

Industry Dynamics and Search in the Labor Market

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Abstract

The paper proposes a model of on- and off-the-job search that combines convex hiring costs and directed search. Firms permanently differ in productivity levels, their production function features constant or decreasing returns to scale, and search costs are convex in search intensity. Wages are determined in a competitive manner, as firms advertise wage contracts (expected discounted incomes) so as to balance wage costs and search costs (queue length). An important assumption is that a firm is able to sort out its coordination problems with their employees in such a way that the on-the-job search behavior of workers maximizes the match surplus. Our model has several interesting features. First, it is close in spirit to the competitive model, with a tractable and unique equilibrium, and is therefore useful for empirical testing. Second, the resulting equilibrium gives rise to an efficient allocation of resources. Third, the equilibrium is characterized by a job ladder: unemployed workers search for low-productivity, low-wage firms. Workers in low-wage firms search for firms slightly higher on the productivity/ ladder, and so forth up to the workers in the second most productive firms who only apply to the most productive firms. Finally, the model rationalizes empirical regularities of on-the-job search and labor turnover. First, job-to job mobility falls with average firm tenure and firm size. Second, wages increase with firm size, and wage growth is larger in fast-growing firms.

1 Introduction

In the real economy, firm-and industry dynamics play an important role. Firms are born, expand and contract. Resources are allocated from less productive to more productive firms, and thereby improve the allocation of resources. There is substantial evidence that reallocation of resources on firms is important for economic growth, and Baily, Hulten and Campbell (1992) argues the about half of overall productivity growth in the U.S. manufacturing in the 80ies can be attributed to this. Existing empirical evidence also shows that industry dynamics is associated with large worker flows, not only in and out of unemployment, but even more importantly as direct job to job movements (Haltiwanger, 1999; Foster et al., 2007; Bartelsman et al; 2005). Lentz and Mortensen (2005, 2006) decompose the effect of firm selection on the growth rate, and then estimate that it accounts for 58 percent of the growth rate

Several recent papers analyze models of industry dynamics (Hopenhayn, 1992; Hopenhayn and Rogerson, 1993, Melitz 2003, Klette and Kortum 2004). However, these papers typically do not take into account that the factor markets, and in particular the labor market, may contain frictions. (An exception here is Lents and Mortensen (2007), who do include a frictional labour market in a Klette-Kortum model of innovation-driven industry dynamics).

This paper studies the joint determination of worker flows and firm dynamics with on the job search. The model contains three key elements. First, it applies the *competitive search equilibrium* concept, initially proposed by Moen (1997). Thus, firms post wages and post a number of vacancies so as to minimize search -and waiting costs. Furthermore, the labor market is endogenously separated into submarkets so that in each submarket, all agents at the same side of the market are identical.

Second, we assume that firms have access to a search technology with *convex hiring costs* (Bertola and Cabalero, 1992; Bertola and Garibaldi, 2001). In the traditional search model (Mortensen and Pissarides, 1994) adjustment costs are linear. Together with constant returns to scale in production, this implies that the size of the firms typically is undefined. Our assumption of convex hiring costs allows firms with different productivity and with constant-returns to scale technology to coexist in the market.

Third, we follow Moen and Rosen (2004) and allow for *efficient contracting*. The contracts are thus designed so as to resolve any agency problems between employers and employees so that their joint income is maximized. In particular, this implies that the workers' on-the-job search behavior maximizes the joint surplus of the worker and the firm. This assumption simplifies the model enormously. Without this assumption, a worker's current wage will influence his search behavior. As shown by Shimer (2006), this opens up for multiple equilibria and generally makes on-the-job search models intractable.

From a theoretical prospective, our model is interesting because it includes, in a simple way, the effects of search frictions for industry dynamics. Furthermore, as wages are set in a competitive fashion, and contracts are efficient, the model is very close in spirit to general equilibrium models. The model gives rise to a (constrained) efficient allocation of resources, and hence is well suited as a benchmark for welfare analysis.

The equilibrium of the model is characterized by a sluggish employment growth toward

a steady-state employment level. Low productivity firms pay low wages, face high turnover rates, and grow slowly towards a steady state with low employment. More efficient firms pay higher wages, post more vacancies, and grow more quickly to a steady state with a higher employment level. The equilibrium features a job ladder: unemployed workers disproportionately search for firms with the lowest productivity. Workers employed in these firms, in turn, search only for firms with higher productivity. Hence our model easily explain a set of stylized facts about industry dynamics and worker flows: 1) productivity differences between firms are large and persistent, 2) workers move from low-wage to high-wage occupations, 3) more productive firms are larger and pay higher wages than less productive firms, 4) job-to-job mobility falls with average firm size and worker tenure, 5) wages increase with firm size, and 6) wages are higher in fast-growing firms .

Pissarides (1994) was the first paper that studied on-the-job search in a Diamond-Mortensen-Pissarides type of matching model. A more recent model of on-the-job search can be found in Kiyotaki and Lagos (2006), who study optimal assignment of workers to jobs in a model where matches differ in quality, but without entry of firms. Delacroix and Shi (2006) analyzes on-the-job search in an urn-ball type of model of the labor market, and also obtain a job ladder in a similar way as we do. However, in their model all agents on both side of the market are homogenous, and firms at most hire one worker, hence their model is ill suited to analyze industry dynamics.

The paper proceeds as follows. Section 2 briefly describes the empirical regularities we are interested and reviews the relevant literature. Section 3 introduces the structure of the model while sections 4 and 5 derive the main formulation of the model for different type of firms. Section 6 introduces the general equilibrium and spells out some key results. Section 7 presents the baseline simulation.

2 A Brief Look at some empirical regularities

We briefly review some of the key empirical regularities linked to industry dynamics and worker flows. We are certainly not meant to be exhaustive in this review, and the selection of facts outlined is closely associated to the theoretical approach we will propose in the rest of the paper.

1. *In any industry there is a large scale reallocation of input and output across producers.* The work of Davis and Haltiwanger (1999) summarize much of the literature on gross job flows that was carried during the last decade; they note that in the United States more than 1 in 10 jobs is created in a given year and more than 1 in 10 jobs is destroyed. Much of this reallocation reflects reallocation within narrowly defined sectors. Note that not only labor is being reallocated but also capital and output (Haltiwanger, 2000). A large fraction of the input and output gross creation is associated with entry of firms and a large fraction output and input gross destruction is associated to firm.
2. *Productivity differences between firms are large and persistent.* Bartelsman and Dome (2000) summarize most of the evidence based on longitudinal micro data on firm level

productivity differential. They clearly argue that the most significant finding of this vast literature is the heterogeneity across establishments and firms in productivity in nearly all industries examined. Bernard et al. (2003) report the distribution across plants of value added per worker relative to the overall mean, and show that a substantial number of plants have productivity either less than a fourth or more than four times the average. These differences are also very persistent over time. Danish data analysed by Mortensen (2007) provide similar patterns.

3. *More productive firms are larger and pay higher wages.* Wage differentials across observationally equivalent workers are both sizable and persistent. The employer size-wage effect is perhaps the strongest such stylized fact: larger firms or plants pay higher wages. The literature includes especially influential work by Krueger and Summers (1988) and Brown and Medoff (1989), and has been surveyed by Oi and Idson (1999).
4. *Job to job mobility falls with worker tenure.* There is a well established relationship between job duration and tenure. Farber (1999) carefully reviews this literature and reports that a monotonically declining survival rates is one of the most robust stylized facts in the labour market. Such monotonicity holds regardless of the reasons beyond job termination and is particularly significant for voluntary job to job movements.
5. *Job to job movements are associated to wage gains.* Bartel and Borjas (1978) early work showed that young men who quit experience significant wage gains compared both to job stayers and to their own wage growth prior to the job change. More recently Light (2005) summarize empirical evidence based on the National Longitudinal Survey of Youth. He shows that the typical worker holds about five jobs in the first 8 years of the career, but that workers vary considerably in their mobility rates. He also reports evidence that workers who change jobs voluntarily through a job to job transition receive significant contemporaneous wage boosts that, on average, are at least as large as the wage gains received by job stayers.
6. *and wages are higher in fast growing firms.* Belzil (2000) uses Danish data and shows that after controlling for individual and business cycle effects, job creation at the firm level is found to increase male wages.
7. *Wage differentials are associated to productivity differentials.* The evidence on this link is more scarce, since dataset able to observe output and wages for a variety of workers are not readily available. Iranzo et al. (2007) using Italian data show that the level of labour productivity is clearly associated to larger wages for both production and non production workers.

3 Structure of the model

The structure of our model is as follows

- Labor is the only factor of production. The labor market is populated by a measure 1 of identical workers. Individuals are neutral, infinitely lived, and discount the future at rate r . Unemployed workers have access to an income flow z , which may denote unemployment benefits, the value of leisure, or the income when self-employed. Workers search for jobs on and off the jobs at no cost (search intensity is given).
- The technology requires an entry cost equal to K . Conditional upon entry, the firm learns its productivity, which may be either low, y_1 , or high, y_2 , $y_1 < y_2$, with probabilities α and $1 - \alpha$, respectively. The productivity of the firm is a fixed effect throughout its life.
- Firms post vacancies and wages to maximize expected profits. Vacancy costs are convex in the number of vacancies posted, so that $c(v_i) = \frac{v_i^2}{2c}$, where c is a constant. We further discuss this assumption at the end of this section.
- Firms die at rate δ and workers exogenously leave the firm at rate s
- Search is directed. Firms face a relationship between the wage they set and the arrival rate of workers, which is derived from the indifference constraint of workers. Firms set wages so as to maximize profits given this relationship.
- Wage contracts are complete, and resolve any agency problems between employers and employees. In particular, the wage contract ensures efficient on-the-job search.

As we will see, in our model at most three submarkets operate in equilibrium. In submarket 1, low productivity firms are hiring from the unemployment pool. In Submarket 2, high productivity firms are hiring from workers employed in low productivity firms. Finally, there may be a submarket 3 in which high-productivity firms hire directly from the unemployment pool. In any submarket i the aggregate number of vacancies is equal to $V_i = f_i v_i$, where f_i is the measure of firms operating in that submarket and v_i is the number of vacancies per firm. If we assume a Cobb Douglas matching function with constant returns to scale and weight β on the workers, the transition rate for workers and for firms is

$$\begin{aligned} p_i &= \theta_i^{1-\beta} \\ q_i &= \theta_i^{-\beta} \end{aligned}$$

where θ_i is the ration of vacancies to searching workers in submarket i . Inverting the first of the previous condition one gets that $\theta_i = p_i^{\frac{1}{1-\beta}}$ so that the transition rate for vacancies can be expressed as

$$q_i = p_i^{-\frac{\beta}{1-\beta}} \quad (1)$$

The model relies on a convex hiring cost to prevent firms from posting an infinite number of vacancies upon entry in the labor market. This assumption can be justified along several dimension.

The measure v is closer to search effort from the firm standpoint rather than to a measure vacancies. When a firm double its search intensity it does not typically double the number of applicants, in a way similar to diminishing returns. The counterpart of this simple reasoning is a convex hiring cost.

Another way to justify convexity relies on labour market frictions linked to firm's optimal scale. Changing firm scale requires a costly look for talent that is not easily available in local markets. Our modeling strategy can be thus seen as a reduced form of this extremely costly search for talent.

At a more technical level, convex hiring costs can be seen as a generalization of Burdet Mortensen (1999) model with on the job search. In their model, the arrival of workers to firms is exogenously set. Our specification allows for some flexibility, and we let firms to increase this arrival rate through search effort. Note also that the structural estimates provided by Yashiv (2000a,b) are fully consistent with a marginal cost increasing in the stock of vacancies.

Finally, most models of endogenous search effort focus on the worker side. In such models (Pissarides, 2000) workers' cost of effort is typically model as convex function with respect to individual effort. Our function is the analogous approach on the firm side.

4 Submarket 1: Low-type firms and unemployed workers

In this section we analyze optimal search behavior of unemployed workers and low-productivity firms.

4.1 Search behavior of unemployed workers

Low-productivity firms only search for unemployed workers. We therefore start by analyzing the search behavior of unemployed workers. The asset valuation of an unemployed worker can be written as

$$rU_1 = z + p_1(W_1 - U_1)$$

where U_1 and W_1 are the continuation values of being unemployed and employed in submarket 1, respectively. In what follows, we define the expected rent from a job to the worker as the net gain obtain to the worker from a move from unemployment to a type 1 job, or similarly as the share of the joint surplus allocated to the worker, and its expression is

$$R_1 \equiv W_1 - U_1$$

Competitive search equilibrium requires that

$$\begin{aligned} rU_1 &= z + pR_1 \\ &= \text{cons} \end{aligned}$$

This equation is key in competitive search equilibrium, as it defines p as a function of R , $p = p(R)$ (for a given U). It follows that

$$rU_1 - z = p(R_1)R_1$$

Taking elasticities with respect to R_1 gives

$$\begin{aligned} 0 &= \varepsilon_{p,R} + 1 \\ \varepsilon_{p,R} &= -1 \end{aligned} \tag{2}$$

where $\varepsilon_{p,R} = \frac{dp}{dR_1} \frac{R}{p}$

4.2 Match surplus

The joint surplus of a firm and one of its workers is defined (in PDV terms) as the continuation value to the firm net of the workers and firms outside option. As the firm is free to open vacancies, the value of an abandoned job is zero, hence

$$S_1 = W_1 + J_1 - U_1 \tag{3}$$

$$= R_1 + J_1 \tag{4}$$

Where J_1 are the continuation values to the job of the the firm. As the wage acts as a pure transfer inside the firm worker pair, the value of the surplus is independent of the actual value of the wage. The net gain if the worker finds a job is given by $W_2 - W_1 - J_1 = W_2 - S_1 - U_1$, hence

$$rS_1 = y_1 - rU_1 + p_2[W_2 - U_1 - S_1] - (\delta + s)S_1 \tag{5}$$

where p_2 is the probability that the worker finds a job in high productivity jobs.

As an example of how to go from equation 3 to equation 5, assume a constant wage $\bar{w}_1$, and write down the asset value equations

$$\begin{aligned} rW_1 &= \bar{w}_1 + p_2[W_2 - W_1] + (\delta + s)[U_1 - W_1] - rU_1 \\ rJ_1 &= y_1 - \bar{w}_1 - (p_2 + \delta + s)J_1 \end{aligned}$$

which together gives (3).

When workers do on-the-job search, they choose between searching in submarkets with different combinations of wages W_2 and job finding rates. This may potentially give rise to excessive on-the-job search if the workers do not take into account that quitting may lead to a negative externality towards the firm as it loses J_1 . In the present model this is not an issue. We assume that the wage contracts are complete, and hence that the workers' on-the-job search behavior maximizes the joint match surplus S_1 . There are various wage contracts that implement this behavior. For example, the worker pays the firm $S_1 - R_1$ up front and then gets a wage equal to y_1 . Alternatively, the worker gets a constant wage and pays a quit fee equal to the continuation value of the firm J_1 if a new job is accepted (see

Moen and Rosen (2004) for more examples). Importantly, the wages paid to the worker in the current job do not influence her on-the-job search behavior.¹

Finally, let R_2 denote the net gain for the worker-firm pair obtained when the workers climbs to firm 2. In that case, the worker gains $W_2 - W_1$, while the firm loses J_1 . Thus

$$\begin{aligned} R_2 &\equiv W_2 - J_1 - W_1 \\ &= W_2 - U_1 - S_1 \end{aligned} \tag{6}$$

(by 3). It follows that we can write S_1 as

$$S_1 = \frac{y_1 - rU_1 + p_2R_2}{r + s + \delta} \tag{7}$$

4.3 The Firm's Maximization Problem

The key firm decision concerns the number of vacancies to be opened and the rent to be paid to each worker. Let $q(R_1)$ denote the relationship between the rents offered to the workers and the arrival rate of workers. The firms then solve

$$\begin{aligned} \text{Max}_{R_1, v_1} &= -\frac{v_1^2}{c} + v_1q(R_1)[S_1 - R_1] \\ \text{s.t.} \quad S_1 &= \frac{y_1 - rU_1 + p_2R_2}{r + \delta + s} \end{aligned} \tag{8}$$

and employment dynamics, contingent on the firm's continued existence is

$$\dot{N}_1 = v_1q(R_1) - (s + p_2 + \delta)N_1$$

As we have already pointed out, the worker's on-the-job search is set so as to maximize S_1 , independently of R . The first order conditions for the firms' maximization problem can thus be written as

$$\begin{aligned} \frac{v_1}{c} &= (S_1 - R_1)q(R_1) \\ -v_1q(R_1) + [S_1 - R_1]v_1q'(R_1) &= 0 \end{aligned}$$

The latter condition easily becomes

$$R_1 = \varepsilon_{q,R}[S_1 - R_1]$$

where $\varepsilon_{q,R} = \frac{dq}{dR} \frac{R}{q}$. The previous expression can be solved for a value of R_1 as

$$R_1 = \frac{\varepsilon_{q,R}S_1}{1 + \varepsilon_{q,R}} \tag{9}$$

¹It follows from this that a worker in a low-type firm will never search for a job in another low-type firm, as these cannot offer a wage that exceeds the productivity in the current firm.

Now

$$\begin{aligned}
\varepsilon_{q_1, R_1} &= -\frac{\beta}{1-\beta} p(R_1)^{-\frac{\beta}{1-\beta}-1} p'(R_1) \frac{R_1}{q} \\
&= -\frac{\beta}{1-\beta} p(R_1)^{-1} p'(R_1) R_1 \\
&= -\frac{\beta}{1-\beta} \varepsilon_{p, R_1}
\end{aligned}$$

Since competitive search implies equation 2, we get

$$\varepsilon_{q, R_1} = \frac{\beta}{1-\beta} \quad (10)$$

Since competitive search implies equation (2), we get

$$\varepsilon_{q, R_1} = \frac{\beta}{1-\beta} \quad (11)$$

Inserted into (9) this gives

$$R_1 = S_1 \beta$$

To repeat, the first order conditions to the firm's maximization problem writes

$$R_1 = S_1 \beta \quad (12)$$

$$\frac{v_1}{c} = (1-\beta) S_1 q(\beta S_1) \quad (13)$$

$$S_1 = \frac{y_1 - rU_1 + p_2 R_2}{r + \delta + s} \quad (14)$$

At the firm level the system solves for v_1 , R_1 and S_1 while p and R_2 are taken as given. The value of a firm that enters the market with zero workers and post v_1 vacancies reads

$$\begin{aligned}
\Pi_1(0, v_1) &= \text{gain from search-costs of vacancies} \\
&= \frac{1}{r + \delta} \left\{ q(\beta S_1) v_1 S_1 (1-\beta) - \frac{v_1^2}{2c} \right\}
\end{aligned}$$

The first term refers to the gain from search. In words, a firm that posts v_1 vacancies filled them with probability q and enjoys a fraction $(1-\beta)$ of the full surplus. The second term refers to the quadratic cost of vacancies. Using the expression for the vacancies obtained in (13), the value of the profits is

$$\begin{aligned}
\Pi_1(0, S_1) &= \frac{1}{r + \delta} \left\{ [S_1(1-\beta)q(\beta S_1)]^2 c - \frac{[S_1(1-\beta)q(\beta S_1)]^2 c^2}{2c} \right\} \\
&= \frac{1}{r + \delta} \frac{[S_1(1-\beta)q(\beta S_1)]^2 c}{2}
\end{aligned}$$

so that a firm that is hiring N workers has

$$\begin{aligned}\Pi_1(N, S_1) &= NS^1 + \Pi_1(0, v_1) \\ \dot{N}_1 &= v_1 q(R_1) - (s + p)N_1\end{aligned}$$

To summarize, firms in this submarket post a measure of vacancies v_1 independently of their employment status. This feature follows directly from the constant returns to scale assumptions. Workers leave the firm exogenously at rate s and the firm may go bankrupt at rate δ . Firms in this submarket face also turnover risk in terms of a direct job transition to a high productivity firm at rate p_2 . In light of the perfect contract structure of our economy, such event is not associated with inefficient separation. Firms in that survive reach a steady state situation in which hiring $v_1 q(p_1)$ exactly match separation $l(s + p_2)$. Note that also in steady state the firm is characterized by continuous job turnover, even though employment does not grow. The wage contract and the associated rent R_1 offered and posted by the firm is also constant throughout the life of the firm and does not feature any transitional dynamics.

5 Submarkets with High Productivity Firms

A high-productivity firm may choose to direct its search towards unemployed workers or workers employed in low-type jobs. We will discuss the two cases in turn

5.1 Employed worker search

In this subsection we derive the optimal wage policy when the high-type firms search for workers employed in low-type firms. We proceed in the same way as in the previous section. Due to efficient contracting, we know that workers in a low-type job will behave so as to maximize S_1 . It follows that for all firms that attract workers, we have that will maximize the value of the match. So that

$$\begin{aligned}(r + \delta + s)S_1 &= y_1 - rU_1 + p_2 R_2 \\ &= \text{const}\end{aligned}$$

which defines $p_2 = p_2(R_2)$. Solving for $p_2 R(p_2)$ and taking elasticities of both sides give

$$\varepsilon_{p_2, R_2} = -1$$

Let J_2 denote the asset value of a filled job, and define S_2 as the

$$\begin{aligned}S_2 &\equiv J_2 + R_2 \\ &J_2 + W_2 - W_1 - J_1\end{aligned}$$

where we have used (6). Note that

$$(r + \delta + s)(J_2 + W_2) = y_2 + (\delta + s)U_1$$

It follows that

$$S_2 = \frac{y_2 - rU}{r + s + \delta} - S_1$$

Firms decide the number of vacancies to opened and the rents to attach to them. In other words, the firms' problem is

$$Max_{R_2, v_2} = -\frac{v_2^2}{c} + v_2 q_2(R_2)[S_2 - R]$$

(note that S_1 is exogenous to the firm). The first order condition with respect to v is

$$\frac{v_2}{c} = (S_2 - R_2)q_2$$

The condition with respect to R_2 is

$$-vq(R_2) + [S_2 - R_2]vq'_2(R_2) = 0$$

this gives a value of R as

$$\begin{aligned} R_2 &= \frac{\varepsilon_{q_2, R_2} S_2}{1 + \varepsilon_{q_2, R_2}} \\ &= \beta S_2 \end{aligned}$$

analogous to the expression for R_1 in (12). To summarize, the first order conditions are given by

$$\begin{aligned} R_2 &= \beta S_2 \\ \frac{v_2}{c} &= (1 - \beta)S_2 q_2(\beta S_2) \\ S_2 &= \frac{y_2 - rU_i}{r + \delta + s} - S_1 \end{aligned}$$

The value of a type 2 firm is

$$\begin{aligned} \Pi_2(0, S_2) &= \frac{1}{r + \delta} \left\{ [S_2(1 - \beta)q(\beta S_2)]^2 c - \frac{[S_2(1 - \beta)q(\beta S_2)]^2 c^2}{2c} \right\} \\ &= \frac{1}{r + \delta} \frac{[S_2(1 - \beta)q(\beta S_2)]^2 c}{2} \end{aligned}$$

To summarize, firms in this submarket post a measure of vacancies v_2 independently of their employment status. The measure of vacancy is proportional to productivity y_2 and is thus likely to be larger than the vacancy posted by firms in submarket 1. Workers leave the firm exogenously at rate s and the firm may go bankrupt at rate δ . In this submarket firms *do not* face the turnover risk associated to a direct job transition. Workers in such job enjoy

the best job in the economy and do not search any longer. Firms in that survive reach a steady state situation in which hiring $v_2 q(p_2)$ exactly match separation ls . Note that also in steady state the firm is characterized by continuous job turnover, even though employment does not grow. The wage contract and the associated rent R_2 offered and posted by the firm is also constant throughout the life of the firm and does not feature any transitional dynamics.

5.2 Unemployed-worker search

Finally we will specify the behavior of high-type firms searching for unemployed workers. Let U_2 denote the continuation value of unemployed workers searching for a high-type job, $\tilde{W}_2$ the continuation value of an employed worker, and define the rent of finding a job as $\tilde{R}_2 \equiv \tilde{W}_2 - U_2$. The indifferent constraint of unemployed workers read

$$\begin{aligned} r\tilde{U}_2 &= z + \tilde{p}_2 \cdot \tilde{R}_2 \\ &= \text{const} \end{aligned}$$

which defines a unique relationship between $\tilde{R}_2$ and $\tilde{p}_2$, which we write $\tilde{p}_2(\tilde{R}_2)$. (Note that when $U_1 = U_2$, which is always the case in equilibrium, this relationship is equal to $p_1(R_1)$ defined above). Write

$$r\tilde{U}_2 - z = \tilde{p}_2(\tilde{R}_2)\tilde{R}_2$$

Taking the elasticity with respect to $\tilde{R}_2$ gives

$$\varepsilon_{p, \tilde{R}^2} = -1$$

Define $\tilde{S}_2$ as

$$\begin{aligned} \tilde{S}_2 &= J_2 + R_2 \\ &= J_2 + W_2 - J_1 - W_1 \end{aligned}$$

Now

$$(r + \delta + s)(J_2 + W_2) = y_2 + (\delta + s)\tilde{U}_2$$

which gives

$$\tilde{S}_2 = \frac{y_2 - r\tilde{U}_2}{r + \delta + s}$$

Again, the key firm decisions concern the number of vacancies to be opened and the rent to be paid to each worker. In other words the firm problem is

$$Max_{R_2, v_2} = -\frac{v_2^2}{c} + v_2 q(\tilde{R}^2)[S_2 - \tilde{R}_2]$$

By proceeding in exactly the same way as above we get that the first order conditions are given by

The three first order condition are

$$\begin{aligned}\tilde{R}_2 &= \beta \tilde{S}_2 \\ \frac{v_2}{c} &= (1 - \beta) S_2 q(\beta S_2) \\ \tilde{S}_2 &= \frac{y_2 - r \tilde{U}_2}{r + \delta + s}\end{aligned}$$

The value of a type 2 firm searching for unemployed workers is

$$\begin{aligned}\tilde{\Pi}_2(0, \tilde{S}_2) &= \frac{1}{r + \delta} \left\{ [\tilde{S}_2(1 - \beta)q(\beta \tilde{S}_2)]^2 c - \frac{[\tilde{S}_2(1 - \beta)q(\beta \tilde{S}_2)]^2 c^2}{2c} \right\} \\ &= \frac{1}{r + \delta} \frac{[\tilde{S}_2(1 - \beta)q(\beta \tilde{S}_2)]^2 c}{2}\end{aligned}$$

6 General Equilibrium

In order to close the model, a set of equilibrium conditions has to be satisfied. The first regards high-productivity firms and unemployed workers. First, productive firms choose optimally what type of workers to search for. In any equilibrium, at least some high-type firms search for employed workers (if not, q_2 would be infinite for any $R_2 > 0$, ensuring an efficient firm infinite profit). Let τ denote the fraction of the firms that search for employed workers. We require that

$$\begin{aligned}\Pi_2 &\geq \tilde{\Pi}_2 \text{ for all } \tau \\ \Pi_2 &= \tilde{\Pi}_2 \text{ if } \tau < 1\end{aligned}$$

Second, unemployed workers must be indifferent as to which type of firms they apply to, hence

$$U_1 = U_2 = U^*$$

Finally, free entry of firms will ensure that the number of firms f adjusts so that there are zero profit *ex ante*, which means that

$$\alpha \text{Max}[\Pi_1; 0] + (1 - \alpha) \text{Max}[\Pi_2; 0] = K$$

Definition 1 *The general equilibrium is given by a vector of value function and job finding rates $\{U^*, p_1^*, p_2^*, \tilde{p}_2^*\}$ and a vector of market quantities $\{u_1, \tilde{u}_2, u_2, n_1, n_2, \tau, f, g\}$ satisfying*

- optimal vacancy and rent posting by firm 1 and 2 in different submarkets
- optimal search for the unemployed and the employed workers

- free entry of firms
- indifference across operating submarkets of firms and workers

When deriving the equilibrium of the model, two different scenarios will be consider, with and without an active third market.

6.1 Equilibrium with All Submarkets

Suppose first that the third market is operative. The determination of $\{U^*, p_1^*, p_2^*, \tilde{p}_2^*\}$ is given by this system

$$\begin{aligned}
\alpha\Pi_1 + (1 - \alpha)\Pi_2 &= k && \text{Free Entry of firm} \\
\Pi_2 &= \tilde{\Pi}_2 && \text{Firm Indifference across submarkets} \\
rU_1 &= r\tilde{U}_2 && \text{Unemployed indifference across submarkets} \\
rU^* &&& \text{Value of Unemployment}
\end{aligned}$$

Or, written out,

$$\begin{aligned}
\alpha \frac{[S_1(1 - \beta)p_1^{-\frac{\beta}{1-\beta}}]^2 c}{2} + (1 - \alpha) \frac{[S_2(1 - \beta)p_2^{-\frac{\beta}{1-\beta}}]^2 c}{2} &= K \\
\frac{[\tilde{S}_2(1 - \beta)\tilde{p}_2^{-\frac{\beta}{1-\beta}}]^2}{2} &= \frac{[S_2(1 - \beta)p_2^{-\frac{\beta}{1-\beta}}]^2}{2} \\
\tilde{p}_2 \tilde{S}_2 &= p_1 S_1 \\
rU &= z + p_1 \beta S_1
\end{aligned}$$

where

$$\begin{aligned}
S_1 &= \frac{y_1 - rU_1 + p_2 \beta S_2}{r + \delta + s} \\
S_2 &= \frac{y_2 - rU_1}{r + \delta + s} - S_1 \\
\tilde{S}_2 &= \frac{y_2 - r\tilde{U}_2}{r + \delta + s}
\end{aligned}$$

This system can the be solved for $\{p_2, \tilde{p}_2, p_1, U\}$. In order to characterize the equilibrium, we utilize that all firms searching in the same submarket offers the same wage. Hence

$$\begin{aligned}
p_1^* &= \left[\frac{(1 - \alpha)fv_1}{u_1} \right]^{1-\beta} \\
p_2^* &= \left[\frac{\tau \alpha f v}{n_1} \right]^{1-\beta} \\
\tilde{p}_2^* &= \left[\frac{(1 - \tau)\alpha f \tilde{v}_2^*}{\tilde{u}_2} \right]^{1-\beta}
\end{aligned}$$

Let n_1 , n_2 and $\tilde{n}_2$ denote the measure of workers working in low-type firms, in high-type firms hiring workers from low type firms, and high-type firms hiring unemployed workers, respectively. Furthermore, let u_1 and $\tilde{u}_2$ denote the measure of unemployed workers searching for low-type and high-type jobs, respectively. Steady state in each submarket requires that inflows equal outflows, or

$$\begin{aligned} p_1 u_1 &= (\delta + s + p_2) n_1 \\ p_2 n_1 &= (\delta + s) n_2 \\ \tilde{p}_2 \tilde{u}_2 &= (\delta + s) \tilde{n}_2 \\ 1 &= u_1 + \tilde{u}_2 + n_1 + n_2 + \tilde{n}_2 \end{aligned}$$

Proposition 2 *The equilibrium satisfies the following property: $W_2 > \widetilde{W}_2 > W_1$.*

Proof: In all submarket, the wage is set so that the indifference curve of the workers and the iso profit constraint of the firms are tangent. It is easy to show that at any given point in W, θ -space, the indifference curve of employed workers is steeper than the indifference curve of unemployed workers, reflecting that unemployed workers are more eager to find jobs. Furthermore, as a type two-firm in equilibrium is indifferent as to which submarket to enter, it follows that $(\widetilde{W}_2, \tilde{\theta}_2)$ and (W_2, θ_2) are at the same iso-profit curve. From a revealed-preference type of argument it follows easily that $W_2 > \widetilde{W}_2$. Furthermore, the iso profit-curve of a high-type firm at any given point in the W, θ -space is flatter than that of the low-type firm, reflecting that high-type firms are more eager to speeding up the hiring process than are low-productivity firms. Since unemployed workers are indifferent as to what firm to apply to, a simple-revealed preferences type of argument delivers that $\widetilde{W}_2 > W_1$.

Corollary 3 *Unemployed workers have no incentives to join the submarket for employed workers.*

Proof: Suppose the unemployed workers obtained $U' \geq U^*$ by searching in the employed-search submarket. Since the indifference curve of the low-type worker is flatter in the (W, θ) space than that of the high-type workers, it follows from a revealed-preference type of argument that a type-2 firm could increase its profits by offering a lower wage, and hence obtain a higher profit than when offering $\widetilde{W}_2$, a contradiction.

We can also show the following result

Proposition 4 *There exists a value α^* such that for any $\alpha \geq \alpha^*$, $\tau \geq 1$, while for all $\alpha \in (0, \alpha^*)$, $0 < \tau < 1$.*

Sketch of proof: For any number $\varepsilon > 0$. Consider a firm that sets $w = y_1 + \varepsilon$. As $\alpha \rightarrow 1$, the arrival rate of workers to this firm goes to infinity, independently of which wage $w \in (y_1, y_2)$ the other high-type firms choose. Thus profits go to infinity. If a high-type firm searches for unemployed workers, the arrival rate of workers to the firm will be bounded, and hence also profit. The claim thus follows. By a similar argument, it also follows that at least some high-type firms search for employed workers as long as $\alpha > 0$.

6.2 Pure Job Ladder Equilibrium

Suppose now that $\alpha > \alpha^*$ so that the equilibrium describe above does not hold because the definition required τ , the proportion of firms hiring directly from the employed to be strictly less than one. We obtained thus a pure job ladder equilibrium since high productivity firms hire only from the employment pool. The determination of $\{U^*, p_1^*, p_2^*, \}$ alongside the distribution of employment across states $\{n_1, n_2, u\}$ and the size of entry f is given by the following 7 equation system

$$\begin{aligned}\alpha\Pi_1 + (1 - \alpha)\Pi_2 &= k \\ rU^* &= z + \beta p_1 S_1 \\ p_1 &= \left[\frac{(1 - \alpha)f v_1}{u_1} \right]^{1-\beta} \\ p_2 &= \left[\frac{\alpha f v}{n_1} \right]^{1-\beta} \\ p_1 u_1 &= (\delta + s + p_2) n_1 \\ p_2 n_1 &= (\delta + s) n_2 \\ u_1 + n_1 + n_2 &= 1\end{aligned}$$

7 Baseline Simulation

Table 1 and 2 report the baseline simulation of the model. The idea of the baseline simulation is mainly to show how an increase in the share of low productivity firms α lead the economy to move toward a pure job ladder equilibrium. Indeed, the only parameter that changes between Tables 1 and 2 is α . The parameters are consistent with most calibration in the existing literature. The pure interest rate r is 0.03, the turnover rate s is 0.02 while the firm bankruptcy rate is 0.01. The productivity level in low type firm is set to a baseline reference value of $y_1 = 1$, while the premium for the high type is 15 percent, so that $y_2 = 1.15$. The flow value of unemployment z is 0.6, not far from the replacement rate observed in real life labour markets. The matching function is Cobb Douglas with an elasticity β equal to 0.5. The parameter of the search cost is 0.06, while the entry cost k is 3.2 a value roughly equal to three times the output produced by a low productivity job.

The equilibrium allocation is described in the central part of Table 1. The job finding rate for unemployed workers p_1 is the largest among the various job finding rates, but the bulk of workers in the labor market is employed in type 2 firms. Indeed, type 2 firms absorb 50 percent of total employment. As a result, the submarket $\hat{2}$, albeit significant, represents a fringe of the entire economy.

The labor market features unemployment flows and job to job flows that are comparable in absolute magnitude, and the *job ladder* mechanism is clearly present in the simulated

economy. Workers start out in low productivity firms and eventually graduate to high type jobs through on the job search. Eventually, firm and match specific shocks at rate δ and s induce another round of job ladder.

The bottom part of the Table 1 features also an important relationship between firm size and firm wages, where the latter are measured in terms of PDV wages. Clearly, high type firms are larger in size and pay higher wage.

Note that in the baseline specification of Table 1 the equilibrium value of τ is very close to one and as a result the submarket $\tilde{2}$ is very small. A small increase in α lead to an equilibrium value of $\tau > 1$, a value that is not consistent with all three submarkets being operative. In other words, as α is increased with respect to the value assigned in Table 1, the economy moves to a pure job ladder equilibrium. This is reported in Table 2. In moving from Table 1 to Table 2 α increases from 0.944 to 0.966, suggesting that α^* in our numerical example is inside this small interval. The economy described in Table 2 does look very similar to that described in Table 1, even though two only submarkets are operative. Note also that the equilibrium value of unemployment U^* does slightly fall as α increases. This is not surprising since in a pure job ladder equilibrium the number of high productivity firms is lower.

Table 1: Baseline Simulation with three submarkets

Parameter	Notation	Value
Pure Discount Rate	r	0.020
Separation Rate	s	0.025
Firm Bankruptcy Rate	δ	0.010
Bargaining Share	β	0.500
entry cost	k	3.100
low type proportion	α	0.944
high type productivity	y_1	1.000
low type productivity	y_2	1.150
unemployed income	z	0.600
search cost parameter	c	0.060
matching function parameter	A	3.000
matching function elasticity	β	0.500
<i>Equilibrium Values</i>		
surplus match low type unemployed 1	S_1	1.282
surplus match high type employed 2	S_2	1.101
surplus match high type unemployed	$\tilde{S}_2$	2.383
unemployment value	U	1.019
unempl. job finding rate in low type	p_1	0.654
on the job finding rate	p_2	0.162
unempl. job finding rate directly to high type	$\tilde{p}_2$	0.352
<i>Equilibrium Quantities</i>		
Unemployment submarket 1	u_1	0.051
Employment in Low productivity type	n_1	0.168
Employment in High productivity type	n_2	0.780
Unemployment submarket $\tilde{2}$	$\tilde{u}_2$	0.000
Total workers in submarket $\tilde{2}$	g	0.001
Number of Firms	f	0.391
Proportion of high type firms in submarket $\tilde{2}$	τ	0.998
<i>Worker Flows</i>		
Unemployment Flows	$u_1 * p_1$	0.033
Job to Job Flows	$n_1 * p_2$	0.027
<i>Firm Size and PDV Wages</i>		
Low type Firm Size	N_1	0.178
High type Firm Size in submarket $\tilde{2}$	$\tilde{N}_2$	6.453
High type Firm Size	N_2	13.966
Wages in Low type Firm	W_1	1.660
Wages in High type Firm in submarket $\tilde{2}$	$\tilde{W}_2$	2.210
Wages in High type Firm in submarket 2	W_2	2.851
<i>Source: Authors' calculation</i>		

8 APPENDIX:

8.1 Notes on specifying the firm problem

The value of the profits are

$$\begin{aligned}
 \Pi &= \int_0^\infty [N(t)y - \int_0^{N(t)} w_a(t) da - \frac{v^2}{2c}] e^{-(r+\delta)t} dt \\
 s.t. \quad N(0) &= N_o \\
 s.t. \quad \dot{N} &= vq(.) - (s+p)N(t)
 \end{aligned}$$

Table 2: Baseline Simulation with two submarkets

Parameter	Notation	Value
Pure Discount Rate	r	0.020
Separation Rate	s	0.025
Firm Bankruptcy Rate	δ	0.010
Bargaining Share	β	0.500
entry cost	k	3.100
low type proportion	α	0.962
high type productivity	y_1	1.000
low type productivity	y_2	1.150
unemployed income	z	0.600
search cost parameter	c	0.060
matching function parameter	A	3.000
matching function elasticity	β	0.500
<i>Equilibrium Values</i>		
surplus match low type unemployed 1	S_1	1.261
surplus match high type employed 2	S_2	1.191
unemployment value	U	1.015
unempl. job finding rate in low type	p_1	0.658
on the job finding rate	p_2	0.142
<i>Equilibrium Quantities</i>		
Unemployment submarket 1	u_1	0.050
Employment in Low productivity type	n_1	0.188
Employment in High productivity type	n_2	0.762
Number of Firms	f	0.396
<i>Worker Flows</i>		
Unemployment Flows	$u_1 * p_1$	0.033
Job to Job Flows	$n_1 * p_2$	0.027
<i>Firm Size and PDV Wages</i>		
Low type Firm Size	N_1	0.195
High type Firm Size	N_2	20.046
Wages in Low type Firm	W_1	1.646
Wages in High type Firm in submarket 2	W_2	2.871
<i>Source: Authors' calculation</i>		

8.2 Low Productivity Firm

Let's think of a model initially with two productivity y_1 is the lowest end and y_2 is the upper end. The expected profits to the firms are

$$\int_0^\infty e^{-(r+\delta)t} \left\{ \tilde{y} - \frac{v^2}{2c} - Rvq(R) + p[W_1^2 - U]N \right\} dt$$

$$\begin{aligned} \pi_1 &= (y_1 - rU)N_t - \frac{v^2}{2c} - Rvq(R) + p[W_1^2 - U]N \\ s.t. \quad \dot{N} &= vq(R) - Ns - pN \end{aligned}$$

where

$$R = W_o^1 - U$$

Define

$$\tilde{y} = y - rU$$

We can write a Hamiltonian associated to the problem

$$H = \tilde{y} - \frac{v^2}{2c} - Rvq(R) + p[W_1^2 - U]N + \lambda[vq(R) - N(s + p)]$$

The controls are R and v and the state is N . The first order condition with respect to v is

$$\frac{v}{c} = (\lambda - R)q$$

this conditions says that the marginal cost of a vacancy is equal to the expected value to the firm.

The condition with respect to R is

$$-vq(R) + [\lambda - R]vq'(R) = 0$$

which becomes

$$R = \varepsilon_{q,R}[\lambda - R]$$

this gives a value of R as

$$R = \frac{\varepsilon_{q,R}\lambda}{1 + \varepsilon_{q,R}}$$

The first order condition for the workers (assuming steady state and thus $\dot{\lambda} = 0$) is

$$\begin{aligned} (r + \delta)\lambda &= \frac{\partial H}{\partial N} \\ (r + \delta)\lambda &= \tilde{y} + p[W^2 - U] - (s + p)N \end{aligned}$$

$$\begin{aligned} (r + \delta + s)\lambda &= \tilde{y} + p(W^2 - U - \lambda) \\ (r + \delta + s + p)\lambda &= \tilde{y} + pR^2 \end{aligned}$$

where

$$R^2 = W^2 - U - \lambda$$

so the three foc are

$$\begin{aligned} \frac{v}{c} &= (\lambda - R)q \\ R &= \frac{\varepsilon_{q,R}\lambda}{1 + \varepsilon_{q,R}} \\ (r + \delta + s)\lambda &= \tilde{y} + pR^2 \end{aligned}$$

At the firm level the system solves for v , R and λ while p and R^2 are taken as given.

8.3 High-productivity firm

It is very similar but it does not have the value of on the job search

$$\begin{aligned}\pi_2 &= (y_2 - rU - (r + s + \delta)\lambda_1)N - \frac{v^2}{2c} - R_2 vq(R_2) \\ s.t. \quad \dot{N} &= vq(R_2) - Ns\end{aligned}$$

where as above

$$R_2 = W_2 - U - \lambda_1$$

Define

$$\tilde{y} = y - rU - (r + s + \delta)\lambda_1$$

So the Hamiltonian reads

$$H = \tilde{y}N - \frac{v^2}{2c} - R_2 vq(R_2) + \lambda_2[vq(R) - Ns]$$

The first order condition with respect to v

$$\frac{v}{c} = (\lambda_2 - R_2)q$$

The condition with respect to R_2 is

$$-vq(R_2) + [\lambda_2 - R_2]vq'(R_2) = 0$$

this gives a value of R as

$$R_2 = \frac{\varepsilon_{q,R_2}\lambda_2}{1 + \varepsilon_{q,R_2}}$$

where one should note that q is a different function than the one of the low productivity firm

The first order condition for the workers (assuming steady state and thus $\dot{\lambda}_2 = 0$) is

$$\begin{aligned}(r + \delta)\lambda_2 &= \frac{\partial H}{\partial N} \\ (r + \delta)\lambda_2 &= \tilde{y}_2 - sN\end{aligned}$$

$$\begin{aligned}(r + \delta + s)\lambda_2 &= \tilde{y} \\ (r + \delta + s)\lambda_2 &= \tilde{y}\end{aligned}$$

9 References

- Bartelsman, Eric J. and Mark Doms (2000). Understanding productivity: Lessons from longitudinal microdata. *Journal of Economic Literature* 38, no. 3: 569: 594
- Bartelsman, Eric; Scarpetta, Stefano and Schivardi, Fabiano, (2005) "Comparative Analysis of Firm Demographics and Survival: Micro-level Evidence for the OECD countries " *Industrial and Corporate Change*, Vol. 14, pp. 365-391, 2005.
- Baily, Martin Neil, Charles Hulten and David Campbell. (1992) "Productivity Dynamics in Manufacturing Establishments," *Brookings Papers on Economic Activity: Microeconomics*, 1992, 187-249.
- Belzil, Christian, (2000). "Job Creation and Job Destruction, Worker Reallocation, and Wages," *Journal of Labor Economics*, University of Chicago Press, vol. 18(2), pages 183-203,
- Bertola and Cabalero, (1994); Cross Sectional Efficiency and Labour Hoarding in a Matching Model of Unemployment, *Review of Economic Studies*, Vol 61(3) 435-457
- Bertola, G. and Garibaldi, P. (2001), "Wages and the Size of Firms in Dynamic Matching Models." *Review of Economic Dynamics* 4 (2001): 335-368
- Davis, Steven J., and John Haltiwanger (1999) "Gross Job Flows," in Ashenfelter and Card.(eds.), *Handbook of Labor Economics*, North-Holland, 1999, 2711-2805.
- Delacroix, Alan and Shi, Shouyong (2006), "Directed Search On the Job and the Wage Ladder", *International Economic Review* 2006, vol 47, 651-699
- Hopenhyn, Hugo. (1992) "Entry, Exit, and Firm Dynamics in Long Run Equilibrium," *Econometrica*, 60(5), 1127-50.
- Hopenhyn, Hugo, and Richard Rogerson. (1993) "Job Turnover and Policy Evaluation: A General Equilibrium Approach," *Journal of Political Economy*, 101(5), 915-38.
- Melitz (2003) The Impact of Trade on Intra-Industry Reallocations and Aggregate Industry Productivity *Econometrica* 71
- Nobuhiro Kiyotaki & Ricardo Lagos, (2006). "A model of job and worker flows," Staff Report 358, *Federal Reserve Bank of Minneapolis*
- Foster Lucia; Haltiwanger, John; and Chad Syverson (2007). "Reallocation, Firm Turnover and Efficiency: Selection on Productivity or Profitability?" *University of Maryland*.
- Klette TJ and S Kortum (2004), " Innovating Firms and Aggregate Innovation", *Journal of Political Economy*, 112, 986–1018
- Lane, Julia; Haltiwanger, John and Spletzer, James (1999) "Productivity Differences Across Employers: The Role of Employer Size, Age, and Human Capital," *American Economic Review*, 94-98.
- Restuccia Diego , and Richard Rogerson (2007): Policy distortions and aggregate productivity with heterogenous plants. *NBER Working Paper* 13018
- Moen, Espen (1997) "Competitive Search Equilibrium". *Journal of Political Economy* 1997, vol 105, 385- 411
- Moen Espen, and Åsa Rosen (2004) "Does poaching distort training?" *Review of Economic Studies*, 2004 vol. 71, 1143-1162.

Lentz, Rasmus, and Mortensen, Dale, (2005) "Productivity Growth and Worker Reallocation" *International Economic Review*, Vol. 46, No. 3, pp. 731-749,

Lentz, Rasmus, and Mortensen, Dale (2006), "An Empirical Model of Growth through Product Innovation". Unpublished paper.

Lentz, Rasmus, and Mortensen, Dale (2007), "Labor Market Friction, Firm Heterogeneity, and Productivity". Unpublished paper.

Mortensen Dale., and Pissarides, Christopher Albert 1994, Job Creation and Job Destruction in the Theory of Unemployment, *Review of Economic Studies* Vol 61(3) 397-415

Oi, Walter and Idosn, Todd (1999) Firm Size and Wages, in *Handbook of Labor Economics*, Vol 3. O. Ashenfleter and D. Card eds, 2165-2214, Elsevier Science, Amsterdam

Pissarides (1994), Search Unemployment with On-the-job-search, *Review of Economic Studies*, Vol 61(3) 457-494

Shi (2006)

Shimer (2006), On-the-job Search and Strategic Bargaining, *European Economic Review*, 50(4): 811-830.

Yashiv, Eran (2000a) The Determinants of Equilibrium Unemployment, *American Economic Review*, Vol. 90 No. 5, 1297-1322

Yashiv, Eran (2000b) Hiring as Investment Behavior, *Review of Economic Dynamics*, Vol. 3, 486-522.