(k)}\right)_k$, which converge to some $\overline{x}_1^*$, $\overline{x}_2^*$.

The threshold pair $\overline{x}_1^*$, $\overline{x}_2^*$ satisfies the indifference conditions (7) in round 1 and (5) in round 2. This is because the expected payoffs of the threshold types $\overline{x}_t^{(k)}$ are non-negative in all rounds k of IEDS by induction, and if the expected payoffs of the threshold types $\overline{x}_t^*$ were positive, then the process of IEDS could continue to eliminate exiting for some types below $\overline{x}_t^*$.

Next, we consider a similar process of IEDS from below. There exist thresholds $\underline{x}_1^{(1)}$ and $\underline{x}_2^{(1)}$ such that investing is dominated for types $x_t^i < \underline{x}_t^{(1)}$, $t = 1, 2$. The existence of $\underline{x}_2^{(1)}$ is again assured by A3b, and the existence of $\underline{x}_1^{(1)}$ is assures by A3c in this case. Iteration of the best response mapping defines a pair of non-decreasing sequences $\left(\underline{x}_1^{(k)}\right)_k$, $\left(\underline{x}_2^{(k)}\right)_k$, which converge to some $\underline{x}_1^*$, $\underline{x}_2^*$, that again satisfy the indifference conditions (7), (5).

It remains to prove that $\underline{x}_t^* = \overline{x}_t^*$ for $t = 1, 2$. For $t = 1, 2$, let $m_t(x_1, x_2)$ be the expected continuation payoff of the threshold type $x_t^i = x_t$ in round t who plays in round $t' > t$ the best response against the fellow-players playing threshold strategy profile with thresholds x_1, x_2. In particular, when constructing $m_1(x_1, x_2)$, the player i is not bound to use threshold x_2 in round 2 but i's fellow-players are. That is, $m_1(x_1, x_2) = D_1 + D_2$ and $m_2(x_1, x_2)$ equals the left hand side of (5). Functions m_t satisfy two monotonicity properties:

[20] The Strengthened Monotonicity condition A4 assures that $\overline{x}_2^{(2)}$ and $\overline{x}_2^{(2)}$ are uniquely defined.

1. Monotonicity in Thresholds in Other Rounds: $m_t(x_1, x_2)$ non-increases in $x_{t'}$, when $t' \neq t$. This follows from the action monotonicity assumption A2.

2. Strict Translation Monotonicity: $m_t(x_1, x_2) < m_t(x_1 + \Delta, x_2 + \Delta)$ for each $\Delta > 0$ and each t. This is established by the following argument: For each $t' \in \{1, 2\}$, type $x^i_{t'} = x_{t'}$ under the threshold profile $(x_t)_t$ has the same belief over actions of fellow-players as type $x^i_{t'} = x^*_{t'} + \Delta$ in the threshold profile $(x_t + \Delta)_t$, because $(x_t + \Delta)_t$ is a translation of $(x_t)_t$. State monotonicity assumption A1 assures that, controlling for i's belief over fellow-players' action, i's expected payoff for investing in round t is non-decreasing in x^i_t. Additionally, A4 assures that it (strictly) increases in x^i_t.

By construction, $\underline{x}^*_t \leq \overline{x}^*_t$ for $t = 1, 2$. Let us suppose there exists t such that $\underline{x}^*_t < \overline{x}^*_t$ and prove that this leads to a contradiction. Let $\Delta > 0$ be the $\max_{t\in\{1,2\}}(\overline{x}^*_t - \underline{x}^*_t)$ and t' be a round where the maximum is attained. Let the threshold pair $(\tilde{x}_1, \tilde{x}_2)$ be a translation of $(\underline{x}^*_1, \underline{x}^*_2)$ by Δ; that is, $\tilde{x}_t = \underline{x}^*_t + \Delta$ for $t = 1, 2$.

By Strict Translation Monotonicity, $m_{t'}(\tilde{x}_1, \tilde{x}_2) > m_{t'}(\underline{x}^*_1, \underline{x}^*_2)$. The threshold pair $(\underline{x}^*_1, \underline{x}^*_2)$ satisfies the indifference conditions, and hence $m_{t'}(\underline{x}^*_1, \underline{x}^*_2) = 0$, and so $m_{t'}(\tilde{x}_1, \tilde{x}_2)$ is strictly positive.

On the other hand, by Monotonicity in Thresholds in Other Rounds, $m_{t'}(\tilde{x}_1, \tilde{x}_2) \leq m_{t'}(\overline{x}^*_1, \overline{x}^*_2)$, because $\tilde{x}_{t'} = \overline{x}^*_{t'}$ and $\tilde{x}_t \geq \overline{x}^*_t$ for $t = 1, 2$ by construction. The threshold pair $(\overline{x}^*_1, \overline{x}^*_2)$ satisfies the indifference conditions, and hence $m_{t'}(\tilde{x}_1, \tilde{x}_2) \leq 0$, which contradicts the finding in the previous paragraph.

Statement (ii): Before proceeding with the actual proof, we notify that this statement can be in a straightforward manner extended to the games with $T \geq 2$ rounds: For each value of the threshold x^*_1 there exists a unique threshold profile x^*_2, ..., x^*_T such that x^*_t satisfies the indifference condition in round t, for $t = 2, \ldots, T$. The proof of this statement is identical to the proof of the statement (i), but applied only to thresholds in rounds 2,..., T, and treating x^*_1 as an exogenous parameter.

Consider again the case of two rounds. Denote $\Delta^* = x^*_2 - x^*_1$. The expected payoff of the threshold type x^*_2 in round 2 is $m_2(x^*_1, x^*_1 + \Delta^*)$. By Strict Translation Monotonicity it increases in x^*_1 and by Weak State Monotonicity A1 and Weak Action Monotonicity A2 it is non-decreasing in Δ^*. Hence, by the implicit function theorem Δ^* solving $m_2(x^*_1, x^*_1+\Delta^*) = 0$ is non-increasing in x^*_1. In the case of many rounds, the argument can be generalized by the translation technique used in the proof of the statement (i). □

Proof of Lemma 3. For simplicity of notations, we denote $a_t = \frac{x^*_t - \theta}{\sigma}$. From (2) and (4) we have

$$l_1(\theta; x_1^*) = \int_{-\infty}^{+\infty} \Big[1 - F_1\Big(\frac{x_1^* - \theta}{\sigma} - \varepsilon_2\Big)\Big] f_2(\varepsilon_2) d\varepsilon_2,$$
$$l_2(\theta; x_1^*, x_2^*) = \int_{\frac{x_2^* - \theta}{\sigma}}^{+\infty} \Big[1 - F_1\Big(\frac{x_1^* - \theta}{\sigma} - \varepsilon_2\Big)\Big] f_2(\varepsilon_2) d\varepsilon_2,$$

which can be rewritten using the above notation as follows:

$$l_1(x_1^* - \sigma a_1; x_1^*) = \int_{-\infty}^{+\infty} [1 - F_1(a_1 - \varepsilon_2)] f_2(\varepsilon_2) d\varepsilon_2 =: \tilde{l}_1(a_1),$$
$$l_2(x_1^* - \sigma a_1; x_1^*, x_1^* + \sigma h) = \int_{a_1 + h}^{+\infty} [1 - F_1(a_1 - \varepsilon_2)] f_2(\varepsilon_2) d\varepsilon_2 =: \tilde{l}_2(a_1, h),$$

where $h = \frac{x_2^* - x_1^*}{\sigma}$. Note that functions $\tilde{l}_1$ and $\tilde{l}_2$ do not depend on σ and on x_1^*, and also that both $\tilde{l}_1(a_1)$ and $\tilde{l}_2(a_1, h)$ are non-increasing in a_1 and $\tilde{l}_2(a_1, h)$ is also non-increasing in h. Moreover,

$$\tilde{l}_2(a_1, h) \leq \tilde{l}_1(a_1),$$
$$[1 - F_1(-h)]\,[1 - F_2(a_1 + h)] \leq \tilde{l}_2(a_1, h) \leq 1 - F_2(a_1 + h)$$

and in the limits we have

$$\lim_{a_1 \to -\infty} \tilde{l}_2(a_1, h) = 1, \qquad \lim_{a_1 \to +\infty} \tilde{l}_1(a_1) = 0,$$
$$\lim_{h \to -\infty} \tilde{l}_2(a_1, h) = \tilde{l}_1(a_1), \qquad \lim_{h \to +\infty} \tilde{l}_2(a_1, h) = 0.$$

On the other hand, $\tilde{l}_2(a_2 - h, h)$ is non-decreasing in h and in the limits we have

$$\lim_{h \to -\infty} \tilde{l}_2(a_2 - h, h) = 0, \qquad \lim_{h \to +\infty} \tilde{l}_2(a_2 - h, h) = 1 - F_2(a_2).$$

The indifference condition 5 in round 2 of the dynamic game is

$$J^\sigma(h) = \mathrm{E}[u_2(\theta, l_1, l_2) \mid x_2^i = x_2^*] =$$
$$= \int_{-\infty}^{+\infty} u_2\left(x_1^* - \sigma a_2 + \sigma h, \tilde{l}_1(a_2 - h), \tilde{l}_2(a_2 - h, h)\right) f_2(a_2) da_2 = 0.$$

By A4, $J^\sigma(h)$ is strictly increasing in h, and h^σ defined in Section 6.1 is the unique solution of $J^\sigma(h) = 0$.

Statement (i): It follows from the inequality $[1 - F_1(-h)][1 - F_2(a_1 + h)] \leq \tilde{l}_2(a_1, h)$ that

$$J^\sigma(h) \geq \int_{-\infty}^{+\infty} u_2(x_1^* - \sigma a_1, \tilde{l}_1(a_1), [1 - F_1(-h)]\,[1 - F_2(a_1 + h)]) f_2(a_1 + h) da_1.$$

We denote $p = 1 - F_1(-h)$ and use the following transformation: $n = [1 - F_1(-h)][1 - F_2(a_1 + h)]$, $a_1 = F_2^{-1}\Big(1 - \frac{n}{p}\Big) - h$. We obtain that:

$$
\begin{aligned}
J^\sigma(h) &\geq \frac{1}{p}\int_0^p u_2\left(x_1^* - \sigma F_2^{-1}\Big(1 - \frac{n}{p}\Big) + \sigma h, 1 - F\left(F_2^{-1}\Big(1 - \frac{n}{p}\Big) - h\right), n\right) dn \geq \\
&\geq \frac{1}{p}\int_0^p u_2\left(x_1^* - \sigma F_2^{-1}(1 - n), 1 - F\left(F_2^{-1}\Big(1 - \frac{n}{p}\Big) - h\right), n\right) dn,
\end{aligned}
$$

where the second inequality holds only for all $h > 0$ to which we can constrain ourself in the rest of the argument.

Now assume by contradiction that there is a sequence $\sigma_k \to 0$ such that $h_k = h^{\sigma_k} \to +\infty$. Without loss of generality we may assume that σ_k is decreasing and h_k is increasing and positive (otherwise, we may choose such subsequences). Then $p_k = 1 - F_1(-h_k) \to 1$ and the above integral converges to $\int_0^1 u_2(x_1^*, 1, n)dn$. This establishes a contradiction, as $J^{\sigma_k}(h_k) = 0$ by the definition, but $\liminf J^{\sigma_k}(h_k) \geq \int_0^1 u_2(x_1^*, 1, n)dn > 0$ by the upper non-triviality condition.

Statement (ii) The inequality $l_1 \leq l_2$ implies that

$$
J^\sigma(h) = \mathrm{E}[u_2(\theta, l_1, l_2) \mid x_2^i = x_2^*] \leq \mathrm{E}[u_2(\theta, l_1, l_1) \mid x_2^i = x_2^*].
$$

Using $l_1(\theta) = 1 - F(-h + a_2)$, we obtain that the right hand side equals

$$
\int_{-\infty}^{+\infty} u_2\left(x_2^* - \sigma a_2, 1 - F(-h + a_2), 1 - F(-h + a_2)\right) f_2(a_2) da_2.
$$

Assume by contradiction that there is a decreasing sequence $\sigma_k \to 0$ such that h_k is decreasing and diverges to $-\infty$. Then the upper bound converges to $\int_0^1 u_2(x_1^*, 0, 0)dn = u_2(x_1^*, 0, 0)$. This establishes a contradiction, as $J^{\sigma_k}(h_k) = 0$ by the definition, but $\limsup J^{\sigma_k}(h_k) \leq u_2(x_1^*, 0, 0) < 0$ by the lower non-triviality condition. $\square$

Proof of Lemma 4. Clearly, $m_{st}(x_1^*)$ is well defined. Moreover, $m_{dyn}(x_1^*)$ is well defined because for each x_1^* there exists unique x_2^* solving the indifference condition in round 2; see Proposition 2, statement (ii).

Observe that $m_{st}(x_1^*)$ is strictly increasing in x_1^* because the threshold belief across fellow-players' actions is identical for each x_1^*, and controlling for the beliefs, the expected payoff for investing strictly increases in signal by Weak State Monotonicy A1 combined with Strengthened Monotonicity A4.

Also $m_{dyn}(x_1^*)$ is strictly increasing in x_1^*. By Proposition 2 statement (ii), $x_2^*(x_1^*) - x_1^*$ is non-decreasing in x_1^*. Hence, if $x_1^* > x_1'^*$ then threshold belief over fellow-players' actions under the threshold x_1^* first order dominates the threshold beliefs under the threshold $x_1'^*$. Hence the expected payoff for investing in round 1 of the threshold type $x_1^i = x_1^*$ is non-decreasing in x_1^* by Weak State Monotonicy A1 and Weak Action Monotonicity A2. The strict monotonicity is implied by Strengthened Monotonicity A4. ☐

Proof of Lemma 7. Denote α_i the inverse function to $\tilde{l}_i$ with respect to a_1. It follows from $l_2 \leq l_1$ that $\alpha_2(n,h) \leq \alpha_1(n)$. By Laplacian property:

$$S_2(x_1^*;\sigma) = \int_0^1 u_2(x_1^* - \sigma\alpha_1(n), n)dn,$$
$$D_2(x_1^*, x_1^* + \sigma h^\sigma; \sigma) = \int_0^1 u_2(x_1^* - \sigma\alpha_2(n, h^\sigma), n)dn.$$

It follows from Morris and Shin (2003) that

$$S_2^*(x_1^*) = \lim_{\sigma\to 0}\int_0^1 u_2(x_1^* - \sigma\alpha_1(n), n)dn = \int_0^1 u_2(x_1^*, n)dn.$$

Thus, it remains to show that

$$D_2^*(x_1^*) = \lim_{\sigma\to 0}\int_0^1 u(x_1^* - \sigma\alpha_2(n, h^\sigma), n)dn = \int_0^1 u_2(x_1^*, n)dn.$$

We know that $\alpha_2(n,h) \leq \alpha_1(n)$ and so we need to find some lower bound for $\alpha_2(n,h)$. Since $\alpha_2(n,h)$ is non-increasing in h, it is sufficient that h^σ is bounded from above when $\sigma \to 0$. This is assured by Lemma 3 under the upper non-triviality condition. ☐

Proof of Proposition 3.

Statement (i): By Lemma 6, $S_1 = D_1$ and by Lemma 5, $S_2 \leq D_2$. Hence we obtain $m_{st}(x_1^*) \leq m_{dyn}(x_1^*)$. By Lemma 4, both functions are strictly increasing and hence the root of $m_{dyn}(\cdot)$ does not exceed the root of $m_{st}(\cdot)$. The existence of the roots is implied by the existence of the dominance regions.

Statement (ii): By A5, $\int_0^1 u_2(x,1,l_2)dl_2$ is continuous in$x$. Hence there exits $\overline{\varepsilon} > 0$ such that the upper indifference condition is satisfied at all $x \in [x_{1,st}^*(0) - \overline{\varepsilon}, x_{1,st}^*(0)]$.

We prove the statement for $\varepsilon \in (0, \overline{\varepsilon}]$. Note that then the statement holds for an $\varepsilon > 0$. By Morris and Shin (2003), $\lim_{\sigma\to 0} m_{st}(x_{1,st}^*(0) - \varepsilon; \sigma) = m_{st}^*(x_{1,st}^*(0) - \varepsilon)$. The function m_{st}^* is assumed to have a unique root and it is non-decreasing so $m_{st}^*(x_{1,st}^*(0) - \varepsilon)$ is strictly negative. Therefore, there exists $\overline{\sigma}_1$ such that for all $\sigma < \overline{\sigma}_1$, $m_{st}(x_{1,st}^*(0) - \varepsilon; \sigma) < -\delta$ for some positive δ. We have that $S_1(x_{1,st}^*(0) - \varepsilon) \geq D_1(x_{1,st}^*(0) - \varepsilon)$ in general and by Lemma 7 $S_2^*(x_{1,st}^*(0) - \varepsilon) = D_1^*(x_{1,st}^*(0) - \varepsilon)$ in the limit $\sigma \to 0$, because the upper non-triviality condition is satisfied at $x_{1,st}^*(0) - \varepsilon$. Hence, there exists $\overline{\sigma}_2$ such that for all $\sigma < \overline{\sigma}_2$, $m_{dyn}(x_{1,st}^*(0) - \varepsilon; \sigma) < -\frac{\delta}{2}$. Then the unique root of $m_{dyn}(\cdot; \sigma)$ is larger than $x_{1,st}^*(0) - \varepsilon$ because $m_{dyn}(\cdot; \sigma)$ is strictly increasing.

Statement (iii): directly follows from the statements (i) and (ii). ☐

Proof of Lemma 8. We divide the expected payoff for action *invest* of the threshold type $x_1^i = x_1^*$ in round 1 into two parts D_1 and C where D_1 is the expected payoff for the first stage of the project and C is the expected continuation payoff for all the later stages.[21] The threshold type $x_1^i = x_1^*$ in round 1 has uniform beliefs over l_1 and hence

$$D_1 = \int_0^1 u_1(\theta, l_1, l_2, \ldots, l_T) dl_1,$$

where θ and $l_2, \ldots, l_T$ are treated as functions of l_1.

We will express C as an expected payoff of a threshold type in round 1 of a related virtual game $\tilde{\Gamma}$ with $T-1$ rounds. The mapping from the original signals to the virtual signals is the following: $\tilde{x}_1^i = \min\{x_1^i - x_1^*, x_2^i - x_2^*\}$ and $\tilde{x}_t^i = x_{t+1}^i$ for $t = 2, \ldots, T-1$. The virtual thresholds are defined as: $\tilde{x}_1^* = \min\{x_1^* - x_1^*, x_2^* - x_2^*\} = 0$ and $\tilde{x}_t^* = x_{t+1}^*$ for $t = 2, \ldots, T-1$.

The state of the world in the virtual game $\tilde{\Gamma}$ is $(\theta, (\tilde{x}_t^i)_{i;\, t=1,\ldots,T-1})$ and the probability over the states is defined by the probability distribution in the original game and by the mapping of $\left(x_t^i\right)_t$ to $\left(\tilde{x}_t^i\right)_t$.

The investment level in round t of the virtual game is denoted as

$$\tilde{l}_t(\theta) = \Pr(\tilde{x}_{t'}^i \geq \tilde{x}_{t'}^* \text{ for all } t' = 1, \ldots, t \ \mid \theta).$$

The payoff function for stage t of the virtual game is

$$\tilde{u}(\theta, \tilde{l}_1, \ldots, \tilde{l}_{T-1}) = u(\theta, l_1(\theta), \tilde{l}_1, \ldots, \tilde{l}_{T-1}),$$

where $l_1(\theta)$ is the investment level in round 1 in the original game and hence, from the perspective of the virtual game it is an exogenous function.

The following three properties are analogous to the three properties in Proof of Lemma 2 and they also follow from the same arguments.

(i) Expected payoff of the threshold type $\tilde{x}_1^i = \tilde{x}_1^*$ in the virtual game $\tilde{\Gamma}$ equals the expected continuation payoff C of the threshold type x_1^* in the original game Γ.

(ii) $\tilde{l}_t = l_{t+1}$ for all $t = 1 \ldots, T-1$.

(iii) The virtual errors $\tilde{x}_t^i - \theta$ in the virtual game $\tilde{\Gamma}$ are independent of θ.

By property (i), C equals the payoff expected by the threshold type $\tilde{x}_1^*$ in the virtual game. By property (iii) the Laplacian property holds for stage 1 of the virtual game, and hence $\tilde{l}_1 \mid \tilde{x}_1^i = \tilde{x}_1^*$ is uniform on $[0, 1]$. By property (ii) the mapping does not distort the

[21] C is D_2 in the case of the game with 2 rounds.

investment levels and hence $C = D_2 + \tilde{C}$ with

$$D_2 = \int_0^1 u_2(\theta, l_1, l_2, \ldots, l_T) dl_2$$

where θ and l_t, $t \neq 2$ are taken as functions of l_2. The expressions for D_t, $t > 2$ are proved by iterative application of the induction argument. $\square$

Proof of Proposition 6. Define $k(\theta, \gamma) = \gamma l_1(\theta) + (1-\gamma) l_2(\theta)$ and denote $\vartheta_k(k, \gamma)$ its inverse function with respect to θ. Then $k(\theta, \gamma)$ is non-decreasing in θ and because $l_1(\theta) \geq l_2(\theta)$ it is also non-decreasing in γ. Thus, $\vartheta_k(k, \gamma)$ is non-increasing in γ. Moreover, it follows from (11) that

$$\begin{aligned} D_1 + D_2 &= \int_{-\infty}^{+\infty} u(\theta, \gamma l_1(\theta) + (1-\gamma) l_2(\theta))\, d[\gamma l_1(\theta) + (1-\gamma) l_2(\theta)] = \\ &= \int_{-\infty}^{+\infty} u(\theta, k(\theta)) dk(\theta) = \int_0^1 u(\vartheta_k(k), k) dk, \end{aligned}$$

which implies that $D_1 + D_2$ is non-increasing in γ. Consequently, the threshold $x^*_{1,dyn}$ is non-decreasing in γ. $\square$

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