Financial Deepening, Inflation and Economic Growth*

Aleksander Berentsen

Department of Economics, University of Basel, Switzerland

Shouyong Shi

Department of Economics, University of Toronto, Canada

January 2008

Abstract

We develop a general equilibrium framework to analyze the relationship between the operation of the financial system, inflation and economic growth. We first investigate the dynamic interactions between financial development and growth by analyzing how financial innovations affect real growth, and, in turn, growth affects the financial system. We then study how inflation affects financial deepening and economic growth. Finally, we discuss policy implications and derive optimal fiscal and monetary policies.

Keywords: Money; Credit, Innovation; Growth.

JEL Classification:

* The second author gratefully acknowledges the financial support from the Bank of Canada Fellowship and the Social Sciences and Humanities Research Council of Canada. The opinion expressed here is my own and it does not represent the view of the Bank of Canada.

1 Introduction

2 The Model

2.1 The Economy

Consider a discrete-time economy with many households. The number of households in each type is large and normalized to one. All households have the same discount factor $\beta \in (0,1)$. Denote $R \equiv (1-\beta)/\beta$. The households of each type are specialized in producing a good which they do not consume; instead, they consume a set of goods produced by other types of households. Goods are perishable between periods. The utility of consuming a particular good produced by other households is $\varepsilon u(c)$, where ε is a taste shock. The cumulative distribution function of ε, denoted as $F(.)$, is assumed to be continuous on the support $[0, \varepsilon_H]$, where $0 < \varepsilon_H < \infty$.[1] Let $\bar{\varepsilon}$ denote the expected value of ε. The taste shocks are independent among different buyers and households and independent over time. For the model to generate balanced growth, we let $u(c) = \ln(c)$.

The cost (disutility) of producing q units of a good is $h(q/a)$, where a is the level of productivity of the household's technology to be described later. The function h is assumed to satisfy $h(0) = 0$, $h' > 0$, $h'' > 0$, and $h'(0) = 0$. For most part of the paper we will utilize the particular functional form that $h(q/a) = (h_0/\eta)(q/a)^\eta$, where $\eta > 1$ and $h_0 > 0$. We pick an arbitrary household as the representative household and use lower-case letters to denote its decisions. The decisions of other households and the aggregate variables are denoted with capital-case letters.

A household consists of a large number of members, who share consumption and regard the household's utility as the common objective.[2] The measure of members in a household is normalized to one. The household divides the members into four groups: a fraction l of members are innovators; a fraction k work in the intermediation/financial sector; and a fraction $(1-k-l)$ participate in the goods market, who are divided further equally into buyers or sellers.[3] This allocation of members to the four activities (or time) is the link

[1] All of our results go through with a non-zero lower bound. Setting the lower bound of ε to zero simplifies the presentation of the results.

[2] The device of a household is used here to maintain tractability, as it enables us to smooth the matching risk within a household and hence to obtain a degenerate distribution of money holdings across households. See Shi (1997).

[3] Although the household can choose the optimal division of the market participants into buyers and sellers, as in Shi (1997), we choose not to model this choice in order to focus on the more important

between inflation and economic growth, which we want to focus on in this paper.

To be more explicit about the taste shocks and the household's allocation, it is useful to represent the members of a household by the points in a square whose width and length are both equal to one. Let a particular member be indexed by (i, j) and let I_i be the set (unit interval) of members with the same index i. The household allocates the members as follows. For each i, the household allocates each member (i, j), where $j \in I_i$, randomly into one of the three groups described above, with probabilities l, k, and $(1-k-l)$, respectively. The members who are allocated to the goods market are evenly divided into sellers and buyers. The members in I_i experience the same taste shock and share consumption among themselves. The shocks are independent between any two subsets of members who differ in i. We will refer to a member who receives ε as the realization of the taste shock as a type-ε member.

The innovators improve the household's production technology. If a is the current productivity of the household's technology in producing goods, then the productivity of the technology in the next period is

$$a_{+1} = a + af(l). \tag{1}$$

The subscripts "+1" indicate the next period throughout the paper. The function f is assumed to satisfy the following properties: $f(0) = 0$, $f(1) < \infty$, $f' > 0$, $f'' < 0$, and $f'(0) = \infty$. One interpretation of a is that it is the stock of human capital, as in Lucas (1988). Like many endogenous growth models, our model generates long-run growth from the non-diminishing marginal productivity of a in the innovation process. We specify the innovation sector, rather than using the so-called AK model, to deliver long-run growth because we want to emphasize the allocation of time between different sectors. The modeling is also convenient for finding identification restrictions in the calibration, as illustrated later.

In the goods market, double coincidence of wants is not possible between any two households. Moreover, agents in the goods market are anonymous so that no record-keeping on the histories of transactions in the goods market is feasible. Thus, a medium is needed for a transaction to take place. This medium is fiat money, a perfectly storable object which is intrinsically worthless. To simplify the description of the transactions in the goods market while preserving the role of money, we assume that each household sells the goods in a separate location, while the buyers of each household are dispersed over all the locations.

allocation of time in this paper – the one between innovation, labor supply for the financial market, and goods-market activities (i.e., the choice of l).

Moreover, we assume that the participants of the goods market take prices as given. This particular pricing mechanism streamlines the characterization of the equilibrium and simplifies the algebra, although alternative pricing mechanisms, such as sequential bargaining (e.g., Shi, 2001) and Nash bargaining (e.g., Lagos and Wright, 2005), can also be considered.

2.2 The first-best allocation

Consider the allocation of a social planner who can dictate all quantities and the allocation of time subject to feasibility constraints. Since there is no need for a credit market, the planner's problem is to choose $P = \{q(\varepsilon), q_s, l\}$ to solve the following problem:

$$W(a) = \max_P \int_0^{\varepsilon_H} \varepsilon u[c(\varepsilon)]\,dF(\varepsilon) - \frac{1-l}{2} h(q_s/a) + \beta W(a_{+1})$$

subject to the following constraints:

$$\begin{aligned} c(\varepsilon) &= \frac{1-l}{2} q(\varepsilon) && (2)\\ a_{+1} &= a + af(l) && (3)\\ q_s &= \int_0^{\varepsilon_H} q(\varepsilon)\,dF(\varepsilon) && (4) \end{aligned}$$

The first term in the welfare function is a household's total utility of consumption and the second term the disutility of production. The first constraint requires that total consumption equals the total amount of goods received by the buyers. The second constraint is the low of motion of productivity. Equation (4) is the feasibility constraint.

Proposition 1 *The social optimum is the unique solution to the following equations:*

$$\begin{aligned} l &= l^* \text{ where } \frac{f'(l^*)(1-l^*)}{1+f(l^*)} = R\left(\frac{\eta-1}{\eta}\right) \text{ and} && (5)\\ q_s &= q_s^*, \quad \text{where } h(q_s^*/a) = \frac{2\bar{\varepsilon}}{(1-l^*)\,\eta}, && (6)\\ q(\varepsilon) &= q^*(\varepsilon) \text{ where } q^*(\varepsilon) = \frac{\varepsilon}{\bar{\varepsilon}} * q_s^*. && (7) \end{aligned}$$

The social optimum requires efficiency along two dimensions, the quantity of goods consumed, $q(\varepsilon)$, and produced, q_s, by each member, and the allocation of the members to each group. The socially efficient rate of growth is

$$g^* = \frac{(1 - l^*) f'(l^*)}{R\left(\frac{\eta - 1}{\eta}\right)} \tag{8}$$

2.3 Financial intermediation

Because the household allocates money to the buyers before the taste shock is realized, individual buyers may have incentive to borrow and lend. We assume that borrowing and lending can only be conducted through intermediaries, called banks. The intermediation/financial sector is perfectly competitive in the sense that there is free entry of banks.

Financial intermediaries have no ability to keep records on transactions in the goods market. Consistent with the earlier assumption on record-keeping, this assumption prevents banks from issuing credit that supersedes money or directly intermediating the trade in the goods market. However, banks are able to keep financial records on monetary loans and repayments, at a cost. So, borrowing and lending are in terms of money. If a buyer fails to repay a loan, the bank can confiscate money holdings of the buyer's household, which ensures that loans are always repaid. Banks take the deposit rate as given and compete in the loan market. Depositors have perfect information about the banks' financial state and trading histories, which induces the banks to always repay the depositors.[4]

To describe the amount and the cost of loans, let $Y(\varepsilon)$ denote the economy-wide average of the amount of borrowing per household by type-ε members. Because $Y(\varepsilon)$ can be negative, in which case it is the amount of lending, we use the indicator function $1_Y : \mathbb{R} \to \{0, 1\}$ to separate the case of borrowing from other cases. That is, $1_Y = 1$ if $Y(\varepsilon) > 0$ and equal to 0 otherwise. Then, the nominal value of loans given out by the bank in a period is:

$$\frac{1 - K - L}{2} \int_0^{\varepsilon_H} \mathbf{1}_Y Y(\varepsilon)\, dF(\varepsilon).$$

Here, the quantity K is the measure of agents per household who work in the intermediation sector, and L is the measure of agents per household who work in the innovation sector.

The bank must hire workers to administer and monitor loans. We assume that the hours needed for intermediation is proportional to the ratio of total amount of loans to the money

[4] Our assumptions of perfect monitoring by the banks on the borrowers and by the depositors on the banks simplify the analysis and enable us to focus on growth. For a relaxation of this assumption, see Berentsen, Camera and Waller (2006).

stock. That is the production function in the intermediation sector satisfies

$$\frac{1-K-L}{2}\int_0^{\varepsilon_H}\left[\mathbf{1}_Y Y(\varepsilon)/M\right]dF(\varepsilon)=\frac{K}{\kappa} \tag{9}$$

where $\kappa > 0$ is the inverse of productivity in the financial sector. We refer to the case $\kappa = 0$ as a perfect loan market.

The wage bill paid to those workers is the cost of operating the loans.[5] Let w denote the nominal wage rate in the economy. Then, the total nominal cost of loans is wK. A financial intermediary covers the cost of loans with a spread between the loan rate, r_ℓ, and the deposit rate, r_d. To separate the borrowing rate from the deposit rate we use the indicator function $1_r : \mathbb{R} \to \{r_d, r_\ell\}$ where $1_r = r_d$ if $Y(\varepsilon) < 0$ and $1_r = r_\ell$ if $Y(\varepsilon) > 0$. Then, profits satisfy

$$\int_0^{\varepsilon_H}\frac{1-K-L}{2}Y(\varepsilon)\,1_r dF(\varepsilon)-wK. \tag{10}$$

Because there is free entry of banks, each bank makes zero profit, and so

$$wK=\int_0^{\varepsilon_H}\frac{1-K-L}{2}Y(\varepsilon)\,1_r dF(\varepsilon). \tag{11}$$

Financial intermediaries take the loan rate and the deposit rate as given. There is no strategic interaction among financial intermediaries or between financial intermediaries and agents. In particular, there is no bargaining over terms of the loan contract.

The sequence of events in a period is as follows. At the beginning of the period, each household chooses l and k and divides its holdings of money among the buyers. Then, buyers and sellers leave the household. Each buyer discovers the value of the taste shock, ε, and chooses to lend to or borrow from intermediaries. Then, the traders move to the goods market. Once trades are completed, all members return home and give their holdings of money and consumption goods to the household. The household allocates consumption evenly among all members. Before the period ends, the household receives a lump-sum monetary transfer from the government.

[5] There can be other cost of loans, which we abstract from in order to focus on the time allocation between different sectors.

2.4 The representative household's decisions

At the beginning of a period, the household chooses the division of the members into innovators, l, workers in the financial sector, k, and goods-market traders, $(1-k-l)$. It also chooses the allocation of money among the buyers, x. These choices are made before the taste shocks are realized. After leaving the household and observing the taste shocks, each buyer chooses the amount of borrowing, $y(\varepsilon)$, in the interest of the household. If $y(\varepsilon) > 0$, it is the amount of borrowing; if $y(\varepsilon) < 0$, it is the amount of lending. In the goods market a buyer trades $pq(\varepsilon)$ units of money for $q(\varepsilon)$ units of goods, and a seller sells q_s units of goods for pq_s units of money. At the end of the period, the household chooses consumption, $c(\varepsilon)$, future holdings of money, m_{+1}, and the future productivity, a_{+1}. We collect these choices of the representative household as:

$$d \equiv [l, k, x, q(\varepsilon), y(\varepsilon), q_s, q(\varepsilon), m_{+1}, a_{+1}].$$

Let m denote a representative household's holdings of money at the beginning of a period. The household's value function is $V(a, m)$. Define

$$\omega \equiv \beta \frac{\partial V(a_{+1}, m_{+1})}{\partial m_{+1}} \quad \text{and} \quad \phi \equiv \beta \frac{\partial V(a_{+1}, m_{+1})}{\partial a_{+1}}.$$

The variable ω is the discounted (shadow) value of money next period and ϕ the discounted value of future productivity.

The representative household's problem is to choose d to solve the following problem:

$$V(a, m) = \max_{d} \int_0^{\varepsilon_H} \varepsilon u[c(\varepsilon)]\, dF(\varepsilon) - \frac{1-k-l}{2} h(q_s/a) + \beta V(a_{+1}, m_{+1})$$

subject to the following constraints:

$$c(\varepsilon) = \frac{1-k-l}{2} q(\varepsilon) \tag{12}$$

$$pq(\varepsilon) \leq x + y(\varepsilon) \tag{13}$$

$$m \geq \frac{1-k-l}{2} x \tag{14}$$

$$a_{+1} = a + af(l) \tag{15}$$

$$m_{+1} - m = \frac{1-k-l}{2} \left\{ pq_s - \int_0^{\varepsilon_H} [pq(\varepsilon) + 1_r y(\varepsilon)]\, dF(\varepsilon) \right\} + wk + T \tag{16}$$

where T is a lump-sum transfer or tax.

In the objective function, the integral is the expected utility of consumption of the members in the subset I_i, i.e., those who have the same index i where the index i is suppressed. (Recall that the members in I_i experience the same taste shock and share consumption among themselves.) Because the shocks are independent across the members who differ in the index i, the integral is also the sum of utilities of consumption in the household. The second term in the objective function is the sum of disutility of production and the last term is the discounted future value. Notice that the amount of production by a seller is independent of the taste shock received by the seller, because the seller sells to other households' buyers.

The constraint (12) reflects the fact that only a fraction $(1-k-l)/2$ of the members in each subset I_i are buyers. The constraint (13) requires that a buyer in the goods market should not spend more money than what he has, which consists of the amount of money allocated by the household and the amount borrowed from the intermediary (in the case of lending, $y(\varepsilon) < 0$). The constraint (14) requires that the total amount of money allocated to the buyers should not exceed the household's holdings at the beginning of the period.[6] The constraint (15) is the law of motion of productivity specified earlier.

The law of motion of the household's holding of money is (16), which can be explained as follows. In the period, the buyers spend money to buy goods and pay interests on loans (or earn interests from lending), the sum of which is given by the integral on the right-hand side of the equation. The sellers receive money as payments for goods, the amount of which is given as pq_s. These terms are multiplied by $(1-k-l)/2$ to convert them into quantities per member in the household. In addition to these terms, the household also receives (monetary) wage payments to the workers who work in the intermediary, wk, and lump-sum monetary transfers from the government, T.

2.5 Definition of the equilibrium and optimal conditions

We focus on the monetary equilibrium which is symmetric in the sense that the decisions are the same for all households. Throughout this paper, monetary policy is such that monetary transfer maintains the gross rate of money growth at a constant level $\gamma \geq \beta$.

[6] In principle, the household can make the allocation of money to the buyers in each subset I_i dependent on the buyer's identity (i, j). However, because the household must allocate money to the buyers before the taste shocks are realized and because the shocks are independent across the subsets I_i, it is optimal for the household to allocate money evenly among all buyers.

With the above focus, a monetary equilibrium consists of the representative household's decisions, d, other household's decisions, D, and interest rates, $r_d \geq 0$ and $r_\ell \geq 0$, which meet the following requirements: (i) d solves the representative household's maximization problem above; (ii) the decisions are symmetric across households: $d = D$; (iii) the loan market and the good markets clear.

Market clearing in the loans market requires that

$$\int_0^{\varepsilon_H} y(\varepsilon)\, dF(\varepsilon) = 0. \tag{17}$$

The market clearing condition in the goods market is:

$$q_s = \int_0^{\varepsilon_H} q(\varepsilon)\, dF(\varepsilon). \tag{18}$$

To understand this condition, recall that the sellers of a household sell goods to other households' buyers whose tastes are distributed according to F. Thus, the total amount of the household's goods demanded by other households is the right-hand side of (18) multiplied by $(1-K-L)/2$. The total amount supplied by the household is the left-hand side of the equation multiplied by $(1-k-l)/2$. Anticipating a symmetric equilibrium, we cancelled these terms on the two sides of the market.

Let us analyze the household's optimal decisions in the symmetric equilibrium. Denote the Lagrangian multiplier of (13) as $\pi(\varepsilon)(1-k-l)/2$. Then, the first-order conditions for $q(\varepsilon)$ and q_s are

$$\varepsilon u'[c(\varepsilon)] = p[\pi(\varepsilon) + \omega], \tag{19}$$

$$ap\omega = h'(q_s/a). \tag{20}$$

The first-order condition for $y(\varepsilon)$ is

$$\pi(\varepsilon) - 1_r \omega = 0, \text{ if } y(\varepsilon) \neq 0. \tag{21}$$

If $y(\varepsilon) > 0$, this condition says that the marginal cost of borrowing is equal to the marginal utility of increased consumption which is brought about by one unit of borrowed money. Similarly, if $y(\varepsilon) < 0$, the marginal benefit of lending is equal to the marginal disutility of decreased consumption resulted from lending.

Let μ be the Lagrangian multiplier of (14), which can be interpreted as the additional value of allocating a unit of money to the current exchange rather than holding it to the next period. This value can be obtained from the optimal choice of money among buyers:

$$\mu = \int_0^{\varepsilon_H} \pi(\varepsilon)\, dF(\varepsilon). \tag{22}$$

The first-order condition for l after some simplification is

$$\phi a f'(l) = \left(\frac{\eta - 1}{2}\right) h(q_s/a) \tag{23}$$

where we have used the form $h(q/a) = (h_0/\eta)(q/a)^\eta$. An increase in the input into innovation increases future productivity by $af'(l)$, whose value in terms of utility is given by the left-hand side of the equation. Such a higher l reduces the number of buyers and the number of sellers by a factor $1/2$ each. The net cost of these reductions, given by the right-hand side of the above equation, comes from the following effects: the reduction in the disutility of production, the fall in consumption for any given $q(\varepsilon)$, the increase in money allocated to each buyer (as there are fewer buyers now), the reduction in the amount of money obtained from selling goods (as there are fewer sellers now) and from interest payments. The first effect is $-h/2$, while the net of the last three effects is $\eta h/2$ after simplification with other first-order conditions.

Alternatively, the household can allocate members to work in the intermediary sector. Because the nominal wage rate is w and money earned by these workers are carried over to the next period, the marginal value of allocating a member to the intermediary sector, measured in terms of utility, is ωw. The net marginal cost is given by the right-hand side of (23). Thus, the first-order condition for k is:

$$\omega w = \left(\frac{\eta - 1}{2}\right) h(q_s/a). \tag{24}$$

The envelope conditions for m and a are

$$\frac{\omega_{-1}}{\beta} = \mu + \omega, \tag{25}$$

$$\frac{\phi_{-1}}{\beta} = \frac{1 - k - l}{2}\left(\frac{\eta}{a}\right) h(q_s/a) + \phi\left[1 + f(l)\right]. \tag{26}$$

These conditions state simply that the current value of an asset (m or a) is equal to the future value of the asset plus the additional value of the asset in the current exchange. Take the condition for a, for example. A marginal unit of productivity results in $f(l)$ units of future productivity, whose value is $\phi f(l)$. In addition, a marginal unit of productivity can save the production cost in the exchange, the value of which can be computed as the first term in the condition for a.

Finally, the household has to decide which types of buyers will enter the loan market. Consumption will be affected by this decision as follows. If a buyer enters the loan market, then from (19)-(21) consumption satisfies

$$c_d(\varepsilon) = \frac{\varepsilon}{p\omega(1+r_d)} \quad \text{and} \quad c_\ell(\varepsilon) = \frac{\varepsilon}{p\omega(1+r_\ell)}, \tag{27}$$

where the index d indicates that the buyer has deposited money in the intermediary and the index ℓ means that he has taken out loans. If a buyer does not entered the loan market and if he is constrained in the goods market, consumption satisfies

$$c_0(\varepsilon) = \frac{m}{p} \tag{28}$$

Here, the index 0 indicates that the buyer does not used the loan market in the period.

Lemma 1 below characterizes the optimal borrowing and lending decisions (see the Appendix for a proof):

Lemma 1 *If $(1+r_\ell)\omega m < \varepsilon_H$, there exist critical values ε_d and ε_l, with $0 \leq \varepsilon_d \leq \varepsilon_\ell < \varepsilon_H$, such that the following is true: if $\varepsilon < \varepsilon_d$, a buyer lends money in the loans market; if $\varepsilon > \varepsilon_\ell$, he borrows money; and if $\varepsilon_d \leq \varepsilon \leq \varepsilon_\ell$, he stays out of the loan market. The critical values solve*

$$\varepsilon_d = (1+r_d)\omega m \tag{29}$$

$$\varepsilon_\ell = (1+r_\ell)\omega m. \tag{30}$$

The optimal borrowing and lending decisions follow the cut-off rules according to the realization of the taste shock. The cut-off levels, ε_d and ε_ℓ, partition the set of taste shocks into three regions. For shocks lower than ε_d, a buyer enters the loan market to lend out money; for shocks higher than ε_l, the buyer enters the loan market to borrow. For intermediate values, the buyer does not use the loan market. The constraint $(1+r_\ell)\omega m < \varepsilon_H$ requires that the buyer with the highest taste-shock is willing to borrow.

Lemma 2 below gives the amount of borrowing/lending by a buyer and the quantity of goods obtained by the buyer (see the Appendix for a proof):

Lemma 2 *In any equilibrium, the amount of borrowing and lending by a buyer with a taste shock ε and the amount of goods purchased by the buyer satisfy:*

$$\begin{array}{lll} y_d(\varepsilon) \leq \left(\frac{\varepsilon}{\varepsilon_d} - 1\right) x, & q_d(\varepsilon) = \frac{x}{p}\frac{\varepsilon}{\varepsilon_d}, & \text{if } \varepsilon \leq \varepsilon_d \\ y_0(\varepsilon) = 0, & q_0(\varepsilon) = x/p, & \text{if } \varepsilon_d \leq \varepsilon \leq \varepsilon_\ell \\ y_\ell(\varepsilon) = \left(\frac{\varepsilon}{\varepsilon_\ell} - 1\right) x, & q_\ell(\varepsilon) = \frac{x}{p}\frac{\varepsilon}{\varepsilon_l}, & \text{if } \varepsilon_\ell \leq \varepsilon. \end{array} \tag{31}$$

Note that the above formula for $y_d(\varepsilon)$ holds as equality if $r_d > 0$. If $r_d = 0$, the household is indifferent in how much to lend for $\varepsilon \leq \varepsilon_d$.

3 Balanced Growth

In this section, we characterize the balanced growth path of the economy. A balanced growth path in this economy is defined as an equilibrium in which productivity, a, grows at a constant rate $g \in (1 + f(0), 1 + f(1))$, and interest rates are positive and finite constants (i.e., $0 < r_d \leq r_\ell < \infty$). It is clear from (15) that

$$g = 1 + f(l). \tag{32}$$

Thus, l is a constant in $(0, 1)$ on the balanced growth path. In the Appendix we prove the following lemma:

Lemma 3 *The balanced growth path has the following properties: (i) l and k are constants in $(0, 1)$ with $l + k < 1$; (ii) $q(\varepsilon)$ and q_s grow at rate g; (iii) the marginal value of money, ω, decreases at rate γ; and (iv) the marginal value of productivity, ϕ, falls at rate g. Interest rates, r_d and r_ℓ, the cut-off levels of shocks for lending and borrowing, ε_d and ε_ℓ, and the* *total value of money, ωm, satisfy the following equations:*

$$\begin{aligned}
\omega m\left(r_{\ell}-r_{d}\right) &= \left[(\eta-1)/2\right] h\left(q_{s}/a\right)\kappa && (33)\\
\varepsilon_{d} &= \left(1+r_{d}\right)\omega m && (34)\\
\varepsilon_{\ell} &= \left(1+r_{\ell}\right)\omega m && (35)\\
\frac{\omega m\gamma}{\beta} &= \bar{\varepsilon}+\int_{0}^{\varepsilon_{d}}\left(\varepsilon_{d}-\varepsilon\right)dF(\varepsilon)+\int_{\varepsilon_{\ell}}^{\varepsilon_{H}}\left(\varepsilon_{\ell}-\varepsilon\right)dF(\varepsilon) && (36)\\
\int_{0}^{\varepsilon_{d}}\left(1-\frac{\varepsilon}{\varepsilon_{d}}\right)dF(\varepsilon) &\geq \int_{\varepsilon_{\ell}}^{\varepsilon_{H}}\left(\frac{\varepsilon}{\varepsilon_{\ell}}-1\right)dF(\varepsilon)\quad (=0\ \textit{if}\ r_{d}>0)\,. && (37)
\end{aligned}$$

Furthermore, output produced by a seller, q_s, and the labor input into innovation, l, and the labor input into the financial sector, k, obey the following equations:

$$\begin{aligned}
\omega m &= \frac{\eta h\left(q_{s}/a\right)(1-k-l)/2}{1+\int_{0}^{\varepsilon_{d}}\left(\frac{\varepsilon}{\varepsilon_{d}}-1\right)dF(\varepsilon)+\int_{\varepsilon_{\ell}}^{\varepsilon_{H}}\left(\frac{\varepsilon}{\varepsilon_{\ell}}-1\right)dF(\varepsilon)} && (38)\\
1-k-l &= \frac{R\left[1+f(l)\right]}{f'(l)}\left(\frac{\eta-1}{\eta}\right) && (39)\\
k &= \kappa\int_{\varepsilon_{\ell}}^{\varepsilon_{H}}\left(\varepsilon/\varepsilon_{\ell}-1\right)dF(\varepsilon) && (40)
\end{aligned}$$

The characterization of the balanced growth path is given by (33)-(40). The most noticeable condition of the balanced growth path is (39), which comes from the envelope condition of a. This condition shows that the goods market and the loan market can affect the long-run growth rate only if they can affect the amount of resources put into intermediation, k. If k is fixed exogenously, then (39) will solve for the input into innovation, l, and hence for the long-run growth rate independently of all other equations. In this case, money growth will have no effect on long-run growth.[7] To emphasize this feature of the balanced growth path, let us express the solution for k from (39) as a function of l:

$$k = k(l).$$

It is easy to show that $k(l)$ is a decreasing function and that $[1-k(l)-l]$ is increasing in l. One unit of increase in the time in innovation induces a reduction in the time input in

[7]This feature survives a number of variations and extensions of the model. For example, if we model the cost of intermediation alternatively as an amount of goods put into intermediation, then long-run growth of the economy will be independent of money growth. To create an effect of the goods market on long-run growth in this case, the innovation technology must use goods as one of the inputs.

intermediation by more than one unit.

We now explore the effects of the growth rate of the money supply. In an economy with a perfect loan market, i.e., with $\kappa = 0$, the effects are summarized below (see the Appendix for a proof):

Proposition 2 *For $\gamma > \beta$ and $\kappa = 0$, the equilibrium allocation $(r_d, r_\ell, \varepsilon_d, \varepsilon_\ell, l, k, q(\varepsilon), q_s, \omega m)$ satisfies $r_d = r_\ell = (\gamma - \beta)/\beta$, $\varepsilon_d = \varepsilon_\ell = \bar{\varepsilon}$, $k = 0$, $\omega m = \bar{\varepsilon}\beta/\gamma < \omega m^*$ and*

$$
\begin{aligned}
l &= l^* && (41)\\
q_s &< q_s^*, \quad \text{where } h(q_s/a) = \frac{\beta}{\gamma} h(q_s^*/a), && (42)\\
q(\varepsilon) &= \frac{\varepsilon}{\bar{\varepsilon}} * q_s. && (43)
\end{aligned}
$$

The balanced growth path depends on money growth as follows: (i) $d\varepsilon_d/d\gamma = 0$ and $d\varepsilon_\ell/d\gamma = 0$; (ii) $dr_d/d\gamma, dr_\ell/d\gamma > 0$; (iii) $dl/d\gamma = dg/d\gamma = 0$; (iv) $d(\omega m)/d\gamma < 0$; (v) $dq_s/d\gamma < 0$.

These effects are expected. Higher inflation generated by higher money growth raises the nominal interest rate, as in (ii), and reduces the total real value of money holdings of a household, as in (iv). Because the loan rate is equal to the deposit rate when the loan market is perfect, the change in the value of money affects the two sides of the loan market equally, and so the cut-off levels for lending and borrowing do not change (see (i)). The negative effect of inflation on the quantity of goods produced, as stated in (v), occurs in this type of models regularly. Finally, because a perfect loan market implies $k = 0$, there is no link between the goods market or the loan market and the innovation process, and hence money growth has no effect on long-run growth (see (iii)).

To understand how the credit market affects the allocation, note that there are two possible consumption inefficiencies in this economy. First, there is an inefficient allocation of consumption across buyers since some buyers are constrained while others are not. As a result, the marginal utilities of consumption are not equalized. This is a extensive margin inefficiency. Second, due to the time cost of holding money, the quantities consumed by all buyers are inefficiently low if $\gamma > \beta$. This is an intensive margin inefficiency. A perfect credit market can overcome the first inefficiency but not the second. Only under the Friedman rule $\gamma = \beta$, the first-best allocation characterized in Proposition 1 can be attained.

When the loan market is imperfect, i.e., when $\kappa > 0$, money growth does affect long-run growth, by affecting the time input in intermediation, k. The following proposition describes the effects of money growth in such an economy (see the Appendix for a proof):

Proposition 3 *With an imperfect loan market, the balanced growth path depends on money growth as follows: (i)* $d\varepsilon_d/d\gamma > 0$ *and* $d\varepsilon_\ell/d\gamma < 0$*; (ii)* $dr_d/d\gamma > 0$ *and* $dr_\ell/d\gamma > 0$*; (iii)* $d(\omega m)/d\gamma < 0$*;(iv)* $dl/d\gamma < 0$ *and* $dg/d\gamma < 0$*; (v)* $dk/d\gamma > 0$*; and (vi)* $dq_s/d\gamma$ *is ambiguous.*

Higher money growth reduces the total real value of money and increases both the deposit rate and the loan rate, as in the economy with a perfect loan market. However, because there is a cost of intermediation, the two sides of the loan market are not affected equally by higher money growth. (to be completed: the explanation for the effects on ε_d and ε_l.)

The time input into intermediation increases with inflation. To explain this effect, it is necessary to examine how the amount of loans, relative to the money stock, changes with inflation. Denote the total nominal amount of loans as L. Using the loan market clearing condition, (37), we can compute the ratio of loans to the money stock as follows:

$$L/M \equiv -\int_0^{\varepsilon_d} \frac{1-k-l}{2}\frac{y(\varepsilon)}{m} dF(\varepsilon) = \int_0^{\varepsilon_d} \left(1-\frac{\varepsilon}{\varepsilon_d}\right) dF(\varepsilon). \tag{44}$$

Because ε_d increases with γ, the loan/money ratio increases with γ. The reason is that the household visits the loan market more often in order to reduce the exposure to the inflation tax. To administer and monitor this increased amount of loans relative to the money stock, the time input into intermediation increases with inflation.

Now we can explain why higher inflation reduces long-run growth. As resources are shifted to the loan market, the time input into innovation falls, which reduces the long-run growth rate.

It is also interesting to consider how the real value of loans for the household changes in γ. This quantity satisfies

$$-\omega \int_0^{\varepsilon_d} \left[(1-k-l)/2\right] y(\varepsilon)\, dF(\varepsilon) = \omega m \int_0^{\varepsilon_d} (1-\varepsilon/\varepsilon_d)\, dF(\varepsilon). \tag{45}$$

The real amount of loans, evaluated at the market price p, is

$$-\int_0^{\varepsilon_d} \left[(1-k-l)/2\right]\left[y(\varepsilon)/(pa)\right] dF(\varepsilon) = \frac{\omega m}{h'(q_s/a)} \int_0^{\varepsilon_d} (1-\varepsilon/\varepsilon_d)\, dF(\varepsilon). \tag{46}$$

The real value of loans (45) and the real amount of loans evaluated at market price (46) will be affected by changes in inflation as follows. On the one hand, as discusses earlier, increasing inflation decreases the real amount of money ωm. This effect reduces these two quantities ceteris paribus. On the other hand, as also discussed earlier, the nominal amount of loans taken out increases, i.e., $\int_0^{\varepsilon_d} (1 - \varepsilon/\varepsilon_d)\, dF(\varepsilon)$ increases. This effect increases these two quantities ceteris paribus. Which effect dominates is a matter of parameter values. There is an additional effect coming through the supply side in (46). Inflation reduces aggregate supply and since $h'(q_s/a)$ is convex this effect will increase the real amount of loans, evaluated at the market price p, ceteris paribus.

4 Quantitative Analysis

4.1 Parameters and descriptions of two experiments

We choose the model period as a quarter and use the following functional forms:

$$u(c) = \ln(c), \quad h\left(\frac{q_s}{a}\right) = \frac{h_0}{\eta}\left(\frac{q_s}{a}\right)^{\eta}, \quad f(l) = 1 + f_0 l^{\delta},$$
$$F(\varepsilon) \text{ is lognormal}$$

The forms of u and h have already been used in previous sections, while the form of f is a natural choice. For the distribution of taste shocks, there is no obvious way to calibrate it, and so we simply report findings for a log-normal distribution. With these functional forms, the parameters to be identified are as follows: (i) preference parameters: (β, η, h_0); (ii) technology parameters: (f_0, δ, κ); (iii) distribution parameters: mean μ and standard deviation σ; and (iv) policy parameters: the money growth rate, γ. This list contains nine parameters.

Table 1 lists the identification restrictions and the identified values of the parameters. Two parameters, (β, γ), are identified in "obvious" ways using the following observations about the postwar US data. The choice $\beta = 0.99$ yields an annual real interest rate of 4 per cent. The quarterly average of inflation gives $\gamma = 1.012$.

Table 1. Calibration and parameter values

parameters	identification restrictions	results
β	the real interest rate	0.99
γ	average quarterly inflation	1.012
κ	quarterly loan rate	0.44
μ	deposit rate	-0.2
σ	average value of ε	0.03
h_0	normalization	1.00
η	R&D expenditure/GDP	1.00238
δ	shopping time/working time	0.0204
f_0	real quarterly growth rate	0.0100238

In the following we discuss the calibration procedure. We first choose σ and μ to get an expected value for the preference chock of $\bar{\varepsilon} = 0.82$. We also explored other values. The changes were small and we therefore only report results for this target. The average quarterly loan rate is $r_\ell^* = 0.025$. We use r_ℓ^* and (35) to express (36) and (37) as functions of ωm and ε_d only. The solution to these two functions then yield a pair $(\omega m, \varepsilon_d)$. We can then use these values to derive r_d from (34). The average quarterly deposit rate is $r_d^* = 0.015$.[8] If $r_d \neq r_d^*$, we modify σ and μ (holding $\bar{\varepsilon} = 0.82$ constant) and repeat this procedure until we get r_d sufficiently close to r_d^*. This procedure thus yields $\varepsilon_d^* = 0.815$, $\varepsilon_\ell^* = 0.823$, and $\omega m^* = 0.802$.

There are four parameters still to be identified, (h_0, η, f_0, δ). While h_0 is normalized to one, the other three parameters are identified jointly with the following restrictions. First, from the time-use diary in Juster and Stafford (1991), we calculate the ratio of the shopping time to working time as 11.16%. Second, from the OECD data reported in Grossman and Helpman (1995, p10), the average ratio of R&D expenditure to GDP is about 2% in the US. To utilize these restrictions we first notice that the measure of agents who innovate is l, the measure of agents who intermediate is k, and the measure of agents who produce is $(1-k-l)/2$. Adding up these measures, we get the total measure of working agents as $(1+k+l)/2$. The measure of buyers is $(1-k-l)/2$, and so the ratio of shopping time to

[8] The sources of data for the loan rate is .The deposit rate is the 3-month deposit certificates interest rate. A certificate of deposit is a time deposit, which is offered by banks, thrift institutions, and credit unions. Deposit certificates are similar to savings accounts in that they are insured and thus virtually risk-free. They are different from savings accounts in that the CD has a specific, fixed term (often three months, six months, or one to five years), and, usually, a fixed interest rate.

working time in the model is $(1-k-l)/(1+k+l)$. Matching this ratio to 0.1116 yields $x^* \equiv k+l = 0.894298$.

Next, compute the real expenditure in innovation as $wl/p = \phi a f'(l)/(p\omega)$. Use (20) and (23) to rewrite this expenditure as $\left(\frac{\eta-1}{2\eta}\right) q_s l$. Dividing it by real GDP, $\left(\frac{1-k-l}{2}\right) q_s$, we get the ratio of R&D expenditure to GDP as $(\eta-1)l/[(1-x)\eta]$. Setting this ratio to 0.02 we can express $l(\eta)$ as a function of η. Next note that (39) also expresses $l(\eta)$ as a function of η. We thus can equate these two functions to get $\eta^* = 1.00238$. Combining (33) and (38) yields $\blacksquare = 0.444$. From (40) we then get $k^* = 0.0043$ and, since $l = x - k$, we have $l^* = 0.8899$.

The last steps are to set $h_0^* = 1$ and, then, to use (38) to get $(q_s/a)^* = 15.09$. Finally, we set the average real quarterly growth rate to 0.01. This number is consistent with the postwar US growth experience. By matching this number we can solve (1) and (39) for $\delta^* = 0.0204$ and $f_0^* = 0.0100238$.

Table 2 reports the values of some endogenous variables on the balanced growth path.

Table 2: Calibration variables

Variables	r_d	r_ℓ	ε_d	ε_ℓ	k	l	ωm	q^s/a
results	0.0152	0.0252	0.815	0.823	0.004	0.889	0.802	15.09

With the identified model, we conduct two experiments to examine the quantitative effects of a financial innovation and monetary policy. In both experiments, we assume that the economy is on the balanced growth path at the beginning of period 0, where the money growth rate is γ_0 and the parameter of the intermediation cost is κ_0. In the first experiment, an unanticipated financial innovation occurs at time 0, which is expected to reduce the intermediation cost as follows:

$$\kappa_t = \kappa^* + (\kappa_0 - \kappa^*)\,\rho^t, \text{ all } t \geq 0, \tag{47}$$

where $\kappa^* < \kappa_0$ and $\rho \in (0,1)$. That is, the financial innovation is expected to reduce the intermediation cost to κ^* on the new balanced growth path, but this reduction may be gradual. The parameter ρ measures the degree of persistence of the old financial intermediation technology. When ρ is close to 0, the reduction in the cost is swift; when ρ is close to 1, the reduction is slow.[9]

[9] To see this, it may be helpful to rewrite the process as $\kappa_{t+1} = (1-\rho)\kappa^* + \rho\kappa_t$, for all $t \geq 0$.

In the second experiment, an unanticipated change in the policy regime occurs at time 0, which is expected to reduce the money growth rate as follows:

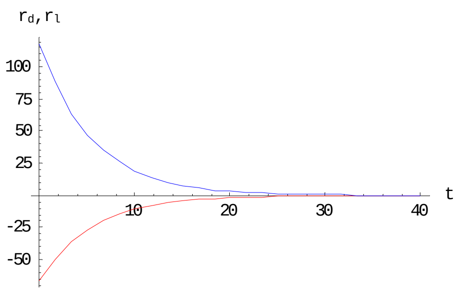
$$\gamma_t = \gamma^* + (\gamma_0 - \gamma^*)\,\rho^t, \text{ all } t \geq 0, \tag{48}$$

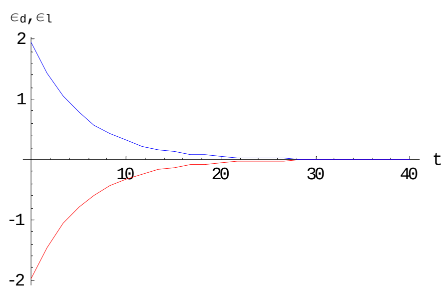
where $\gamma^* < \gamma_0$ and $\rho \in (0, 1)$. Again, γ^* is the money growth rate on the new balanced growth path and ρ is the degree of persistence of the old policy.

In both experiments, we will compute the transition of the economy to the new balanced growth path. Let x be an endogenous variable in the list $(\varepsilon_d, \varepsilon_\ell, r_d, r_\ell, k, l, g)$ and let x^* be its value on the balanced growth path. We will present the percentage deviation of x from its level on the balanced growth path, computed as $x_d(t)/x^*$ where $x_d(t) \equiv x_d(t) - x^*$. We choose the initial values to be $\kappa_0 = 5\kappa^*$ and $\gamma_0 = 1 + 5\,(\gamma^* - 1)$. Since $(\kappa_0 - \kappa^*)$ and $(\gamma_0 - \gamma^*)$ are small, the dynamics around the balanced growth path can be approximated by the corresponding linearized system, which is obtained by differentiating (D) with respect to κ_t and γ_t, and evaluating the derivatives along the balanced growth path. Given the initial conditions, the system can be solved using standard techniques such as those in Blanchard and Khan (1980).

4.2 Propagation of a cost shock in the financial sector

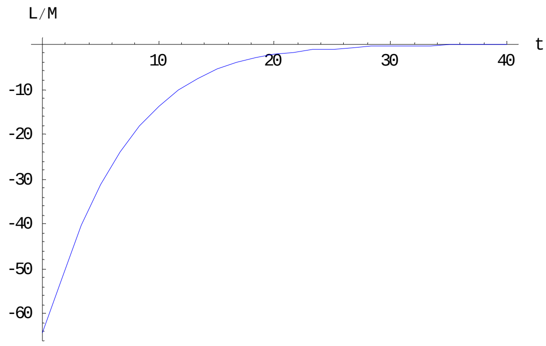
Let us first examine the propagation of a cost shock to κ. We set $\rho = 1/1.2$. Figures x depict the percentage changes of ε_d and ε_ℓ and Figure x r_d and r_ℓ in response to a decrease in κ at $t = 0$. It is obvious that such a decrease reduces the wedge between the two interest rates over time. In particular, the deposit rate is increasing and the borrowing rate is decreasing. The changes of the interest rates are not symmetric. The borrowing rate decreases by approximately 125% while the deposit rate increases by 70%. In all figures that follow the base is the steady state. For example, in the first figure the deposit rate is initially 70% smaller than the steady state interest rate, i.e., the calibrated interest rate.





During the transition ε_ℓ is decreasing and ε_d increasing. Mirrowing the interest rates, the borrowing bound decreases by approximately 2 per cent, while the deposit bound increases by 2 per cent.

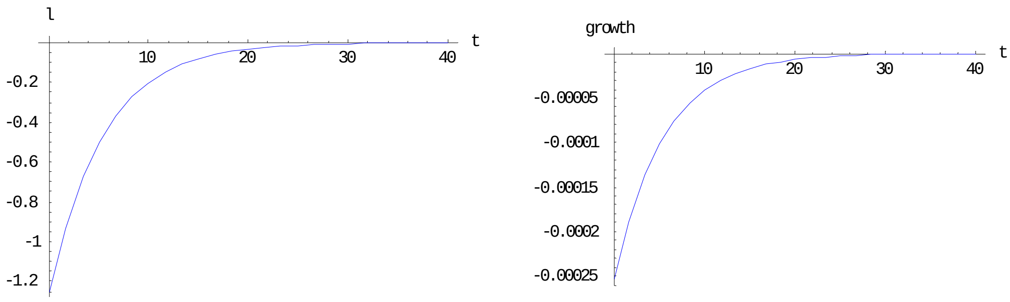
The reduced interest rate spread induces the household to visit the financial sector more often. As a consequence the ratio of the total amount of loans to the money stock, $(L/M)_t$, is increasing until the balanced growth path is attained where $\kappa_t = \kappa^*$. We call this process *financial deepening*. The increase is approximately 70 per cent as displayed in Figure x.



What is the effect on l and g. From a theoretic point of view the effect is ambiguous for the following reason. When κ_t decreases, participation in the financial market increases and, consequently, $(L/M)_t$ increases. As a result the quantity

$$\Psi \equiv \kappa_t \frac{\eta}{\eta - 1}(L/M)_t \tag{49}$$

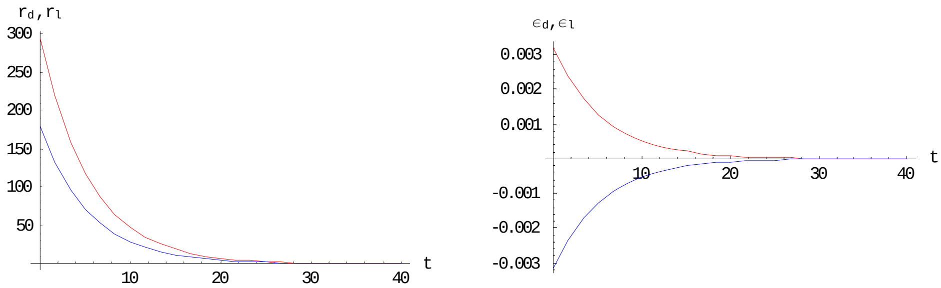
can either increase or decrease. Since Ψ is the right-hand side of equation (40) the implications for l and g are that they can either increase or decrease. However, as Figures x and x show, both l and g are increasing, meaning that the direct effect of the decrease in κ_t dominates the increase of $(L/M)_t$. Labor input into innovation increases by roughly 1.2%. The growth rate increases by 0.0002%. Thus, financial deepening yields a higher growth rate. Although the growth effect is small. The increase in growth arises because the household reallocates working time from the financial sector to the innovation sector.



4.3 Propagation of a monetary policy shock

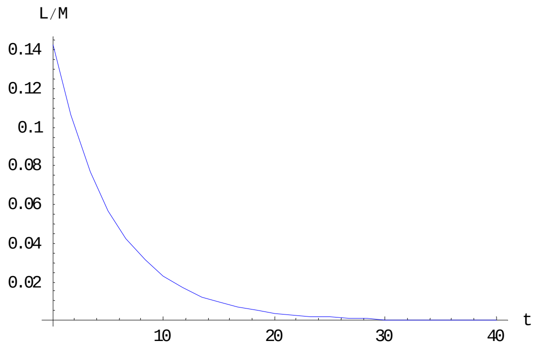
Let us next examine the propagation of a cost shock to γ. We set $\rho = 1/1.2$.

Figures x depict the percentage changes of ε_d and ε_ℓ and Figure x r_d and r_ℓ in response to a decrease in γ at $t = 0$. In contrast to the cost shock, both the deposit rate and the borrowing rate are decreasing. The changes of the interest rates are not symmetric. The borrowing rate decreases by approximately 300% and the deposit rate by 175%. Note that, since κ is constant in this simulation, the wedge between the two interest rates is constant over time.

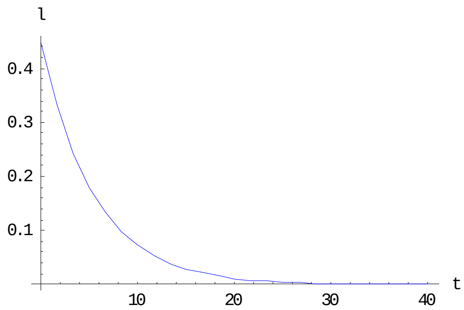


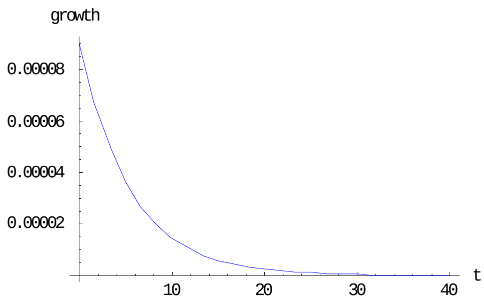
During the transition ε_ℓ is increasing and ε_d decreasing meaning that less agents visit the credit market over time. Mirrowing the interest rates, the changes of the bounds are not symmetric. The borrowing bound increases by approximately 0.003% while the deposit bound decreases by 0.003%.

The ratio of the total amount of loans to the money stock, $(L/M)_t$, is decreasing. The decrease is approximately 0.14% as displayed in Figure x.

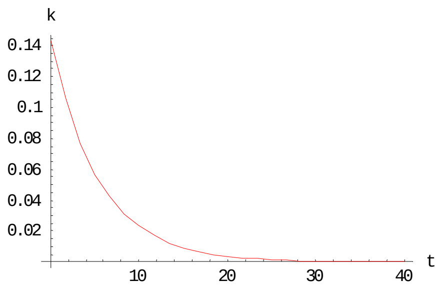


Figures x and x show, both l and g are decreasing. Labor input into innovation decreases by roughly 0.45%. The growth rate decreases by 0.00008%.





Finally, what is the effect on k? Figure x shows that k is decreasing by 0.14% per cent.



This explains why inflation has a positive effect on growth. Inflation reduces the benefit of trading. This drives agents either into the innovation sector or the financial sector. Since few people move to the financial sector, most people move into the innovation sector which increases the growth rate during the transition.

Appendix

Proof of Proposition 1. The first-order conditions and the envelope condition for a are

$$\begin{aligned} q(\varepsilon) &: \varepsilon u'[c(\varepsilon)]\frac{1-l}{2} = \pi && (50)\\ q_s &: \frac{1-l}{2}h'(q_s/a)/a = \pi && (51)\\ l &: -\frac{1}{2}\int_0^{\varepsilon_H}\varepsilon u'[c(\varepsilon)]q(\varepsilon)\,dF(\varepsilon) + \frac{1}{2}h(q_s/a) + \phi a f'(l) = 0 && (52)\\ a &: \frac{1-l}{2}h'(q_s/a)(q_s/a)/a + \phi[1+f(l)] = \phi_{-1}/\beta && (53)\end{aligned}$$

Combine (50) and (51) to get

$$\varepsilon u'[c(\varepsilon)] = h'(q_s/a)/a \tag{54}$$

Integrate (54) and use (4) to get

$$\bar{\varepsilon} = \frac{1-l}{2}\eta h(q_s/a) \tag{55}$$

Rewrite (53) as follows

$$a\phi = \frac{1-l}{2[1+f(l)]R}h'(q_s/a)(q_s/a)$$

Use this equation and (4) to rewrite (52) as follows

$$\frac{(1-l)f'(l)}{[1+f(l)]} = R\left(\frac{\eta-1}{\eta}\right) \tag{56}$$

Equation (56) yields l^*. The left-hand side is decreasing in l, approaches $+\infty$ as $l \to 0$ and approaches 0 as $l \to 0$. Hence, a unique value $l^* \in (0,1)$ exist. It then follows from (55) that a unique quantity q_s/a exist. Then, from (54) and (55), we get

$$q(\varepsilon) = (\varepsilon/\bar{\varepsilon})\,q_s.$$

■

Proof of Lemma 1. The household is indifferent between borrowing and not borrowing for a taste shock ε_ℓ that solves

$$\varepsilon_\ell u[c_\ell(\varepsilon_\ell)] - \omega[pq_\ell(\varepsilon_\ell) + r_\ell y(\varepsilon_\ell)][(1-k-l)/2] = \varepsilon_\ell u[c_0(\varepsilon_\ell)] - \omega p c_0(\varepsilon_\ell)$$

Using (27), (28), $a = A$ and $y(\varepsilon) = pq_\ell(\varepsilon) - x$ we can write this equation as follows

$$\varepsilon_\ell \ln\left[\frac{\varepsilon_\ell}{(1+r_\ell)\,\omega m}\right] = \varepsilon_\ell - (1+r_\ell)\,\omega m$$

The unique solution to this equation is $\varepsilon_\ell = (1+r_\ell)\,\omega m$.

For the lending decision we need to distinguish between $r_d > 0$ and $r_d = 0$. If $r_d > 0$, a household is indifferent between lending and not lending for a taste shock ε_d that solves

$$\varepsilon_d u\left[c_d(\varepsilon_d)\right] - \omega\left[pq_d(\varepsilon_d) + r_d y(\varepsilon_d)\right]\left[(1-k-l)/2\right] = \varepsilon_d u\left[c_0(\varepsilon_d)\right] - \omega p c_0(\varepsilon_d)$$

Again by using (27), (28), $a = A$ and $y(\varepsilon) = pq_d(\varepsilon) - x$ we can write it as follows

$$\varepsilon_d \ln\left[\frac{\varepsilon_d}{(1+r_d)\,\omega m}\right] = \varepsilon_d - (1+r_d)\,\omega m$$

The unique solution to this equation is $\varepsilon_d = (1+r_d)\,\omega m$. Note that if $r_d > 0$ buyers who do not go to the loan market are constraint in the good market.

If $r_d = 0$, the household is only lending for taste shocks that satisfy $m/p \geq \varepsilon/(p\omega)$, where $\varepsilon/(p\omega)$ is efficient consumption for taste shock ε. In this case he indifferent between any amount $y(\varepsilon)$ that satisfies $pq(\varepsilon) - y(\varepsilon) \leq x$. Then, (27), (28) implies

$$\varepsilon_d = \omega m.$$

The critical value ε_ℓ still solves $\varepsilon_\ell = (1+r_\ell)\,\omega m$. Finally, note that in both cases we need $\varepsilon_\ell = (1+r_\ell)\,\omega m \leq \varepsilon_H$. ■

Proof of Lemma 2. If $r_d > 0$, buyers spend all their money in the goods market. Consequently, we have

$$pq(\varepsilon) = x + y(\varepsilon) \tag{57}$$

For $\varepsilon \leq \varepsilon_d$ and $\varepsilon \geq \varepsilon_\ell$ we have $c_j(\varepsilon) = \frac{\varepsilon}{p\omega(1+r_j)}$, $j = d,\ell$, and so, from Lemma 1, $c_j(\varepsilon) = \frac{\varepsilon m}{p\varepsilon_j}$, implying that $q_j(\varepsilon) = \frac{\varepsilon}{\varepsilon_j}\frac{x}{p}$. Use this expression for $q_j(\varepsilon)$ to write (57) as follows: $p\frac{\varepsilon}{\varepsilon_j}\frac{x}{p} = x + y_j(\varepsilon)$. Then, solve for $y_j(\varepsilon) = x\left(\frac{\varepsilon}{\varepsilon_j} - 1\right)$. For $\varepsilon_d < \varepsilon < \varepsilon_\ell$, $y(\varepsilon) = 0$ and so $q_0(\varepsilon) = x/p$.

If $r_d = 0$, buyers with $\varepsilon \leq \varepsilon_d$ hold more money than what they need for efficient consumption. Accordingly, we have $c_d(\varepsilon) = \frac{\varepsilon}{p\omega}$. For $\varepsilon_d \leq \varepsilon \leq \varepsilon_\ell$ we still have $c_0(\varepsilon) = m/p$ and for $\varepsilon_\ell \leq \varepsilon$ $c_\ell(\varepsilon) = \frac{\varepsilon}{p\omega(1+r_\ell)}$. Using Lemma 1, we can rewrite these expressions as follows: $q_d(\varepsilon) = \frac{\varepsilon x}{p\varepsilon_d}$, $q_0(\varepsilon) = x/p$ and $q_\ell(\varepsilon) = \frac{\varepsilon x}{p\varepsilon_\ell}$. Use the expressions for $q_j(\varepsilon)$, $j = d,\ell$, to write

(57) as follows: $p\frac{\varepsilon}{\varepsilon_d}\frac{x}{p} = x + y_d(\varepsilon)$. Then, solve for $y_d(\varepsilon) = x\left(\frac{\varepsilon}{\varepsilon_d} - 1\right)$. Along the same lines, one can show that $y_\ell(\varepsilon) = x\left(\frac{\varepsilon}{\varepsilon_\ell} - 1\right)$.

To verify the properties in the lemma, suppose first that output grows at the rate g. Since output is proportional to $q(\varepsilon)$, then $q(\varepsilon)$ grows at the rate g. Similarly, consumption grows at the rate g. Consider $pq(\varepsilon) = x + y(\varepsilon)$: since $x + y(\varepsilon)$ grows at rate γ and since $q(\varepsilon)$ grows at the rate g p must grow at the rate $\gamma - g$. Then from the seller's first order condition $ap\omega = h'(q_s/a)$ we can see that ω decreases at rate γ. From (20) it is clear that ϕ must decrease at the rate g.

We first derive (40). Use Lemma 1 and (9) to get (40). To derive (38) use Lemma 1 to write (18) as follows

$$q_s = \int_0^{\varepsilon_d} q_d(\varepsilon)\, dF(\varepsilon) + \int_{\varepsilon_d}^{\varepsilon_\ell} q_0(\varepsilon)\, dF(\varepsilon) + \int_{\varepsilon_\ell}^{\varepsilon_H} q_\ell(\varepsilon)\, dF(\varepsilon)$$

Then using the expressions for $q(\varepsilon)$ in (31) yields $q_s = x/p\left[1 + \int_0^{\varepsilon_d}(\varepsilon/\varepsilon_d - 1)\, dF(\varepsilon) + \int_{\varepsilon_\ell}^{\varepsilon_H}(\varepsilon/\varepsilon_\ell - 1)\, dF\right.$ Note that if $r_d > 0$ we get $q_s = x/p$ since in this case $\int_0^{\varepsilon_d}(\varepsilon/\varepsilon_d - 1)\, dF(\varepsilon) + \int_{\varepsilon_\ell}^{\varepsilon_H}(\varepsilon/\varepsilon_\ell - 1)\, dF(\varepsilon) = 0$. If $r_d = 0$ we get $q_s < x/p$ since in this case $\int_0^{\varepsilon_d}(\varepsilon/\varepsilon_d - 1)\, dF(\varepsilon) + \int_{\varepsilon_\ell}^{\varepsilon_H}(\varepsilon/\varepsilon_\ell - 1)\, dF(\varepsilon) \leq 0$. The reason, why $q_s < x/p$, is that not all money is used for purchases. Then, use (20) to get (38).

We next derive (33)-(37). To get (33) use Lemma 1 to rewrite (11) as follows

$$wk = (r_\ell - r_d)\int_{\varepsilon_\ell}^{\varepsilon_H}[(1 - k - l)/2]\, y_\ell(\varepsilon)\, dF(\varepsilon)$$

Using Lemma 2 yields

$$wk = (r_\ell - r_d)\, m\int_{\varepsilon_\ell}^{\varepsilon_H}(\varepsilon/\varepsilon_\ell - 1)\, dF(\varepsilon)$$

Using (40) yields

$$w\kappa = (r_\ell - r_d)\, m$$

Finally, using (24) yields

$$\omega m(r_\ell - r_d) = [(\eta - 1)/2]\, h(q_s/a)\, \kappa$$

Equations (34) and (35) are implied by Lemma 1.

To derive (36), use Lemma 1 and (22) to rewrite (25) as follows

$$\frac{\omega_{-1}}{\beta}=\int_0^{\varepsilon_d}\frac{\varepsilon u'\left[c_d\left(\varepsilon\right)\right]}{p}dF\left(\varepsilon\right)+\int_{\varepsilon_d}^{\varepsilon_\ell}\frac{\varepsilon u'\left[c_0\left(\varepsilon\right)\right]}{p}dF\left(\varepsilon\right)+\int_{\varepsilon_\ell}^{\varepsilon_H}\frac{\varepsilon u'\left[c_\ell\left(\varepsilon\right)\right]}{p}dF\left(\varepsilon\right).$$

Then use (27) and (28) and replace $\frac{\omega_{-1}}{\omega}=\gamma$ to get

$$\frac{\gamma}{\beta}=\int_0^{\varepsilon_d}\left(1+r_d\right)dF\left(\varepsilon\right)+\int_{\varepsilon_d}^{\varepsilon_\ell}\varepsilon/\omega m dF\left(\varepsilon\right)+\int_{\varepsilon_\ell}^{\varepsilon_H}\left(1+r_\ell\right)dF\left(\varepsilon\right)$$

Multiply through by ωm and use (34) and (35) to get (36).

To derive (37) use Lemma 1 to rewrite (17) as follows:

$$\int_0^{\varepsilon_d}y_d\left(\varepsilon\right)dF\left(\varepsilon\right)+\int_{\varepsilon_d}^{\varepsilon_\ell}y_0\left(\varepsilon\right)dF\left(\varepsilon\right)+\int_{\varepsilon_\ell}^{\varepsilon_H}y_\ell\left(\varepsilon\right)dF\left(\varepsilon\right)=0.$$

Then use Lemma 2 to replace $y\left(\varepsilon\right)$ and rearrange to get (37).

To derive (39) rewrite (23) and (26) as follows

$$2\phi a = h\left(q_s/a\right)\left(\eta-1\right)/f'\left(l\right) \tag{58}$$

$$\frac{\phi_{-1}}{\beta} = \left[\left(1-k-l\right)/2\right]h'\left(q_s/a\right)\left(q_s/a\right)/a+\phi\left[f\left(l\right)+1\right] \tag{59}$$

Then, use (58) to substitute $2\phi a$ in (59) and rearrange to get

$$\frac{\phi_{-1}}{\phi\beta}=\frac{\left(1-k-l\right)f'\left(l\right)\eta}{\eta-1}+f\left(l\right)+1$$

Then, since $\phi_{-1}/\phi=g=f\left(l\right)+1$ we get (39). ■

Proof of Proposition 2. If $\kappa=0$, from (33)-(37) $r_d=r_\ell=\left(\gamma-\beta\right)/\beta$, $\varepsilon_d=\varepsilon_\ell=\bar{\varepsilon}$ and $\omega m=\bar{\varepsilon}\beta/\gamma$. Furthermore, from (40), $k=0$. Use this information to rewrite the remaining equations as follows.

$$\bar{\varepsilon}\beta/\gamma = \eta h\left(q_s/a\right)\left(1-l\right)/2 \tag{60}$$

$$\frac{f'\left(l\right)\left(1-l\right)}{1+f\left(l\right)} = R\left(\frac{\eta-1}{\eta}\right) \tag{61}$$

Equation (61) is identical to (5). Hence $l=l^*$. It the follows from (60) that

$$h(q_s^*/a)=\left(\gamma/\beta\right)h\left(q_s/a\right)$$

Hence $q_s \leq q_s^*$. Finally, existence of an unique equilibrium is established by the same arguments as in the proof of Proposition (x).

We now explore the dependence on money growth. (i) and (ii) are obvious. Then, (iii) follows from (39) since $k = 0$ and so l is independent of γ. Finally, (iv) follows from (36) since $\omega m = (\beta/\gamma)\,\bar{\varepsilon}$ and (v) from and (38). ■

Proof of Proposition 3. To be done ■

References

[1] Berentsen, A., Camera, G. and C. Waller, 2006, "Money, credit and banking," *Journal of Economic Theory*, forthcoming.

[2] Blanchard, O.J. and C.M. Kahn, 1980, "The solution of linear difference models under rational expectations," *Econometrica* 48, 1305-1311.

[3] Bureau of Labor Statistics, 2006, "Occupational employment and wages, May 2005," United States Department of Labor, Washington D.C..

[4] Grossman, G.M. and E. Helpman, 1995, *Innovation and Growth in the Global Economy*, MIT press, Cambridge, Massachusetts.

[5] Ireland, P., 1994, "Money and growth: an alternative approach," *American Economic Review* 84, 47-65.

[6] Juster, F.T. and F. Stafford, 1991, "The allocation of time: empirical findings, behavioral models, and problems of measurement," *Journal of Economic Literature* 24, 471 522.

[7] Lagos, R. and R. Wright, 2005, "A unified framework for monetary theory and policy evaluation," *Journal of Political Economy* 113, 463-484.

[8] Lucas, R.E., Jr., 1988, "On the mechanism of economic development," *Journal of Monetary Economics* 22, 3-42.

[9] Shi, S., 1997, "A divisible search model of fiat money," *Econometrica* 65, 75-102.

[10] Shi, S., 2001, "Liquidity, bargaining, and multiple equilibria in a search monetary model," *Annals of Economics and Finance* 2, 325-351.