

Exit Options and Dividend Policy under Liquidity Constraints (Preliminary)*

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Abstract

We introduce a liquidity constraint to the standard model of a firm with stochastic cash flow and an irreversible exit decision. A firm with no cash holdings and negative cash flow is forced to exit regardless of its option value. This creates a precautionary motive for holding cash, which has a liquidity cost because the return on cash is below the discount rate of interest. We characterize the optimal exit and dividend policy and show how it can be solved numerically. We show that the firm almost surely exits voluntarily to preempt forced exit. Numerical results show that the optimal policy is very sensitive to a small liquidity cost. We also study the implications of the liquidity constraint on an industry with idiosyncratic productivity shocks and endogenous entry. Equilibrium inefficiency has three channels: 1) cross-section of firm productivity is distorted as some firms exit too early and others too late, 2) turnover rate is distorted, 3) precautionary cash holdings are socially inefficient. The effect of the liquidity constraint on average productivity is in principle ambiguous. Monte-Carlo simulation shows that when entry costs are low the liquidity constraint lowers average productivity, as old but currently unproductive firms that have succeeded in accumulating a lot of cash then continue at inefficiently low productivity levels.

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1 Introduction

In the standard entry-exit problem of a firm with a stochastic cash flow, the optimal policy requires the firm to accept negative cash flows indefinitely. Our main question is how firms should behave if they have a limited capability of paying for losses. We also study the effect of liquidity constraints on industry equilibrium, assuming that firms can enter the industry by paying a one-time investment cost but, once in, have only a finite ability to sustain losses.

In the standard problem the firm does not exit immediately when cash flow becomes negative due to the potential of bouncing back and earning significant profits in the future. The option to exit later is valuable and must be balanced against the expected near-term losses. The efficient policy is described by a path-independent negative threshold level of cash flow below which the firm exits. However, the value of continuation is partly due to future paths where the cash flow remains negative for arbitrarily long periods of time. It seems realistic in many contexts that it would be difficult to keep raising more funds for a firm with a long history of losses. As soon as there is a limit to a firm's ability to sustain losses the problem changes in a fundamental way.

We model the liquidity constraint as the complete inability to raise new funds. The firm has an initial stock of cash that it can augment with retained earnings. A firm without cash and with a negative cash flow is forced to exit immediately, regardless of the option value of continuation, so the firm has an incentive to hoard cash in order to avoid such inefficient exit in the future. This precautionary saving comes at a cost when the cash held inside the firm earns interest at a rate below the discount rate. Therefore, if the firm is sufficiently safe from forced exit—with sufficiently high cash flow and/or cash stock—it is optimal to pay out some of the cash to the owners. The problem of a liquidity constrained firm thus generates the optimal dividend policy in addition to the optimal exit policy. The optimal policy is defined with respect to two state variables: cash flow and cash stock.

The difference between the discount rate and the return on firm's cash holdings can be due to a liquidity premium for the cash holdings—which represents a liquid asset that has to be available at short notice in case the cash flow turns negative—or due to a tax advantage for the owners from being able to invest outside the firm. Our numerical results show that a very small liquidity premium has a large impact on optimal firm behavior. The liquidity constrained world is not approximated by the unconstrained model even when the liquidity cost is trivial. We also analyze the relation of the optimal exit and payout policies with the

variance and the drift of the cash flow process.

The natural motivation for liquidity constraints is asymmetric information: it may be hard for a firm/manager to credibly convey to owners/outside investors that it has upside potential. However, we do not explicitly model the mechanics behind the liquidity constraint. Our model has no other imperfections, such as agency problems. The optimal policy maximizes the value of the firm to its owners, taking as given the lack of further cash injections by the owners. This is a special case of a more general model where raising new funds is costly; in effect we assume that the cost is prohibitive.

The obvious effect of liquidity constraint on industry equilibrium is that it results in too much selective pressure. Setups where entrepreneurs have limited ability to pay for an initial entry cost result in an inefficiently high threshold for entry, so the average productivity of existing firms is increased by a pre-entry liquidity constraint. In our model a similar overselectivity shows up as the weeding out of marginally productive firms that should survive a temporary negative cash flow due to option value but that exit due to insufficient funds (more accurately, as we'll see, to preempt such forced exit). However, the post-entry liquidity constraint creates an opposing underselectivity effect that can dominate. The underselectivity is due to a price distortion in industry equilibrium, which attracts firms with sufficient cash to stay on even when their productivity falls below the efficient exit threshold. We show that when the entry cost is sufficiently low, a liquidity constraint lowers the average productivity of firms in the industry.

Our model builds on the literature of real options. The seminal paper by McDonald and Siegel (1986) models the optimal timing of investment under uncertain cash flow. Dixit (1989) analyzes the firm's optimal entry and exit decisions in the same framework. A large number of extensions to various directions is summarized in Dixit and Pindyck (1993). Our paper extends this line of research further by adding a liquidity constraint to the problem.

Two other papers address the effects of liquidity constraints on optimal exercise of real options. Boyle and Guthrie (2003) analyze the optimal timing of investment, when uncertain cash flow prior to the investment affects the firm's ability to finance the investment. Our paper, in contrast, focuses on post-investment cash flow uncertainty and its effects on dividends and exit. Mello and Parsons (2000) analyze the optimal hedging policy of a financially constrained firm. Even though it is not their main focus, the model shares with ours the property that past earnings affect the optimal exit policy.¹

¹Examples of other real options papers that add external constraints on firms' behavior include Miao and Wang (forthcoming) and Grenadier and Wang (2007).

Liquidity constraints have also been analyzed in other contexts. Deaton (1991) analyzes optimal lifetime consumption, and shows that the liquidity constraint leads to precautionary saving demand. The main difference to our setup is that consumers do not face an exit decision. Precautionary saving in Deaton (1991) results from the convexity of marginal utility of consumption, whereas in our model agents are risk neutral and precautionary saving results from the threat of forced exit.

Hopenhayn (1992) presents the seminal analysis of steady-state equilibrium of a competitive industry with idiosyncratic uncertainty and entry and exit. Caballero and Pindyck (1996) analyze the effects of both idiosyncratic and aggregate uncertainty on industry dynamics. Miao (2005) analyzes the optimal capital structure in such a setting. Our model introduces financial constraints in such market level analysis.

We first characterize the problem of an individual firm in section 2, and then solve the optimal policy in section 3. The model of a competitive industry is presented in section 4 where we analyze impact of liquidity constraints on industry equilibrium with the help of simulations.

2 The Problem of the Firm

The firm faces a stochastic revenue x that follows geometric Brownian motion:

$$dx = \mu x dt + \sigma x dw, \quad (1)$$

where dw is the increment of a standardized Wiener process (i.e., with mean zero and variance dt). The firm earns a profit flow $\pi = x - c$ where the fixed cost c is a positive constant. The firm uses a positive discount rate $\rho > \mu$. Exit is irreversible and without an additional exit cost. (The entry decision will only show up in industry equilibrium.)

There are two fundamentally different cases. An unconstrained firm can accumulate negative profits indefinitely if needed. The problem of an unconstrained firm is described by the standard real option model of optimal exit. The sole decision is to choose the exit threshold for x . There is no meaningful decision for when (if at all) to retain cash or pay dividends. This is the efficient benchmark for our analysis.

A constrained firm has to worry about its ability to cover negative profits, because it is forced to exit if it has no cash while it faces a negative cash flow. The optimal exit policy depends both on revenue x and cash stock s . The firm's cash stock is augmented by the profit flow and by the interest earned on the cash stock at rate $r \leq \rho$. If $r < \rho$ then the cash

held by the firm is less productive than other assets available to the owners, so the firm faces a meaningful decision of how to pay dividends. The downside of paying dividends is that the reduced cash stock lowers the capability to cover any future losses. For simplicity we assume that the liquidity constraint is very stark in the sense that it is not possible to inject more cash into the firm. This is a special case of a more complex model where raising more funds is costly; in effect we assume that the cost is prohibitive.

2.1 Unconstrained Firm

The unconstrained firm will exit if the cash flow becomes too negative. The value function for the unconstrained firm is defined by the familiar differential equation:

$$\rho V = x - c + \mu x V_x + \frac{\sigma^2}{2} x^2 V_{xx} \quad (2)$$

(see e.g. Dixit and Pindyck 1993, Chapter 7). This ODE has a well known closed-form solution. The optimal exit threshold is

$$x^* = \frac{\beta}{1 + \beta} \left(1 - \frac{\mu}{\rho}\right) c, \quad (3)$$

$$\text{where } \beta = -\frac{1}{2} + \frac{\mu}{\sigma^2} + \sqrt{\left(\frac{1}{2} - \frac{\mu}{\sigma^2}\right)^2 + \frac{2\rho}{\sigma^2}} > 0.$$

The unconstrained value function is

$$V^*(x) = \begin{cases} \frac{x^*}{\beta(\rho - \mu)} \left(\frac{x^*}{x}\right)^\beta + \frac{x}{\rho - \mu} - \frac{c}{\rho} & \text{for } x \geq x^*, \\ 0 & \text{for } x < x^*. \end{cases} \quad (4)$$

2.2 Constrained Firm

The constrained firm has an initial cash stock that is exogenous to the problem. The cash stock earns interest at rate $r \leq \rho$. When the firm is not paying dividends, the cash flow is the sum of the profit flow and the interest income flow

$$\frac{ds}{dt} = x - c + rs. \quad (5)$$

The firm is forced to exit if $x \leq c$ and $s = 0$. If the firm chooses to exit when $s > 0$ then the remaining cash stock is paid out as the liquidation value.

The firm may at any point in time choose from three policy options. First, the firm may exit, which is irreversible, and results in the exit value s . Second, the firm may continue

while paying a positive dividend to the owners. Third, the firm can continue without paying dividend. The key to solving the firm's problem is in dividing the (x, s) –space into regions in each of which one of the three policy options is optimal. It will become clear later that the firm will never venture inside the dividend region, but will pay dividends at a rate that keeps it at the boundary between the continuation and dividend regions.

We define the value of the constrained firm $V(x, s)$ as gross of the cash stock, so the value at the time of exit is $V(x, s) = s$. Consider the value function $V(x, s)$ in the continuation region. Using Ito's lemma, we can write the differential dV as:²

$$dV = V_s ds + V_x dx + \frac{1}{2} V_{xx} (dx)^2. \quad (6)$$

Taking the expectation and letting dt be small yields:

$$E(dV) = V_s ds + V_x \mu x dt + \frac{1}{2} V_{xx} \sigma^2 x^2 dt,$$

where ds is from (5). The Bellman equation is $V(x, s) = E(V + dV) / (1 + \rho dt)$, or

$$\rho V dt = E(dV) = (x - c + rs) V_s dt + \mu x V_x dt + \frac{\sigma^2}{2} x^2 V_{xx} dt. \quad (7)$$

This leads to the following PDE in the continuation region:

$$\rho V = (x - c + rs) V_s + \mu x V_x + \frac{\sigma^2}{2} x^2 V_{xx}. \quad (8)$$

Note that this PDE does not contain a cash flow term. The reason is that, in the continuation region, the cash flow between the firm and its owners is zero: Positive cash flow adds to the cash stock and negative flow subtracts from it. This PDE does not have a closed-form solution.

The fixed boundary condition comes from forced exit:

$$V(x, 0) = 0 \text{ for } x \leq c. \quad (9)$$

Also, as x gets sufficiently high then the prospect of forced exit becomes remote and is discounted towards zero, so

$$\lim_{x \rightarrow \infty} V(x, s) = V^*(x) + s. \quad (10)$$

²In the Appendix we show how this derivation is done more intuitively, if in a lengthier fashion, using the limiting case of a discrete-time binary process.

We will next characterize the conditions for the optimal policy of the constrained firm. The optimal policy is described by the boundaries of the continuation region in (x, s) -space. In terms of the PDE, these are free boundaries. Figure 1 shows a schematic view of the optimal policy regions. In addition to the continuation region, the possible states $\{x, s\}$ that the firm may visit under the optimal policy also include the half-line $s = 0, x \in (\hat{x}_{\max}, \infty)$ where current cash flow is so high that the firm does not hold any cash and profits are immediately given away as dividends.

Recall that x^* is the optimal exit threshold of the unconstrained firm, and use $s^* \equiv -(x^* - c)/\rho$ to denote the associated worst-case realization for the present value of profits.

Proposition 1 *Let $\sigma > 0$, $\rho > \mu$, and $\rho > r \geq 0$. Then the optimal policy of the liquidity constrained firm is defined by three constants: $x_{\min} \in (x^*, c)$, $x_{\max} \in (c, \infty)$, and $\tilde{s}_{\max} \in (0, s^*)$, and two continuous functions: $\tilde{s} : [x_{\min}, c] \rightarrow [0, \tilde{s}_{\max}]$, which is a bijection and strictly decreasing, and $\hat{s} : [x_{\min}, x_{\max}] \rightarrow [0, \hat{s}_{\max}]$, which is a surjection, strictly increasing at $x = \hat{x}_{\min}$ and strictly decreasing at all $x > c - r\hat{s}(x)$. Furthermore, $\hat{s}_{\max} \in (\tilde{s}_{\max}, s^*)$, and $\hat{s}(x) > \tilde{s}(x)$ at all $x \in (x_{\min}, c]$. It is optimal to:*

- Continue (without paying dividends) at all $\{x, s\}$ in the set surrounded by $\hat{s}$, $\tilde{s}$, and the $\{s = 0\}$ -line.
- Pay a lump dividend $s - \hat{s}(x)$ when $x \in [x_{\min}, x_{\max}]$ and $s > \hat{s}(x)$, and a lump dividend s when $x \geq x_{\max}$.
- Exit otherwise (i.e., when $x < x_{\min}$, or when $x \in [x_{\min}, c]$ and $s < \tilde{s}(x)$).

Proof A sketch is given in the appendix.

The special case of no liquidity premium is qualitatively different.

Proposition 2 *Let $\sigma > 0$, $\rho > \mu$, and $r = \rho > 0$. Then the optimal policy of the liquidity constrained firm is defined by a constant $s^* = -(x^* - c)/\rho$, and a strictly decreasing function $\tilde{s} : [x^*, c] \rightarrow [0, s^*]$. It is optimal to:*

- Continue (without paying dividends) for all $\{x, s\}$ inside the open set bordered by $\tilde{s}$, and the $s = 0$ and $s = s^*$ lines.
- Exit when $x < x^*$ or when $x \in [x^*, c]$ and $s < \tilde{s}(x)$.

The firm is indifferent between paying any lump dividend less than $s - \hat{s}(x)$ when $x \geq x^*$ and $s > \hat{s}(x)$.

Proof A sketch in the appendix.

Next we discuss the optimal policy and its meaning in more detail.

[Figure 1 here]

Exit Policy

The constrained firm certainly exits whenever the unconstrained would, i.e., when $x \leq x^*$. In addition, it is forced to exit when it has no cash to cover the current loss, i.e., when $(x \leq c, s = 0)$. Clearly the firm would never exit while current profits are positive ($x > c$). Now consider a firm with a very small s and with x slightly above x^* . This firm could in principle continue. However, as $ds/dt < 0$, the firm is just about to run out of cash and be forced to exit at the next instant. For sufficiently small s the firm is so unlikely to bounce back to a positive cash flow before s hits zero that it is better off exiting immediately and just taking the remaining s . We denote the optimal exit policy by $\tilde{s}(x)$, which gives the level of cash stock at which the firm exits given $x \leq c$. For $x < c$, the lower is x the higher the s required for continuation to be optimal, so $\tilde{s}'(x) < 0$ in $x \in (\hat{x}_{\min}, c)$. We call exiting when $x > x^*$ and $s > 0$ *precautionary exit*.

Inside the continuation region the value of the firm must exceed s . At the exit policy the firm is indifferent between taking the exit value s and continuation value, so

$$V(x, \tilde{s}(x)) = s. \quad (11)$$

Smooth pasting at the exit policy requires

$$V_s(x, \tilde{s}(x)) = 1, \quad (12)$$

$$V_x(x, \tilde{s}(x)) = 0. \quad (13)$$

The only way in which a firm following the optimal policy can extinguish all funds is to hit the zero-flow-zero-stock point for cash, $\{x, s\} = \{c, 0\}$. Thus the constrained firm will experience a forced exit with probability 0.³ All exit by liquidity constrained firms is precautionary.

³The firm's position in (x, s) -space cannot evolve along the boundaries of the continuation region because, if $s = 0$ and $x > c$ then $ds > 0$, and if $x < c$ then the firm exits if it hits the boundary $\{x, \tilde{s}(x)\}$.

Dividend policy

Suppose first that $r < \rho$ so that holding cash is costly. The benefit of holding cash is that it may allow the firm to avoid a forced exit in the future when the option value of continuation would still be positive. This benefit is bounded above by $V^*(c)$, the unconstrained continuation value at the zero profit flow. Since the cost of holding cash increases without bound in s there exists, for any x , some s high enough so that it is better to stop accumulating cash. We call this threshold the dividend policy $\hat{s}(x)$. For sufficiently high x the possibility of forced exit is so remote that it is not worth holding on to any cash. We denote the threshold above which it is optimal to not hold any cash $\hat{x}_{\max}$.

At the boundary, cash is equally valuable inside as it is outside the firm, where one dollar is of course worth one dollar. This value matching condition

$$V_s(x, \hat{s}(x)) = 1 \quad (14)$$

defines the right boundary of the continuation region. The associated smooth-pasting condition requires⁴

$$V_{ss}(x, \hat{s}(x)) = 0, \quad (15)$$

$$V_{xs}(x, \hat{s}(x)) = 0. \quad (16)$$

The firm is constrained at the margin only in the continuation region: There having a dollar more would increase the value of the firm by more than a dollar, so $V_s(x, s) > 1$.

When the firm hits the dividend boundary from inside it pays out just enough cash to not cross the boundary. There are only two times when the firm would pay a lump-sum dividend. If the firm starts at $s > \hat{s}(x)$ then it immediately gives the excess $s - \hat{s}(x)$ as a lump sum dividend, and if it exits then it pays out the remaining s as the liquidation value. Note that if a firm that enters the industry at revenue level $x = x_0$ has the opportunity to choose the initial cash stock then $s_0 = \hat{s}(x_0)$ is the optimal choice.

Consider now the special case in which $r = \rho$. Hoarding cash inside the firm is now costless, so as long as there is a positive probability of being forced to exit in the future it is strictly optimal to not pay any dividends. No matter how high x , falling below x^* remains possible. However, in this case the firm would become irreversibly unconstrained if it were to accumulate so much cash that it could use the interest income from the cash stock to

⁴In terms of Dumas (1991), the dividend is “an infinitesimal regulator” while exit is a discrete regulator so there must be “super-contact” at $\hat{s}(x)$.

cover the worst-case losses under the optimal unconstrained policy. The worst-case loss under the unconstrained policy is $x^* - c$ forever, so this *escape level of cash* is

$$s^* = -\frac{x^* - c}{r}. \quad (17)$$

This means that a fixed boundary condition

$$V(x, s^*) = V^*(x) + s^* \quad (18)$$

now replaces the free boundary $\hat{s}(x)$ seen in the $r < \rho$ case. For $s \geq s^*$ the firm is indifferent between paying dividends or not and $V(x, s) = V^*(x) + s$. Since the firm is unconstrained at s^* and the unconstrained exit threshold is x^* , now $\hat{s}(x^*) = s^*$ and $\hat{x}_{\min} = x^*$.

3 Solving the Firm's Optimal Policy

The PDE defined by (8) and the various free boundary conditions cannot be solved analytically. To solve the firm's problem we turn to a discrete-time approximation of the problem and solve it numerically. In the binomial process approximation of geometric Brownian motion the evolution of x is governed by

$$x(t + \Delta) = \begin{cases} x(t) e^{\sigma\sqrt{\Delta}} & \text{with probability } q = \frac{1}{2} \left(1 + \frac{\mu + \frac{\sigma^2}{2}}{\sigma} \sqrt{\Delta} \right) \\ x(t) e^{-\sigma\sqrt{\Delta}} & \text{with probability } 1 - q \end{cases} \quad (19)$$

where Δ is the length of the time period.⁵ The evolution of the cash stock is now

$$s(t + \Delta) = (s(t) - \delta(t))(1 + r\Delta) + (x(t) - c)\Delta \quad (20)$$

where $\delta(t) \in [0, s^+(t)]$ is the dividend paid at time t . The dividend cannot be so high as to make the cash holdings negative at any point in time, so maximum feasible dividend, restricted by $\min\{s(t + \Delta), s(t)\} \geq 0$, is $s^+(t) \equiv s(t) + \min\{0, (x(t) - c)\Delta / (1 + r\Delta)\}$.

The value function of the firm, stated in recursive form, is

$$V(x(t), s(t) | t) = \max \left\{ s(t), \max_{s^+ \geq \delta \geq 0} \left\{ \delta + \frac{1}{1 + \rho\Delta} [EV(x(t + \Delta), s(t + \Delta) | t + \Delta)] \right\} \right\} \quad (21)$$

⁵Cox, Ross and Rubinstein (1979) show that geometric Brownian motion is the limit of this binary process when $\Delta \rightarrow 0$.

where $s(t + \Delta)$ is from (20).

The recursion in (21) clearly satisfies Blackwell's sufficient conditions so $V(x, s|t)$ is a contraction mapping. Thus it can be solved by iterating backwards in time: Starting from an arbitrary $V_T(x, s|T)$ the value function converges to the unique solution $V(x, s)$.⁶

3.1 Comparative Statics of Optimal Policy

Next we investigate how the optimal policy depends on the parameters r, μ, σ . We do this comparison by varying one parameter at a time from a set of baseline parameters, $r = 0.05$, $\rho = 0.1$, $\mu = 0$, $\sigma = 0.25$. The results are depicted in Figure 2. The solid lines mark the borders of the continuation region in the liquidity constrained case, and the dashed line marks the optimal exit threshold in the unconstrained case.

The left hand panel of Figure 2 shows the impact of varying the return on firm's cash holdings, r . As r gets smaller it becomes costlier to hold cash so continuation is everywhere less attractive and the continuation region shrinks. Interestingly, the optimal dividend policy is extremely sensitive to r for values near ρ . The case $r = \rho = 0.1$ results in the escape level of cash $s^* = 3.24$ from (17). This is much higher than the highest cash holdings that the firm would ever keep even at $r = 0.0999$.⁷ The optimal policy approaches the policy of the limiting case $r = \rho$ very slowly. This is understandable because the limiting case is qualitatively different. The value $\hat{x}_{\max}$ above which the firm optimally holds no cash is finite for all $r < \rho$, but if $r = \rho$ then the firm will not stop holding cash no matter how high x . The strictly decreasing dividend boundary $\hat{s}$ that hits the x -axes at a finite value $\hat{x}_{\max}$ has as its limiting value when $r \rightarrow \rho$ the horizontal half-line that begins at $\{x^*, s^*\}$. The high sensitivity of optimal policy to r near ρ means that, even when firms can earn approximately the same return on their cash holdings as is the discount rate, a model where the liquidity cost is assumed away can not generate firm behavior that approximates the optimal behavior.

The top right panel of Figure 2 shows the relation of the optimal policy and the volatility of the cash flow process. As is typical, higher volatility makes it optimal to accept bigger losses because it increasing the upside potential while the downside is still protected by

⁶A natural starting point for the backward induction is $V(x, s|T) = s$. This means that the problem is turned into a finite-horizon problem with forced exit in the last period. By increasing T the value function at $t = 0$ converges to that of the infinite horizon problem.

⁷The numerical solution converges extremely slow when r is near ρ . This limits our ability to solve the optimal policy for values of r close to but strictly below ρ .

the exit option. In terms of the optimal policy, the increased option value shows up as an enlarged continuation region. This is already visible in the unconstrained problem, where the exit threshold x^* is decreasing in σ . In the constrained problem, the dividend boundary shifts out to the right already because, at any given x , higher volatility also increases the risk of facing forced exit within any given period of time.

The bottom right panel shows the effect of varying μ , the percentage drift of the cash flow process. Higher μ increases the option value at any given level of losses, as the firm is more likely to bounce back to positive profits within any given period of time. However, as higher μ also makes the firm safer at any given point—by making it less likely that forced exit would threaten it within any given time—it is not obvious that a higher μ should also shift out the dividend boundary. However, we have found no counterexamples to this result.

[Figure 2 here]

Think of the firm's cash stock as a stockpile of dollars, and the firm as using the last-in-first-out principle in managing this stockpile (i.e., the firm only ever touches the top of the pile). Given that there is a liquidity cost of holding on to cash it may seem surprising that the firm runs out of cash with probability zero. After all, at every point in time, the bottom dollar incurs the same liquidity cost as any other dollar. Why keep a dollar that is “almost surely” never used—why not lower the stockpile by a dollar? Why not use an otherwise similar policy that saves the liquidity cost on the last dollar but sometimes allows the cash to run out? To understand the answer, note that the “last” dollar would only be called upon in the vicinity of $\{c, 0\}$, i.e. when the flow profit is zero and the cash stock is down to the last dollar. At this point the marginal contribution of cash to the value of the firm is extremely high. The numerical solution is necessarily based on a discrete model, but in the continuous case, V_s would approach infinity at $\{c, -\varepsilon\}$ because an “epsilon” more of cash would allow the firm to continue, while without cash it is forced to quit and take zero value. Note that by surviving to $\{c, \varepsilon\}$ the firm enters a region where $ds > 0$ so being able to continue immediately gives a significant chance of drifting away from the danger zone. Figure 3 shows selected derivatives of the value function, including V_s in the top-left panel⁸. Similarly, the cross-partial V_{sx} exhibits an extreme reversal from large positive to large negative values near $\{c, 0\}$, the zero-profit zero-cash point. A slightly better cash

⁸The figure is calculated under the baseline parameters, but it is qualitatively similar as long as $r < \rho$.

flow leads to an extremely large increase in the marginal value of cash, as it saves the firm with very high likelihood to the left of that point, where $ds < 0$. To the right of the point, the firm is drifting toward safety as $ds > 0$ so it is suddenly much less likely that the firm would end up needing the “last” dollar and the impact of better cash flow on the marginal value of cash is extremely negative.

[Figure 3 here]

4 Industry Equilibrium

In this section we consider the impact of a liquidity constraint on a whole industry where firms face the problem studied in the previous two sections. We now assume that revenue depends on firm-specific output or “productivity” z and an endogenous industry-specific output price p , so that $x = pz$. Productivity z follows geometric Brownian motion $dz = \mu z dt + \sigma z dw$, with the shocks dw independent across firms. New firms of productivity $z_0 > 0$ can be established by paying an entry cost ϕ . In the constrained case new firms enter with an initial cash stock s_0 . (We treat the initial value s_0 as a parameter of the problem; the value z_0 amounts to setting the units of measurement for z and can be chosen without loss of generality.) The industry faces a downward sloping demand curve for its output. Price p is determined by the entry condition of new firms: It must adjust to eliminate expected rents to entrants. We assume that firms are atomistic so there is no aggregate uncertainty. Therefore p will be a constant from firms’ point of view which in effect just face the revenue process (1). Our main goal is to analyze the effect of the liquidity constraint on the steady-state distributions of productivity z and firm lifetimes T .

If the drift in productivity is sufficiently high then firms have a positive probability of escaping to high values of z and living forever. This possibility would have no implications for the characterization of the optimal firm policy seen above, but it is not sensible from the point of view of population dynamics because then all firms would be infinitely old in stationary equilibrium. Thus, to study the steady state, we now assume that there is an exogenous “death rate” λ at which firms are forced to exit with their current cash stock as the exit value. In steady state, both the dying and the endogenously exiting firms are replaced by new firms at $\{z_0, s_0\}$. The risk of exogenous death changes the firm’s optimal

policy slightly compared to how it was presented in Section 2: The firms discount the future at rate $\lambda + \rho$ instead of ρ and, in addition, the Bellman equation of the constrained firm results in a term λs on the right hand side of (8).⁹

As no firm lives forever and there is no aggregate uncertainty it is intuitive that there exists a stationary distribution for z . (The details of the stationary distribution are shown in the Appendix.) However, a further restriction

$$\lambda > \mu + (1/2)\sigma^2 \quad (22)$$

is needed for economic sense, because otherwise the mean output of firms in steady state is not finite. Intuitively, if the drift and the variance of the process for z are too high relative to the exogenous death rate then the vanishing mass of firms that escape to the right tail begin to dominate industry output.

4.1 Equilibrium price

Unconstrained firms Entering firms must expect zero rents over the entry cost, so the equilibrium price is determined by

$$p^* = \{p \text{ st. } V^*(pz_0) = \phi\}. \quad (23)$$

Even though the value function has a closed-form solution (4) the equilibrium price can only be solved implicitly. It is intuitive (and straightforward to show) that p^* is increasing in ϕ and has a strictly positive limit as $\phi \rightarrow 0$. Price p^* also pins down the exit threshold for z because we know the exit threshold for revenue x^* from (3); thus z^* is solved from $x^* = p^*z^*$. Therefore z^* is also decreasing in ϕ . From the point of view of efficiency in steady state, ϕ is the cost of replacing an existing firm with a firm of type z_0 . When this replacement is less costly it makes sense to have a higher “replacement threshold” z^* .

The limiting case $\phi \rightarrow 0$ is somewhat pathological. Then $z^* = z_0$ and newborn firms exit immediately with probability one. Nevertheless the industry manages to produce a flow of output at average cost c/z_0 . The immediate exit follows from the property of Brownian motion that z will visit both sides of the initial point in a vanishingly short time.

⁹The simulation results are based on optimal firm policies that have been adjusted to take into account $\lambda > 0$.

Constrained Firms For constrained firms the value at entry must cover both the entry cost and the initial cash injection.

$$V(pz_0, s_0) = \phi + s_0. \quad (24)$$

A liquidity constraint is of course bound to reduce welfare. Due to perfect competition, welfare in our model is purely a matter of consumer surplus and the impact on p captures the impact on welfare.¹⁰

As in the unconstrained case, from firms' point of view p is fixed and the problem can be described in terms of revenue. It follows that, for given parameters of the firm's problem, the policy regions are fixed in $\{x, s\}$ -space. Varying p merely affects how the values of z map to the x -axes. While z_0 and s_0 are parameters of the problem, p must adjust so that the value function at the point where new firms enter, $\{pz_0, s_0\}$ matches the value $\phi + s_0$.

Impact of Liquidity Constraint on Industry Equilibrium

We know that a liquidity constraint must reduce welfare so p will be increasing in the severity of the constraint. But is the welfare loss due to less productive firms, too short firm lifetimes (leading to excessive entry costs), or both? We next compute the steady state distribution of $\{z, s\}$ by creating firm spells in a Monte Carlo simulation that adhere to the optimal policies computed in the previous section and face an output price determined by (23) and (24).

The severity of the liquidity constraint is captured by three parameters: The entrants' initial (post-entry) cash stock s_0 , the liquidity premium $\rho - r$, and the entry cost ϕ . Having either more cash to start with or a lower liquidity cost obviously result in a less harsh liquidity constraint; in the extreme case of $s_0 = s^*$ and $r = \rho$ the firm becomes unconstrained. The role of ϕ is more subtle. By making it more costly to replace existing firms with new firms of type z_0 a higher ϕ increases the equilibrium option value of continuation: The value of an entering firm must be exactly ϕ higher than the value of a firm at the exit threshold. (In the liquidity constrained case, this comparison is between two firms with the cash level s_0 .) Therefore if ϕ is small then the exit threshold must be close to the initial point z_0 , which in turn means that many of the firms will exit soon after entry. Likewise,

¹⁰We do not model the exact shape of the demand curve, as we are not interested in the aggregate level of output, but rather in its distribution between firms. The industry as a whole has constant returns to scale: To double the steady state output, the mass of firms would double but the distribution of z would be unchanged.

for high ϕ the exit threshold must be far below the productivity level of an entrant. Thus ϕ captures how far entrants are from the exit threshold and how acute is the threat of forced exit for newly established liquidity constrained firms.

The liquidity constraint has three channels for causing distortions. Average productivity of firms in steady state may be either too high or too low. The average lifetime of firms can also be too high or too low, impacting the total steady state entry costs in the industry. (The turnover rate of firms is the inverse of average firm lifetime.) Finally, the cash stock held by the industry incurs a liquidity cost.

While it is clear that the liquidity constraint must increase p , the direction of the distortion on average productivity is not obvious. The liquidity constraint could decrease both average productivity and average lifetime. More interestingly, it can increase productivity while decreasing longevity, or vice versa.¹¹ Recall from Section 2 that, holding p fixed, the liquidity constraint will cause some firms to exit when $x > x^*$; as these exiting firms are relatively unproductive this tends to increase the average productivity of survivors compared to the efficient benchmark, and makes firms shorter-lived thus increasing the steady-state flow of entry costs in the industry. However, in equilibrium, p is not fixed but in fact higher under the liquidity constraint. This means that the lowest type to stay in the industry, $\hat{z}_{\min}$, can be less productive than the lowest type that survives in the unconstrained case. Some unproductive but cash-rich firms $z < z^*$ do not exit due to the price distortion, so the impact on average productivity is ambiguous. (By the same token, the effect of the liquidity constraint on lifetimes is also ambiguous.) Figure 7 shows this trade-off in (z, s) -space. The dotted region shows firms that are more productive than the efficient exit threshold z^* but that exit due to the liquidity constraint. The shaded region covers firms that would exit in the efficient solution but that stay in under the liquidity constraint. These distortions on firm survival have opposing impacts on $\bar{z}$, and the total impact could in principle go either way.

The results from the simulations are collected in Figures 4 and 5. The simulations were calculated using the baseline parameters $\mu = 0$, $\sigma = 0.25$, $\rho = 0.1$, $r = 0.05$, $\lambda = 0.1$, $z_0 = 1$, $c = 1$. Figure 4 shows the levels of selected steady state outcomes as a function of the entry cost ϕ , for several values of the starting cash s_0 in the constrained case (solid

¹¹We have not been able to rule out the logical possibility that firms could be both longer-lived and more productive in the constrained case. This would require the liquidity cost alone to dominate the benefits of higher productivity and lower entry costs. However, this result did not come up under any of the parameter combinations that we used in the simulations.

lines), and in the unconstrained case (dashed line). Figure 5 shows the associated levels under the liquidity constraint relative to the efficient benchmark.

[Figure 4 here]

[Figure 5 here]

The top left panel shows the impact of the liquidity constraint on output price. Smaller values of s_0 and higher values of ϕ imply a harsher liquidity constraint so both naturally increase the price.

Mean productivity is shown in the top right panel. Higher ϕ makes it costlier to replace unproductive firms with a new firm of productivity z_0 so the mean is always decreasing in ϕ . The relation with s_0 is more subtle. When ϕ is small then the entrants are close to the exit threshold; in the unconstrained case this means that there will be almost no firms in the industry with $z < z_0$ so average productivity is very high. However, in the unconstrained case, even though the exit threshold of a firm with the entry level of cash s_0 is very close to z_0 , firms that manage to accumulate more cash before falling to below z_0 in productivity will stay in the industry even with a relatively low z . This is why, for small levels of ϕ , the usual result, by which a liquidity constraint provides too much selective pressure and weeds out marginal firms, fails and mean productivity is in fact lower in the constrained case. This is a case of “survival of the fattest:” Currently unproductive but previously successful and therefore cash-rich firms stay in the industry and crowd out entry by more productive firms.¹²

The bottom panels show the average firm age and lifetime in steady state. Higher ϕ increases lifetimes across the board by making turnover more costly. Average lifetime is also reduced by a lower s_0 as more firms run out of cash early on. However, average firm age is higher in the constrained case if s_0 and ϕ are sufficiently low. Figure 6 shows the distributions of lifetimes and ages. The top panel shows the lifetime distributions with different values of s_0 together with the unconstrained case. With low s_0 liquidity constraint is in effect shifting the lifetime distribution to the left, but as s_0 is increased, the distribution approaches the unconstrained case. The bottom panel shows the cross-section age distributions with different values of s_0 . Also here, the lower values of s_0 shift distributions to the left.

¹²Zingales (1998) finds evidence of similar phenomenon in the trucking industry.

5 Conclusion

We have analyzed the problem of a liquidity constrained firm that faces a stochastic cash flow. The firm may be forced to exit due to inability to absorb a negative cash flow, even when the possibility to rebound into profits conveys option value that would make continuation (socially) optimal. To prevent such inefficient exit, the firm engages in precautionary saving out of retained earnings, and to preempt it the firm will exit before actually running out of cash. The optimal policy includes both an exit policy and a dividend policy, which depend on current cash flow and cash holdings.

Traditional pre-entry liquidity constraints tend to increase the average productivity of firms in market equilibrium, because the standard for profitable entry is set too high. We showed that post-entry liquidity constraint lead to the opposite phenomenon, where unproductive firms that have a lot of cash (from earlier success) do not exit soon enough and end up reducing the average productivity below the efficient benchmark level. We showed that this latter “survival of the fattest” effect dominates when entry costs are sufficiently low.

Appendix

Deriving equation (8) from a binary process

Cox, Ross, and Rubinstein (1979) showed how option models can be simplified by interpreting the Geometric Brownian Motion (GBM) as the limiting case of a binary process where the time interval dt between jumps goes to zero. Applied to our model, the binary draws are relative changes $x(t + dt)/x(t)$ of either $e^{\sigma\sqrt{dt}}$ or $e^{-\sigma\sqrt{dt}}$. The upward jump has probability

$$q = \frac{1}{2} \left(1 + \frac{\mu - \sigma^2/2}{\sigma} \sqrt{dt} \right) \quad (25)$$

where μ is the drift of the limiting GBM, and $\mu - \sigma^2/2$ is the drift of $\log(x)$. The corresponding change in cash holdings is $ds = (x - c + rs) dt$.

Consider the Bellman equation for a firm in state (x, s) in the interior of the continuation region. The value of the firm is the discounted expectation of the value after dt has passed:

$$V(x, s) = \frac{1}{1 + \rho dt} \left[qV(xe^{\sigma\sqrt{dt}}, s + ds) + (1 - q)V(xe^{-\sigma\sqrt{dt}}, s + ds) \right]. \quad (26)$$

Now write $V = V(x, s)$, and let $dt \rightarrow 0$. Then:

$$\begin{aligned} V(xe^{\sigma\sqrt{dt}}, s + ds) &= V + V_x(xe^{\sigma\sqrt{dt}} - x) + V_s ds + \\ &\quad \frac{1}{2} \left(V_{xx}(xe^{\sigma\sqrt{dt}} - x)^2 + 2V_{xs}(xe^{\sigma\sqrt{dt}} - x) ds + V_{ss}(ds)^2 \right) + \dots \end{aligned} \quad (27)$$

and analogously for $V(xe^{-\sigma\sqrt{dt}}, s + ds)$. Using the series definition $e^{\sigma\sqrt{dt}} = 1 + \sigma\sqrt{dt} + \frac{\sigma^2 dt}{2} + \dots$, we can write

$$\begin{aligned} xe^{\sigma\sqrt{dt}} - x &= x \left(\sigma\sqrt{dt} + \frac{\sigma^2 dt}{2} + \dots \right), \\ xe^{-\sigma\sqrt{dt}} - x &= x \left(-\sigma\sqrt{dt} + \frac{\sigma^2 dt}{2} + \dots \right). \end{aligned} \quad (28)$$

Therefore the value conditional on an upwards jump is

$$\begin{aligned}
V\left(xe^{\sigma\sqrt{dt}}, s+ds\right) &= V + V_x x \left(\sigma\sqrt{dt} + \frac{\sigma^2 dt}{2} + \dots\right) + V_s ds \\
&\quad + \frac{1}{2} \left(V_{xx}^2 x^2 \left(\sigma\sqrt{dt} + \frac{\sigma^2 dt}{2} + \dots\right)^2 \right. \\
&\quad \left. + 2V_{xs} x \left(\sigma\sqrt{dt} + \frac{\sigma^2 dt}{2} + \dots\right) ds + V_{ss} (ds)^2 \right) + \dots \\
&= V + V_x x \left(\sigma\sqrt{dt} + \frac{\sigma^2 dt}{2} + \dots\right) + V_s ds \\
&\quad + \frac{1}{2} \left(V_{xx}^2 x^2 (\sigma^2 dt + \dots) + 2V_{xs} x \left(\sigma\sqrt{dt} + \frac{\sigma^2 dt}{2} + \dots\right) ds + \dots \right).
\end{aligned} \tag{29}$$

An analogous expression holds for $V\left(xe^{-\sigma\sqrt{dt}}, s+ds\right)$. Ignoring all terms with dt of order higher than 1, we get

$$\begin{aligned}
V\left(xe^{\sigma\sqrt{dt}}, s+ds\right) &= V + V_x x \left(\sigma\sqrt{dt} + \frac{\sigma^2 dt}{2}\right) + V_s (x - c + rs) dt + \frac{1}{2} V_{xx}^2 x^2 \sigma^2 dt \\
V\left(xe^{-\sigma\sqrt{dt}}, s+ds\right) &= V + V_x x \left(-\sigma\sqrt{dt} + \frac{\sigma^2 dt}{2}\right) + V_s (x - c + rs) dt + \frac{1}{2} V_{xx}^2 x^2 \sigma^2 dt
\end{aligned} \tag{30}$$

Combined with (25) and simplifying, the Bellman equation (26) can now be written as

$$\begin{aligned}
\rho V dt &= \frac{1}{2} \left[V_x x \left(\sigma\sqrt{dt} + \frac{\sigma^2 dt}{2}\right) + V_s (x - c + rs) dt + \frac{1}{2} V_{xx}^2 x^2 \sigma^2 dt \right] + \frac{1}{2} (\mu - \sigma^2/2) x V_x dt \\
&\quad + \frac{1}{2} \left[V_x x \left(-\sigma\sqrt{dt} + \frac{\sigma^2 dt}{2}\right) + V_s (x - c + rs) dt + \frac{1}{2} V_{xx}^2 x^2 \sigma^2 dt \right] + \frac{1}{2} (\mu - \sigma^2/2) x V_x dt \\
&= V_s (x - c + rs) dt + \left(\mu - \frac{\sigma^2}{2} + \frac{\sigma^2}{2} \right) x V_x dt + \frac{1}{2} \sigma^2 x^2 V_{xx}^2 dt
\end{aligned} \tag{31}$$

Finally, dividing by dt yields (8).

Sketch of Proof for Proposition 1

This is a proof sketch. A formal proof is under work.

Let us consider in turn the properties of the regions in the state-space, where it is optimal to exit, continue, and pay dividends, respectively.

First, consider optimal exit. Given an arbitrary point (x, s) , define regions $(x, s)^+ \equiv \{(x', s') \mid x' > x, s' > s\} \setminus (x, s)$ and $(x, s)^- \equiv \{(x', s') \mid x' < x, s' < s\} \setminus (x, s)$. It is clear that there must be some state points where it is strictly optimal to exit (e.g. when an unconstrained firm would find it optimal to exit) and points where it is strictly optimal

not to exit (e.g. all points where current profit flow is positive). Then there must also be points where the firm is indifferent between exiting and staying. Take a point (x, s) where the firm is indifferent between exiting and staying, in which case the value of the firm is s . Then starting from any $(x', s') \in (x, s)^+$, the firm could do the following: pay immediately a lumpy dividend $s' - s$, then replicate the cash flow obtained with optimal policy from (x, s) , and since $x' > x$, this results with a greater final wealth upon exit. This means that the value of the firm at (x', s') must be strictly greater than s' , so it is strictly non-optimal to exit. The same logic implies that if the firm were to continue at some $(x', s') \in (x, s)^-$, its value would be strictly below s' , thus it must be strictly optimal to exit for all points in $(x, s)^-$. We also know from the construction of the model that $(c - \varepsilon, 0)$ must be in the exit region for any $\varepsilon > 0$, but $(c, 0 + \varepsilon)$ can not be in the exit region for any $\varepsilon > 0$. This means that the points of indifference must form a decreasing continuous curve in the (x, s) plane that ends in $(c, 0)$. Also, it can be shown that for s high enough, this indifference curve must be vertical and run to infinity: for high enough s the cost of carrying an extra dollar must at some point go below its marginal value, therefore for high s an additional dollar will not change the optimal exit behavior (the firm would pay it immediately out as dividends anyway). This means that there must be some $x_{\min} < c$ such that the downward sloping part of the indifference curve can be written as a strictly decreasing function $\tilde{s} : [x_{\min}, c] \longrightarrow [0, \tilde{s}_{\max}]$, and the vertical part as the set $\{(x, s) \mid x = x_{\min}, s > \hat{s}_{\max}\}$.

Second, consider the region where it is optimal to continue without paying dividends. Start with points slightly above $\hat{s}$. As shown above, it must be strictly non-optimal to exit at any $(x, \hat{s}(x) + \varepsilon)$, $x \in (x_{\min}, c)$, $\varepsilon > 0$. If ε is small enough, it can not be optimal to pay dividends either, because this would bring the firm to the exit region and would be essentially the same as exiting. Therefore, the continuation region must include some connected region above $\hat{s}$. Next, let us show that the continuation region can not contain any other set of points disconnected from this region. For if it did, this disconnected part should be bounded below by the dividend region. But then, the only way to escape from this region is to pay voluntarily a discrete payment: either in the form of a lump dividend or voluntary exit (the forced exit point $(c, 0)$ must be disconnected from this region). Then the firm could pay a small part of that lumpy payment ahead of time without affecting its future, which would save on liquidity premiums. Having such a disconnected continuation region can thus not be optimal. The continuation region must be a connected set that includes points just next to the exit threshold.

Finally, consider the dividend region. Clearly it is optimal to pay dividends when s and/or x are sufficiently high (for then the prospects of running out of cash are distant). As x increases, at some point it is not optimal to keep any cash (let us denote the lowest such x -value by $x_{\max}$). A crucial part is to show in addition this: if (x, s) is in the dividend region, then also (x, s') must be in the dividend region for any $s' > s$ (this is intuitive, but yet to be proved formally). Given this, the boundary between the continuation region and dividend region is some continuous function $\hat{s} : [x_{\min}, x_{\max}] \longrightarrow [0, \hat{s}_{\max}]$.

The final part of this proof is to establish the properties of $\hat{s}$ stated in the proposition. Note that $(x_{\min}, \tilde{s}_{\max})$ is the upper left corner of the continuation region, where exit and dividend thresholds meet. We first want to show that $(x_{\min} + \varepsilon, \tilde{s}_{\max})$ is in the continuation region by showing that $V_s(x_{\min} + \varepsilon, \tilde{s}_{\max}) > 1$, where ε is an arbitrarily small positive number.

Differentiate both sides of (8) with respect to s to get another equation that must hold in the continuation region (including at the exit and dividend thresholds):

$$\rho V_s = rV_s + (x - c + rs) V_{ss} + \mu x V_{xs} + \frac{\sigma^2}{2} x^2 V_{xss}. \quad (32)$$

At the exit and dividend thresholds we have $V_s = 1$ and in the dividend threshold we have in addition $V_{ss} = V_{xs} = 0$. Combining these facts, note that in the intersection point $(x_{\min}, \tilde{s}_{\max})$ equation (32) reduces to

$$(\rho - r) = \frac{\sigma^2}{2} x^2 V_{xss},$$

which implies $V_{xss}(x_{\min}, \tilde{s}_{\max}) > 0$. In addition, smooth pasting conditions imply that $V_s(x_{\min}, \tilde{s}_{\max}) = 1$ and $V_{xs}(x_{\min}, \tilde{s}_{\max}) = 0$. Expanding V_s we get

$$V_s(x_{\min} + \varepsilon, \tilde{s}_{\max}) = V_s(x_{\min}, \tilde{s}_{\max}) + V_{xs}(x_{\min}, \tilde{s}_{\max}) \cdot \varepsilon + \frac{1}{2} V_{xss}(x_{\min}, \tilde{s}_{\max}) \cdot \varepsilon^2 + O(\varepsilon^3) > 1$$

for ε small enough. This means that $(x_{\min} + \varepsilon, \tilde{s}_{\max})$ must be in the continuation region rather than in the dividend region. It follows that $\hat{s}'(x_{\min}) > 0$.

Finally, take some point $(x', \hat{s}(x'))$, where $x' > c - r\hat{s}(x')$. At that point we have $ds > 0$. Consider then a firm at point (x'', s'') , where $x'' > x'$ and $s'' > \hat{s}(x')$. Since $ds > 0$, the firm could pay out a small dividend, and still ensure that its cash holding is at least $\hat{s}(x')$ at the next moment when x falls to x' . Thus, it must be strictly optimal to pay dividends at (x'', s'') . This means that $\hat{s}(x)$ must be decreasing for $x > c - r\hat{s}(x)$.

5.1 Sketch of Proof for Proposition 2

The properties of the exit region follow from the same reasoning as in Proposition 1, except that now $\hat{x}_{\min} = x^*$ and $\tilde{s}_{\max} = s^*$. If the firm continues and $s < s^*$ it does not pay any dividend because there is a positive probability of forced exit and holding cash is costless. If $s > s^*$ then the firm cannot become liquidity constrained even under the worst-case realization of the optimal unconstrained policy. There is thus neither cost nor benefit to holding cash in excess of s^* .

Unconstrained Case: Stationary distribution of z

Denote $y \equiv \log z$. The exit threshold is $y^* = \log z^*$ and new firms are born as $y_0 > y^*$. Dixit and Pindyck (1993) show¹³ that when z evolves according to GBM and newborn firms have density $g(y)$ then the stationary density, $f(y)$, must satisfy

$$\frac{1}{2}\sigma^2 f''(y) - \mu f'(y) - \lambda f(y) + g(y) = 0 \quad (33)$$

and the boundary conditions $f(y^*) = 0$ and $\lim_{y \rightarrow \infty} f(y) = 0$. In our setup $g(y)$ is positive at y_0 and zero elsewhere, so the density $f(y_0)$ is a free parameter. The point y_0 splices the differential equation into two regions, with the $f(y_0) = f_0$ as a boundary condition in the middle. (f is finite but not differentiable at y_0). The value of f_0 can be solved from the requirement that total probability density integrates to one. Combining the boundary conditions with (33) yields the closed-form solution:

$$f(y) = \begin{cases} 0, & y \leq y^* \\ \frac{\lambda e^{\frac{y(\mu-\gamma)}{\sigma^2}} \left(e^{\frac{2y\gamma}{\sigma^2}} - e^{\frac{2y^*\gamma}{\sigma^2}} \right)}{\gamma \left(e^{\frac{y_0(\mu+\gamma)}{\sigma^2}} - e^{\frac{y^*(\mu+\gamma)}{\sigma^2}} \right)}, & y^* < y \leq y_0 \\ \frac{2\lambda e^{-\frac{(y-y_0)(\gamma-\mu)}{\sigma^2}}}{\gamma \left(1 + \coth\left(\frac{(y_0-y^*)\gamma}{\sigma^2}\right) - e^{\frac{-(y_0-y^*)\mu}{\sigma^2}} \operatorname{csch}\left(\frac{(y_0-y^*)\gamma}{\sigma^2}\right) \right)}, & y_0 < y \end{cases} \quad (34)$$

where $\gamma \equiv \sqrt{\mu^2 + 2\lambda\sigma^2}$.

The steady state is not economically sensible unless it results in a finite mean for $z = e^y$. Here $E[e^y | y > y_0] < \infty$ is a necessary and a sufficient condition for the finite mean. Canceling out the terms that are independent of y , and substituting γ into (34), the finite

¹³See chapter 8, section 4.c.

mean requirement is

$$\int_{y_0}^{\infty} e^{y - \frac{(y-y_0)(\sqrt{\mu^2 + 2\lambda\sigma^2} - \mu)}{\sigma^2}} dy < \infty. \quad (35)$$

This holds if the exponent goes to $-\infty$ as y increases without limit, i.e., if¹⁴

$$\sigma^2 - \sqrt{\mu^2 + 2\lambda\sigma^2} + \mu < 0 \iff \quad (36)$$

$$\lambda > \mu + \frac{\sigma^2}{2}. \quad (37)$$

The associated distribution of firm lifetimes is

$$F_T(T|z^*, z_0) = 1 - e^{-\lambda T} \left(1 - \Phi \left(\frac{\log(z^*/z_0) - \alpha T}{\sigma\sqrt{T}} \right) - \left(\frac{z_0}{z^*} \right)^{-\frac{2\alpha}{\sigma^2}} \Phi \left(\frac{\log(z^*/z_0) + \alpha T}{\sigma\sqrt{T}} \right) \right) \quad (38)$$

where $\alpha \equiv \mu - (1/2)\sigma^2$ and Φ is the standard normal CDF.

Constrained Case: Stationary Distribution of (z, s)

The stationarity proof in the unconstrained case is sufficient for the stationarity of the distribution of z in the constrained process. As s is endogenously bounded by the optimal dividend policy and, firm by firm, depends deterministically on the history of z , the fact that z has a stationary distribution suffices for the stationarity of the joint distribution (z, s) . However, now the optimal policy has no closed form solution, so we generate firm spells (histories $\{z_t, s_t\}_t$) with the given parameters and the optimal firm policy using the discrete-time approximation of the GBM process (19). Thanks to $\lambda > 0$, all spells end by some finite $T_{\max}$. The stationary distribution is calculated from this artificial data.

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¹⁴This is slightly different from Dixit and Pindyck (1993) because there the exit threshold is an upper bound for the stochastic variable. There the drift must not be too negative or else too many firms escape to the left tail; hence their equivalent condition has different signs.

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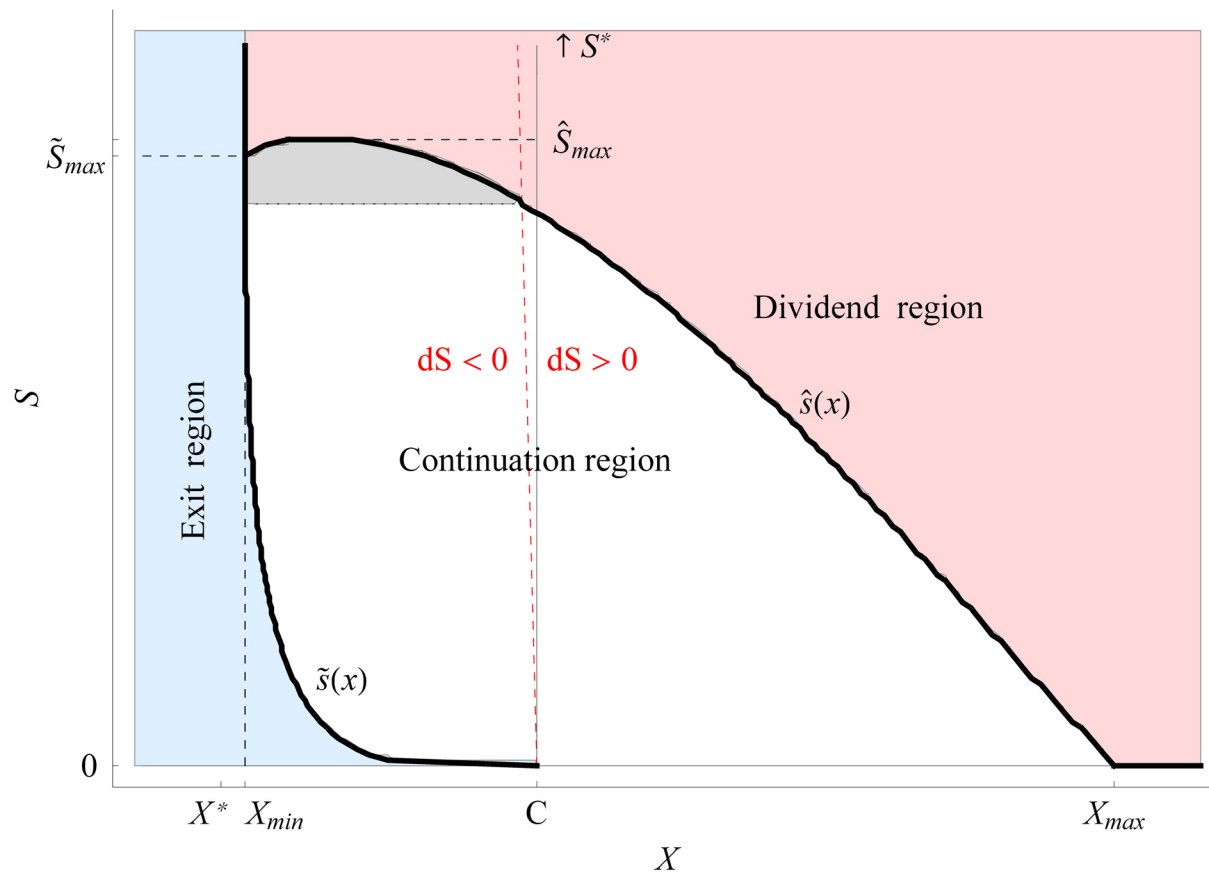


Figure 1. Optimal policy regions of a liquidity constrained firm.

Figure 2. Optimal policy of a liquidity constrained firm: Comparative statics.

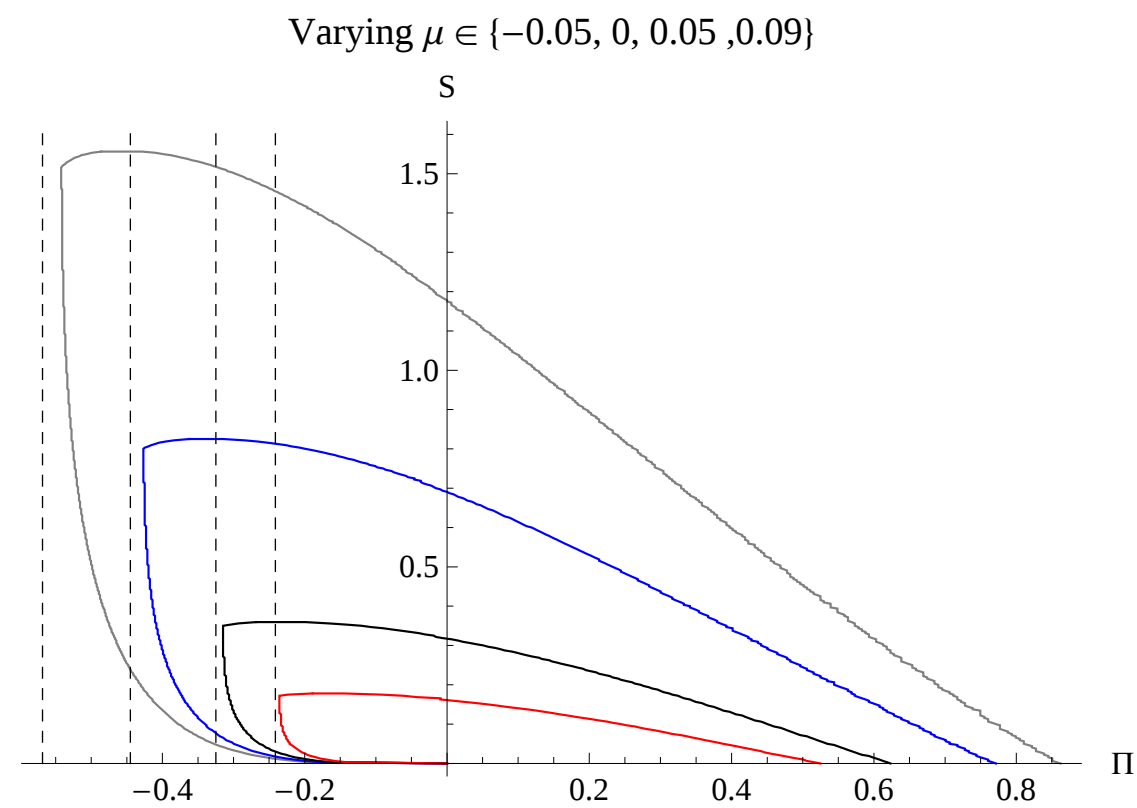
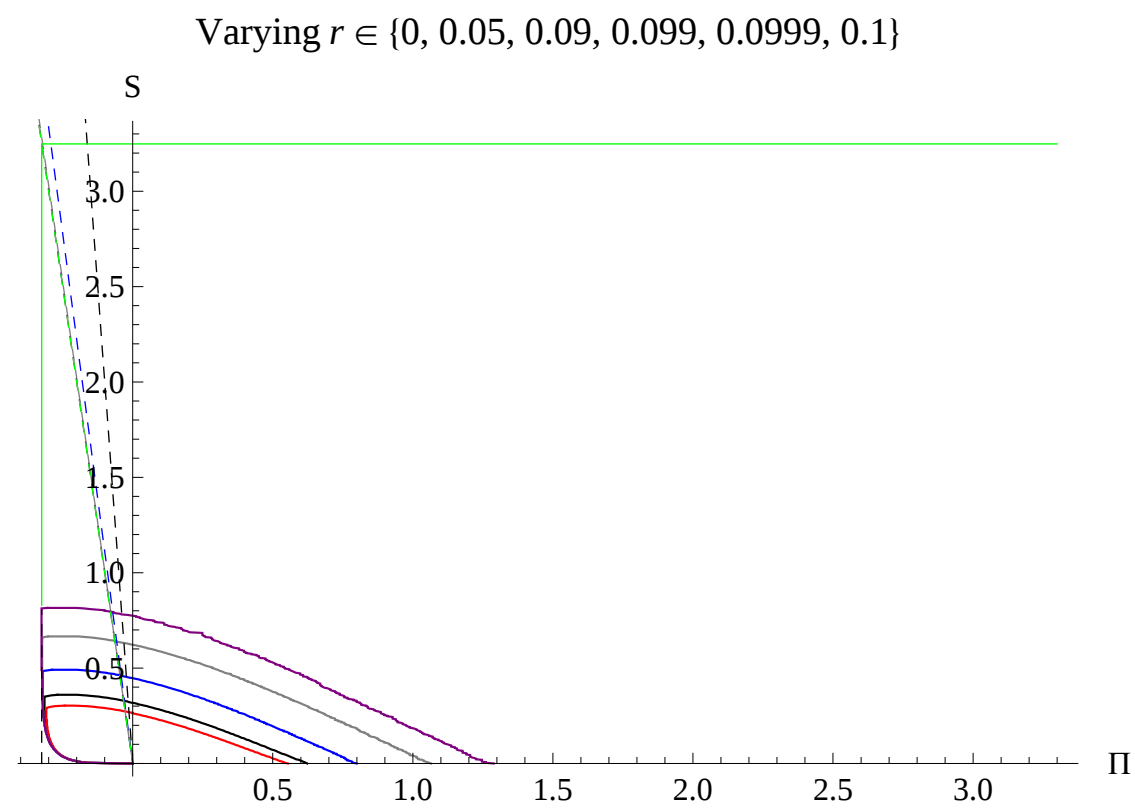
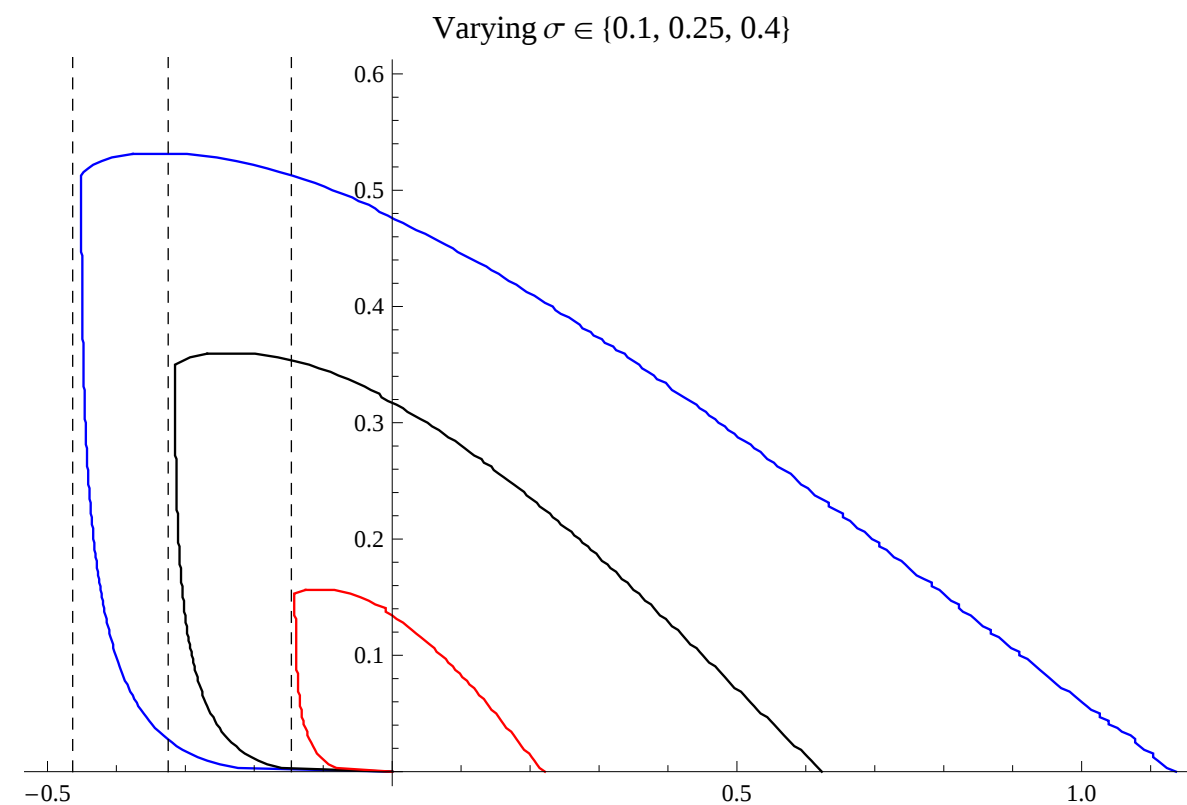
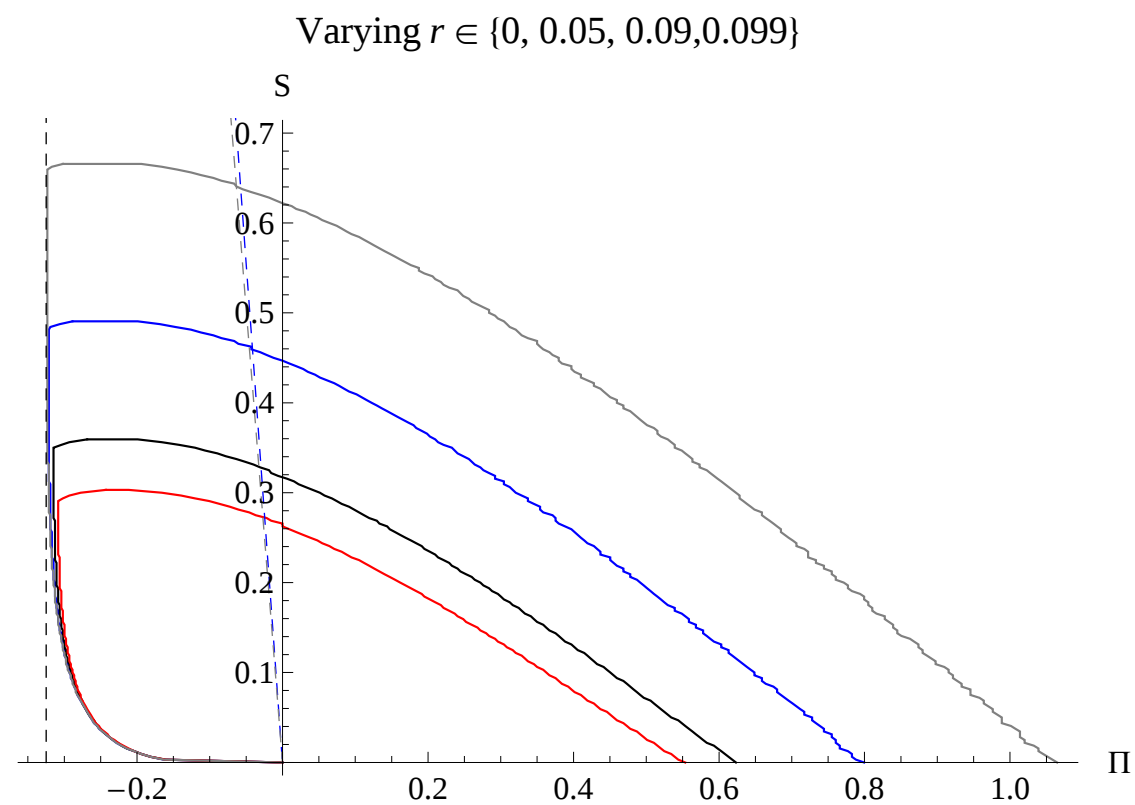
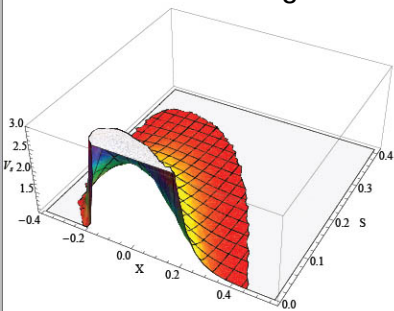
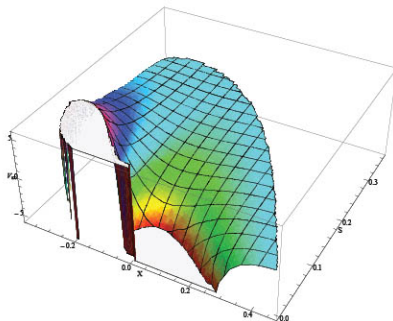


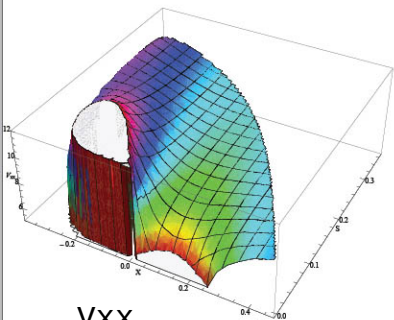
Figure 3.



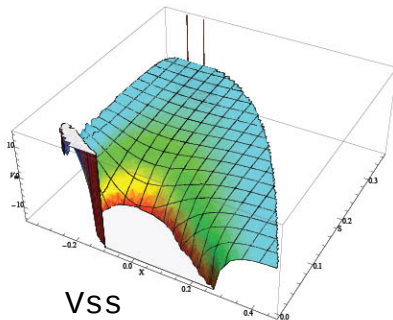
V_s



V_{sx}



V_{xx}



V_{ss}

Figure 4.
Steady state levels
baseline parameters,
 $s_0 = 0$ (blue), 0.1 (green), 0.5 (red)

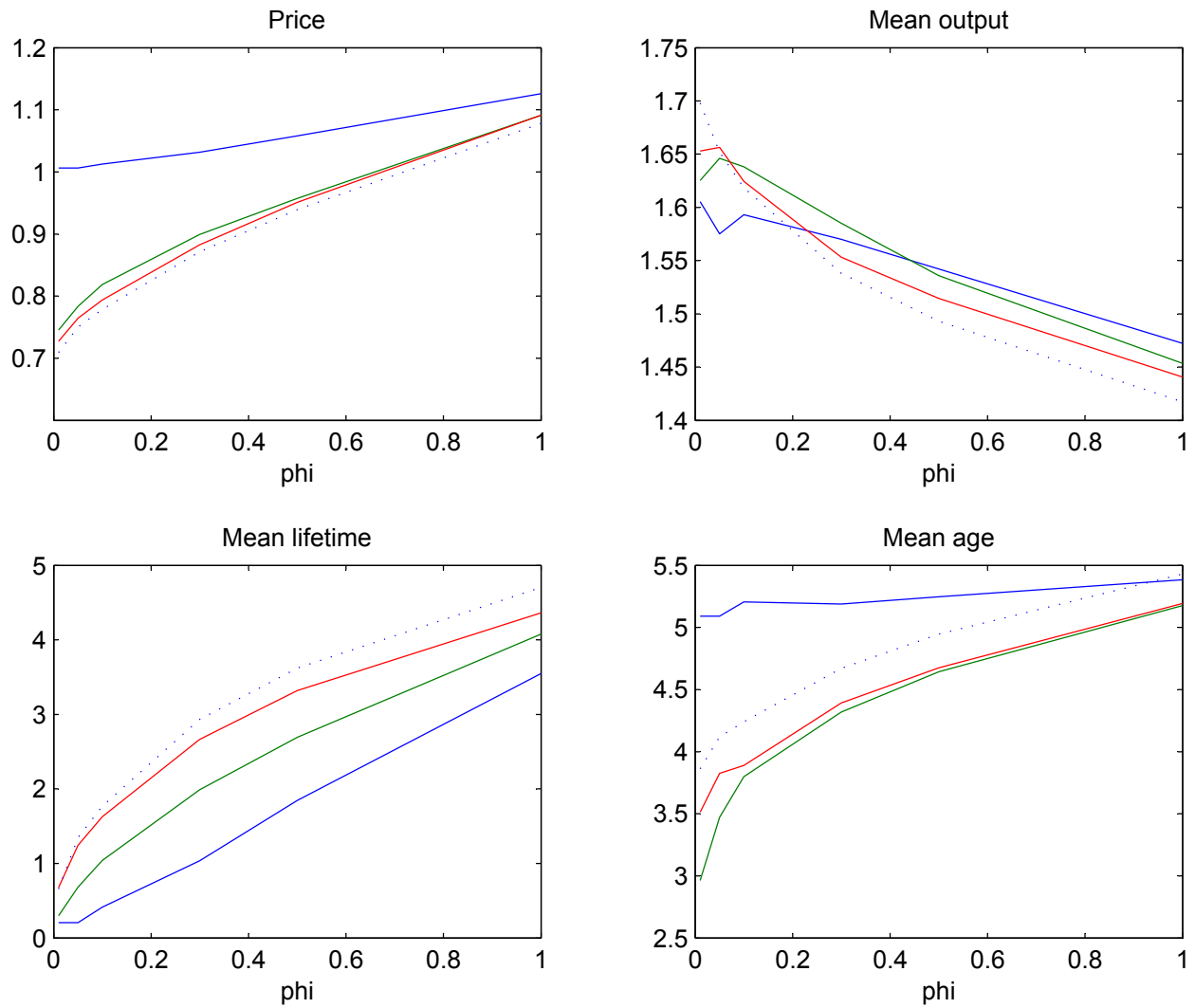


Figure 5. Percentage difference between the steady state levels in constrained and unconstrained cases. Baseline parameters, $s_0 = 0$ (blue), 0.1 (green), 0.5 (red).

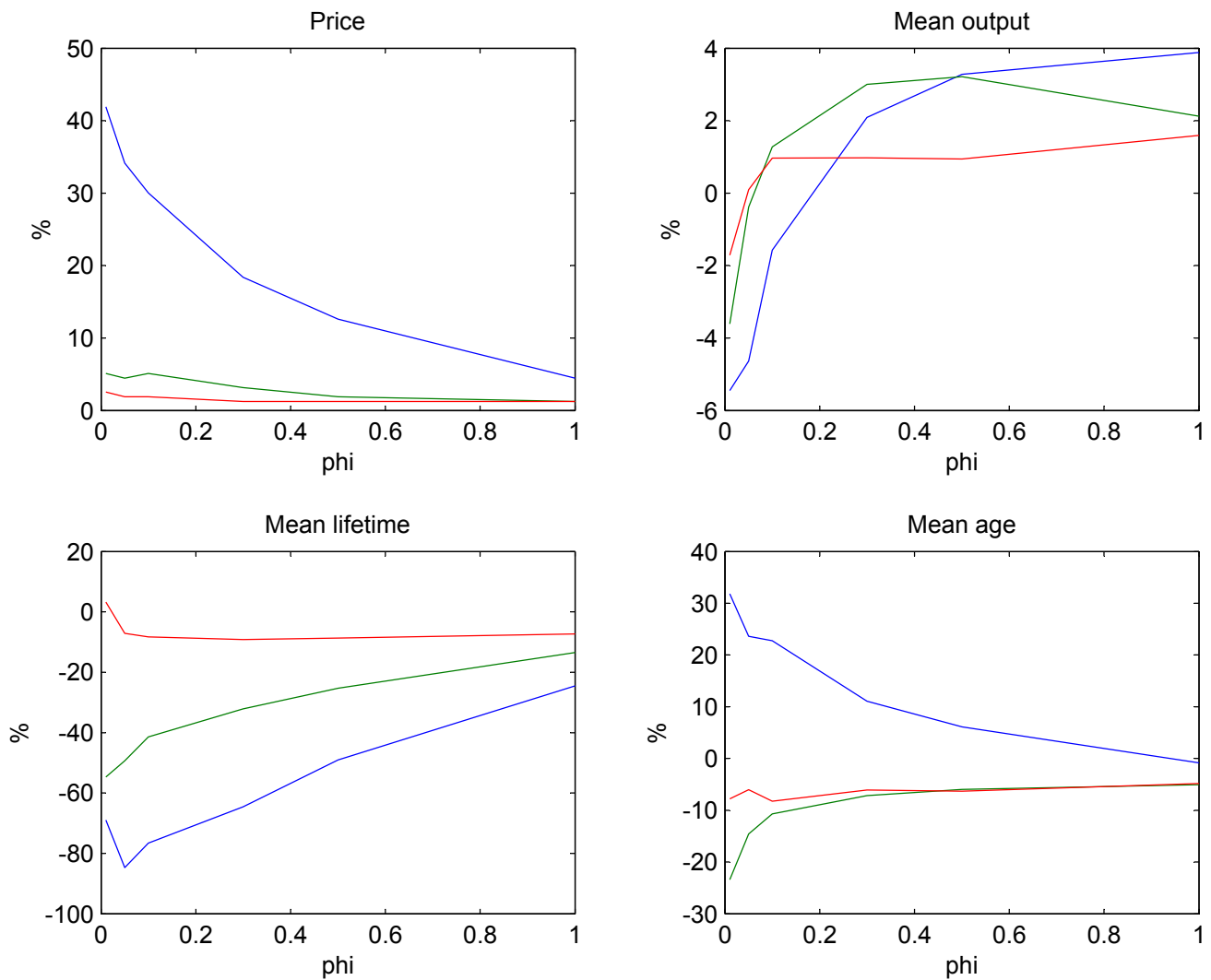


Figure 6. Steady state age and lifetime distributions.

