Buyer heterogeneity and competing mechanisms*

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Abstract

Burdett, Shi and Wright (2001) offer a directed search model where the buyers decide which seller to visit after observing the price each seller posts, and showed that there exists a unique symmetric equilibrium. Coles and Eeckhout (2003) showed that there is a continuum of symmetric equilibria, if the sellers are allowed to post general mechanisms and not only fixed prices. I show that many of the equilibria that Coles and Eeckhout identify, including the fixed price equilibria suggested by Burdett, Shi and Wright (2001), are not robust to introducing heterogeneous buyers with two possible types. In this case only ex-post efficient equilibria exist, i.e. a buyer with a lower valuation can never win against a buyer with a higher valuation if they visit the same seller. This suggests that mechanisms like auctions that utilize buyer competition in an efficient manner may endogenously arise when sellers post competing mechanisms. When the type space is continuous instead of having two possible types, the ex-post efficiency result is not maintained. If the sellers' strategy space is unrestricted, then any allocation respecting sequential rationality of the buyers, can be implemented in equilibrium. However, posting simple fixed price mechanisms never constitute an equilibrium, while posting (second-price or first-price) auctions may if the distribution function is convex.

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1 Introduction

Burdett, Shi and Wright (2001) offer a directed search model where the buyers decide which seller to visit after observing the price each seller posts. If a shop is visited by more than one buyer, then each of those buyers obtains the good with the same probability. While they obtain a unique symmetric equilibrium, Coles and Eeckhout (2003) show that there is a continuum of symmetric equilibria if the sellers can post general mechanisms and not only prices. Some of these equilibria resemble the posted price case considered by Burdett, Shi and Wright (2001), because the price a buyer pays does not depend on the number of other buyers in the same store. Other equilibria are more similar to auctions: if more than one buyer showed up, then the price is close to the value of the object. Coles and Eeckhout (2003) argue that without further restrictions it is impossible to decide which of these equilibria prevails.

This paper assumes that the sellers do not know the reserve values of the buyers that can be either low (l) or high (h). In this case, many of the equilibria described by Coles and Eeckhout (2003) disappear and in equilibrium each store sells to the buyer with the highest valuation who visited the store (ex-post efficiency). Therefore, many of the equilibria (including the one with fixed price mechanisms) identified by Coles and Eeckhout (2003) are not robust to small changes in valuations.

The intuition for this result is simple. Suppose that seller j posts an ex-post *in*efficient mechanism in equilibrium. Then an ex-post efficient mechanism, conducted in three steps, increases the revenue of j. In Step 0 the buyers who visited seller j report their types. In Step 1 the original mechanism is played ignoring the announcements in Step 0. In Step 2 there is a possible retrading depending on the reports from Step 0: if in Step 1, a low type obtains the object when a buyer with high type is also present at j, then the low type buyer (re)sells the object to j for an amount l and the high type buyer buys it (from j) for an amount h, giving the seller an extra profit of $h - l > 0$. It is a (weakly) dominant strategy for each buyer to report truthfully in Step 0 and then the buyers make the same utility at seller j as in the case when j used the original mechanism. Therefore, it remains an equilibrium in the the buyers' stage game to use the same actions (together with reporting the type truthfully if a buyer visited j) as in the case when seller j did not deviate. Then the suggested deviation is obviously profitable, because j obtains an extra revenue of $h - l$ when ex-post inefficiency would have occurred without the deviation of j.

However, there may be other equilibria in the stage game of the buyers and the buyers may coordinate in a way that hurts the deviator. We show that the deviator can offer a more complicated mechanism that eliminates all the equilibria of the buyers' stage game that are unfavorable for him. To derive this mechanism, we follow the approach of the unique implementation literature and for each buyer define lotteries whose payoffs depend on the decisions of the other buyers.[1]

Virag (2007) argues that auctions constitute natural equilibria even in the original complete information model. He shows that if the sellers can collude on equilibria that maximize their profits, then only such equilibria exist where a buyer's payment is increasing in the number of other buyers visiting the same seller. This suggests that posting auction-like mechanisms may be a means to achieve collusion between sellers. However, there is a crucial difference between the collusion based argument of that paper and the incomplete information analysis presented here. Under collusion, the key is that the price must increase in the number of bidders who visited the same store, while under incomplete information the key is that the mechanism is ex-post efficient. Obviously, both features can be obtained by using an open auction, but there are many non-auction mechanisms that satisfy one of the two features. But an equilibrium in auctions is compelling, because it combines both of these properties (higher price when more buyers visit and favoring a consumer with higher valuation).

When the analysis is extended to the case of a continuous type space the results change dramatically. Suppose that the sellers post *arbitrary* (but identical) mechanisms $\gamma_1 = \gamma_2 = ... = \gamma_n$ and the buyers play a symmetric Bayesian equilibrium in the stage game induced by those mechanisms. Then the resulting allocation can be induced as an equilibrium outcome of the entire game. The intuition is that by posting appropriate lotteries, seller j can pin down the equilibrium visiting probabilities he receives from all types

[1] An alternative equilibrium selection device is to require that when the buyers face two situations in which there is a one-to-one correspondence between the two sets of equilibria, and equilibrium number k in situation one yields the same payoffs (for all buyers and sellers) as equilibrium number k in situation two, then the buyers coordinate on equilibria that correspond to each other in the two situations. Once this requirement is made, the simple mechanism described above is a profitable deviation. In other words, all that is required is that once a deviation is made the buyers are not out to punish the deviator.

of the buyers. With a continuous type space this implies that equilibrium utilities are pinned down as well. Therefore, no individual seller can induce different visiting probabilities and buyer utilities and thus no profitable deviation exists. Although, the constructed equilibrium relies on lotteries that are not practical, there is no consensus in the literature on what constitutes the class of "reasonable" mechanisms.

At the end of the paper we discuss a result that starts addressing this issue: we show that it is not an equilibrium when all sellers post fixed-price mechanisms and show that if there are two buyers and two sellers and the distribution of the types is convex, then all sellers posting ex-post efficient mechanisms (second-price auctions) forms an equilibrium. This observation suggests that if the set of admissible mechanisms is restricted in an appropriate manner, then many of the ex-post inefficient equilibria can be ruled out, but not the ones that are ex-post efficient. However, it is beyond the scope of this paper to study what restrictions are needed to recover the efficiency result of the two-type case.

In Section 2 we set up the basic two-type model and analyze it in Section 3. In Section 4 we turn to the continuous type space model. First, we show the multiplicity of equilibria (Section 4.1) and then provide results about fixed-price and auction mechanisms (Section 4.2). Many of the proofs are in the Appendices.

1.1 Literature review

This work belongs to the directed work literature initiated by Peters (1991) and built upon by Burdett, Shi and Wright (2001) and Coles and Eeckhout (2000, 2003).[2] Similarly to Coles and Eeckhout (2003), we allow sellers to post more general mechanisms than just prices and also introduce buyer heterogeneity as in Coles and Eeckhout (2000). The main difference between our paper and the last one is that we concentrate on symmetric equilibria and allow for *any* mechanism to be posted, while Coles and Eeckhout (2000) concentrate on a class of direct mechanisms and show that in that class there is only asymmetric equilibria where different types of buyers sort themselves between the different sellers. Also, that paper concentrates on the two-buyer, two-seller case mostly, while our results do not depend on the number of agents. Julien, B. , Kennes, J. and I. King (2005) consider a setup similar to ours, but restricts attention to the case when sellers post auctions without reserve price and show efficient sorting of buyers.

This paper is also related to a growing literature on competing mechanisms with private information. McAfee (1993) and Peters (1997) show that in a framework similar to ours, as the market becomes large there is an equilibrium with second price auctions and each seller posts a reserve price equal to his marginal cost. Peters and Severinov (1997, 2004) also study competing auctions, the latter work with an application to Internet auctions. Burguet and Sakovics (1999) consider competing second price auctions where the choice variable is the reserve price. Our contribution is to allow a more general class of mechanism in small markets and to characterize the properties of the equilibrium.

At a more abstract level, Epstein and Peters (1999) and Martimort and Stole (2002) provide revelation principle like results for competing mechanism. Our paper does not follow this branch of literature, since we constrain the class of mechanisms available for the sellers inasmuch that a seller cannot condition his mechanism on the mechanism offered by the other sellers.

2 Model

2.1 Basic setup and admissible selling mechanisms

There are m buyers and n sellers, with $n, m \geq 2$. Each seller has a unit supply of and each buyer has a unit demand for the indivisible homogenous good. The game starts with the sellers simultaneously posting their selling mechanisms, $\gamma^1, \gamma^2, ..., \gamma^n \in \Gamma$. After observing the mechanisms posted, each buyer decides which seller to visit and what action to take at that seller. Finally, each seller allocates his object and collects the payments from the buyers based on the actions of the buyers and the mechanism he posted.

We now turn to the description of the set of admissible mechanisms Γ. The restriction imposed is that the sellers cannot explicitly condition their mechanisms on the mechanisms of the other sellers. To start the formal description let $q^j \subset \{1, 2, ..., m\}$ be the set of buyers who visited seller j. Mechanism γ^j specifies

[2] Some applications of directed search models include labor economics (e.g. Julien, Kennes and King (2000), Montgomery (1991) and Shi (2006)) and industrial economics models (e.g. Green and Newbery (1992)).

- the action space ς_i^j for buyer $i \in q^j$ at seller j,[3][4]
- the probability that buyer $i \in q^j$ obtains the object $p_i^j(q^j) : \underset{i \in q^j}{\times} \varsigma_i^j \to [0,1]$,
- the expected payment buyer $i \in q^j$ makes $e_i^j(q^j) : \underset{i \in q^j}{\times} \varsigma_i^j \to \mathbb{R}$.

We define the *fixed price mechanism* used by Burdett, Shi and Wright (2001). Such a mechanism specifies that each buyer has an equal probability of receiving the object at a predetermined price. Formally, $p_i^j = \frac{1}{c(q^j)}$ where $c(q^j)$ is the number of elements in q^j. The payment is $e_i^j = c(q^j)x^j$ where x^j is the fixed price set by seller j.

The valuations of the buyers are private information and are distributed i.i.d. and $v_i \in \{l, h\}$ with probability p and $1-p$. The buyer's utility is $v_i - t$ if he obtains the object and pays t and $-t$ if he does not obtain the object and pays t. Each buyer maximizes his expected utility and each seller maximizes his expected revenue. An allocation is *ex-post efficient* if seller $j \in \{1, 2, ..., n\}$ allocates the object to a buyer with a high type if such a buyer visited seller j. (Note that this definition allows the seller to retain the object if only low types visited.)

2.2 Strategies, expected payoffs and equilibrium

A pure strategy for seller j is a mechanism $\gamma^j \in \Gamma$. The actions of the sellers $\gamma = (\gamma^1, \gamma^2, ..., \gamma^n)$ induce a Bayesian game B_γ for the buyers, where each buyer has two possible types l or h and the action space for buyer i is

$$\Sigma_i(\gamma) = \{(r_i, a_i) : r_i \in \{1, 2, ..., n\}, a_i \in \varsigma_i^{r_i}\}.$$

In this Bayesian game B_γ a pure strategy for buyer i is an action choice for both types, i.e. a choice from $\Sigma_i(\gamma) \times \Sigma_i(\gamma)$. The set of mixed strategies for i in B_γ is a random action pair $\mu_i(\gamma) \in \alpha_i(\gamma) = \Delta(\Sigma_i(\gamma)) \times \Delta(\Sigma_i(\gamma))$ where Δ is the set of all probability distributions on $\Sigma_i(\gamma)$. Let $\alpha(\gamma) = \overset{m}{\underset{i=1}{\times}} \alpha_i(\gamma)$, $\alpha_{-i}(\gamma) = \underset{k \neq i}{\times} \alpha_k(\gamma)$ and let $\mu_{-i}(\gamma) \in \alpha_{-i}(\gamma)$ denote the strategies of all buyers other than i in B_γ. Let μ_i denote the strategy of buyer i in the entire game, i.e. a selection that chooses $\mu_i(\gamma)$ for all γ and let $\mu = (\mu_1, \mu_2, ..., \mu_m)$ be the strategy profile of all buyers.

Now, we turn to the definition of an equilibrium $(\gamma, \mu) = (\gamma^1, ..., \gamma^n, \mu_1, ..., \mu_m)$. Our equilibrium concept is a Perfect Bayesian Equilibrium in undominated strategies with the restriction that the sellers use pure strategies. For the buyers the equilibrium condition is that for all γ, the buyers play a Bayesian equilibrium in B_γ. Let $u_i(\gamma, \mu_{-i}(\gamma), \sigma_i(v_i), v_i)$ denote the expected utility of buyer i with type v_i in game B_γ if he takes action $\sigma_i(v_i) \in \Sigma_i(\gamma)$ and the other buyers employ strategies $\mu_{-i}(\gamma)$. (Note, that the vector $(\gamma, \mu_{-i}(\gamma), \sigma_i(v_i))$ determines the probability that buyer i obtains an object and also the expected payment he makes and thus pins down his utility as well.[5]) In equilibrium it must hold that for all $i \in \{1, 2, ..., m\}, \gamma \in \Gamma^n, v_i \in \{l, h\}$, $\mu_{-i}(\gamma) \in \alpha_{-i}(\gamma)$ and $s_i \in \Sigma_i(\gamma)$

$$u_i(\gamma, \mu_{-i}(\gamma), \sigma_i(v_i), v_i) \geq u_i(\gamma, \mu_{-i}(\gamma), s_i, v_i)$$

if $\sigma_i(v_i) \in \Sigma_i(\gamma)$ is in the support of the strategy of buyer i in game B_γ when buyer i has type v_i.

Now, we turn to the sellers' best reply condition. Let γ^{-j} denote the actions of all sellers other than j. Let $w^j(\gamma^j, \gamma^{-j}, \mu(\gamma))$ denote the expected revenue of seller j if strategies γ^j, γ^{-j} are used by the sellers and strategies $\mu(\gamma) \in \alpha(\gamma)$ are used by the buyers. Again, given these strategies the expected revenues of the sellers can be calculated in a straightforward manner. Then the equilibrium condition is such that for all $g^j \in \Gamma$ it holds that

$$w^j(\gamma^j, \gamma^{-j}, \mu(\gamma)) \geq w^j(g^j, \gamma^{-j}, \mu(g^j, \gamma^{-j})).$$

[3] We assume that when participating in the mechanism, a buyer does not know which other buyers visited seller j unless the mechanism announces it explicitly.

[4] It is not sufficient to allow seller j to post a simple direct mechanism where the action space of each buyer $i \in q^j$ is to report a type $v_i^j \in \{l, h\}$. As Calzolari and Pavan (2006) prove it in a single agent contracting framework, the agent needs to be able to report all of their payoff relevant private informations to the mechanism designer. Since the buyers know the equilibrium visiting probabilities of the other buyers, they need to be able to report it to the seller.

[5] Since it is not necessary for our purposes to calculate these values for the general case, we omit this cumbersome exercise.

2.3 Symmetry assumptions

To capture the search frictions and anonymity on large markets we follow the directed search literature initiated by Burdett, Shi and Wright (2001) and make several symmetry assumptions. First, since the buyers are ex-ante identical, we assume that any two buyers who show up at the same location must be treated identically, a natural anonymity requirement especially in large markets. Such an anonymity requires that $\varsigma_i^j = \varsigma_{i'}^j$ for all $i, i' \in \{1, 2, ..., m\}$. In addition the following must hold: take two situations where the set of players who showed up are q^j and $\widehat{q}^j$. Let $\tau^j = (\tau_i^j)_{i \in q^j}$ and $\widehat{\tau}^j = (\widehat{\tau}_i^j)_{i \in \widehat{q}^j}$ be the actions taken by the buyers who visited seller j in the two situations. If there exists a permutation $\pi : \{1, 2, ..., m\} \to \{1, 2, ..., m\}$ such that

$$i \in q^j \Longleftrightarrow \pi(i) \in \widehat{q}^j$$

and for all $i \in q^j$

$$\tau_i^j = \widehat{\tau}_{\pi(i)}^j,$$

then anonymity requires that

$$p_i^j(\tau^j, q^j) = p_{\pi(i)}^j(\widehat{\tau}^j, \widehat{q}^j)$$

and

$$e_i^j(\tau^j, q^j) = e_{\pi(i)}^j(\widehat{\tau}^j, \widehat{q}^j).$$

In words, the name of the agent should not matter in the mechanism offered by seller j.

Second, to emphasize the role of anonymity and the coordination problems between buyers we restrict attention to cases where the buyers use the same (possible mixed) strategies. Formally, this requires

$$\mu_1 = \mu_2 = ... = \mu_m = \widetilde{\mu}.$$

Third, we concentrate on equilibria where the sellers post the same mechanism, i.e. $\gamma^1 = \gamma^2 = ... = \gamma^n = \widetilde{\gamma}$ and each buyer visits each seller with probability $r_i = \frac{1}{n}$ when the sellers post identical mechanisms, regardless of whether a buyer's type is l or h.

3 Analysis

3.1 An ex-post efficient equilibrium

Before turning to the main result that rules out equilibria that are not ex-post efficient, we prove that an (ex-post efficient) equilibrium exists. For simplicity we concentrate on the case when there are two buyers and two sellers and the two types have equal probability, but the result can be easily extended to the general case. In the proposed equilibrium each buyer reports his type to the seller and a high type (with value h) always wins against a low type. When two high or two low types visited a particular seller then the object is allocated to each of them with equal probability. The payments are as follows: when both report low type the winner pays $t = l$, when both report high type the winner pays $q = h$. When a buyer is alone at the seller (and thus wins), then he needs to pay $r = \frac{h+3l}{8}$ and when a buyer with a high type competes with a low type, then he pays $s = \frac{h+l}{2}$.

Proposition 1 *If h is close enough to l, then it is a symmetric equilibrium if both sellers post the mechanism above, each buyer - regardless of his type - visits the two sellers with equal probability and makes a truthful announcement of his type.*

Proof. Since the two sellers post the same mechanisms the buyers are obviously indifferent between visiting the two sellers. A buyer with low type does not have an incentive to report h, since it is only profitable for such a buyer to win, when he pays less than l. A buyer with high type has potentially an incentive to misreport his type when the other buyer reported l, because in this case he can lower his payment. However, then he only wins with probability $\frac{1}{2}$ and his expected utility is $\frac{h-l}{2}$. When he reports h and the other buyer reports l, then he wins for sure and his utility is $h - s = \frac{h-l}{2}$. Therefore, there is never any incentive for misrepresenting. One also needs to make sure that $r \leq l$, otherwise low types would not participate. This condition boils down to $h \leq 5l$.

We turn now to the problem of the seller. Suppose that seller j deviates and the probability a low or a high type visits him is $p(l)$ and $p(h)$ respectively. These probabilities are sufficient to provide a tight upper bound for the revenue of seller j: the revenue of seller j can be written as the difference between the total surplus he generates and the total utility he provides to the buyers who visit him, which can be written as

$$R_j \leq \widehat{R}_j = (p(h) - \frac{p^2(h)}{4})h + (p(l) - \frac{p^2(l)}{4} - \frac{p(l)p(h)}{2})l - p(l)U(l) - p(h)U(h),$$

where $U(l)$ and $U(h)$ are the rents the two types make when visiting seller j.[6] Therefore, it is sufficient to pin down the rents each type of the buyer makes after this deviation to obtain an upper bound $\widehat{R}_j$ for the revenue.

There are several cases to study depending on $p(l)$ and $p(h)$. If a probability is zero, then the utility of that type does not affect the revenue of seller j and thus it does not have to be calculated. When $p(l) \in (0,1)$, then a buyer with type l makes an equal utility by visiting seller j and visiting the other seller. His utility at seller j can then be calculated as his utility at the other seller, which can be written as

$$U(l) = \frac{p(l) + p(h)}{2}(l - r).$$

When $p(h) \in (0,1)$ similar considerations yield

$$U(h) = \frac{p(l) + p(h)}{2}(h - r) + \frac{1 - p(l)}{2}(h - s).$$

Finally, when $p(l) = 1$ or $p(h) = 1$ visiting the other sellers may yield a lower utility for the buyer than visiting j and thus

$$U(l) \geq \frac{p(l) + p(h)}{2}(l - r)$$

and

$$U(h) \geq \frac{p(l) + p(h)}{2}(h - r) + \frac{1 - p(l)}{2}(h - s).$$

Therefore,

$$R_j \leq \widehat{R}_j \leq R_j^* = (p(h) - \frac{p^2(h)}{4})h + (p(l) - \frac{p^2(l)}{4} - \frac{p(l)p(h)}{2})l -$$
$$-\{p(l)\frac{p(l) + p(h)}{2}(l - r) - p(h)[\frac{p(l) + p(h)}{2}(h - r) + \frac{1 - p(l)}{2}(h - s)]\}.$$

Moreover, in the case when seller j does not deviate, it holds $R_j = \widehat{R}_j = R_j^*$ and thus if one cannot find a deviation that increases this "modified revenue" R_j^*, then there cannot be a profitable deviation at all. The final step is thus to solve $\max\limits_{p(l),p(h)\in[0,1]} R_j^*$. This is a simple convex program and thus the first order condition is sufficient for optimality. Under the chosen values of r and s the first order condition holds at $p(l) = p(h) = \frac{1}{2}$, which then concludes the proof. ■

It is interesting to note, that there is no symmetric equilibrium where the sellers post a second price auction with a reserve price, i.e. where $s = l$. This illustrates that the ex-post efficient equilibrium characterized here cannot be implemented as a simple second price (or open) auction with a reserve price. However, such auctions augmented by appropriately defined lotteries constitute an equilibrium as Proposition 5 shows. Moreover, Coles and Eeckhout (2000) show that there are equilibria that resemble to posting second price auctions where the buyers use asymmetric strategies.

3.2 A preliminary result

A goal of the paper is to show that in any equilibrium in the two-type case an *ex-post efficient* allocation occurs. First, we provide a simple result that highlights the intuition why ex-post efficiency must be a

[6] This formula uses the fact that given $p(l), p(h), U(l)$ and $U(h)$ the seller has the highest revenue if he always sells the object whenever there is a buyer visiting and the mechanism is also ex-post efficient.

property of the equilibrium. Let $\widetilde{\varsigma}$ denote the (common) strategy space of the buyers and for all $i \in q^j$ define $(\widetilde{p}_i(\widetilde{\tau}^j, q^j), \widetilde{e}_i(\widetilde{\tau}^j, q^j))$ as the allocation vector when the set of players who visited seller j is q^j and the actions taken after arriving to seller j are described by vector $\widetilde{\tau}^j$. Suppose that the equilibrium does not satisfy ex-post efficiency and define the mechanism $\overline{\gamma}$ that consists of two stages:

Stage 1: First, let each buyer present report a type v_i^j. Then implement the mechanism $\widetilde{\gamma}$ characterized by functions $\widetilde{\varsigma}$ and $\widetilde{p}_i, \widetilde{e}_i$ ignoring the extra type announcement for now.

Stage 2: If the object is allocated to a bidder who reported type l (or not sold) and there is a buyer who reported type h, then the seller buys the object from the current owner (if any) at price l and resells it to a buyer who reported high type at a price h.

The following result shows that if a single seller deviates to $\overline{\gamma}$, then there exists an equilibrium in the buyers' stage game that provides a higher revenue for the deviator than the revenue from playing $\widetilde{\gamma}$:

Proposition 2 *If seller j posts $\overline{\gamma}$ and all other sellers post $\widetilde{\gamma}$, then it is an equilibrium in the buyers' stage game if the buyers*

i) use the same visiting probabilities as in the case when j posts $\widetilde{\gamma}$

ii) employ the same action when visiting seller j as in the case when j posts $\widetilde{\gamma}$

iii) report the type truthfully at j when j posts $\overline{\gamma}$, which is a weakly dominant action.

If the buyers indeed choose these actions, then the revenue of j exceeds his revenue when he posts $\widetilde{\gamma}$.

Proof. First, note that if a buyer with high type reports l, then he may need to sell the object at a price lower than his valuation (l), while if a low type reports h, then he may need to buy the object at a price higher than his valuation (h). Moreover, there are no circumstances under which lying is profitable and therefore it is a weakly dominated strategy. But if every buyer who visits seller j reports truthfully, then Stage 2 does not affect the payoffs of the buyers. Therefore, assuming that all buyers report truthfully at seller j, $\overline{\gamma}$ induces a stage game for the buyers that is payoff equivalent to the stage game for the buyers if seller j posts $\widetilde{\gamma}$. Therefore, for the buyers the same strategies (together with truthful revelation at seller j) are still part of an equilibrium. If each buyer indeed follows the same strategies as in the case of no deviation from seller j, then this deviation of seller j improves upon his revenue, since in Stage 1 he achieves the same revenue as without the deviation and in Stage 2 he achieves an extra positive expected revenue if the original mechanism $\widetilde{\gamma}$ did not induce an ex-post efficient allocation at seller j. ■

3.3 Equilibrium and ex-post efficiency

The proof of Proposition 2 implies that if there is a unique (symmetric) equilibrium in the buyers' stage game when all sellers post $\widetilde{\gamma}$, then there is a unique (symmetric) equilibrium in undominated strategies in the buyers' stage game when seller j deviates to $\overline{\gamma}$. Moreover, the equilibrium in the buyers' stage game after the deviation is such that seller j has a higher revenue than without his deviation. Therefore, posting $\overline{\gamma}$ is a profitable deviation for seller j. Hence, the following Corollary of Proposition 2:

Corollary 3 *There is no symmetric equilibrium in undominated strategies where the sellers post fixed price mechanisms.*

However, if there are multiple equilibria in the buyers' stage game when all sellers posted mechanism $\widetilde{\gamma}$, then if seller j deviates to $\overline{\gamma}$, then the buyers may switch to different strategies that constitute an equilibrium in their stage game, which may make the deviation unprofitable for seller j. Therefore, the deviator needs to make sure that the buyers do not coordinate on an equilibrium in the buyers' stage game that hurts the deviator. The next Theorem shows that the deviator can achieve the desired equilibrium in the buyers' stage game:

Theorem 4 *If all sellers other than j post mechanism $\widetilde{\gamma}$ and the resulting allocation is not ex-post efficient, then seller j can find a mechanism γ_j, such that the unique equilibrium in undominated strategies in the buyers' stage game is such that*

i) seller j is visited with probability $\frac{1}{n}$ by each type of all buyers

ii) the utility of both types of the buyers is the same as in the equilibrium in the buyers' stage game when all sellers posted $\widetilde{\gamma}$

iii) the mechanism γ_j induces an ex-post efficient equilibrium allocation at seller j

iv) such a deviation is profitable for seller j.

Proof. See Appendix. ■

The main idea of the proof is that if buyer i visits seller j, then he is able to buy any (possibly negative) quantity of a lottery, whose payoff depends on the type announcement and visiting decision of buyer $i+1$. If buyer $i+1$ does not make the visiting choice described in i), then buyer i does not have a best reply, since he can make an infinitely large expected utility by buying ∞ or $-\infty$ units of the lottery. Therefore, in any equilibrium of the buyers' stage game i) needs to hold. An ex-post efficient mechanism (similar to the one considered in Proposition 2) that induces a unique equilibrium in the buyers' stage game is defined. This mechanism is such that it is weakly dominant for a buyer to report truthfully at seller j. This report can be used to construct the lotteries above. By choosing the payments of this mechanism appropriately, one can make sure that the utilities for the buyers are the same as without the deviation. Finally, points i) - iii) together with quasi-linearity of the payoffs imply point iv).

This result continues to be true even if the strategies of the buyers and the sellers do not respect *any* of the symmetry assumptions as long as each seller is visited by both types of each buyer with positive probability. If each seller j obtains visits with positive probability from either low or high type customers only, then the deviation γ_j (described in the proof in Appendix 1) is no longer profitable, and more importantly, ex-post efficiency loses its bite.[7]

Note, that our construction with the lotteries involves some (out of equilibrium) beliefs for the buyers for which there is no optimal strategy. This feature plays the role of a powerful equilibrium selection device in the buyers' stage game. However, even if one imposes a (large enough) upper bound on the quantity of lotteries bought, a similar result can still be established without relying on (even out of equilibrium) buyers' beliefs that preclude optimal strategies to exist. The same comment is applicable to the construction used in the continuous type case below.

4 The case of continuous type space

4.1 Equilibrium and indeterminacy

A main limitation of Theorem 4 is the assumption that there are only two possible types for the buyers. When this assumption is relaxed, the results change in an important way. Even with three possible types there may be equilibria that are not ex-post efficient, because now there is more constraints that a profitable deviation need to satisfy. When the type space is continuous, the probability of receiving an object uniquely determines the utilities of the buyers, which dramatically constraints the possibility for a profitable deviation.

To fix ideas assume that each buyer's valuation is distributed on $v_i \in [a, b]$ according to a continuous and strictly positive density function f and a corresponding distribution function F. To simplify analysis we consider only mechanisms that sell the object with probability one when at least one buyer visited a particular seller. The next Proposition shows that one cannot place any more restriction on the equilibrium allocation beyond the requirement that the buyers play a Bayesian equilibrium in their stage game[8].

Proposition 5 *Take any strategy profile where the buyers use $(\mu_1, \mu_2, ..., \mu_m)$ and the sellers use $(\gamma^1, \gamma^2, ..., \gamma^n)$ respecting the symmetry requirements of Section 2.3, such that the buyers play a Bayesian Equilibrium in their stage game given any mechanisms the sellers posted and each seller sells his object if at least one buyer visited his store. Then there exists an equilibrium of the (entire) game that implements the allocation resulting from strategies $(\mu_1, \mu_2, ..., \mu_m)$, $(\gamma^1, \gamma^2, ..., \gamma^n)$.*

Proof. The idea of the proof is simple: with the use of appropriately defined lotteries no seller can profitably induce visiting probabilities that are different from the ones induced by $(\mu_1, \mu_2, ..., \mu_m)$, $(\gamma^1, \gamma^2, ..., \gamma^n)$. Then

[7] However, one can show that there is no equilibrium where all sellers post fixed price mechanisms and the set of sellers visited by buyers with high and low types are disjoint. In this case seller j can post lotteries similar to the ones considered in the proof of Theorem 4 that make sure that both types visit j with positive probability and then the deviation becomes profitable for the same reason as in Proposition 2. But as Coles and Eeckhout (2000) show, asymmetric equilibria exist where the buyers sort themselves to visiting different sellers, with the sellers posting more complicated mechanisms than a simple fixed price mechanism.

[8] Recall that an allocation in this game consists of the probabilities of a purchase and an expected payment for each type of each buyer, and a probability of selling and an expected revenue for each seller.

using a strategy profile that is similar to $(\gamma^1, \gamma^2, ..., \gamma^n)$, the sellers induce the desired equilibrium allocation. The proof follows this insight and is conducted in three steps:

Step 1: Each seller, in addition to some extra components described in Step 2, posts lotteries that are similar to the ones used in the proof of Theorem 4. Seller j offers the following lottery for player i: it pays $x_i^j \alpha_{i+1}^j$ to i if $i+1$ does not visit and asks $x_i^j(1-\alpha_{i+1}^j)$ from i if $i+1$ visits j where $x_i^j \in \mathbb{R}$ is a choice a variable of i. If the probability of i visiting j were not exactly α_{i+1}^j, then i does not have a best choice, since by choosing x_i^j as large or as low as possible he could make a payment arbitrarily large. This implies that if j posts such a lottery, then in any equilibrium $i+1$ visits j with probability α_{i+1}^j. Note that this variable $\alpha_{i+1}^j \in [0,1]$ can be chosen in any way seller j likes and thus any visiting probabilities can be induced in equilibrium. Now, suppose that buyer $i+1$ is also required to report a type. Since this type announcement will not affect his own allocation, it is incentive compatible for $i+1$ to report his type $v_{i+1} \in [a,b]$ truthfully. Let $Q \subset [a,b]$ be any measurable set and define a lottery that pays zero when $i+1$ did not show up, and pays $y_i^j \psi_{i+1}^j(Q)$ when he showed up and his reported type is not in Q and pays $-y_i^j(1-\psi_{i+1}^j(Q))$ when the reported type is Q where $y_i^j \in \mathbb{R}$ is a choice a variable of i. For the same reason as above such a lottery pins down the probability that $v_{i+1} \in Q$ when $i+1$ visits j. Since such a lottery is defined for all measurable sets, the type distribution of $i+1$ conditional on visiting j is also pinned down. This conditional type distribution together with the ex-ante visiting probability pinned down by the first lottery, pins down the probability of visiting j for (almost) all types of $i+1$. If each seller offers a similar lottery to each buyer, then the equilibrium visiting decisions of the buyers are pinned down and can be made equal to the one in the symmetric case where all types of each buyer visit each seller with probability $\frac{1}{n}$. Assuming that the sellers use the appropriate lotteries, the same visiting probabilities that were induced by $(\mu_1, \mu_2, ..., \mu_m)$ can be implemented. Moreover, each buyer makes a zero profit from each lottery no matter which seller he visits.

Step 2: Suppose that, in addition to those lotteries, the sellers offer the mechanisms $(\gamma^1, \gamma^2, ..., \gamma^m)$ to determine the allocation of the object and the payments, ignoring the type announcements, which are thus used only to perform the lotteries. Since the same visiting probabilities are induced as was induced without the extra lotteries, it follows that using the same strategies at any given seller constitutes an equilibrium in the buyers' stage game.

Step 3: Suppose that each seller $l \neq j$ posts mechanism γ^l in addition to lotteries described in Step 1. Then Step 1 implies that seller j cannot deviate (from posting γ^j) in a way that induces different visiting probabilities at sellers other than j, which has several consequences:

- As it was argued in Step 2, it is an equilibrium in the buyers' stage game to use the same actions upon visiting any seller other than j. In what follows we assume that the buyers indeed play this equilibrium when visiting a seller other than j.[9]
- This implies that after the deviation of seller j the utility of all types of each buyer when visiting a seller other than j is the same as when j posted γ^j and the buyers followed strategies $(\mu_1, \mu_2, ..., \mu_m)$.
- Since all types of each buyer attained a strictly positive utility before the deviation (with the possible exception of the very lowest type a), it must hold that all types achieve a strictly positive utility after the deviation. Thus no types ever stay home, which means that the probability of each type visiting j is pinned down once we know the visiting probabilities at sellers other than j.
- Since all types of each buyer visit each seller with a positive probability it means that all types of each buyer achieve the same utility when visiting j as when visiting any other seller. Therefore, the second bullet point implies that the utility of all types of each buyer when visiting seller j after the deviation is the same as when j posted γ^j and the buyers followed strategies $(\mu_1, \mu_2, ..., \mu_m)$.
- In this continuous type case this implies that the winning probabilities of all types of each buyer at seller j are the same with or without the deviation of seller j, since the slope of the rent each buyer makes as a function of his type is equal to his probability of receiving the object.

[9] One way to make sure that this happens is if the sellers post mechanisms that have unique equilibria given the set of buyers who visited. Such a uniqueness can be obtained by using further lotteries that are similar to the ones used in the proof of Theorem 4.

- This implies that the total surplus generated at j is the same with or without the deviation.
- Finally, since the revenue is equal to the total surplus minus the rent of the buyers at j and both components are the same with or without the deviation, it follows that the revenue is the same as well. The deviation is not profitable.

■

To emphasize the generality of the result we can lift the symmetry assumptions, and only assume that all types of each buyer visit each seller with a positive probability in equilibrium. The above proof goes through without any significant modification. Also, the above proof implies that even if indirect mechanisms are allowed a profitable deviation still does not exist.

The key difference between the case of two types and the case of continuously many types is that in the latter case the utilities achieved pin down also the probabilities of each type obtaining the object as it is highlighted in Step 3. By posting appropriate lotteries the sellers pin down the equilibrium utilities of the buyers, which then implies that the winning probabilities and thus the expected revenues are pinned down as well. On the other hand, when there are two possible types only, the winning probabilities can be chosen independently of the utilities achieved by specifying ex-post trade between the different types. An interesting consequence of Proposition 5 is the fact that the set of equilibrium allocations does not change in a smooth manner. On one hand, even if an arbitrarily low probability is assigned to types on (a, b) there is an equilibrium described above. However, when that probability is exactly zero (thus we are in the two-type case) only equilibria that are ex-post efficient survive.

4.2 Fixed price and auction mechanisms

Proposition 5 shows that there is very little limitation on the set of equilibrium allocations beyond the requirement that buyers behave in a sequentially rational manner. One may argue that this result obtains only because the sellers are allowed to use mechanisms that do not look practical. It is interesting to ask whether one can obtain restrictions on the set of equilibrium allocations by restricting the class of mechanisms that can be posted by the sellers. To partially address this question, we show it in Appendix 2 that if all other sellers post a fixed-price mechanism, then there exists a profitable deviation for each seller j, while the identified deviation is no longer profitable if the other sellers post (second-price) auctions. This observation suggests that mechanisms that are inefficient will not be posted in equilibrium if "implausible additions" like lotteries are excluded. To further discuss the efficiency properties of the equilibrium we consider a set-up similar to Burguet and Sakovics (1999) who study competing auctions with two sellers. We further assume that there are also two buyers, that the density function f is positive and increasing on $[a, a+1]$ and that function $x + \frac{F}{f}$ is increasing in x.[10] We show below that if the opponent posts an auction, then there is no profitable deviation from posting an auction even if arbitrary mechanisms are allowed.

In what follows we assume that seller 2 posts an auction with a reserve price. Suppose that seller 1 posts a general mechanism and let $p(z)$ be the probability that type $z \in [a, a+1]$ visits seller 1. Let $W_i(z)$ $(i = 1, 2)$ be the probability that type z wins if he visits seller i and let $u_i(z)$ be the expected utility from visiting seller i. Then the following Lemma holds:

Lemma 6 *There exists an $h \leq a+1$ such that $p(z) = 1$ for all $z \leq h$ and for all $z > h$ at least one of the following holds for almost all $z \in [h, a+1]$:*
i) $p(z) = 0$
ii) $W_1(z) = W_2(z)$ and $u_1(z) = u_2(z)$.

Proof. See Appendix 4. ■

To obtain an intuition behind Lemma 6 note, that h is the type above which seller 2 is visited with positive probability. If for type $z \geq h$ it holds that $p(z) \in (0, 1)$, then it must be true (at every continuity

[10] This condition is a sufficient condition that helps in establishing a technical result, Lemma 9. Its content is intuitive: recall that in standard mechanism design the virtual utility $x - \frac{1-F}{f}$ represents the amount of money that can be extracted from a buyer when the utility of the *lowest* type is given. In our game, because of the presence of a competing seller, sometimes the utility of the *highest* type is given, in which case we arrive at our modified virtual utility $x + \frac{F}{f}$. For details, see the proof of Lemma 9.

point of W_1 and W_2) that $W_1(z) = W_2(z)$, otherwise it would not be possible that the probability of winning is increasing in the types. Then the only nontrivial case is when $p(z) = 1$. However, the only way it can hold that $p(z) = 1$ for all $z \in [c, d]$ is if W_1 is constant in z. To see this, note that otherwise it would hold that

$$W_2(d) = W_1(d) > W_1(c) = W_2(c),$$

which contradicts the definition of an efficient auction and the fact that no types between c and d ever visit seller 2. But since W_2 is also constant on $[c, d]$, the two functions must be equal on that interval. The property characterized in the above Lemma holds only for ex-post efficient mechanisms. The significance of this result is that it implies that for all $z \geq h$ it holds that $p(z)W_1(z) = p(z)W_2(z)$, which makes it possible to calculate the revenue of seller 1 using only the probabilities with which each type visits each seller. In effect, we can calculate the utilities of the bidders at seller 1 and the total welfare of seller 1's mechanism by referring to the (known) mechanism of seller 2. This procedure enormously reduces the complexity of seller 1's problem.

Let π be the probability that a buyer visits seller 1 or

$$\pi = \int_a^{a+1} p(z)f(z)dz.$$

Using Lemma 6 one can calculate the expected revenue of seller 1 as follows:

Proposition 7 *The expected revenue of seller* 1 *can be written as*

$$R_1 = \int_a^h f(z)[zW_1(z) - u_1(z)]dz +$$

$$+r\pi(\pi - F(h)) + \int_h^{a+1} \int_h^z f(z)f(t)p(z)t(1 - p(t))dtdz. \tag{1}$$

Proof. See Appendix 4. ■

In Appendix 4 we also show that for all $z \geq h$, the visiting probabilities are restricted as follows:

$$\int_z^{a+1} f(x)p(x) \int_x^{a+1} f(t)(2p(t) - 1)dtdx \leq 0. \tag{2}$$

This constraint follows from certain feasibility constraints described by Maskin and Riley (1984). If for some $z \geq h$ it is satisfied as an equality, then it means that at seller 1 a buyer with type $v < z$ can win only if no type higher than z has visited seller 1. The violation of the constraint would mean that for types above z we try to assign so high winning probabilities that are infeasible even if those types never lose against any types less than z.

Case 1: Let us assume first that a profitable deviation exists where $\pi \leq \frac{1}{2}$. If a deviation is such that $\pi \leq \frac{1}{2}$ is induced, then $h = a$ and $p(z) = 0$ for all $z \leq v^*$ such that

$$F(v^*) = 1 - 2\pi.$$

To see this, let $d = \inf_z p(z) > 0$ and suppose that $d < v^*$. Then it holds that $W_1(d) \geq 1 - \pi$ and $W_2(d) = \pi + F(d)$. By Lemma 6, it follows that $W_1(d) = W_2(d)$ or

$$\pi + F(d) \geq 1 - \pi,$$

which implies the claim.

Therefore, it follows that $h = a$ and thus

$$R_1 = r\pi^2 + \int_a^{a+1} \int_a^z f(z)f(t)p(z)t(1 - p(t))dtdz. \tag{3}$$

Letting

$$P(t) = \int_a^t f(x)p(x)dx,$$

in Appendix 4 we show that this revenue can be rewritten as

$$R_1 = K(\pi) + \int_a^{a+1} \int_a^t f(z)P(z)dzdt - \frac{1}{2}\int_a^{a+1} P^2(z)dz + \pi \int_a^{a+1} P(z)dz. \tag{4}$$

The idea of the rest of the proof is to show that, given the desired probability of visit $\pi \leq \frac{1}{2}$, for seller 1 it is the most profitable to post an auction. As we know it from Burguet and Sakovics (1999) an auction that induces $\pi \leq \frac{1}{2}$ is such that for all $z \leq v^*$

$$p^a(z) = 0$$

and for all $z > v^*$ it holds that

$$p^a(z) = 0.5.$$

Let $P^a(z) = \int_a^z f(t)p^a(t)dt$. Now, we are ready to prove the main result for the case of $\pi \leq \frac{1}{2}$:

Lemma 8 *For any admissible visiting probability function that generates a total visiting probability of $\pi \leq \frac{1}{2}$ seller 1 could increase his revenues by inducing visiting probability function p^a defined above.*

Proof. Inspecting (4) reveals that since π is fixed, we only need to prove for all admissible functions and $t \in [a, a+1]$

$$\int_a^t f(z)P^a(z)dz \geq \int_a^t f(z)P(z)dz \tag{5}$$

and

$$\pi \int_a^{a+1} P^a(z)dz - \int_a^{a+1} \frac{(P^a(t))^2}{2} dt \geq \pi \int_a^{a+1} P(z)dz - \int_a^{a+1} \frac{P^2(t)}{2} dt. \tag{6}$$

In Appendix 4 we prove formula (5) using the key feasibility condition (2) and then we establish (6) as well. ■

Case 2: Now, we turn to the case when $\pi > \frac{1}{2}$. In this case it holds that $a \leq h \leq a + \pi$. For types $z \geq h$ it still holds that $p(z)W_1(z) = p(z)W_2(z)$ and the calculations are similar to Case 1. However, for $z < h$ one cannot calculate $W_1(z)$ just from knowing function p and thus seller 1 needs to specify the allocation used for types less than h. There are two constraints regarding that allocation. First, by monotonicity for all $z < h$

$$W_1(z) \leq W_2(h) = \pi. \tag{7}$$

Second the allocation must be feasible, which (using the results of Maskin and Riley (1984)) implies that for all $z \leq h$

$$\int_a^z f(z)W_1(z)dz \geq \int_a^z f(z)(1 - \pi + F(z))dz. \tag{8}$$

Let

$$\Delta = \int_a^h f(z)W_1(z)dz - \int_a^h f(z)(1 - \pi + F(z))dz \geq 0 \tag{9}$$

denote the slack the seller specifies for the winning probability of types less than h. If $\Delta = 0$, then it means that types below h never win against type above h at seller 1, while if $\Delta > 0$ low types sometimes win against high types. The next Lemma takes Δ as given and specifies the optimal winning probabilities for types lower than h:

Lemma 9 *For any $\pi > \frac{1}{2}$ and $\Delta \geq 0$ and $h > a$ the revenue maximizing choice of allocation for seller 1 involves for all $z \leq q (\leq h)$*

$$W_1(z) = 1 - \pi + F(z)$$

and for all $z \in [q, h]$

$$W_1(z) = \pi.$$

Moreover, q is uniquely determined by π, Δ and h.

Proof. See Appendix 4. ■

The content of the Lemma is that given the constraints (7) and (8), the seller sells the object in an efficient manner, i.e. he allocates it always to the highest type who visited him if both buyers report types not more than h. This is fairly intuitive, because for types $z \leq h$ seller 1 is not restricted by the mechanism posted by seller 2 (since those types *strictly* prefer visiting seller 1) and thus the problem of seller 1 is similar to the mechanism design problem with one seller. Then under the assumption that $x + \frac{F}{f}$ is increasing, the efficient allocation is also revenue maximizing. The significance of Lemma 9 is that it describes $\int_a^h f(z)[zW_1(z) - u_1(z)]dz$, the term in R_1 that was not pinned down by the visiting probabilities and thus allows us to calculate R_1. Once that calculation is performed, similar steps as in Case 1 can be implemented to obtain the following result:

Lemma 10 *For any $\pi \geq \frac{1}{2}$ the most profitable mechanism to post is a second price auction with $r_1 \leq r$.*

Proof. See Appendix 4. ■

Putting our results together implies the following Proposition:

Proposition 11 *If f and $x + \frac{F}{f}$ are both increasing, then for any auction the opponent posts the best reply of seller 1 is to post an auction himself.*

Such a result holds whenever f and $x + \frac{F}{f}$ are both increasing and thus it seems to be a fairly general characteristic of competing mechanisms. When the density function is decreasing, the virtual utilities, $x - \frac{1-F}{f}$ may be non-monotonic and thus even in the case of one seller, the optimal mechanism may not be an (efficient) auction. To construct an equilibrium for the game where the sellers are allowed to post arbitrary mechanisms, consider the game of Burguet and Sakovics (1999) where the sellers are allowed to post auctions only. If there exists a pure strategy equilibrium of this game, then the above Proposition implies that it is also an equilibrium of the game where the sellers are allowed to post arbitrary mechanisms. However, as Burguet and Sakovics (1999) show, only a mixed strategy equilibrium exists under their assumptions on the distribution of valuations. Unfortunately, it is hard to establish that against a *mixture* of auctions an auction still constitutes a best response. To tackle this issue, Virag (2007b) assumes that the lower bound of the possible types is large enough and proves the existence of an equilibrium in pure strategies. Using the results of that paper, the following Theorem characterizes the symmetric equilibrium in (second-price) auctions when a is large enough:

Theorem 12 *If f and $x + \frac{F}{f}$ are both increasing, and $a \geq \frac{1}{2f(a)}$, then there is a symmetric equilibrium in the game where the sellers are allowed to post arbitrary mechanisms. In this equilibrium both sellers post second-price auctions with reserve price $r = \frac{a}{2}$ and the buyers visit each seller with equal probability and bid their valuations at the visited seller.*

Proof. Appendix 4 contains the proof using techniques from Virag (2007b). ■

5 Conclusion

A seminal paper in the literature of directed search, Burdett, Shi and Wright (2001) assumed that all sellers post a fixed price. Coles and Eeckhout (2003) argued that although such an equilibrium exists when sellers are allowed to post arbitrary mechanisms, but there are a continuum of other equilibria arises.

In Section 3 we have shown that if incomplete information is introduced to this standard directed search model with two possible types for the buyers, then many of the equilibria that Coles and Eeckhout (2003) find in the complete information case are eliminated. In particular, only ex-post efficient equilibria survive and no equilibrium with fixed price mechanisms exists. If auctions are interpreted as selling mechanism where buyer-competition implies that a buyer with high valuation always wins (as it happens in an English auction), then our result implies that only equilibria where the sellers post auctions survive.

In Section 4 we show that this result does not extend to the case of more than two types, and when the type space is continuous very little restriction can be placed on the set of equilibrium allocations. The key factor allowing this indeterminacy is that the payment of a given buyer may be very sensitive to the actions of the other buyers, which may make it very costly for each seller to change the visiting probabilities of the buyers. To reduce this sensitivity one may require that the payment of each buyer does not depend on the actions of other buyers "too much". In general, the more restrictions one places on the extent that a buyer's payment depends on the actions of the other buyer's, the more (inefficient) equilibria one can eliminate. However, ruling out such dependence entirely prevents sellers to use standard mechanisms like second price auctions. While it is beyond the scope of this paper to study what restrictions are needed to recover the efficiency results of the two-type case, it is possible to provide some restrictions on the mechanisms posted. For example, we show that if all other sellers post fixed-price mechanisms, then there exists a profitable deviation for seller j, while if the sellers post auctions the identified deviation is no longer profitable and indeed there exists an equilibrium in auctions under some restrictions on the distribution of valuations.

6 Appendix 1

Proof of Theorem 4:
Proof. It is easy to see, that the first three results imply the last one, because then the total surplus generated at seller j increases, but the buyer's utility who show up at seller j remains the same, which implies the result through quasilinearity of the utility functions. In Step 1 we define the lottery components of the mechanism γ_j that rule out any equilibria with different visiting probabilities at seller j. Step 2 makes it sure that the utility level of the two types of the buyers do not change. Result iii) is a direct consequence of the construction used in Step 2.

Formally, the mechanism offered by seller j, γ_j needs three reports from a buyer who visits seller j: $(x_i, y_i, v_i^j) \in \mathbb{R} \times \mathbb{R} \times \{l, h\}$.

Step 1: Mechanism γ_j offers two lotteries to each buyer i who showed up. Each buyer i receives amount $(n-1)x_i$ if buyer $i+1$ (buyer 1 when $i = n$) showed up at seller j, but needs to pay x_i if buyer $i+1$ did not show up, where $x_i \in \mathbb{R}$ is a choice variable of buyer i. Since buyer i is risk neutral, he would like to choose x_i as big (small) as possible if the probability that $i+1$ shows up is greater (less) than $1/n$. Therefore, there is an optimal decision for i only if this probability is equal to $1/n$, which implies that in equilibrium the probability is $1/n$. Let $\beta_{i+1}^j(v)$ be the probability that $i+1$ visits j if his type is $v \in \{l, h\}$. Then it follows that

$$p\beta_{i+1}^j(l) + (1-p)\beta_{i+1}^j(h) = \frac{1}{n}. \tag{10}$$

The second lottery makes the payment of bidder i dependent of the type report of $i+1$. In Step 2, we show that it is a (weakly) dominant strategy for $i+1$ to report truthfully and thus buyer i believes that the type distribution of buyer $i+1$ is the same as the distribution of his report. This lottery pays zero if $i+1$ did not show up and pays $y_i(1-p)$ $(y_i p)$ if $i+1$ showed up and reported type l (h), where $y_i \in \mathbb{R}$ is again a choice variable of buyer i. Therefore, a necessary condition for having an optimal decision is that the probability that $i+1$ has type l conditional on showing up at seller j is p. Using Bayes rule this implies that

$$\frac{p\beta_{i+1}^j(l)}{p\beta_{i+1}^j(l) + (1-p)\beta_{i+1}^j(h)} = p. \tag{11}$$

Equations (10) and (11) imply that $\beta_{i+1}^j(l) = \beta_{i+1}^j(h) = \frac{1}{n}$ must hold in equilibrium, just as required by point i).

Step 2: Mechanism γ_j is defined as follows: in the first stage, the lotteries defined in Step 1 are implemented after observing which buyers showed up at seller j and what types those buyers reported. Then the object is assigned randomly to one of the bidders who visited seller j and this tentative owner pays an amount d. If this tentative owner reported type l and there is another bidder who reported type h (if there is more than one buyer with high type, then one is chosen randomly), then the tentative owner sells the object to a buyer with type h (the final owner) at price $w \in (l, h)$. Otherwise, the tentative owner keeps the good and no further transaction occurs. It is a weakly dominant strategy for each buyer to report his type truthfully and thus ex-post efficiency holds.

The only point left to prove is that d and w can be chosen so that the (expected) utility of both types of the buyers are equal to their (expected) utility without the deviation on the part of seller j. The key idea is that there are two variables (d, w) to choose to match two utility levels (one for each type). Let $u(l)$ and

$u(h)$ denote the utility levels of the two types in the equilibrium played when all sellers posted $\widetilde{\gamma}$. Let $\Pr(l)$ and $\Pr(h)$ be the probability that a buyer with type l or h obtains the object in the equilibrium played.[11] Then the incentive conditions preventing mimicking the other type imply that

$$\Pr(h) \geq \Pr(l)$$

and

$$\Pr(l)(h-l) \leq u(h)-u(l) \leq \Pr(h)(h-l). \tag{12}$$

Let $\widehat{u}(v)$ be the utility of type $v \in \{l,h\}$ when seller j deviates to γ_j. Since in equilibrium a buyer with type $v \in \{l,h\}$ visits seller j with positive probability ($\frac{1}{n}$), this utility can be calculated as the utility from visiting seller j. The probability that a buyer becomes the tentative owner conditional on visiting seller j can be calculated as

$$\alpha = \sum_{i=1}^{m} \frac{1}{i} \binom{m-1}{i-1} [\frac{1}{n}]^{i-1} [\frac{n-1}{n}]^{m-i}.$$

A buyer with type l becomes the final owner if and only if he is the tentative owner and there is no buyer with type h who visited seller j. This happens with probability

$$\theta(l) = \sum_{i=1}^{m} \frac{1}{i} \binom{m-1}{i-1} [\frac{p}{n}]^{i-1} [\frac{n-1}{n}]^{m-i}.$$

Then the utility of type l can be written as

$$\widehat{u}(l) = \theta(l)(l-d) + (\alpha - \theta(l))(w-d). \tag{13}$$

One needs to choose d and w such that

$$\theta(l)(l-d) + (\alpha - \theta(l))(w-l) = u(l).$$

Finally, one needs to make sure that

$$\widehat{u}(h) - \widehat{u}(l) = u(h) - u(l).$$

Let $\theta(h)$ be the probability that a buyer with h who visited seller j obtains the object, which can be written as

$$\theta(h) = \sum_{i=1}^{m} \frac{1}{i} \binom{m-1}{i-1} [\frac{1-p}{n}]^{i-1} [1-\frac{1-p}{n}]^{m-i}.$$

The utility of the high type is

$$\widehat{u}(h) = \alpha(h-d) + (\theta(h) - \alpha)(h-w).$$

Therefore,

$$\widehat{u}(h) - \widehat{u}(l) = [\theta(h)h - \theta(l)l] - (\theta(h) - \theta(l))w.$$

If $w = l$, then

$$\widehat{u}(h) - \widehat{u}(l) = \theta(h)(h-l).$$

and if $w = h$ then

$$\widehat{u}(h) - \widehat{u}(l) = \theta(l)(h-l).$$

Since $\widehat{u}(h) - \widehat{u}(l)$ is continuous in w, thus for any $\theta \in (\theta(l), \theta(h))$ there exists $w \in (l,h)$ such that

$$\widehat{u}(h) - \widehat{u}(l) = \theta(h-l).$$

Formula (12) implies that

$$u(h) - u(l) = \delta(h-l)$$

[11] Since the buyers use the same strategies these probabilities are the same for all buyers.

with $\delta \in [\Pr(l), \Pr(h)]$. Note, that the visiting probabilities are the same in the two situations (i.e. when seller j did or did not deviate) and also that after the deviation the object is allocated in an ex-post efficient manner. These two facts together with the symmetry requirements imply that[12]

$$\theta(h) > \Pr(h)$$

and

$$\theta(l) > \Pr(l).$$

Therefore, $\delta \in (\theta(l), \theta(h))$ and it is possible to choose $w \in (l, h)$ such that

$$\widehat{u}(h) - \widehat{u}(l) = \delta(h - l) = u(h) - u(l).$$

Finally, one can choose d to satisfy equation (13), which concludes the proof. ■

7 Appendix 2

In this Appendix, we show that there is a profitable deviation when all other sellers post a fixed price mechanism. To establish this result, we take the setup of Section 4 of the main text and assume that each buyer's valuation is distributed on $v_i \in [a, b]$ according to a continuous and strictly positive density function f and a corresponding distribution function F. We assume that $b - a$ is small enough and thus when the sellers are restricted to post fixed prices, the symmetric equilibrium is such that $p < a$ and thus all buyers are served. I show below that if all other sellers post a fixed price $p < a$, then seller j can profitably deviate by posting a hybrid mechanism $M(r)$ that is run differently depending on the number of buyers who visited seller j. When there are at least three buyers, then the mechanism is a fixed price mechanism with price p. When there is only one buyer at j he receives the object at price $r(< p)$ and when there are two buyers the mechanism becomes a second price auction with no reserve price. Define $\underline{v}$ and $\overline{v}$ as

$$F(\underline{v}) = \frac{1}{2n}$$

and

$$F(\overline{v}) = 1 - \frac{1}{2n}.$$

The following theorem proves that his deviation increases revenues:

Theorem 13 *For every $p < a$ there exists an $r < p$ such that if seller j posts $M(r)$ and all other sellers post the fixed price mechanism p, then any symmetric equilibrium in undominated strategies of the stage game of the buyers is such that*

i) for all i and $l \neq j$

$$\Pr(i \text{ visits } l \mid v_i \in [\underline{v}, \overline{v}]) = \frac{1}{n-1}$$

and

$$\Pr(i \text{ visits } j \mid v_i \in [\underline{v}, \overline{v}]) = 0$$

ii) buyers with type less than $\underline{v}$ or higher than $\overline{v}$ visit seller j with probability 1

iii) each buyer who visited j submits a bid equal to his valuation

iv) the expected revenue of seller j is higher than when he posted the fixed price mechanism p.

Proof. First, we prove that in the buyers' stage game there exists an equilibrium like that. The proof that in the buyers' stage game every equilibrium is such is relegated to Appendix 8. Beside the visiting decision each buyer needs only to decide about the bid submitted when he visits j. However, it is obviously optimal for each buyer to bid his valuation when he visits j. To prove that the visiting decisions described in points i) and ii) constitute an equilibrium one needs to check two conditions. The first necessary condition,

[12] The intuitition is simple: after the deviation a high type receive the object at seller j if at least one buyer with type h visited seller j, while a low type obtains the object only if no high type showed up. On the other hand, in the original situation ex-post efficiency was violated.

monotonicity is implied by the incentive constraints of the buyers in the same way as in standard auction models:

Monotonicity condition: If $\widehat{v} > \widetilde{v}$ *then for all* i

$$W(\widehat{v}) = Pr(i \text{ receives an object} \mid v_i = \widehat{v}) \geq$$

$$W(\widetilde{v}) = Pr(i \text{ receives an object} \mid v_i = \widetilde{v}).$$

If the buyers employ the strategies described above, then the probability of winning W is constant for all $v \in [\underline{v}, \overline{v}]$. When the types compared both visit seller j monotonicity holds obviously and thus it is sufficient to check that for all $v < \underline{v}$ it holds that $W(v) \leq W(\underline{v})$ and for all $v > \overline{v}$ it holds that $W(v) \geq W(\overline{v})$. Since W is increasing in v for $v < \underline{v}$ the first condition simplifies to

$$\lim_{v \nearrow \underline{v}} W(v) \leq W(\underline{v})$$

and since W is increasing in v for $v > \overline{v}$ the second becomes

$$\lim_{v \searrow \overline{v}} W(v) \geq W(\overline{v}).$$

We show that

$$\lim_{v \nearrow \underline{v}} W(v) = \lim_{v \searrow \overline{v}} W(v) = W(\underline{v}) = W(\overline{v}),$$

which establishes monotonicity. To prove this, notice that seller j is visited by each seller with probability $F(\underline{v}) + 1 - F(\overline{v}) = \frac{1}{n}$, just like any other seller. Let k be the number of other buyers showing up at the seller visited by buyer i. At the other sellers the probability of winning conditional on k is $\frac{1}{k+1}$. Therefore, for any type the winning probability at a seller other than j is

$$(1 - \frac{1}{n})^{m-1} + (m-1)\frac{1}{n}(1 - \frac{1}{n})^{m-2}\frac{1}{2} + ...$$

At seller j this conditional probability is $\frac{1}{k+1}$ when $k \neq 1$. When $k = 1$ a buyer at seller j wins if and only if the other buyer has a lower type than him. For a buyer with a type close to $\underline{v}$, it is equal to $\frac{F(\underline{v})}{F(\underline{v})+1-F(\overline{v})} = \frac{1}{2}$ and thus the unconditional probability of winning for such a type is

$$\lim_{v \nearrow \underline{v}} W(v) = (1 - \frac{1}{n})^{m-1} + (m-1)\frac{1}{n}(1 - \frac{1}{n})^{m-2}\frac{1}{2} + ...$$

as well. Since type $\underline{v}$ visits a seller other than j it holds that

$$W(\underline{v}) = (1 - \frac{1}{n})^{m-1} + (m-1)\frac{1}{n}(1 - \frac{1}{n})^{m-2}\frac{1}{2} + ...$$

To conclude the proof of the monotonicity condition, note that a buyer with close to type $\overline{v}$ who visits seller j has the same probability of winning as a type close to $\underline{v}$, since they beat the same set of types when $k = 1$, i.e. types that are lower than $\underline{v}$.

The second necessary condition states that since type $\underline{v}$ has the same winning probability at j and at any other seller and he is indifferent between those visiting seller j or any other seller, it must hold that the expected payment conditional on winning is the same for him:

Payment condition: When visiting j *the expected payment of type* $\underline{v}$ *conditional on winning is equal to* p.

When visiting j the payment is p if there are at least two other buyers visiting j. Therefore, the payment condition is equivalent to the condition that the expected payment of type $\underline{v}$ conditional on winning and having at most one other buyer visiting j is equal to p. When no other buyer visits the payment is r. When one other buyer visits and the buyer with type $\underline{v}$ wins, then he needs to pay the bid of the loser, which is equal to the type of the loser. The (conditional) expected value of this type is $E[v \mid v \leq \underline{v}]$. Therefore, the payment condition becomes:

$$\frac{r(1 - \frac{1}{n})^{m-1} + E[v \mid v \leq \underline{v}](m-1)\frac{1}{2n}(1 - \frac{1}{n})^{m-2}}{(1 - \frac{1}{n})^{m-1} + (m-1)\frac{1}{2n}(1 - \frac{1}{n})^{m-2}} = p. \tag{14}$$

Calculating r from this equation pins down the mechanism $M(r)$.

The above argument implies that visiting seller j and bidding $\underline{v}$ or visiting any other seller yields the same probability of winning and the same expected payment. Therefore, type $\underline{v}$ is indifferent between visiting j or any other seller. On the other hand, types lower than $\underline{v}$ strictly prefer to bid lower than $\underline{v}$ when visiting seller j and therefore they also strictly prefer visiting j and bidding $v < \underline{v}$ to visiting any other seller (which is payoff-equivalent to visiting j and bidding $\underline{v}$). A similar argument implies that types greater than $\overline{v}$ do not want to visit a seller other than j either.

Finally, we show that posting $M(r)$ is a profitable deviation for seller j. Let $\pi(2)$ be the probability that at least two buyers visited a seller.[13] The expected revenue of seller j from posting the fixed price mechanism p is

$$p[m\frac{1}{n}(1-\frac{1}{n})^{m-1} + \binom{m}{2}\frac{1}{n^2}(1-\frac{1}{n})^{m-2}] + p\pi(2). \tag{15}$$

The expected revenue from posting $M(r)$ is

$$rm\frac{1}{n}(1-\frac{1}{n})^{m-1} + \binom{m}{2}\frac{1}{n^2}(1-\frac{1}{n})^{m-2}E[\min\{v_1,v_2\} \mid v_1,v_2 \in [a,\underline{v}) \cup (\overline{v},b]] + p\pi(2). \tag{16}$$

Formula (14) implies that it yields a higher revenue to post $M(r)$ if and only if

$$E[\min\{v_1,v_2\} \mid v_1,v_2 \in [a,\underline{v}) \cup (\overline{v},b]] > E[v \mid v \leq \underline{v}].$$

To prove that this holds, note that

$$E[v \mid v \leq \underline{v}] = \underline{v} - \int_0^{\underline{v}} \frac{F(x)}{F(\underline{v})}dx$$

and that

$$E[\min\{v_1,v_2\} \mid v_1,v_2 \in [a,\underline{v}) \cup (\overline{v},b]] =$$

$$= \frac{1}{4}E[\min\{v_1,v_2\} \mid v_1,v_2 \in (\overline{v},b]] + \frac{1}{2}E[\min\{v_1,v_2\} \mid v_1 \in [a,\underline{v}), v_2 \in (\overline{v},b]] +$$

$$+\frac{1}{4}E[\min\{v_1,v_2\} \mid v_1,v_2 \in [a,\underline{v})] >$$

$$> \frac{1}{4}\underline{v} + \frac{1}{2}E[v \mid v \leq \underline{v}] + \frac{1}{4}(\underline{v} - \int_0^{\underline{v}} \frac{2F(x)}{F(\underline{v})} - \frac{F^2(x)}{F^2(\underline{v})}dx) =$$

$$= \underline{v} - \int_0^{\underline{v}} \frac{F(x)}{F(\underline{v})}dx + \frac{1}{4}\int_0^{\underline{v}} \frac{F^2(x)}{F^2(\underline{v})}dx > E[v \mid v \leq \underline{v}],$$

which completes the proof. ■

The crucial feature of the profitable deviation is the improvement in ex-post efficiency by providing a higher probability of winning for buyers who value the object more. The idea is similar to the one in the two type case: by improving ex-post efficiency the deviating seller can capture a higher revenue.

8 Appendix 3

In this Appendix we prove that every equilibrium in the buyers' stage game is as described in Theorem 13.
Proof. Define $W(v)$ as the probability that type v of a buyer obtains an object after seller j's deviation. Incentive conditions imply that this probability is increasing and therefore, it is a well defined (single valued) function for almost every $v \in [0,1]$. Moreover, suppose that type x and type y visits a seller other than j. Then it must hold that $W(x) = W(y)$ in equilibrium, since otherwise the strategy of one type yields a higher winning probability with an equal price p to be paid upon winning, which provides a profitable deviation for the type who wins with lower probability. Consequently, for all $z \in [x,y]$, $W(z) = W(x)$ holds. Since for those types who visit seller j the winning probability is strictly increasing, it follows that there exist $\widehat{v}$ and

[13] This probability is the same for seller j as for any other seller, since each buyer visits each seller with probability $\frac{1}{n}$

$\widetilde{v}$, such that types $[\widehat{v}, \widetilde{v}]$ visit sellers other than j and types $v \in (0, \widehat{v})$ and $v \in (\widetilde{v}, 1)$ visit seller j allowing that $\widehat{v} = 0$ or $\widetilde{v} = 1$. By construction, $W(\widehat{v}) = W(\widetilde{v})$ holds and one can show that

$$\lim_{v \nearrow \widehat{v}} W(v) = \lim_{v \searrow \widetilde{v}} W(v) = W(\widehat{v}) = W(\widetilde{v}) \tag{17}$$

holds.[14] Therefore, W is a continuous function that is strictly increasing on $[0, \widehat{v})$ and $(\widetilde{v}, 1]$ and constant in between.

Case 1: $\widehat{v} = 0$ and $\widetilde{v} < 1$

In this case only buyers with type greater than $\widetilde{v}$ visit seller j. Consider buyer type $\widetilde{v}$ who visited j. He wins only if he does not need to compete in an auction. He pays $r < p$ when no other buyer showed up and p if at least two other did, thus his payment conditional on winning is strictly less than p. His probability of winning is at least as much as that of any type less than $\widetilde{v}$, which is the probability of winning at any seller other than j. Consequently, by visiting j and bidding $\widetilde{v}$ a buyer needs to pay less and wins more often than when visiting any other buyer than j, which cannot happen in equilibrium.

Case 2: $\widetilde{v} = 1$ and $\widehat{v} > 0$

In this case only buyers with type less than $\widehat{v}$ visit seller j. Monotonicity implies that visiting a seller other than j yields at least as high a probability of winning as visiting j and bidding $\widehat{v}$ (the bid of type $\widehat{v}$). If $F(\widehat{v}) \leq \frac{1}{n}$ held, then this could not be the case, since type $\widehat{v}$ would win as often at j as at any other seller when the number of other visitors is not equal to one and would win for sure when one other buyer visited j. Therefore, $F(\widehat{v}) > \frac{1}{n}$ must hold and thus $\widehat{v} > \underline{v}$, where $F(\underline{v}) = \frac{1}{2n}$. Consider now type $\underline{v}$ who visits seller j. Monotonicity implies that he wins less often than by visiting any seller other than j. By showing that his expected payment conditional on winning is larger than p, I establish a contradiction which rules out this case. The expected payment when at least two other buyers visited is equal to p and thus we only need to characterize the expected payment conditional on having at most one other buyer visiting seller j, which is equal to

$$\frac{r(1-F(\widehat{v}))^{m-1} + E[v \mid v \leq \underline{v}](m-1)\frac{1}{2n}(1-F(\widehat{v}))^{m-2}}{(1-F(\widehat{v}))^{m-1} + (m-1)\frac{1}{2n}(1-F(\widehat{v}))^{m-2}} =$$

$$= \frac{r(1-F(\widehat{v})) + E[v \mid v \leq \underline{v}](m-1)\frac{1}{2n}}{(1-F(\widehat{v})) + (m-1)\frac{1}{2n}} >$$

$$> \frac{r(1-\frac{1}{n}) + E[v \mid v \leq \underline{v}](m-1)\frac{1}{2n}}{(1-\frac{1}{n}) + (m-1)\frac{1}{2n}} = p.$$

The inequality follows from the fact that $r < E[v \mid v \leq \underline{v}]$ and $1 - F(\widehat{v}) > 1 - \frac{1}{n}$, while the equality follows from (14).

Case 3: $\widehat{v} = 0$ and $\widetilde{v} = 1$

In this case no one ever visits seller j, but then it is profitable to visit him and buy the object for sure at a reduced (compared to p) price r.

Case 4: $\widehat{v} = \widetilde{v}$

In this case everyone always visits seller j, which cannot be an equilibrium in the buyers' stage game as it can be easily shown.

Case 5: $\widehat{v} > 0$ and $\widetilde{v} < 1$

In this case we need to show that $\widehat{v} = \underline{v}$ and $\widetilde{v} = \overline{v}$ to conclude the proof. Let $N(x)$ be the probability of obtaining the object at a fixed price mechanism if each buyer visits the seller with probability x, which is a decreasing function of x. Let $\widetilde{s} = F(\widehat{v}) + 1 - F(\widetilde{v})$ be the visiting of seller j and then the visiting probability for any other seller in a symmetric equilibrium is $\frac{1-\widetilde{s}}{n-1}$. Monotonicity implies that the probability of obtaining the object at seller j when bidding $\widehat{v}$ is equal to the probability of obtaining the object at a seller other than j. This condition can be written as

$$N(\frac{1-\widetilde{s}}{n-1}) = N(\widetilde{s}) + (m-1)\widetilde{s}(1-\widetilde{s})^{m-2}(\frac{F(\widehat{v})}{\widetilde{s}} - \frac{1}{2}).$$

[14] This follows from the argument given in the main text.

The last term on the right hand side reflects that when visiting seller j and having exactly one competitor a buyer with type $\widehat{v}$ wins with probability $\frac{F(\widehat{v})}{\widetilde{s}}$ and not with probability $\frac{1}{2}$. Then $\widetilde{s} = \frac{1}{n}$ implies $F(\widehat{v}) = \frac{\widetilde{s}}{2} = \frac{1}{2n}$ and thus $\widehat{v} = \underline{v}$ and thus $\widetilde{v} = \overline{v}$. Moreover,

$$\widetilde{s} > \frac{1}{n} \Longrightarrow F(\widehat{v}) > \frac{\widetilde{s}}{2} > \frac{1}{2n}$$

and

$$\widetilde{s} < \frac{1}{n} \Longrightarrow F(\widehat{v}) < \frac{\widetilde{s}}{2} < \frac{1}{2n}.$$

Suppose that $\widetilde{s} > \frac{1}{n}$ and thus $\widehat{v} > \underline{v}$. In this case type $\underline{v}$ visits seller j. Rewriting (14) with a slight modification yields

$$\frac{r(1-\frac{1}{n}) + E[v \mid v \leq \underline{v}](m-1)\frac{1}{2n}}{(1-\frac{1}{n}) + (m-1)\frac{1}{2n}} = p. \tag{18}$$

Since type $\underline{v}$ wins with a probability lower than if he visited a seller other than j (this follows from monotonicity) his expected payment conditional on winning must be less than p, otherwise he would visit another seller. Therefore,

$$\frac{r(1-\widetilde{s})^{m-1} + E[v \mid v \leq \underline{v}](m-1)\frac{1}{2n}(1-\widetilde{s})^{m-2}}{(1-\widetilde{s})^{m-1} + (m-1)\frac{1}{2n}(1-\widetilde{s})^{m-2}} < p$$

or

$$\frac{r(1-\widetilde{s}) + E[v \mid v \leq \underline{v}](m-1)\frac{1}{2n}}{(1-\widetilde{s}) + (m-1)\frac{1}{2n}} < p. \tag{19}$$

Comparing (18) and (19) and noting that $E[v \mid v \leq \underline{v}] > a > p > r$ it follows that $1 - \widetilde{s} > 1 - \frac{1}{n}$ or $\widetilde{s} < \frac{1}{n}$ contradicting our initial hypothesis.

When $\widetilde{s} < \frac{1}{n}$ then $\widehat{v} < \underline{v}$ and in the equilibrium characterized by cutoffs $\underline{v}$ and $\overline{v}$, type $\widehat{v}$ visits seller j and his expected payment conditional on winning is less than p. In the purported equilibrium characterized by cutoffs $\widehat{v}$ and $\widetilde{v}$, the payment condition implies that his expected payment conditional on winning is equal to p. Using the two conditions a contradiction follows, the details are similar to the calculations above and are omitted. ∎

9 Appendix 4

Proof of Lemma 6:
Proof. Let $h = \inf_x \{p(z) < 1 \text{ for all } x \in [z, z+\varepsilon] \text{ for some } \varepsilon > 0\}$ or $h = a+1$ if $p(z) = 1$ for almost all z. Suppose that $h < a+1$ and that for $c > h$ and $z \in [c, d]$ it holds that $p(z) = 1$, but for all $z \in (d, d+\varepsilon)$ it holds that $p(z) < 1$ for a small enough ε.[15] Assume that $p(z) > 0$ on $(d, d+\varepsilon)$. (The case where p takes zero as well can be handled similarly.) Since seller 2 offers an auction it holds for all $z \in [c, d]$ that

$$W_2(z) = W_2(d) = W_2(c).$$

Since W_1 is increasing, it is almost everywhere continuous and thus $W_1^*(d) = \lim_{x \searrow d} W_1(x)$ exists. Moreover, $W_1^*(d) = W_2(d)$, otherwise there exists $x, y \in (d, d+\varepsilon)$ such that $x > y$ and

$$W_1(x) < W_2(y) \text{ or } W_1(y) > W_2(x),$$

which would violate that a higher type obtains a higher probability of winning regardless of which seller he visits. Therefore, monotonicity of the equilibrium allocation implies that for all $z \in [c, d]$ it holds that

$$W_1(z) = W_2(c) = W_2(d) = W_2(z).$$

Therefore, both mechanisms offer identical winning probabilities for bidders who report $z \in [c, d]$. Now, we prove that $u_1(z) = u_2(z)$. First, note that $u_1(d) = u_2(d)$ and thus the expected payment of type d is the same at seller 1 and seller 2, because $W_1(d) = W_2(d)$ as well. But the expected payment, $e_1(z)$ $(i = 1, 2)$ is constant on $[c, d]$ because W_1 is constant there. However, the rule of the auction (either first- or second-price) implies that $e_2(z)$ is also constant on $[c, d]$ and thus for all $z \in [c, d]$

$$e_1(z) = e_2(z),$$

which now implies that indeed $u_1(z) = u_2(z)$.

If for any z it holds that $p(z) \in (0, 1)$, then indifference implies that $u_1(z) = u_2(z)$. Moreover, if $p(z) \in (0, 1)$ holds for all $z \in [e, f]$ then $W_1(z) = W_2(z)$ otherwise monotonicity of the equilibrium winning probabilities in the type would fail. ■

Corollary 14 *The expected revenue of seller* 1 *is*

$$R_1 = \int_a^h f(z) z W_1(z) dz - \int_a^h f(z) u_1(z) dz + \int_h^{a+1} f(z) z p(z) W_2(z) dz - \int_h^{a+1} f(z) p(z) u_2(z) dz.$$

Proof. Using quasilinearity of the utility functions implies that

$$R_1 = \int_a^h f(z) z W_1(z) dz - \int_a^h f(z) u_1(z) dz + \int_h^{a+1} f(z) z p(z) W_1(z) dz - \int_h^{a+1} f(z) p(z) u_1(z) dz.$$

From this the result immediately follows using Lemma 6. ■

Proof of Proposition 7:
Proof. By construction, (using that the reserve price $r < a$ if a is high enough)

$$W_2(z) = \pi + \int_a^z f(t)(1 - p(t)) dt. \tag{20}$$

By construction, for all $z \leq h$

$$u_2(z) = \pi(z - r), \tag{21}$$

since such a low type can only win the auction if the other buyer did not show up. Standard auction theoretic arguments imply that for all $z \geq h$

$$u_2(z) = \pi(h - r) + \int_h^z W_2(z) dz.$$

[15] Note, that such a $d < a+1$ exists because the highest type needs to visit the auction with positive probability otherwise monotonicity of the equilibrium winning probabilities in the type would fail.

After some rearrangements this becomes

$$u_2(z) = \pi(z - r) + \int_h^z f(t)(1 - p(t))(z - t)dt. \tag{22}$$

Finally, using Corollary 14 and substituting in (20) and (22) implies the result after collecting terms. ■

The derivation of formula (2):

Proof. First, rewriting (20) implies

$$W_2(z) = 1 - \int_z^{a+1} f(t)(1 - p(t))dt. \tag{23}$$

Consequently,

$$\int_a^{a+1} (1 - p(z))f(z)W_2(z)dz = 1 - \pi - \frac{(1 - \pi)^2}{2}. \tag{24}$$

Second, using the results of Maskin and Riley (1984) implies[16] that for all $z \in (a, a+1]$

$$\int_z^{a+1} p(x)f(x)W_1(x)dx \leq \int_z^{a+1} p(x)f(x)[1 - \int_x^{a+1} f(t)p(t)]dx \tag{25}$$

and

$$\int_a^{a+1} p(z)f(z)W_1(z)dz = \int_a^{a+1} p(z)f(z)[1 - \int_z^{a+1} f(x)p(x)dx]dz = \pi - \frac{\pi^2}{2}. \tag{26}$$

Using (23) Lemma 6 implies that for all $z \geq h$

$$\int_z^{a+1} p(x)f(x)W_1(x)dx = \int_z^{a+1} p(x)f(x)W_2(x)dx =$$

$$= \int_z^{a+1} p(x)f(x)(1 - \int_x^{a+1} f(t)(1 - p(t))dt)dx \leq \int_z^{a+1} p(x)f(x)[1 - \int_x^{a+1} f(t)p(t)]dx$$

or for all $z \geq h$

$$\int_z^{a+1} f(x)p(x) \int_x^{a+1} f(t)(2p(t) - 1)dtdx \leq 0. \tag{27}$$

■

Derivation of formula (4):

Proof. Using that (2) holds as an equality at $z = h = a$, Fubini's theorem implies that

$$\int_a^{a+1} \int_a^x f(x)p(x)f(t)dxdt = \int_a^{a+1} \int_x^{a+1} 2p(x)p(t)f(x)f(t)dtdx = \pi^2.$$

Letting $P(z) = \int_a^z f(t)p(t)dt$ the last formula becomes

$$\int_a^{a+1} f(z)P(z)dz = \pi^2. \tag{28}$$

Note, that after integration by parts and using (28):

$$\alpha = \int_a^{a+1} \int_a^z p(z)f(z)tp(t)f(t)dtdz = \int_a^{a+1} p(z)f(z)zP(z)dz - \int_a^{a+1} p(z)f(z) \int_a^z P(t)dtdz =$$

$$= \frac{\pi^2(a+1)}{2} - \int_a^{a+1} \frac{P^2(z)}{2}dz - \int_a^{a+1} p(z)f(z) \int_a^z P(t)dtdz.$$

[16] The formula below, (25) gives an upper bound for how often types above z can win. This probability is obviously maximized if they never lose against any type lower than z, in which case the probability is given by the right hand side of (25). Formula (26) follows from the fact that seller 1 sells the object for sure when a is large enough.

Further modifications yield that

$$\alpha = \frac{\pi^2(a+1)}{2} - \frac{1}{2}\int_a^{a+1} P(z)\int_a^z p(t)f(t)dtdz - \int_a^{a+1} p(z)f(z)\int_a^z P(t)dtdz =$$

$$= \frac{\pi^2(a+1)}{2} + \frac{1}{2}\int_a^{a+1} P(z)\int_a^z p(t)f(t)dtdz - \int_a^{a+1}\int_a^z [P(z)f(t)p(t) + p(z)f(z)P(t)]dtdz.$$

Another integration by parts implies that

$$\int_a^{a+1}\int_a^z [P(z)f(t)p(t)+p(z)f(z)P(t)]dtdz = \frac{\int_a^{a+1}\int_a^{a+1}[P(z)f(t)p(t) + p(z)f(z)P(t)]dtdz}{2} = \pi\int_a^{a+1} P(z)dz.$$

Thus

$$\alpha = \frac{\pi^2(a+1)}{2} + \frac{1}{2}\int_a^{a+1} P^2(z)dz - \pi\int_a^{a+1} P(z)dz.$$

Let

$$\beta = \int_a^{a+1}\int_a^z p(z)f(z)tf(t)dtdz = \int_a^{a+1} tf(t)\int_t^{a+1} p(z)f(z)dzdt =$$

$$= \pi\int_a^{a+1} tf(t)dt - \int_a^{a+1} tf(t)P(t)dt = \pi\int_a^{a+1} tf(t)dt - \pi^2(a+1) + \int_a^{a+1}\int_a^t f(z)P(z)dz,$$

after integration by parts and using (28). Therefore,

$$\gamma = \beta - \alpha = C(\pi) + \int_a^{a+1}\int_a^t f(z)P(z)dz + \pi\int_a^{a+1} P(z)dz - \frac{1}{2}\int_a^{a+1} P^2(z)dz.$$

Then using formula (3) concludes the proof. ■

Derivation of formula (5):

Proof. It follows from (2) that for any admissible visiting probabilities $p(z) = 0$ for $z \le v^*$ and for all $z \ge a$

$$\int_z^{a+1} f(x)p(x)\int_x^{a+1} f(t)(2p(t)-1)dtdx \le 0. \tag{29}$$

By construction, for all $z \ge v^*$

$$\int_a^z f(t)P^a(t)dt = \int_{v^*}^z f(t)\frac{F(t)-F(v^*)}{2}dt = \frac{(F(z)-F(v^*))^2}{4},$$

after an integration by parts. For all $z < v^*$ it holds that $\int_a^z f(t)P^a(t)dt = \int_a^z f(t)P(t)dt = 0$. Therefore, it is necessary and sufficient for the claim to prove that all $z \ge v^*$

$$\int_a^z f(t)P(t)dt \le \frac{(F(z)-F(v^*))^2}{4}.$$

After an integration by parts and rearrangements, it follows from (28) and (29) that for any admissible visiting probabilities

$$\int_z^{a+1} f(x)p(x)\int_x^{a+1} f(t)(2p(t)-1)dtdx = \int_z^{a+1}\int_z^t f(x)p(x)f(t)(2p(t)-1)dxdt =$$

$$= 2\int_z^{a+1}\int_z^t f(x)f(t)p(x)p(t)dxdt - \int_z^{a+1}\int_z^t f(x)f(t)p(x)dxdt =$$

$$= (\pi - P(z))^2 - \int_z^{a+1} f(t)(P(t)-P(z))dt =$$

$$= (\pi - P(z))^2 + P(z)(1 - F(z)) - \int_z^{a+1} f(t)P(t)dt =$$

$$= (\pi - P(z))^2 + P(z)(1 - F(z)) - \pi^2 + \int_a^z f(t)P(t)dt \leq 0.$$

Therefore, it follows that

$$\int_a^z f(t)P(t)dt \leq P(z)(2\pi - P(z) - (1 - F(z))).$$

Function $P(z)(2\pi - P(z) - (1 - F(z)))$ is maximized (with respect to $P(z)$) at $P(z) = \frac{F(z)-F(v^*)}{2}$ and thus

$$P(z)(2\pi - P(z) - (1 - F(z))) \leq \frac{(F(z) - F(v^*))^2}{4},$$

which in turn implies that

$$\int_a^z f(t)P(t)dt \leq P(z)(2\pi - P(z) - (1 - F(z))) \leq \frac{(F(z) - F(v^*))^2}{4},$$

concluding the derivation. ■

Derivation of inequality (6):

Proof. First, I prove that if for all z it holds that

$$\int_a^z P^a(x)dx \geq \int_a^z P(x)dx, \tag{30}$$

then for all admissible visiting probability functions

$$-\frac{1}{2}\int_a^{a+1} (P^a(z))^2 dz + \pi \int_a^{a+1} P^a(z)dz \geq -\frac{1}{2}\int_a^{a+1} P^2(z)dz + \pi \int_a^{a+1} P(z)dz \tag{31}$$

and thus (4) and (5) imply that the revenue is higher when the visiting probability function is p^a than it is p. To prove this statement note that after an integration by parts, it follows that under (30)

$$\int_a^{a+1} P^a(t)P(t)dt - \int_a^{a+1} (P^a(t))^2 dt =$$

$$= (\pi \int_a^{a+1} P(x)dx - \pi \int_a^{a+1} P^a(x)dx) + (\int_a^{a+1} p^a(x)f(x) \int_a^x P^a(t)dtdx - \int_a^{a+1} p^a(x)f(x) \int_a^x P(t)dtdx) \geq$$

$$\geq \pi \int_a^{a+1} P(x)dx - \pi \int_a^{a+1} P^a(x)dx.$$

Since it holds that

$$\int_a^{a+1} 2P^a(t)P(t)dt - \int_a^{a+1} (P^a(t))^2 dt - \int_a^{a+1} P^2(t)dt \leq 0,$$

it follows that

$$\frac{\int_a^{a+1} P^2(t)dt}{2} - \frac{\int_a^{a+1} (P^a(t))^2 dt}{2} \geq \pi \int_a^{a+1} P(x)dx - \pi \int_a^{a+1} P^a(x)dx,$$

which is equivalent to (6).

Now, I prove that indeed $\int_a^z P^a(x)dx \geq \int_a^z P(x)dx$. Because of formula (28) and the fact that it holds as an equality for $z = a+1$, it follows that one can sign the function $P(z) - P^a(z)$ in the following way. First, note that it is a continuous function and can thus only cross zero for at most at countable many points. So, there exists a sequence $x_1 \leq x_2 \leq x_3 \leq ... \in [a, a+1]$ such that $P(z) \leq P^a(z)$ if $z \in [x_{2k-1}, x_{2k}]$ and $P(z) \geq P^a(z)$ if $z \in [x_{2k}, x_{2k+1}]$ for all positive integers k. Therefore, for $\int_a^z P^a(x)dx \geq \int_a^z P(x)dx$ to hold for all z, it is sufficient that it holds for $z = x_{2k}$ for all positive integers k. First, let $k = 1$. Then using that f is increasing,

$$\int_a^{x_1} (P^a(x) - P(x))f(x)dx \leq f(x_1) \int_a^{x_1} (P^a(x) - P(x))dx$$

and

$$\int_{x_1}^{x_2}(P(x)-P^a(x))f(x)dx \geq f(x_1)\int_{x_1}^{x_2}(P(x)-P^a(x))dx.$$

Using (28) and the last two formulas, it follows that

$$f(x_1)\int_{x_1}^{x_2}(P(x)-P^a(x))dx \leq \int_{x_1}^{x_2}(P(x)-P^a(x))f(x)dx \leq$$

$$\leq \int_{a}^{x_1}(P^a(x)-P(x))f(x)dx \leq f(x_1)\int_{a}^{x_1}(P^a(x)-P(x))dx,$$

which yields that

$$\int_a^{x_2} P(x)dx \leq \int_a^{x_2} P^a(x)dx.$$

Similar calculations establish the result for x_{2k} and $k > 1$, which concludes the proof for the $\pi \leq \frac{1}{2}$ case. ■

Proof of Lemma 9:

Proof. First, note that changing the winning probabilities for those low types does not affect the revenues achieved from types above h. The reason is that because $u_1(h) = u_2(h) = \pi(h-r)$ and for all $z \geq h$ it holds that

$$u_1(z) = u_2(z) = \pi(h-r) + \int_h^z W_2(z)dz = \pi(h-r) + \int_h^z (1 - \int_z^{a+1} f(x)(1-p(x)dx)dz.$$

Thus the utilities of types above h are given as well as their winning probabilities, using that for all $z \geq h$ it holds that

$$p(z)f(z)W_1(z) = p(z)f(z)W_2(z) = p(z)f(z)(1 - \int_z^{a+1} f(x)(1-p(x)dx).$$

Therefore, the component of the revenue in (1) that pertains to types above h is all fixed.

After this observation, the problem becomes a one-seller mechanism design problem where the seller faces a convex distribution of types on $[a,h]$. The utility type h receives is equal to $\overline{u} = \pi(h-r)$ and for all $z \leq h$

$$u_1(z) = \overline{u} - \int_z^h W_1(z).$$

Then the revenue from types below h can be written as

$$\int_a^h (zW_1(z) - u_1(z))f(z)dz = -\overline{u}F(h) + \int_a^h W_1(z)f(z)[z + \frac{F(z)}{f(z)}]dz$$

after integration by parts. By assumption $x + \frac{F}{f}$ is increasing and thus the last formula implies that one would like to increase the probability of winning for high types as much as possible given the constraints (7) and (8), which establishes the result. Finally, the value of q can be calculated using that

$$\int_a^q f(z)(1-\pi+F(z))dz + \int_q^h f(z)\pi dz = \int_a^h f(z)W_1(z)dz = \int_a^{a+1} f(z)p(z)W_1(z)dz - \int_h^{a+1} f(z)p(z)W_1(z)dz =$$

$$= \pi - \frac{\pi^2}{2} - \int_h^{a+1} f(z)p(z)W_2(z)dz = \pi - \frac{\pi^2}{2} - \int_h^{a+1} f(z)p(z)(1 - \int_z^{a+1} f(x)(1-p(x)dx)dz. \qquad (32)$$

■

Proof of Lemma 10:

The proof is conducted in Steps. First, we calculate the revenue of seller 1 for any given choice of function p for a given π, Δ and h:

Lemma 15 *There exists a function E such that the (expected) revenue achieved by seller 1 is the following:*

$$R_1 = E(\Delta, h, \pi) + \int_a^{a+1}\int_a^t f(z)P(z)dzdt - \frac{1}{2}\int_a^{a+1} P^2(z)dz + \pi\int_a^{a+1} P(z)dz.$$

Proof. Using Lemma 9 and formula (1) and after similar steps as in Case 1, we arrive at the above formula. ■

The above revenue formula is similar to formulas (4) from Case 1, except that now there are two more choice parameters involved, h and Δ. However, once we fix those parameters we can see that if one can find a probability function p^a and a corresponding distribution function P^a such that $\int_a^{a+1} f(x)p^a(x)dx = \pi$ and for any admissible distribution function P and for all $z \geq h$ it holds that

$$\int_h^z f(x)P^a(x)dx \geq \int_h^z f(x)P(x)dx, \tag{33}$$

then - for a given π, Δ and h - the revenue is maximized.[17] Such a function is proposed in the next Lemma:

Lemma 16 *Let $\overline{v}$ and n_2 be such that*

$$F(\overline{v}) = 2\pi - 1$$

and

$$F(n_2) = 2F(n_1) - F(\overline{v}).$$

There exists an $n_1 \geq \max\{h, \overline{v}\}$ and an $n_2 \geq n_1$ such that an appropriate p^a function is such that for all $z \leq n_1$ it holds that $p^a(z) = 1$, for all $z \in [n_1, n_2]$ it holds that $p^a(z) = 0$ and for all $z \geq n_2$ it holds that $p^a(z) = \frac{1}{2}$.

Proof. By construction $n_1 \geq h$ and thus it is sufficient to establish (33) for all $z \geq n_2$.[18] By extending (2) to the case when $h > a$ it holds that for all $z \geq h$ and for all admissible probability functions p

$$\int_z^{a+1} f(x)p(x) \int_x^{a+1} f(t)(2p(t) - 1)dtdx \leq 0.$$

For the region where $z \geq n_2$ the calculations are very similar to Case 1, as presented in the derivation of formula (5). ■

The structure of the suggested p^a function is such that it does not allow types less than n_1 to win at seller 1 when a higher type showed up. This means that if we changed the value of h to n_1 and changed the value of Δ to 0[19], then we would face a similar problem as in the original case. This can be established formally to obtain the following result:

Corollary 17 *Let*

$$F(n_2^*) = 2F(h) - F(\overline{v}).$$

For a fixed π one can restrict attention to the case where $\Delta = 0$ and the probability functions are such that $p^a(z) = 1$ for all $z \leq h$, $p^a(z) = 0$ for all $z \in [h, n_2^*]$ and $p^a(z) = \frac{1}{2}$ for $z \geq n_2^*$ with $h \geq \overline{v}$.

Proof. Let $\widetilde{h}$ denote the old value of h. Once h is changed to n_1 and Δ to 0, then for all $z \leq n_1$ the function W_1 is characterized by Lemma 9. According to that Lemma for all $z \leq q$ it holds that

$$W_1(z) = 1 - \pi + F(z)$$

and for $z \in [q, n_1]$ it holds that $W_1(z) = \pi$.

Now, take the situation with the original values of h and $\Delta (\geq 0)$, i.e. let $h = \widetilde{h}$. By Lemma 9 for all $z \leq q$

$$W_1(z) = 1 - \pi + F(z)$$

[17] The proof here follows the same lines as in Case 1: we can establish (6) using (5) in a similar manner. The details are omitted.

[18] Since $p^a(z) = 1$ for $z \leq n_1$ (33) must hold for such values of z. Also, since $p^a(z) = 0$ for all $z \in [n_1, n_2]$, if (33) holds at $z = n_2$ then it must hold for all $z \in [n_1, n_2]$. To see this, note that $P^a(z) - P(z)$ is decreasing on $[n_1, n_2]$. If $P^a(n_2) \geq P(n_2)$ then for all $z \in [n_1, n_2]$, $P^a(z) \geq P(z)$ and thus $\int_h^z f(x)P^a(x)dx - \int_h^z f(x)P(x)dx$ is increasing and thus positive on the interval. If $P^a(n_2) < P(n_2)$ then $\int_h^z f(x)P^a(x)dx - \int_h^z f(x)P(x)dx$ is increasing and then decreasing on $[n_1, n_2]$. However, the difference is positive on the two ends n_1 and n_2, therefore it must be positive on the entire interval.

[19] Changing Δ to 0 indicates efficiency in the sense that ypes less than h can never win against types higher than h.

and for all $z \in [q, \widetilde{h}]$ it holds that $W_1(z) = \pi$. By construction $p^a(z) = 1$ for all $z \in [\widetilde{h}, n_1]$ and thus those types do not visit the auction of seller 2. Thus $W_2(n_1) = \pi$ and then monotonicity of the probability of winning in the types implies that for all $z \in [h, n_1]$ it holds that $W_1(z) = \pi$. These observations imply that the visiting probability functions are the same in the two situations and the allocation is also the same, thus the revenue of seller 1 is equal as well. ■

Finally, one can calculate the revenue of this mechanism that depends only on two parameters π and h. Fixing π, the problem becomes maximizing $R_1(h)$ with respect to h subject to the constraint that $h \geq \overline{v}$. Moreover, it follows from Burguet and Sakovics (1999) that the case when $h = \overline{v}$ pertains to seller 1 choosing an auction with a reserve price r_1 that is less than r, the reserve price of seller 2. Therefore, one only needs to prove that R_1 is maximized at $h = \overline{v}$. The following proof establishes this and thus implies that when $\pi > \frac{1}{2}$ the best deviation is an auction:

Claim 18 *Suppose that* $\Delta = 0$ *and* $\pi \geq \frac{1}{2}$. *Then* $\overline{v} \in \{\arg\max_h R_1(h)$ *s.t.* $h \geq \overline{v}\}$.

Proof. The proof below uses techniques from standard one-seller mechanism design. The proof of Lemma 9 implies that if one ignores constraints (7) and (8), and implements an efficient outcome for types $z \leq h$, then the revenue increases. In the modified mechanism the probability that type h wins is $W_1^{\text{mod}}(h) = F(h) + 1 - \pi = \frac{1+F(n_2^*)}{2} = W_1^{\text{mod}}(n_2)$ and thus monotonicity of the winning probability still holds. Moreover, $W_1^{\text{mod}}(\overline{v}) = 1 - \pi + 2\pi - 1 = \pi$ and $u_1^{\text{mod}}(\overline{v}) = \pi(\overline{v} - r)$ holds and for it holds that $u_1^{\text{mod}}(a) = \pi(\overline{v} - r) - \int_a^{\overline{v}} (1 - \pi + F(t))dt$ just like in the auction where $h = \overline{v}$. To summarize, we compare two mechanisms: the modified mechanism with $h > \overline{v}$ and an efficient outcome, and the (efficient) auction with $h = \overline{v}$. Both mechanisms can be considered as second price auctions with reserve price r_1, just the distribution of types participating are different. By construction, the type distribution in the auction with $h = \overline{v}$ stochastically dominates the type distribution of the (modified) mechanism with $h > \overline{v}$. Since the revenue is equal to the expectation of the maximum of r_1 and the second highest type participating (0 if only one participates) in both of those mechanisms, it implies that the auction with $h = \overline{v}$ has higher revenue, concluding the proof. ■

Proof of Theorem 12:

Proof. The game where both sellers post auctions with reserve prices was analyzed by Burguet and Sakovics (1999). Using their techniques, but assuming that a is positive, one can calculate a pure strategy equilibrium. First, let $r_1 \leq r$ be the reserve price charged by seller 1. Then all types $z \in [a, x]$ visit seller 1 for sure and types $z \geq x$ visit the two sellers with equal probability. Type x is indifferent between visiting the two sellers. Since his probability of winning is such that

$$W_1(x) = W_2(x) = \frac{1 + F(x)}{2},$$

it follows that the expected payment at the two sellers is the same for type x. At seller 2 type x is the lowest type who ever shows up, thus he pays only the reserve price r. At seller 1 he pays the reserve price r_1 if the other buyer does not show up, which is with probability $\frac{1-F(x)}{2}$. If the other buyer shows up and has type higher than x, then the buyer with type x does not win. Finally, if the other buyer has type less than x and shows up (which is with probability x), then the buyer pays the expected value of the type of the other buyer, which is $\frac{\int_a^x zf(z)dz}{F(x)}$. Putting these together yields the incentive constraint

$$\frac{1 - F(x)}{2} r_1 + F(x) \frac{\int_a^x zf(z)dz}{F(x)} = \frac{1 + F(x)}{2} r. \tag{34}$$

Now, we turn to the revenue of seller 1 when choosing r_1. The probability that exactly one buyer shows is $\frac{1-F^2(x)}{2}$, in which case the revenue is r_1. The other cases can be covered similarly using that the revenue when two bidders visit is the type of the weaker bidder:

$$R_1 = \frac{1 - F^2(x)}{2} r_1 + [F^2(x) \frac{\int_a^x z2f(z)(1 - F(z))dx}{F^2(x)} + \tag{35}$$

$$+F(x)(1 - F(x)) \frac{\int_a^x zf(z)dz}{F(x)} + \frac{(1 - F(x))^2}{4} \frac{\int_x^{a+1} z2f(z)(1 - F(z))dx}{(1 - F(x))^2}].$$

Substituting from (34) yields

$$R_1 = \frac{(1+F(x))^2}{2}r - (1+F(x))\int_a^x zf(z)dz +$$

$$+(1-F(x))\int_a^x zf(z)dz + \frac{1}{4}\int_x^{a+1} z2f(z)(1-F(z))dx.$$

In a symmetric equilibrium $x = 0$ and thus $\frac{\partial R_1}{\partial x} \mid_{x=0}= 0$ follows, which is equivalent to $r = \frac{a}{2}$. The second order condition holds for all $a \geq 0$.

Now, we analyze the case where $r_1 \geq r$. In this case types lower than $a+y$ visit seller 2 and types higher than y visit the two sellers with equal probability. The equations corresponding to (34) and (35) are

$$\frac{1-F(a+y)}{2}r + \int_a^{a+y} xf(x)dx = \frac{1+F(a+y)}{2}r_1$$

and

$$R_1 = \frac{1-F^2(a+y)}{2}r_1 + \frac{1}{4}\int_{a+y}^{a+1} 2xf(x)(1-F(x))dx.$$

Using the last two equations imply that

$$R_1 = \frac{(1-F(a+y))^2}{2}r + (1-F(a+y))\int_a^{a+y} xf(x)dx + \frac{1}{4}\int_{a+y}^{a+1} 2xf(x)(1-F(x))dx.$$

The first order condition $\frac{\partial R_1}{\partial y} \mid_{y=0}= 0$ is again equivalent to $r = \frac{a}{2}$. For the second order condition, note that

$$\omega(y) = \frac{\frac{\partial R_1}{\partial y}}{f(a+y)} = (1-F(a+y))(\frac{a+y}{2} - r) - \int_a^{a+y} xf(x)dx.$$

Then

$$\omega'(y) = \frac{1}{2} - [\frac{F(a+y)}{2} + f(a+y)(\frac{3}{2}(a+y) - r)] = \frac{1}{2} - \beta(y)$$

where $\beta'(y) \geq 0$. Therefore, if $\omega'(0) \leq 0$ then for all $y \geq 0$ it holds that $\omega(y) \leq 0$ and thus $\frac{\partial R_1}{\partial y} \leq 0$, which is sufficient for the (global) second order condition to hold. But, after using $r = \frac{a}{2}$

$$\omega'(0) = \frac{1}{2} - f(a)a \leq 0.$$

■

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