

The Interplay Between Incentives and Communication Complexity in Mechanism Design

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The attached short paper outlines the main project. However, the talk will be based on an extended version of the paper, which is not yet written up but for which I have most of the results. Therefore, I also attach slides of a recent talk.

COMMUNICATION COMPLEXITY AND MECHANISM DESIGN

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Abstract

The curse of the Revelation Principle is that it leads us to focus on unrealistic mechanisms in which agents report all private information to the principal, who then makes all decisions centrally. This is not true when communication costs are introduced. Then partial communication of information, sequential back-and-forth conversations, and decentralization of decisions become part of optimal mechanisms. This paper looks at the interplay between the incentive constraints and the communication constraints. When can they be separated, so that e.g. one can separately study the incentive compatibility of a social choice rule and the minimum-cost communication protocol that realizes the social choice rule? (JEL: D82, C72)

1. Introduction

The Revelation Principle in mechanism design is both a blessing and a curse. It is a blessing because it so greatly simplifies the design of optimal mechanisms and allows one to abstract from the details of the mechanism and focus on the social choice rule to be implemented. It is a curse because direct mechanisms provide such an unrealistic picture of decision-making in organizations. Only in the simplest of cases, such as private-values auctions of a single good, does it ring true that the privately-informed agents reveal all their information to a central authority, which then determines all decisions according to a pre-specified contract.

The obvious explanation of why organizations are run by intricate and indirect flows of partial information is that it would be too complex for everyone to communicate all information to one person and for this one person to process and use this information. The integration of mechanism design and complexity considerations using formal models of complexity has been labeled *algorithmic mechanism design* in the field of theoretical computer science, where this area of research has recently received considerable attention.

There are two aspects to this complexity: communication complexity and computational complexity. Both capture certain limits on human brainpower. In the case of communication, the main bottlenecks are not the networks, mail systems, or other infrastructures that transmit information from one human to another, but rather the processing

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ability of the humans, particularly on the receiving end.

This note looks at communication complexity in mechanism design. One motive for studying communication (rather than computation) complexity is convenience. Communication is modeled by a message game—just like our mechanisms—and so we do not have to add new machinery to the description of a mechanism; we only add a measure of communication complexity. In contrast, to model computation, we need to bring in models of information processing that are not currently part of mechanism design and that tend to be unfamiliar to economists. The two parts do not fit neatly together.

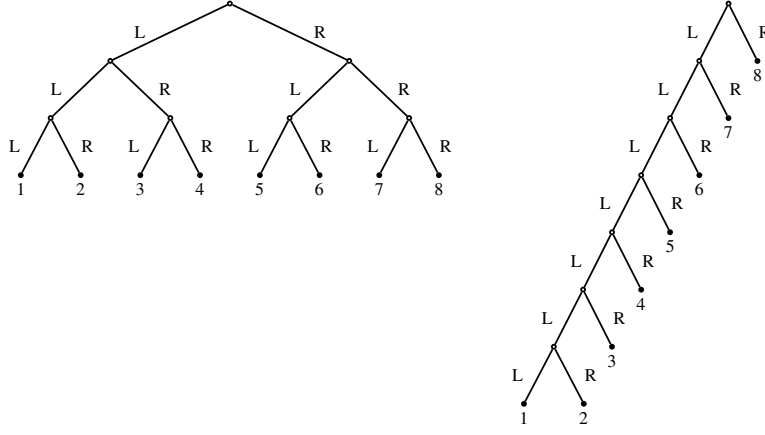
The main theme of this note is to consider when communication complexity and incentive constraints interact in an interesting (and complicated) way. We show that, for ex post Nash implementation, there is a nice separation between the checking of incentive compatibility constraints of a social choice function and the design of a communication-efficient protocol to compute the social choice function. This separation does not exist for the other notions of incentive compatibility that we consider. One implication of the separation is that (when it holds) incentive constraints do not increase the communication complexity of computing a social choice function. However, this implication is not as strong as it sounds. For example, in a model with quasilinear preferences, if the principal does not care about the transfers, then the cost of computing a decision rule for the non-transfer part of an outcome is increased when there are incentive constraints—due to the need to compute also transfers that make the social choice function incentive compatible (Fadel and Segal (2006)).

Besides the point outlined in that paragraph, this note is also intended to be a brief introduction to the modeling of communication complexity in mechanism design. The focus is on formal models of such complexity, and we do not attempt to review the papers that have already studied mechanism design with communication constraints. There are quite a few, though still a small part of the mechanism design literature. Seminal papers include Green and Laffont (1986, 1987), Melumad, Mookherjee, and Reichelstein (1992, 1997), and Laffont and Martimort (1998). See Poitevin (2000) and Mookherjee (2006) for surveys. Nor will we attempt to survey the literature on communication complexity that does not take into account incentives (which includes both work in computer science and early work on message space size in economics). A recent contribution in this area that is also a good introduction and guide to the literature is Nisan and Segal (2006).

2. Elements of a model

Let's fix notation for some standard components of static mechanism design. ("Static" refers to the fact that agents receive private information once and decisions are made at just one point in time, as opposed to a flow of information and decisions. The mechanisms may still involve dynamic message games.) Exogenous parts of the model including the following.

- Set $N = \{1, \dots, n\}$ of agents.
- $\forall i \in N$: set Θ_i of "types". (Let $\Theta = \Theta_1 \times \dots \times \Theta_n$.)
- Set X of outcomes.
- Preferences $x \succsim_{\theta}^i x'$ for each i that depend on the state $\theta \in \Theta$.

FIGURE 1. Two binary trees for communicating $\theta \in \{1, \dots, \Theta\}$.

- (If Bayesian) utilities $u_i: X \times \Theta \rightarrow \mathbb{R}$.
- (If Bayesian) probability measure p on Θ .

We then determine endogenously the following.

- A social choice function $f: \Theta \rightarrow X$.
- A mechanism, which is an extensive game form Γ with terminal histories assigned to outcomes in X . Actions in the mechanism are called messages.
- A plan of action for player i specifies the message that player i sends at each information set. Let S_i be the set of plans of action, with typical element s_i .
- A strategy σ_i for player i specifies each message as a function of the player's type, i.e., $\sigma_i: \Theta_i \rightarrow S_i$. Let Σ_i be the set of strategies.
- For a profile s of plans of action, let $\lambda(s)$ be the outcome at the terminal node reached when players use s .
- A pair (Γ, σ) , where Γ is a mechanism and σ is a strategy profile, is called a *protocol*. A protocol (Γ, σ) “computes” a s.c.f. f , defined by $f(\theta) = \lambda(\sigma_1(\theta_1), \dots, \sigma_n(\theta_n))$.

3. One-way communication complexity: bits

This note will not get into the finer details of measuring communication complexity. We will merely outline the most common measure when communication is discrete: bits.

For this purpose, suppose that there is a single player, so that Θ is that player's type space. Suppose that a player reports the value of $\Theta = \{1, \dots, 8\}$. Communication is represented by a binary tree with 8 leaves, as in Figure 1.

There are different trees that can be used to communicate θ . Each tree can be represented by a binary tree with leaves Θ , and so the set of possible communication protocols is the set $\mathcal{T}(\Theta)$ of such binary trees. For $T \in \mathcal{T}(\Theta)$, let $\delta(\theta, T)$ be the depth of leaf θ in T .

If we measure complexity by maximum path length (“worst-case”, or $\max_{\theta \in \Theta} \delta(\theta, T)$), then complexity is minimized by a balanced binary tree (left side of Figure 1). If instead we measure complexity by the expected path length, $\sum_{\theta \in \Theta} p(\theta) \delta(\theta, T)$, then for non-uniform distribution on the set of types, we may want to use an imbalanced tree (such as the one on the right side of Figure 1), such that the most likely types require shorter messages.

4. The advantage of interactive communication

In a direct mechanism, all players simultaneously report their entire types to a central authority. There is none of the back-and-forth dialogue that exists in organizations.

It is easy to illustrate the advantage of dialogue when we want to reduce communication complexity. For example, suppose that there are two players, 1 and 2. The players must choose which of two paths to go down to find a hidden treasure, *Left* or *Right*. Player 2 has two clues, a blue card and a red card, each of which has as the name of a path. Player 1 knows which of the two cards has the name of the path with the treasure. Thus, $\Theta_1 = \{Blue, Red\}$ and $\Theta_2 = \{Left, Right\}$ ². If the two players simultaneously announce their messages, then to reveal enough information to determine the correct path each must reveal its type, which amounts to one bit of information from player 1 and two bits from player 2. One bit can be economized by having player 1 first announce his information—which of player 2’s clues contains the name of the path—and then player 2 announces only the name of the path written on that clue.

5. Incentives and complexity

The main purpose of this note is to consider the interaction between communication complexity and incentive compatibility.

What would it mean for there to be no interaction? This would be true if

1. an economist can examine a social choice function and determine whether it is incentive compatible (in some sense) without considering the details of a communication protocol, and then
2. a computer scientist can design a minimum-complexity communication protocol (with some restrictions) that realizes the social choice function, without worrying about incentive constraints.

The separation we are looking for is a converse of the Revelation Principle, as illustrated in Figure 2 and now explained. Fix a s.c.f. f .

- Let (Γ, σ) be a protocol that computes f .
- Let (Γ^D, σ^D) be the direct mechanism with truthful revelation that computes f .

Given some notion of incentive compatibility, the Revelation Principle says that (Γ^D, σ^D) is IC if (Γ, σ) is IC. Suppose that a converse holds: If (Γ^D, σ^D) is IC then (Γ, σ) is IC compatible. Then we get the separation alluded to above. The economist checks the IC constraints of f by looking at a direct mechanism with truthful revelation. The computer

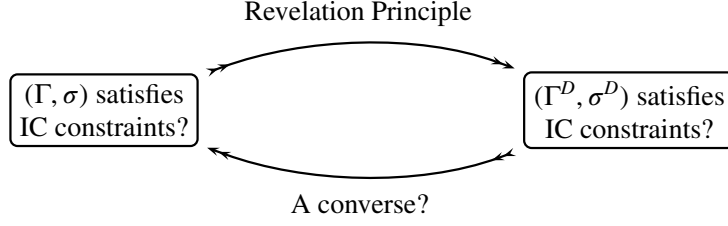


FIGURE 2. “Separation” is a converse of the Revelation Principle.

scientists then designs the communication-efficient protocol. According to the converse, whichever protocol the computer scientists comes up with will satisfy the IC constraints.

A further consequence of the separation is that incentive constraints do not increase the communication complexity of computing a given s.c.f. f . The IC constraints merely limit the set of feasible social choice functions.

6. Counter example: Dominant strategy implementation

We start with a counterexample, which helps illustrate the above.

Consider a private-values environment and suppose that we are interested in dominant-strategy implementation. That is, the incentive constraint that the economist imposes on a protocol is that each player’s plan of action for each of the player’s types be a dominant strategy for the player in the extensive-form game.

Suppose there are two players, 1 and 2. Suppose that $\Theta_1 = \Theta_2 = \{H, T\}$ and $X = \{X_{HH}, X_{HT}, X_{TH}, X_{TT}\}$. Let f be the s.c.f. such that $f(\theta_1, \theta_2) = X_{\theta_1\theta_2}$. Suppose that player 1’s preferences are as follows:

- When $\theta_1 = H$: $X_{HT} \succ_{1H} X_{TT} \succ_{1H} X_{HH} \succ_{1H} X_{TH}$.
- When $\theta_1 = T$: $X_{TH} \succ_{1T} X_{HH} \succ_{1T} X_{TT} \succ_{1T} X_{TH}$.

Then f is DS incentive compatible. The DS-IC constraints are

- $X_{HT} \succ_{1H} X_{TT}$ and $X_{HH} \succ_{1H} X_{TH}$;
- $X_{TH} \succ_{1T} X_{HH}$ and $X_{TT} \succ_{1T} X_{TH}$.

Consider the mechanism in Figure 3. The strategy profile “each player chooses his type” yields the s.c.f. f , and it has 2 bits of communication—the minimum. However, for player 1 when of type H , choosing H is not a best response to player’s 2 plan of action “ H following H and T following T ”. For player 1, choosing T leads to the outcome X_{TT} whereas choosing H leads to the outcome X_{HH} , yet $X_{TT} \succ_{1H} X_{HH}$.

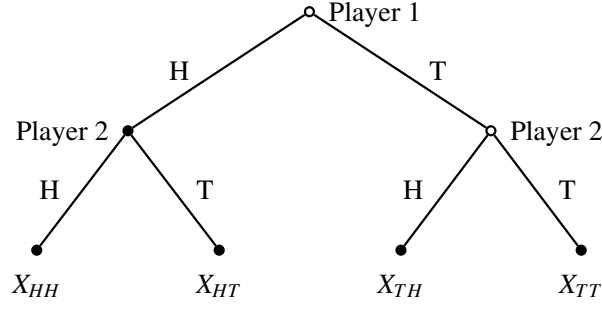


FIGURE 3. A counterexample for dominant-strategy implementation.

7. Ex-post implementation

For direct mechanisms in a private-values setting, ex-post Nash equilibrium is the same as dominant-strategy equilibrium. This is not true for indirect mechanisms.

In games of complete information, dominant-strategy equilibrium takes care of a robustness consideration: it does not require the coordination that leads to common knowledge of expectations or of equilibrium strategies. In games of incomplete information, dominant-strategy equilibrium takes care of another robustness consideration: that beliefs may not be common knowledge or, in the case of mechanism design, may not be known to the mechanism-designer. Ex-post Nash equilibrium takes care only of this second one: in contrast to Bayesian Nash equilibrium, it is belief-free.

Consider, in our previous example, the plan of action by player 2 to which player 1's strategy was not a best response. That plan of action was not chosen by any type of player 2 given the protocol, yet dominant-strategy equilibrium requires that 1's action be a best response to it. In contrast, in ex-post Nash, we only require that player 1's plan of action be a best response to those plans of action chosen by some type of player 2. This is true in the previous example.

Let's review what we are trying to show. Ex-post Nash equilibrium in a direct mechanism means that, for any type of player 2, player 1's payoff from choosing his type is as high as his payoff from reporting a different type. In ex-post Nash equilibrium in a general mechanism, the condition is that, for the plan of action of any type of player 2, player 1's payoff from choosing the plan of action of his type is as high as choosing any other plan of action. The latter follows from the former as long as "choosing any other plan of action" is the same, in terms of outcomes, as "choosing the plan of action of some other type of player 1". That is, as long as deviating in the indirect mechanism means acting as if one were of a different type.

This is true (we will show) as long as all leaves in the extensive game form of the protocol are reached for some profile of types. We call a protocol with this property *parsimonious*.

DEFINITION 1. A protocol is *parsimonious* if every leaf is reached for some state.

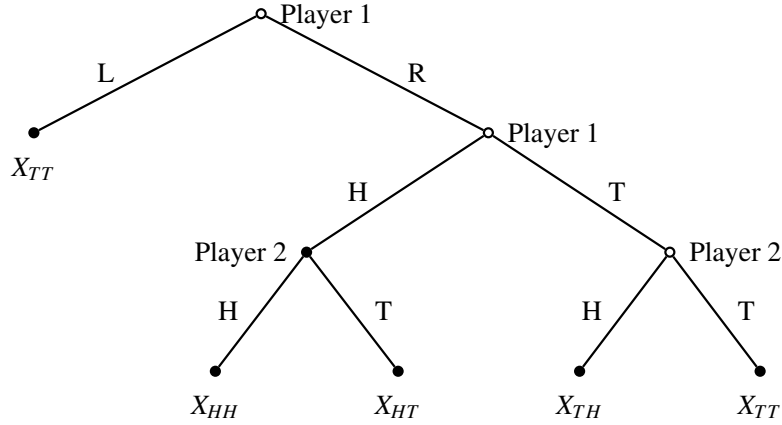


FIGURE 4. A non-parsimonious protocol.

We illustrate it with a counterexample, which is a modification of our previous example. The extensive game form is shown in Figure 4. The strategies are the following.

- Player 1. Type H : R then H . Type T : R then T .
- Player 2. Type H : Always H . Type T : Always T .

This protocol yields the same social choice function as in the previous example. It is not parsimonious because the left-most leaf is not reached for any profile of types. It does not satisfy the ex-post Nash IC constraints: for when player 1 is type H and player 2 is type H , choosing L (leading to outcome X_{TT}) is better than following the protocol (leading to outcome X_{HH}).

A non-parsimonious protocol can always be “pruned” to yield a parsimonious protocol that yields the same social choice function. First eliminate all unreached edges (and the final node of each such edge) in the extensive game form (this can be done in one step, rather than iteratively). This may leave some nodes with a single edge leading from them (each one is part of an information set at which the player has a single possible message and hence does actually not have any choice to make); each such edge can be eliminated, collapsing the two nodes connected by the edge. In a game with perfect recall (to which we restrict attention), such pruning naturally preserves the information structure of the game.

The pruned game is simpler by several measures. First, it may reduce (but never increase) the worst-case communication complexity. Second, it reduces the average communication complexity for any probability measure on the profiles of types. Third, it reduces the complexity of describing the protocol. Thus, a computer scientist designing minimum-complexity protocols would naturally restrict her attention to parsimonious protocols.

We have been using examples with private values, but our separation result for ex-post implementation holds without such an assumption.

PROPOSITION 1. *Suppose a protocol is parsimonious and suppose it realizes a s.c.f. that is ex post implementable. Then the strategy profile is an ex post Nash equilibrium of the mechanism.*

Proof. Let (Γ, σ) be the protocol and let f be the s.c.f. realized by the protocol. Let i be one of the players. The ex-post IC constraints for player i for the s.c.f. f are

$$\forall \theta_i, \theta_{-i}, \theta'_i : \quad f(\theta_i, \theta_{-i}) \succsim_{(\theta_i, \theta_{-i})}^i f(\theta'_i, \theta_{-i}). \quad (1)$$

The ex-post IC constraints for player i for an indirect protocol are

$$\forall \theta_i, \theta_{-i}, s'_i : \quad \lambda(\sigma_i(\theta_i), \sigma_{-i}(\theta_{-i})) \succsim_{(\theta_i, \theta_{-i})}^i \lambda(s'_i, \sigma_{-i}(\theta_{-i})). \quad (2)$$

Let's examine equation (2) for a particular θ , θ_{-i} , and s'_i . Assuming that the indirect protocol computes f , the l.h.s. of equation (2) is $f(\theta_i, \theta_{-i})$. Now consider the r.h.s. Since the protocol is parsimonious, there is some state θ' such that the terminal node $\lambda(s'_i, \sigma_{-i}(\theta_{-i}))$ is reached. That is,

$$\lambda(s'_i, \sigma_{-i}(\theta_{-i})) = \lambda(\sigma_i(\theta'_i), \sigma_{-i}(\theta'_{-i})) \quad (3)$$

If two pairs (s_i, s_{-i}) and (s'_i, s'_{-i}) of plans of action reach the same terminal node, then (s_i, s'_{-i}) and (s'_i, s_{-i}) also reach that node. (Each pair reaches the terminal node if and only if, at each information set on the path to that node, each individual plan of action chooses the action that continues along that path.) Thus,

$$\lambda(\sigma_i(\theta'_i), \sigma_{-i}(\theta'_{-i})) = \lambda(\sigma_i(\theta'_i), \sigma_{-i}(\theta_{-i})). \quad (4)$$

Again, since the indirect mechanism computes f ,

$$\lambda(\sigma_i(\theta'_i), \sigma_{-i}(\theta_{-i})) = f(\theta'_i, \theta_{-i}). \quad (5)$$

Now use equations (3), (4), and (5) to replace the r.h.s. of equation (2) by $f(\theta'_i, \theta_{-i})$. Remember also that the l.h.s. is $f(\theta_i, \theta_{-i})$. Thus, this IC is equivalent to

$$f(\theta_i, \theta_{-i}) \succsim_{(\theta_i, \theta_{-i})}^i f(\theta'_i, \theta_{-i}).$$

This is just one of the IC constraints for the direct mechanism, in equation (1), which are assumed to hold. $\square$

8. Nash and SPE implementation

Recall that the information structure in Nash and SPE implementation is that all agents have the same information. Therefore, without incentive problems, efficient communication is for one agent to announce the desired outcome. Incentive constraints make this impossible. There is no separation and incentive constraints may greatly increase the communication complexity of computing a social choice function. SPE is probably better than Nash, however, because SPE takes advantage of interactive communication.

9. Bayesian mechanism design

Nash and SPE implementation can be viewed as a special case of Bayesian mechanism design (with perfect correlation of types), so we already have examples showing that incentives can increase communication complexity. Fadel and Segal (2006) contains further examples. Fadel and Segal refer to the lack of separation as a need to “hide” information due to the incentive constraints, which thereby raises communication complexity. We leave unexplored in this short note the quantitative impact and whether there are subclasses of problems for which the separation holds.

10. Transferable utility

The separation does not mean that incentives constraints do not increase communication complexity given some objective of the principle. The principle may have economic objectives with indifference classes of social choice functions. Within an indifference class, the different social choice functions may have different communication complexities. Even with separation, the incentive constraints may make infeasible the low-complexity social choice functions, thereby raising the communication cost of achieving a target indifferent class.

This is a rather abstract introduction that frames the following leading and motivating example: when there is transferable (quasilinear) utility and the principle does not care about the transfers.

Specifically, suppose that the outcomes space can be decomposed as $X = \hat{X} \times \mathbb{R}^n$, where $\mathbb{R}^n$ are profiles of transfers. Call $g: \Theta \rightarrow \hat{X}$ an *allocation*. It is common in such settings that the principle cares only about the allocation. The transfers are merely a means to align incentives. The need to compute transfers at best has no effect on complexity but will typically add to the communication complexity. Hence, we can ask, for a given allocation, how much does communication complexity rise due to the need to compute transfers in order to satisfy incentive constraints? Fadel and Segal (2006) study precisely this question, for ex post Nash implementation.

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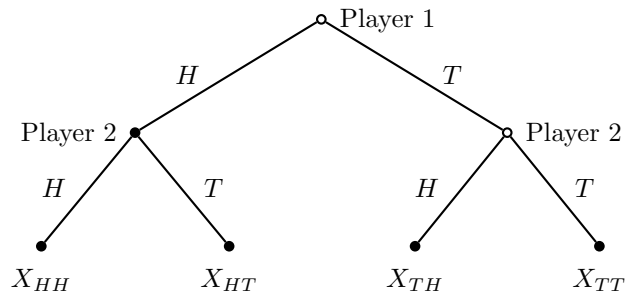


Figure 3: A counterexample for dominant-strategy implementation.

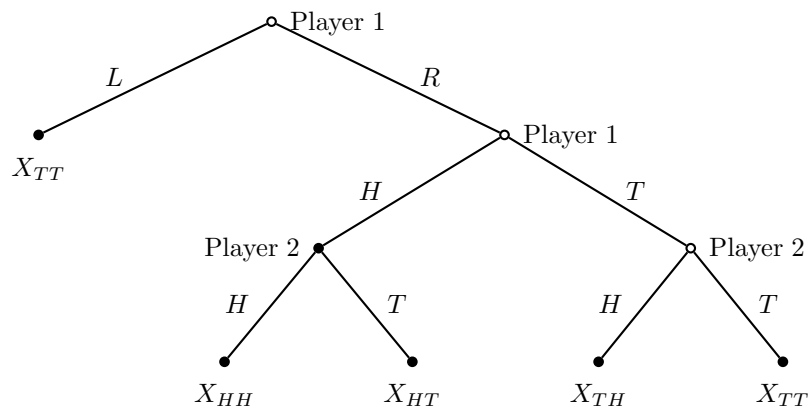


Figure 4: A non-parsimonious protocol.

1 **Communication complexity and mechanism design**

**The interplay between incentive constraints and
communication complexity in mechanism design**

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2 **The curse of the revelation principle**

The RP is a

- **blessing**
 - simplifies the design of optimal mechanisms;
 - allows one to abstract away from the details of the mechanism.
- **curse**
 - direct mechanisms provide an unrealistic picture of decision-making in organizations.

3

Why are direct revelation mechanisms so rare?

Obvious first-order explanation:

they are too complex

(i.e., too much information has to be communicated and processed)

4

Communication vs. computational complexity

Advantage of modeling communication complexity

From the point of view of economists:

No new machinery — communication is represented by an extensive game form and strategies

(Only new tools involve measures of complexity and new tricks for proving theorems.)

5 Advantage of interactive communication: Example

Players 1 and 2 must choose a path—**Left** or **Right**—to find a treasure.

Player 2 holds two clues—**Blue** and **Red**—each has the name of a path.

Player 1 knows which clue has the name of the correct path.

$$\Theta_1 = \{\text{Blue}, \text{Red}\}$$

$$\Theta_2 = \{\text{Left}_B, \text{Right}_B\} \times \{\text{Left}_R, \text{Right}_R\}$$

Simultaneous messages: Each must reveal his type. 1 bit from player 1 and 2 bits from player 2.

Sequential messages: Player 1 announces which of player 2's clues contains the name of the path; player 2 announces only the name of the path written on that clue. 1 bit from each player.

6 Example: A simple resource allocation problem

Divide K units of a homogeneous good among 2 people.

Player i 's valuation: $v_i: \{0, \dots, K\} \rightarrow \mathbb{R}$

- $v_i(0) = 0$ (normalization)
- Monotone
- Decreasing marginal valuation: $mv_i(k) \geq mv_i(k-1)$, where $mv_i(k) = v_i(k) - v_i(k-1)$.

An allocation is $k \in \{0, \dots, K\}$, denoting quantity that player 1 gets.

Realize scf $f(v_1, v_2) \in \arg \max v_1(k) + v_2(K-k)$.

7

Protocol 1: Players make simultaneous announcements.

Protocol 2: Player 1 announces $(v_1(1), \dots, v_1(K))$; player 2 announces optimal k .

8

Protocol 3: (Loosely)

- 1 announces $mv_1(K/2)$
- 2 says “I want more” if $mv_2(K/2 + 1) > mv_1(K/2)$; otherwise “I want less”.
- If 2 says “I want more”, lower 1’s allocation to $K/4$. 1 announces $mv_1(K/4)$.
- ...

9

Set up

1. From standard economics, define mechanism design problem.
2. Useful terms:
 - Mechanism Γ + strategy profile σ = **protocol** (Γ, σ)
(let $\mathcal{P}$ be set of protocols)
 - Each protocol (Γ, σ) **computes** a scf f .
3. Complexity measure $c : \hat{\mathcal{P}} \rightarrow \mathbb{R}$

10

Separation btwn incentives and comm. complexity

What would it mean for there to be **no** interaction between incentives and communication complexity?

An economist can check IC constraints of a scf (without considering the details of a communication protocol)

and then ...

a computer scientist can design a minimum-complexity communication protocol that realizes the scf, without worrying about IC constraints.

11

More specifically

Let (Γ, σ) be protocol that computes scf f .

Let (Γ^D, σ^D) be the direct mechanism with truthful revelation that computes the same f .

(Γ, σ) satisfies
IC constraints?

(Γ^D, σ^D) satisfies
IC constraints?

12

More specifically

Let (Γ, σ) be protocol that computes scf f .

Let (Γ^D, σ^D) be the direct mechanism with truthful revelation that computes the same f .

Revelation Principle

(Γ, σ) satisfies
IC constraints?

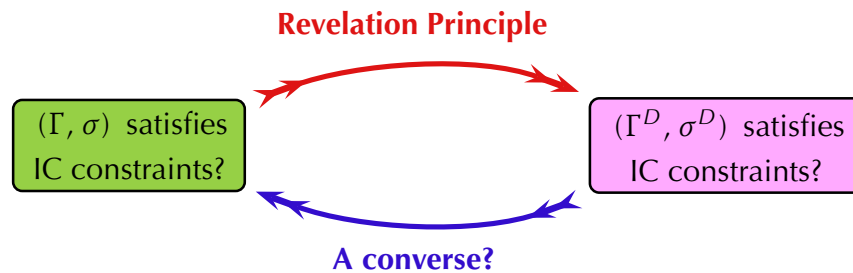
(Γ^D, σ^D) satisfies
IC constraints?

13

More specifically

Let (Γ, σ) be protocol that computes scf f .

Let (Γ^D, σ^D) be the direct mechanism with truthful revelation that computes the same f .



Affirmative answer implies:

- Separation.
- Incentive constraints do not increase communication complexity.

14

The game plan

For various types of mechanism design (IC constraints):

1. Dominant-strategy
2. Ex-post Nash
3. Nash implementation
4. SPE implementation
5. Bayesian

... either

- prove that separation holds, or
- provide counterexample showing that it does not.

If not, do IC constraints increase communication complexity?

15

Fixing notation for the environment

Exogenous:

Players: $i = 1, \dots, n$
 Types of player i : Θ
 "States": $\Theta = \Theta_1 \times \dots \times \Theta_n$
 Set of outcomes: X
 Preferences: $x \succsim_{\theta}^i x'$

If Bayesian:

Utilities: $u_i : X \times \Theta \rightarrow \mathbb{R}$
 Prior: p

16

... and mechanisms and protocols

Endogenous:

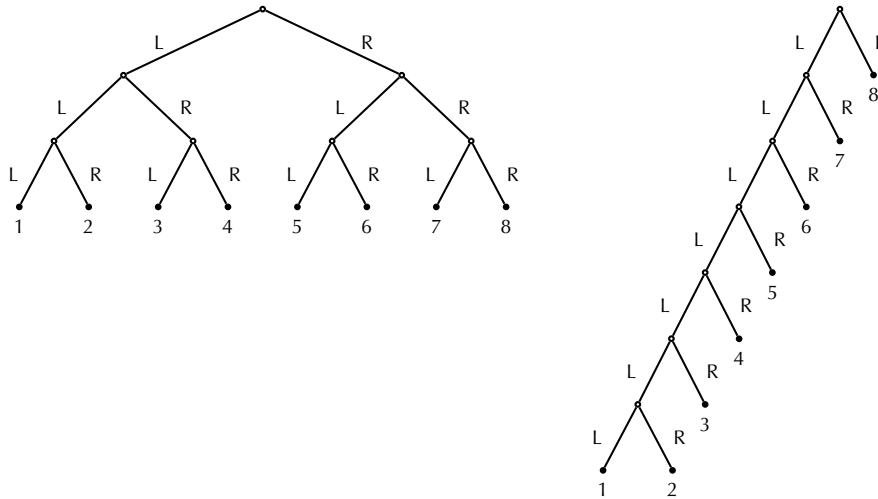
- **Social choice function:** $f : \Theta \rightarrow X$.
- **Mechanism:** extensive game form Γ .
- (Pure) **plan of action** for player i : $s_i \in S_i$.
- A **strategy** for i : $\sigma_i : \Theta_i \rightarrow S_i$.
- For a profile s of plans of actions, let $\lambda(s)$ be the **outcome** at the terminal node **reached when players use** s .
- Each protocol (Γ, σ) **computes** a scf $f(\theta) = \lambda(\sigma_1(\theta_1), \dots, \sigma_n(\theta_n))$.

17

One-way communication complexity: bits

Player reports value of $\Theta = \{1, \dots, 8\}$.

Communication represented by binary tree with 8 leaves.



18

Counterexample for DSE implementation

Players 1 and 2. Private values.

Types and Outcomes:

$$\Theta_1 = \{B, R\}$$

$$\Theta_2 = \{B, R\}$$

$$X = \{BB, BR, RB, RR\}$$

SCF:

$\theta_1 \backslash \theta_2$	B	R
B	BB	BR
R	RB	RR

19

Counterexample for DSE implementation

Players 1 and 2. Private values.

Types and Outcomes:

$$\Theta_1 = \{B, R\} \quad \Theta_2 = \{B, R\} \quad X = \{BB, BR, RB, RR\}$$

SCF:

$\theta_1 \backslash \theta_2$		B	R
		B	R
B		BB	BR
R		RB	RR

Player 1's preferences when $\theta_1 = B$.

20

Counterexample for DSE implementation

Players 1 and 2. Private values.

Types and Outcomes:

$$\Theta_1 = \{B, R\} \quad \Theta_2 = \{B, R\} \quad X = \{BB, BR, RB, RR\}$$

SCF:

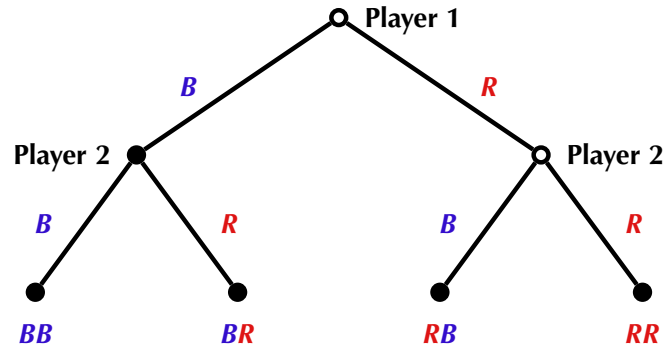
$\theta_1 \backslash \theta_2$		B	R
		B	R
B		BB	BR
R		RB	RR

Player 1's preferences when $\theta_1 = R$.

21

A protocol that computes f

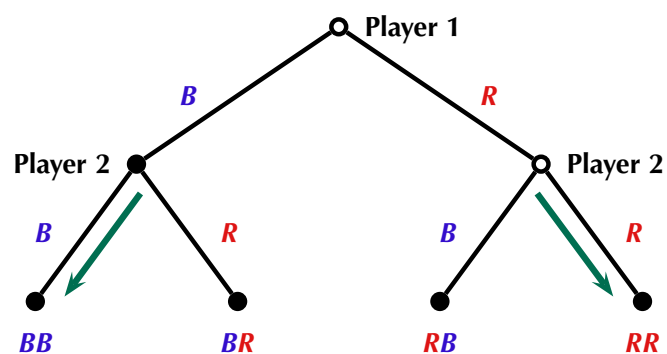
Mechanism:



Strategy profile: "each player reports his type"

22

But not DS equilibrium

When $\theta_1 = B$, reporting B is not best response to ...

Player 1's
preferences
when $\theta_1 = B$:

$\theta_1 \backslash \theta_2$	B	R
B	BB	BR
R	RB	RR

23

But a partial positive result for DS-IC

Proposition: Assume private values. Suppose a protocol

1. realizes a DS-IC scf and
2. uses all possible strategies for each player.

Then it is DS-IC.

Proof: 1's DS-IC constraints for indirect protocol: $\forall \theta_1, s_2, s'_1 :$

$$\lambda(\sigma_1(\theta_1), s_2) \succsim_{\theta_1}^1 \lambda(s'_1, s_2)$$

24

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$$\begin{array}{ccc} \lambda(\sigma_1(\theta_1), s_2) & \succsim_{\theta_1}^1 & \lambda(s'_1, s_2) \\ \parallel & & \parallel \\ \lambda(\sigma_1(\theta_1), \sigma_2(\theta_2)) & & \lambda(\sigma_1(\theta'_1), \sigma_2(\theta_2)) \end{array}$$

25

But a partial positive result for DS-IC

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 \lambda(\sigma_1(\theta_1), s_2) & \succsim_{\theta_1}^1 & \lambda(s'_1, s_2) \\
 \parallel & & \parallel \\
 \lambda(\sigma_1(\theta_1), \sigma_2(\theta_2)) & & \lambda(\sigma_1(\theta'_1), \sigma_2(\theta_2)) \\
 \parallel & & \parallel \\
 f(\theta_1, \theta_2) & \checkmark & f(\theta'_1, \theta_2)
 \end{array}$$

26

Example where DS-IC increases comm. complexity?

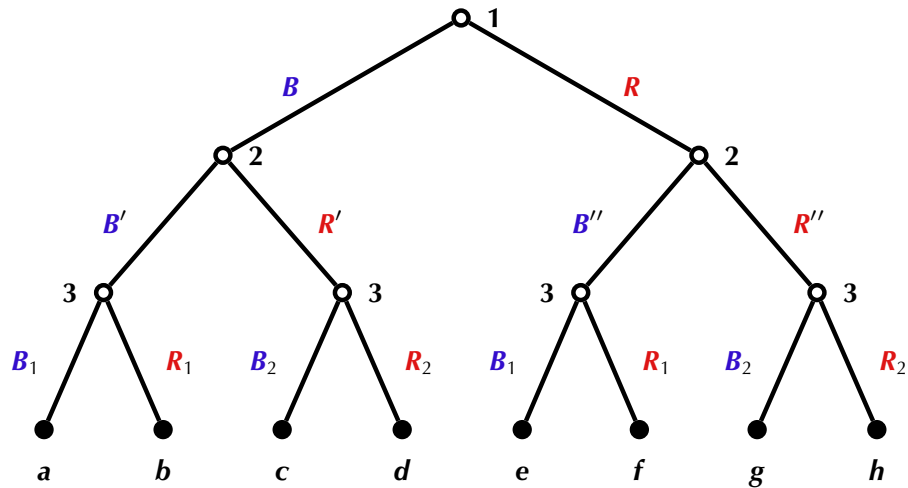
As starting point, we need a protocol ...

1. Not all strategies are used;
2. Yet it (strictly) minimizes communication complexity.

27

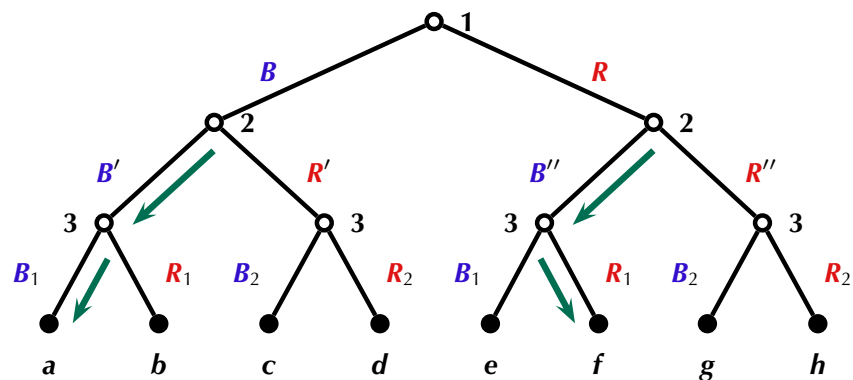
Example with private values and 3 players

$$\Theta_1 = \{B, R\} \quad \Theta_2 = \{B', R'\} \times \{B'', R''\} \quad \Theta_3 = \{B_1, R_1\} \times \{B_2, R_2\}$$



28

Player 1's DS-IC constraints

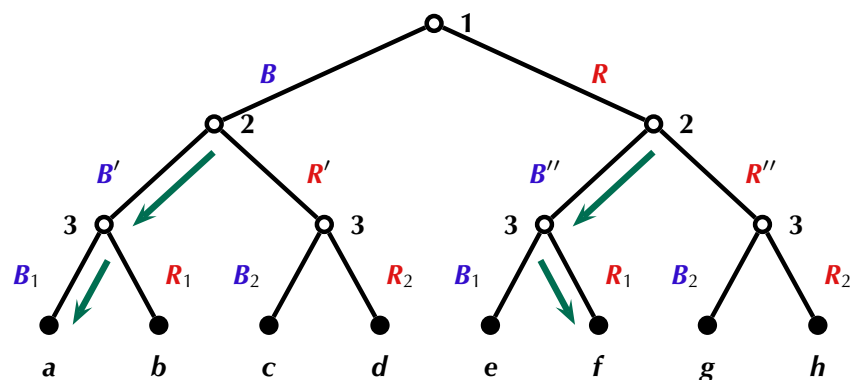


DS-IC constraints for scf:	$a \succ e$	$b \succ f$	$c \succ g$	$d \succ h$
	$a \succ g$	$b \succ g$	$c \succ e$	$d \succ e$
	$a \succ h$	$b \succ h$	$c \succ f$	$d \succ f$

Eg: $b \succ c \succ d \succ f \succ a \succ e \succ g \succ h$

29

But protocol is not DS-IC if $f \succ a$.



DS-IC	$a \succ e$	$b \succ f$	$c \succ g$	$d \succ h$
constraints	$a \succ g$	$b \succ g$	$c \succ e$	$d \succ e$
for scf:	$a \succ h$	$b \succ h$	$c \succ f$	$d \succ f$

Eg: $b \succ c \succ d \succ f \succ a \succ e \succ g \succ h$

30

Ex post Nash implementation

Problem with DSE is that we have to check best replies by 1 against plans that, in the indirect mechanism, are not chosen by any type of player 2.

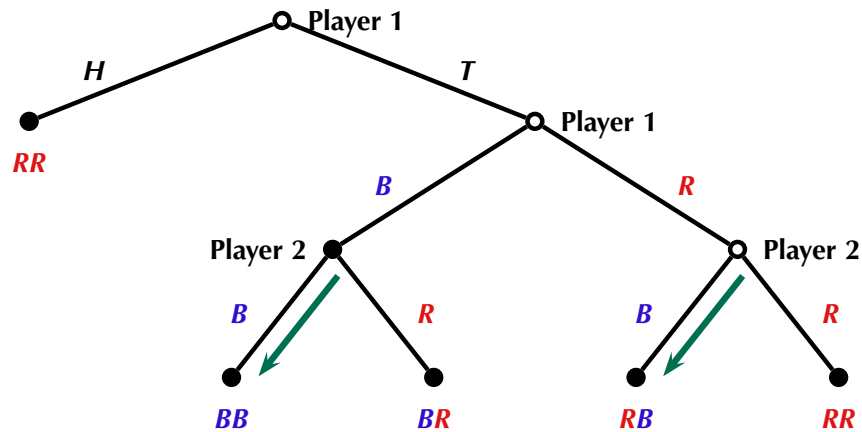
Ex post Nash equilibrium eliminates this problem.

What is left to do

Check that deviations by player 1 are limited (effectively) to pretending he is different type.

31

Counterexample (absurd)



Player 1's
preferences
when $\theta_1 = B$:

θ_2	B	R
B	BB	BR
R	RB	RR

32

Positive result for ex post Nash implementation

Prune superfluous edges (that are never reached) so that all leaves are reached for some profile of types.

Definition: A protocol is **parsimonious** if all leaves are reached for some profile of types.

Proposition: Suppose a protocol is parsimonious and suppose it realizes a scf that is ex-post Nash implementable. Then the strategy profile is an ex post Nash equilibrium of the mechanism.

(No assumption of private values.)

33

Proof**1's ex-post Nash IC constraints for scf**

$$\forall \theta_1, \theta_2, \theta'_1 :$$

$$f(\theta_1, \theta_2) \succeq_{(\theta_1, \theta_2)}^1 f(\theta'_1, \theta_2)$$

1's Ex-post Nash IC constraints for protocol:

$$\forall \theta_1, \theta_2, s'_1 :$$

$$\lambda(\sigma_1(\theta_1), \sigma_2(\theta_2)) \succeq_{(\theta_1, \theta_2)}^1 \lambda(s'_1, \sigma_2(\theta_2))$$

34

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$$\lambda(\sigma_1(\theta_1), \sigma_2(\theta_2)) \succeq_{(\theta_1, \theta_2)}^1 \lambda(s'_1, \sigma_2(\theta_2))$$

$$\parallel$$

$$f(\theta_1, \theta_2)$$

35

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1's Ex-post Nash IC constraints for protocol:

$$\forall \theta_1, \theta_2, s'_1 :$$

$$\begin{array}{ccc}
\lambda(\sigma_1(\theta_1), \sigma_2(\theta_2)) & \succsim_{(\theta_1, \theta_2)}^1 & \lambda(s'_1, \sigma_2(\theta_2)) \\
\parallel & & \parallel \\
f(\theta_1, \theta_2) & & \lambda(\sigma_1(\theta'_1), \sigma_2(\theta'_2))
\end{array}$$

36

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$$\forall \theta_1, \theta_2, \theta'_1 :$$

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1's Ex-post Nash IC constraints for protocol:

$$\forall \theta_1, \theta_2, s'_1 :$$

$$\begin{array}{ccc}
\lambda(\sigma_1(\theta_1), \sigma_2(\theta_2)) & \succsim_{(\theta_1, \theta_2)}^1 & \lambda(s'_1, \sigma_2(\theta_2)) \\
\parallel & & \parallel \\
f(\theta_1, \theta_2) & & \lambda(\sigma_1(\theta'_1), \sigma_2(\theta'_2)) \\
& & \parallel \\
& & \lambda(\sigma_1(\theta'_1), \sigma_2(\theta_2))
\end{array}$$

37

Proof**1's ex-post Nash IC constraints for scf** $\forall \theta_1, \theta_2, \theta'_1 :$

$$f(\theta_1, \theta_2) \succsim_{(\theta_1, \theta_2)}^1 f(\theta'_1, \theta_2)$$

1's Ex-post Nash IC constraints for protocol: $\forall \theta_1, \theta_2, s'_1 :$

$$\begin{array}{ccc}
 \lambda(\sigma_1(\theta_1), \sigma_2(\theta_2)) & \succsim_{(\theta_1, \theta_2)}^1 & \lambda(s'_1, \sigma_2(\theta_2)) \\
 \parallel & & \parallel \\
 f(\theta_1, \theta_2) & & \lambda(\sigma_1(\theta'_1), \sigma_2(\theta'_2)) \\
 & & \parallel \\
 & & \lambda(\sigma_1(\theta'_1), \sigma_2(\theta_2)) \\
 & & \parallel \\
 & & f(\theta'_1, \theta_2)
 \end{array}$$

38

Proof**1's ex-post Nash IC constraints for scf** $\forall \theta_1, \theta_2, \theta'_1 :$

$$f(\theta_1, \theta_2) \succsim_{(\theta_1, \theta_2)}^1 f(\theta'_1, \theta_2)$$

1's Ex-post Nash IC constraints for protocol: $\forall \theta_1, \theta_2, s'_1 :$

$$\begin{array}{ccc}
 \lambda(\sigma_1(\theta_1), \sigma_2(\theta_2)) & \succsim_{(\theta_1, \theta_2)}^1 & \lambda(s'_1, \sigma_2(\theta_2)) \\
 \parallel & & \parallel \\
 f(\theta_1, \theta_2) & & \lambda(\sigma_1(\theta'_1), \sigma_2(\theta'_2)) \\
 & & \parallel \\
 & & \lambda(\sigma_1(\theta'_1), \sigma_2(\theta_2)) \\
 \vdots & & \parallel \\
 f(\theta_1, \theta_2) & \checkmark & f(\theta'_1, \theta_2)
 \end{array}$$

39

Negative result for Nash and SPE-implementation

Recall: Information structure in Nash and SPE-implementation is that all agents have same information.

Therefore, without incentive problems, efficient communication is for one agent to announce desired outcome. Incentive constraints make this impossible.

Nash usually worse than SPE because SPE takes advantage of interactive communication.

40

Example

Version of Solomon's predicament

An object goes to one of two players, Alice and Bob.

One player's valuation is v_H and the other's is v_L , with $v_H > v_L$.

Players know who has v_H but mechanism designer does not.

Outcome is (j, m_a, m_b) , where j is person who gets object ($j \in \{\emptyset, a, b\}$ and m_a and m_b are punishments for Alice and Bob, resp.

Player i 's utility is $v_i - m_i$ if $i = j$, and $-m_i$ otherwise.

$$\text{SCF: } f(v_1, v_2) = \begin{cases} (a, 0, 0) & v_a = v_H \\ (b, 0, 0) & v_b = v_H \end{cases}$$

41

Static mechanism with smallest message space

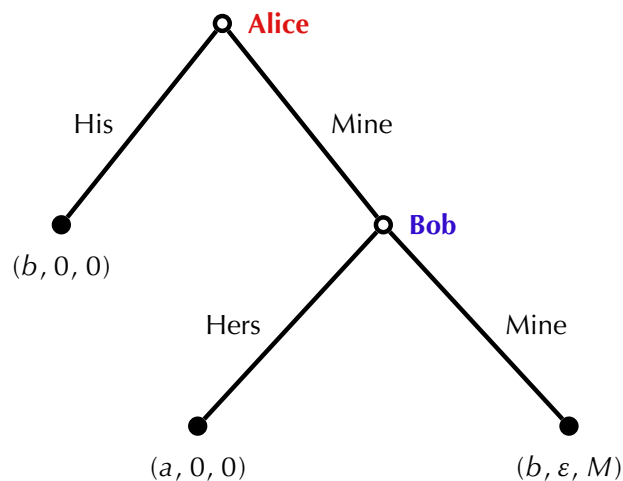
(among those that satisfy full Nash implementation)

		Bob		
		Mine	Hers	Mine+
Alice	Mine	$(\emptyset, \varepsilon, \varepsilon)$	$(a, 0, 0)$	(b, ε, M)
	His	$(b, 0, 0)$	$(\emptyset, \varepsilon, \varepsilon)$	$(\emptyset, 0, 0)$
	Mine+	(a, M, ε)	$(\emptyset, 0, 0)$	$(\emptyset, 2\varepsilon, 2\varepsilon)$

$$\varepsilon \text{ small; } M = \frac{v_H + v_L}{2}$$

42

SPE - less communication, still more than needed



But (given *full* implementation) refinement of SPE helps. Nash requires more communication.

43

Bayesian mechanism design

Suppose that types, though correlated, are not perfectly so.

Take previous example. With high probability $v_a \neq v_b$, but $v_a = v_b$ is possible.

Previous mechanism implements SCF

$$f(v_L, v_L) = (b, 0, 0)$$

$$f(v_L, v_H) = (b, 0, 0)$$

$$f(v_H, v_L) = (a, 0, 0)$$

$$f(v_H, v_H) = (b, \varepsilon, M)$$

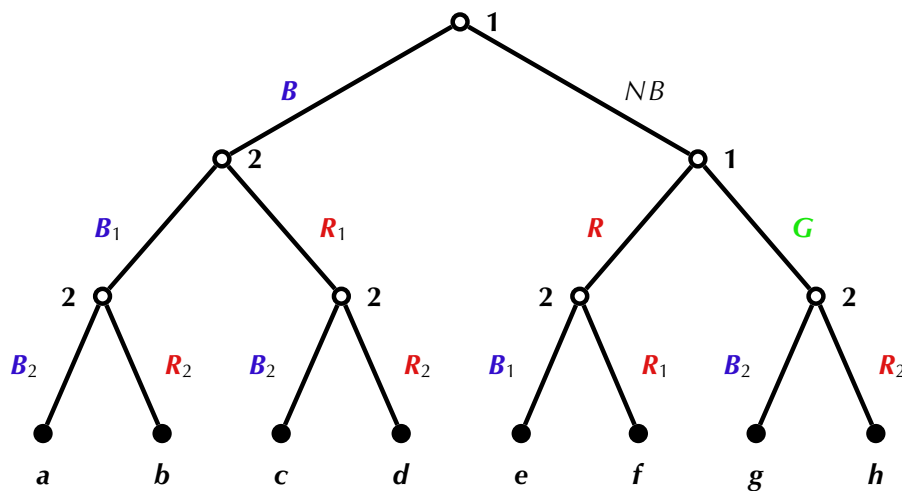
This is a minimum-communication complexity protocol for this SCF.

(Note: Direct mechanism does not work if we want full implementation, but indirect protocol has full implementation for PBE.)

44

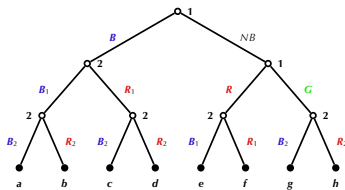
Counterexample for Bayesian implementation

$$\Theta_1 = \{B, R, G\} \quad \Theta_2 = \{B_1, R_1\} \times \{B_2, R_2\}$$



45

2's Bayesian IC constraints when type $B_1 B_2$

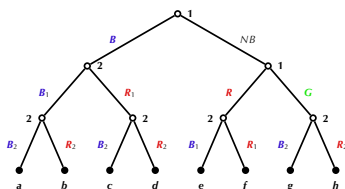


Private values, ind. types.
Player 1's types equally likely.

SCF :	θ_1
θ_2	B R G
$B_1 B_2$	a e g
$B_1 R_2$	b e h
$R_1 B_2$	c f g
$R_1 B_2$	d f h

46

2's Bayesian IC constraints when type $B_1 B_2$



Private values, ind. types.
Player 1's types equally likely.

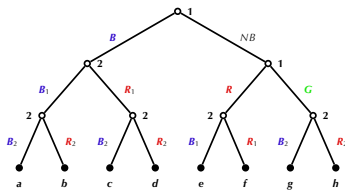
SCF :	θ_1
θ_2	B R G
$B_1 B_2$	a e g
$B_1 R_2$	b e h
$R_1 B_2$	c f g
$R_1 B_2$	d f h

1's IC constraints
for SCF:

Lottery on these
should be better than
lotteries on these

47

2's Bayesian IC constraints when type $B_1 B_2$



Private values, ind. types.
Player 1's types equally likely.

SCF :	θ_1
θ_2	B R G
$B_1 B_2$	a e g
$B_1 R_2$	b e h
$R_1 B_2$	c f g
$R_1 B_2$	d f h

1's IC constraints
for SCF:

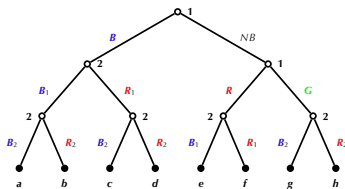
Lottery on these
should be better than
lotteries on these

But in the perfect-information protocol,
we need "statewise dominance".

E.g., $e \succsim_{B_1 B_2}^2 f$.

48

2's Bayesian IC constraints when type $B_1 B_2$



Private values, ind. types.
Player 1's types equally likely.

SCF :	θ_1
θ_2	B R G
$B_1 B_2$	a e g
$B_1 R_2$	b e h
$R_1 B_2$	c f g
$R_1 B_2$	d f h

1's IC constraints
for SCF:

Lottery on these
should be better than
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But in the perfect-information protocol,
we need "statewise dominance".

E.g., $e \succsim_{B_1 B_2}^2 f$.

Payoffs:	θ_1
θ_2	B R G
$B_1 B_2$	2 0 1
$B_1 R_2$	0 0 0
$R_1 B_2$	0 1 1
$R_1 B_2$	0 1 0

49

Summary

Separation holds for ex post Nash.

Does not hold for the others, and IC constraints increase communication complexity:

1. Dominant-strategy
2. Nash implementation
3. SPE implementation
4. Bayesian

We also saw examples of how “conversation” reduces communication complexity.

50

Transferable (quasilinear) utility

Outcomes $X = \hat{X} \times \mathbb{R}^n$, where $\mathbb{R}^n$ are profiles of transfers.

Call $g: \Theta \rightarrow \hat{X}$ an **allocation**.

Many scf's induce the same allocation; they differ by the transfers, and each of them has different communication complexity.

Without incentive problems, no transfers needed. But transfers (e.g., VCG) are needed to obtain incentive compatibility, and the computation of the transfers imposes additional complexity.

Thus, even with ex-post Nash implementation, incentives may—almost surely will—change the complexity of realizing a given allocation.

See Fadel and Segal (2006): The Communication Cost of Selfishness