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Accounting for Low-Frequency Variation in Q

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Abstract

The distinction between firm value and the quantity of physical capital in the U.S. has grown increasingly large over the last 30 years. Why? We ask to what extent 'valuation effects' are responsible. We append a model of industry equilibrium to the 'growth option' model of what a firm is. Growth options are positive net-present-value investment opportunities which certain firms have access to upon the arrival of a new technology. The model of industry equilibrium puts restrictions on the value of the growth options due to the entry and exit of firms. These restrictions make it possible to ask questions about the aggregate scale of the growth-option component of firm value and, therefore, to examine the relationship between the value of the U.S. corporate sector and the value of its physical capital.

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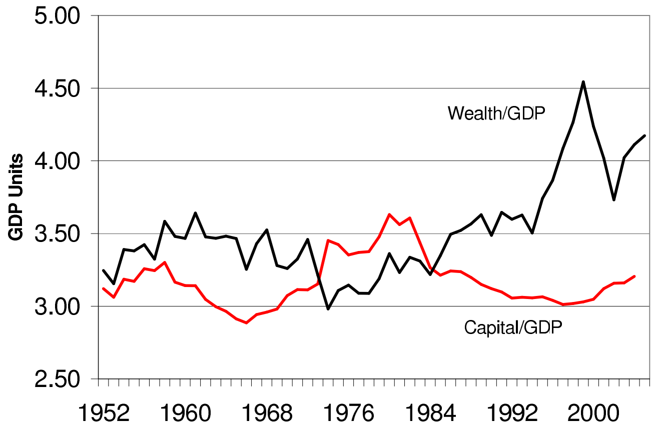
1 Introduction

The distinction between wealth and capital has grown increasingly large for the U.S. economy. Figure 1 plots the post-war capital-to-output and wealth-to-output ratio.

Figure 1*
Wealth and Capital in the U.S.

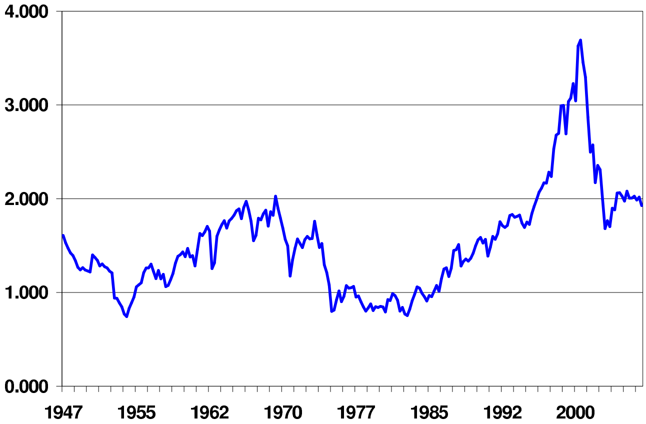


Understanding what is driving this wedge is important for many questions. Is the U.S. current account deficit a disaster in the making? Are baby-boomers savings too little? We don't ask these questions here. But we do try to make some progress in understanding what's driving U.S. wealth accumulation.

Closer inspection of Figure 1 (notes available upon request to Chris Telmer) indicates that almost all of the increase in U.S. household net worth is attributable to either housing or the value of incorporated businesses. We do not address housing here. We focus on the value of the U.S. corporate sector. That it is a driving force for Figure 1 is apparent from data on Tobin's Q, the ratio of the market value of the U.S. corporate sector to the replacement cost of its physical capital.

*Annual data, 1952-2004. The capital stock is defined the sum of private and government residential and nonresidential fixed assets — equipment, software and structures — plus the stock of consumer durables. Source: Table 1.1 from the Fixed Assets Tables of the BEA. Wealth is defined as U.S. household net worth (market value). Source: Flow of Funds Table B102. Net worth inclusive of the government looks almost the same (should do this in the future).

Figure 2[†]
Tobin's Q



What is driving Q? Most answers fall into one of three categories: bubbles, capital accumulation, or valuation effects. While we are not rationality zealots, we will ignore bubbles here. We are interested in low-frequency variation in firm value for which bubbles seem an inadequate explanation. Hall (2001) provides further argument against bubbles in this context.

The capital accumulation explanation — Hall (2001), McGrattan and Prescott (2005) — is that the variation in Q reflects the accumulation and decumulation of intangible or organizational capital of some sort. It would be foolish to deny that intangible capital is an important ingredient in the valuation of U.S. firms. Nevertheless, the 1970s and the post 2001 period pose a challenge for this explanation. What does it mean for the value of intangible capital to be negative in the 1970s? Is the rapid fall in the value of the stock market in 2001 best thought of as an equally rapid increase in the depreciation rate of intangible capital?

We explore an alternative explanation: valuation effects. If firm value is defined as the quantity of its capital multiplied by the price of its capital, then we ask if variation in the price of capital can generate low-frequency variation in Q similar to what we see in the data. A useful way to explain what we do is by defining firm value, V, as the sum of the value of its "assets in place," V_a, and the value of its future investment opportunities, V_g:

$$V = V_a + V_g \ .$$

[†]Quarterly data, 1952Q1-2006Q1. Replacement cost of capital is computed by taking fixed investment of non-farm, non-financial businesses — Table F102 from the Flow of Funds — and accumulating-up the capital stock assuming an annual depreciation rate of 10%. Initial conditions and some scaling assumptions taken from Robert Hall. Market value for non-farm, non-financial U.S. corporations is computed by adding the market value of equity to the market value of debt. Data are from the Flow of Funds and incorporate an adjustment for difference between the book value of debt and the market value of debt developed in Hall (2001) . Thanks to Robert Hall, Hanno Lustig and Stijn Van Nieuwerburgh for providing their data and code for the latter.

For example, in the Real Business Cycle (RBC) model without adjustment costs $V_g = 0$ and $V_a = K$, where K is the capital stock of the firm. If adjustment costs are incorporated, V_g reflects the value of future changes in the quantity of capital.

The starting point for our paper is the "growth options" approach to firm value inherent in textbook treatments of corporate finance (*i.e.*, the "real options" approach) as well as recent papers by Berk, Green, and Naik (1999), Carlson, Fisher, and Giammarino (2004) Gomes, Kogan, and Zhang (2003), Obreja (2007), Panageas and Yu (2006) and others. In these papers V_g derives from a firm's inherent ownership claim over "projects" which have a positive net-present-value (NPV). That is, a firm is an institution which receives an option to invest in a project each period. The strict positivity of V_g derives from the fact that the firm will only exercise this option for positive NPV projects. These models are silent on where these positive NPV projects come from and why firms are lucky enough to receive them. The models are also absent any sort of scale decision.

For our question — what explains the difference between the *size* of the corporate capital stock and its value? —- these issues cannot be avoided. Therefore, our point of departure is to append a model of industry equilibrium onto the growth options model. Our model is a simplified version of Panageas and Yu (2006) appended with a model of industry equilibrium similar to Jovanovic and MacDonald (1994).

2 Model

Time is discrete and indexed with t. The driving force of the model is an increasing sequence of technological epochs indexed with $n \in (-\infty..., -1, 0, 1, ...\infty)$. At each date t, a technological improvement — a change from epoch n to epoch $n+1$ — occurs with probability λ. In the language of Jovanovic and MacDonald (1994) — who cite the original source as Schumpeter (1934) — the arrival of a new epoch represents a fundementally new *invention* which then induces *innovation*: the adaptation of the invention for some sort of commercial use. Our model is one in which a new epoch arrives and some firms are better able to take advantage of it than other firms. These 'early entrants' invest in the new technology and earn an economic rent in that the investment has a positive net-present-value (NPV). These rents are manifest in firm value which exceeds the replacement cost of existing capital. As time passes other firms enter, the price of output falls, and the NPV for the marginal entrant is driven to zero.

There are two types of firms, indexed by $j = 1, 2$. Firms represent ownership of a collection of 'trees' that have been planted in the past. Each firm has the right to plant one tree per technological epoch, n. Trees associated with previous epochs cannot be planted in the current epoch. This planting decision is the only decision that the firm makes. It works as follows. At the onset of epoch n each firm receives an *i.i.d.* idiosyncratic shock ε, with realized value denoted ε_{nj}. A newly planted tree yields fruit

$$y_{njt} = A^n z_t \varepsilon_{njt} \quad , \tag{1}$$

where z_t is an aggregate shock — a technological shock which affects all firms but is stationary and higher frequency than the epochs — and $A > 1$ so that larger values of n represent technological improvements. The decision to plant depends on revenue and planting cost. Planting cost — the cost of paying a gardener — varies with the aggregate shock and is denoted $w(z_t)$. Revenue depends on an industry wide inverse demand function $D(y_t)$, where y_t is total fruit produced at date t by firms 1 and 2.

We use y_{jt} to denote the total fruit production for firm j at date t during epoch n:

$$y_{jt} = \sum_{k=\infty}^{n} A^k z_t \varepsilon_{kjt} 1_{kj} \quad , \tag{2}$$

where the indicator function 1_{kj} equals one if the firm planted a tree during epoch $k < n$ and zero otherwise. Total industry output is $y_t = y_{1t} + y_{2t}$.

2.1 Model Timeline

1. A new epoch is identified with a new invention as well as the birth of a new industry. Thus, we will assume that the arrival of a new invention coincides with the beginning of a new epoch and a new industry. For simplicity, we shift down the time within an epoch so that the beginning of the epoch happens at time t=0.

2. Not all inventions are the same. We will assume that inventions have the potential of being high-quality inventions, but they don't have to. Whether an invention is high-quality or not will be decided at a later time t=T, when a refinement on the invention will be issued. The likelihood that an invention is of high quality is ρ.

3. All firms are allowed to enter costlessly in the new industry. All new entrants will have primitive know-how (PKH).

4. Among the PKH firms some will acquire low know-how (LKH) with probability β (per period). LKH firms innovate and earn some profits, as a result. We denote these profits with π^l, and they are a function of the price of the output product as well as the output given in equation (1). While their profits are the same, LKH firms might innovate in different ways. These heterogeneity in innovation will only play a role at time t=T, when a positive refinement on the invention is likely to arrive. However, LKH firms could have a competitive advantage over the rest of the firms when the refinement comes.

5. At time t=T, a refinement on the invention will be issued. This refinement confirms whether the invention is high quality or not. If the invention is high-quality firms which already innovate (LKH firms) have the potential to generate superior profits. In order to generate superior profits a firm needs to have high know-how (HKH).

6. A firm is deemed HKH if at the refinement time t=T, its innovation process is compatible with the direction in which the invention is high quality. While we do not model specifically how an innovation process can lead to HKH, we will assume that among the firms which currently innovate (LKH firms) a fraction r (per period) of these firms acquire HKH firms from time T onwards. HKH firms generate profits of π^h.

7. We depart from JM's model by assuming that HKH firms have an additional advantage over the rest of the firms. They are endowed with an option to scale up their production and therefore extract potentially larger rents. We constrain the actual scale to a coarse grid so we can prevent HKH firms from adjusting immediately to the optimal production scale. HKH firms have a

strong incentive to scale up as soon as possible because this way they will be able to extract potentially high rents before other LKH firms acquire HKH. With a coarse scaling grid this might be difficult, especially when the technology is not very productive, and so they are forced to wait. Each HKH firm receives only one such option.

8. Finally, at the refinement time t=T, new entrants are allowed into the industry. These new entrants are assumed to have PKH initially. To acquire HKH and be able to enjoy the perks of this class of firms, new entrants have to acquire LKH first. We will assume that this happens at a rate of r^{ϕ}, per period. Notice that new entrants with PKH or LKH as well as incumbents with LKH have to acquire very quickly HKH or they won't be able to extract any economic rents. If HKH firms scale up quickly, the rest of the firms will not be able to generate much economic rents and they might be forced to exit the industry. These forces have a very rapid impact on the composition of the industry after the refinement, and the industry will reach a stationary equilibrium, eventually.

2.2 Capital Accumulation Dynamics

According to the above timeline firms have two occasions to invest in new capital: First, when they originally enter the industry, and second, when they have HKH and they exercise the option to scale up.

When firms enter the industry they have to reallocate capital from their existing technologies to the new technology which is an attribute of the new industry. The level of capital allocated for the new technology ensures that, initially, firms are indifferent between participating/not participating in the new industry. The trade-off here is the following: While the existing assets of the firms become less productive as epochs change, participating in a new epoch can be risky because the invention underlying the new epoch might not be of high-quality and/or the synergies between the innovation process and the refinement might not align in reasonable time (that is acquiring HKH could take a while).

If firms manage to acquire HKH and they want to exercise the option to scale up they will cover these scaling costs with capital reallocated from the old assets first and then new capital. Thus, to the extent that a HKH firm wants to scale up to a larger scale of production, this firm might end up converting all its old assets into new assets. More importantly, if after this there's still an imbalance between the current scale and the targeted scale, this firm will actually demand new capital to support its growth plan. This particular scenario will lead to an increase in the actual asset base of the firm.

If participating firms decide to exit the industry at some point, the capital allocated to new assets is used to scale up the existing old assets. In other words, the capital is being reallocated back from new assets to old assets.

2.3 Firms

The economy consists of a sequence of epochs. At a given point in time a firm consists of a portfolio of technologies, some of which originate from past epochs which have reached a stationary state while the remaining technologies originate from the current epochs which have not reached a stationary state but in which the firm participates.

The opportunity cost for this firm means investing a unit of capital in the technologies originating from stationary epochs rather than investing it in a new technology originating from an epoch which has not reached its stationary state. As time passes and new epochs become stable, this opportunity cost evolves.

2.4 Model Dynamics

For now we'll assume that all HKH firms are identical. That is HKH firms are equally productive and have the same level of capital, which we denote with k^h.

In what follows we'll import the notation in JM '94. Since we contribute to the JM model by adding capital accumulation and scaling options, we'll focus on describing these in more details.

We assume that a HKH firm has the option to scale its production with I, where I can take values on the following grid $0 = I_1 < I_2 < ... < I_m$. The cost for implementing the new scale $k + I$ is a function of the scale upgrade I and the macro-economic conditions, $\Phi(I, z)$. HKH firms can time their investment according to the realization of the aggregate shock z.

Suppose t = T+1 (that is the first period after the refinement has occurred). Following PY '07 we focus on strategies of the form: Invest I_n the first time z_s crosses some threshold $\bar{z}_n(t)$. That is invest at $\tau_n(t) = \inf\{s \geq t \,|\, z_s \geq \bar{z}_n(t)\}$. Clearly, $\bar{z}_n(t)$ is chosen to maximize:

$$\sum_{s=t}^{\tau_n} \gamma^{s-t} k_t^h E_t[\pi^h(z_s, p_s)] + \sum_{s \geq \tau_n + 1} \gamma^{s-t} [k_t^h + I_n] E_t[\pi^h(z_s, p_s)] \\ - E_t[\gamma^{\tau_n - t} \Phi(k_t^h, I_n, z_{\tau_n(t)})] \tag{3}$$

or

$$GO_n(z_t, p_t) = \sum_{s \geq \tau_n + 1} \gamma^{s-t} I_n E_t[\pi^h(z_s, p_s)] - E_t[\gamma^{\tau_n - t} \Phi(I_n, z_{\tau_n(t)})] \tag{4}$$

GO_n is the value of the growth option to scale up the production by I_n, as of time t. Implicit in this formula is the fact that the level of capital changes only when the option is exercised. The depreciation is assumed to be 0.

HKH firms invest the first time z_s crosses the threshold $\bar{z}_{n^*}(t)$, where n^* is given by:

$$n^* = \text{argmax}_n GO_n(z_t, p_t) \tag{5}$$

Thus, $\tau_{n^*}(t)$ denotes the time when HKH firms exercise their options. The aggregate output Y_s is given by $Y_s = n_s^h k_s^h y_s^h + n_s^l k_s^l y_s^l$, where:

$$\begin{aligned} y_s^h &= A^N(1+\mu_N)z_s \\ y_s^l &= A^N z_s \end{aligned} \tag{6}$$

are the output of HKH and LKH firms, respectively, per unit of capital. $\mu > 0$ captures the ability of the HKH firms to transform a unit of input capital into more output.

After the refinement, the industry composition evolves as follows:

$$\begin{aligned} n_T^h &= 0 \\ n_{T+1}^h &= r n_T^l \\ n_{s+1}^h &= n_s^h + r n_s^l \text{ for any } s > T \\ n_{T+1}^l &= (1-r)n_T^l + r^\Phi n_T^\Phi \end{aligned} \tag{7}$$

where n_T^Φ denotes the number of new entrants at time T.

After t=T, the evolution of the number of LKH firms depends on how fast the output price trends down.

- The exercise of the scale-up options by the HKH firms at time $\tau_{n^*}(t)$ induces a sudden increase in output at $\tau_{n^*}(t) + 1$ inducing some LKH firms to exit. After the shake-out the price of the output stabilizes at some constant level p^*.
- Starting with time $\tau_{n^*}(t) + 1$, a fraction r of the surviving LKH firms will acquire HKH driving the output price bellow p^* once again. The industry stabilizes by shaking out few more of the unconverted LKH firms. This process continues until the industry composition shifts towards HKH firms only.
- The LKH firms which acquire HKH after $\tau_{n^*}(t)+1$ will still receive the option to scale up. Since the output price is now sufficiently depressed, exercising this option will lower this price even further leading to very limited gains. WLOG, in equilibrium we can have these options be worth 0.

Let τ^* be the first time LKH firms start exiting the industry after the refinement. According to the equilibrium described above, the number of LKH firms after T+1 evolves as follows:

$$\begin{aligned} n^l_{s+1} &= (1-r)n^l_s, \text{ for any } T+1 < s < \tau^* \\ n^l_{s+1} &= (1-r)n^l_s - x_{s+1}, \text{ for any } s \geq \tau^* \end{aligned} \tag{8}$$

where x_{s+1} is the number of LKH firms exiting the industry at time t = s+1.

The equilibrium quantities and prices depend on whether : 1. $\tau^* > \tau_{n^*}$ or 2. $\tau^* \leq \tau_{n^*}$. We study each of these cases separately.

Case 1: $\tau^* > \tau_{n^*}$

According to the equilibrium it follows that in this case, $\tau^* = \tau_{n^*} + 1$. Let $s = \tau^*$.

As mentioned in the previous section LKH firm are heterogeneous in their opportunistic cost. A LKH firm decides to exit the industry if the PV of the profits from the technologies in which this LKH firm has invested up to this point exceeds the PV f the profits generated by the new technology. Since both old and new technologies respond to the same aggregate shock, the exit decision will be based on whether the output price falls bellow a reservation price $p^{*,i}$, where i is an index for the LKH firms. This reservation price is clearly firm specific and it can differ from the equilibrium price p^*. The relation betwen the two is given by:

$$\begin{aligned} \sum_{u\geq s} E_s[\gamma^{u-s} k\pi^{a,i}_u] &= k\pi^l(z_s, p^{*,i}) \\ &\quad + \gamma E_s\left[rV^h_1(k, z_{s+1}, p^*) + (1-r)\sum_{u\geq s+1} \gamma^{u-(s+1)} k\pi^{a,i}_u \right] \end{aligned} \tag{9}$$

where $\pi^{a,i}_u$ is the opportunity cost at time u and $V^h(k, z_{s+1}, p^*)$ is the value of a HKH firm at time $s+1$:

$$V^h_1(k, z_{s+1}, p^*) = \sum_{u\geq s+1} \gamma^{u-(s+1)} kE_{s+1}[\pi^h(z_u, p^*)] + GO_{n^*}(z_{s+1}, p^*) \tag{10}$$

In equilibrium, $GO_{n^*}(z_{s+1}, p^*) = 0$ and so the previous equation becomes:

$$V^h_1(k, z_{s+1}, p^*) = \sum_{u\geq s+1} \gamma^{u-(s+1)} kE_{s+1}[\pi^h(z_u, p^*)] \tag{11}$$

Under the assumptions that $\pi^l(z_u, p) = y^l_u p$, and $\pi^h(z_u, p) = y^h_u p$, equation (10) reduces to:

$$\sum_{u\geq s} \gamma^{u-s} kE_s[\pi^{a,i}_u] = ky^l_s p^{*,i} + \gamma \sum_{u\geq s+1} \gamma^{u-(s+1)} E_s\left[rky^h_u p^* + (1-r)k\pi^a_u \right] \tag{12}$$

Note that for LKH firm i, $\pi^{a,i}$ captures the profit of the old technologies originating from epochs which reached their stationary state and in which the firm has participated. This means that

$$\pi^{a,i}(z_u) = \sum_{j<N} \delta_j(i) y_u^j p_j^* \tag{13}$$

where $y_u^j = A^j(1+\mu_j)z_u$, p_j^* is the stationary price for epoch j, and $\delta_j(i) = 1$, if the firm participated in epoch j, and 0, otherwise.

Thus, equation (10) becomes:

$$\begin{aligned} k \sum_{j<N} \delta_j(i) A^j (1+\mu_j) p_j^* \sum_{u \geq s} \gamma^{u-s} E_s[z_u] = k A^N z_s p^{*,i} \\ + \gamma \left[r A^N (1+\mu_N) p^* + (1-r) \sum_{j<N} \delta_j(i) A^j (1+\mu_j) p_j^* \right] k \sum_{u \geq s+1} \gamma^{u-(s+1)} E_s[z_u] \end{aligned} \tag{14}$$

Since $E_s[z_u] = z_s F(u-s)$, the above equation becomes:

$$\begin{aligned} \sum_{j<N} \delta_j(i) A^j (1+\mu_j) p_j^* \sum_{u \geq s} \gamma^{u-s} F(u-s) = A^N p^{*,i} \\ + \gamma \left[r A^N (1+\mu_N) p^* + (1-r) \sum_{j<N} \delta_j(i) A^j (1+\mu_j) p_j^* \right] \sum_{u \geq s+1} \gamma^{u-(s+1)} F(u-s) \end{aligned} \tag{15}$$

This can be further simplified to:

$$\begin{aligned} \sum_{j<N} \delta_j(i) A^j (1+\mu_j) p_j^* \left[1 + r \sum_{u \geq 1} \gamma^u F(u) \right] &= A^N p^{*,i} \\ &+ r A^N (1+\mu_N) p^* \sum_{u \geq 1} \gamma^u F(u) \end{aligned} \tag{16}$$

The index or type i of a LKH firm takes values in the set $\{0, ..., 2^{N-1}\}$ (i.e. the collection of technologies in the portfolio of firm i can be mapped into a subset of $\{0, 1, ..., N-1\}$). Let ν_t^i denote the mass of LKH firms of type i at time t. These masses re-adjust whenever there is new entry or exit so that $\sum_i \nu_t^i = 1$.

Every period a fraction r of the LKH firms acquire HKH. For tractability we will assume that these firms are drawn from each type i. That is, every period a fraction r of type i firms acquire HKH, so that a total fraction of LKH firms $r = \sum_i r\nu_t^i$ acquires HKH.

For convenience we re-arrange these reservation prices so that $p^{*,1}$ tops the list, followed by $p^{*,2}$, and so on. At $s = \tau^*$, before the LKH start exiting, the output price p_s drops bellow the reservation price of some of the LKH types. That is $p_s \leq p^{*,1}$.

Let n^h_{s-1} be the mass of HKH firms at time $s-1 = \tau_{n^*}$ (the firms which exercise their options). Let n^l_{s-1} be the mass of LKH firms at that time. Between $s-1$ and s a fraction r of LKH will acquire HKH. Thus the total output at time s, Y_s, can be expressed as:

$$Y_s = \left[n^h_{s-1}[k+I](1+\mu_N)A^N + rn^l_{s-1}(1+\mu_N)kA^N + n^l_s kA^N\right] z_s \tag{17}$$

and the output price p_s can be expressed as:

$$\begin{aligned} p_s &= z_s^{\eta} Y_s^{-\eta} \\ &= \left[n^h_{s-1}[k+I](1+\mu_N) + rn^l_{s-1}(1+\mu_N)k + n^l_s k\right]^{-\eta} A^{-N\eta} \end{aligned} \tag{18}$$

Here $I = I_{n^*}$ is the optimal scale upgrade by the HKH firms, while $n^l_s = (1-r)n^l_{s-1}$.

Since $p_s \leq p^{*,1}$, a fraction x of the LKH firms will exit rising the output price to the stationary equilibrium price $p_s(x) = p^*$, where $p_s(x)$ is defined just as p_s except for n^l_s being replaced by $(1-x)n^l_s$. In equilibrium there exists a type m so that

$$\nu^1_s + ... + \nu^m_s > x \geq \nu^1_s + ... + \nu^{m-1}_s \tag{19}$$

Then $p^* = p_s(x) = p^{*,m}$. Replacing $p^{*,m}$ with p^* in equation (17), we can solve for p^*:

$$p^* = \frac{1 + r\sum_{u\geq 1}\gamma^u F(u)}{1 + r(1+\mu_N)\sum_{u\geq 1}\gamma^u F(u)} \sum_{j<N} \delta_j(m) A^{j-N}(1+\mu_j)p^*_j \tag{20}$$

From this equation and the definition of $p_s(x)$ we can infer x. Then we can check whether the condition in equation (20) is satisfied. This process pins down the stationary equilibrium price p^*.

<u>Case 2:</u> $\tau^* \leq \tau_{n^*}$ (i.e. exit occurs before HKH firms exercise)

In this case LKH firms exit before HKH firms exercise. Nevertheless, in equilibrium the choice of scale upgrade by the HKH firms will be such that exit occurs after the exercise as well. This means that in this case the output price will stabilize temporarily between τ^* and τ_{n^*} to some value $\bar{p}^*$. After the exercise, the price stabilizes to p^*, just as in the case considered previously.

Let $s = \tau^*$. We can compute the reservation prices of the LKH firms, $\bar{p}^{*,i}$, as follows:

$$
\begin{aligned}
\sum_{u\geq s} E_s[\gamma^{u-s}k\pi_u^{a,i}] &= k\pi^l(z_s,\bar{p}^{*,i}) \\
&\quad + \gamma E_s\left[rV_2^h(k,z_{s+1},\bar{p}^*,p^*) + (1-r)\sum_{u\geq s+1}\gamma^{u-(s+1)}k\pi_u^{a,i}\right]
\end{aligned} \tag{21}
$$

where $V_2^h(k,z_t,\bar{p}^*,p^*)$ is given by:

$$
\begin{aligned}
V_2^h(k,z_t,\bar{p}^*,p^*) &= E_t\left[\sum_{u=t}^{\tau_{n^*}}\gamma^{u-t}k\pi^h(z_u,\bar{p}^*)\right] \\
&\quad + E_t\left[\sum_{u>\tau_{n^*}}\gamma^{u-t}k\pi^h(z_u,p^*)\right] + GO_{n^*}(z_t,p^*)
\end{aligned} \tag{22}
$$

and

$$
GO_{n^*}(z_t,p^*) = E_t\left[\sum_{s\geq\tau_{n^*}+1}\gamma^{s-t}I_{n^*}\pi^h(z_s,p^*) - \gamma^{\tau_{n^*}-t}\Phi(I_{n^*},z_{\tau_{n^*}})\right] \tag{23}
$$

After the refinement at time T, the growth options GO_n are evaluated at every point in time. As such, the exercise time τ_{n^*} is a function of the time when the growth options are evaluated. In particular, if the options are valued at time $s+1$, it must be that $\tau_{n^*} = \tau_{n^*(s+1)}$.

Using the definition of the profit functions we can re-express V^h as:

$$
\begin{aligned}
V_2^h(k,z_t,\bar{p}^*,p^*) &= kA^N(1+\mu_N)\bar{p}^* E_t\left[\sum_{u=t}^{\tau_{n^*}}\gamma^{u-t}z_u\right] \\
&\quad + E_t\left[\sum_{u>\tau_{n^*}}\gamma^{u-t}k\pi^h(z_u,p^*)\right] + GO_{n^*}(z_t,p^*)
\end{aligned} \tag{24}
$$

Then the reservation prices solve:

$$
\begin{aligned}
\sum_{u\geq s} E_s[\gamma^{u-s}k\pi_u^{a,i}] &= kA^N z_s\bar{p}^{*,i} + kA^N(1+\mu_N)\bar{p}^* rE_s\left[\sum_{u=s+1}^{\tau_{n^*(s+1)}}\gamma^{u-s}z_u\right] \\
&\quad + rE_s\left[\sum_{u>\tau_{n^*(s+1)}}\gamma^{u-s}k\pi^h(z_u,p^*) + \gamma GO_{n^*(s+1)}(z_{s+1},p^*)\right] \\
&\quad + (1-r)E_s\left[\sum_{u\geq s+1}\gamma^{u-s}k\pi_u^{a,i}\right]
\end{aligned} \tag{25}
$$

Just as in the previous case, we re-arrange the reservation prices so that $\bar{p}^{*,1}$ tops the list. The output Y_s before LKH firms start exiting can be expressed as:

$$Y_s = \left[n_s^h(1+\mu_N)kA^N + n_s^l kA^N\right] z_s \tag{26}$$

and the output price p_s can be expressed as:

$$p_s = z_s^\eta Y_s^{-\eta} = \left[n_s^h(1+\mu_N) + n_s^l\right]^{-\eta} k^{-\eta} A^{-N\eta} \tag{27}$$

Since $p_s \leq \bar{p}^{*,1}$, a fraction $\bar{x}$ of the LKH firms will exit rising the output price to the stationary equilibrium price $p_s(\bar{x}) = \bar{p}^*$, where $p_s(\bar{x})$ is defined just as p_s except for n_s^l being replaced by $(1-\bar{x})n_s^l$. In equilibrium there exists a type $\bar{m}$ so that

$$\nu_s^1 + ... + \nu_s^{\bar{m}} > x \geq \nu_s^1 + ... + \nu_s^{\bar{m}-1} \tag{28}$$

Then $\bar{p}^* = p_s(\bar{x}) = \bar{p}^{*,\bar{m}}$. Replacing $\bar{p}^{*,\bar{m}}$ with $\bar{p}^*$ in equation (26), we can solve for $\bar{p}^*$:

$$\begin{aligned}\bar{p}^* = & \frac{1 + r\sum_{u\geq 1}\gamma^u F(u)}{z_s + r(1+\mu_N)E_s\left[\sum_{u=s+1}^{\tau_{n^*(s+1)}}\gamma^{u-s}z_u\right]} z_s \sum_{j<N}\delta_j(\bar{m})A^{j-N}(1+\mu_j)p_j^* \\ & - \frac{rE_s\left[\sum_{u>\tau_{n^*(s+1)}}\gamma^{u-s}k\pi^h(z_u,p^*) + \gamma GO_{n^*(s+1)}(z_{s+1},p^*)\right]}{kA^N z_s + rkA^N(1+\mu_N)E_s\left[\sum_{u=s+1}^{\tau_{n^*(s+1)}}\gamma^{u-s}z_u\right]}\end{aligned} \tag{29}$$

From this equation and the definition of $p_s(\bar{x})$ we can infer $\bar{x}$. Then we can check whether the condition in equation (29) is satisfied. This process pins down the stationary equilibrium price $\bar{p}^*$. To see how $\bar{p}^*$ compares to p^* we can rewrite the above formula as follows:

$$\begin{aligned}\bar{p}^* = & \frac{z_s + r(1+\mu_N)z_s\sum_{u\geq 1}\gamma^u F(u)}{z_s + r(1+\mu_N)E_s\left[\sum_{u=s+1}^{\tau_{n^*(s+1)}}\gamma^{u-s}z_u\right]} \frac{\sum_{j<N}\delta_j(\bar{m})A^{j-N}(1+\mu_j)p_j^*}{\sum_{j<N}\delta_j(m)A^{j-N}(1+\mu_j)p_j^*} p^* \\ & - \frac{rE_s\left[\sum_{u>\tau_{n^*(s+1)}}\gamma^{u-s}k\pi^h(z_u,p^*) + \gamma GO_{n^*(s+1)}(z_{s+1},p^*)\right]}{kA^N z_s + rkA^N(1+\mu_N)E_s\left[\sum_{u=s+1}^{\tau_{n^*(s+1)}}\gamma^{u-s}z_u\right]}\end{aligned} \tag{30}$$

which further simplifies to:

$$\begin{aligned}\bar{p}^* = & qp^* + q\frac{rE_s\left[\sum_{u>\tau_{n^*(s+1)}}\gamma^{u-s}k\pi^h(z_u,p^*)\right]}{kA^N z_s + rkA^N(1+\mu_N)E_s\left[\sum_{u=s+1}^{\tau_{n^*(s+1)}}\gamma^{u-s}z_u\right]} \\ & - \frac{rE_s\left[\sum_{u>\tau_{n^*(s+1)}}\gamma^{u-s}k\pi^h(z_u,p^*) + \gamma GO_{n^*(s+1)}(z_{s+1},p^*)\right]}{kA^N z_s + rkA^N(1+\mu_N)E_s\left[\sum_{u=s+1}^{\tau_{n^*(s+1)}}\gamma^{u-s}z_u\right]}\end{aligned} \tag{31}$$

where

$$q = \frac{\sum_{j<N} \delta_j(\bar{m}) A^{j-N}(1+\mu_j) p_j^*}{\sum_{j<N} \delta_j(m) A^{j-N}(1+\mu_j) p_j^*} \tag{32}$$

Due to the fact that LKH firms with higher outside opportunity costs are likely to exit first, it must be the case that $q > 1$. The above becomes:

$$\begin{aligned} \bar{p}^* = qp^* + \frac{r\left[q-1\right] E_s \left[\sum_{u>\tau_{n^*(s+1)}} \gamma^{u-s} k\pi^h(z_u, p^*)\right]}{kA^N z_s + rkA^N(1+\mu_N) E_s \left[\sum_{u=s+1}^{\tau_{n^*(s+1)}} \gamma^{u-s} z_u\right]} \\ - \frac{r\gamma E_s \left[GO_{n^*(s+1)}(z_{s+1}, p^*)\right]}{kA^N z_s + rkA^N(1+\mu_N) E_s \left[\sum_{u=s+1}^{\tau_{n^*(s+1)}} \gamma^{u-s} z_u\right]} \end{aligned} \tag{33}$$

Note that if $I_1 = ... = I_m = 0$, then $GO_{n^*(s+1)} = 0$ and $\bar{p}^* \geq qp^* > p^*$ as we would expect. If we increase $I_m > 0$ slightly this relation should still hold.

2.5 Cost Structure and the Optimal Number of New Entrants

At the refinement there will be an influx of new entrants, say n_T^ϕ. Some of these firms have already a portfolio of technologies. The rest are brand new firms owning no technologies. To participate in the industry, new entrants have to acquire LKH first. This LKH can be acquired now for a fixed cost, F^L. Those who afford this cost install k units of capital in the new technology. The cost of installing these units of capital is $C(k)$. The value of the new entrant's investment is:

$$\begin{aligned} V^l(k, z_T) &= k\pi^l(z_t, p_T) + (1-r)\gamma E_T \left[V^l(k, z_{T+1}) 1_{\{p_{T+1} > p^{*,i}\}}\right] \\ &+ (1-r)\gamma E_t \left[\sum_{s \geq T+1} \gamma^{s-(T+1)} k\pi_s^{a,i} 1_{\{p_{T+1} \leq p^{*,i}\}}\right] + \gamma E_T \left[rV^h(k, z_{T+1})\right] \end{aligned} \tag{34}$$

where V^h is given by:

$$\begin{aligned} V^h(k, z_t) = \sum_{s=t}^{\bar{\tau}} E_t \left[\gamma^{s-t} k\pi^h(z_s, p_s)\right] \\ + E_t \left[\gamma^{\bar{\tau}-t} \left\{V_1^h(k, z_{\bar{\tau}}) 1_{\{\bar{\tau} < \tau_{n^*}\}} + V_2^h(k, z_{\bar{\tau}}) 1_{\{\bar{\tau} \geq \tau_{n^*}\}}\right\}\right] \end{aligned} \tag{35}$$

Here $\bar{\tau} = \min\{\tau^*, \tau_{n^*}\}$ denotes the first time that either the industry experiences exit, after the refinement, or that HKH firms exercise their options - whichever occurs first.

To compute V^l we need to know the industry composition (n_t^h, n_t^l) at the refinement time $t = T$. This is given by:

$$
\begin{aligned}
n_T^h &= r n_{T-1}^l \\
n_T^l &= (1-r) n_{T-1}^l + n_T^\phi
\end{aligned}
\tag{36}
$$

Assume n_{T-1}^l is known for now. A firm i with PKH will participate in the epoch at time $t = T$ as long as:

$$
V^{l,i}(k, z_T) \geq F^L + C(k) + E_T \left[\sum_{s \geq T} \gamma^{s-T} k \pi_s^{a,i} \right] \tag{37}
$$

Brand new firms which own no technologies can enter the industry as well. The optimal mass n_T^ϕ of such firms solves:

$$
V^{l,0}(k, z_T) = F^L + C(k) + F^\phi \tag{38}
$$

where F^ϕ is a fixed set-up cost incurred by firms which own no technologies. This cost is explained a bit in more detailed bellow.

Before the refinement, the value of a LKH firm, U^l (we differentiate between the value of a LKH firm before and after the refinement, just as in JM '94), is given by:

$$
U^l(k, z_t) = k\pi^l(z_t, p_t) + \gamma E_t \left[\rho V^l(k, z_{t+1}) + (1-\rho) U^l(k, z_{t+1}) \right] \tag{39}
$$

At time t=0, the value of firm i with PKH can be computed as:

$$
\begin{aligned}
U^{\phi,i}(k, z_0) = {} & \gamma\beta E_0 \left[\rho V^{l,i}(k, z_1) + (1-\rho) U^{l,i}(k, z_1) \right] \\
& + (1-\beta)\gamma E_0 \left[\sum_{s \geq 1} \gamma^{s-1} k \pi_s^{a,i} \right]
\end{aligned}
\tag{40}
$$

Firm i will participate in the new epoch as long as:

$$
U^{\phi,i}(k, z_0) \geq C(k) + E_0 \left[\sum_{s \geq 0} \gamma^s k \pi_s^{a,i} \right] \tag{41}
$$

Participation in the new epoch is costlessly for incumbent firms (firms which already have a portfolio of technologies) but it is costly for brand new firms which own no technology. We can interpret this cost as a set-up cost, which can be passed up by the incumbents since they already have some structures in place. Then, the number of brand new firms entering the epoch, n_0^ϕ, solves:

$$
U^{\phi,0}(k, z_0) = C(k) + F^\phi \tag{42}
$$

Clearly, $n_0^\phi = n_0^\phi(k)$. All the prices and quantities computed so far (directly or indirectly) depend on the initial scale of production k, which we assume it to be homogeneous across all the participants. We choose k so as to maximize the ex-ante aggregate value of the economic rents generated by the new technology:

$$\max_k \sum_i U^{\phi,i}(k, z_0) - C(k) - E_0 \left[\sum_{s \geq 0} \gamma^s k \pi_s^{a,i} \right] \tag{43}$$

subject to the constraint that

$$U^{\phi,0}(k, z_0) = C(k) + F^\phi \tag{44}$$

We conclude this section with few issues which address the model's internal consistency. First, why don't outside firms participate in an industry which has already reached its stationary equilibrium? We assume that in order to participate in a mature industry a firm needs to acquire HKH for a fee. If this fee is relatively high (which could be in the interest of the industry incumbents) new entrance in this industry can be deterred. Second, where does capital come from? We can assume that firms which already own some assets (technologies) can re-alocate the initial scale of production k from the old technologies to the new ones. Firms with no assets have to use external capital. This could further justify the higher entry cost for this type of firms (that is the additional cost F^ϕ incurred by firms with no assets).

2.6 Implementation

We now implement the equilibrium with the following characteristics:

1. When a new epoch arrives, all existing firms - the ones which own already technologies - participate and a mass of brand new firms enter as well. All these firms have PKH.

2. Before the refinement, PKH firms acquire LKH costlessly, at a rate of β

3. Once the refinement arrives, at stopping time T, an additional mass of brand new firms is allowed to enter the industry. All the PKH firms which have not acquired LKH by the refinement as well as the new entrants have the chance to acquire LKH for a cost (a take it or leave it offer), F^L. New entrants have to also pay the additional set-up cost F^ϕ. After this time, LKH can no longer be acquired.

4. Also at the refinement, a fraction of the LKH firms will acquire costlessly HKH. In addition to improved cash flows, HKH firms have also an option

to scale up their production. If a LKH firm acquires HKH after the existing HKH firms have exercised their scale option, the scale option of the new HKH firm will be worth zero.

5. There is no exit before the HKH firms exercise their scale options. In the period following the exercise time, there will be a major shake-out and the industry will reach an equilibrium.

Suppose for now that HKH firms have only one growth option, I. The value of a HKH firm at time $t = T$ if it were to exercise at time $u \geq T$ becomes:

$$\begin{aligned} V^h(k, z_T; u) = &\sum_{s=T+1}^{u} \gamma^{s-T} E_T[k\pi^h(z_s, p_s)] \\ &+ E_T\left[\sum_{s \geq u+1} \gamma^{s-(u+1)}[k+I]\pi^h(z_s, p^*) - \gamma^{u-T}\Phi(I, z_u)\right] \end{aligned} \tag{45}$$

Since the industry composition is known at time $T+1$ we can easily forecast the industry composition up to time u:

$$\begin{aligned} n_s^l &= (1-r)n_{s-1}^l \\ n_s^h &= n_{s-1}^h + rn_{s-1}^l \end{aligned} \tag{46}$$

which in turn allows us to easily forecast the output price p_s up to time $u+1$ as:

$$p_s = \left[n_s^l k y_s^l + n_s^h k y_s^h\right]^{-\eta} z_s^\eta = \left[n_s^l + (1+\mu_N)n_s^h\right]^{-\eta}\left[kA^N\right]^{-\eta} \tag{47}$$

The value of a HKH firm at time T can then be computed as:

$$V^h(k, z_T) = \max_{u \geq T+1} V^h(k, z_T; u) = V^h(k, z_T; u(T)) \tag{48}$$

Notice that in order to compute $V^h(k, z_T)$ we need to computed p_s, p_s^* and $p_s^{*,i}$ for any $s \geq T$. These prices however do not depend on z_s but rather on the industry composition $(\nu_s^1, ..., \nu_s^M)$ at time s. This industry composition evolves in a predictable fashion as long as r is relatively small and no exit occurs before exercise.

If $\Phi(I, z_u) = z_u\Phi(I)$, then the formula in equation (46) becomes:

$$\begin{aligned} V^h(k, z_T; u) = z_T\Big\{&kA^N(1+\mu_N)\sum_{s \geq T+1}^{u} \gamma^{s-T}F(s-T)p_s] \\ &+ [k+I]A^N(1+\mu_N)\sum_{s \geq u+1} \gamma^{s-T}F(s-T)p^* - \gamma^{u-T}F(u-T)\Phi(I)\Big\} \end{aligned} \tag{49}$$

Now, we're going to make few simplifying assumptions in the original model to get a bit more tractability. First, suppose that there is an outside opportunity which everybody can default to. The per-period profit from this outside opportunity is $\pi^{a,0}(k, z_s) = A^N k z_s (1+\mu_0) p_0^* = k z_s \pi^{a,0}$. Note that this profit depends on the epoch N.

The firms which already own portfolios of technologies have a second outside opportunity, namely, investing in their own portfolio. Thus, these firms will choose to default on the outside opportunity which is the most profitable.

Under these assumptions few things change. First, the stationary price p^* can be computed much simpler as:

$$p^* = \frac{1 + r\sum_{u\geq 1}\gamma^u F(u)}{1 + r(1+\mu_N)\sum_{u\geq 1}\gamma^u F(u)}\pi^{a,0} \tag{50}$$

The reservation prices of the participating firms can be computed as:

$$\begin{aligned} p^{*,i} = \max &\left\{\pi^0 A^{-N}, \sum_{j<N}\delta_j(i)A^{j-N}(1+\mu_j)p_j^*\right\}\left[1 + r\sum_{u\geq 1}\gamma^u F(u)\right] \\ &- r(1+\mu_N)p^*\sum_{u\geq 1}\gamma^u F(u) \end{aligned} \tag{51}$$

Notice that all the reservation prices are at least as high as the stationary price $p^{*,i} \geq p^*$, for any i. This formulas simplify the analysis quite a bit because now none of these prices are time dependent.

Going back to equation (50), assume for a moment that time is continuous rather than discrete. Then:

$$\begin{aligned} V^h(k, z_T; u) = z_T\Big\{ & kA^N(1+\mu_N)\int_{T+1}^{u}\gamma^{s-T}F(s-T)p_s ds \\ & + [k+I]A^N(1+\mu_N)\int_u^\infty \gamma^{s-T}F(s-T)p^* ds - \gamma^{u-T}F(u-T)\Phi(I)\Big\} \end{aligned} \tag{52}$$

Differentiating with respect to u yields:

$$kA^N(1+\mu_N)[\gamma f]^{u-T}p_u - [k+I]A^N(1+\mu_N)[\gamma f]^{u-T}p^* - \ln(\gamma f)[\gamma f]^{u-T}\Phi(I) = 0 \tag{53}$$

where $F(u-T) = f^{u-T}$. This formula simplifies to:

$$p_u = \frac{k+I}{k}p^* + \frac{1}{kA^N(1+\mu_N)}\ln(\gamma f)\Phi(I) = \bar{p}(I) \tag{54}$$

Since the output price essentially declines after the refinement, the above condition can be met as long as the industry composition is such that:

$$p_T \geq \frac{k+I}{k}p^* + \frac{1}{kA^N(1+\mu_N)}\ln(\gamma f)\Phi(I) \tag{55}$$

The fact that the exercise time does not depend on when the valuation is made (i.e. T or T+1, so on) is a direct consequence of the structure of the investment cost Φ. Note that we made no assumption about exit in determining the condition in equation (55). There will be no exit before exercise as long as the reservation price for the LKH firms with the highest opportunity cost is bellow $\bar{p}(I)$. That will happen if the following inequality holds:

$$\begin{aligned} \sum_{j<N}\delta_j(1)A^j(1+\mu_j)p_j^*\left[1+r\sum_{u\geq 1}\gamma^u F(u)\right] &\leq A^N\bar{p}(I) \\ +rA^N(1+\mu_N)\frac{k+I}{k}p^*\sum_{u\geq 1}\gamma^u F(u) &- r\Phi(I)F(1) \end{aligned} \tag{56}$$

This simplifies to:

$$\begin{aligned} &\sum_{j<N}\delta_j(1)A^j(1+\mu_j)p_j^*\left[1+r\sum_{u\geq 1}(f\gamma)^u\right] \leq \\ &A^N\frac{k+I}{k}p^*\left[r(1+\mu_N)\sum_{u\geq 1}(f\gamma)^u+1\right] - \Phi(I)\left[rf - \frac{1}{kA^N(1+\mu_N)}\ln(\gamma f)\right] \end{aligned} \tag{57}$$

which further becomes:

$$\begin{aligned} \sum_{j<N}\delta_j(1)A^j(1+\mu_j)p_j^* &\leq A^N\frac{k+I}{k}\pi^0 \\ &- \Phi(I)\frac{1-\gamma f}{1-(1-r)\gamma f}\left[rf - \frac{1}{kA^N(1+\mu_N)}\ln(\gamma f)\right] \end{aligned} \tag{58}$$

At the same time, condition (56) can be rewritten as:

$$p_T \geq \frac{k+I}{k}\frac{1-(1-r)\gamma f}{1-[1-(1+\mu_N)r]\gamma f}\pi^0 + \frac{1}{kA^N(1+\mu_N)}\ln(\gamma f)\Phi(I) \tag{59}$$

Equations (59)-(60) impose restrictions on the parameter space. The question is whether there exists a subset of the parameter space which satisfies these two conditions. Note that if we choose our parameters so that $\gamma f < 1$, but close to 1.

Then the negative term on the RHS of (59) is small. At the same time, for any r we have:

$$\frac{1-(1-r)\gamma f}{1-[1-(1+\mu_N)r]\gamma f} < 1 \tag{60}$$

Under these parameter restrictions, note that if we increase I sufficiently high and if Φ has a quadratic component, then the inequalities in (59) and (60) could be satisfied.

In what follows I will assume that we can find parameters that satisfies the constraints in (59) and (60). In this parameter subspace there will be no exit prior to the exercise time, in equilibrium. This simplifies quite a bit the analysis. Let's now value a LKH firm at the refinement time T:

$$\begin{aligned} V^{l,i}(k,z_T) = {} & k\gamma E_T[\pi^l(z_{T+1},p_{T+1})] + \sum_{s=T+2}^{u} \gamma^{u-T} E_T\left[rV^h(k,z_s) + (1-r)k\pi^l(z_s,p_s)\right] \\ & + r\gamma^{u+1-T} E_T\left[rV^h(k,z_{u+1})\right] + (1-r)\gamma^{u+1-T} E_T\left[\Pi^{a,i}(z_{u+1},k)\right] \end{aligned} \tag{61}$$

where $\Pi^{a,i}(z_{u+1},k)$ captures the value of the outside opportunity for LKH firms of type i and

$$\begin{aligned} V^h(k,z_s) = {} & \sum_{s'=s}^{u} \gamma^{u-s} E_s\left[k\pi^h(z_{s'},p_{s'})\right] + [k+I] \sum_{s\geq u+1} \gamma^{s-T} E_s[\pi^h(z_{s'},p^*)] \\ & - \gamma^{u-T} E_s\left[\Phi(I,z_u)\right] \end{aligned} \tag{62}$$

Firm i participates as long as:

$$\begin{aligned} & k\gamma E_T[\pi^l(z_{T+1},p_{T+1})] + \sum_{s=T+2}^{u} \gamma^{u-T} E_T\left[rV^h(k,z_s) + (1-r)k\pi^l(z_s,p_s)\right] \\ & + r\gamma^{u+1-T} E_T\left[rV^h(k,z_s)\right] \geq \\ & F^L + C(k) + \Pi^{a,i}(k,z_T) - (1-r)\gamma^{u+1-T} E_T\left[\Pi^{a,i}(z_{u+1},k)\right] \geq \\ & F^L + C(k) + \frac{1-(1-r)(\gamma f)^{u+1-T}}{1-\gamma f} k z_T \pi^{a,i} \end{aligned} \tag{63}$$

This inequality tells us that if firms of type i participate then all firms with lower types will participate as well. Suppose at time T there are n^l_T LKH firms of which $n^h_{T+1} = rn^l_T$ firms will acquire HKH. Let $(\nu^1, ..., \nu^M)$ denote the distribution of existing PKH firms at time T. These firms will have to decide whether to acquire LKH for F^L and enter the industry.

Assume that type 1 firms are the highest type willing to enter. Then the output price is determined by:

$$p_{T+1} = [rn^h_T(1+\mu_N) + (1-r)n^l_T + \nu^1]^{-\eta}(kA^N)^{-\eta} \tag{64}$$

Check whether (68) holds for $i = 1$. If it doesn't stop. Have ν^0 new firms entering the industry so that (68) holds with equality for $i = 0$, where

$$p_{T+1} = [rn_T^h(1+\mu_N) + (1-r)n_T^l + \nu^0]^{-\eta}(kA^N)^{-\eta} \tag{65}$$

If it does, assume that type 2 firms are the highest type willing to enter. Then the output price is determined by:

$$p_{T+1} = [rn_T^h(1+\mu_N) + (1-r)n_T^l + \nu^1 + \nu^2]^{-\eta}(kA^N)^{-\eta} \tag{66}$$

Check whether (64) holds for $i = 2$. If it doesn't stop. Have ν^0 new firms entering the industry so that (64) holds with equality for $i = 0$, where

$$p_{T+1} = [rn_T^h(1+\mu_N) + (1-r)n_T^l + \nu^1 + \nu^0]^{-\eta}(kA^N)^{-\eta} \tag{67}$$

If it does, continue as before.

Before the refinement, the value of a LKH firm, $U^{l,i}$ (we differentiate between the value of a LKH firm before and after the refinement, just as in JM '94), is given by:

$$U^{l,i}(k, z_t) = k\pi^l(z_t, p_t) + \gamma E_t\left[\rho V^{l,i}(k, z_{t+1}) + (1-\rho)U^{l,i}(k, z_{t+1})\right] \tag{68}$$

At time t=0, the value of firm i with PKH can be computed as:

$$\begin{aligned} U^{\phi,i}(k, z_0) &= \gamma\beta E_0\left[\rho V^{l,i}(k, z_1) + (1-\rho)U^{l,i}(k, z_1)\right] \\ &+ (1-\beta)\gamma E_0\left[(1-\rho)U^{\phi,i}(k, z_1)\right] \\ &+ \rho E_0\left[\left[\chi V^{l,i}(k, z_1) + (1-\chi)\Pi^{a,i}(k, z_1)\right]\right] \end{aligned} \tag{69}$$

where χ equals 1 if the firm acquires LKH at the refinement and 0 otherwise. Firm i will participate in the new epoch as long as:

$$U^{\phi,i}(k, z_0) \geq C(k) + \Pi^{a,i}(k, z_0) \tag{70}$$

Participation in the new epoch is costlessly for incumbent firms (firms which already have a portfolio of technologies) but it is costly for brand new firms which own no technology. Then, the mass of brand new firms entering the epoch, ν_0^{ϕ}, solves:

$$U^{\phi,0}(k, z_0) = C(k) + F^{\phi} + \Pi^{a,0}(k, z_0) \tag{71}$$

At time $t = 0$, we would like all firms to participate in the new epoch. This is equivalent to the fact that the firm with the highest opportunity costs is participating. So, at time $t = 0$, we only need to check (71) for the firm with the highest opportunity cost. Then, we can choose ν_0^{ϕ} so that (72) holds. We treat k at this time as an extra degree of freedom.

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