Isotone Recursive Methods for Stationary Markovian Equilibrium in Overlapping Generations Models with Stochastic Nonclassical Production*

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Abstract

Based on an order-theoretic approach, we derive sufficient conditions for the existence, characterization, and computation of minimal state space Markovian equilibrium decision processes (MEDPs) and stationary Markov equilibrium (SME) for a large class of stochastic overlapping generations models. In contrast to previous work, our focus is exclusively on constructive fixed point methods. In addition, we extend results obtained in earlier work to the case of more general reduced-form stochastic production technologies that allow for a broad set of equilibrium distortions such as public policy distortions, social security, monetary equilibrium, and production nonconvexities. Our order-based methods provide monotone iterative algorithms the uniformly converge to extremal Markovian equilibrium decision proceesses. Further, those methods can be tied to the computation of extremal equilibrium invariant distributions. Our methods select equilibrium that avoid many of the problems associated with the existence of indeterminacies that have been well-documented in previous work. We study cases where MEDPs are unique continuous functions, as well as provide the first results in the literature on the existence and computation of minimal state space MEDPS and SME for the case where capital income is not monotone. We show that our results on minimal state space MEDPs extend to the case of n-period stochastic OLG models. Finally, we conclude with examples common in macroeconomics such

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as models with social security, we provide a brief discussion of equilibrium comparative statics, and show how some of our results extend to settings with unbounded state spaces.

1 Introduction

Building on the seminal work of Samuelson [38] and Diamond [21], the overlapping generations (OLG) model has become a workhorse for many areas of applied dynamic general equilibrium theory over over the last three decades. These models have been used to study a wide array of issues in macroeconomics and public finance, including policy issues in intergenerational risk sharing and social security, human capital formation and public education, economic growth and infrastructure or environmental degradation, macroeconomic fluctuations, public finance, and monetary economics. In recent years, with a few notable exceptions, a majority of the work studying OLG models has employed numerical characterizations of "minimal state space" Markovian equilibrium as their primary method of analysis.[1] Yet, in the theoretical literature, little is known about the existence and computation of minimal state space Markovian Equilibrium.decision processes (MEDPs) in even the simpliest of OLG models. Indeed, negative results concerning the existence of minimal state space MEDPs are known for some stochastic OLG models (e.g., see Kubler and Polemarchakis [30]), and provide a source of concern.

Even when positive theoretical results on the existence of stationary Markov equilibrium (SME) have been provided, often those results employ the topological constructions found in the work of Mertens and Parathasarthy [?] and Duffie, Geanokopokis, MasColell, and McClennan [23]. Unfortuantely, even when establishing the existence of SME, these approaches prove difficult to related directly to the existence of minimal state space MEDPs. Given that in most applied work, minimal state space MEDPs are important as they are relatively simple to compute and characterize, this drawback of this topological approach is important. Further, as these topological methods are typically based on nonconstructive fixed point arguments (where continuity is studied in the context of weak topologies), they prove difficult to relate to the rigorous analysis of approximate solutions and successive approximations. Finally, often these topological approaches are not amenable to identifying conditions under which equilibrium comparison results are available for a large set of Markovian equilibrium with respect to ordered perturbations of the space of economies.

This paper takes a new approach to address some of these issues in a broad collection of OLG models with stochastic production. The technologies for

[1] By "minimal state space" Markovian equilibrium, we mean Markovian equilibrium decision processes that are defined only on current period states variables (as opposed to, for example, the Markovian equilibrium typically constructed using the methods in Duffie et. al. [23]). In the work in macroeconomics that has its roots in Lucas and Prescott [?], Prescott and Mehra [?], and Stokey, Lucas, and Prescott [40], such minimal state space equilibria are often referred to as "recursive" equilibria.

the economy are allowed to be (in reduced-form) either classical or nonclassical, thereby allowing many interesting economies as special cases including economies with social security, public taxation, and binding cash-in-advance constraints. Given that sequential equilibrium in OLG models are well-known to be suboptimal (even without such additional market frictions), our methods rely exclusively on an equilibrium Euler equation approach and make no appeals to Negishi-like methods. As our focus is in minimal state space MEDPs, we take a very different approach that that advocated in Duffie et. al. [23], which has been implenented in a series of important recent papers in Wang [44][45]. Our approach combines the "little k, big K" recursive equilibrium formulation standard in recursive competitive equilibrium theory (ala Lucas and Prescott [**?**]) with an order-based fixed point theorem to identify classes of equilibria where minimal state space MEDPs exist and can be sharply characterized (as in the work of Coleman [11][**?**], Greenwood and Huffman [26], and Datta, Mirman, and Reffett [13], and Mirman, Morand, and Reffett [**?**]). Therefore, our methods, in this sense, provide a systematic equilibrium selection procedure for minimal state space MEDPs within the potentially much large set of MEDPs and SME that can be identified using the methods of Duffie et. al. [23]. In some cases, under very common assumptions used in the applied literature (e.g., under capital income monotonicity assumptions and strictly concave utility), we prove that the unique minimal state space MEDPs within a collection of equicontinuous decision rules. We further provide an explicit successive approximation algorithm that converges uniformly to this unique MEDPs, and in principle uniform error bounds could easily be constructed for standard methods for constructing approximate solutions. The successive approximation scheme is *globally* stable, and using it, we also show how to construct the corresponding extremal invariant distributions (which we call stationary Markov equilibrium or SME). Equally as important, we are able to provide the first results in the literature on existence and computation of minimal state space MEDPs in the *absence* of capital monotonicity. In particular, we show for this case, there exists a complete lattice of upper-semi continuous and lower-semi continuous isotone whose extremal elements can be computed using successive approximation schemes converging (in both order and topology) to the extremal MEDPs and associated extremal SME.

There are a number of important results on the existence and characterization of Markovian equilibrium in OLG models with production. Galor and Ryder[25] obtained some of the first results concerning the existence, uniqueness, and stability of steady state for a deterministic setup with classical production. The work of Galor and Ryder [25] has been extended to a class of OLG models with stochastic "classical" production (i.e., a production setting identical to that used in the work of Brock and Mirman [9]) in a series of papers by Wang [44] [45] and Hauenschild [27]. For economies with identical and independently distributed (iid) production shocks and classical production, Wang[44] obtains sufficient conditions for the existence of a globally unique and stable non-trivial SME using a topological approach. Wang [45] notes that problems arise in economies without capital income monotonicity because of multiple equilibria and equilibrium indeterminacies. Hauenschild [27] extends some of Wang's results to economies with pay-as-you-go financed social security. Finally Wang [45] provides some existence results along the lines of Duffie & al. [23] for economies with stochastic classical production with Markov shocks. A key implication of Wang's work (along with the storyline in the paper of Kubler and Polemarchakis [30]) is that when sunspots and multiplicities are considered, the state space that is used to construct SME can potentially become very large.[2] As such solutions are difficult to compute, this provides one motivation for identifying conditions under which minimal state space MEDPs exist.

Our paper improves on the results of Wang[44] in many ways. For all cases, we show how to allow for very general reduced-form "nonclassical" production technologies that allow for taxation, valued fiat money, transfers and social security, production nonconvexities. This is true even the the case of capital income monotonicity. Further, in all cases, we provide explicit successive approximation results for fixed point operators that can become the basis of both numerical solution methods, and sharp equilibrium comparative static results. Interesting, we show that even in the case where capital income is monotone, minimal state space MEDPs exist outside this narrow class of continuous MEDPs. In particular, we show MEDPs that are only bounded and measurable also exist. Importantly, for the case of economies *without* capital income monotonicity, we show our isotone method work also. In particular, we provide algorithms to construct extremal semi continuous measurable MEDPs and associated SME. The equilibrium in this case we construct are (i) minimal state space, and (ii) very "well-behaved" (i.e., associated with equilibrium investment decision rules that are isotone.) Our methods in all cases differ substantially from the "correspondence" approach advocated in Wang [44] and Duffie et. al. [23], and are in principle easily implemented using existing numerical methods. We should mention, though, when discussing SME, our focus is on invariant distributions as in Stokey et. al. [40] and Hopenhayn and Prescott [28], as opposed to ergodic distributions in Wang [44]. Finally, we show how our methods can be extended directly to n-period lifecycle models. .

It is important to realize that *all* of the results concerning MEDP and SME are obtained in settings where: (i) the economy is endowed with a very simple form of behavioral heterogeneity (i.e., all agents of each generation are identical), and (ii) there is a single asset that agents can access to save (namely capital).

[2] There are numerous pioneering papers in OLGs models which we have not mentioned directly in our remarks. Aside from the paper by Diamond, see Balasko and Shell [3][4], Okuno and Zilcha [35], Zilcha [?], Dechert and Yamamoto [20], Demange and Laroque [17][18], and Chattopadhyay and Gottardi [10], and Barbie, Hagedorn, and Kaul [5]). Many of these papers discuss the important question of how to define dynamic efficiency in an OLG environment, and often these papers have a much more complicated structure (e.g., many assets each period and incomplete markets, many good each period, etc.)

We note that although some of these papers study economies with or without stochastic production, none of this related work addresses the questions addressed in this paper (i.e., the construction and usefulness of monotone methods in the study of MEDPs and SME in OLG models production subjected to *iid* shocks from both a theoretical and numerical point of view).

Much of the work in the existing applied literature uses such a simple stochastic OLG model. For cases with additional complications such as within period heterogeneity, multiple commodities each period (and therefore complications associated with relative prices), multiple assets, little is known. Indeed, Kubler and Polemarchakis [30] provide some very interesting negative results concerning the existence of minimal state space Markovian equilibrium for economies with multiple types of agents born each period, multiple commodities, and many assets. The economies considered in the work of Kubler and Polemarchakis are much more complicated than the environments considered in this paper (and in all the existing work on stochastic OLG models with production). In this sense, we are trying to develop techniques and basic results for simple OLG models with stochastic production that we feel have a chance to be generalized to some more general settings. Given the recent positive results on existence of MEDP using Abreau, Pierce, and Stachetti (APS) approach in Miao and Santos [31], we believe our methods have the potential to be integrated with this APS approach to address more complicated versions of our model.[3]

Finally, we should mention that the present paper is related to an emerging literature on monotone and mixed-monotone recursive methods starting with the pioneering work of Coleman [11][12], and continuing with the papers by Greenwood and Huffman [26], Datta, Mirman, and Reffett [13], Morand and Reffett [33], and Datta et. al [15], and Mirman, Morand, and Reffett [**?**]. One interesting aspect of the work here is we also develop monotone methods for studying the existence of SME, which none of these papers study explicitly.

The paper is organized as follows. In section two, we detail the class of economies under consideration and provide some preliminary results. In section three, we study the set of Markovian equilibrium investment decisions and present algorithms to construct extremal Markovian equilibrium investment decisions. In section four we address the existence and construction of extremal stationary Markov equilibrium. Section five discusses extensions to the n-period case. In the last section, discuss some applications and additional results.

2 Setup and preliminary results

Throughout the paper X and Z are compact intervals of $\mathbb{R}^n_+$, $X^* = X \setminus \{0\}$, $S = X \times Z$, $S^* = X^* \times Z$, and $\mathcal{B}(S)$ is the Borel algebra corresponding to S (note that $\mathcal{B}(S) = \mathcal{B}(X) \times \mathcal{B}(Z)$). All compact subsets of $\mathbb{R}^n$ are endowed with the standard pointwise partial order $\leq$ and the usual topology on $\mathbb{R}^n$. In practice we consider $X = [0, k_{\max}] \subset \mathbb{R}$ where $k_{\max} > 0$, $Z = [z_{\min}, z_{\max}] \subset \mathbb{R}$ where $0 < z_{\min} \leq z_{\max}$, although most results can be generalized to compact intervals of $\mathbb{R}^n_+$.

[3] Using the ASP approach, Mia and Santos prove existence of MEDP in the space of *all* measurable mappings. Such general result comes at the expense of losing important characterization of MEDP that prove useful in constructing associated SME, and these theoretical results remain to be tied to numerical implementations. See Reffett [36] for a discussion of how Miao and Santo's APS method relate to the methods developed in this paper.

2.1 Some results from lattice theory

A *partial order* $\leq$ on a set X is a reflexive, transitive, and antisymmetric relation, in which case $(X, \leq)$ is a partially ordered set, or poset. An *upper (resp. lower) bound* of $A \subset X$ is an element $u \in X$ (resp. $v \in X$) such that $\forall x \in A$, $u \geq x$ (resp. $v \leq x$). If x^u is an upper bound of A, and $x^u \leq y$ for any other upper bound y of A, then x^u is the least upper bound or supremum of A, denoted $\vee A$ (similarly one can define the greatest lower bound or infimum of A, denoted $\wedge A$). A chain C is a subset of X that can be linearly ordered, that is, $\forall (p, p') \in C$ $p \leq p'$ or $p' \leq p$. The poset $(X, \leq)$ is a *lattice* if $\forall (x,x') \in X \times X$, both $x \wedge x'$ and $x \vee x'$ exist and belong to X. If $B \subset X$ and $(B, \leq)$ is a lattice, then we say that B is a sublattice of X. A lattice $(X, \leq)$ is *complete* if $\vee B$ and $\wedge B$ exist for any $B \subset X$. If every chain $C \subset X$ is complete, then X is referred to as a *chain complete poset*. A set C is *countable* if it is either finite or there is a bijection from the natural numbers onto C. If every chain $C \subset X$ is countable and complete, then X is referred to as a *countable chain complete poset*. Finally, the countable chain $\{x_n\}$ with $x_i \leq x_j$ order converges to x iff $\vee\{x_n\} = x$. Showing that a partially ordered set is a complete lattice sometimes requires much less work that the definition of completeness would have us believe, as demonstrated in the following theorem (from Davey and Priestley [16]).

Theorem 1 *The poset $(P, \leq)$ is a complete lattice if and only if P has a top (resp. bottom) element and for any $P' \subset P$, $\wedge_P P'$ (resp. $\vee_P P'$) exists (in P).*

Given any bounded function $w : S \to \mathbb{R}^+$, the set $W = \{h : S \to \mathbb{R}^+, 0 \leq h \leq w\}$ endowed with the pointwise partial order $\leq$ is clearly a complete lattice. If w is isotone (i.e., non-decreasing), then the subset $H \subset W$ of isotone functions is also a complete lattice (i.e. a complete sublattice of W). In addition, if w is continuous in k for each $z \in Z$ (in the usual topology on $\mathbb{R}$), the sets $H_k^u = \{h \in H,\ h$ upper semicontinuous in $k \in X$ for each $z \in Z\}$ and $H_k^l = \{h \in H,\ h$ lower semicontinuous in k for each $z \in Z\}$ endowed with the pointwise partial order are complete sublattices of H, as established in the following Proposition. In the rest of this paper, we will identify W as the set of "bounded functions", H as the set of bounded isotone functions, and H_k^u as the set of bounded isotone upper semicontinuous functions.

Proposition 2 *The poset $(H_k^u, \leq)$ is a complete lattice.*

Proof. Given any $B \subset H_k^u$, denote $g(s) = \sup_{h \in B} h(s)$. Clearly $0 \leq g \leq w$, g is isotone, and $g(., z)$ is usc for any given z. Thus g is an upper bound of B, and it is easy to see that it is the least upper bound. Consequently, $g = \vee B$. Since w is the top element of H_k^u, the result follows from Theorem xx above. ■

Measurability[4] will turn out to be a very useful property, but the set of measurable functions in H is not a complete lattice (although it is a countable chain complete poset). Nevertheless, we have the following important result:[5]

[4] We are only concerned with Borel joint measurability.

[5] Note also that any $h \in H_k^l \cap H_k^u$ is a Caratheodory function and therefore measurable.

Proposition 3 *Any $h \in H_k^u$ (resp. H_k^l) is measurable.*

Proof. Since X is a compact interval of $\mathbb{R}$, denote by $\{x_0, x_1, ...\}$ a countable dense subset of X. Given any $\alpha \in \mathbb{R}$, we claim that:

$$\begin{aligned}\{s &\in S,\ h(s) \leq \alpha\} \\ &= \\ \bigcap_{n=1}^{\infty} \bigcup_{m=0}^{\infty}]x_m - 1/n, x_m] \times \{z &\in Z,\ h(x_m, z) < \alpha + 1/n\}.\end{aligned}$$

This property implies that h is measurable: Indeed, since h is increasing in z for each k, it is $\mathcal{B}(Z)$-measurable for each k which implies that $\{z \in Z,\ h(x_m, z) < \alpha + 1/n\} \in \mathcal{B}(Z)$, and that $\{s \in S,\ h(s) \leq \alpha\} \in \mathcal{B}(S)$. We prove now the stated claim. First, consider (k, z) such that $h(k, z) \leq \alpha$. Since h is usc and increasing in k (for each z), it is necessarily right continuous at k. As a result:

$$\forall n \in \mathbb{N},\ \exists m \text{ such that } x_m - 1/n < k < x_m \text{ and } h(x_m, z) < \alpha + 1/n.$$

Thus:

$$\forall n \in \mathbb{N},\ \exists m \text{ such that } (k, z) \in]x_m - 1/n, x_m] \times \{z \in Z,\ h(x_m, z) < \alpha + 1/n\},$$

which implies that:

$$\forall n \in \mathbb{N},\ (k, z) \in \bigcup_{m=0}^{\infty}]x_m - 1/n, x_m] \times \{z \in Z,\ h(x_m, z) < \alpha + 1/n\},$$

and therefore that:

$$(k, z) \in \bigcap_{n=1}^{\infty} \bigcup_{m=0}^{\infty}]x_m - 1/n, x_m] \times \{z \in Z,\ h(x_m, z) < \alpha + 1/n\}.$$

Reciprocally, suppose that for all $n \in \mathbb{N}$, (k, z) belongs to $\bigcup_{m=0}^{\infty}]x_m - 1/n, x_m] \times \{z \in Z,\ h(x_m, z) < \alpha + 1/n\}$. This implies that for all n, there exists $m(n)$ such that $k \in]x_{m(n)} - 1/n, x_{m(n)}]$ and $h(x_{m(n)}, z) < \alpha + 1/n$. By construction the sequence $\{x_{m(1)}, x_{m(2)}, ...\}$ converges to k and $x_{m(n)} \geq k$, so by continuity from the right at k of $h(., z)$, $h(x_{m(n)}, z)$ converges to $h(k, z)$ and necessarily $h(k, z) \leq \alpha$. Finally, we note that a similar result holds if h is lsc rather than usc, since it can be shown that:

$$\begin{aligned}\{(k, z) &\in S,\ h(k, z) \geq \alpha\} \\ &= \\ \bigcap_{n=1}^{\infty} \bigcup_{m=0}^{\infty} [x_m, x_m + 1/n[\times \{z &\in Z,\ h(x_m, z) > \alpha - 1/n\}.\end{aligned}$$

■

The other type of partially ordered sets of interest for this paper are sets of probability measures, in which we will prove existence of stationary Markov equilibria. First, we denote by $\Lambda(X, \mathcal{B}(X))$ the set of probability measures defined on the measurable space $(X, \mathcal{B}(X))$, which we endowed with the stochastic order $\geq_s$ defined as follows:

$$\mu \geq_s \mu' \text{ if } \int_X f(k)\mu(dk) \geq \int_X f(k)\mu'(dk),$$

for every increasing, and bounded function $f : X \to \mathbb{R}_+$, in which case we say that μ stochastically dominates μ'.

Proposition 4 *The poset* $(\Lambda(X, \mathcal{B}(X)), \geq_s)$ *is a complete lattice with minimal and maximal elements* δ_0 *and* $\delta_{k_{\max}}$.

Proof. It is easy to show that the set $\mathbb{D}(X)$ of functions $F : X \to [0, 1]$, that are increasing, upper semicontinuous, and satisfy $F(b) = 1$, is a complete lattice when endowed with the pointwise order. $\mathbb{D}(X)$ has maximal and minimal elements (respectively, the function $F(k) = 1$ for all $k \in X$, and the function $G(k) = 1$ if $k = b$ otherwise $G(k) = 0$), and is in fact the set of probability distributions over the compact set X. It is well-known that to any probability measure $\mu \in \Lambda(X, \mathcal{B}(X))$ corresponds a unique distribution function $F_\mu \in \mathbb{D}(X)$ and vice versa, and $\mu \geq_s \mu'$ is equivalent to $F_\mu \leq F_{\mu'}$ (see, for instance, Stokey & al. [40]).[6] $(\Lambda(X, \mathcal{B}(X)), \geq_s)$ is thus isomorphic to $(\mathbb{D}(X), \leq)$, and is therefore a complete lattice with minimal element the singular probability measure δ_0, and maximal element the singular probability measure $\delta_{k_{\max}}$.$\blacksquare$

The space $\Lambda(X, \mathcal{B}(X))$ is also endowed with the weak topology under which a sequence of probability measures $\{\mu_n\}$ in $\Lambda(X, \mathcal{B}(X))$ is said to weakly converge to $\mu \in \Lambda(X, \mathcal{B}(X))$ if for all continuous functions $f : X \to \mathbb{R}$:

$$\lim_{n\to\infty} \int_X f(k)\mu_n(dk) = \int_X f(k)\mu'(dk), \tag{CV}$$

in which case we write $\mu_n \Longrightarrow \mu$, and call μ the weak limit of the sequence $\{\mu_n\}$. It is easy to see that an increasing sequence $\{\mu_n\}$ in $(\Lambda(X, \mathcal{B}(X)), \geq_s)$ necessarily converges to its supremum, that is:

$$\mu_n \Rightarrow \mu = \vee\{\mu_n\}.$$

Similarly, for a decreasing sequence $\{\mu_n\}$:

$$\mu_n \Rightarrow \mu = \wedge\{\mu_n\}.$$

Just as we defined the stochasic order on $\Lambda(X, \mathcal{B}(X))$, we also define the stochastic order on the set $\Lambda(S, \mathcal{B}(S))$ of probability measures defined on the measurable space $(S, \mathcal{B}(S))$. It is well know that this set is not a complete lattice, albeit a chain complete lattice (for a proof, see HP92).

[6] This is not true if $X \subset \mathbb{R}^l$ with $l \geq 2$, and this is one fundamental reason why the argument in this paper cannot be trivially generalized to economies with Markov shocks. See more on this at the end of Section 4 of the paper.

Proposition 5 $(\Lambda(S, \mathcal{B}(S)), \geq_s)$ *is a chain complete lattice with minimal and maximal elements* $\delta_{(0,z_{\min})}$ *and* $\delta_{(k_{\max},z_{\max})}$.

The proofs of existence of equilibria in this paper are based on an extension of Tarski's fixed point theorem. Recall that Tarski's theorem combines the isotonicity of a self map F on the complete lattice $(P, \geq)$ to prove the existence of a complete lattice of fixed point of F. Although Tarski's fixed point theorem is not constructive, we show below (in a result related to Theorem 4.2 in Dugundji and Granas [22]), that the additional property of order continuity of F permits the construction of specific recursive sequences order converging to the extremal fixed points of F. Recalling that a countable chain $\{x_n\}$ with $x_i \leq x_j$ in the poset $(P, \leq)$ order converges to $x \in P$ iff $\vee\{x_n\} = x$, order continuity is defined as follows:

Definition 6 *Let* $(P, \geq)$ *be a poset. A function* $F : (P, \geq) \to (P, \geq)$ *is order continuous if for any countable chain* $C \subset P$ *such that* $\vee C$ *and* $\wedge C$ *both exist,*

$$\vee\{F(C)\} = F(\vee C) \text{ and } \wedge \{F(C)\} = F(\wedge C).$$

As a result of order continuity, which implies isotonicity since $u \leq v$ implies $\vee\{F(u), F(v)\} = F(\vee\{u, v\}) = F(v)$, and thus $F(u) \leq F(v)$, successive iterations on F starting from extremal elements order converge to extremal fixed points, a property we exploit in the following theorem.

Theorem 7 *Let* $(P, \geq)$ *be a complete lattice with maximal element* $p_{\max}$ *and minimal element* $p_{\min}$. *(a). If* $F : (P, \geq) \to (P, \geq)$ *is isotone, then the set of fixed points of* F *is a non-empty complete lattice with maximal and minimal elements. (b). If* $F : (P, \geq) \to (P, \geq)$ *is order continuous and there exists* $a \in P$ *such that* $F(a) \geq a$, *then* $\vee\{F^n(a)\}_{n\in\mathbb{N}}$ *is the minimal fixed point of* F *in the order interval* $[a, p_{\max}]$. *(c). If* $F : (P, \geq) \to (P, \geq)$ *is order continuous and there exists* b *such that* $b \geq F(b)$, *then* $\wedge\{F^n(b)\}_{n\in\mathbb{N}}$ *is the maximal fixed point in order interval* $[p_{\min}, b]$.

Proof. (a). This is essentially Tarski's fixed point theorem (see Tarski [42]). Consider the set $Q = \{x \in P,\ x \leq F(x)\}$. Since $p_{\min} \in Q$, it is nonempty. Consider a chain C in Q, and $u = \vee C$. Then $c \leq u$ for all $c \in C$, so that by isotonicity of F, $c \leq F(c) \leq F(u)$ for all $c \in C$, which implies that $F(u) \geq u$. Thus $u \in C$, and every chain in Q has an upper bound. By Zorn Lemma, Q has a maximal element, which we denote q. Since $q \leq F(q)$, $F(q) \leq F^2(q)$ so $F(q) \in Q$, which implies that $F(q) = q$, and q is clearly the maximal fixed point in P since any fixed point must belong to Q. Considering $Q = \{x \in P,\ F(x) \leq x\}$ and following a symmetric argument proves the existence of a minimal fixed point. (b). Suppose $F : (P, \geq) \to (P, \geq)$ is order continuous. Since $F(a) \geq a$ and F isotone, $\forall n \in N$, $F^{n+1}(a) \geq F^n(a)$, and $\{F^n(a)\}_{n\in\mathbb{N}}$ is a countable chain. P is a complete lattice so $\vee\{F^n(a)\}_{n\in\mathbb{N}}$ exists, and if F is order continuous, $F(\vee\{F^n(a)\}_{n\in\mathbb{N}}) = \vee\{F(a)\} = \vee\{F^n(a)\}_{n\in\mathbb{N}}$ so that $\vee\{F^n(a)\}_{n\in\mathbb{N}}$ is a fixed point of F. Consider any $d \in P$ such that $F(d) = d$.

Since $d \geq a$ and F is isotone, it is easy to see that $\forall n \in N$, $d \geq F^n(a)$ which implies that d is an upper bound of $\{F^n(a)\}_{n\in\mathbb{N}}$. Thus $d \geq \vee\{F^n(a)\}_{n\in\mathbb{N}}$ and $\vee\{F^n(a)\}_{n\in\mathbb{N}}$ is thus the least fixed point of F in $[a, p_{\max}]$, and thus the unique minimal fixed point. (c). Follows a similar argument to that in (b). ■

In view of the above proof, the hypothesis of order continuity in (b) and (c) can clearly be weakened to that of order continuity along monotone recursive F-sequences, that is, sequences of the form $\{x, F(x), ..., F^n(x), ...\}$ where either $x \leq F(x)$ or $x \geq F(x)$ and as long as F is isotone.[7] Therefore:

Corollary 8 *The preceding theorem holds with $F : (P, \geq) \to (P, \geq)$ isotone and order continuous along monotone F-sequences and isotone.*

2.2 Assumptions on the economic primitives

Consider the standard OLG model in which agents live for two periods, with preferences represented by a standard utility function for which we assume in IV a complementarity condition between consumption when young (denoted as c_1) and consumption when old (denoted as c_2). Note that we allow for non-time separability in lifetime consumption, so Wang [44] is a special case covered by Assumption 1.

Assumption 1. The utility function $u : X \times X \to \mathbb{R}$ is:

I. twice continuously differentiable;

II. strictly increasing in each of its arguments and jointly concave;

III. satisfies $\forall c_2 > 0$, $\lim_{c_1\to 0^+} u_1(c_1, c_2) = +\infty$ and $\forall c_1 > 0$, $\lim_{c_2\to 0^+} u_2(c_1, c_2) = +\infty$;

IV. has increasing differences in (c_1, c_2) (i.e., $u_{12} \geq 0$).

As in Wang [44] and Hauenschild [27], we assume that the production shocks are *iid* with compact support.

Assumption 2. The random variable z_t follows an iid process characterized by the probability measure denoted γ. The support of γ is the compact set $Z = [z_{\min}, z_{\max}] \subset \mathbb{R}$ with $z_{\max} > z_{\min} > 0$.

Production of the single consumption/capital good admits a nonclassical specification in its reduced form, [8]denoted $F(k, n, K, N, z)$, with constant returns to scale in private inputs (k, n) and with (K, N) being the social inputs . The following assumptions on F, including the monotonic properties of the equilibrium wage rate and rental rate of capital, are adapted from the literature on nonoptimal stochastic growth (e.g., Coleman [11] and Greenwood and Huffman [26]), and are completely standard. Anticipating $n = 1 = N$ in equilibrium (since households do not value leisure), we state some of our assumptions in terms of this restriction on equilibrium labor supply.[9]

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[7] Order continuity along monotone recursive F-sequences does not imply that F is increasing. Consider for instance $F : [0, 1] \to [0, 1]$ such that $F(x) = 1 - x$. F is clearly continuous along the monotone recursive F-sequence $\{1/2, 1/2, 1/2, ...\}$, but F is not increasing.

[8] See Greenwood and Huffman [26] for an extensive discussion of "reduced-form" production technologies.

[9] Stachurski [39] studies the interesting case of threshold externalities in an OLG model. We do not consider this case in this paper.

Assumption 3. The production function $F(k, n, K, N, z) : X \times [0, 1] \times X \times [0, 1] \times Z \to \mathbb{R}_+$ is:

I. twice continuously differentiable in its first two arguments, and continuous in all arguments;

II. increasing in all arguments, strictly increasing and strictly concave in its first two arguments;

IIIa. $r(k, z) = F_1(k, 1, k, 1, z)$ is decreasing and continuous in k, and $\lim_{k\to 0} r(k, z) = +\infty$;

IIIb. $w(k, z) = F_2(k, 1, k, 1, z)$ is increasing and continuous in k, and $\lim_{k\to 0^+} w(k, z) = 0$;

IV. such that there exists a maximal sustainable capital stock $k_{\max}$ (i.e., $\forall k \geq k_{\max}$ and $\forall z \in Z$, $F(k, 1, k, 1, z) \leq k_{\max}$, and $\forall k \leq k_{\max}$, $\exists z \in Z$, $F(k, 1, k, 1, z) \geq k_{\max}$), and with $F(0, 1, 0, 1, z) = 0$.

Also, IV implies that the set of feasible capital stock can be restricted to be in the compact interval $X = [0, k_{\max}]$ (as long as we place the initial date zero capital stocks $k_0 = K_0$ in X), and also places restrictions on the amount of nonconvexity we can allow.

Finally, we will make an additional assumption leading to sharper properties of the MEDP.[10]

Assumption 3'. Both $r(k, z)$ and $w(k, z)$ are continuous and increasing in z for all k.

3 Markov Equilibrium Decision Policies

INTRODUCTION

In this section, the construction of extremal non trivial MEDP is shown to result from the application of the Euler equation method, a method which has been traditionnally reserved to the study of infinitely-lived agent models with non-classical production (see, for instance, Coleman[11], Greenwood and Huffman[26], Datta & al [13], or Morand and Reffett [33]). From the Euler equation we construct an isotone non-linear operator which maps a complete lattice of policies into itself.[11] As a direct consequence of Tarski's fixed point theorem, the set of fixed points of this operator is a non-empty complete lattice, and by construction all fixed points but the trivial 0 are MEDP. The construction of extremal MEDP by successive iterations of the operator relies on the order continuity of the operator.

3.1 The Euler equation method

A young agent in period t earns the competitive wage w and must decide what amount y to invest, thus saving for period $t + 1$ consumption, while the rest

[10] The monotonicity assumption is standard; it can be shown that the continuity assumption may be weakened to upper semicontinuity.

[11] The non-linear operator in this paper differs from the one in the infinitely-lived agent models cited above.

is consumed immediately. To make this decision the agent postulates a law of motion $h \in W$ for the capital stock which he uses to compute the expected return on his capital investment. In addition the competitive wage and return on capital are obtained from the firms optimization problem, that is, $w(k,z) = F_2(k,1,k,1,z)$ and $r(k,z) = F_1(k,1,k,1,z)$. Thus, given $s \in S^*$ and $h \in W$, a young agent seeks to solve:

$$\max_{y \in [0,w(s)]} \int_Z u(w(s) - y, r(h(s), z')y)\gamma(dz'),$$

Using the Euler equation associated with the agent's maximization problem we define a MEDP as follows:

Definition 9 *A MEDP is a function bounded function h such that, for all $s \in S^*$, $0 < h(s)$ and:*

$$\begin{aligned} & \int_Z u_1(w(s) - h(s), r(h(s), z')h(s))\gamma(dz') \\ = \quad & \\ & \int_Z u_2(w(s) - h(s), r(h(s), z')h(s))r(h(s), z')\gamma(dz'). \end{aligned} \tag{E}$$

We restrict our search for equilibrium decision policies (i.e., $k = K$ and $n = N = 1$) to those that are Markovian (i.e., function of the current state), thus justifying our term "Markovian Equilibrium Decision Policy" or MEDP. Following the Euler equation method, we define a nonlinear operator A as follows:

Definition 10 *Given any $h \in W$, define A as follows: If $h(s) > 0$, then $Ah(s)$ is defined as the unique solution*[12] *for y to:*

$$\begin{aligned} & \int_Z u_1(w(s) - y, r(h(s), z')y)\gamma(dz') \qquad \text{(E')} \\ = & \int_Z u_2(w(s) - y, r(h(s), z')y)r(y, z')\gamma(dz'), \end{aligned}$$

and $Ah(s) = 0$ whenever $h(s) = 0$.

A bounded function h is a MEDP if and only if it is a non-zero fixed point of the operator A, so that existence, characterization, and construction of extremal MEDP is basically the study of the set of non-trival fixed points of A. Three properties of A will be relevant here. First A is an isotone operator on several complete lattices, which implies by Tarski's theorem that there are fixed point of A,and that the set of fixed points has the specific order structure of a complete lattice. Second, A can be shown to be order continuous along

[12] It is easy to verify the existence of a unique solution under Assumption 1.

monotone sequences, with the consequence that the result of the previous section on construction of extremal fixed points of A apply. And finally, under some minimum assumption, A can be show to map strictly up a small order interval to the right of 0, enabling us to set aside the trivial fixed point 0 and to construct minimal MEDP.

Proposition 11 *(Isotonicity). Under Assumptions 1, 2, 3, the operator A is an isotone self map on $(W, \leq)$. With the addition of Assumption 3', A is an isotone self map on $(H, \leq)$ and on H_k^u (resp. H_k^l).*

Proof. By construction A maps W into itself, and it is easy to verify that $Ah \geq Ah'$ whenever $h \geq h'$. Clearly Ah is isotone is k whenever h is, and Assumption 3' is sufficient for preservation of isotonicity in z, thus making A an isotone map on $(H, \leq)$. Consider $h \in H_k^u$, and therefore right continuous at every $k \in [0, k_{\max}[$ given any z. Since the unique solution to (E') can be expressed as a continuous function of h, w, and r, Assumption 1, 2, 3, 3' imply that Ah is right continuous in k as well, and therefore also usc in k since increasing. Thus A is an isotone self map on H_k^u (and on H_k^l by a similar argument).[13]
■

As a consequence of Tarski's fixed point theorem, we have the following results:

Proposition 12 *(Fixed points of A). Under Assumptions 1, 2, 3 the set of fixed points of A in $(W, \leq)$ is a non-empty complete lattice. Under Assumptions 1, 2, 3, 3' the set of fixed points of A in $(H, \leq)$ (resp. $(H_k^u, \leq)$ and $(H_k^l, \leq)$) is a non-empty complete lattice.*

Proposition 13 *(Order continuity) Under Assumptions 1,2, 3 (3') $A : (W, \geq) \to (W, \geq)$ (resp. $(H, \geq) \to (H, \geq)$) is order continuous along any monotone sequence.*

Proof. We prove that for an increasing sequence $\{g_n\}$ in $(H, \leq)$,

$$\sup(\{Ag_n(s)\}) = A(\sup\{g_n(s)\}).$$

For such a sequence and for all $s \in S$, the sequence of real numbers $\{g_n(s)\}$ is increasing and bounded above (by $w(s)$), thus $\lim_{n\to\infty} g_n(s) = \sup\{g_n(s)\}$. For the same reason $\lim_{n\to\infty} Ag_n(s) = \sup\{Ag_n(s)\}$. By definition, for all $n \in \mathbb{N}$, and all $s \in S^*$:

$$\begin{aligned} & \int_Z u_1(w(s) - Ag_n(s), r(g_n(s), z')Ag_n(s))\gamma(dz') \\ = & \\ & \int_Z u_2(w(s) - Ag_n(s), r(g_n(s), z')Ag_n(s))r(Ag_n(s), z')\gamma(dz') \end{aligned}$$

[13] The same argument applies for semicontinuity in z given any k. As a result, under Assumptions 1, 2, 3, 3' A is an increasing self map on the set of isotone and (jointly) continuous functions.

The functions u_1 are u_2 continuous (Assumption 1), r is continuous in its first argument (Assumption 3), hence taking limits when n goes to infinity, we have:

$$\begin{aligned} & \int_Z u_1(w(s) - \sup\{Ag_n(s)\}, r(\sup\{g_n(s)\}, z') \sup\{Ag_n(s)\})\gamma(dz') \\ = & \\ & \int_Z u_2(w(s) - \sup\{Ag_n(s)\}, r(\sup\{g_n(s)\}, z') \sup\{Ag_n(s)\}) r(\sup\{Ag_n(s)\}, z')\gamma(dz'), \end{aligned}$$

which implies that $A(\sup\{g_n(s)\}) = \sup\{Ag_n(s)\}$. A symmetric argument can easily be made for a decreasing sequence $\{g_n\}$ noting that in this case, the sequences of real numbers $\{g_n(s)\}$ is decreasing and bounded below by 0, therefore $\lim_{n\to\infty} g_n(s) = \inf\{g_n(s)\}$.■

To exclude economies in which 0 may be the only MEDP, we show in the Lemma below that the following assumption is suuficient to demonstrate that A maps strictly up an order interval immediately to the right of 0. This property will leads to the construction by successive approximation of minimal MEDP

Assumption 4. *(i)* $\lim_{k\to 0^+} r(k, z_{\max})k = 0$, *and (ii)* $\forall c_1 > 0$, $\lim_{c_2\to 0} u_2(c_1, c_2) = \infty$.

Lemma 14 *Under assumption 1, 2, 3, 4 there exists $h_0 \in H_k^l$ and continuous in z such that (i) $\forall s \in S^*$, $Ah_0(s) > h_0(s) > 0$, and (ii) $\forall h \in]0, h_0]$, $Ah > h$ on S^*.*

Proof. See Appendix B RE-WRITE THIS APPENDIX.■

Proposition 15 *(Construction of extremal MEPD). Under Assumptions 1, 2, 3, 3' and 4, $h_{\min} = \vee\{A^n h_0\} \in H_k^l$ is the minimal MEDP in W (and thus also in in H and in H_k^l), and $h_{\max} = \wedge\{A^n w\} \in H_k^u$ is the maximal MEDP in W (and thus in H and in H_k^u).*

Proof: As a consequence of (i) in the lemma of the previous section, A maps the complete lattice $[h_0, w] \subset W$ into itself, and therefore A must have a fixed point greater than h_0. By (ii) in the lemma of the previous section, there cannot be any other strictly positive fixed point in W smaller than h_0. By Theorem 4 in Section 2, the minimal fixed point of A in $W \cap [h_0, w]$ (an thus the minimal MEDP in W) must therefore be $h_{\min} = \vee\{A^n h_0\}$. It is also the minimal MEDP in H with the addition of Assumption 3'. Note that:

$$h_{\min}(s) = \vee\{A^n h_0\}(s) = \lim_{n\to\infty} A^n h_0(s) = \sup\{A^n h_0(s)\}.$$

Further, $h_{\min} \in H_k^l$ since it is the upper envelope of a family of of elements of H_k^l, and is therefore the minimal MEDP in H_k^l. Similarly, the maximal MEDP in $(W, \leq)$ is obtained as the inf (pointwise limit) of a decreasing sequence beginning at w. That is, is:

$$h_{\max}(s) = \wedge\{A^n w\}(s) = \lim_{n\to\infty} A^n w(s) = \inf\{A^n w(s)\},$$

which implies that $h_{\max} \in H_k^u$ since it is the lower envelope of a family of elements of H_k^u.

Corollary 16 *The function $g_{\min} : S \to X$ defined as:*

$$g_{\min}(s) = \inf_{k'>k}\{\sup\{A^n h_0(k', z)\}\} = \inf_{k'>k}\{\vee\{A^n h_0\}(k', z)\} \text{ for all } s = (k, z) \in [0, k_{\max}[\times Z$$

and $g_{\min}(k_{\max}, z) = \vee\{A^n h_0\}(k_{\max}, z)$ is the minimal MEDP in H_k^u.

Proof. By construction $g_{\min} \in H_k^u$, $g_{\min}(., z)$ and $q(., z) = \vee\{A^n h_0\}(., z)$ differ at most at the discontinuity points of $\vee\{A^n h_0\}(., z)$, and $g_{\min}(., z)$ is the smallest usc function greater than $\vee\{A^n h_0\}(., z)$. In addition, since $\vee\{A^n h_0\} \in H_k^l$, for any $s \in S$, $g_{\min}(s) = \lim_{k' \to k^+} \vee\{A^n h_0\}(k', z)$. For any $s = (k, z) \in [0, k_{\max}[\times Z$, and for all $k' > k$, by definition of $q(., z)$:

$$\begin{array}{l} \displaystyle\int_Z u_1(w(k', z) - q(k', z), r(q(k', z), z')q(k', z))\gamma(dz') \\ = \\ \displaystyle\int_Z u_2(w(k', z) - q(k', z), r(q(k', z), z')q(k', z))r(q(k', z), z')\gamma(dz'). \end{array}$$

Both functions u_1 and u_2 are continuous and r is continuous in its first argument so taking limits when $k' \to k^+$ on both sides of the previous equality implies that:

$$\begin{array}{l} \displaystyle\int_Z u_1(w(s) - g_{\min}(s), r(g_{\min}(s), z')g_{\min}(s))\gamma(dz') \\ = \\ \displaystyle\int_Z u_2(w(s) - g_{\min}(s), r(g_{\min}(s), z')g_{\min}(s))r(g_{\min}(s), z')\gamma(dz'), \end{array}$$

which proves that, $Ag_{\min}(s) = g_{\min}(s)$.■ ■

Similarly, it is straightforward to modify $g_{\max} \in H_k^u$ to construct the maximal isotone bounded lower semicontinuous MEDP, so we state the following result without proof.

Corollary 17 *The function $g_{\max} : S \to X$ defined as for all $s \in S^*$,:*

$$g_{\max}(s) = \sup_{k'<k}\{\wedge\{A^n w\}(k', z)\},$$

and $g_{\max}(0, z) = 0$, is the maximal MEDP in H_k^l.

We now summarize our findings in the following proposition.

Proposition 18 *Under Assumptions 1, 2, 3, 3', (i) the set of bounded MEDP is a non-empty complete lattice, (ii) there exists a countable set $\{h^n\}_{n\in\mathbb{N}}$ of measurable elements of W such that any bounded MEDP h satisfies $h(s) \in cl\{h^1(s), h^2(s), ...\}$ for each s, (iii) the minimal bounded MEDP $h_{\min} = \vee\{A^n h_0\}$ belongs to H_k^l and is measurable, and (iv) the maximal bounded MEDP $h_{\max} = \wedge\{A^n w\}$ belongs to H_k^u and is measurable.*

Proof. (i). It is precisely the set of fixed points of A in $(W, \leq)$ minus 0. (ii) The function $p : S^* \times X \to R$ defined as:

$$p(s,y) = -\left| \int\limits_Z [u_1(w(s) - y, r(y,z')y) - u_2(w(s) - y, r(y,z')y)r(y,z')]\gamma(dz') \right|$$

is continuous, and the correspondence $\Psi : S \to \mathbb{R}$ defined as $\Psi(s) = [0, w(s)]$ is non-empty, compact valued and measurable. By definition, MEDP are the (non-zero) maximizers of p, that is the set of MEDP is . As a consequence of the measurable maximum theorem (see for instance, AB), the correspondence Φ defined as $\Phi(s) = \arg\max_{y\in\Psi(s)} p(s,y)$ is measurable, nonempty and compact valued. By Castaing's theorem (see AB), this implies that there exists a countable sequence $\{h^n\}_{n\in\mathbb{N}}$ of measurable selectors from Φ satisfying:

$$\forall s \in S,\ \Phi(s) = cl\{h^1(s), h^2(s), ...\}.$$

(iii) and (iv) were established above. ■

Proposition 19 *Under Assumptions 1, 2, 3, 3', 4 the set of bounded isotone MEDP is a non-empty complete lattice.*

Proposition 20 *Under Assumptions 1, 2, 3, 3', 4 the set of bounded upper semicontinuous isotone MEDP is a non-empty complete lattice.*

3.2 Uniqueness of MEDP under capital income monotonicity

Under the additional assumption of capital income monotonicity, the only case discussed in Wang[44], we prove the existence of a unique bounded isotone h^*. Since it is both minimal and maximal MEDP, by our previous findings it is upper semicontinuous and lower semicontinuous at the same time, and therefore continuous. In addition, we prove that the Markovian equilibrium consumption decision policy corresponding to this MEDP is also isotone, and thus also continuous (if both $w - h^*$ and h^* are isotone when w is continuous, they both must also be continuous.[14]

Proposition 21 *Under Assumption 1, 2, 3, 3', and 4, if $r(y,z)y$ is increasing in y for all $z \in Z$ (an hypothesis we call "capital income monotonicity") then there exists a unique bounded isotone MEDP h^* in H. The corresponding Markovian equilibrium consumption policy, $w - h^*$ is also isotone, which implies that both h^* and $w - h^*$ are continuous.*

[14] In fact, both are Lipschitz continuous functions, an important property when considering numerical implementations of our methods since Lipschitz continuous functions can be approximated with greater accuracy and convergence rates than merely continuous functions.

Proof: Under capital income monotonicity, for all $s \in S^*$, it is easy to see that the following equation in y:

$$\begin{aligned}&\int_Z u_1(w(s)-y, r(y,z')y)\gamma(dz')\\ =&\\ &\int_Z u_2(w(s)-y, r(y,z')y)r(y,z')\gamma(dz').\end{aligned}$$

has a unique solution. Note that the function h^* is therefore the maximal and minimal MEDP, and thus usc and lsc in k, i.e., continuous in k. By definition, for all $s \in S^*$:

$$\begin{aligned}&\int_Z u_1(w(s)-h^*(s, r(h^*(s),z')h^*(s))\gamma(dz')\\ =&\\ &\int_Z u_2(w(s)-h^*(s), r(h^*(s),z')h^*(s))r(h^*(s),z')\gamma(dz').\end{aligned} \tag{E''}$$

Suppose there exists $s=(k,z) \in X^* \times Z$ such that $w(k,z)-h^*(k,z)$ decreases with an increase in k. Then, for all $z' \in Z$, the expression:

$$u_1(w(k,z)-h^*(k,z), r(h^*(k,z),z')h^*(k,z))$$

increases with k under the assumption of capital income monotonicity, and given that $h^*(k,z)$ is increasing in k, $u_{12} \geq 0$ and $u_{11} \leq 0$. However, for all $z' \in Z$, the expression:

$$u_2(w(k,z)-h^*(k,z), r(h^*(k,z),z')h^*(k,z))r(h^*(k,z),z')$$

necessarily decreases with an increase in k. Thus LHS and RHS in equation (E") above move in opposite direction when k increases, which is impossible. As a result, $w(k,z)-h^*(k,z)$ must be increasing in k. The same argument works to show that $w(k,z)-h^*(k,z)$ must be increasing in z. Finally, under the assumption that w is continuous, if both the equilibrium investment and the equilibrium consumption policies are increasing in s, they both necessarily must be continuous in s.

We emphasize that w and $w-h^*$ are in fact Lipschitz continuous functions since their gradient fields are all bounded by the variation in the wage rate in equilibrium, an important property when considering numerical implementations of our methods since Lipschitz continuous functions can be approximated with greater accuracy and convergence rates than merely continuous functions. We also note that capital income isotonicity is not necessary for uniqueness of MEDP. Consider, for instance, preferences represented by:

$$\ln(c_t)+\ln(c_{t+1}).$$

The maximization problem of an agent is:

$$\max_{y\in[0,w(s)]}\left\{\ln(w(s)-y)+\int_Z \ln(r(h(s),z')y)\gamma(dz')\right\},$$

and the corresponding first order condition is:

$$(w(s) - y) = y.$$

Thus, irrespective of the production function, there exists a unique Markovian equilibrium decision policy (the function $h = .5w$).■

4 Existence and construction of stationary Markov equilibria

We define a stationary Markov equilibrium as a "non-trivial" (i.e., not all mass is concentrated at $k = 0$) invariant distribution, in line with the work of Hopenhayn and Prescott [28] and Futia[24], and in contrast to Wang[44][45] who follows the path of Duffie & al.[23] and focuses on ergodic distributions. Our main contribution is to establish algorithms converging to extremal invariant probability measures corresponding to any increasing and measurable MEDP h. The invariants measures are precisely the fixed points of a stochastic operator mapping the chain complete lattice $\Lambda(S, B(S))$ into itself, and most everything follows from the order continuity of this operator. When the MEDP is also continuous (which is the case under capital income monotonicity), In this case the order continuous operator maps the complete lattice $\Lambda(X, B(X))$ into itself, and the set of SME is a non-empty complete lattice. We treat the case of h continuous first before proceeding with the general case.

4.1 SME associated with a continuous, increasing and measurable MEDP

Recall that any measurable bounded MEDP h induces a Markov process for the capital stock represented by the transition function P_h defined as:

$$\begin{aligned} \forall A \in \; & \mathcal{B}(X),\, P_h(k, A) = \Pr\{h(k, z) \in A\} = \gamma(\{z \in Z,\ h(k, z) \in A\}) \\ = \; & \int_Z \chi_A(h(k, z))\gamma(dz). \end{aligned}$$

That is, $P_h(k, A)$ is the probability that the capital stock is in the set A one period after being equal to k.[15] If we denote by μ_t the probability measure associated with the random variable k_t, then μ_{t+1} is defined by applying the operator $T_h^* : (\Lambda(X, \mathcal{B}(X)), \geq_s) \to (\Lambda(X, \mathcal{B}(X)), \geq_s)$ to μ_t in the following manner:

$$\forall B \in \mathcal{B}(X),\, \mu_{t+1}(B) = T_h^* \mu_t(B) = \int P_h(k, B)\mu_t(dk). \tag{M1}$$

Thus, $\mu_{t+1}(B)$ is the probability that the k_{t+1} lies in the set B if k_t is drawn according to the probability measure μ_t. We define a stationary Markov equilibria (SME) as simply a non-trivial fixed point of T_h^*.

[15] The $\mathcal{B}(S)$-measurability of h implies that P_h is indeed a transition function since for each $k \in X$, $P_h(k, .)$ is a probability measure, and for each A, $P_h(., A)$ is a measurable function.

Definition 22 *Given a measurable MEDP h, a stationary Markov equilibrium (SME) is a probability measure $\mu \in \Lambda(X, \mathcal{B}(X))$ distinct from δ_0 such that:*

$$\forall B \in \mathcal{B}(X),\ \mu(B) = T_h^* \mu(B) = \int P_h(k, B)\mu(dk).$$

It is easy to verify that if h is isotone and continuous, then the Markov operator T_h^* is an isotone self map on $(\Lambda(X, \mathcal{B}(X)), \geq_s)$ and P_h has the Feller property (see, for instance Exercise 8.10 in SLP), or, equivalently, that T_h^* is a weakly continuous operator (see, for instance, Exercise 12.7 in SLP). These two properties imply that T_h^* is order continuous along any monotone sequence. Indeed, if the sequence $\{\mu_n\}$ is increasing, then $\mu_n \Rightarrow \mu = \vee\{\mu_n\}$, so that $T_h^*(\mu_n) \Rightarrow T_h^*(\vee\{\mu_n\})$ by weak continuity. Since T_h^* is an isotone operator, the sequence $\{T_h^*(\mu_n)\}$ is also increasing and therefore $T_h^*(\mu_n) \Rightarrow \vee\{T_h^*(\mu_n)\}$. By uniqueness of the limit, $T_h^*(\vee\{\mu_n\}) = \vee\{T_h^*(\mu_n)\}$, which proves continuity along any increasing sequence. The existence and computational results below follow directly from the fixed point theorem of Section 2.

Proposition 23 *(Fixed points of T_h^*) For any continuous and increasing MEDP h, the set of fixed points of T_h^* is a non-empty complete lattice with maximal and minimal elements, respectively $\wedge\{T_h^{*n}\delta_{k_{\max}}\}$ and $\vee\{T_h^*\delta_0\}$.*

Since our definition of SME excludes δ_0, the previous result does not necessarily imply the existence of a SME. Indeed, suppose for instance that:

$$\forall (k, z) \in S^*,\ 0 < h^*(k, z) < k.$$

It is then easy to see that given any initial distribution of capital stock, in the long run the capital stock will be 0. The only fixed point of $T_{h^*}^*$ is δ_0 and the set of SME is therefore empty, a case taking place for instance when $w(k, z) < k$ for all (k, z) in S^*.: $\forall (k, z) \in S^*$, $w(k, z) < k$. Thus one needs to think of sufficient conditions under which the set of SME is non-empty (see, for instance Wang[44]) but it is most useful to express any such condition in terms of restrictions on the primitives of the problem (unlike in Wang[44]). While condition (I) in Assumption 5 below is necessary, condition (II) is sufficient for the existence of a specific element h_0 of H to be maped up strictly by A. It implies that the isotone operator A maps the order interval $[h_0, w] \subset H$ (a complete lattice when endowed with the pointwise order) into itself, so that A must have a fixed point in this interval. Since under the assumption of capital income monotonicity, the fixed point h^* of A in H is unique it must be that:

$$\forall k \in [0, k_0] \text{ and } \forall z \in Z,\ h^*(k, z) > h_0(k, z)(> k).$$

Given this property of h^*, we show that there exist a fixed point of $T_{h^*}^*$ that is distinct from δ_0. The argument is the following: Consider any measure μ_0 with support in $[0, k_0]$ and distinct from δ_0 (we write $\mu_0 >_s \delta_0$). Since h^* maps up strictly every point in $[0, k_0]$, μ_0 is mapped up strictly by the operator $T_{h^*}^*$. By isotonicity of $T_{h^*}^*$ the sequence $\{T_{h^*}^{*n}\mu_0\}$ is increasing, and by order continuity along monotone sequences of $T^*_{h^*}$ it weakly converges to a fixed point of $T^*_{h^*}$. Clearly by construction this fixed point is strictly greater than δ_0. The rest of this section formalizes this argument.

Assumption 5: *Assume that:*

(I). There exists a right neighborhood Δ of 0 such that for all $k \in \Delta$ and all $z \in Z$, $w(k,z) \geq k$.

(II). The following inequality holds:

$$\begin{aligned} & \lim_{k \to 0^+} u_1(w(k, z_{\min}) - k, r(k, z_{\max})k) \\ < & \\ & \lim_{k \to 0^+} u_2(w(k, z_{\min}) - k, r(k, z_{\max})k) r(k, z_{\min}). \end{aligned}$$

Note that for log separable utility, condition.(II) is equivalent to:

$$\lim_{k \to 0^+} (w(k, z_{\min})/k) > 2,$$

and with the traditional Cobb-Douglass production function with multiplicative shocks, it is trivially satisfied (and so is condition (I)). For a polynomial utility of the form $u(c_1, c_2) = (c_1)^{\eta_1}(c_2)^{\eta_2}$ the condition is equivalent to:

$$\lim_{k \to 0^+} (w(k, z_{\min})/k) > [1 + \frac{\eta_1 r(k, z_{\max})}{\eta_2 r(k, z_{\min})}],$$

also trivially satisfied with Cobb-Douglass production and multiplicative shocks.

We can now state a key proposition that extends the uniqueness result in Coleman [12] and Morand and Reffett [33] obtained for infinite horizon economies to the present class of OLG models under assumption 5.

Proposition 24 *Under Assumption 5, the set of SME associated with an isotone continuous MEDP h is a non-empty complete lattice. The maximal SME is $\wedge\{T^{*n}_{h^*}\delta_{k_{\max}}\}$, and there exists $k_0 \in X$ such that the minimal SME is $\vee\{T^{*n}_{h^*}\delta_{k'}\}$ for any $0 < k' \leq k_0$.*

Proof: The proof is in two parts. Part 1 establishes the existence of h_0 that is mapped up strictly by the operator A, and Part 2 shows the existence of a probability measure μ_0 that is mapped up $T^*_{h^*}$, where h^* is the unique MEDP.

Part 1. By continuity of all functions in k, the inequality in Assumption 5 must be satisfied in a right neighborhood of 0. That is, there exists of $\Theta =]0, k_0] \subset \Delta$ such that, $\forall k \in \Theta$:

$$\begin{aligned} & u_1(w(k, z_{\min}) - k, r(k, z_{\max})k) \\ < & \\ & u_2(w(k, z_{\min}) - k, r(k, z_{\max})k) r(k, z_{\min}). \end{aligned}$$

Consequently, $\forall k \in \Theta =]0, k_0]$:

$$\begin{aligned}
& \int_Z u_1(w(k,z) - k, r(k,z')k) G(dz') \\
\leq & \\
& u_1(w(k, z_{\min}) - k, r(k, z_{\max})k) \\
< & \\
& u_2(w(k, z_{\min}) - k, r(k, z_{\max})k) r(k, z_{\min}) \\
\leq & \\
& \int_Z u_2(w(k,z) - k, r(k,z')k) r(k,z') G(dz').
\end{aligned}$$

Next, consider the function $h_0 : X \times Z \to X$ defined as:

$$h_0(k,z) = \begin{array}{c} 0 \text{ if } k = 0,\ z \in Z \\ k \text{ if } 0 < k \leq k_0,\ z \in Z \\ k_0 \text{ if } k \geq k_0,\ z \in Z \end{array}.$$

We prove now that $Ah_0 > h_0$. First, consider $0 < k \leq k_0$, $z \in Z$, and suppose that $Ah_0(k,z) \leq h_0(k,z) = k$. Then:

$$\begin{aligned}
& \int_Z u_1(w(k,z) - k, r(k,z')k) G(dz') \\
< & \\
& \int_Z u_2(w(k,z) - k, r(k,z')k) r(k,z') G(dz') \\
\leq & \\
& \int_Z u_2(w(k,z) - Ah_0(k,z), r(k,z') Ah_0(k,z)) r(Ah_0(k,z), z') G(dz'),
\end{aligned}$$

where the first inequality stems from the result just above, and the second from $u_{22} \leq 0$, $u_{12} \geq 0$ and r decreasing in its first argument. By definition of Ah_0, this last expression is equal to:

$$\int_Z u_1(w(k,z) - Ah_0(k,z), r(k,z') Ah_0(k,z)) G(dz').$$

Thus, we have $Ah_0(k,z) \leq k$ and:

$$\begin{aligned}
& \int_Z u_1(w(k,z) - k, r(k,z')k) G(dz') \\
< & \\
& \int_Z u_1(w(k,z) - Ah_0(k,z), r(k,z') Ah_0(k,z)) G(dz').
\end{aligned}$$

which contradicts the hypothesis that $u_{11} \leq 0$ and $u_{12} \geq 0$. It must therefore be that for all $k \in]0, k_0]$ and all $z \in Z$, $Ah_0(k,z) > h_0(k,z) = k$, that is A maps

h_0 strictly up at least in the interval $]0, k_0]$. Finally, for $k > k_0$, since Ah_0 is increasing in its first argument:

$$Ah_0(k, z) \geq Ah_0(k_0, z) > h_0(k_0, z) = k_0 = h_0(k, z).$$

We have therefore proven that A maps h_0 up (strictly). The order interval $[h_0, w]$ in $(H, \leq)$ is a complete lattice when endowed with the pointwise order, and by isotonicity of A, there must exists a fixed point of A in that interval. Under capital income isotonicity, $h^* \in [h_0, w]$.

Part 2. Consider any probability measure in $(\Lambda(X, \mathcal{B}(X)), \geq_s)$ but with support in the compact interval $[0, k_0]$ and distinct from δ_0. We show next that $T^*_{h^*}\mu_0 \geq_s \mu_0$. Consider any $f : X \to \mathbb{R}_+$ measurable, increasing and bounded, we have:

$$\begin{aligned} & \int [\int f(k') P_{h^*}(k, dk')] \mu_0(dk) = \int [\int f(h^*(k, z)) \lambda(dz)] \mu_0(dk) \\ = & \\ & \int_{[0,k_0]} [\int_Z f(h^*(k, z)) \lambda(dz)] \mu_0(dk) + \int_{[k_0, k_{\max}]} [\int f(h^*(k, z)) \lambda(dz)] \mu_0(dk) \\ \geq & \\ & \int_{[0,k_0]} f(k) \mu_0(dk) \end{aligned}$$

since $h^*(k, z) > k$ on $[0, k_0]$. Note that if f is strictly positive on $[0, k_0]$ then the last inequality is strict.

We have just demonstrated that $T^*_{h^*}\mu_0 \geq_s \mu_0$ and that $T^*_{h^*}\mu_0$ is distinct from μ_0, so we write $T^*_{h^*}\mu_0 >_s \mu_0$ ($>_s \delta_0$). By order continuity along any monotone sequence of $T^*_{h^*}$, necessarily the increasing sequence $\{T^{*n}_{h^*}\mu_0\}$ converges weakly to a fixed point of $T^*_{h^*}$ strictly greater than δ_0. In addition, it is easy to see that there cannot be any fixed point of $T^*_{h^*}$ with support in $[0, k_0]$ other than δ_0 so that the minimal non-trivial (i.e., distinct from δ_0) fixed point of $T^*_{h^*}$, which is by definition the minimal SME, can be constructed as the limit of the sequence $\{T^{*n}_{h^*}\mu_0\}$, where $\mu_0 = \delta_{k'}$ for any $0 < k' \leq k_0$. This completes the proof that the set of SME is the non-empty complete lattice of fixed points of $T^*_{h^*}$ minus δ_0, and that the maximal SME and minimal SME can be obtained as claimed.■

Corollary: Under capital income monotonicity....

4.2 Existence and construction of extremal SME for non-continuous MEDP

Continuity of h, however, is not necessary for T^*_h to be order continuous along recursive monotone T^*_h-sequences. Indeed, recall the result in Morand and Reffett[34] that in an OLG model with Markov shocks associated with the transition function Q, if Q is increasing and satisfies Doeblin's condition (D), then the $\mathcal{B}(S)$-measurability of any isotone MEDP h is sufficient for T^*_h is order continuous along recursive monotone T^*_h-sequences, a property leading to the construction of extremal SME by successive approximation (see REF).

We use this result by considering iid shocks as a special case of Markov shocks for which the transition function is defined as $Q(z,B)=\gamma(B)=\Pr(z'\in B$ given $z)$. Recall that a Markov transition function Q satisfies Doeblin's condition (D) is there exists $\delta\in\Lambda(Z,B(Z))$ and $\theta<1$ and $\eta>0$ such that:

$$\forall B\in\mathcal{B}(Z),\ \delta(B)\geq\theta \text{ implies that } \forall z\in Z,\ Q(z,B)\geq\eta.$$

In particular, since $Q(z,B)=\gamma(B)$, then any $\theta=\eta<1$ and $\delta=\gamma$ show that iid shock trivially satisfy Doeblin's condition. Morand and Reffett[34] also show that if Q satisfies Doeblin's condition (D), then the transition function P_h corresponding to any $B(S)$-measurable MEDP h and defined by:

$$\forall A\times B\in\mathcal{B}(S),\ P_h(x,z;A,B)=\{\begin{array}{c} Q(z,B) \text{ if } h(x,z)\in A \\ 0 \text{ otherwise.}\end{array}$$

also satisfies Doeblin's condition (D). Consequently, by Theorem 11.9 in Stokey & al., the n-average of any recursive T_h^*-sequence converges in the total variation norm, and therefore weakly converges, to a fixed point of the isotone T_h^*. Next, we show that the poset $(\Lambda(S,B(S)),\leq_s)$ is countable chain complete and that any monotone recursive T_h^*-sequence weakly converges. By uniqueness of the limit, the weak limit of such sequence is also the limit of the average n-sequence, and is a fixed point of T_h^*. This precisely proves that T_h^* is order continuous along recursive monotone T_h^*-sequences, and an application of Theorem 4 gives the following important result.

Proposition 25 *Under Assumptions 1, 2, 3, 3', 4, 5 and if shocks satisfy Doeblin's condition (D), for any $\mathcal{B}(S)$-measurable MEDP h in H, there exists a non-empty set of SME with maximal and minimal elements respectively given by $\gamma_{\max}(h)=\wedge\{T_h^{*n}\delta_{(k_{\max},z_{\max})}\}$ and $\gamma_{\min}(h)=\vee\{T_h^{*n}\mu_0\}$, where $\mu_0=\delta_{(k',z_{\min})}$ for any $0<k'\leq k_0$, k_0 constructed from Assumption 5.*

Finally, for economies satisfying Assumption 4, by our results in the previous section of the paper, there exist minimal and maximal MEDP $h_{\min}$ and $h_{\max}$ in H, and both are $\mathcal{B}(S)$-measurable. Necessarily, any other MEDP h in H satisfies $h_{\min}\leq h\leq h_{\max}$. By the comparative statics result of Proposition 24 above,

$$T_{h_{\min}}^*\mu_0\leq T_h^*\mu_0,$$

and recursively,

$$\gamma_{\min}(h_{\min})=\vee\{T_{h_{\min}}^{*n}\mu_0\}_{n\in N}\leq\vee\{T_h^{*n}\mu_0\}_{n\in N}=\gamma_{\min}(h).$$

By a similar argument:

$$\gamma_{\max}(h_{\max})=\wedge\{T_{h_{\max}}^{*n}\delta_{(k_{\max},z_{\max})}\}_{n\in\mathbb{N}}\geq\wedge\{T_h^{*n}\delta_{(k_{\max},z_{\max})}\}_{n\in\mathbb{N}}=\gamma_{\max}(h),$$

and this proves that $\gamma_{\max}(h_{\max})$ and $\gamma_{\min}(h_{\min})$ are the greatest and least SME, respectively. We state this very general result in the last proposition of the paper.

Proposition 26 *Under Assumptions 4 and 5, and when shocks satisfy Doeblin's condition (D), the set of SME is nonempty and there exist a greatest and a least SME, respectively* $\gamma_{\max}(h_{\max}) = \wedge\{T^{*n}_{h_{\max}}\delta_{(k_{\max},z_{\max})}\}_{n\in\mathbb{N}}$ *and* $\gamma_{\min}(h_{\min}) = \vee\{T^{*n}_{h_{\min}}\mu_0\}_{n\in N}$ *where* $\mu_0 = \delta_{(k',z_{\min})}$ *for any* $0 < k' \leq k_0$, k_0 *constructed from Assumption 5.*

5 Extensions

We consider here two applications of our monotone methods. Comparative statics and 3 period OLG model.

5.1 Multi-period OLG models

5.2 Comparative statics.

Our second example shows that the results of Hauenschild [27] that incorporates a social security system in the overlapping generation model of Wang [44] can easily be derived from our setup. This example thus illustrates the power of monotone methods to generate (weak) comparative statics results. Recall that in Hauenschild [27], a Markovian equilibrium investment policy is a function h satisfying:

$$\begin{aligned} & \int_Z u_1((1-\tau)w(k,z) - h(k,z), r(h(k,z),z')h(k,z) + \tau w(h(k,z),z')))\gamma(dz') \\ = & \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\text{(B1)} \\ & \int_Z u_2((1-\tau)w(k,z) - h(k,z), r(h(k,z),z')h(k,z) + \tau w(h(k,z),z'))) \\ & r(h(k,z),z')\gamma(dz'). \end{aligned}$$

Consider the following equation in y :

$$\begin{aligned} & \int_Z u_1((1-\tau)w(k,z) - y, r(h(k,z),z')y + \tau w(y,z')))\gamma(dz') \\ = & \\ & \int_Z u_2((1-\tau)w(k,z) - y, r(h(k,z),z')y + \tau w(y,z')))r(y,z')\gamma(dz'). \end{aligned}$$

For any $(k,z) \in X \times Z$ and $h \in E$, denote $Ah(k,z)$ the unique solution to this equation. It is easy to see that, in addition to being an order continuous isotone operator mapping E into itself, A is also isotone in $-\tau$. Consequently, an increase in τ generates a decrease (in the pointwise order) of the extremal Markovian equilibrium investment policies $h_{\tau,\max}$ and $h_{\tau,\min}$.

Next, recall that any equilibrium investment policy h induces a Markov process for the capital stock defined by the following transition function P_h:

For all $A \in \mathcal{B}(X)$, $P_h(k,A) = \Pr\{h(k,z) \in A\} = \lambda(\{z \in Z,\ h(k,z) \in A\})$.

Consider two Markovian equilibrium policies $h' \geq h$ and their respective transition functions $P_{h'}$ and P_h. For any $k \in X$ and any function $f : X \to \mathbb{R}_+$ bounded, measurable and increasing:

$$\int f(k')P_{h'}(k, dk') = \int f(h'(k, z))\lambda(dz) \geq \int f(h(k, z))\lambda(dz) = \int f(k')P_h(k, dk').$$

Thus, for any $\mu \in \Lambda(X, B(X))$:

$$\begin{array}{rcl} \int f(k')T^*_{h'}\mu(dk') & = & \int [\int f(k')P_{h'}(k, dk')]\mu(dk) \\ & \geq & \\ \int [\int f(k')P_h(k, dk')]\mu(dk) & = & \int f(k')T^*_h\mu(dk'), \end{array}$$

which establishes that $T^*_{h'}\mu \geq T^*_h\mu$. Thus the natural ordering on the set of taxes τ induces an ordering by stochastic dominance of the corresponding extremal stationary Markov equilibria in the following way:

$$\tau' \geq \tau \text{ implies } h_{\tau,\max} \geq h_{\tau',\max} \text{ implies } \lim_{n\to\infty} T^{*n}_\tau \delta_{k\max} \geq_s \lim_{n\to\infty} T^{*n}_{\tau'} \delta_{k\max}.$$

In particular, considering $\tau = 0$, the (Hauenschild Proposition 2)..

6 Appendix B.

Lemma 27 *Under Assumption 4, for all $k \in X^*$, there exists a right neighborhood $\Omega =]0, \overline{k}]$ with $0 < \overline{k} \leq w(k, z_{\min})$ and $M > 0$ such that, for all $x \in \Omega$,*

$$u_2(w(k, z_{\min}) - x, r(x, z_{\max})x) > M.$$

Proof. If $\lim_{x\to 0^+} r(x, z_{\max})x = 0$ then for all $k \in X^*$:

$$\lim_{x\to 0^+} u_2(w(k, z_{\min}) - x, r(x, z_{\max})x) = u_2(w(k, z_{\min}), \lim_{x\to 0^+} r(x, z_{\max})x) = \infty.$$

The expression $u_2(w(k, z_{\min}) - x, r(x, z_{\max})x)$ can therefore made arbitrarily large in a right neighborhood of 0, and the existence of Ω thus follows.■

Lemma 28 *For all $s = (k, z) \in S^*$, there exists $h_0(s) \in]0, w(s)[$ such that:*

$$\begin{array}{rl} & \int_Z u_1(w(s) - h_0(s), r(h_0(s), z')h_0(s))G(dz') \\ < & \\ & \int_Z u_2(w(s) - h_0(s), r(h_0(s), z')h_0(s))r(h_0(s), z')G(dz'). \end{array} \quad \text{(E0)}$$

In addition, h_0 can be chosen increasing in k for each z, constant in z (and therefore continuous and increasing in z) for each k.

Proof. Fix $k \in X^*$. For all $z \in Z$:

$$\lim_{x \to 0^+} \int_Z u_1(w(k,z) - x, r(x,z')x) G(dz')$$
$$=$$
$$\int_Z u_1(w(k,z), 0) G(dz')$$
$$\leq$$
$$u_1(w(k, z_{\min}), 0).$$

Thus there exists a right neighborhood of 0, denoted $\Psi =]0, \overline{x}]$, such that, for all $x \in \Psi$:

$$\int_Z u_1(w(k,z) - x, r(x,z')x) G(dz')$$
$$<$$
$$.5 u_1(w(k, z_{\min}), 0).$$

Next, for $x \in \Omega$:

$$\int_Z u_2(w(k,z) - x, r(x,z')x) r(x,z') G(dz')$$
$$\geq$$
$$\int_Z u_2(w(k, z_{\min}) - x, r(x, z_{\max})x) r(x,z') G(dz')$$
$$\geq$$
$$\int_Z M r(x,z') G(dz'),$$

where the first inequality stems from $u_{12} \geq 0$ and u_2 decreasing, and the second from the Lemma above. This last expression can be made arbitrarily large, independently of z, by choosing x in Ω sufficiently close to 0. That is, it is always possible to choose x^* sufficiently small in $\Omega \cap \Psi$ so that:

$$\int_Z M r(x^*, z') F(dz') \geq .5 u_1(w(k, z_{\min}), 0). \tag{E1}$$

Pick such an x^* and set $\delta_0(k,z) = x^*$ for all $z \in Z$. By construction, any $x \in]0, \delta_0(s)]$ satisfies:

$$\int_Z u_1(w(s) - x, r(x,z')x) G(dz')$$
$$<$$
$$.5 u_1(w(k, z_{\min}), 0)$$
$$\leq$$
$$\int_Z M r(x,z') G(dz')$$
$$\leq$$
$$\int_Z u_2(w(s) - x, r(x,z')x) r(x,z') G(dz').$$

That is, by construction, we have, for all $x \in]0, \delta_0(s)]$:

$$\begin{aligned} & \int_Z u_1(w(s) - x, r(x, z')x)G(dz') \\ < & \\ & \int_Z u_2(w(s) - x, r(x, z')x)r(x, z')G(dz'). \end{aligned} \tag{E2}$$

We repeat the same operation for each k in X^*, thus constructing a function $\delta_0 : S \to X$, setting $\delta_0(0, z) = 0$. By construction, for each $k \in X$, $\delta_0(k, z)$ is constant in z, and therefore increasing in z. In addition, any function smaller (pointwise) than δ_0 also satisfies (E2). In particular, the function $p_0 : X \times Z \to X$ defined as:

$$p_0(k, z) = \min_{k' \geq k} \{\delta_0(k', z)\}.$$

satisfies (E2), is increasing in k for all z, and constant in z for all k (and thus continuous in z for all k). Finally, the function h_0 defined as follows:

$$h_0(k, z) = \left\{ \begin{array}{c} \sup_{0 < k' < k} p_0(k', z) \text{ for } (k, z) \in X^* \times Z \\ 0 \text{ for } k = 0,\ z \in Z \end{array} \right\}$$

is smaller than p_0 (and therefore than δ_0, hence it satisfies (E2)), increasing in k for all z, constant in z for all k, and lower semicontinuous in k for all z.■

Proposition 29 *$\forall s \in S^*$, $h_0(s) \geq h(s) > 0$ implies that $Ah(s) > h(s) > 0$.*

Proof. Suppose that there exists $s \in S^*$ such that $Ah(s) \leq h(s)$. Then:

$$\begin{aligned} & \int_Z u_1(w(s) - h(s), r(h(s), z')h(s))G(dz') \\ < & \\ & \int_Z u_2(w(s) - h(s), r(h(s), z')h(s))r(h(s), z')G(dz') \\ \leq & \\ & \int_Z u_2(w(s) - Ah(s), r(h(s), z')Ah(s))r(Ah(s), z')G(dz'), \end{aligned}$$

where the first inequality stems from $(E2)$ (since $0 < x = h(s) \leq h_0(s) \leq \delta_0(s)$) and the second from $u_{22} \leq 0$, $u_{12} \geq 0$ and r decreasing in its first argument. By definition of Ah, this last expression is equal to:

$$\int_Z u_1(w(s) - Ah(s), r(h(s), z')Ah(s))G(dz').$$

Summarizing, we have:

$$\begin{aligned} & \int_Z u_1(w(s) - h(s), r(h(s), z')h(s))G(dz') \\ < & \\ & \int_Z u_1(w(s) - Ah(s), r(h(s), z')Ah(s))G(dz'). \end{aligned}$$

which is contradicted by the hypothesis that $u_{11} \leq 0$ and $u_{12} \geq 0$. Thus, necessarily, $Ah(s) > h(s)$. In particular, A maps h_0 strictly up.■

7 Appendix C: Proof of Theorem 25

Let $\mathbf{X}(\mathrm{S}_e) = [0, g(S_e)] \subset \mathbf{K}^{n-1}$. Consider the extend-real valued mapping Z : $\mathbf{X}(S_e) \times \mathbf{S}_e \times [0, g) \rightarrow \bar{\mathbf{R}}^{n-1} = R^{n-1} + \{-\infty, \infty\}$

$$Z(x, S_e, h) = \Psi_2(x, S_e) - \Psi_1(x, S_e, h),$$

where, for each $j = \{1, 2, ..., n-2\}$,

$$\Psi_{1j} = \beta \int_{\Theta} u'_{j+1}(g_{j+1}(x, x, z') - h_{j+1}(x, x, z'))\bar{r}(x, z')G(dz')$$

$$\Psi_{2j} = u'_j(g_j(S_e) - x_j), j = 1, 2, ..., n-2$$

and for $j = n-1$,

$$\Psi_{1,n-1} = \beta \int_{\Theta} u'_n(r(h(S_e))x_{n-1})\bar{r}(x, z')G(dz')$$

$$\Psi_{2n-1} = u'_{n-1}(g_{n-1}(S_e) - x_{n-1}), j = 1, 2, ..., n-2$$

where $g_1 = w \cdot z_1$.

Let $t = (K_e, h) \in \mathbf{T} := \mathbf{K}_{+}^{n-1} \times \hat{\mathbf{K}}_{+}^{n-1} \times [0, g,]$, and $\mathring{\mathbf{T}} = \mathbf{K}_{++}^{n-1} \times \hat{\mathbf{K}}_{++}^{n-1} \times [0, g)$ be the interior of $\mathbf{T}$. Consider the nonlinear operator $A : \mathbf{H}(\mathbf{S}_e) \rightarrow \mathbf{H}'(\mathbf{S}_e)$, where $\mathbf{H}'$is a subset of the set of bounded functions $\mathbf{B(S}_e)$ on $\mathbf{S}_e$, where $A(h)(S_e)$ defined as follows

$$A(h)(S_e) = x(t, z) \in x^*(t, z),\ t \in \mathbf{T}, z \in Z,\ x(t, z) \text{ isotone in } t, \text{ each } z \in Z,$$

where, the correspondence $x^*(t, z)$ is defined as follows: at $t \in \mathring{\mathbf{T}}$,

$$x^*(t, z) = x \in \{x | Z(x, t, z) \leq 0, \text{ and } Z_j = 0 \text{ if } x_j > 0\},$$

at $t \in \mathbf{T} \backslash \mathring{\mathbf{T}}$, $g_k = h_k$, any $S_e \in \mathbf{S}_e, k \in \mathbf{J}_k \subset \mathbf{J}$,

$$x^*(t, z) = \{(x_{-k}, g_k), x_{-k} \in Z_{-k}(x_{-k}, g_k, t, z) \leq 0, Z_j = 0 \text{ if } x_j > 0, j \in \mathbf{J} \backslash \mathbf{J}_k,$$

elsewhere,

$$x^*(t, z) = 0,$$

Proof of Lemma 24 :

(i). $\mathbf{H}$ is an equicontinuous collection defined on a compact set $\mathbf{S}_e$. This is because as $g - h$ is increasing in K_e, h increasing in K_e, Z discrete, and g continuously differentiable, the pointwise variation for any $h \in \mathbf{H}$ on $\mathbf{S}_e$ is bounded by a uniformly continuous, locally Lipschitz function (namely, the gradient of vector of income processes g). The range of each element h is also pointwise bounded in the interval $[0, g]$. Therefore, by the Arzela-Ascoli theorem, $\mathbf{H}$ is relatively compact in $\mathbf{C}(\mathbf{S}_e)$. As $\mathbf{H}(\mathbf{S}_e)$ is also closed in $\mathbf{C}(\mathbf{S}_e)$, $\mathbf{H}$ is compact in $\mathbf{C}(\mathbf{S}_e)$.

(ii). Let $C \subset \mathbf{H}(\mathbf{S}_e)$ a chain in the pointwise partial ordered on $\mathbf{H}$. As $\mathbf{H}$ is a compact metric space, by a result in Amann [**?**] (XXX), $\vee C \in \mathbf{H}$ and $\wedge C \in \mathbf{H}$. Therefore, $\mathbf{H(S}_e)$ is chain complete.■

(NOTE: If H is a lattice, $\mathbf{H}$ is subcomplete in C(S). Need to check)

Under Assumptions $3'$ and 7, as g is locally Lipschitz in K_e on the interior of $\mathbf{K}_e$, Z is denumerable, $\mathbf{H}$ consists of locally Lipschitz continuous functions on $\mathbf{K}_e \times Z$. It is useful mention two lemmata before we prove the existence proposition.

Lemma 30 *Under assumptions $1'$, $3'$, 6, 7,for fixed $z \in \mathbf{Z}$,$t \in \mathbf{T}, h \in [0, g)$, $Z_j(x, K_e, h, z)$ is (i) decreasing in h; (ii) decreasing in K_e; and (iii) is increasing in x.*

Proof: (i) Fix (x, K_e, z). Let $h' \geq h$.The concavity assumptions in Assumption $1'$ imply Ψ_{1j} is isotone in h. Therefore, $Z_j(x, K, h'_j, z) \leq Z_j(x, K, h_j, z)$.Further, if $h' \geq h$, $h' \neq h$, as $u_j(c)$ strictly concave, this inequality is strict. Therefore, Z_j is decreasing in h.

(ii) Fix (x, h, z), and take $K'_e \geq K_e$. Noting $h \in \mathbf{H}(\mathbf{S}_e)$, Ψ_{1j} is isotone in K_e.Therefore, $Z_j(x, K', h, z) \leq Z_j(x, K, h, z)$. As g is strictly increasing in K_e by Assumptions $3'$ and 7, and Ψ_2 is strictly decreasing in K_e, and this inequality is strict. Therefore, Z is decreasing in K_e.

(iii) For each (K_e, h, z),when $x' \geq x$, Ψ_2 antitone in x by Assumptions $1'$, $3'$, and 6. Ψ_1 is isotone by Assumption $1'$, and $h \in \mathbf{H}(\mathbf{S}_e)$; therefore, $Z(x', K, h, \theta) \geq Z(x, K, h, \theta)$. If $x' \geq x, x' \neq x$, by strict concave of $u_j(c)$ in Assumption $1'$, this inequality is strict. Therefore, Z is increasing in x.■

We mention a lemma proven in Datta et. al. [15], which is a corollary to a theorem in Höft ([**?**], Theorem 2.2). Define the nonempty correspondence $y(x) : X \to 2^Y \backslash \varnothing$, where $\mathbf{X}$ and $\mathbf{Y}$ are complete lattices. Let $C \subset \mathbf{X}$ be a chain, $g(x) : C \to \mathbf{Y}$ an isotone map such that $g(x) \in y(x)$ (e.g., $g(x)$ an isotone selection in $y(x)$). Define $\vee C = x_{0\vee}$ (respectively, $\wedge C = x_{0\wedge}$). We say a nonempty valued correspondence $y(x)$ is *ascending upward* in the weak-induced set order; (i.e., for any $x_1 \leq x_2$, $y_1 \in y(x_1)$, there exists a $y_2 \in y(x_2)$ such that $y_1 \leq y_2$. A dual definition for *ascending downward* is sufficient can be used for chain faithful downward. It is *ascending* if it is ascending both upwards and downwards.

The isotone selection result is the following:

Lemma 31 *Let $(\mathbf{X}, \geq_X)$ and $(\mathbf{Y}, \geq_Y)$ be complete lattices, $y^*(x) : \mathbf{X} \to 2^{\mathbf{Y}} \backslash \varnothing$ nonempty , chain subcomplete-valued, and ascending upward or downward such* *that (a)* $\wedge \mathbf{Y} \in y(\wedge \mathbf{X})$ *and (b)* $\vee \mathbf{Y} \in y(\vee \mathbf{Y})$*. Then* $y^*(x)$ *admits an isotone selection.*

We now use the conditions in Lemma 24 to check the conditions of Lemma 25 (which guarantees the operator $A(h)(S_e)$ is well defined).

First, define the equilibrium Kuhn-Karash-Tucker (KKT) multipliers in equilibrium for each $t \in \mathring{\mathbf{T}}$. Recalling the definition of the mapping $Z(x, K_e, h, z) = Z(x, t, z)$, define a new extended-real valued mapping $\phi_j(x, t, z) : \mathbf{K}_+^{n-1} \times \mathbf{T}_+ \times Z \rightarrow \bar{\mathbf{R}}^{n-1}$ for $j \in J$ as follows:

$$\phi_j(x, t, z) = \Psi_{j2}(x, t, z) - \Psi_{j1}(x, t, z)$$

where, for each $j \in \mathbf{J}$

$$\Psi_{1j} = \beta \int_{\Theta} u_j'(g_{j+1}(x, x, z') - h_{j+1}(x, x, z'))\bar{r}(x, z')G(dz')$$

and

$$\Psi_{2j} = u_j'(g_j(S_e) - x).$$

The mapping $\phi(x, t, z) = [\phi_1, \phi_2, ..., \phi_{n-1}]$ is well-defined.[16] For fixed $(t, z) \in \mathring{\mathbf{T}} \times \mathbf{Z}$, we have the following properties for $\phi(x, t, z)$:

(a) ϕ_j is continuous and strictly increasing in x_j on $(0, g_j]$, ϕ_j is upper-semicontinuous and non-decreasing in x on $[0, g_j]$;

(b) for $x = \mathbf{0},$ $\phi_j(\mathbf{0}, t, z) = -\infty$ (by the Inada condition in Assumption $1'$); and

(c) $\phi_{J_1}(x_{-J_1}, \mathbf{g}_{J_1}, t, z) = \infty$ (as for any subset of indices of agents, say $J_1 \subset J$, where $x_{J_1} = g_{J_1}$, the Inada condition in Assumption $1'$ gives the result).

Define the KKT multipliers in equilibrium as $\phi \cdot I_x(x = 0)$, where the j^{th} component of the KKT is given by $\phi_j(x, K, h, z) \cdot I_{x_J}(x_j = 0)$. Here, $I_x(x_j = 0)$ is an indicator variable for the condition $x_j = 0$. Define the extended-valued mapping $\hat{\Psi}$ that embeds Z and the KKT multipliers $\phi \cdot I_x(x = 0)$ into a single system of functional inequalities: namely, $\hat{\Psi}(x, t, z) : \mathbf{K}^{n-1} \times \mathring{\mathbf{T}} \times \mathbf{Z} \rightarrow \bar{\mathbf{R}}^{n-1}$:

$$\begin{aligned}
\hat{\Psi}(x, t, z) &= \Psi_2(x, t, z) - \hat{\Psi}_1(x, t, z) \\
\hat{\Psi}_1 &= \beta \int_{\Theta} u'(g(x, x, z') - h(x, x, z')))\bar{r}(x, z')\gamma(dz') + \phi(x, t, z)I_x(x = 0), \\
\Psi_2 &= u'(g(K_e, z) - x),
\end{aligned}$$

where $I_x(x = 0)$ is the indicator variable on vector of equations whose components indicate for each equation $j \in \mathbf{J}$, if household j is borrowing constrained, i.e., $I_{x_j} = 1$ when $x_j = 0$.

[16] We are careful not to compare differences or sums of infinity and negative infinity. Conventions will not be needed.

We make a few observations about the system $\hat{\Psi}(x,t,z)$.Under assumptions $1'$,$3'$, 6, and 7, noting the results in Lemma 1, and given the definition of ϕ, the system $\hat{\Psi}(x,t,z)$ is well-defined. Further, observe that we have the following four statements concerning the structure of the system $\hat{\Psi}$ are the case: for any fixed $(t,z) \in \mathring{\mathbf{T}} \times \mathbf{Z}$, we have

(x.i) as $x_j \to g_j(K_e,z), \hat{\Psi}_j \to \infty$ for any $j \in \mathbf{J}$;

(x.ii) as $x \to g(K_e,z)$, the system $\hat{\Psi} \to \infty$ (both of these as $I_x = \mathbf{0}$, and $Z \to \infty$);

(x.iii) partitioning $x = (x_{-j}, x_j)$ for some component $j \in J_1 \subset \mathbf{J}$ where x_{-j} deletes x_j from x, when $x_{-j} \to \mathbf{0}_{-j}$, $x_j > \mathbf{0}_j$, $\hat{\Psi}_{-j}(x,t,z) \leq 0$; and, finally

(x.iv) when additionally $x_j \to \mathbf{0}_j$ (implying, therefore, as the vector $x \to \mathbf{0}$), by the definition of ϕ (noting Assumption $3'$ and $h \in [0,g)$), we have $\lim_{x\to\mathbf{0}} \hat{\Psi}(x,t,z) \leq \mathbf{0})$ [17].

We are now ready to prove the Theorem.

Proof of Theorem:

The proof takes place in sequence of steps. We start with five observations concerning $A(h)$.

Step (i) $A(h)$ is well-defined

Claim A: The correspondence $x^* : \mathbf{T} \times \mathbf{Z} \to \mathbf{2}^{\mathbf{K}^J} \backslash \varnothing$ *is nonempty for each* $(t,z) \in \mathbf{T} \times \mathbf{Z}$.

Proof: We first prove $x^*(t,z)$ is nonempty on $\mathring{\mathbf{T}} \times \mathbf{Z}$. First note that for any $(t,z) \in \mathring{\mathbf{T}} \times \mathbf{Z}$, by Lemma 1(iii) and the remarks above, $\hat{\Psi}$is strictly increasing and continuous in x on $(0, g(K_e,z)]$, and nondecreasing and upper-semicontinuous on $[0, g(K_e,z)]$ with $\hat{\Psi}_j = 0$ whenever $x_j = 0$. Therefore, for each $(t,z) \in \mathring{\mathbf{T}} \times \mathbf{Z}$, there is a pair of vectors $(x_l(t,z), x^T(t,z)) \in [0, g(K_e,z)] \times [0, g(K_e,z)]$ such that

(a) $\hat{\Psi}(x_l(t,z),t,z) \leq 0$;

(b) $\hat{\Psi}(x^T(t,z),t,z) \geq 0$, and

(c) $x_l(t,z) \leq x^T(t,z)$ in the standard component product Euclidean order on R^J, $x^T(t,z) < g$.

As $\hat{\Psi}$ is strictly increasing and continuous jointly in x on $(0, g(K,z)]$ and nondecreasing and upper-semicontinuous on all of the order interval $[0, g(K,z)]$, it is is nondecreasing in x_j on $[0, g_j(K,z)]$ for fixed $x^j = (x_1, x_2, ...x_{j-1}, x_{j+1,}..., x_J)$, $x = (x_j, x^j)$ each $j \in J$, the mapping $\hat{\Psi}(x,t,z)$, therefore, satisfies the semicontinuity conditions of the generalized intermediate value theorem in arbitrary product spaces of Guillerme ([**?**], Theorem 3) on $[x_L(t,z), x^T(t,z)]$. Therefore,

[17] Notice, Assumption $3(\mathrm{i})'$ is *not* needed for our fixed point construction. Given the definition of ϕ, our operator is well-defined. Assumption Three guarantees that we have a non-trivial Markov equilibrium with non-zero production.

we conclude there exists a $x^*(t,z)$ such that g $\hat{\Psi}(x^*(t,z),t,z) = 0$; that is, correspondence $x^*(t,z)$ is nonempty on the suborder interval $[x_l(t,z), x^T(t,z)] \subset [0, g(K_e, z)]$ with $x^T(t,z) < g$.

For any $(t,z) \in \mathbf{T} \times Z \backslash \mathring{\mathbf{T}} \times \mathbf{Z}$, with $h_k = g_k$ by a similar argument, we have $x_k^*(t,z) = g_k$, and $x_{-k}^*(t,z)$ is constructed exactly as when $t \in \mathring{\mathbf{T}}$ (i.e., only using the block of functional inequalities $Z_{-k}(x_{-k}, g_k, t, z) \leq 0, Z_j = 0$ when $x_j^* > 0$. Elsewheere, $x^*(t,z) = 0$.

Therefore, $x^*(t,z) : \mathbf{T} \times \mathbf{Z} \rightarrow \mathbf{K}^J$ is non-empty for each $(t,z) \in \mathbf{T} \times \mathbf{Z}$.■

Remarks. First, observe as $\hat{\Psi}$ strictly increasing in x on $[0, g(K_e, z)]$, and noting the definition of $x_Z^*(t,z)$, $x_Z^*(t,z)$ *antichain*-valued for each $(t,z) \in \mathbf{T} \times \mathbf{Z}$ (where the antichain structure is relative to the componentwise order on R_+^{n-1}).[18]

Second, fix $\theta \in \Theta$, and consider $x(t,z) \in x_Z^*(t,z)$ for $t \in \mathbf{T}_+$. By the definition x_Z^*, for any $x(t,z) \in x_Z^*(t,z)$, we have $\hat{\Psi}(x(t,z),t,z) = 0$. Consider $t' \geq t, t \neq t', t$ and t' in $\mathring{\mathbf{T}}$. By Lemma 1, as $\hat{\Psi}$ is strictly antitone in t, we have $-\infty < \hat{z} = \hat{\Psi}(x(t,z),t',z) < 0$. Define the directed net $G(x(t,z)) = \{x | x \geq x(t,z), \hat{\Psi}(x,t,z) = z \geq 0 > \hat{z}, z \text{ finite}\}$. Compute the point $x^T = \sup G(x(t,z)) \in [0, g(K,z)]$, which exists in $[0, g(K,z]$ as $[0, g(K,z]$ is a complete sublattice of R_+^{m-1}. Compute $z^T = \hat{\Psi}(x^T, t', z) \geq z \geq 0$ where the inequality $z^T \geq z$ follows from the definition of join $\vee G(x(t,z))$ and the fact that $\hat{\Psi}$ is strictly increasing in x. By the isotonicity and continuity of $\hat{\Psi}$ in x, define $\hat{G}(x(t,z)) = \{x | z^T \geq \hat{\Psi}(x,t,z) \geq 0 > \hat{z}\}$ where, by construction, $x^T = \sup \hat{G} \subset [0,g]$. Also, note that, $\hat{G}(x(t,z))$ is nonempty (as certainly we have $x(t,z) \in G(x(t,z))$, and therefore, so is $\hat{G}(x(t,z))$.

Claim B: The correspondence $x^*(t,z) : \mathbf{T} \times \mathbf{Z} \rightarrow \mathbf{2}^{\mathbf{K}^J} \backslash \varnothing$ is ascending.

Proof: For $t \in \mathring{\mathbf{T}}$, by Lemma 1, as $\hat{\Psi}$ is (i) strictly increasing and continuous in x on $[0,g)$ and nondecreasing and upper semicontinuous in x on all of $\mathbf{R}_+^J$, (ii) the order interval $[0, g(K,z)]$ is convex (and, therefore, connected), (iii) by lemma 1(i)-(ii), and the argument directly above, $\hat{\Psi}(x^*(t,z),t',z) \leq 0$ for $t' \geq t$. Following the argument in Claim A, and the argument above, let the lower solution be $x_L(t',z) = x^*(t,z)$.By lemma 24(iii), noting there exists an upper solution $x^T(t',z)$ such that $\hat{\Psi}(x^T, t', z) \geq 0$, we can again apply the theorem of Guillerme ([**?**], Theorem 3) to $\hat{\Psi}(x,t,z)$ relative to the connected suborder interval $[x_L(t',z), x^T(t',z)] = [x(t,z), x^T(t',z)] \subset [0, g(K,z]$. We conclude that for any $x(t,z) \in x^*(t,z)$, there exists a element $x(t',z) \in x^*(t',z)$ such that $x(t,z) \leq x(t',z)$. This proves $x^*(t,z)$ is ascending upward in $\mathring{\mathbf{T}}$.

A similar argument shows that for any $x(t',z) \in x^*(t',z)$, there exists and $x(t,z) \in x^*(t,z)$ such that $x(t,z) \leq x(t',z)$. This proves $x^*(t,z)$ is ascending downwards in $\mathring{\mathbf{T}}$.

Noting the definition of $x^*(t,z)$ that extends $x^*(t,z)$ onto all of $\mathbf{T} \times \mathbf{Z}$, $x^* = 0$, we have $x^* : \mathbf{T} \times \mathbf{Z} \rightarrow \mathbf{K}^{n-1}$ ascending upward. Noting the condition is trivial

[18] Let X be a partially ordered set. We say a set $X_1 \subset X$ is an *antichain* if no two elements of X_1 are ordered. Antichains are necessarily chain complete.

when both $t, t' \in \mathbf{T}\backslash\mathring{\mathbf{T}}$, then the last case to consider has for any $t^{'} \in \mathring{\mathbf{T}}, t' \geq t$. Then, $x^*(t,z)$ is a singleton with $x^*(t,z) \leq x^*(t',z)$ for all $t \in \mathbf{T}\backslash\mathring{\mathbf{T}}$, fixed $z \in \mathbf{Z}$. We conclude $x^*(t,z) : \mathbf{T}_+ \times \mathbf{Z} \to \mathbf{K}^{n-1}$ is a nonempty correspondence that is ascending upward on $\mathring{\mathbf{T}} \times \mathbf{Z}$.

A dual argument shows $x^* : \mathbf{T} \times \mathbf{Z} \to \mathbf{K}^{n-1}$ is ascending downward on $\mathbf{T}\times\mathbf{Z}$.

Fix $z \in \mathbf{Z}$, and let $C \subset \mathbf{T}_+$ be a chain. Suppressing the notation for z for the moment, let $g(t) : C \to \mathbf{K}^{n-1}$ be an isotone map. Let $t_{0\vee} = \vee C$ (respectively, $t_{0\wedge} = \wedge C$), which exist as $\mathbf{T}_+$ is chain complete (actually, a complete lattice). Assume that $t_{0\vee} \neq \vee\mathbf{T}$ (resp, $t_{0\wedge} \neq \wedge\mathbf{T}$). For such a C, by $x^*(t,z)$ ascending upwards (A) and downwards (B),for any $t \leq t_{0\vee}$ (respectively, any $t_{0\wedge} \leq t$),there exists a $k_{0\vee} \in x^*(t_{0\vee})$ (respectively, $k_{0\wedge} \in x^*(t_{0\wedge})$ such that $g(t) \leq k_{0\vee}$ (respectively, $k_{0\wedge} \leq g(t)$) for all $t \in C \subset \mathbf{T}$ (noting the definition of $x^*(t,z)$ for any $t \in \mathbf{T}_+\backslash\mathring{\mathbf{T}}_+$). Therefore the correspondence $x^*(t,z)$ is chain faithful upward and chain faithful downward.

Now, let $t_{0\vee} = \vee\mathbf{T}$ (resp, $t_{0\wedge} = \wedge\mathbf{T}$). As $\vee\mathbf{K}^{n-1} \in x^*(t_{0\vee})$ (resp, $\wedge\mathbf{K}^{n-1} \in x^*(t_{0\wedge})$, the correspondence $x^*(t,z)$ is chain-faithful upwards and chain-faithful downwards. Therefore, by Höft's equivalence lemma (Lemma 2 above), as $\mathbf{T}_+$ and $\mathbf{K}^{n-1}$ are both complete lattices (and, therefore, chain complete), $x^*(t,z)$ is chain-faithful. ■

Claim C: $x^*(t,z) : \mathbf{T} \times \mathbf{Z} \to \mathbf{2}^{\mathbf{K}^J}\backslash\varnothing$ *is chain subcomplete valued for each* (t,z).

Proof: Recalling the remark above, $x^* : \mathring{\mathbf{T}} \times \mathbf{Z}$ is antichain-valued. It is, therefore, chain subcomplete-valued on $\mathring{\mathbf{T}} \times \mathbf{Z}$. Given the definition of the extension of $x^*(t,z)$ to $\mathbf{T} \times \mathbf{Z}$, we conclude that $x^* : \mathbf{T} \times \mathbf{Z} \to \mathbf{K}^J$ is antichained valued. Therefore $x^*(t,z)$ is chain subcomplete for each $(t,z) \in \mathbf{T} \times \mathbf{Z}$. ■

Then by Claims A-C, $x^*(t,z)$ satisfies the hypotheses of Lemma 25, therefore, $x^*(t,z)$ admits an isotone selection. $A(h)(S_e)$ is well-defined.■

Step (ii) $A(h) \in \mathbf{H}$

We note that by part (i) of the proposition, $A(h)(K,z)$ isotone in K with $A(h)(0,z) = 0$ for all z by definition. We, therefore, have when $K' \geq K$ and $K' \neq 0$, for equations $j = 1, 2, ..., n-2$

$$-\hat{\Psi}_1(A(h)(K', K', z, h)) + \Psi_2(A(h)(K,z)) \geq -\hat{\Psi}_1(A(h)(K,K,z,h)) + \Psi_2(A(h)) = 0$$

as $h \in \mathbf{H}$ (and, therefore, h such that $u'(g-h)(K_e,z)$ is antitone in K_e), and r decreasing in K. For equation $j = n-1$,by the assumption $u'(rx)r$ is increasing in r, we again have $\hat{\Psi}_{1n-1}$ increasing in K_e. Therefore $A(h)$ must be such that the vector of marginal utilities $u'(g-A(h))(K_e,z)$ decreasing in K_e, each $z \in Z$. This is true, in particular, when $K = \hat{K}$.

Further, as $\Psi_{j2}(x)$ is only a function of $g_j - A(h)_i$ for each j, by the concavity of the vector of marginal utilities $u(c)$, we have $A(h)(K_e,z)$ such that for each j, $g_j - A(h_j)$ is isotone in K_e, each z. Therefore as both $A(h)$ and $g - A(h)$ are isotone in K_e, and g is uniformly continuous in K, $A(h)$ is an element of an equicontinuous set of functions on the compact set $\mathbf{S}_e$). Therefore $A(h) \subset \mathbf{H}$.

Step (iii) $A(h)$ completely continuous on **H.**

Continuity of Ah follows from an standard argument for operators on compact subsets of $\mathbf{C}(\mathbf{S}_e)$ where $\mathbf{S}_e$ is compact (e.g., Coleman [11], Proposition 4). Further, as $A(H_1) = \{A(h)|h \in H_1 \subset \mathbf{H}\} \subset \mathbf{H}$ is relative compact for all bounded subsets $H_1 \subset \mathbf{H}, \mathbf{H}$ compact, the closure of $A(H_1)$ is compact. Therefore $A(h)$ is continuous and compact in h, i.e., completely continuous on **H**.

Step (iv) $A(h)$ isotone on **H.**

Consider any $h' \geq h$, with $h, h' \in \mathbf{H}$. Since Ψ is increasing in h and $\Psi(A(h), K, z, h) = 0$ by definition of the operator A, then $Z(A(h), K, z, h') \geq 0$. Consequently, with Ψ is strictly increasing in its first argument, by an argument similar to (i) for the existence of monotone selection, it must be that $A(h')(K, z) \geq A(h)(K, z)$ for all (K, z).

Step (v) $A(h)$ order continuous on **H.**

Consider an increasing sequence $h_{i\in\mathbf{N}} \subset \mathbf{H}$, $h^u = \sup(h_{i\in\mathbf{N}})$ where $\boldsymbol{N}$ denotes the natural numbers. The point h^u is well-defined as **H** is a complete lattice and therefore countably chain complete. As $A(h)$ is topologically continuous (in the uniform metric topology), we have additionally for $h_{i\in\boldsymbol{N}}$ the result that when $h_i \to h$ (uniformly) in **H**, $A(h_{i\in\boldsymbol{N}}) \to A(h^u)$ (uniformly) in **H**. Further, as **H** is equicontinuous, we have $\vee h_{i\in N} = h = h^u$ (see, for example, Heikkilä and Reffett [**?**], Lemma 4.1). Since Ah isotone, $A(h_{i\in\boldsymbol{N}}\ )$ is a countable chain. By topological continuity of $A(h)$, as $A(h)$and the fact that $\vee h_{i\in N} = h^u = h$, we have $\vee A(h_i) = A(h^u) = A(\lim h_{i\in\boldsymbol{N}})$. As $h^u = \vee(h_{i\in\boldsymbol{N}})$, we have $\vee(A(h_{i\in\boldsymbol{N}})) = A(\vee h_{i\in\boldsymbol{N}})$. As dual argument works for a decreasing chain $h_{i\in\boldsymbol{N}} \to h^u$ for $A(\wedge h_{i\in N}) = \wedge A(h_{i\in\boldsymbol{N}}) = \wedge A(h^u)$. Therefore $A(h)$ is order continuous in **H**.■

Proof of Theorem:

(i) The set **H** is a nonempty complete lattice, but (iv) the operator A is a isotone transformation of **H**. Therefore, by the main theorem in Tarski ([42], Theorem 1), the fixed pont set E is nonempty complete lattice.

(ii) and (iii). By Step (iii), we have $A(h)$ completely continuous. By Steps (iv) and (v), we have $A(h)$ is order continuous (and, therefore, necessarily isotone). Note $A(g) = g$ by definition, and therefore g is the greatest fixed point in **H**. As the hypotheses of a lemma in Vulikh ([**?**], XX) are satisfied for $A(h)$, the successive approximations $\lim_n A^n(0) = \sup_n A^n(0)$ converge to a least fixed point $h_l^* = \inf E \geq 0$. Given that $A(h)$ is completely continuous, and the equicontinuity of **H,** this convergence is uniform (see Amann ([**?**], Corollary 6.2)..■

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